Stochastic second order hyperbolic cauchy-problems as continuum limits of discrete difference equations

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Abstract

We consider (damped) stochastic wave type equations on $\mathbb{R}_{\geq 0} \times \mathbb{T}^q$,

$$\partial_{tt} u(t,x) = Au(t,x) + \partial_t Du(t,x) + f(t,x) + \sigma(t)W(t,x),$$

where by wave-type we mean that the equations are second order in time and the spatial differential operator A, aside from a reasonable lower order perturbation, is a essentially $(-1)^{m-1}\Delta^m$ ($m \in \mathbb{N}^*$) and σW is an additive space-time white noise.

The aim of this thesis is to study how these equations emerge as limiting equations for discrete equations of the form

$$\partial_{tt} X_N(t,l) = A_N X_N(t,l) + \partial_t D_N X_N(t,l) + g_N(t,l,X_N(t,l)) + \sigma B_N(t,l),$$

whose spatial variables live on an equidistant, cubic, Bravais lattice, that is a discretization of the spatial torus $\mathbb{T}^q$. Where $(A_N)^*_{N\in\mathbb{N}}$ and $(D_N)_{N\in\mathbb{N}^*}$ constitute sequences of discrete difference quotient operators, whose continuous realizations converge to A respectively D in the strong operator topology and B_N is a spatial L_1-average of W, with respect to the step size of the lattice.

We first consider linear discrete equations, here $(g_N)_{N\in\mathbb{N}} = (f_N)^*_{N\in\mathbb{N}}$ is a sequence approximating f, in a space of continuous functions with values in a suitable Sobolev space and we show that, maximal regularity aside, the solutions of the discrete equations approximate the solution of the continuous equation, and in turn get approximated by spatially averaged, orthogonal projections of the continuous solution. Thus proving a Cauchy-Born rule under the influence of space-time white noise.

While in a second step we consider renormalized continuum limits of non-linear equations. Here $g_N(t,l,X_N(t,l)) = f_N(t,l) + b(X_N(t,l))$, is the sum of the function f_N from the linear part and a polynomial non-linearity, rescaled to compensate the divergence in the limit, that will either disappear in the limit or spawn a linear term, depending on the strength of the rescaling compared to the non-linearity's speed of divergence. Here we determine a candidate for the scaling under which the renormalization into a linear term occurs and show that for stronger rescaled non-linearities the term vanishes in the limit, for odd powered non-linearities a small gap between the critical scaling and the vanishing scalings can be observed.

Notation

As basis of all further notation we will use the following generic notations and conventions. $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ denote generic Banach spaces, $(H, \langle\cdot,\cdot\rangle_H)$ is a generic Hilbert space and $U \subset \mathbb{R}^q$ an open, non-empty domain in $\mathbb{R}^q$. Given any space of E-valued functions or distributions $X(U,E)$, we denote the real valued version $X(U,\mathbb{R})$ by $X(U)$. And unless needed for clarification the measure and sigma-field will be omitted in the L_p spaces and all derived function spaces.

General Notations

$\mathbb{N}$	$\{0,1,2\dots\}$ the natural numbers
$\mathbb{N}^*$	$\{1,2\dots\}$ the natural numbers without zero
$\mathbb{N}^q$	set of all multi indices
$\mathbb{Z}$	integers
$\mathbb{Z}^q$	set of all Fourier indices
$(\mathbb{Z}^q)^p$	set of all Fourier multi indices
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{R}^q$	euclidean vector space
$\mathbb{C}$	complex numbers
$\mathbb{K}$	placeholder field for $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{B}_{\mathbb{K}^q}(c,r)$	open $\mathbb{K}^q$ ball with center $c \in \mathbb{K}$ and radius $r > 0$
$\mathbb{B}_{\mathbb{Z}^q}(c,r)$	$\mathbb{B}_{\mathbb{K}^q}(c,r) \cap \mathbb{Z}^q$
$\mathbb{T}^q$	$[-\pi,\pi)^q$ the q-dimensional torus
Q_N^q	$\left\{-\pi, \frac{-\pi+1}{N}, \dots, \frac{\pi-2}{N}, \frac{\pi-1}{N}\right\}^q$ equidistant grid discretization of $\mathbb{T}^q$
$\hat{Q}_N^q$	$\{-N, -N+1, -N+2, \dots, N-2, N-2\}$ the Fourier dual of Q_N^q
$C^q(l)$	$\bigtimes_{j=1}^{q} \left[l_j, l_j + \frac{\pi}{N}\right]$ averaging cube extended from $l \in Q_N^q$
S_n	Permutation group of $\{1,2,\dots,n\}$ for $n \in \mathbb{N}$
$n!!$	double factorial of $n \in \mathbb{N}$
e_j	canonical basis vector in $\mathbb{R}^q$ for $j \in \{1,\dots,q\}$
A^T	transposed of $A \in \mathbb{R}^{q\times q}$
$\overline{\lim}$	$\limsup$

Specific Distributions, Functions, Operators and Maps

$\langle x \rangle$	$(1+\langle x,x\rangle_{\mathbb{R}^q})^{1/2}$ standard Fourier weight
$\hat{u}, \mathcal{F}u$	Fourier transform
$\mathcal{F}^{-1}u$	inverse Fourier transform
$\hat{u}^d, \mathcal{F}^d u$	discrete Fourier transform
$(\mathcal{F}^d)^{-1}u$	inverse discrete Fourier transform
$f * g$	convolution of (generalized) functions
$f *_d g$	discrete convolution of functions
$\mathrm{supp}(u)$	support of a function or distribution
Id_E	Identity map in E
Id	Identity map, when the space is clear
$\mathbb{1}_M$	indicator function of a set (sub-)set M
π_j	$[x \mapsto x_j]$ the j-th coordinate projection
$\lfloor x \rfloor$	the floor function $[x \mapsto \max\{j \in \mathbb{Z} \mid j \leq x\}]$
δ_{x_0}	Dirac δ distribution at $x_0 \in \mathbb{R}^q$
$\delta_{x,y}$	Kronecker delta
Δ	Laplace operator
$\Delta_{N,d}$	discrete Laplace operator
$\tilde{T}_{h,j}$	right translation operator, shifting by $h \in \mathbb{R}$ in direction e_j, over the continuum
$T_{h,j}$	discrete right translation operator, shifting periodically by a multiple of the underlying grids step size in direction e_j
$\tilde{\partial}_{h,j}$	partial difference quotient operator, with error $h \in \mathbb{R}$, in direction e_j
$\partial_{h,j}$	discrete partial difference quotient operator, with error $h \in \mathbb{R}$ being a multiple of the underlying grids step size, in direction e_j
$w(N,j)$	$\begin{cases} 1, & j=0 \\ \frac{\exp(\frac{ij\pi}{N})-1}{\frac{ij\pi}{N}}, & j \neq 0 \end{cases}$ averaging weight of the j-th Fourier coefficient for $j \in \mathbb{Z}$
$w(N,j,q)$	$\prod_{l=1}^{q} w(N,j_l)$ averaging weight of the j-th Fourier coefficient for $j \in \mathbb{Z}^q$
$\beta_{q,m}(N)$	$\begin{cases} 1, & q<2m, \\ \log(N), & q=2m, \\ N^{q-2m}, & q \geq 2m+1, \end{cases}$ Scaling function for dimension q and operator order $2m$
$\beta_q(N)$	$\beta_{q,1}(N)$
$f(x) = \mathcal{O}(g(x))$	Landau Big-$\mathcal{O}$ notation

Spaces of Functions and Distributions

$C(U)$	continuous functions
$C_c(U)$	continuous functions with compact support
$C^n(U)$	n-times continuously differentiable functions
$C_c^n(U)$	$C^n(U) \cap C_c(U)$
$BC(\mathbb{R}_{\geq 0})$	Bounded and continuous functions mapping $\mathbb{R}_{\geq 0}$ to $\mathbb{R}$
$C^\infty(U)$	$\bigcap_{n \in \mathbb{N}} C^n(U)$
$C_c^\infty(U)$	$\bigcap_{n \in \mathbb{N}} C_c^n(U)$
$C([0,T],E)$	continuous. E-valued functions on $[0,T],\ T>0$
$\mathcal{D}(U)$	space of testfunctions
$\mathcal{D}'(U)$	space of all distributions
$\mathcal{S}(U)$	Schwartz space of tempered functions
$\mathcal{S}'(U)$	Schwartz space of tempered distributions
$\mathcal{L}_0(U,\mathcal{B}(U),\lambda^q,\mathbb{K})$	$\mathcal{B}(U)-\mathcal{B}(\mathbb{K})$-measurable functions
$L_p([0,T],E)$	Bochner-Lebesgue space of p-integrable E- valued functions
$L_p(U)$	usual Lebesgue space for $p \geq 0$
$L_{p,loc}(U)$	$\{f \in \mathcal{L}_0(U,\mathcal{B}(U),\lambda^q,\mathbb{K}) \mid \mathbb{1}_K f \in L_p(U)\ \forall K \Subset U\}$ locally p-integrable functions
$W_p^n(U)$	classical Sobolev space of integer order n and integrability $p \geq 1$
$H^m(U)$	classical Sobolev Hilbert space of integer order
$H^s(U)$	classical Sobolev Hilbert space of fractional order $s \in \mathbb{R}$
$H^s(\mathbb{T}^q)$	Fourier Sobolev Hilbert space of fractional order $s \in \mathbb{R}$ over the torus $\mathbb{T}^q$
$h^s(Q_N^q)$	periodic discrete Sobolev Hilbert space over the torus grid Q_N^q
$h^s(\hat{Q}_N^q)$	Fourier dual of $h^s(Q_N^q)$
$l_2(Q_N^q)$	discrete periodic l_2 space
$l_2(\hat{Q}_N^q)$	Fourier dual of $l_2(Q_N^q)$

Functional Analysis Notation

V^*	algebraic dual space of a vector space V
V'	topological dual space of a vector space V
$(A, \mathrm{dom}(A))$	linear operator A with domain $\mathrm{dom}(A)$
$(A^*, \mathrm{dom}(A^*))$	Hilbert space adjoint operator of A, with domain $\mathrm{dom}(A^*)$
$\|x\|_{D(A)}$	$\|x\|_E + \|Ax\|_F$ graph norm of an Operator $A: E \to F$
$D(A)$	$(\mathrm{dom}(A), \|\cdot\|_{D(A)})$
$\mathcal{L}(E,F)$	Set of bounded linear Operators mapping E into F
$\mathcal{L}(E)$	$\mathcal{L}(E,E)$
$\mathcal{A}(E,F)$	Set of closed linear Operators mapping E into F
$\mathcal{A}(E)$	$\mathcal{A}(E,E)$
$\mathcal{G}(E,M,\omega)$	Set of generators of C_0-semigroups
$\mathcal{G}(E,1,0)$	Set of generators of contraction semigroups on E
$\mathcal{G}(E)$	Set of all generators of C_0-semigroups on E
$\mathcal{A}_N(A)$	Set of all sequences, of discrete operators, admissible for a PDO A
$\omega(\mathbb{T})$	growth rate of the operator semigroup $\mathbb{T}$
$ran(A)$	image of the operator A
$\rho(A)$	resolvent set of the operator A
$\sigma(A)$	spectrum of the operator A
$R(\lambda, A)$	resolvent Map
$p(A,\xi)$	The polynomial describing the differential operator A and its eigenvalues

P_M	orthogonal projector onto a subspace $M \subset H$
$P^\perp$	$Id_H - P$ complementary projector to the projector P on H
P_N^a	Projector onto $\mathrm{span}\{\phi_j \mid j \in [-N, N+1]^q\}$, with respect to the orthonormal basis $\Phi = \{\phi_j\}_{j \in \mathbb{Z}^q}$, as defined below
P_δ^a	$P^a_{\lfloor N^\delta \rfloor}$ for $\delta \in (0,1)$
Φ	A generic orthonormal basis in the general theory. And everywhere else $\Phi = \{\varphi_j \mid j \in \mathbb{Z}^q\} = \{e^{i\langle \cdot, j\rangle_{\mathbb{R}^q}} : \mathbb{R}^q \to \mathbb{C} \mid j \in \mathbb{Z}^q\}$ the canonical orthonormal basis of $L_2(\mathbb{T}^q)$.
$\Phi_{N,d}$	$\{e^{i\langle \cdot, j\rangle_{\mathbb{R}^q}} : \hat{Q}_N^q \to \mathbb{C} \mid j \in Q_N^q\}$ orthonormal basis of $l_2(Q_N^q)$
$\hat{\Phi}_{N,d}$	$\{e^{i\langle \cdot, j\rangle_{\mathbb{R}^q}} : Q_N^q \to \mathbb{C} \mid j \in \hat{Q}_N^q\}$ orthonormal basis of $l_2(\hat{Q}_N^q$
$A(x,D)$	Partial differential operator in the differential D, and with coefficients in x
$A(D)$	Partial differential operator in the differential D, with constant coefficients
$SOT-\lim$	limit with respect to the string operator topology

$E \hookrightarrow\hookrightarrow F$	E is compactly embedded into F
$K \Subset M$	the set K is compactly contained in the set M

Multi and Fourier Indices

$\lvert\alpha\rvert$	$\sum_{j=1}^{q} \alpha_j$ Length of the multi index $\alpha \in \mathbb{N}^q$
x^α	$\prod_{j=1}^{q} x_j^{\alpha_j}$ for $\alpha \in \mathbb{N}^q$ and $x \in \mathbb{R}^q$
$\partial^\alpha u$	$\frac{\partial^{\lvert\alpha\rvert}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_q}^{\alpha_q}} u$ for $\alpha \in \mathbb{N}^q$
$D^\alpha u$	$(-i\partial)^\alpha u$ or $\partial^\alpha u$
$\mathcal{M}(\mathbb{Z}^q, j, p)$	Set of al Fourier multi indices $\alpha \in (\mathbb{Z}^q)^p$ of length p with j matches
$M_{\tilde{j}}^{j}(\alpha)$	$\mathbb{1}_{\mathcal{M}(\mathbb{Z}^q, j, \tilde{j})}$ match function for Fourier multi indices $\alpha \in (\mathbb{Z}^q)^p$
$\tilde{m}(\alpha)$	Match number of a Fourier multi index $\alpha \in (\mathbb{Z}^q)^p$
$\alpha(k)$	$\left(\alpha, k - \sum_{j=1}^{p} \alpha_j\right)$ for any $\alpha \in (\mathbb{Z}^q)^p$
u_α	$\prod_{j=1}^{p} U_{\alpha_j}$ for any $\alpha \in (\mathbb{Z}^q)^p$

Probability and Measure Theory

$X \sim \mathcal{N}(\mu, \Sigma^2)$	X is Gaussian with mean μ and covariance Σ^2
$X \sim \mathcal{CN}(\mu, \Sigma^2, \Gamma)$	X is complex Gaussian with mean $\mathbb{E}[X] = \mu$, $\mathbb{E}\left[XX^T\right] = \Sigma^2$ and $\mathbb{E}\left[\lvert X\rvert^2\right] = \Gamma$
$\mathcal{P}_T$	The predictable-σ-algebra
$\mathcal{P}_T(\Omega, E)$	Set of all E valued predictable processes
$\mathcal{H}_0$	counting measure
$\{X \neq Y\}$	$\{(\omega, t) \in \Omega \times T \mid X_t(\omega) \neq Y_t(\omega)\}$
$\xrightarrow{\mathbb{P}}$	convergence in probability

Contents

Introduction

„[...], *the real understanding involves, I believe, a synthesis of the discrete and continuous,* [...].“[1)]

The aim of this dissertation is to study stochastic partial differential equations (SPDEs) of the form

$$\partial_{tt}u(t,x) = Au(t,x) + \partial_t Du(t,x) + f(t,x) + \sigma(t)W(t,x), \tag{SPDE}$$

on the torus $\mathbb{T}^q = (\mathbb{R}/2\pi\mathbb{Z})^q$, as limiting equations of corresponding discrete equations

$$\partial_{tt}X_N(t,l) = A_N X_N(t,l) + \partial_t D_N X_N(t,l) + g_N(t,l,X_N(t,l)) + \sigma B_N(t,l), \tag{SPDQE}$$

on the equidistant, discrete, defectless, cubic lattices $\frac{\pi}{N}\{-N,\ldots,N-1\}^q$.
Here the spatial partial differential operator (PDO) A is essentially an even order, self-adjoint, negative PDO (think some power Δ^p, $p \in \mathbb{N}^*$ of the Laplace operator), the damping operator D is either a non-positive constant or a positive multiple of the Laplacian, the forcing f is continuous in time while taking values in some Sobolev- Hilbert-space and $\sigma(t)W(t,x)$ is an additive space time white noise.
While on the discrete side A_N, D_N, are discrete operators (polynomials in the translation operators with shift size equal to the step size of the lattice) with continuous realizations $\tilde{A}_N, \tilde{D}_N$, that converge to A and D in the strong operator topology respectively and $\sigma B_N(t,l)$ is a spatial L_1-average of $\sigma(t)W(t,x)$. In the first part, where we deal with linear equations, g_N is a function f_N independent of X_N approximating f uniformly in time, whereas in part two an additional polynomial non-linear term, rescaled by some power of N, is added to the forcing f_N, from the linear part. Rigorous definitions and conditions will be given at a later point in this thesis.

This type of equation first appeared in the 1960s [Gro67], as infinite dimensional Itō-equations, while the study of SPDEs as such is commonly associated with the 1970s. From thereon three main approaches towards SPDEs were developed, the semigroup approach ([DPZ14]), the variational approach ([PR07] and [GM11]) and the martingale measure approach that goes back to Walsh ([Wal86]) and was later extended (see e.g. [D+99]), as the original theory can not integrate the stochastic wave equations Green's function for dimensions $q \geq 2$ (see [DKM+09]), and studied in various schools.(Italian, Russian, etc.) For a more detailed, though self-proclaimed biased, account on the history of SPDEs see [Zam20]. Now we have to choose which of the two approaches to model our space time white noise we use, while these approaches and their stochastic integrals are (essentially) equivalent (see [DQS11],[Jet86]), working with the Da Prato-Zabczyk approach, has the advantages that, in some sense, it is a linear extension of the deterministic semigroup approach and comes with a Fourier series expansion, which will prove very useful in the proof of our main results, while simultaneously working in the deterministic setting.

Now in the linear setting, we prove a Cauchy-Born rule, with noise, for interacting particle systems, on a cubic lattice $\frac{\pi}{N}\{-N,\ldots,N-1\}$, in the linearized regime, with interaction potential V. Indeed (SPDQE) is the result of introducing a macroscopic time $\frac{\pi}{N}t$ to the newton equation of motion, for the aforementioned particle system, whose interaction potential V is encoded by the discrete difference operator corresponding to A_N, with corresponding damping, forcing and noise.

1) [Lov10]

To visualize this consider the following simplified situation, without damping, forcing and noise, of a particle system, of $2N$ particles, in one dimension, with nearest neighbour interactions according to the potential $V(x) = \frac{1}{2}\left(x - \frac{\pi}{N}\right)^2$. Assuming $\frac{\pi}{N}i$ to be the equilibrium position of the particle labelled by $i \in \{-N, \dots, N-1\}$ the Newton-Equation of motion read:

$$\frac{d^2 X_N(t,l)}{dt^2} = X_N(t,l+1) - 2X_N(t,l) + X_N(t,l-1)$$

Now introducing the macroscopic time $\tau = \frac{\pi}{N}t$ yields

$$\frac{d^2 X_N(\tau,l)}{d\tau^2} = \Delta_{N,d} X(\tau,l).$$

This corresponds exactly to the situation $A_N = \Delta_{N,d}$ and $A = \Delta$, and for suitably regular initial data the deterministic version of this parts main result gives the first result of [BLL12]. Due to the regularity or rather the lack thereof their proof using a factorization argument doesn't carry over and we instead we combine the mild solution with Fourier analysis to extend the result such that it can be applied towards arbitrarily irregular initial data and a stochastic forcing. In the deterministic setting these kind of results have seen extensive investigation covering a wider range of lattice types, including defects, and potentials than this dissertation, (see for example [BLL12],[BS16],[OT13] and [BLBL07]) because of their applications in chemistry and material sciences.

In the second half due to irregular nature of space-time white noise ([Ver10]) once the dimension reaches the order of the differential Operator A, its regularizing ability will be exhausted and the solutions of (SPDE), will no longer be function valued and instead become generalized stochastic processes ([Sch18]), taking values in a space of (Schwartz) distributions. Whilst this has no effect on the treatment of linear equations using the semigroup approach, aside from changing the Hilbert space from a Lebesgue space to a Sobolev space, the impact on non-linear equations is considerably higher. As already remarked by Schwartz [Sch54] a simultaneous faithful extension of multiplication and derivation, that covers all distributions, is impossible, this raises once more the question if a non-linear SPDE is even well-posed. For parabolic (and sometimes elliptic) SPDEs an answer to this was given in the form of the fields-medal earning contributions, by Hairer [Hai14], while hyperbolic SPDE's like the stochastic wave [GKO18b],[GKO18a] or sine-Gordon equation [HS16] are treated according to the paracontrolled approach, using Wick renormalizations and the Da-Prato-Debussche decomposition. Our situation may be comparably easier, as only the discrete equations are non-linear and the divergence only occurs in the limit, however this does not prevent us from modifying these tools to our needs. Similar results to ours have already been obtained for parabolic [HM+18] and elliptic [MSS06] equations, therefore there now exists a treatment of each type of the trinity.

So far we have discussed the contents of this thesis in the context of other works, in the fields of (stochastic) analysis and continuum limits, but before we outline the structure of this thesis, let us summarize what was achieved in this thesis:

In part one we are solving (SPDE) for any negative partial differential Operator of order $2m$ ($m \in \mathbb{N}^*$), perturbed by any differential operator of at most order m, in any space dimension $q \in \mathbb{R}$, either undamped, damped by a constant rate or strongly damped by a multiple of a standard Laplacian, for any regularity, that the regularity of the data, forcing and space time white noise allow.
We then do the same with (SPDQE) for partial differential quotient operators A_N and D_N, whose continuum realizations converge to A respectively D in the strong operator topology, and g_N approximating f.

Now the aforementioned Cauchy-Born rule can be obtained in the following way:
We represent both our partial difference quotient operators and a linearized interaction potentials, in the corresponding Newton equations, in terms of translation operators (shifts), by introducing

the macroscopic time $\tau := \sqrt{\frac{\pi}{N}}^m t$ in the Newton equations and multiplying the equation with $\left(\frac{\pi}{N}\right)^m$, those equations will coincide. By applying one of our main results, Theorem 3.3.1 or Theorem 3.3.1, we obtain the particle systems governing continuum equation. Among the partial differential equations that can appear as continuum limits, of these discrete systems, are the (stochastic) wave, plate and telegraph equation.

In part two g_N not only consists of an approximation of the forcing term f, but also a non-linear function $b(Z_N)$, that is either a scaled polynomial or the composition of a a scaling function (in N) and a polynomial, in the solution Z_N of the, now non-linear, equation (SPDQE). By converting them into discrete convolutions, of the linear solutions Fourier coefficients, we compute the scaling of the moments of X_N, in N, such that we can derive a critical scaling power of N, for which $b(X_N)$ stays bounded between two constants independent of N ($\mathbb{P}$-a.s.), and a subcritical level for which $b(X_N)$ vanishes, for $N \to \infty$. To obtain the lower bound for the scaling we need an additional ellipticity condition on the partial differential operator A.
We then want to turn this knowledge about the linear solution into knowledge about the non-linear solution Z_N. By differentiating their difference, twice in time, we obtain a random partial differential equation for their difference, in which the non-linearity can be understood as a forcing. Using a Gronwall estimates and our moment bounds, we can show that the solution of the non-linear equation scales like the solution of the linear equation, and converges to the same linear continuum solution. For even ordered polynomials this works in the full subcritical regime, while for even ordered monomials the linear solution recreates a $h^s(Q_N^q)$-norm, when computing the scaling, as we have no a priori knowledge whether the same happens for the non-linear solution we are left with a gap of $\frac{q-2}{2}$, were q is our dimension, in the subcritical regime were we can't say anything about Z_N, based on our knowledge about X_N.

Finally we give an overview of how this dissertation is organized. This thesis consist of two parts, not counting the preliminaries, as well as an appendix.

The preliminaries will recall important results on (stochastic) partial differential equations, which will reappear later in the main part of this dissertation, as well as most of the notation we will utilize. While, in the interest of keeping this thesis reasonably self-contained, we aim at a well-rounded presentation of the topics covered in this part, their treatment is far from comprehensive, as most of them have given rise to various monographs on themselves, a selection of which will be provided in the respective sections.

In part I we will develop the continuum limit theory, for linear equations, with the main results as well as some examples of wave-type being given in chapter 3. While the regularity w.r.t. time of the mild solution is the same in the deterministic and the stochastic case (as usually with a $\mathbb{P}$-a.s. in the stochastic case), this doesn't carry over to the continuum limit as we would need a general result allowing us to interchange the expectation operator with a supremum in time, e.g. a bound for the moments of said supremum by the supremum of the errors moments. Nevertheless we obtain convergence for all and not just one moment in time. The discrete versions of these continuum limit theorems on the other hand are merely the second side of the respective same coin, showing, to say it with [Lov10], that not only the large finite approximates the infinite[2)], but also conversely an approximation of the large finite can be obtained from the continuum. The subsequent chapters are dedicated to various steps of the main results proofs. In Chapter4 we give a proof of existence, of a mild solution, that is a modified version of the proof for the wave equation, using the spectral calculus for self-adjoint operators instead of the integration by parts. The fact that our differential operators and their discrete counterparts act as multiplication operators, under the respective Fourier transform, allows to determine a representation of the semigroup on the Fourier space, via explicitly solving an ordinary second order oscillation differential equation. In Chapter 5 we will then develop the error estimates that we combine in Chapter 6 to complete the proof of our main results. When developing these error estimates the high frequency Fourier modes have to be

2) While both perspectives are valid, this will be the mindset assumed throughout part I.

treated different from the low frequency Fourier modes, therefore we introduce a Fourier cutoff, by virtue of an orthogonal projection in the Fourier space.

In the second part we turn to nonlinear discrete equations, with a polynomial nonlinearity, that is rescaled to avoid an immediat universal blow up of the limiting equation. Using the „Da Prato-Debusche trick“we establish the linear solution as a basis for a priori knowledge and, by its gaussianity, this leads to the assumption that the limiting equation will again be linear. By the linear continuum limit, established in part I, we can obtain the moment bounds, controlling the nonlinearoty evaluated at the linear solution, that give us our assumed critical scaling, for the emergence of a renormalization term, by computing with the continous solution. As we require our inital data and forcing now to be actual functions and not distributions our critical scaling depends solely on the stochastic convolution term. As we assume a bound on the symbol of our differentialoperator A, that lets it behave similar to the linear wave propagator in [GKO18b] and [GKO18a], it is not surprising that our stochastic convolution scales in a way, that generalizing their findings. Finally we prove critically of the scaling, by showing that for stronger rescaling of the non-linearity, it actually collapses to zero. For even powers this result is optimal, in that the non-linearity collapses for any scaling larger than our critical candidate. For odd non-linearitys on the other hand there is a gap depending on the spatial dimension, as the discrete convolutions for the linear solution reproduce a Sobolev norm, which is an effect we a priori can't assume for the non-linear solution.

The appendix serves as a place to present some results that either appear in a more standalone way, atleast in the context of this thesis, or are basic computations outsourced from the main part.

Some of the proofs contained in this work are not fully justified. As a consequence, while we keep the usual nomenclature for easy readability, the mathematical results stated as ›Theorems ‹, ›Corollaries ‹and ›Lemma ‹are to be understood as (well-founded) conjectures and their ›Proofs ‹as evidence that substantiate these statements.

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Preliminaries

1 Functional Analysis and PD(Q)E

In this chapter we will recall some functional analytic facts related to abstract Cauchy problems, that take the form of partial differential equations, if the equation is posed in a space of (generalized) functions over a continuum, respectively partial difference quotient equations, if the functions are defined on a discrete space. While the aim here is to present all the necessary tools in a meaningful context, the treatment will be far from depletive, for that we provide further references. A universal reference (atleast for the continuum theory) will be the first monograph in the trilogy by Niles Jacob [Jac01], with more specific references given at the beginning of each section.

1.1 Operatorsemigroups

Let $A : \mathrm{dom}(A) \subset E \to E$ a linear operator and $u_0 \in E$. Consider the **Abstract Cauchy-Problem**

$$\begin{cases} \partial_t u = Au, \\ u(0) = u_0. \end{cases} \tag{ACP}$$

A solution of (ACP) is a function $u \in C^1((0,\infty), E) \cap C([0,\infty), E)$, such that $u(t) \in dom(A)$ for all $t > 0$ and u solves (ACP) pointwise in E. If $A \in \mathcal{L}(E)$ the solution is given by

$$e^{tA}u_0 := \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} u_0.$$

But for unbounded operators, like differential operators this is generally not defined, as it is not even clear if $u_0 \in dom(A^k)$ for every $k \in \mathbb{N}_0$.

Naturally we reference Hille [HP96] and Yosida [Yos94], further references for this section are [Eva10], [Paz12] and [BR84].

1.1.1 Strongly Continuous Semigroups and their Generators

The Operators that generate C_0-semigroups can be found in the following class of potentially unbounded operators, though not every closed operator generates a C_0-semigroup, the required spectral properties will be specified when we give the classical characterization results.

Definition 1.1.1 (Closed Operator)
*A linear Operator $A : dom(A) \subset E \to F$ is called **closed** , if for every sequence $(x_n)_{n\in\mathbb{N}} \subset dom(A)$ we have:*

$$\left.\begin{aligned} x_n &\to x \text{ in } E \\ Ax_n &\to y \text{ in } F \end{aligned}\right\} \Longrightarrow x \in dom(A) \text{ with } Ax = y.$$

Notation 1.1.2 (Closed Operator)

$$\begin{aligned} \mathcal{A}(E,F) :=& \{A : dom(A) \subset E \to F \text{ linear operator} \mid A \text{ is closed}\}. \\ \mathcal{A}(E) :=& \mathcal{A}(E,E). \end{aligned}$$

Remark 1.1.3
Note that $\mathcal{A}(E,F)$ in general is not a vector space as it is generally not closed under addition.

Nevertheless as long as a perturbation stays bounded, the sum stays closed as the following theorem shows.

Theorem 1.1.4
Let $A \in \mathcal{A}(E,F)$ and $B \in \mathcal{L}(E,F)$, then $A+B \in \mathcal{A}(E,F)$.

Proof
Let $(x_n)_{n\in\mathbb{N}} \subset \mathrm{dom}(A+B) = \mathrm{dom}(A)$ and $(x,y) \in E \times F$, such that

$$x_n \to x \text{ in } E \text{ and } (A+B)x_n \to y \text{ in } F.$$

As $B \in \mathcal{L}(E,F)$, $Bx_n \to Bx$ holds and therefore,

$$\left.\begin{aligned} x_n \to x \text{ in } E \\ Ax_n \to y - Bx \text{ in } F \end{aligned}\right\} \overset{A\in\mathcal{A}(E)F}{\Longrightarrow} x \in \mathrm{dom}(A) \text{ with } Ax = y - Bx.$$

This is $x \in \mathrm{dom}(A+B)$ with $(A+B)x = y$, and therefore $A+B \in \mathcal{A}(E,F)$. ■

Definition 1.1.5
*For a linear Operator $A : dom(A) \subset E \to F$ we define $D(A) := (dom(A), \|\cdot\|_{D(A)})$, where $\|\cdot\|_{D(A)}$ is the **Graph norm** (of A) is defined via*

$$\|x\|_{D(A)} := \|x\|_E + \|Ax\|_F \quad \forall\, x \in dom(A).$$

Now for a linear operator and its domain equipped with the graph norm we have the following properties.

Theorem 1.1.6
Let $A : dom(A) \subset E \to F$ a linear Operator. Then

(i) $D(A)$ *is a normed vector space* $D(A) \hookrightarrow E$,

(ii) $A \in \mathcal{L}(D(A),F)$,

and

$$A \in \mathcal{A}(E,F) \Longleftrightarrow D(A) \text{ is a Banachspace.}$$

Now we define C_0-semigroups, the generalization of the exponential e^{tA},while it is common (though informal) to carry over the expression e^{tA}, we will instead use the more formal $T(t)$ notation.

Definition 1.1.7 (C_0-Semigroup)
Let $\mathbb{T} := \{T(t) \mid t \geq 0\} \subset \mathcal{L}(E)$ a one parameter family of bounded operators.
*$\mathbb{T}$ is called a **strongly continuous** or C_0 **semigroup**, if*

- *$T(0) = Id_E$. (normalization)*
- *$T(t+s) = T(t)T(s)$ for all $t,s \geq 0$. (semigroup property)*
- *$\mathbb{T}$ is strongly continuous in $t_0 = 0$, i.e. the map $[t \mapsto T(t)x] : \mathbb{R}_{\geq 0} \to E$ is continuous in $t_0 = 0$ for every $x \in E$.*

Next we give a basic growth estimate on semi groups, that will be used in the estimates for the high-frequency terms, in the proof of our linear main results.

Lemma 1.1.8 (Evolution Estimate)
Let $\mathbb{T} := \{T(t) \mid t \geq 0\}$ a C_0-semi group, then there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\|_{\mathcal{L}(E)} \leq Me^{\omega t}.$$

The previous estimate immediately inspires the following definitions.

Definition 1.1.9
Let $\mathbb{T} := \{T(t) \mid t \geq 0\}$ a C_0 semigroup.

(a) The ***growth rate*** *$\omega(\mathbb{T})$ of $\mathbb{T}$ is defined as*

$$\omega(\mathbb{T}) := \inf\{\omega \in \mathbb{R} \mid \sup_{t\geq 0} e^{-\omega t} \|T(t)\|_{\mathcal{L}(E)} < \infty\}.$$

(b) $\mathbb{T}$ is called a ***contraction semigroup*** *if Lemma 1.1.8 holds with $\omega = 0$ and $M = 1$.*

Another important consequence of Lemma 1.1.8 is that strong continuity means that $\mathbb{T} : \mathbb{R}_{\geq 0} \to \mathcal{L}(E)$ is continuous with respect to the strong operator topology.

Corollary 1.1.10
Let $\mathbb{T} := \{T(t) \mid t \geq 0\}$ a C_0-semigroup. Then for every $x \in E$ we have $[t \mapsto T(t)x] \in C([0,\infty), E)$.

Now we define what it means for an operator to generate a C_0-semigroup. (We define it in an intuitive way as the right-sided time derivative of the semigroup evaluated at $t = 0$ on a suitable domain.)

Definition 1.1.11 (Infinitesimal Generator)
Let $\mathbb{T} := \{T(t) \mid t \geq 0\}$ a C_0-semigroup. Define

$$dom(A) := \left\{ x \in E \mid \lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exist in } E \right\}$$

and

$$Ax := \lim_{t\searrow 0} \frac{T(t)x - x}{t} \quad \text{for all } x \in dom(A).$$

The linear Operator $(dom(A), A)$ is called the ***infinitesimal generator*** *(or* ***generator*** *for short) of $\mathbb{T}$.*

The following well-known theorem gives some properties describing the interaction of an operator and the semigroup generated by it, which is used in solving (ACP).

Theorem 1.1.12
Let $\mathbb{T} := \{T_1(t) \mid t \geq 0\}$ a C_0-semigroup with generator A.

(i) For all $t \geq 0$ and all $x \in E$ we have:

$$\lim_{h\searrow 0} \frac{1}{h} \int_t^{h+t} T(s)x \, ds = T(t)x.$$

(ii) For all $t \geq 0$ and all $x \in E$ we have:

$$\int_0^t T(s)x \, dx \in dom(A) \quad \text{and} \quad A\left(\int_0^t T(s)x \, dx \right) = T(t)x - x.$$

(iii) For all $t \geq 0$ and all $x \in dom(A)$ we have: $T(t)x \in dom(A)$ and

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax.$$

A direct consequence of this theorem is that we can establish a one to one relation between semigroups and their generators.

Corollary 1.1.13
Let $\mathbb{T}_1 := \{T_1(t) \mid t \geq 0\}$ and $\mathbb{T}_2 := \{T_2(t) \mid t \geq 0\}$ C_0 semigroups with generator A_1 respectively A_2.
Then $A_1 = A_2 \Longrightarrow \mathbb{T}_1 = \mathbb{T}_2$.

Notation 1.1.14 (Infinitesimal Generators)

$$\begin{aligned}\mathcal{G}(E,M,\omega) :=& \{A \mid A \text{ generates a } C_0-\text{semigroup on } E \text{ such that } \|T(t)\|_{\mathcal{L}(E)} \le Me^{\omega t} \text{ for all } t \ge 0\}.\\ \mathcal{G}(E,1,0) :=& \{A \mid A \text{ generates a contraction semigroup on } E.\}.\\ \mathcal{G}(E) :=& \bigcup_{M\ge 1,\omega\ge 0} \mathcal{G}(E,M,\omega).\end{aligned}$$

Now we return to the Abstract Cauchy-Problem, from the start of this section and show how we can solve it if the Operator A generates a C_0-semigroup. Both mild and classical solutions of our abstract Cauchy Problem will be discussed.

Definition 1.1.15 (Mild Solution)
*Let $\mathbb{T} = \{T(t) \mid t \ge 0\}$ a C_0-semigroup on E with generator A and $u_0 \in E$. The **mild solution** of (ACP) $u \in C([0,\infty), E)$ is defined via*

$$u(t) := T(t)u_0.$$

As we want to include forcing terms we extend the previous initial value problem and definition in the following way. Let $\{0\} \ne J \subset [0,\infty)$ an interval containing zero, $A : \mathrm{dom}(A) \subset E \to E$ a linear operator, $f \in C(J,E)$ and $u_0 \in E$. Consider the **inhomogenous Abstract Cauchy-Problem**

$$\begin{cases}\partial_t u = Au + f, & t \in J \setminus \{0\},\\ u(0) = u_0.\end{cases} \tag{iACP}$$

Definition 1.1.16 (Mild Solution)
*Let $\mathbb{T} = \{T(t) \mid t \ge 0\}$ a C_0-semigroup on E with generator A and $u_0 \in E$. The **mild solution** of (iACP) $u \in C(J,E)$ is defined via the variation of parameters formula*

$$u(t) := T(t)u_0 + \int_0^t T(t-s)f(s)\,ds \quad \forall\, t \in J.$$

As as our equations will be second order in time the following adjustments are in order. Let $\{0\} \ne J \subset [0,\infty)$ an interval containing zero, $A : \mathrm{dom}(A) \subset E \to E$ a linear operator, $f \in C(J,E)$ and $u_0 \in E$. Consider the **inhomogenous Abstract Cauchy-Problem**

$$\begin{cases}\partial_t tu = Au + B\partial_t u + f, & t \in J \setminus \{0\},\\ u(0) = u_0.\end{cases} \tag{iACP2}$$

Definition 1.1.17 (Mild Solution)
A mild solution of (iACP2) is given by the first coordinate $u_1 = \pi$ of the mild solution u of

$$\partial_t \begin{pmatrix} u_1 \\ \partial_t u_1 \end{pmatrix} = \begin{pmatrix} 0 & Id \\ A_{d,N} & B \end{pmatrix} \begin{pmatrix} u_1 \\ \partial_t u_1 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{1.1.1}$$

1.1.2 Generator Charakterisations

Now to find a mild solution to solve (ACP) for a given Operator A, we need to know does A generate a C_0-semi group or not. As the definition of a generator requires knowledge of the semi group, it is not useful in determining if a given operator A, for example a differential operator, generates a C_0-semi group. For this we present the classical characterisation results, that allow us to determine are we dealing with the generator of a C_0 semi group or not, based on the operator A alone.

The Hille Yoshida Theorem

The first characterisation result is the one by Hille and Yosida from around 1948, that was later generalized by Feller, Miyadera and Phillips (see [BR84]), though we don't need it and only included it for historical reasons, therefore we give the „original"version.

Theorem 1.1.18 (Hille Yoshida)
Let $A: dom(A) \subset E \to E$, $M \geq 1$ and $\omega \in \mathbb{R}$. Then $A \in \mathcal{G}(E, M, \omega)$ if and only if $A \in \mathcal{A}(E)$ is densely defined with $(\omega, \infty) \in \rho(A)$ and

$$\|(\lambda id_E - A)^{-n}\|_{\mathcal{L}(E)} \leq \frac{M}{(\lambda-\omega)^n} \quad for \;\; all \; \mathbb{R} \ni \lambda > \omega \quad \forall n \in \mathbb{N}.$$

Moreover $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ with

$$\|(\lambda id_E - A)^{-n}\|_{\mathcal{L}(E)} \leq \frac{M}{(\operatorname{Re}\lambda-\omega)^n} \quad for \;\; all \; \operatorname{Re}\lambda > \omega \quad \forall n \in \mathbb{N}.$$

Proof
Confirm [Paz12, Section 1.3 Theorem 3.1] or [HP96, page 363]. ■

The following well known perturbation theorem is of interest as it allows for our odd order terms to appear in our general linear results, as long as they are of sufficiently small order and therefore only a bounded perturbation.

Theorem 1.1.19 (Perturbation Theorem)
Let $A \in \mathcal{G}(E, M, \omega)$ and $B \in \mathcal{L}(E)$, then $A + B \in \mathcal{G}(E, M, \omega + M\|B\|_{\mathcal{L}(E)})$.

The Lumer Philips Theorem

The second result is the theorem of Lumer and Phillips, which is especially useful in the Hilbert space setting we are working with. As it is based on the concept of dissipativity we will briefly introduce what it means for the operator A to be dissipative and provide an alternative characterization of dissipativity in Hilbert spaces, before stating the Lumer-Phillips theorem.

Definition 1.1.20 (Dissipativ Operator)
A linear Operator $A : dom(A) \subset H \to H$

*(i) is called **dissipativ**, if $\operatorname{Re}\langle Ax, x\rangle_H \leq 0$ for every $x \in dom(A)$.*

*(ii) is called **m-dissipativ**, if it is dissipativ and there is a $\lambda_0 > 0$ such that $im(\lambda_0 Id - A) = H$.*

Theorem 1.1.21 (Lumer Philips)
Let $A \in \mathcal{A}(E)$ a densely defined Operator. Then $A \in \mathcal{G}(E, 1, 0)$ if and only if A is m-dissipativ.

Proof
Confirm [Paz12, Section 1.4 Theorem 4.3] or [Yos94, page 250]. ■

As the sufficient conditions rely on the adjoint operator of A we briefly recall the definition of the adjoint operator of an unbounded operator.

Definition 1.1.22 ((Self-)Adjoint Operator)
Let $A : dom(A) \subset H \to H$ a linear Operator.

(i) The linear Operator $(dom(A^*), A^*)$, defined via

$$\begin{aligned} dom(A^*) :=& \{y \in H \mid there \;\; exists \;\; a \; z \in H : \; \langle Ax, y\rangle_H = \langle x, z\rangle_H \; for \;\; all \; x \in dom(A)\}, \\ A^* y :=& z, \end{aligned}$$

*is called the **adjoint operator** (of A) .*

*(ii) A is called **self-adjoint**, if $A = A^*$.*

Using the Hilbert space characterization of dissipativity we can now formulate the following sufficient conditions for A to generate a C_0 semigroup.

Corollary 1.1.23
Let $A : dom(A) \subset H \to H$ *a linear Operator.*
If A *is densely defined, self-adjoint and dissipative, then* $A \in \mathcal{G}(H, 1, 0)$.

Corollary 1.1.24
Let $A \in \mathcal{A}(H)$ *densely defined. If* A *and* A^* *are dissipative then* $A \in \mathcal{G}(H, 1, 0)$.

1.2 Fourier Methods in Hilbert spaces

In this section we will discuss the Fourier analytical techniques, that will allow us to obtain an explicit representation of of solution semi groups and develop error estimates between our discrete and continuous solutions. The first section gives a general introduction about Fourier series in Hilbert spaces. While the second and third subsection respectively, apply those results for the continuous respectively discrete Sobolev Hilbert spaces. In the final subsection 1.2.4 we will introduce a discretization, via integral averages, and the corresponding continuation, constant interpolation, that allow us to interchange the discrete and continuous Fourier coefficients and basis. From this we derive a „discrete to continuous norm equivalence“

1.2.1 General Theorie

We will begin with the abstract framework, i.e. some nice structural properties of Hilbert spaces that can be found for example in [Yos94] and [Wer06].

Definition 1.2.1 (Orthonormal system)
Let $(H,\langle\cdot,\cdot\rangle_H)$ a Hilbert space. A subset $\Phi \subset H$ is called an

- ***orthonormal system**, if and only if $\langle\varphi,\psi\rangle_H = \delta_{\varphi,\psi}$.*
- ***complete orthonormal system** or **orthonormal basis** if and only if it is an orthonormal system and $\Phi^\perp = \{0\}$.*

Definition 1.2.2
Let $(H,\langle\cdot,\cdot\rangle_H)$ a Hilbert space $x,y \in H$ and $M \subset H$.

*(i) We say x is **orthogonal** to y and write $x \perp y$ if $\langle x,y\rangle_H = 0$.*

(ii) $M^\perp := \{y \in H \mid \langle x,y\rangle_H = 0 \text{ for all } x \in M\}$.

Theorem/Definition 1.2.3 (Orthogonal Projections)
*Let $(H,\langle\cdot,\cdot\rangle_H)$ a Hilbert space and $M \subset H$ a closed linear subspace. Then $M^\perp$ is also a closed linear subspace of H called the **orthogonal complement** of M and every $x \in H$ admits a unique decomposition*

$$x = m + n,$$

*with $m \in M$ and $n \in M^\perp$. m is called the **orthogonal projection** of x upon M denoted by $m := P_M x$. $P_M \in \mathcal{L}(H)$ is called the **(orthogonal) projector** upon M.*

Proof
Confirm [Yos94, page 82].

Next we will introduce the projectors that we will use to separate the low frequency and high frequency in the error estimates.

Definition 1.2.4 (Projections)
Let $(H,\langle\cdot,\cdot\rangle_H)$ a Hilbert space with a countable orthonormal basis $\Phi = \{\varphi_j \mid j \in \mathbb{Z}^q\}$.

(i) For $N \in \mathbb{N}$ define $M_s(N) := span(\varphi_j \mid j \in \mathbb{Z}^q \cap [-N,N]^q)$ and the symmetric projector $P_N^s := P_{M_s(N)}$.

(ii) For $N \in \mathbb{N}$ define $M_a(N) := span(\varphi_j \mid j \in \mathbb{Z}^q \cap (-N,N]^q)$ and the asymmetric projector $P_N^a := P_{M_a(N)}$.

(iii) For $N \in \mathbb{N}$ and $\delta \in (0,1)$ define $P_\delta^a := P^a_{\lfloor N^\delta \rfloor}$ and $P_\delta^s := P^s_{\lfloor N^\delta \rfloor}$.

Remark 1.2.5
The symmetric Projector $P_N^s := P_{M_s(N)}$ is also called the canonical Galerkin Projector.

Finally we will show how we give every element of a given (separable) Hilbert space a series expansion with respect to a given orthonormal basis, once we linked our discrete and continuous complete orthonormal systems in Subsection 1.2.4 this will establish the framework in which we compute our error estimates.

Definition 1.2.6 (Fourier series)
Let $(H, \langle\cdot,\cdot\rangle_H)$ a Hilbert space and $\Phi \subset H$ an orthonormal system.

- *$\langle x, \varphi\rangle_H$ is called the **Fourier coefficient** of x with respect to $\varphi \in \Phi$.*
- *$\sum_{\varphi\in\Phi} \langle x, \varphi\rangle_H \varphi$ is called the **Fourier series** of x with respect to Φ.*

Theorem 1.2.7 (Bessel Inequality)
Let $\Phi \subset H$ be an orthonormal system, then for every $x \in H$ the following inequality holds:

$$\sum_{\varphi\in\Phi} |\langle x, \varphi\rangle_H|^2 \leq \|x\|_H^2. \tag{1.2.1}$$

Theorem 1.2.8
Let $(H, \langle\cdot,\cdot\rangle_H)$ a Hilbert space and $\Phi \subset H$ a countable orthonormal system, i.e. $\Phi = \{\varphi_j \in H \mid j \in I\}$, for a countable set I. Then

(a) For all $x \in H$ the Fourier series

$$\sum_{j\in I} \langle x, \varphi_j\rangle_H \varphi_j$$

converges unconditionally. And for $M := \overline{span(\Phi)} = \Phi^{\perp\perp}$ one has

$$P_M(x) = \sum_{j\in I} \langle x, \varphi_j\rangle_H \varphi_j.$$

(b) The following statements are equivalent:

(i) Φ is an orthonormal basis,

(ii) $P_M(x) = x$ for all $x \in H$,

(iii) For every $x, y \in H$ one has

$$\langle x, y\rangle_H = \sum_{j\in\mathbb{Z}} \langle x, \varphi_j\rangle_H \overline{\langle y, \varphi_j\rangle_H}.$$

Theorem 1.2.9 (Parseval Identity)
Let $(H, \langle\cdot,\cdot\rangle_H)$ a Hilbert space and $\Phi \subset H$ an orthonormal system. Then for every $x \in H$:

$$\|x\|_H^2 = \langle x, x\rangle_H = \sum_{\varphi\in\Phi} |\langle x, \varphi\rangle_H|^2. \tag{1.2.2}$$

Lemma 1.2.10
Let $(H, \langle\cdot,\cdot\rangle_H)$ an infinite dimensional Hilbert space. The following statements are equivalent:

(i) H is separable.

(ii) There exists a countable complete orthonormal system.

(iii) All complete orthonormal systems are countable.

1.2.2 Lebesgue and Sobolev Spaces on the Torus

Now we will apply the previously established general theory to the continuum case on the Torus $\mathbb{T}^q$.

Definition/Notation 1.2.11 (Multi-index)
*Let $n \in \mathbb{N}$. A **multi-index** is a n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, for which we use the following notations:*

$$|\alpha| := \sum_{j=1}^{n} \alpha_j, \quad \alpha! := \prod_{j=1}^{n} (\alpha_j!), \quad x^\alpha := \prod_{j=1}^{n} x_j^{\alpha_j}, \quad \text{and} \quad \partial^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f.$$

Notation 1.2.12
For $x \in \mathbb{R}^q$ we write $\langle x \rangle = (1 + |x|^2)^{1/2}$.

Definition 1.2.13
The Hilbert space $(L_2(\mathbb{T}^q), \langle \cdot, \cdot \rangle_{L^2(\mathbb{T}^q)})$ is defined as

$$L_2(\mathbb{T}^q) := \left\{ f \in \mathcal{L}_0(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q), \lambda^q, \mathbb{C}) \mid f(x + 2\pi e_i) = f(x) \text{ for a.e } x \in \mathbb{R}^q \text{and } \|f\|_{L_2(\mathbb{T}^q)} < \infty \right\}$$

with

$$\|f\|_{L_2(\mathbb{T}^q)} = \frac{1}{(2\pi)^{q/2}} \int_{\mathbb{T}^q} |f(y)|^2 \, dy$$

and the inner product given by:

$$\langle f, g \rangle_{L_2(\mathbb{T}^q)} := \frac{1}{(2\pi)^q} \int_{\mathbb{T}^q} f(y) \overline{g}(y) \, dy.$$

Now we need an orthonormal basis, therefore recall the classical one on the torus.

Lemma 1.2.14
The family $\Phi := \{\varphi_j(\cdot) \mid j \in \mathbb{Z}^q\} \subset L_2(\mathbb{T}^q)$, with

$$\varphi_j(x) := e^{i \langle x, j \rangle_{\mathbb{R}^q}} \text{ for all } x \in \mathbb{T}^q, \text{ for every } j \in \mathbb{Z}^q,$$

defines an orthonormal basis of $(L_2(\mathbb{T}^q), \langle \cdot, \cdot \rangle_{L_2(\mathbb{T}^q)})$.

Using the continuous Plancherel Theorem we can use the Fourier transform and our orthonormal basis to reformulate the definition of the Sobolev Hilbert spaces on the torus $\mathbb{T}^q$ in the following way.

Definition 1.2.15 (Real Order Sobolev Spaces)
For $s \in \mathbb{R}$ the Hilbert space $(H^s(\mathbb{T}^q), \langle \cdot, \cdot \rangle_{H^s(\mathbb{T}^q)})$ is defined as

$$H^s(\mathbb{T}^q) := \{ f \in L_2(\mathbb{T}^q) \mid \|f\|_{H^s(\mathbb{T}^q)} := \sqrt{\langle f, f \rangle_{H^s(\mathbb{T}^q)}} < \infty \},$$

where

$$\langle f, g \rangle_{H^s(\mathbb{T}^q)} := \sum_{k \in \mathbb{Z}^q} \langle k \rangle^s \hat{f}(k) \overline{\hat{g}(k)}.$$

Remark 1.2.16

(i) $(H^0(\mathbb{T}^q), \langle\cdot,\cdot\rangle_{H^0(\mathbb{T}^q)}) = (L_2(\mathbb{T}^q), \langle\cdot,\cdot\rangle_{L_2(\mathbb{T}^q)})$.

(ii) If $s \in \mathbb{N}$ Definition 1.2.15 coincides with the classical Sobolev space definition.

(iii) The Hilbert spaces in Definition 1.2.15 are called ***Besselpotential spaces****.*

Lemma 1.2.17
For every $s \in \mathbb{R}$ the family $\Phi_s := \{\langle j\rangle^{-s/2}\varphi_j(\cdot) \mid j \in \mathbb{Z}^q\} \subset H^s(\mathbb{T}^q)$, with

$$\varphi_j(x) := e^{i\langle x,j\rangle_{\mathbb{R}^q}} \text{ for all } x \in \mathbb{T}^q, \text{ for every } j \in \mathbb{Z}^q,$$

defines an orthonormal basis of $(H^s(\mathbb{T}^q), \langle\cdot,\cdot\rangle_{H^s(\mathbb{T}^q)})$.

Proof
Follows by Lemma 1.2.14 and the definition of the Sobolev spaces $H^s(\mathbb{T}^q)$. ■

1.2.3 Lebesgue and Sobolev spaces on the discrete Torus

In this section we introduce the discrete Fourier counterparts of our discrete lattice and the corresponding function spaces. Then we can introduce the discrete Fourier transform on these function spaces, for which we show properties analogous to the continuous Fourier transform. When it comes to the discrete Sobolev spaces we take inspiration from [Ste91], adjusting some ideas to our finite grid setting. Let us begin with some motivation Now we are looking for suitable operators that can be defined on spaces of discrete functions and approximate our partial differential operators in a way compatible with our underlying discrete structure. For this we recall the centralized differential quotients from elementary analysis:

$$\begin{aligned}\frac{\partial}{\partial x_j} f(x) &= \lim_{h\to 0} \frac{f(x+e_jh) - f(x-e_jh)}{2h},\\ \frac{\partial^2}{\partial x_j^2} f(x) &= \lim_{h\to 0} \frac{f(x+e_jh) - 2f(x) + f(x-e_jh)}{h^2}.\end{aligned}$$

Leaving aside the limit for a moment we define.

Definition 1.2.18 (Shift Operator)
For $h, s \in \mathbb{R}$ and $j \in \{1.\dots,q\}$ we define the ***left shift operator*** *$\tilde{T}_{h,j}$ on $H^s(\mathbb{R}^q)$ via*

$$\tilde{T}_{h,j}u(\cdot) := u(\cdot + e_jh).$$

With this one can rewrite the previous differential quotients into

$$\begin{aligned}\frac{\partial}{\partial x} f(x) &= \lim_{h\to 0} \frac{1}{2h}(\tilde{T}_{h,j} - \tilde{T}_{-h,j})f(x),\\ \frac{\partial^2}{\partial x^2} f(x) &= \lim_{h\to 0} \frac{1}{h^2}(\tilde{T}_{h,j} - 2Id + \tilde{T}_{-h,j})f(x).\end{aligned}$$

Thus shifting the limit from the function onto the operators. This begs the question on how to interpret this operator limit. The answer to this question is given by the following topology.

Definition 1.2.19 (Strong Operator Topology)
The ***strong operator topology*** *on $\mathcal{L}(E,F)$ is the topology induced by the family of seminorms $\{\|\cdot(e)\|_F\}_{e\in E}$.*

Notation 1.2.20
Let $T \in \mathcal{L}(E,F)$ and $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(E,F)$ that converges with and denote the convergence with respect to the strong operator topology by

$$T = SOT - \lim_{n\to\infty} T_n \quad or \quad T_n \xrightarrow{SOT} T.$$

Now we define how to extend the above formula to general differentials and thereby differential operators.

Definition 1.2.21 (Differential quotients)
*For $h, s \in \mathbb{R}$, $m \in \mathbb{N}$ and $j \in \{1\ldots, q\}$ we define the **differential quotient operator** $\tilde{\partial}^m_{h,j}$ on $H^s(\mathbb{R}^q)$ via*

$$\tilde{\partial}^m_{h,j} := \begin{cases} \frac{\tilde{T}^{m/2}_{h,j}(Id-\tilde{T}^{-1}_{h,j})^m}{h^m}, & m \text{ even,} \\ \frac{(\tilde{T}_{h,j}-\tilde{T}^{-1}_{h,j})\tilde{T}^{(m-1)/2}_{h,j}(Id-\tilde{T}^{-1}_{h,j})^{(m-1)}}{2h^m}, & m \text{ odd.} \end{cases}$$

Definition 1.2.22 (Partial Differential Quotient Operator)
*A linear **partial differential quotient operator** (or **PDQO** for short) $\tilde{A}_h(x, \partial_h)$ of order $m(\in \mathbb{N}_0)$, in the difference quotient $\tilde{\partial}_h$, is a polynomial of degree m in $\tilde{\partial}_h$, i.e. given a function $u \in H^s(\mathbb{R}^q)$ one has:*

$$\tilde{A}_h(x, \tilde{\partial}_h)u := \sum_{|\alpha|\leq m} a_\alpha(x)\tilde{\partial}^\alpha_h u.$$

Theorem 1.2.23
Let $A(x, \partial) = \sum_{|\alpha|\leq m} a_\alpha(x)\tilde{\partial}^\alpha$ a partial differential operator and $\tilde{A}_h(x, \tilde{\partial}_h) := \sum_{|\alpha|\leq m} a_\alpha(x)\tilde{\partial}^\alpha_h$ the corresponding partial differential quotient operator, then

$$A(x, \partial) = SOT - \lim_{h\to 0} \tilde{A}_h(x, \tilde{\partial}_h).$$

Now we will introduce the corresponding discrete difference quotient operators by matching the scaling parameter h with the step size of our lattice. Most notably we will introduce our version of the discrete Laplacian (as this term is also frequently used for the corresponding discrete difference operator which has the wrong scaling for our purposes), as this will allow to define the discrete counterparts of the Sobolev spaces H^s.

Definition 1.2.24 (Discrete Shift Operator)
*For $N \in \mathbb{N}$, $s \in \mathbb{R}$, $j \in \{1\ldots, q\}$ and $h \in \mathbb{R}$, a positive multiple of Q^q_N's step size $\frac{\pi}{N}$, we define the **discrete left shift operator** $T_{h,j}$ on $h^s(Q^q_N)$ via*

$$T_{h,j}u(\cdot) := u(\cdot + he_j).$$

Definition 1.2.25 (Discrete Differential Quotients)
*For $N, m \in \mathbb{N}$, $s \in \mathbb{R}$, $j \in \{1\ldots, q\}$ and $h \in \mathbb{R}$, a positive multiple of Q^q_N's step size $\frac{\pi}{N}$, we define the **discrete differential quotient operator** $\partial^m_{h,j}$ via*

$$\partial^m_{h,j} := \begin{cases} \frac{T^{m/2}_{h,j}(Id-T^{-1}_{h,j})^m}{h^m}, & m \text{ even,} \\ \frac{(T_{h,j}-T^{-1}_{h,j})T^{(m-1)/2}_{h,j}(Id-T^{-1}_{h,j})^{(m-1)}}{2h^m}, & m \text{ odd.} \end{cases}$$

The corresponding multi index notation is given in the following way.

Notation 1.2.26
For $N \in \mathbb{N}$, $h \in \mathbb{R}$, a positive multiple of Q^q_N's step size $\frac{\pi}{N}$, and $\alpha \in \mathbb{Z}^q$ we set

$$T^\alpha_h := T^{\alpha_1}_{h,1} \ldots T^{\alpha_q}_{h,q}$$

and for $\alpha \in \mathbb{N}^q$ we set

$$\partial^\alpha_h := \partial^\alpha_{h,1} \ldots \partial^\alpha_{h,q}.$$

And as we do in the continuum case with D and ∂, we will use the generic notation ∂_h for the discrete differentials $\partial_{h,1}, \ldots, \partial_{h,q}$

Definition 1.2.27 (discrete Partial Difference Quotient Operator)
A linear ***partial difference quotient operator*** *(or* ***PDQO*** *for short) $A_{N,d}(x,\partial_h)$ of order $m(\in \mathbb{N}_0)$, in the discrete differential ∂_h, is a polynomial of degree m in ∂_h^m, for $h \in \mathbb{R}$, a positive multiple of Q_N^q's step size $\frac{\pi}{N}$. I.e. given a function $u : Q_N^q \to \mathbb{K}$ one has:*

$$A_{N,d}(x,\partial_h)u := \sum_{|\alpha|\leq m} a_\alpha(x)\partial_h^\alpha u.$$

Example/Definition 1.2.28
The discrete Laplacian $\Delta_{N,d}$ on $l_2(Q_N^q)$ is for each $N \in \mathbb{N}$ given by

$$\Delta_{N,d} = \sum_{j=1}^{q} \partial^2_{\frac{\pi}{N},j} = \sum_{j=1}^{q} \frac{T_{\frac{\pi}{N},j}(Id - T^{-1}_{\frac{\pi}{N},j})^2}{\left(\frac{\pi}{N}\right)^2}.$$

As in the continuum case, we need knowledge on the adjoint of our operators to solve the discrete equations via semigroup theory. With the following Lemma we show that the symmetric difference quotients mirror the behaviour of the corresponding differential operators, when adjoining.

Lemma 1.2.29
For $N \in \mathbb{N}$, a multi index $\alpha \in \mathbb{N}^q$ and $h \in \mathbb{R}$, a positive multiple of Q_N^q's step size $\frac{\pi}{N}$ we have:

$$\left(T^\alpha_{\frac{\pi}{N},j}\right)^* = T^{-\alpha}_{\frac{\pi}{N}} \qquad \text{and} \qquad \left(\partial^\alpha_{\frac{\pi}{N}}\right)^* = (-1)^{|\alpha|}\partial^\alpha_{\frac{\pi}{N}}.$$

We begin the main content part of this section with the definition of our discrete cubic lattice and its Fourier dual . Followed by the definition of the respective l_2-spaces defined on them.

Definition 1.2.30
Let $q, N \in \mathbb{N}^*$ and define:

- $Q_N^q := \left\{-\pi, \dots, -\frac{\pi}{N}, 0, \frac{\pi}{N}, \dots, \frac{(N-1)}{N}\pi\right\}^q$,
- $\hat{Q}_N^q := \{-N, \dots, -1, 0, 1, \dots, N-1\}^q$.

Definition 1.2.31
Let $N, q \in \mathbb{N}^*$ we define:

$$l_2(Q_N^q) := \left\{f : \frac{\pi}{N}\mathbb{Z}^q \to \mathbb{K} \mid f(j) = f(j + 2\pi e_i) \text{ for all } j \in Q_N^q \text{ and } 1 \leq i \leq q\right\}$$

and equip it with the inner product

$$\langle f, g\rangle_{(0,N,d)} := \langle f, g\rangle_{l_2(Q_N^q)} := \frac{1}{(2N)^q} \sum_{j \in Q_N^q} f(j)\overline{g(j)} \quad \text{for all } f, g \in l_2(Q_N^q)$$

and the corresponding norm

$$\|f\|_{l_2(Q_N^q)} := \langle f, f\rangle^{1/2}_{l_2(Q_N^q)} \quad \text{for all } f, g \in l_2(Q_N^q).$$

Definition 1.2.32
Let $N, q \in \mathbb{N}^*$ we define:

$$l_2(\hat{Q}_N^q) := \left\{\hat{f}^d : \mathbb{Z}^q \to \mathbb{K} \mid \hat{f}^d(j) = \hat{f}^d(j + 2Ne_i) \text{ for all } j \in \hat{Q}_N^q \text{ and } 1 \leq i \leq q\right\}$$

and endow it with the inner product

$$\langle \hat{f}^d, \hat{g}^d\rangle_{(\hat{0},N,d)} := \langle \hat{f}^d, \hat{g}^d\rangle_{l_2(\hat{Q}_N^q)} := \frac{1}{(2N)^q} \sum_{j \in \hat{Q}_N^q} \hat{f}^d(j)\overline{\hat{g}^d(j)} \quad \text{for all } \hat{f}^d, \hat{g}^d \in l_2(\hat{Q}_N^q)$$

and the corresponding norm

$$\left\|\hat{f}^d\right\|_{l_2(\hat{Q}_N^q)} := \langle \hat{f}^d, \hat{f}^d\rangle^{1/2}_{l_2(\hat{Q}_n^q)} \quad \text{for all } \hat{f}^d \in l_2(\hat{Q}_N^q).$$

Now we notice the identities.

Lemma 1.2.33

(i) For all $k, k' \in \hat{Q}_N^q$:

$$\langle e^{i\langle \cdot, k\rangle_{\mathbb{R}^q}}, e^{i\langle \cdot, k'\rangle_{\mathbb{R}^q}}\rangle_{l_2(Q_N^q)} = \delta_{k,k'}.$$

(ii) For all $j, j' \in Q_N^q$:

$$\langle e^{i\langle j, \cdot\rangle_{\mathbb{R}^q}}, e^{i\langle j', \cdot\rangle_{\mathbb{R}^q}}\rangle_{l_2(\hat{Q}_N^q)} = \delta_{j,j'}.$$

With these we find a complete orthonormal system for each of our discrete spaces.

Corollary 1.2.34

(i) For every $N \in \mathbb{N}^*$ the family $\Phi_{N,d} := \{\varphi_j^d(\cdot) \mid j \in \hat{Q}_N^q\} \subset l_2(Q_N^q)$, with

$$\varphi_j(k) := e^{i\langle k, j\rangle_{\mathbb{R}^q}} \text{ for all } k \in Q_N^q, \text{ and } j \in \hat{Q}_N^q,$$

defines an orthonormal basis of $(l_2(Q_N^q), \langle \cdot, \cdot \rangle_{l_2(Q_N^q)})$.

(ii) For every $N \in \mathbb{N}^*$ the family $\hat{\Phi}_{N,d} := \{\hat{\varphi}_j^d(\cdot) \mid j \in Q_N^q\} \subset l_2(\hat{Q}_N^q))$, with

$$\hat{\varphi}_j^d(k) := e^{i\langle k, j\rangle_{\mathbb{R}^q}} \text{ for all } k \in \hat{Q}_N^q, \text{ and } j \in Q_N^q,$$

defines an orthonormal basis of $(l_2(\hat{Q}_N^q), \langle \cdot, \cdot \rangle_{l_2(\hat{Q}_N^q)})$.

Using that by Corollary 1.2.36 $\Delta_{N,d}$ is negative, we get that $Id - \Delta_{N,d}$ is positive and therefore the following definition makes sense for all $s \in \mathbb{R}$, as the fractional operator $(Id - \Delta_{N,d})^{s/2}$ can be defined for all $s \in \mathbb{R}$. Confirm also [Ste91].

Definition 1.2.35
For $N, q \in \mathbb{N}^*$ and $s \in \mathbb{R}$ define:

$$h^s(Q_N^q) := \left\{ f : \frac{\pi}{N}\mathbb{Z}^q \to \mathbb{K} \mid f(j) = f(j + 2\pi e_i) \text{ for all } j \in \frac{\pi}{N}\mathbb{Z}^q \text{ and } 1 \le i \le q \right\}$$

with the inner product

$$\langle f, g \rangle_{h^s(Q_N^q)} := \langle (Id - \Delta_{N,d})^{s/2} f, (1 - \Delta_{N,d})^{s/2} g \rangle_{l_2(Q_N^q)},$$

and corresponding norm

$$\|f\|_{h^s(Q_N^q)} := \langle f, g \rangle_{h^s(Q_N^q)}^{1/2}.$$

Corollary 1.2.36
The discrete Laplacian $\Delta_{N,d}$ is a negative self-adjoint operator. As by Lemma 1.2.29

$$\Delta_{N,d}^* = \left(\sum_{j=1}^q \partial_{\frac{\pi}{N},j}^2 \right)^* = \sum_{j=1}^q \left(\partial_{\frac{\pi}{N},j}^2 \right)^* = \sum_{j=1}^q (-1)^2 \partial_{\frac{\pi}{N},j}^2 = \Delta_{N,d}.$$

And using that by Corollary 1.2.34 a complete orthonormal basis of $l_2(Q_N^q)$ is given by

$$\{e^{i\langle \cdot, k\rangle_{\mathbb{R}^q}} : Q_N^q \to \mathbb{C} \mid k \in \mathbb{Z}^q\}$$

we compute

$$\begin{aligned}\Delta_{N,d}e^{i\langle j,k\rangle_{\mathbb{R}^q}} &= \sum_{j=1}^{q}\partial^2_{\frac{\pi}{N},j}e^{i\langle j,k\rangle_{\mathbb{R}^q}} = \left(\sum_{j=1}^{q}\frac{e^{i\langle\frac{\pi}{N}e_j,k\rangle_{\mathbb{R}^q}}+e^{-i\langle\frac{\pi}{N}e_j,k\rangle_{\mathbb{R}^q}}-2}{\left(\frac{\pi}{N}\right)^2}\right)e^{i\langle j,k\rangle_{\mathbb{R}^q}}\\ &=-4\left(\sum_{j=1}^{q}\left(\frac{e^{i\langle\frac{\pi}{2N}e_j,k\rangle_{\mathbb{R}^q}}-e^{-i\langle\frac{\pi}{2N}e_j,k\rangle_{\mathbb{R}^q}}}{\frac{\pi}{N}2i}\right)^2\right)e^{i\langle j,k\rangle_{\mathbb{R}^q}}\\ &=-4\left(\sum_{j=1}^{q}\frac{N^2}{\pi^2}\sin^2\left(\frac{\pi}{N}k_j\right)\right)e^{i\langle j,k\rangle_{\mathbb{R}^q}}\end{aligned}$$

therefore the discrete Laplacians eigenvalues λ_k are, for all $k\in\mathbb{Z}^q$, given by

$$\lambda_k=-4\sum_{j=1}^{q}\frac{N}{\pi}\sin^2\left(\frac{\pi}{N}k_j\right)$$

,with the Fourier basis as corresponding eigenfunctions. Thus showing that it is a negative operator.

As for their Fourier counterparts, as in the continuous case we use the Eigenvalues of the respective Laplacian to define the Fourier weights, leading to the following definition of the discrete Fourier Sobolev spaces.

Definition 1.2.37
For $N,q\in\mathbb{N}^*$ and $s\in\mathbb{R}$ define:

$$h^s(\hat{Q}^q_n):=\{f:\mathbb{Z}^q\to\mathbb{K}\mid f(j)=f(j+2Ne_i)\ \text{for all } j\in\mathbb{Z}^q\ \text{and } 1\le i\le q\}$$

and equip it with the inner product

$$\langle f,g\rangle_{h^s(\hat{Q}^q_n)}:=\langle\langle 2\frac{N}{\pi}\tilde{g}(\cdot)^s f,\langle 2\frac{N}{\pi}\tilde{g}(\cdot)^s g\rangle_{l_2(\hat{Q}^q_N)},$$

where $\tilde{g}:\mathbb{R}^q\to\mathbb{R}^q$ is the map defined via

$$k=\begin{pmatrix}k_1\\ \vdots\\ k_q\end{pmatrix}\mapsto\begin{pmatrix}\sin\left(\frac{k_1\pi}{2N}\right)\\ \vdots\\ \sin\left(\frac{k_1\pi}{2N}\right)\end{pmatrix}$$

With the corresponding norm given by

$$\|f\|_{h^s(\hat{Q}^q_N)}:=\langle f,f\rangle^{1/2}_{h^s(\hat{Q}^q_n)}.$$

Remark 1.2.38
Unlink the continuous $H^s(\mathbb{T}^q)$ spaces, these finite discrete Sobolev spaces are all equivalent, for fixed N, with the equivalence constant depending on N, in such a way that they diverge for N to infinity.

Corollary 1.2.39

(i) For every $N\in\mathbb{N}^*$ and every $s\in\mathbb{R}$ the family $\Phi_{N,s,d}:=\{(1-\Delta_{N,d})^{-s/2}\varphi^d_j(\cdot)\mid j\in\hat{Q}^q_N\}\subset l_2(Q^q_N)$, with

$$\varphi_j(k):=e^{i\langle k,j\rangle_{\mathbb{R}^q}}\ \text{for all } k\in Q^q_N,\ \text{and } j\in\hat{Q}^q_N,$$

defines an orthonormal basis of $(h^s(Q^q_N),\langle\cdot,\cdot\rangle_{h^s(Q^q_N)})$.

(ii) For every $N\in\mathbb{N}^*$ and every $s\in\mathbb{R}$ the family $\hat{\Phi}_{N,s,d}:=\{\langle 2\frac{N}{\pi}s(\frac{\cdot\pi}{2N})\rangle^{-s/2}\hat{\varphi}^d_j(\cdot)\mid j\in Q^q_N\}\subset l_2(\hat{Q}^q_N))$, with

$$\hat{\varphi}^d_j(k):=e^{i\langle k,j\rangle_{\mathbb{R}^q}}\ \text{for all } k\in\hat{Q}^q_N,\ \text{and } j\in Q^q_N,$$

defines an orthonormal basis of $(h^s(\hat{Q}^q_n),\langle\cdot,\cdot\rangle_{h^s(\hat{Q}^q_n)})$, where $\tilde{g}$ is the eigenvalue function from Definition 1.2.37

The Discrete Fouriertransform

Now we can introduce the discrete counterparts of the Fourier transform and the related results, that is the Plancherel and Parseval theorem and, for the treatment of the non-linearities, we will prove that, like the Fourier transform, the discrete Fourier transform interchanges multiplication with the respective convolution.

Definition 1.2.40 (Discrete Fouriertransform)
Given a function $f \in l_2(Q_N^q)$ one defines the map $\hat{f}^d \in l_2(\hat{Q}_N^q)$ via

$$\hat{f}^d(k) = \langle f, \varphi_k^d \rangle_{l_2(Q_N^q)} \quad \text{for all } k \in \hat{Q}_N^q.$$

*$\hat{f}^d$ is called the **discrete Fourier transform** of y and the map $\mathcal{F}^d : l_2(Q_N^q) \to l_2(\hat{Q}_N^q)$ defined via $\mathcal{F}^d := \hat{(\cdot)}^d := [y \mapsto \hat{f}^d]$ is called the **discrete Fourier transform**.*

Remark 1.2.41
Note that for a periodic function $f \in l_2(Q_N^q)$, the Fourier coefficients are $2N$-periodic, by definition.

Theorem 1.2.42 (Discrete Fourierinversion)
For all $j \in Q_N^q$ we have

$$f(j) = \sum_{k \in \hat{Q}_N^q} \hat{f}^d(k) e^{i\langle j,k \rangle_{\mathbb{R}^q}} = (2N)^q \langle \hat{f}^d, \overline{\hat{\varphi}_j^d} \rangle_{l_2(\hat{Q}_N^q)}. \tag{1.2.3}$$

Proof
Follows by Theorem 1.2.8 applied with Definition 1.2.40 for the complete orthonormalsystem from Corollary 1.2.34. ■

Furthermore a discrete Parseval theorem and Plancherel theorem can be obtained.

Theorem 1.2.43 (discrete Parseval)
Let $f, g : Q_N^q \to \mathbb{K}$ then

$$(2N)^q \langle \hat{f}^d, \hat{g}^d \rangle_{h^s(\hat{Q}_N^q)} = \langle f, g \rangle_{h^s(Q_N^q)}.$$

Corollary 1.2.44 (discrete Plancherel)
Let $f \in l_2(Q_N^q)$ then

$$(2N)^q \left\| \hat{f}^d \right\|_{h^s(\hat{Q}_N^q)} = \|f\|_{h^s(Q_N^q)}.$$

Proof
Choose $f \equiv g$ in Theorem 1.2.43. ■

The final property of the discrete Fourier transform we want to show, will allow us to compute the discrete Fourier transform of a product as the discrete convolution of the factors Fourier transforms, this will enable us to carry over results from the linear theory into the nonlinear setting.

Definition 1.2.45 (Discrete Convolution)

- *The **discrete convolution** on $l_2(Q_N^q)$ is defined via*

$$(f *_d g)(n) := \langle f, \overline{g(n - \cdot)} \rangle_{l_2(Q_N^q)} \quad \text{for all } f, g \in l_2(Q_N^q).$$

- *The **discrete convolution** on $l_2(\hat{Q}_N^q)$ is defined via*

$$(\hat{f}^d *_d \hat{g}^d)(n) := \langle \hat{f}, \overline{\hat{g}(n - \cdot)} \rangle_{l_2(\hat{Q}_N^q)} \quad \text{for all } \hat{f}^d, \hat{g}^d \in l_2(\hat{Q}_N^q).$$

Theorem 1.2.46 (Convolution/Multiplication)
For all $n \in \hat{Q}_N^q$ the following equalities hold:

(i) $(\hat{f}^d *_d \hat{g}^d)(n) = \frac{1}{(2N)^q}\widehat{fg}^d(n)$,

(ii) $(\hat{f}^d \hat{g}^d)(n) = \widehat{f *_d g}^d(n)$.

Remark 1.2.47
The different prefactors are due to uneven distribution of the prefactor in the definition of the discrete Fourier transform.

1.2.4 Discrete to Continous and Back

With both the continuous and discrete Fourier theory established, we will use this section to lay the groundwork for our continuum limit theory, by either lifting the discrete solution to the spatial continuum, via a constant extension, or discretize the continuous solution, using a spatial L_1-average. Subsequently we will express the Fourier coefficients, of the continuation respectively discretization, by the original (generalized) functions Fourier coefficients, and use this to prove an „equivalence“of the continuous and discrete norms.

Definition 1.2.48
For $q, N \in \mathbb{N}$ and $l \in Q_N^q$ we define the cube

$$C^q(l) := \bigtimes_{j=1}^{q} \left[l_j, l_j + \frac{\pi}{N}\right].$$

Now we will use these cubes as a basis for our averaging and interpolation schemes.

Definition 1.2.49
For a function $f : \mathbb{T}^q \to \mathbb{K}$ and $N, q \in \mathbb{N}$ the ***discretization operator*** $[\cdot]_{N,q}$ *is defined via*

$$[f]_{N,q}(l) := \left(\frac{N}{\pi}\right)^q \int_{C^q(l)} v(y)\, dy, \quad \textit{for all } l \in Q_N^q. \tag{1.2.4}$$

Definition 1.2.50
Let $N, q \in \mathbb{N}$ and $f : Q_N^q \to \mathbb{K}$ we define the ***continuation*** $\tilde{f} : \mathbb{T}^q \to \mathbb{K}$ *via*

$$\tilde{f}(x) = \sum_{j \in Q_N^q} f(j) \mathbb{1}_{C^q(j)}(x),$$

and the ***continuation operator*** $[\cdot]^{\sim}_{N,q} : l_2(Q_N^q) \to L_2(\mathbb{T}^q)$ *via*

$$[f]^{\sim}_{N,q} = \tilde{f}.$$

We use the $\sim$ to indicate that the continuation and discretization are in an almost inverse relation, as every non constant continuous function on the torus is returned as a step function, it is evident that this is not a full inversion. As we will see aside from the following weights, these operations will leave the Fourier coefficients of (generalized) functions.

Definition 1.2.51

- *For $N \in \mathbb{N}$ and $j \in Q_N^1$ define*

$$w(N, j) := \begin{cases} 1 & j = 0 \\ \frac{\exp\left(\frac{ij\pi}{N}\right) - 1}{\frac{ij\pi}{N}} & j \neq 0. \end{cases}$$

- *For $q \in \mathbb{N}$, $N \in \mathbb{N}^q$ and $j \in Q_N^q$ define*

$$w(N, j, q) := \prod_{l=1}^{q} w(N, j_l).$$

At various point throughout this thesis we will need both upper and lower bounds for these weight functions. While upper bounds are easy to come by, bounding them from below, away from zero, requires a „cut off "away from $\pm N$ (respectively $N + 2N\mathbb{Z}$ for the periodic copies), this can be done as follows.

Lemma 1.2.52
For every $N \in \mathbb{N}$ and $\alpha \in (0,1)$ there exists a constant $C_\alpha > 0$, such that for every $j \in \hat{Q}^1_{\lfloor \alpha N \rfloor}$

$$C_\alpha \leq |w(N,j)| \leq 1, \tag{1.2.5}$$

with the upper bound holding for all $j \in \hat{Q}^1_N$.

Proof
For the lower bound we note that

$$\frac{|\exp(ix) - 1|}{|x|} = 0 \Longleftrightarrow x \in 2\pi\mathbb{Z} \setminus \{0\},$$

therefore we get for all $j \in \hat{Q}^1_{\lfloor \alpha N \rfloor}$

$$|w(N,j)| \geq \min_{x \in \overline{\mathbb{B}}_{\mathbb{R}}(0,\alpha N)} \frac{|\exp(ix) - 1|}{|x|} > 0.$$

As for the upper bound, here we use that $\max_{x\in[-1,1]} |\sin(x)| < 1$, to deduce that for all $x \in \mathbb{T}^1$ we have

$$\frac{|\exp(ix) - 1|}{|x|} = \frac{2\sin\left(\frac{x}{2}\right)}{|x|} \leq 1.$$

Therefore we get $|w(N,j)| \leq 1$ for all $j \in \hat{Q}^1_N$. ∎

Lemma 1.2.53
Let $N \in \mathbb{N}$ and $f \in P^a_N(H^s(\mathbb{T}^q))$ with continuum Fourier-coefficients f_j. Then for every $j \in \hat{Q}^q_N$, the j-th Fourier coefficient is given by:

$$\widehat{[f]_{N,q}(\cdot)}^d(m) = f_j w(N,j,q).$$

Proof
Let $j \in \hat{Q}^q_N$, by definition of the discretization and the discrete Fourier transform, the j-th discrete Fourier coefficient is given by

$$\begin{aligned}
&\widehat{[f]_{N,q}(\cdot)}^d(j)\\
=&\left(\frac{1}{2\pi}\right)^q \sum_{p\in Q^q_N} \int_{C^q(p)} f(y)\, dy\; e^{-i\langle p,j\rangle}\\
=&\left(\frac{1}{2\pi}\right)^q \sum_{p\in Q^q_N} \int_{C^q(p)} \sum_{k\in\hat{Q}^q_N} f_k e^{i\langle k,y\rangle}\, dy\; e^{-i\langle p,j\rangle}.
\end{aligned}$$

Replacing f by its Fourier series, the averaging integral applies only to the continuous Fourier basis and we compute

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^q \sum_{p\in Q^q_N} \int_{C^q(p)} \sum_{k\in\hat{Q}^q_N} f_k e^{i\langle k,y\rangle}\, dy\; e^{-i\langle p,j\rangle}\\
=&\left(\frac{1}{2\pi}\right)^q \sum_{p\in Q^q_N} \sum_{k\in\hat{Q}^q_N} f_k e^{-i\langle p,j\rangle} \int_{C^q(p)} e^{i\langle k,y\rangle}\, dy
\end{aligned}$$

$$=\left(\frac{1}{2\pi}\right)^q \sum_{p\in Q_N^q}\sum_{k\hat{Q}_N^q} f_k e^{-i\langle p,j\rangle} \prod_{k_l\neq 0}\left(\frac{\exp\left(i(p_l+\frac{\pi}{N})k_l\right)-\exp\left(ip_lk_l\right)}{ik_l}\right)\prod_{k_l=0}\left(\frac{\pi}{N}\right)$$

$$=\left(\frac{1}{2\pi}\right)^q \sum_{p\in Q_N^q}\sum_{k\hat{Q}_N^q} f_k e^{-i\langle p,j\rangle_{\mathbb{R}^q}} e^{i\langle p,k\rangle_{\mathbb{R}^q}} \prod_{k_l\neq 0}\left(\frac{\exp\left(i\frac{\pi}{N}k_l\right)-1}{ik_l}\right)\prod_{k_l=0}\left(\frac{\pi}{N}\right)$$

Identifying the product term, as our previously defined weight function, we compute the $l_2(Q_N^q)$ inner product, with respect to p, using that the p-dependent functions are elements of our discrete orthonormal basis:

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^q \sum_{p\in Q_N^q}\sum_{k\hat{Q}_N^q} f_k e^{-i\langle p,j\rangle_{\mathbb{R}^q}} e^{i\langle p,k\rangle_{\mathbb{R}^q}} \prod_{k_l\neq 0}\left(\frac{\exp\left(i\frac{\pi}{N})k_l\right)-1}{ik_l}\right)\prod_{k_l=0}\left(\frac{\pi}{N}\right)\\
=&\left(\frac{1}{2\pi}\right)^q \sum_{p\in Q_N^q}\sum_{k\in\hat{Q}_N^q} f_k e^{i\langle p,k\rangle} e^{-i\langle p,j\rangle} w(N,j,q)\left(\frac{\pi}{N}\right)^q\\
\overset{L1.2.33}{=}&\left(\frac{1}{2N}\right)^q \sum_{k\in\hat{Q}_N^q} f_k (2N)^q \delta_{k,j} w(N,j,q)\\
=&f_j w(N,j,q).
\end{aligned}$$

■

Notation 1.2.54
Let $m\in\mathbb{Z}^q$ and $j\in\hat{Q}_N^q$ then we write $j=m \bmod_{Q_N^q}$ if $j_i-m_i \bmod 2N=0$, for $i=1,\dots,q$.

Lemma 1.2.55
Let $N\in\mathbb{N}$, $f\in h^s(Q_N^q)$. Then for $m\in\mathbb{Z}^q$ and $m \bmod_{\hat{Q}_N^q}=j$ we have

$$\widehat{[f]^{\sim}_{N,q}(\cdot)}(m)=\hat{f}^d(j)w(N,-j,q).$$

Proof
Let $m\in\mathbb{Z}^q$, with $m \bmod_{\hat{Q}_N^q}=j$. Using the periodicity of e^{-ix} we have $e^{-i\langle p,m\rangle}=e^{-i\langle p,j\rangle}$ and thus , obtain the periodicity of the discrete Fourier coefficient, from the computation below.

$$\begin{aligned}
&\widehat{[f]^{\sim}_{N,q}(\cdot)}(m)\\
=&\left(\frac{1}{2\pi}\right)^q\int_{\mathbb{T}^q}[f]^{\sim}_{N,q}(x)e^{-i\langle m,x\rangle_{\mathbb{R}^q}}\,dx\\
=&\left(\frac{1}{2\pi}\right)^q\int_{\mathbb{T}^q}[f]^{\sim}_{N,q}(x)e^{-i\langle j,x\rangle_{\mathbb{R}^q}}\,dx.\\
=&\widehat{[f]^{\sim}_{N,q}(\cdot)}(j).
\end{aligned}$$

Together with the series expansion of the step function $[f]^{\sim}_{N,q}$, we can again shift the averaging integral onto the orthonormal basis and compute:

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^q\int_{\mathbb{T}^q}[f]^{\sim}_{N,q}(x)e^{-i\langle j,x\rangle_{\mathbb{R}^q}}\,dx\\
=&\left(\frac{1}{2\pi}\right)^q\int_{\mathbb{T}^q}\sum_{k\in Q_N^q}f(k)\mathbb{1}_{C^q(k)}(x)e^{-i\langle j,x\rangle_{\mathbb{R}^q}}\,dx\\
=&\left(\frac{1}{2\pi}\right)^q\sum_{k\in Q_N^q}f(k)\int_{C^q(k)}e^{-i\langle j,x\rangle_{\mathbb{R}^q}}\,dx
\end{aligned}$$

$$=\left(\frac{1}{2\pi}\right)^q \sum_{k\in Q_N^q} f(k)\prod_{k_l\neq 0}\left(\frac{\exp\left(-i(k_l+\frac{\pi}{N})j_l\right)-\exp\left(-ij_lk_l\right)}{-ij_l}\right)\prod_{k_l=0}\left(\frac{\pi}{N}\right)$$
$$=\left(\frac{1}{2\pi}\right)^q \sum_{k\in Q_N^q} f(k)e^{-i\langle j,k\rangle_{\mathbb{R}^q}}\prod_{k_l\neq 0}\left(\frac{\exp\left(-i\frac{\pi}{N}j_l\right)-1}{-ij_l}\right)\prod_{k_l=0}\left(\frac{\pi}{N}\right)$$

Identifying the product term, as our previously defined weight function, though this time with a different sign in j, we compute the $l_2(Q_N^q)$ inner product, with respect to k, or rather identify it as the definition of the discrete Fourier transform to conclude:

$$\begin{aligned}&\left(\frac{1}{2\pi}\right)^q \sum_{k\in Q_N^q} f(k)e^{-i\langle j,k\rangle_{\mathbb{R}^q}}\prod_{k_l\neq 0}\left(\frac{\exp\left(-i\frac{\pi}{N}j_l\right)-1}{-ij_l}\right)\prod_{k_l=0}\left(\frac{\pi}{N}\right)\\ =&\left(\frac{1}{2N}\right)^q \sum_{k\in Q_N^q} f(k)e^{-i\langle j,k\rangle_{\mathbb{R}^q}}w(N,-j,q)\\ =&\hat{f}^d(j)w(N,-j,q).\end{aligned}$$

■

Outsourcing from the proof of our upcoming „discrete to continuous norm equivalence“, we relate the discrete and continuous Fourier weights, with the following inequality chain.

Lemma 1.2.56
For every $N\in\mathbb{N}$ and every $j\in\hat{Q}_N^q$, we have:

$$\frac{1}{10}\langle j\rangle \leq \langle 2\frac{N}{\pi}\tilde{g}(\frac{j\pi}{2N})\rangle \leq \langle j\rangle, \tag{1.2.6}$$

where $g\tilde{g}$ is the discrete eigenvalue function from Definition 1.2.37.

Proof
First we note, that by the definition of the Fourier weights $\langle j\rangle := (1+|j|^2)^{1/2}$, the proof can be reduced to the one dimensional case, i.e. showing that

$$\frac{j_l^2}{10}\leq\left(\frac{2N}{\pi}\right)^2\sin^2\left(\frac{j_l\pi}{2N}\right)\leq j_l^2 \quad \text{for all } 1\leq l\leq q.$$

For this we use that for every $N\in\mathbb{N}$ and every $j\in\hat{Q}_N^q$, we have $-N+1\leq j_1,\dots,j_q\leq N$, thus we can substitute $\frac{j_l}{N}$ by $x\in[-1,1]$, for $1\leq l\leq q$, and only have to show

$$\frac{(\pi x)^2}{10}<4\sin^2\left(\frac{\pi x}{2}\right)<(\pi x)^2, \tag{1.2.7}$$

as (1.2.7) multiplied by $\frac{N^2}{\pi^2}$ gives the claim. The upper bound in (1.2.7) follows as

$$4\sin^2\left(\frac{\pi x}{2}\right)\leq\min\{4,(\pi x)^2\},$$

by its power series representation. As for the lower bound we estimate for $x\in\mathbb{B}_{\mathbb{R}}(0,\frac{2}{\pi})$

$$4\sin^2\left(\frac{\pi x}{2}\right)\geq 4\left(\left(\frac{\pi x}{2}\right)^2-\left(\frac{\pi x}{2}\right)^4\frac{1}{3}\right)\geq\frac{2}{3}(\pi x)^2\geq\left(\frac{\pi x}{\sqrt{10}}\right)^2$$

whereas for $x\notin\mathbb{B}_{\mathbb{R}}(0,\frac{2}{\pi})$, we get with a slightly different intermediate estimate:

$$4\sin^2\left(\frac{\pi x}{2}\right)\geq 4\left(\frac{\pi x}{2}\right)^2=(\pi x)^2\geq\left(\frac{\pi x}{\sqrt{10}}\right)^2.$$

■

In this section and chapters final result we prove an equivalence result, connecting the norms over the discrete and continuous torus, not only will this give us the second linear continuum limit theorem for free, once we have proven one, but it will also play an important role in proving that the limiting equation, in the sub critical case, is the limiting equation, of the corresponding linear, discrete equation.

Corollary 1.2.57
Let $f \in H^s(\mathbb{T}^q)$. For all $\alpha \in (0,1)$, such that $\alpha N \in \mathbb{N}$, there exist constants $C, C_\alpha > 0$ such that

$$C_\alpha \left\| P^a_{\alpha N} f \right\|^2_{H^s(\mathbb{T}^q)} \leq \left\| [P^a_N f]_{N,q} \right\|^2_{h^s(Q^q_N)} \leq C \|f\|^2_{H^s(\mathbb{T}^q)}.$$

Proof
By definition,the discrete Plancherel Theorem and Lemma 1.2.53 we have

$$\left\| [P^a_N f]_{N,q} \right\|^2_{h^s(Q^q_N)} = \sum_{j \in \hat{Q}^q_N} \langle 2\frac{N}{\pi} s(\frac{j\pi}{2N}) \rangle^{2s} |\hat{f}^d(j)|^2 = \sum_{j \in \hat{Q}^q_N} \langle 2\frac{N}{\pi} s(\frac{j\pi}{2N}) \rangle^{2s} |f_j|^2 |w(N,j,q)|^2.$$

Now note that

$$|w(N,j,q)|^2 \leq \prod_{l=1}^{q} |w(N,j_l)|^2.$$

Therefore by the weight bounds in Lemma 1.2.52 we get:

$$\left| \frac{e^{i\alpha\pi} - 1}{i\alpha\pi} \right| \sum_{j \in \hat{Q}^q_{\alpha N}} \langle 2\frac{N}{\pi} s(\frac{j\pi}{2N}) \rangle^{2s} |f_j|^2 \leq \left\| [P^a_N f]_{N,q} \right\|^2_{h^s(Q^q_N)} \leq \sum_{j \in \hat{Q}^q_N} \langle 2\frac{N}{\pi} s(\frac{j\pi}{2N}) \rangle^{2s} |f_j|^2.$$

The following estimates, or more precisely the involved constants, depend on the sign of s, therefore we have to analyse $s \geq 0$ and $s < 0$ separately. For $s \geq 0$ we obtain, by Lemma 1.2.56:

$$\left| \frac{e^{i\alpha\pi} - 1}{i\alpha\pi} 10^{-2s} \right| \sum_{j \in \hat{Q}^q_{\alpha N}} \langle j \rangle^{2s} |f_j|^2 \leq \left\| [P^a_N f]_{N,q} \right\|^2_{h^s(Q^q_N)} \leq \sum_{j \in \hat{Q}^q_N} \langle j \rangle^{2s} |f_j|^2 \leq \|f\|_{H^s(\mathbb{T}^q)}.$$

Whereas for $s < 0$ we get, by Lemma 1.2.56:

$$\left| \frac{e^{i\alpha\pi} - 1}{i\alpha\pi} \right| \sum_{j \in \hat{Q}^q_{\alpha N}} \langle j \rangle^{2s} |f_j|^2 \leq \left\| [P^a_N f]_{N,q} \right\|^2_{h^s(Q^q_N)} \leq 10^{-2s} \sum_{j \in \hat{Q}^q_N} \langle j \rangle^{2s} |f_j|^2 \leq 10^{-2s} \|f\|_{H^s(\mathbb{T}^q)}.$$

Setting $C := \max\{1, 10^{-2s}\}$ and $C_\alpha := \min\left\{1, \left|\frac{e^{i\alpha\pi}-1}{i\alpha\pi}\right| 10^{-2s}\right\}$, we obtain the claimed „equivalence“. ∎

2 Probability Essentials

After covering the deterministic theory we now need to understand the stochastic convolution

$$\int_0^T S(t-s)\sigma(s)\, dW(s),$$

where S is the semigroup generated by our partial differential operator A and W is the infinite dimensional Wiener-process associated with our noise term $\sigma\dot{W}$. We begin with some general theory on stochastic processes, before extending the concept of a Gaussian process into infinite dimensions. We then define the stochastic integral with respect to infinite dimensional Wiener-processes and state the important **Itô-isometry**. Finally we extend the analytical concept of mild solutions to SPDE's. For a comprehensive introduction into this theory we refer the reader to the paper [Tap13b] or the monographs [PR07], [GM11] and [DPZ14].

2.1 Stochastic Processes

Following [JS13] though admitting more general index sets we introduce some basics on stochastic processes.

Definition 2.1.1 (Stochastic Basis)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $(T, \leq)$ a totally ordered set or $T = \mathbb{R}^n$, for some $n \in \mathbb{N}$.

*(i) A family $\mathbb{F} := \{\mathcal{F}\}_{t\in T}$ of sub-σ-algebras (of $\mathcal{F}$) is called **filtration**, if it is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s \leq t$. The tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called **stochastic basis** .*

*(ii) If $\mathcal{F}$ is $\mathbb{P}$-complete a filtration $\mathbb{F} := \{\mathcal{F}\}_{t\in T}$ is called **normal** if it is right-continuous, i.e.*

$$\mathcal{F}_t = \mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s,$$

and for all $A \in \mathcal{F}$: $\mathbb{P}(A) = 0$ implies $A \in \mathcal{F}_0$.

*(iii) A stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called **complete** or said to **satisfy the usual conditions**, if the filtration $\mathbb{F}$ is normal.*

Remark 2.1.2
From now on we will always assume that our stochastic basis satisfies the usual conditions.

Definition 2.1.3
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a stochastic basis and $(T, \leq)$ a totally ordered set or $T = \mathbb{R}^n$, for some $n \in \mathbb{N}$.

- *The subsets of $\Omega \times T$ are called **random sets**.*
- *A random set A is called **evanescent**, if*

$$\mathbb{P}(\{\omega \mid \text{there exists a } t \in T \text{ such that } (\omega, t) \in A\}) = 0.$$

- *Two E-valued processes X and Y are said to be **indistinguishable** if the random set $\{X \neq Y\} := \{(\omega, t) \in \Omega \times T \mid X_t(\omega) \neq Y_t(\omega)\}$ is evanescent.*

Definition 2.1.4 (Stochastic Process)
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a stochastic basis and $(T, \leq)$ a totally ordered set or $T = \mathbb{R}^n$, for some $n \in \mathbb{N}$.

(i) An (E-valued) ***stochastic process*** *(or* ***process*** *for short) is a family $X = \{X_t\}_{t \in T}$ of mappings*

$$X_t : \Omega \to E \quad \textit{for all } t \in T.$$

(ii) If $X = \{X_t\}_{t \in T}$ is a process and $T \subset \mathbb{R}^q$ for some $q \in \mathbb{N}$, then X is called a ***random field****.*

(iii) A stochastic process is called ***adapted to the filtration*** *$\mathbb{F}$ (or* ***adapted*** *for short), if X_t is $\mathcal{F}_t$-measurable for every $t \in T$.*

Definition 2.1.5 (Previsible-σ-Algebra)
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a stochastic basis and $(T, \leq)$ a totally ordered set or $T = \mathbb{R}^n$, for some $n \in \mathbb{N}$.

(i) The ***predictable-σ-algebra*** *is defined as*

$$\mathcal{P}_T := \sigma(\{X : \Omega \times T \to E \mid X \textit{ is adapted and leftcontinous}\}),$$

(ii) If a process X is $\mathcal{P}_T - \mathcal{B}(E)$–measurable as a map $X : \Omega \times T \to E$, X is called ***predictable****. And we denote the set of all predictable process by*

$$\mathcal{P}_T(\Omega, E) := \{X : \Omega \times T \to E \mid X \textit{ is a predictable process}\}.$$

An important class of stochastic processes are martingales, as they are the first class of candidates for stochastic integrators, later though not in this thesis generalized to semimartingales.

Definition 2.1.6 (Martingale)
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a stochastic basis and $(T, \leq)$ a totally ordered set or $T = \mathbb{R}^n$, for some $n \in \mathbb{N}$.
A stochastic process X is called a ($\mathbb{F}$-)Martingale, if $\mathbb{P}$ almost all of its paths are càdlàg, such that every X_t is integrable and for every $s \leq t$ we have:

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s.$$

An important property of martingales and one our mild solutions will sadly lack is the following.

Theorem 2.1.7 (Maximal Inequality)
Let $p > 1$ and $\{M(t)\}_{t \in [0,T]}$ a right-continuous H-valued $\mathbb{F}$–martingale, then

$$\left(\mathbb{E}\left[\sup_{t \in [0,T]} \|M(t)\|_H^p \right] \right)^{1/p} \leq \frac{p}{p-1} \left(\mathbb{E}\left[\|M(T)\|_H^p \right] \right)^{1/p}.$$

Proof
Confirm [PR07, Theorem 2.2.7]. ■

2.2 Gaussian Processes in Infinite Dimensions

Now additionally following [Ver10] and [Sch18], we will discuss the processes that we will use to model the noise term in our stochastic equations. But before that we will discuss an important property of Gaussian processes, that we will need for our Moment bounds in the non-linear setting. (The property itself is a well-known exercise, but as I didn't find a source a brief sketch is given.)

Lemma 2.2.1
For a Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ and $p \in \mathbb{N}$ we have:

$$\mathbb{E}[X^p] = \sum_{k=0}^{\lfloor p/2 \rfloor} \binom{p}{2k} (2k-1)!! \sigma^{2k} \mu^{p-2k}.$$

Proof
First we consider $X \sim \mathcal{N}(0,\sigma^2)$. Using (iterated) Integration by parts we get for p odd

$$\mathbb{E}[X^p] = (p-1)!!(\sigma^2)^{(p-1)}\mathbb{E}[X] = 0,$$

and for p even

$$\mathbb{E}[X^p] = (p-1)!!(\sigma^2)^{(p/2-1)}\mathbb{E}[X^2] = (p-1)!!(\sigma^2)^{(p/2-1)}\sigma^2 = (p-1)!!(\sigma^2)^{p/2}.$$

Now for $X \sim \mathcal{N}(\mu,\sigma^2)$ we set $X = Z + \mu$, with $Z \sim \mathcal{N}(0,\sigma^2)$. Then we compute using the binomial theorem

$$\mathbb{E}\left[X^p\right] = \mathbb{E}\left[(Z+\mu)^p\right] = \sum_{k=0}^{p}\binom{p}{k}\mathbb{E}\left[Z^k\right]\mu^{p-k}$$

using the previous computations we obtain

$$\mathbb{E}\left[X^p\right] = \sum_{k=0}^{\lfloor p/2 \rfloor}\binom{p}{2k}(2k-1)!!\sigma^{2k}\mu^{p-2k}.$$

Which proofs the assertion. ∎

Wiener Processes in Infinite Dimensions

Following for example [PR07] or [GM11] we first introduce the infinite dimensional counterpart of the classical Wiener process before we will use it to model space-time white noise.

Definition 2.2.2 (Q-Wiener Process)
*Let Q a trace class operator on H. A H-valued stochastic process $\{W(t)\}_{t\in[0,T]}$, on a stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, is called a Q-**Wiener process**, if*

(i) $W(0) = 0$ *($\mathbb{P}$-almost surely)*

(ii) *W has $\mathbb{P}$-almost surely continuous trajectories,*

(iii) *for any finite sequence $0 = t_0 < t_1 < \ldots < t_n \leq T$ ($n \in \mathbb{N}$) the increments*

$$W(t_1), W(t_2) - W(t_1), \ldots W(t_n) - W(t_{n-1})$$

are independent random variables.

(iv) *the increments obey the following Gaussian laws:*

$$\mathbb{P} \circ (W(t) - W(s))^{-1} = N(0,(t-s)Q) \quad \textit{for all } 0 \leq s \leq t \leq T.$$

Lemma 2.2.3
Let Q be a trace-class operator on H with an orthonormal basis of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$ and corresponding eigenvalues $\{\lambda_k^2\}_{k\in\mathbb{N}}$. If $\{W(t)\}_{t\in T}$ is a Q-Wiener process, then there exists a sequence of independent real- or complex-valued Brownian motions such that

$$W(t) = \sum_{k\in\mathbb{N}} \lambda_k^2 \beta_k(t) e_k.$$

Definition 2.2.4 (Cylindrical-Q-Wiener Process)
Let $Q \in \mathcal{L}(H)$, not necessarily of trace-class, with an orthonormal basis of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$ and corresponding eigenvalues $\{\lambda_k^2\}_{k\in\mathbb{N}}$ and $\{W(t)\}_{t\in T}$ a $H-$valued stochastic process. If there exists a sequence of independent real- or complex-valued Brownian motions such that

$$W(t) = \sum_{k\in\mathbb{N}} \lambda_k^2 \beta_k(t) e_k,$$

*converges in a larger Hilbert space U, we call $\{W(t)\}_{t\in T}$ a **cylindrical-Q-Wiener process**.*

Lemma 2.2.5
Let $Q \in \mathcal{L}(H)$, not necessarily of trace-class, with an orthonormal basis of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$ and corresponding eigenvalues $\{\lambda_k^2\}_{k\in\mathbb{N}}$. If $\{W(t)\}_{t\in T}$ is a Q-Wiener process, the there exists a sequence of real-valued Brownian motions such that

$$W(t) = \sum_{k\in\mathbb{N}} \lambda_k^2 \beta_k(t) e_k.$$

White Noise

Now we will introduced what informally could be called the derivative of our previously established, infinite dimensional Wiener processes, by defining space- time white noise as generalized processes, thereby turning this informal notion into a formal one.

Definition 2.2.6 (Noise)
Let W be a H-valued cylindrical-Q-Wiener-process, supported by a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, whose representation with respect to the complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ of H is given by

$$W(t) = \sum_{k\in\mathbb{N}} \lambda_k^2 \beta_k(t) e_k,$$

*with a family of independent real-or complex-valued Brownian motions $\{\beta_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k^2\}_{k\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$, then the corresponding **space time white noise** is given by*

$$\dot{W}(t) = \sum_{k\in\mathbb{N}} \lambda_k \dot{\beta}_k(t) e_k.$$

Where the time derivatives $\dot{\beta}_k$, are to be understood in the sense of generalized Brownian motions and $H \subset \mathcal{D}(\mathbb{T}^q)$.

Definition 2.2.7
Let $\xi := \{\sigma \dot{W}\}_{t\in T}$ a noise process.

*(i) ξ is called **additive noise** if σ is independent of $\dot{W}$.*

*(ii) ξ is called **multiplicative noise** if $\sigma = \sigma(\dot{W})$.*

Example 2.2.8
By choosing $H = L_2(\mathbb{T}^q)$ and $U = H^s(\mathbb{T}^q)$ in 2.2.6, for $s < \frac{-d}{2}$, we obtain analogous to / generalizing [Ver10], space time white noise.

Remark 2.2.9
Another approach to modeling and was introduced for example by [Wal86] and [HKPS93] uses the Bochner-Minlos Theorem to construct an infinite dimensional Gaussian probability measure.

Stochastic Integration in Infinite Dimensions

For the standard bottom up construction of the stochastic integral with respect to Q- respectivly cylindrical-Wiener processes we refer the reader to [Tap13b] or the monographs [PR07], [GM11] and [DPZ14]. Nevertheless we want to list two important properties of the stochastic integral and a likewise important consequence of them, which will be used frequently in the proofs of our main results.

Theorem 2.2.10 (Itō-Isometry)
Let $\Phi \in L_2([0,T]\times\Omega, \mathcal{P}_T, dt\otimes\mathbb{P}_T, L_2(Q^{1/2}), U)$, then

$$\mathbb{E}\left[\left\|\int_0^T \Phi(s)\, dW(s)\right\|_U^2\right] = \mathbb{E}\left[\int_0^T tr(\Phi(s)^*\Phi(s))\, ds\right]$$

Proof
Confirm [PR07, Proposition 2.3.5]. ■

Theorem 2.2.11 (Stochastic Fubini Theorem)
Let $(E,\mathcal{E},\mu)$ a finite measure space, with positive measure μ and $\Phi(\cdot,\cdot,\cdot)$ a $\mathcal{P}_T \times \mathcal{E} - \mathcal{B}(\mathcal{L}_2(H,U))$-measurable mapping. Assume moreover

$$\int_E \left(\mathbb{E}\left[\int_0^t tr(\Phi(t,\omega,x)^* \Phi(s,\omega,x))\, ds \right] \right)^{1/2} d\mu(x) < \infty.$$

then, almost surely

$$\int_E \int_0^t \Phi(s,x)\, dW(s)\, d\mu(x) = \int_0^t \int_E \Phi(s,x)\, d\mu(x)\, dW(s).$$

Proof
Confirm for example [DPZ14]. ■

Corollary 2.2.12
Let β a complex Wiener-process, i.e $\beta_t \sim \mathcal{CN}(0,0,t-s)$ and $\beta = \beta_{\mathrm{Re}} + i\beta_{\mathrm{Im}}$, where $\beta_{\mathrm{Re}}, \beta_{\mathrm{Im}}$ are real independent Wiener-process with $\beta_{\mathrm{Re},t}, \beta_{\mathrm{Im},t} \sim \mathcal{N}(0,t/2)$. Let further $f \in C([0,T])$ then for every $2 \le p \in \mathbb{N}$ there exists a constant $C(p) > 0$:

$$\mathbb{E}\left[\left| \int_0^T f(s)\, d\beta(s) \right|^{2p} \right] = C(p) \left(\int_0^T f^2(s)\, ds \right)^p.$$

We now want to consider the stochastic partial differential equation

$$\partial_t u(t,x) = Au(t,x) + f(t,x,u) + \sigma(t)\dot{W}(t,x) \tag{2.2.1}$$

on the separable Hilbert space H, where H will usually be a Sobolev-Hilbert space $H^s(\mathbb{T}^q)$ and reintroduce the core concepts, of a mild solution.

Definition 2.2.13 (Mild Solution)
*A H-valued predictable process $\{u(t)\}_{t\in[0,T]}$, is called a **mild solution** of the initial value problem corresponding to (2.2.1), with $u(0) = \xi$, if*

$$u(t) = S(t)\xi + \int_0^t S(t-s) f(s,u(s))\, dS + \int_0^t S(t-s)\sigma(s)\, dW(s)$$

holds $\mathbb{P}$-almost surely for each $t \in [0,T]$.

As we are dealing with a 2nd order evolution equation we need to extend this definition to account for the higher order as we did in the deterministic case. We now consider the stochastic partial differential equation

$$\partial_t ttu(t,x) = Au(t,x) + B\partial_t ttu(t,x) + f(t,x,u) + \sigma(t)\dot{W}(t,x) \tag{2.2.2}$$

on the separable Hilbert space H, where H will usually be a Sobolev-Hilbert space $H^s(\mathbb{T}^q)$ and reintroduce the core concepts, of a mild solution.

Definition 2.2.14 (Mild Solution)
Let u a mild solution of the first order system corresponding to (2.2.2),then we call $u_1 = \pi_1(u)$ a mild solution of (2.2.2), where π_1 is the first coordinate projection.

Part I

Central Results

3 Continuum Limit Theorems

While the previous part of this thesis was all about compiling established results, for future usage and to provide a contextual framework for the upcoming results, this chapter jumps directly in medias res, with the presentation of of the continuum limit theorems, that are the central results of this thesis. In Section 3.1 frequently used terminology is recalled[1] respectively introduced and the general framework, in which both the linear and non-linear continuum limit theorems are formulated, is laid out. This entails in particular the general assumptions of the main results and supporting results. In Section 3.3 and Section 3.3 the continuum limit theorems for linear respectively non-linear equations are given.

3.1 General Framework

The deterministic and stochastic processes that are approximated, as well as those that are used to approximate, are continuous in forward time (i.e. their index sets are subsets of $[0,\infty)$) and take one of two types of spaces, of (generalized) functions as their state space. The state spaces are either one of the Sobolev Hilbert spaces $H^r(\mathbb{T}^q)$, on the torus $\mathbb{T}^q = [-\pi,\pi)$, or a discrete l_2-space , on one of the grids $\{Q_N^q\}_{N\in\mathbb{N}^*}$, with an additional norm weight based on the discrete Laplacian , denoted by $h^s(Q_N^q)$. Where $Q_N^q = \{-\pi,\dots,-\frac{\pi}{N},0,\frac{\pi}{N},\dots,\frac{(N-1)}{N}\pi\}^q$ denotes the equidistant $(2N)^q$-point discretization of the Torus $\mathbb{T}^q$. $\mathbb{T}^q$ as underlying spatial structure, will be referred to as the „continuum torus"conversely the grids $\{Q_N^q\}_{N\in\mathbb{N}^*}$ will be referred to as „discrete tori ", the respective settings will be adressed as „continuum case / setting"and „discrete case / setting". The same linguistic distinction is applied to all the mathematical objects (equations, processes, spaces. etc.) related to one of the aforementioned settings. Both the approximated and approximating processes arise from solving[2] (stochastic) wave-type equations, that is second order[3] (stochastic) evolution equations, generated by an evolution operator E of the form,

$$E = \partial_{tt} - A,$$

where A is a partial differential operator (continuum case) respectively a symmetric[4] partial difference quotient operator (discrete case), of order $2m$ acting w.r.t. the spatial variables. In the continuum setting this leads to the initial value problem

$$\begin{cases} \partial_{tt}u(t,x) = Au(t,x) + Du(t,x) + f(t,x) + \sigma(t)W(t,x), & (t,x)\in\mathbb{R}_+\times\mathbb{T}^q,\\ u(0,\cdot) = u_0(\cdot), & \text{in } \mathbb{T}^q,\\ \partial_t u(0,\cdot) = u_1(\cdot), & \text{in } \mathbb{T}^q, \end{cases} \tag{EQ1}$$

while in the discrete setting this spawns a family of initial value problems of the form:

$$\begin{cases} \partial_{tt}X_N(t,l) = A_{d,N}X_N(t,l) + \partial_t D_N X_N(t,l) + b(X_N,N) + f_N(t,l) + \sigma(t)B_N(t,l), & (t,l)\in\mathbb{R}_+\times Q_N^q,\\ X_N(0,\cdot) = X_{N,0}(\cdot), & \text{in } Q_N^q,\\ \partial_t X_N(0,\cdot) = X_{N,1}(\cdot), & \text{in } Q_N^q, \end{cases} \tag{DEQ1}$$

1) If already established in Part I.

2) For a direct comparison the states of both processes have to be in the same setting, therefore a discretization or interpolation is needed.

3) In time.

4) This refers to the type of difference.

which introduces the additional term $b(X_N, N)$, that will be omitted[5)], in case of linear discrete equations, and otherwise represents the scaled polynomial non-linearity. The terms $\partial_t D_N X_N$ and $\partial_t Du$ are the respective discrete and continuum damping terms, with damping operators D_N and D of the form

$$D = -\rho_1 + \rho_2 \Delta \ \text{ and } \ D = -\rho_1 + \rho_2 \Delta_{N,d}, \ \ \rho_1, \rho_2 \geq 0. \tag{DMP}$$

This gives rise to the following classification.

Definition 3.1.1
The partial differential / difference quotient equations in (EQ1) respectively (DEQ1) and thereby the initial value problems (EQ1) respectively (DEQ1) are called:

(i) ***undamped****, if $\rho_1 = 0 = \rho_2$,*

(ii) ***damped****, if $\rho_1 > 0$,*

(iii) ***strongly damped****, if $\rho_2 > 0$.*

Regarding the nature of the stochastic integrals the following can be observed.

Remark 3.1.2
As the volatility function σ is deterministic, there is no difference in understanding the partial differential / difference quotient equations in (EQ1) and (DEQ1) in the Itō or the Stratonovic sense.

3.2 General Assumptions

The following will be assumed throughout the rest of this thesis unless specified otherwise[6)] for the equations EQ1 respectively DEQ1.

(A1) $u_0 \in H^{r+m}(\mathbb{T}^q), u_1 \in H^r(\mathbb{T}^q)$, and $f \in C(\mathbb{R}_{\geq 0}, H^r(\mathbb{T}^q))$ for some $r \in \mathbb{R}$.

(A2) For every $N \in \mathbb{N}^*$ we have $X_{N,0} \in h^{r+m}(Q_N^q), X_{N,1} \in h^r(Q_N^q)$, $f_N \in C(\mathbb{R}_{\geq 0}, h^r(Q_N^q))$ such that

$$\begin{aligned}
\tilde{X}_{N,0} &\overset{N\to\infty}{\longrightarrow} u_0 \ \text{ in } \ H^{r+m}(\mathbb{T}^q), \\
\tilde{X}_{N,1} &\overset{N\to\infty}{\longrightarrow} u_1 \ \text{ in } \ H^{r}(\mathbb{T}^q) \ \text{ and} \\
\tilde{f}_N &\overset{N\to\infty}{\longrightarrow} f \ \text{ in } \ C([0,T], H^{r}(\mathbb{T}^q)) \ \ \forall T > 0.
\end{aligned}$$

5) As its contributions can be covered by A_N and f_N in the linear case.

6) This will be especially relevant w.r.t. the initial data and forcing as we have to require a in general stronger spatial regularity

(A3) $A = p(A, \partial)$ is a partial differential operator of order $2m$ of the form

$$A = (-1)^{m-1}\Delta^m + \sum_{i=1}^{m-1} b_i(-1)^{i-1}\Delta^i + \sum_{|\alpha|=1}^{m} a_\alpha \partial^\alpha,$$

for positive coefficients a_α and b_i.

(A4) The corresponding discrete partial difference quotient operator A is obtained by plugging the discrete differential quotients ∂_h into the same polynomial that encodes A, i.e. $(A_{d,N})_{N\in\mathbb{N}^*} = p(A, \partial_h)$.

(A5) W is a space-time white noise supported on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with volatility function $\sigma \in BC(\mathbb{R}_{\geq 0})$.

(A6) For every $N \in \mathbb{N}^*$ B_N is the Gaussian process given by the following L_2-averages of the asymmetric Galerkin projection of W

$$B_N(t, l) := \langle P_N^\alpha W(t, x), \mathbb{1}_{C^q(l)} \rangle_{L_2(\mathbb{T}^q)} \quad \forall\, l \in Q_N^q.$$

(A7) The respective damping operators are defined according to (DMP).

Remark 3.2.1
In the stochastic case there is no difference in understanding (EQ1) and (DEQ1) in the Itō or the Stratonovic sense, as the volatility function σ is deterministic.

Lets also recall the following notations/definitions from the preliminaries.

Remark 3.2.2
We recall that P_N^a is the Fourier cutoff projector from Definition 1.2.4, from now on always with respect to the complete orthonormal system Lemma 1.2.14.

Remark 3.2.3
The general assumptions are formulated with real world data in mind. If one has freedom of choice when it comes to the initial data and forcing, the „most natural choices" would be

$$X_{N,0} := \langle P_N^a u_0(x), \mathbb{1}_{C^q(l)} \rangle_{L_2(\mathbb{T}^q)}, \quad X_{N,1} := \langle P_N^a u_1(x), \mathbb{1}_{C^q(l)} \rangle_{L_2(\mathbb{T}^q)} \quad \textit{and} \quad f_N := \langle P_N^a f(t,x), \mathbb{1}_{C^q(l)} \rangle_{L_2(\mathbb{T}^q)},$$

as they are exactly the discretized, projections of the corresponding continuum quantities.

3.3 Main Results

Now we are going to state our continuum limit theorems, while the proofs will be done over the next three chapters. In Chapter 4 we will proof the existence of the mild solutions and develop a representation theory for the associated operator semi-groups. In Chapter 5 we will use the semi-group representation to derive error estimates, which we combine in Chapter 6 into the respective three lemmata, that prove the statement of our main results for one specific type of damping.

3.3.1 Linear Main Results

Now we present our linear main results. Even though the stochastic mild solutions, found in Chapter 4, have the same regularity in time as their deterministic counterparts, the same does not hold true for the continuum limit. I.e. even though the stochastic mild solutions u and $\{X_N\}_{N\in\mathbb{N}^*}$ are Gaussian they are no martingales, therefore we lack the Maximal Inequality to exchange the expectation and the supremum, from the norm. Therefore we state our stochastic continuum limit theorems in the Bochner-Lebesgue spaces $L_p([0,T], H^s(\mathbb{T}^q))$, where we can interchange the expectation and temporal integration using the Stochastic Fubini Theorem, allowing us to use the Itō Isometry, when computing our error estimates.

Theorem 3.3.1 (Stochastic Continuum Limit Theorem)
Let $A, D, f, W, u_0, u_1, (A_{d,N})_{N\in\mathbb{N}^*}, (D_N)_{N\in\mathbb{N}^*}, (f_N)_{N\in\mathbb{N}^*}, (B_N)_{N\in\mathbb{N}^*}, (X_{N,0})_{N\in\mathbb{N}^*}, (X_{N,1})_{N\in\mathbb{N}^*}$ and σ satisfy the general assumptions (A1)-(A8), for $r < -\frac{q}{2}$. Then for every $s \leq r$ (EQ1) has a unique stochastic strong, mild solution $u \in H^{s+m}(\mathbb{T}^q)$, (DEQ1) has a unique stochastic strong, mild solution $X_N \in h^{s+m}(Q_N^q)$ for every $N \in \mathbb{N}^*$ and for all $T > 0$ and every $p > 1$ we have:

$$\mathbb{E}\left[\left\|u(t,x) - \tilde{X}_N(t,x)\right\|^2_{L_p([0,T],H^{s+m}(\mathbb{T}^q))}\right] \overset{N\to\infty}{\longrightarrow} 0.$$

and

$$\mathbb{E}\left[\left\|\partial_t u(t,x) - \partial_t \tilde{X}_N(t,x)\right\|^2_{L_p([0,T],H^{s}(\mathbb{T}^q))}\right] \overset{N\to\infty}{\longrightarrow} 0.$$

And for $\sigma \equiv 0$ we obtain

$$\left\|u(t,x) - \tilde{X}_N(t,x)\right\|^2_{C([0,T],H^{s+m}(\mathbb{T}^q))} \overset{N\to\infty}{\longrightarrow} 0$$

and

$$\left\|\partial_t u(t,x) - \partial_t \tilde{X}_N(t,x)\right\|^2_{C([0,T],H^{s}(\mathbb{T}^q))} \overset{N\to\infty}{\longrightarrow} 0,$$

even for every regularity $r \in \mathbb{R}$ of the input data.

Proof
This follows by Lemma 6.2.2. ∎

This time discretizing the asymmetric Galerkin Projection $P_N^a u$ (onto the cube $[-N, N-1]^q$), of the continuum solution u, instead of continuating X_N, we obtain the following continuum limit theorem in the discrete norms.

Theorem 3.3.2 (Stochastic Discrete Continuum Limit Theorem)
Under the assumptions of Theorem 3.3.1, for every $s \leq r < -\frac{q}{2}$ (EQ1) has a unique stochastic strong, analytically mild solution $u \in H^{s+m}(\mathbb{T}^q)$, (DEQ1) has a unique stochastic strong, mild solution $X_N \in h^{s+m}(Q_N^q)$ for every $N \in \mathbb{N}^*$, such that for every $T > 0$ and every $p > 1$ we have:

$$\mathbb{E}\left[\left\|[P_N^a u]_{N,q}(t,l) - X_N(t,l)\right\|^2_{L_p([0,T],h^{s+m}(Q_N^q))}\right] \overset{N\to\infty}{\longrightarrow} 0.$$

and

$$\mathbb{E}\left[\left\|[P_N^a \partial_t u]_{N,q}(t,l) - \partial_t X_N(t,l)\right\|^2_{L_p([0,T],h^{s}(Q_N^q))}\right] \overset{N\to\infty}{\longrightarrow} 0.$$

And for $\sigma \equiv 0$ we obtain

$$\left\|[P_N^a u]_{N,q}(t,l) - X_N(t,l)\right\|^2_{C([0,T],h^{s+m}(Q_N^q))} \overset{N\to\infty}{\longrightarrow} 0$$

and

$$\left\|[P_N^a \partial_t u]_{N,q}(t,l) - \partial_t X_N(t,l)\right\|^2_{C([0,T],h^{s}(Q_N^q))} \overset{N\to\infty}{\longrightarrow} 0.$$

even for every regularity $r \in \mathbb{R}$ of the input data.

Proof
Follows by Theorem 3.3.1 and and the „norm equivalence“in Corollary 1.2.57 ∎

Remark 3.3.3
The bound for the spatial regularity, depending on the dimension, is exactly as one would expect, based on the regularity of Gaussian space time white noise, when formally viewing it as a forcing term, of the same spatial regularity.

4 Existence of Mild Solutions and Semigroup Representation

In this chapter we proof the existence of mild solutions of both the discrete and continuous equations and develop a representation theory for them, by transforming them into first order evolution systems of the form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix},$$

with an operator matrix M. In Section 4.2 we proof that M generates a C_0 semigroup and deduce the existence of a unique mild solution for the corresponding equation. Finally we use the Fourier multiplier nature of our semigroups to develop a representation for them under Fourier transform Section 4.3, which we will use in Chapter 5 to show the convergence of the low frequency Fourier modes.

4.1 Mild Solutions

To use the semigroup approach, for first order evolution equations, the usual trick (confirm [Paz12] or [DPZ14]) to transform the equations from second order equations into first order systems is used, i.e. for the equation

$$\partial_{tt} u(t,x) = Au(t,x) + Du(t,x) + f(t,x) + \sigma(t)W(t,x)$$

we get, setting $v = \partial_\tau u$, And the system corresponding to (EQ1) for $D = \rho_2\Delta - \rho$ is given by:

$$\begin{cases} \partial_t \begin{pmatrix} u(t,x) \\ \partial_t u(t,x) \end{pmatrix} = \begin{pmatrix} 0 & Id \\ A & \rho_2\Delta \end{pmatrix} \begin{pmatrix} u(t,x) \\ \partial_t u(t,x) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t,x) \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma(t)\dot{W}(t,x) \end{pmatrix}, & (t,x) \in \mathbb{R}_+ \times \mathbb{T}^q, \\ \begin{pmatrix} u(0,\cdot) \\ \partial_t u(0,\cdot) \end{pmatrix} = \begin{pmatrix} u_0(\cdot) \\ u_1(\cdot) \end{pmatrix} & \text{in } \mathbb{T}^q. \end{cases} \tag{Sys1}$$

And the system corresponding to (DEQ1) for $D = \rho_2\Delta_{N,d} - \rho_1$ is given by:

$$\begin{cases} \partial_t \begin{pmatrix} X_N(t,l) \\ \partial_t X_N(t,l) \end{pmatrix} = \begin{pmatrix} 0 & Id \\ A_{d,N} & \rho_2\Delta_{N,d} \end{pmatrix} \begin{pmatrix} X_N(t,l) \\ \partial_t X_N(t,l) \end{pmatrix} + \begin{pmatrix} 0 \\ f_N(t,l) \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma(t)\dot{B}_N(t,l) \end{pmatrix}, & (t,l) \in \mathbb{R}_+ \times Q_N^q, \\ \begin{pmatrix} X_N(0,\cdot) \\ \partial_t X_N(0,\cdot) \end{pmatrix} = \begin{pmatrix} X_{N,0}(\cdot) \\ X_{N,1}(\cdot) \end{pmatrix} & \text{in } Q_N^q. \end{cases} \tag{DSys1}$$

4.2 Mild Solutions

With our equations now transformed into first order evolution systems, with respect to time, we will show that for every $r \in \mathbb{R}$ the $D(L^{1/2}) \times H^r(\mathbb{T}^q)$ realization of our operator matrix, generates a C_0 semigroup and that gives us a mild solution of our evolution equation. (The same holds true for the $D(L_{N,d}^{1/2}) \times h^r(Q_N^q)$ realizations of our discrete operators). Here $D(L^{1/2})$ (resp. $D(L_{N,d}^{1/2})$) again denote the formal domain of the square root of L (resp. $L_{N,d}$) equipped with the graph norm. The square root exists, by a standard argument from spectral analysis, and will be an integral tool in the proofs of the generator property, as it allows us to extend the integration by parts argument, used for the wave equation or rather the Laplacian, when showing dissipativity of the operator matrix, to our more general situation.

Though before proving the infinitesimal generator property of our operator matrices, we recall the following embeddings, that will ensure that all of them are densely defined. While the continuum embeddings are the well-established Sobolev embeddings, the discrete embeddings follow by the equivalence of the discrete spaces remarked in Remark 1.2.38.

Remark 4.2.1
The following continuous embeddings hold:

(i)

$$H^{r+2m}(\mathbb{T}^q) \hookrightarrow D(L) \hookrightarrow H^r(\mathbb{T}^q).$$

(ii)

$$H^{r+m}(\mathbb{T}^q) \hookrightarrow D(L^{1/2}) \hookrightarrow H^r(\mathbb{T}^q).$$

(iii)

$$h^{r+2m}(Q_N^q) \hookrightarrow D(L_{N,d}) \hookrightarrow h^r(Q_N^q).$$

(iii)

$$h^{r+m}(Q_N^q) \hookrightarrow D(L_{N,d}^{1/2}) \hookrightarrow h^r(Q_N^q).$$

Theorem 4.2.2
With A as in $(A3)$ the operator matrix in (Sys1) generates a C_0 Semigroup $\{S_3(t)\}_{t\geq 0}$ on $D(L^{1/2}) \times H^r(\mathbb{T}^q)$, for every $r \in \mathbb{R}$.

Proof
Note that, by Theorem 1.1.19 again, it suffices to show that the linear operator

$$\begin{cases} D(M_3) := \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in D(L) \times D(L^{1/2}) \,\middle|\, \Delta g_2 \in H^r(\mathbb{T}^q) \right\}, \\ M_3 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \begin{pmatrix} 0 & Id \\ -L & \rho_2 \Delta \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \forall \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in D(M_3), \end{cases}$$

generates a C_0-semigroup. As $H^{r+2m}(\mathbb{T}^q) \subset D(L)$ and $H^{r+m+2}(\mathbb{T}^q) \subset D(L^{1/2})$ M_1 is densely defined and closed, as $-L, \Delta \in \mathcal{A}(H^r(\mathbb{T}^q))$ as a self-adjoint operators and $Id \in \mathcal{L}(D(L^{1/2}))$.
Now for $g = (g_1, g_2)^T \in \mathrm{dom}(M_3)$ we compute:

$$\begin{aligned} \langle M_3 g, g \rangle_{D(L^{1/2}) \times H^r(\mathbb{T}^q)} =& \langle L^{1/2} g_1, L^{1/2} g_2 \rangle_{H^r(\mathbb{T}^q)} + \langle -L g_1, g_2 \rangle_{H^r(\mathbb{T}^q)} + \langle \rho_2 \Delta g_2, y_2 \rangle_{H^r(\mathbb{T}^q)} \\ =& \langle L^{1/2} g_1, L^{1/2} g_2 \rangle_{H^r(\mathbb{T}^q)} - \langle L^{1/2} g_1, L^{1/2} g_2 \rangle_{H^r(\mathbb{T}^q)} + \langle \rho_2 \Delta g_2, g_2 \rangle_{H^r(\mathbb{T}^q)} \\ =& + \langle \rho_2 \Delta g_2, g_2 \rangle_{H^r(\mathbb{T}^q)} \leq 0. \end{aligned}$$

Therefore by Corollary 1.1.20 M_3 is dissipative. Note that the adjoint operator of M_3 is given by

$$\begin{cases} D(M_3^*) = \mathrm{dom}(M_3), \\ M_3^* \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \begin{pmatrix} 0 & -Id \\ L & \rho_2 \Delta \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \forall \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in D(M_3), \end{cases}$$

and compute for $g = (g_1, g_2)^T \in \mathrm{dom}(M_3)$:

$$\begin{aligned} \langle M_3 g, g \rangle_{D(L^{1/2}) \times H^r(\mathbb{T}^q)} =& \langle -L^{1/2} g_2, L^{1/2} g_1 \rangle_{H^r(\mathbb{T}^q)} + \langle L g_1, g_2 \rangle_{H^r(\mathbb{T}^q)} + \langle \rho_2 \Delta g_2, y_2 \rangle_{H^r(\mathbb{T}^q)} \\ =& - \langle L^{1/2} g_1, L^{1/2} g_2 \rangle_{H^r(\mathbb{T}^q)} + \langle L^{1/2} g_1, L^{1/2} g_2 \rangle_{H^r(\mathbb{T}^q)} + \langle \rho_2 \Delta g_2, g_2 \rangle_{H^r(\mathbb{T}^q)} \\ =& \langle \rho_2 \Delta g_2, g_2 \rangle_{H^r(\mathbb{T}^q)} \leq 0. \end{aligned}$$

Thus M_3^* is also dissipative and the assertion follows by Corollary 1.1.24. ■

Corollary 4.2.3
For initial data and forcing as in $(A1)$ and noise as in $(A5)$ the unique stochastic strong mild solution of (Sys1), by definition, is given by

$$\begin{pmatrix} u(t,x) \\ \partial_t u(t,x) \end{pmatrix} = S_3(t) \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} + \int_0^t S_3(t-s) \begin{pmatrix} 0 \\ f(s,x) \end{pmatrix} ds + \int_0^t S_3(t-s) \begin{pmatrix} 0 \\ \sigma(s) \end{pmatrix} dW(s,x).$$

Proof
This follows by Theorem 4.2.2. ∎

Corollary 4.2.4
For $\sigma \equiv 0$, initial data and forcing as in $(A1)$ and noise as in $(A5)$ the unique mild solution of (Sys1), by Definition, is given by

$$\begin{pmatrix} u(t,x) \\ \partial_t u(t,x) \end{pmatrix} = S_3(t)\begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} + \int_0^t S_3(t-s)\begin{pmatrix} 0 \\ f(s,x) \end{pmatrix} ds.$$

Proof
This follows by Theorem 4.2.2. ∎

Theorem 4.2.5
For every $N \in \mathbb{N}^*$ and $A_{N,d}$ as in $(A4)$ the operator matrix in (DSys1) generates a C_0 semigroup $\{S^d_{N,3}(t)\}_{t\geq 0}$ on $D(L^{1/2}_{N,d}) \times h^r(Q^q_N)$, for every $r \in \mathbb{R}$.

Proof
Note that, by Theorem 1.1.19 again, it suffices to show that the linear operator

$$\begin{cases} D(M_{N,3}) := \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in D(L_{N,d}) \times D(L^{1/2}_{N,d}) \;\middle|\; \Delta g_2 \in h^r(Q^q_N) \right\}, \\ M_{N,3}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \begin{pmatrix} 0 & Id \\ -L & \rho_2 \Delta_{N,d} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \forall \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in D(M_{N,3}), \end{cases}$$

generates a C_0-semigroup. As $h^{r+2m}(Q^q_N) \subset D(L_{N,d})$ and $h^{r+m+2}(Q^q_N) \subset D(L^{1/2}_{N,d})$ $M_{N,3}$ is densely defined and closed, as $-L_{N,d}, \Delta_{N,d} \in \mathcal{A}(h^r(Q^q_N))$ as a self-adjoint operators and $Id \in \mathcal{L}(D(L^{1/2}_{N,d}))$. Now for $g = (g_1, g_2)^T \in \operatorname{dom}(M_{N,3})$ we compute:

$$\begin{aligned} \langle M_{N,3}g, g\rangle_{D(L^{1/2}_{N,d})\times h^r(Q^q_N)} &= \langle L^{1/2}_{N,d}g_1, L^{1/2}_{N,d}g_2\rangle_{h^r(Q^q_N)} + \langle -L_{N,d}g_1, g_2\rangle_{h^r(Q^q_N)} + \langle \rho_2\Delta_{N,d}g_2, g_2\rangle_{h^r(Q^q_N)} \\ &= \langle L^{1/2}_{N,d}g_1, L^{1/2}_{N,d}g_2\rangle_{h^r(Q^q_N)} - \langle L^{1/2}_{N,d}g_1, L^{1/2}_{N,d}g_2\rangle_{h^r(Q^q_N)} + \langle \rho_2\Delta_{N,d}g_2, g_2\rangle_{h^r(Q^q_N)} \\ &= + \langle \rho_2\Delta_{N,d}g_2, g_2\rangle_{h^r(Q^q_N)} \leq 0. \end{aligned}$$

Therefore by Corollary 1.1.20 $M_{N,3}$ is dissipative. Note that the adjoint operator of $M_{N,3}$ is given by

$$\begin{cases} D(M^*_{N,3}) = \operatorname{dom}(M_{N,3}), \\ M_{N,3}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \begin{pmatrix} 0 & -Id \\ LN,d & \rho_2 \Delta_{N,d} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \forall \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in D(M_{N,3}), \end{cases}$$

and compute for $g = (g_1, g_2)^T \in \operatorname{dom}(M_{N,3})$:

$$\begin{aligned} \langle M_{N,3}g, g\rangle_{D(L^{1/2}_{N,d})\times h^r(Q^q_N)} &= \langle -L^{1/2}_{N,d}g_2, L^{1/2}_{N,d}g_1\rangle_{h^r(Q^q_N)} + \langle L_{N,3}g_1, g_2\rangle_{h^r(Q^q_N)} + \langle \rho_2\Delta_{N,d}g_2, g_2\rangle_{h^r(Q^q_N)} \\ &= -\langle L^{1/2}_{N,d}g_1, LN,d^{1/2}g_2\rangle_{h^r(Q^q_N)} + \langle L^{1/2}_{N,d}g_1, L^{1/2}_{N,d}g_2\rangle_{h^r(Q^q_N)} + \langle \rho_2\Delta_{N,d}g_2, g_2\rangle_{h^r(Q^q_N)} \\ &= \langle \rho_2\Delta_{N,d}g_2, g_2\rangle_{h^r(Q^q_N)} \leq 0. \end{aligned}$$

Thus $M^*_{N,3}$ is also dissipative and the assertion follows by Corollary 1.1.24. ∎

Corollary 4.2.6
For every $N \in \mathbb{N}^*$ initial data and forcing as in $(A2)$ and noise as in $(A6)$ the unique stochastic strong mild solution of (DSys1), by definition, is given by

$$\begin{pmatrix} X_N(t,l) \\ \partial_t X_N(t,l) \end{pmatrix} = S^d_{N,3}(t)\begin{pmatrix} X_{N,0}(l) \\ X_{N,1}(l) \end{pmatrix} + \int_0^t S^d_{N,3}(t-s)\begin{pmatrix} 0 \\ f_N(s,l) \end{pmatrix} ds + \int_0^t S^d_{N,3}(t-s)\begin{pmatrix} 0 \\ \sigma(s) \end{pmatrix} dB_N(s,l).$$

Proof
This follows by Theorem 4.2.5. ∎

Corollary 4.2.7
For every $N \in \mathbb{N}^*$ and $\sigma \equiv 0$, initial data and forcing as in $(A2)$ and noise as in $(A6)$ the unique mild solution of (DSys1), by definition, is given by

$$\begin{pmatrix} X_N(t,l) \\ \partial_t X_N(t,l) \end{pmatrix} = S^d_{N,1}(t) \begin{pmatrix} X_{N,0}(l) \\ X_{N,1}(l) \end{pmatrix} + \int_0^t S^d_{N,3}(t-s) \begin{pmatrix} 0 \\ f_N(s,l) \end{pmatrix} ds + \int_0^t S^d_{N,3}(t-s) \begin{pmatrix} 0 \\ \sigma(s) \end{pmatrix} dB_N(s,l).$$

Proof
This follows by Theorem 4.2.5. ∎

4.3 Semigroup Representation

Now that we transformed our equation into an evolution system and established the existence of a C_0-semigroup, generated by the systems evolution operator, and a corresponding mild (or even classical) solution, the next step is to develop a representation theory that can be used to derive error estimates in the low frequency regime. Therefore, in this section, we will exploit the fact that our semigroup acts as a family of Fourier multiplication operators, with respect to the canonical complete orthonormal system of the product space $H^{s+m}(\mathbb{T}^q) \times H^s(\mathbb{T}^q)$, that is

$$\mathcal{F} S_l(t) \begin{pmatrix} \varphi_j \\ \varphi_k \end{pmatrix} = \begin{pmatrix} s^l_{11}(j,t)\varphi(j) + s^l_{12}(k,t)\varphi_k \\ s^l_{21}(j,t)\varphi(j) + s^l_{22}(k,t)\varphi_k \end{pmatrix} \quad j,k \in \mathbb{Z}^q \text{ and } l \in \{1,2,3\}$$

(and similarly for the discrete semigroups, under the discrete Fourier transform, in the corresponding discrete spaces $h^{s+m}(Q^q_N) \times h^s(Q^q_N)$) Thus by applying the respective Fourier transform to the systems from the previous sections, they decouple into a(n) (in)finite family, of first order ordinary differential equation systems, that we explicitly solve to obtain the coefficients $s_{11}(j), s_{12}(j), s_{21}(j)$ and $s_{22}(j)$, of each semigroup, for all $j \in \mathbb{Z}^q$. Again the computations for the damped cases are similar yet have to be done separately, due to the symbol of the (discrete) Laplacian appearing in the strongly damped case, while the undamped cases could be included as either $\rho_1 = 0$ or $\rho_2 = 0$, in one of the damped cases, we keep them distinct for citation and illustration purposes.

To keep the subsequent representation results lucid, we introduce the following abbreviated notation for some square root terms, in the (discrete) symbols of our respective operators and damping terms. We will see that said terms are precisely the eigenvalues of the ordinary differential equation systems, for our semigroups Fourier multiplier coefficients, or equivalently the roots of the corresponding second order differential equations characteristic polynomial. While they might appear unassuming on a first glance, these square roots formulae contain quite a bit of subtlety, as they are ambiguously used to represent any suitable branch[1)] of the square root function applicable to the respective, potentially complex valued evaluation of the symbol term underneath.

Notation 4.3.1
For (i) we have $k \in \mathbb{Z}^q$, while for (ii) we have $k \in \hat{Q}^q_N$, and denote:

(i) $R_3(k) := \sqrt{\left(\frac{\rho_2 p(\Delta,k)}{2}\right)^2 + p(A,k)},$

(ii) $\tilde{R}^d_{N,3}(k) := \sqrt{\left(\frac{\rho_2 p_{N,d}(\Delta_{N,d},k)}{2}\right)^2 + p_{N,d}(A_{N,d},k)}.$

1) And, as long as we are consistent in applying the same branch to each individual pair of evaluated discrete and continuum symbols, all of our low-frequency term estimates work out, while all other estimates are independent of the explicit representation.

Lemma 4.3.2
For $j \in \mathbb{Z}^q$ and $t > 0$ one has

$$\mathcal{F}S_3(t)\begin{pmatrix}\varphi_j\\0\end{pmatrix} = \begin{cases}\begin{pmatrix} e^{\frac{\rho_2 p(\Delta,j)}{2}t}\left(\cosh(R_3(j)t) - \frac{\rho_2 \sinh(R_3(j)t)}{2R_3(j)}\right)\varphi_j \\ e^{\frac{\rho_2 p(\Delta,j)}{2}t}\left(\frac{p(A,j)\sinh(R_3(j)t)}{R_3(j)}\right)\varphi_j\end{pmatrix}, & R_3(j) \neq 0,\\ \begin{pmatrix} e^{\frac{\rho_2 p(\Delta,j)}{2}t}\left(1 - \frac{\rho_2 p(\Delta,j)}{2}t\right)\varphi_j \\ -e^{\frac{\rho_2 p(\Delta,j)}{2}t}\frac{(\rho_2 p(\Delta,j))^2}{4}t\varphi_j\end{pmatrix}, & R_3(j) = 0,\end{cases}$$

and

$$\mathcal{F}S_3(t)\begin{pmatrix}0\\\varphi_j\end{pmatrix} = \begin{cases}\begin{pmatrix} e^{\frac{\rho_2 p(\Delta,j)}{2}t}\frac{\sinh(R_3(j)t)}{R_3(j)}\varphi_j \\ e^{\frac{\rho_2 p(\Delta,j)}{2}t}\left(\cosh(R_3(j)t) + \frac{\rho_2 p(\Delta,j)\sinh(R_3(j)t)}{2R_3(j)}\right)\varphi_j\end{pmatrix}, & R_3(j) \neq 0,\\ \begin{pmatrix} e^{\frac{\rho_2 p(\Delta,j)}{2}t}t\varphi_j \\ e^{\frac{\rho_2 p(\Delta,j)}{2}t}\left(1 + \frac{\rho_2 p(\Delta,j)}{2}t\right)\varphi_j\end{pmatrix}, & R_3(j) = 0.\end{cases}$$

Proof
The reduction of (Sys1) to a deterministic, homogeneous system yields

$$\begin{cases}\partial_t\begin{pmatrix}u(t,x)\\\partial_t u(t,x)\end{pmatrix} = \begin{pmatrix}0 & Id\\ A & \rho_2\end{pmatrix}\begin{pmatrix}u(t,x)\\\partial_t u(t,x)\end{pmatrix}, & (t,x) \in \mathbb{R}_+ \times \mathbb{T}^q,\\ \begin{pmatrix}u(0,\cdot)\\\partial_t u(0,\cdot)\end{pmatrix} = \begin{pmatrix}u_0(\cdot)\\u_1(\cdot)\end{pmatrix} & \text{in } \mathbb{T}^q.\end{cases} \tag{4.3.1}$$

By transforming (4.3.1) and separating by Fourier coefficients, we obtain an family of differential systems for the Fourier coefficients, namely for every $l \in \mathbb{Z}^q$ the corresponding Fourier coefficient u_l and its first derivative, with respect to t, satisfy

$$\begin{cases}\partial_t\begin{pmatrix}u_l(t)\\\partial_t u_l(t)\end{pmatrix} = \begin{pmatrix}0 & Id\\ p(A,l) & \rho_2 p(\Delta,l)\end{pmatrix}\begin{pmatrix}u_l(t)\\\partial_t u_l(t)\end{pmatrix}, & t > 0,\\ \begin{pmatrix}u_l(0)\\\partial_t u_l(0)\end{pmatrix} = \begin{pmatrix}u_{0,l}\\u_{1,l}\end{pmatrix}.\end{cases} \tag{4.3.2}$$

By choosing

$$\begin{pmatrix}u_0(\cdot)\\u_1(\cdot)\end{pmatrix} = \begin{pmatrix}\varphi_j\\0\end{pmatrix} \quad \text{respectivly} \quad \begin{pmatrix}u_0(\cdot)\\u_1(\cdot)\end{pmatrix} = \begin{pmatrix}0\\\varphi_j\end{pmatrix}$$

the claim follows, as for each $l \in \mathbb{Z}^q$ the solution of (4.3.2) is given by

$$\exp\left(\begin{pmatrix}0 & Id\\ p(A,l) & \rho_2 p(\Delta,l)\end{pmatrix}\right)\begin{pmatrix}\delta_{l,j}\\0\end{pmatrix} \quad \text{respectivly} \quad \exp\left(\begin{pmatrix}0 & Id\\ p(A,l) & \rho_2 p(\Delta,l)\end{pmatrix}\right)\begin{pmatrix}0\\\delta_{l,j}\end{pmatrix},$$

and $\frac{\rho_2 p(\Delta,l)}{2} \pm R_3(l)$ are the eigenvalues of

$$\begin{pmatrix}0 & Id\\ p(A,l) & \rho_2 p(\Delta,l)\end{pmatrix}.$$

■

Lemma 4.3.3
For $N \in \mathbb{N}$, $j \in \hat{Q}_N^q$ and $t > 0$ one has

$$\mathcal{F}^d S_{N,3}^d(t)\begin{pmatrix}\varphi_{j,N,d}\\ 0\end{pmatrix} = \begin{cases}\begin{pmatrix} e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t}\left(\cosh(\tilde{R}_{N,3}^d(j)t) - \frac{\rho_2 \sinh(\tilde{R}_{N,3}^d(j)t)}{2\tilde{R}_{N,3}^d(j)}\right)\varphi_{j,N,d} \\ e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t}\left(\frac{p_{N,d}(A_{N,d},j)\sinh(\tilde{R}_{N,3}^d(j)t)}{\tilde{R}_{N,3}^d(j)}\right)\varphi_{j,N,d}\end{pmatrix}, & \tilde{R}_{N,3}^d(j) \neq 0,\\ \begin{pmatrix} e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t}\left(1 - \frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t\right)\varphi_{j,N,d} \\ -e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t}\frac{(\rho_2 p_{N,d}(\Delta_{N,d},j))^2}{4}t\varphi_{j,N,d}\end{pmatrix}, & \tilde{R}_{N,3}^d(j) = 0,\end{cases}$$

and

$$\mathcal{F}^d S_{N,3}^d(t)\begin{pmatrix}0\\ \varphi_{j,N,d}\end{pmatrix} = \begin{cases}\begin{pmatrix} e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t}\frac{\sinh(\tilde{R}_{N,3}^d(j)t)}{\tilde{R}_{N,3}^d(j)}\varphi_{j,N,d} \\ e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t}\left(\cosh(\tilde{R}_{N,3}^d(j)t) + \frac{\rho_2 p_{N,d}(\Delta_{N,d},j)\sinh(\tilde{R}_{N,3}^d(j)t)}{2\tilde{R}_{N,3}^d(j)}\right)\varphi_{j,N,d}\end{pmatrix}, & \tilde{R}_{N,3}^d(j) \neq 0,\\ \begin{pmatrix} e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t}t\varphi_{j,N,d} \\ e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t}\left(1 + \frac{\rho_2 p_{N,d}(\Delta_{N,d},j)}{2}t\right)\varphi_{j,N,d}\end{pmatrix}, & \tilde{R}_{N,3}^d(j) = 0.\end{cases}$$

Proof
First we reduce (DSys1) to the deterministic, homogeneous system

$$\begin{cases}\partial_t\begin{pmatrix}X_N(t,l)\\ \partial_t X_N(t,l)\end{pmatrix} = \begin{pmatrix}0 & Id\\ A_{d,N} & \rho_2\Delta_{N,d}\end{pmatrix}\begin{pmatrix}X_N(t,l)\\ \partial_t X_N(t,l)\end{pmatrix}, & (t,l) \in \mathbb{R}_+ \times Q_N^q,\\ \begin{pmatrix}X_N(0,\cdot)\\ \partial_t X_N(0,\cdot)\end{pmatrix} = \begin{pmatrix}X_{N,0}(\cdot)\\ X_{N,1}(\cdot)\end{pmatrix} & \text{in } Q_N^q,\end{cases} \tag{4.3.3}$$

as this again fully determines the semigroup. By discrete transforming (4.3.3) and separating by Fourier coefficients, we obtain an family of differential systems for the discrete Fourier coefficients, namely for every $m \in \hat{Q}_N^q$ the corresponding Fourier coefficient X_{N_m} and its first derivative, with respect to t, satisfy

$$\begin{cases}\partial_t\begin{pmatrix}X_{N,m}(t)\\ \partial_t X_{N,m}(t)\end{pmatrix} = \begin{pmatrix}0 & Id\\ p_{N,d}(A_{N,d}) & \rho_2 p_{N,d}(\Delta_{N,d},m)\end{pmatrix}\begin{pmatrix}X_{N,m}(t)\\ \partial_t X_{N,m}(t)\end{pmatrix}, & \tau > 0,\\ \begin{pmatrix}X_{N,m}(0)\\ \partial_t X_{N,m}(0)\end{pmatrix} = \begin{pmatrix}X_{N,m,0}\\ X_{N,m,1}\end{pmatrix}.\end{cases} \tag{4.3.4}$$

By choosing

$$\begin{pmatrix}X_{N,0}(\cdot)\\ X_{N,1}(\cdot)\end{pmatrix} = \begin{pmatrix}\varphi_{j,N,d}\\ 0\end{pmatrix} \quad \text{respectivly} \quad \begin{pmatrix}X_{N,0}(\cdot)\\ X_{N,1}(\cdot)\end{pmatrix} = \begin{pmatrix}0\\ \varphi_{j,N,d}\end{pmatrix}$$

the claim follows, as for each $m \in \hat{Q}_N^q$ the solution of (4.3.4) is given by

$$\exp\left(\begin{pmatrix}0 & Id\\ p_{N,d}(A_{N,d},m) & \rho_2 p_{N,d}(\Delta_{N,d},m)\end{pmatrix}\right)\begin{pmatrix}\delta_{m,j}\\ 0\end{pmatrix}$$

respectively

$$\exp\left(\begin{pmatrix}0 & Id\\ p_{N,d}(A_{N,d},m) & \rho_2 p_{N,d}(\Delta_{N,d},m)\end{pmatrix}\right)\begin{pmatrix}0\\ \delta_{m,j}\end{pmatrix}$$

and $\frac{\rho_2 p_{N,d}(\Delta_{N,d},m)}{2} \pm \tilde{R}_{N,3}^d(m)$ are the eigenvalues of

$$\begin{pmatrix}0 & Id\\ p_{N,d}(A_{N,d},m) & \rho_2 p_{N,d}(\Delta_{N,d},m)\end{pmatrix}.$$

■

Now we use these representations of the discrete semigroups to derive the representations of their continuations we need for our estimates.

Corollary 4.3.4
For $i \in \{1,2,3\}$, $N \in \mathbb{N}$, $j,k \in \hat{Q}_N^q$ and $\tau > 0$ one has

$$\mathcal{F}\tilde{S}_{N,i}^d(\tau)\begin{pmatrix}\varphi_j\\ \varphi_k\end{pmatrix} = \mathcal{F}^d S_{N,i}^d(\tau)\begin{pmatrix}\left(\prod_{h=1}^q\left(\frac{e^{i\frac{\pi j_h}{N}}-1}{i\frac{j_h\pi}{N}}\right)\right) & 0\\ 0 & \left(\prod_{h=1}^q\left(\frac{e^{i\frac{\pi k_h}{N}}-1}{i\frac{k_h\pi}{N}}\right)\right)\end{pmatrix}\begin{pmatrix}\varphi_{j,N,d}\\ \varphi_{k,N,d}\end{pmatrix}.$$

Proof
This follows directly by Lemma 1.2.55 ∎

5 Error Estimates

As the result of the previous chapter we have our mild continuum solution u and the discrete mild solutions $\{X_N\}_{N\in\mathbb{N}^*}$, with corresponding constant interpolations $\{\tilde{X}_N\}_{N\in\mathbb{N}^*}$, as well as a representation for the semigroups guiding their evolution. In this chapter, using their respective mild formulations, we study the difference of u and $\tilde{X}_N$ and combining estimates from functional analysis, most notably from the areas of Fourier analysis and the theory of C_0-semigroups, we obtain bounds, that vanish as N goes to infinity, thereby providing the desired convergence.

In Section 5.1 we prepare the subsequent error estimates, done in Section 5.2 and Section 5.3, by decomposing the difference of u and $\tilde{X}_N$, using their variation of parameters form. The resulting new terms can be categorized in the following three ways, terms where the continuum semigroup acts on a difference of the discrete and continuum data, as well as terms where the difference of the semigroups acts on either the high-frequency or the low-frequency parts of the data.

As both the terms with error in the data and the high-frequency terms can be treated by bounding the (difference of the) semigroup(s) and using that it was applied to something already vanishing, for $N \to \infty$, both types will be treated in Section 5.2. While the low-frequency terms are treated, in Section 5.3, using Taylor approximations of the coefficient functions, in the Fourier representation of the semigroup.

5.1 Decomposition & Frequency Cutoff

In a first step we write the solutions in their variation of parameters form and by adding expansions of zero and using triangular inequality we obtain one set of error terms, where the continuum semigroup is applied to the difference of the continuation of the discrete data and continuum data (resp. forcing and noise) and one set, where the difference of the continuum semigroup and the continuation of the discrete semigroup acting on the discrete data (resp. forcing and noise).

The first terms we leave as is, while we split the second set of terms, into a high-frequency and a low-frequency part each, by expanding the identity into the cutoff projector P^a_δ[1)], introduced in Chapter 1.2, and its complementary projector $(P^a_\delta)^\perp = Id - P^a_\delta$. This makes for a total of six, respectively nine, terms we need to estimate in the following sections.

So far, when stating our main results, we considered the norms of the errors $u - \tilde{X}_N$ and $\partial_t u - \partial_t \tilde{X}_N$ individually, while the mild solution, of the Abstract Cauchy Problem, as derived in Chapter 4, consists of both u and $\partial_t u$, with the underlying semigroup acting on the product space $H^{r+m}(\mathbb{T}^q) \times H^r(\mathbb{T}^q)$. Therefore beginning with the deterministic errors we bound the sums of the individual norms, as they appear in our main results, by the norm of the Abstract Cauchy Problems mild solutions in the product space, with the first decomposition happening immediately, while the insertion of the different expansions of zero is done in subsequent Lemmata.

1) Recall that while the cutoff projectors like the canonical Galerkin projectors map into a finite dimensional space, their image $\mathrm{span}\{\varphi_j \mid j \in \mathbb{Z}^q \cap [-\lfloor n^\delta \rfloor, \lfloor n^\delta - 1 \rfloor]^q\}$ is based on the asymmetrical cubes $\mathbb{Z}^q \cap [-n, n-1]^q$, whereas the canonical Galerkin projectors map into the finite dimensional spaces $\mathrm{span}\{\varphi_j \mid j \in \mathbb{Z}^q \cap [-n, n]^q\}$, based on symmetrical cubes.

Lemma 5.1.1
Under the assumptions of Theorem 3.3.1, let $u \in H^{s+m}(\mathbb{T}^q)$ the unique mild solution of (EQ1) and for every $N \in \mathbb{N}^*$ let $X_N \in h^{s+m}(Q_N^q)$ the unique mild solution of (DEQ1). Then depending on the damping and $T > 0$ there exists a constant $C > 0$, such that for every $N \in \mathbb{N}^*$:

$$\begin{aligned}
&\left\|u - \tilde{X}_N\right\|^2_{C([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_t u - \partial_t \tilde{X}_N\right\|^2_{C([0,T],H^{s}(\mathbb{T}^q))} \\
\leq &C \sup_{t\in[0,T]} \left\| S(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \tilde{S}_N^d(t)\begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,i} \end{pmatrix} \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
&+ C \sup_{t\in[0,T]} \left\| \int_0^t S(t-t')\begin{pmatrix} 0 \\ f(t',\cdot) \end{pmatrix}\, dt' - \int_0^t \tilde{S}_N^d(t-t')\begin{pmatrix} 0 \\ \tilde{f}_N(t',\cdot) \end{pmatrix}\, dt' \right\|^2_{H^s(\mathbb{T}^q)\times H^{s+m}(\mathbb{T}^q)}
\end{aligned}$$

Proof
By interchanging summation and the supremum we get

$$\begin{aligned}
&\left\|u - \tilde{X}_N\right\|^2_{C([0,T],H^{s+m}(\mathbb{T}^q)))} + \left\|\partial_t u - \partial_t \tilde{X}_N\right\|^2_{C([0,T],H^{s}(\mathbb{T}^q)))} \\
\leq &2 \sup_{t\in[0,T]} \left\| \begin{pmatrix} u(t,\cdot) \\ \partial_t u(t,\cdot) \end{pmatrix} - \begin{pmatrix} \tilde{X}_N(t,\cdot) \\ \partial_t \tilde{X}_N(t,\cdot) \end{pmatrix} \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}.
\end{aligned}$$

Now the claim follows by inserting the respective mild formulations and the triangular inequality.■

Now, in case of the stochastic solutions, we do the same for the expectation of the norms, albeit with the additional step of reducing the deterministic parts of the solution to those appearing in Lemma 5.1.1, so we can reuse the estimates done for those terms and only need additional estimates for the stochastic convolution terms.

Lemma 5.1.2
Under the assumptions of Theorem 3.3.1, let $u \in H^{s+m}(\mathbb{T}^q)$ the unique, stochastic strong, mild solution of (EQ1) and for every $N \in \mathbb{N}^*$ let $X_N \in h^{s+m}(Q_N^q)$ the unique stochastic strong, mild solution of (DEQ1). Then depending on the damping, every $T > 0$ and every $p > 1$, there exists a constant $C(T,p) > 0$, such that for every $N \in \mathbb{N}^*$:

$$\begin{aligned}
&\mathbb{E}\left[\left\|u - \tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_t u - \partial_t \tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s}(\mathbb{T}^q))}\right] \\
\leq &C(T,p) \sup_{t\in[0,T]} \left\| S(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \tilde{S}_N^d(t)\begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,i} \end{pmatrix} \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
&+ C(T,p) \sup_{t\in[0,T]} \left\| \int_0^t S(t-t')\begin{pmatrix} 0 \\ f(t',\cdot) \end{pmatrix}\, dt' - \int_0^t \tilde{S}_N^d(t-t')\begin{pmatrix} 0 \\ \tilde{f}_N(t',\cdot) \end{pmatrix}\, dt' \right\|^2_{H^s(\mathbb{T}^q)\times H^{s+m}(\mathbb{T}^q)} \\
&+ C(T,p) \int_0^T \mathbb{E}\left[\left\| \int_0^t S(t-t')\begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix}\, dW(t',\cdot) - \int_0^t \tilde{S}_N^d(t-t')\begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix}\, d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^s(\mathbb{T}^q)}\right] dt.
\end{aligned}$$

Proof
Let $v \in H^{s+m}(\mathbb{T}^q)$ the unique mild solution of (EQ1) with $\sigma \equiv 0$ and for every $N \in \mathbb{N}^*$ let $Y_N \in h^{s+m}(Q_N^q)$ the unique mild solution of (DEQ1) with $\sigma \equiv 0$, then we have by triangular inequality and monotony and linearity of the expectation:

$$\begin{aligned}
&\mathbb{E}\left[\left\|u - \tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_r u - \partial_r \tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s}(\mathbb{T}^q))}\right] \\
\leq &C(p)\mathbb{E}\left[\left\|v - \tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_\tau v - \partial_t \tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^{s}(\mathbb{T}^q))}\right] \\
&+ C(p)\mathbb{E}\left[\left\| \int_0^t S_i(t-t')\begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix}\, dW(t',\cdot) - \int_0^t \tilde{S}_{N,i}^d(t-t')\begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix}\, d\tilde{B}_N(t',\cdot) \right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))}\right].
\end{aligned}$$

Then by Fubini's Theorem we can exchange expectation and the L_p-norm and obtain:

$$\begin{aligned} &C(p)\mathbb{E}\left[\left\|\int_0^t S_i(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\,dW(t',\cdot)-\int_0^t \tilde{S}^d_{N,i}(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\,d\tilde{B}_N(t',\cdot)\right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))}\right] \\ \le &C(T,p)\int_0^T \mathbb{E}\left[\left\|\int_0^t S_i(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\,dW(t',\cdot)-\int_0^t \tilde{S}^d_{N,i}(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\,d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^s(\mathbb{T}^q)}\right]dt. \end{aligned} \tag{5.1.1}$$

Furthermore as v and $(Y_N)_{N\in\mathbb{N}}$ are deterministic we can drop the expectation and estimate

$$\begin{aligned} &\mathbb{E}\left[\left\|v-\tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))}+\left\|\partial_t v-\partial_t\tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))}\right] \\ =&\left\|v-\tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))}+\left\|\partial_t v-\partial_t\tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))} \\ \le &C(T,p)\left[\left\|v-\tilde{Y}_N\right\|^{2}_{C([0,T],H^{s+m}(\mathbb{T}^q))}+\left\|\partial_t v-\partial_t\tilde{Y}_N\right\|^{2}_{C([0,T],H^s(\mathbb{T}^q))}\right]. \end{aligned}$$

Now applying Lemma 5.1.1 together with the estimate (5.1.1) proves the claim. ∎

The next three lemmata deal with the next decomposition step, of one of the right-hand side terms, from the previous two lemmata, by introducing suitable expansions of zero, beginning with the initial data term.

Lemma 5.1.3
Under the assumptions of Theorem 3.3.1, let $u\in H^{s+m}(\mathbb{T}^q)$ the unique mild solution of (EQ1) and for every $N\in\mathbb{N}^*$ let $X_N\in h^{s+m}(Q^q_N)$ the unique mild solution of (DEQ1). Then for every $i\in\{1,2,3\}$, depending on the damping and every $T>0$, we have for every $N\in\mathbb{N}^*$:

$$\begin{aligned} &\sup_{t\in[0,T]}\left\|S_i(t)\begin{pmatrix}u_0\\ u_1\end{pmatrix}-\tilde{S}^d_{N,i}(t)\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,i}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\ \le &\sup_{t\in[0,T]}\left\|S_i(t)\left(\begin{pmatrix}u_0\\ u_1\end{pmatrix}-\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right)\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}+\sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}. \end{aligned}$$

Proof
By adding „zero“, in the form of the following expansion:

$$0=-S_i(t)\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}+S_i(t)\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix},$$

using triangular inequality and linearity of the semigroup we compute:

$$\begin{aligned} &\sup_{t\in[0,T]}\left\|S_i(t)\begin{pmatrix}u_0\\ u_1\end{pmatrix}-\tilde{S}^d_{N,i}(t)\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,i}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\ \le &\sup_{t\in[0,T]}\left\|S_i(\tau)\left(\begin{pmatrix}u_0\\ u_1\end{pmatrix}-\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right)\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}+\sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \end{aligned}$$

∎

Now we repeat this procedure for the forcing term.

Lemma 5.1.4
Under the assumptions of Theorem 3.3.1, let $u \in H^{s+m}(\mathbb{T}^q)$ the unique mild solution of (EQ1) and for every $N \in \mathbb{N}^*$ let $X_N \in h^{s+m}(Q_N^q)$ the unique mild solution of (DEQ1). Then for every $i \in \{1,2,3\}$, depending on the damping and every $T > 0$, we have for every $N \in \mathbb{N}^*$:

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ f(t',\cdot) \end{pmatrix} dt' - \int_0^t \tilde{S}^d_{N,i}(t-t') \begin{pmatrix} 0 \\ \tilde{f}_N(t',\cdot) \end{pmatrix} dt' \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
\leq &\sup_{t\in[0,T]} \left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ f(t',\cdot) - \tilde{f}_N(t',x) \end{pmatrix} dt' \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
&+ \sup_{t\in[0,T]} \left\| \int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t')) \begin{pmatrix} 0 \\ \tilde{f}_N(t',\cdot) \end{pmatrix} dt' \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}
\end{aligned}$$

Proof
By adding „zero “in the form of the following expansion:

$$0 = -\int_0^t S_i(t-t') \begin{pmatrix} 0 \\ \tilde{f}_N(t',\cdot) \end{pmatrix} dt' + \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ \tilde{f}_N(t',\cdot) \end{pmatrix} dt',$$

using triangular inequality and linearity of the semigroup we compute:

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ f(t',\cdot) \end{pmatrix} dt' - \int_0^t \tilde{S}^d_{N,i}(t-t') \begin{pmatrix} 0 \\ \tilde{f}_N(t',\cdot) \end{pmatrix} dt' \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
\leq &\sup_{t\in[0,T]} \left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ f(t',\cdot) - \tilde{f}_N(t',\cdot) \end{pmatrix} dt' \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
&+ \sup_{t\in[0,T]} \left\| \int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t')) \begin{pmatrix} 0 \\ \tilde{f}_N(t',\cdot) \end{pmatrix} dt' \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}
\end{aligned}$$

■

And finally also expand the noise term.

Lemma 5.1.5
Under the assumptions of Theorem 3.3.1, let $u \in H^{s+m}(\mathbb{T}^q)$ the unique, stochastic strong, mild solution of (EQ1) and for every $N \in \mathbb{N}^*$ let $X_N \in h^{s+m}(Q_N^q)$ the unique stochastic strong, mild solution of (DEQ1). Then for every $i \in \{1,2,3\}$, depending on the damping, every $T > 0$ and every $p > 1$, we have for every $N \in \mathbb{N}^*$:

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[\left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} dW(t',\cdot) - \int_0^t \tilde{S}^d_{N,i}(t-t') \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt \\
\leq &\int_0^T \mathbb{E}\left[\left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d(W(t',\cdot) - \tilde{B}_N(t',\cdot)) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dr \\
&+ \int_0^T \mathbb{E}\left[\left\| \int_0^r (S_i(t-t') - \tilde{S}^d_{N,i}(t-t')) \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt.
\end{aligned}$$

Proof
By adding zero in the form of the following expansion:

$$0 = \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d\tilde{B}_N(t',\cdot) - \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d\tilde{B}_N(t',\cdot).$$

Using triangular inequality and linearity of the semigroup, as well as the monotonicity of the expectation and integral we compute:

$$\int_0^T \mathbb{E}\left[\left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} dW(t',\cdot) - \int_0^t \tilde{S}^d_{N,i}(t-t') \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt$$

$$\leq \int_0^T \mathbb{E}\left[\left\|\int_0^t S_i(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix} d(W(t',\cdot)-\tilde{B}_N(t',\cdot))\right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\right] dt$$
$$+\int_0^T \mathbb{E}\left[\left\|\int_0^t (S_i(t-t')-\tilde{S}^d_{N,i}(t-t'))\begin{pmatrix}0\\ \sigma(t')\end{pmatrix} d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\right] dt.$$

∎

Now we further decompose the second terms by splitting them into a high frequency part and a low frequency part. For this we recall the cutoff projector P^a_δ, that by Definition 1.2.4 is the orthogonal projector onto, the finite dimensional space, $\text{span}\{\varphi_j \mid j \in \mathbb{Z}^q \cap [-\lfloor N^\delta \rfloor, \lfloor N^\delta - 1 \rfloor]^q\}$, for $\delta \in (0,1)$, where $\Phi = \{\varphi_j \mid j \in \mathbb{Z}^q\}$ denotes the canonical orthonormal basis of $H^s(\mathbb{T}^q)$, for every $s \in \mathbb{R}$. And its complementary projector $(P^a_\delta)^\perp = Id - P^a_\delta$. And we will make use of the following convention: When we write

$$\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \quad \text{or} \quad \left\|(P^a_\delta)^\perp\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}$$

those are to be understood in the sense of matrix vector multiplication. I.e

$$\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} = \left\|(S_i(t)-\tilde{S}^d_{N,i}(t))\begin{pmatrix}P^a_\delta & 0\\ 0 & P^a_\delta\end{pmatrix}\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}$$

with the respective realizations on the diagonal and

$$\left\|(P^a_\delta)^\perp\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} = \left\|\begin{pmatrix}(P^a_\delta)^\perp\tilde{X}_{N,0}\\ (P^a_\delta)^\perp\tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}$$

with the respective realizations in each component.

Lemma 5.1.6
Under the assumptions of Theorem 3.3.1, let $u \in H^{s+m}(\mathbb{T}^q)$ the unique mild solution of (EQ1) and for every $N \in \mathbb{N}^*$ let $X_N \in h^{s+m}(Q^q_N)$ the unique mild solution of (DEQ1). Then for every $i \in \{1,2,3\}$, depending on the damping, every $\delta \in (0,1)$ and every $T > 0$, we have that for every $N \in \mathbb{N}^*$:

$$\sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}$$
$$\leq C\sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}\left\|\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}$$
$$+C\sup_{t\in[0,T]}\left\|S_i(t)-\tilde{S}^d_{N,i}(t)\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}\left\|(P^a_\delta)^\perp\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}$$

Proof
Inserting an Identity and expanding it as $Id = (P^a_\delta)^\perp + P^a_\delta$, then splitting into the high Fourier modes („$(P^a_\delta)^\perp$“) and the low ones („P^a_δ“), using triangular inequality, we obtain:

$$\sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}$$
$$=\sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))((P^a_\delta)^\perp+P^a_\delta)\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}$$
$$\leq \sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))(P^a_\delta)^\perp\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}$$

$$+\sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))P^a_\delta\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}.$$

By using the definition of the operator norm and $\tilde{X}_{N,0}\in H^{s+m}(\mathbb{T}^q)\subset H^{s+m}(\mathbb{T}^q)$ we obtain

$$\begin{aligned}
&\le \sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))(P^a_\delta)^\perp\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\\
&\quad+\sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))P^a_\delta\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\\
&\le \sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}\left\|\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\\
&\quad+\sup_{t\in[0,T]}\left\|S_i(t)-\tilde{S}^d_{N,i}(t)\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}\left\|(P^a_\delta)^\perp\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}.
\end{aligned}$$

■

Using the same procedure once more we get the frequency decomposition of the forcing term.

Lemma 5.1.7
Under the assumptions of Theorem 3.3.1, let $u\in H^{s+m}(\mathbb{T}^q)$ the unique mild solution of (EQ1) and for every $N\in\mathbb{N}^*$ let $X_N\in h^{s+m}(Q^q_N)$ the unique mild solution of (DEQ1). Then for every $i\in\{1,2,3\}$, depending on the damping, every $\delta\in(0,1)$ and every $T>0$, we have that for every $N\in\mathbb{N}^*$:

$$\begin{aligned}
&\sup_{t\in[0,T]}\left\|\int_0^t (S_i(t-t')-\tilde{S}^d_{N,i}(t-t'))\begin{pmatrix}0\\ \tilde{f}_N(t',\cdot)\end{pmatrix}\,dt'\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\\
&\le \int_0^T\sup_{t\in[0,T]}\left\|(S_i(t-t')-\tilde{S}^d_{N,i}(t-t'))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}\left\|\tilde{f}_N(t',\cdot)\right\|^2_{H^s(\mathbb{T}^q)}\,dt'\\
&\quad+\int_0^T\sup_{t\in[0,T]}\left\|(S_i(t)-\tilde{S}^d_{N,i}(t))\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}\left\|(P^a_\delta)^\perp\tilde{f}_N(t',\cdot)\right\|^2_{H^s(\mathbb{T}^q)}\,dt'
\end{aligned}$$

Proof
Inserting an Identity and expanding it as $Id=(P^a_\delta)^\perp+P^a_\delta$, then splitting into the high Fourier modes („$(P^a_\delta)^\perp$“) and the low ones („P^a_δ“), using triangular inequality, we obtain:

$$\begin{aligned}
&\sup_{t\in[0,T]}\left\|\int_0^t (S_i(t-t')-\tilde{S}^d_{N,i}(t-t'))\begin{pmatrix}0\\ \tilde{f}_N(t',\cdot)\end{pmatrix}\,dt'\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\\
&=\sup_{t\in[0,T]}\left\|\int_0^t (S_i(t-t')-\tilde{S}^d_{N,i}(t-t'))((P^a_\delta)^\perp+P^a_\delta)\begin{pmatrix}0\\ \tilde{f}_N(t',\cdot)\end{pmatrix}\,dt'\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\\
&\le \sup_{t\in[0,T]}\int_0^t\left\|(S_i(t-t')-\tilde{S}^d_{N,i}(t-t'))P^a_\delta\begin{pmatrix}0\\ \tilde{f}_N(t',\cdot)\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\,dt'\\
&\quad+\sup_{t\in[0,T]}\int_0^t\left\|(S_i(t-t')-\tilde{S}^d_{N,i}(t-t'))(P^a_\delta)^\perp\begin{pmatrix}0\\ \tilde{f}_N(t',\cdot)\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\,ds
\end{aligned}$$

By using the definition of the operator norm and $\tilde{f}_N(t')\in H^{s+m}(\mathbb{T}^q)\subset H^{s+m}(\mathbb{T}^q)$, for every $t'\in[0,T]$, we obtain

$$\sup_{t\in[0,T]}\int_0^t\left\|(S_i(t-t')-\tilde{S}^d_{N,i}(t-t'))P^a_\delta\begin{pmatrix}0\\ \tilde{f}_N(t',\cdot)\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\,dt'$$

$$
\begin{aligned}
&+ \sup_{t\in[0,T]} \int_0^t \left\| (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))(P^a_\delta)^\perp \begin{pmatrix} 0 \\ \tilde{f}_N(t',\cdot) \end{pmatrix} \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} dt' \\
\leq & \int_0^T \sup_{t\in[0,T]} \left\| (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))P^a_\delta \right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \left\| \tilde{f}_N(t',\cdot) \right\|^2_{H^s(\mathbb{T}^q)} dt' \\
&+ \int_0^T \sup_{t\in[0,T]} \left\| (S_i(t) - \tilde{S}^d_{N,i}(t)) \right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \left\| (P^a_\delta)^\perp \tilde{f}_N(t',\cdot) \right\|^2_{H^s(\mathbb{T}^q)} dt'.
\end{aligned}
$$

■

As for the second split of the noise term, here we keep extraction of the semigroup until later, for the high-frequency part we do it when we compute the error estimate and for the low-frequency part we do it when proving the individual continuum limits in Chapter 6.

Lemma 5.1.8
Under the assumptions of Theorem 3.3.1, let $u \in H^{s+m}(\mathbb{T}^q)$ the unique, stochastic strong, mild solution of (EQ1) and for every $N \in \mathbb{N}^*$ let $X_N \in h^{s+m}(Q^q_N)$ the unique stochastic strong, mild solution of (DEQ1). Then for every $i \in \{1,2,3\}$, depending on the damping, every $T > 0$, every $\delta \in (0,1)$ and every $p > 1$, we have that for every $N \in \mathbb{N}^*$:

$$
\begin{aligned}
&\int_0^T \mathbb{E}\left[\left\| \int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t')) \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt \\
\leq & \mathbb{E}\left[\left\| \int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))P^a_\delta \sigma(t')\, d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] \\
&+ \mathbb{E}\left[\left\| \int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))(P^a_\delta)^\perp \sigma(t')\, d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right].
\end{aligned}
$$

Proof
Again inserting the identity expansion $Id = (P^a_\delta)^\perp + P^a_\delta$ and triangular inequality we compute:

$$
\begin{aligned}
&\int_0^T \mathbb{E}\left[\left\| \int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t')) \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt \\
= & \int_0^T \mathbb{E}\left[\left\| \int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))((P^a_\delta)^\perp + P^a_\delta) \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt \\
\leq & \int_0^T \mathbb{E}\left[\left\| \int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))P^a_\delta \sigma(t')\, d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt \\
&+ \int_0^T \mathbb{E}\left[\left\| \int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))(P^a_\delta)^\perp \sigma(t')\, d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt.
\end{aligned}
$$

■

5.2 Elementary Functional Analytic Estimates

In this section we compute error estimates for the terms, in which either the continuous semigroup is applied to a difference directly in the data (initial data, forcing or noise) or the error stems from a difference of the continuous semigroup and the discrete semigroups continuous counterpart, applied to the high-frequency Fourier modes.
The common theme of all these estimates will be that we use the evolution estimate for semigroups, the central tool of this section, to bound the operator norm of the semigroup by a constant, at most

depending on time, and then use convergence of the semigroups arguments. Therefore unlike the estimates for the lower Fourier modes in the upcoming Section 5.3, the estimates in this section are identical, for all three semigroup pairs, and therefore done simultaneously.

The first estimates are for the deterministic terms in which the respective continuous semigroup is applied to a difference of the continuous data and the respective constant interpolation of the discrete data. Here we use the aforementioned evolution estimate and the convergence assumed in $(A2)$ to obtain the following two estimates.

Lemma 5.2.1
Under the assumptions of Theorem 3.3.1 for every $i \in \{1,2,3\}$, every $T > 0$ and every $\varepsilon > 0$, there exists a $N(\varepsilon) \in \mathbb{N}^*$ such that for every $N \geq N(\varepsilon)$:

$$\sup_{t\in[0,T]} \left\| S_i(t)\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix}\right)\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} < \varepsilon.$$

Proof
Let $\varepsilon > 0$. Using Lemma 1.1.8 and then bounding the exponential depending on the sign of ω_i we compute:

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\| S_i(t)\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix}\right)\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
\leq& C \sup_{t\in[0,T]} M_i e^{2\omega_i t} \left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
\leq& \underbrace{C M_i \max\{1, e^{\omega_i T}\}}_{=:C_i(T)} \left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}.
\end{aligned}$$

And by the, in (A2), assumed convergence of the initial data there exists $N(\varepsilon) \in \mathbb{N}^*$ such that for every $N \geq N(\varepsilon)$:

$$\left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \leq \frac{\varepsilon}{C_i(T)}.$$

■

Lemma 5.2.2
Under the assumptions of Theorem 3.3.1 for every $i \in \{1,2,3\}$, every $T > 0$ and every $\varepsilon > 0$, there exists a $N(\varepsilon) \in \mathbb{N}^*$ such that for every $N \geq N(\varepsilon)$:

$$\sup_{t\in[0,T]} \left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ (f(t',\cdot) - \tilde{f}_N(t',\cdot)) \end{pmatrix} \, dt' \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^(\mathbb{T}^q)s)} < \varepsilon.$$

Proof
Let $\varepsilon > 0$. Again using Lemma 1.1.8 and then bounding the exponential depending on the sign of ω_i we compute:

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ (f(t',\cdot) - \tilde{f}_N(t',\cdot)) \end{pmatrix} \, dt' \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^(\mathbb{T}^q)s)} \\
\leq& C \int_0^T \sup_{t\in[0,T]} \left\| S_i(t-t')\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \left\| f(t',\cdot) - \tilde{f}_N(t',\cdot)\right\|^2_{H^s(\mathbb{T}^q)} \, dt' \\
\leq& C \int_0^T \sup_{t\in[0,T]} M_i e^{2\omega_i(t-t')} \left\| f(t',\cdot) - \tilde{f}_N(t',\cdot)\right\|^2_{H^s(\mathbb{T}^q)} \, dt'
\end{aligned}$$

$$\leq \underbrace{CM_i \max\{1, e^{\omega_i T}\}}_{=:C_i(T)} \int_0^T \left\| f(t',\cdot) - \tilde{f}_N(t',\cdot) \right\|^2_{H^s(\mathbb{T}^q)} \, dt'.$$

And by the, in (A2), assumed uniform convergence in time of the forcing terms, there exists a $N(\varepsilon) \in \mathbb{N}^*$ such that

$$\left\| f(t',\cdot) - \tilde{f}_N(t',\cdot) \right\|^2_{H^s(\mathbb{T}^q)} < \frac{\varepsilon}{C_i(T)T}$$

for all $t' \in [0,T)$ for all $N \geq N(\varepsilon)$. ■

Now that we have all the estimates we need for the deterministic case we want do do the same for the stochastic case,as by Lemma 5.1.2 the estimates for the initial data and forcing part of the mild solutions, can be traced back to the respective deterministic estimate, all we need in addition is the following estimate for the difference of the stochastic convolution terms, here we use the convergence assumed in our general assumption $(A6)$.

Lemma 5.2.3
Under the assumptions of Theorem 3.3.1 for every $i \in \{1,2,3\}$, depending on the damping, every $T > 0$, every $\delta \in (0,1)$ and every $p > 1$, there exists a $N(\varepsilon) \in \mathbb{N}^*$ such that:

$$\int_0^T \mathbb{E}\left[\left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d(W(t',\cdot) - \tilde{B}_N(t',\cdot)) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt < \varepsilon,$$

for all $N \geq N(\varepsilon)$.

Proof
In a first step perform the following decomposition:

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[\left\| \int_0^t S_i(t-t') \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} d(W(t',\cdot) - \tilde{B}_N(t',\cdot)) \right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \right] dt \\
\leq& C(T) \int_0^T \|S_i(t)\|^{2p}_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \mathbb{E}\left[\left\| \int_0^t 1 \, d(W(t',\cdot) - \tilde{B}_N(t',\cdot)) \right\|^{2p}_{H^s(\mathbb{T}^q)} \right] dt \\
\leq& C(T) \int_0^T \|S_i(t)\|^{2p}_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \mathbb{E}\left[\left\| \int_0^t P_N^a \, d(W(t',\cdot) - \tilde{B}_N(t',\cdot)) \right\|^{2p}_{H^s(\mathbb{T}^q)} \right] dt \\
+& C(T) \int_0^T \|S_i(t)\|^{2p}_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \mathbb{E}\left[\left\| \int_0^t (P_N^a)^\perp \, d\tilde{B}_N(t',\cdot) \right\|^{2p}_{H^s(\mathbb{T}^q)} \right] dt \\
+& C(T) \int_0^T \|S_i(t)\|^{2p}_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \mathbb{E}\left[\left\| \int_0^t (P_N^a)^\perp \, dW(t',\cdot) \right\|^{2p}_{H^s(\mathbb{T}^q)} \right] dt.
\end{aligned}$$

Where right in the first estimate we moved $\|\sigma\|_\infty$ in our constant $C(T)$. Using once again

$$\sup_{t\in[0,T]} \|S_i(t)\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \leq \underbrace{C(T)M_i \max\{1, e^{\omega_i T}\}}_{=:C_i(T)},$$

the claim follows as by the Itō-Isometry

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[\left\| \int_0^t P_N^a \, d(W(t',\cdot) - \tilde{B}_N(t',\cdot)) \right\|^2_{H^s(\mathbb{T}^q)} \right] dt \\
\overset{\text{Itō-Isom.}}{=}& \int_0^T t^p \left(\sum_{k\in\mathbb{Z}^q\setminus[-N,N-1]^q} \langle k\rangle^{2s} \prod_{j=1}^q \left(\frac{e^{i\frac{\pi k_j}{N}} - 1}{i\frac{k_j \pi}{N}} - 1 \right)^2 \right)^p dt
\end{aligned}$$

$$\leq CT^p \left(\sum_{k\in\mathbb{Z}^q\setminus[-N,N-1]^q} \langle k\rangle^{2s} \right)^p \overset{N\to\infty}{\longrightarrow} 0,$$

as well as

$$\int_0^T \mathbb{E}\left[\left\| \int_0^t (P_N^a)^\perp \, d\tilde{B}_N(t',\cdot) \right\|_{H^s(\mathbb{T}^q)}^{2p} \right] dt$$

$$\overset{\text{Itō-Isom.}}{=} \int_0^T t^p \left(\sum_{k\in\mathbb{Z}^q\setminus[-N,N-1]^q} \langle k\rangle^{2s} \prod_{j=1}^q \left(\frac{e^{i\frac{\pi k_j}{N}}-1}{i\frac{k_j\pi}{N}} - 1 \right)^2 \right)^p dt$$

$$\leq CT^p \left(\sum_{k\in\mathbb{Z}^q\setminus[-N,N-1]^q} \langle k\rangle^{2s} \right)^p \overset{N\to\infty}{\longrightarrow} 0,$$

and finally

$$\int_0^T \mathbb{E}\left[\left\| \int_0^t (P_N^a)^\perp \, dW(t',\cdot) \right\|_{H^s(\mathbb{T}^q)}^{2p} \right] dt \overset{\text{Itō-Isom.}}{=} \int_0^T t^p \left(\sum_{k\in\mathbb{Z}^q\setminus[-N,N-1]^q} \langle k\rangle^{2s} \right)^p dt$$

$$= CT^p \left(\sum_{k\in\mathbb{Z}^q\setminus[-N,N-1]^q} \langle k\rangle^{s} \right)^p \overset{N\to\infty}{\longrightarrow} 0.$$

∎

Now for the second type of terms we at first return to the deterministic case. This time we are dealing with the difference of the continuous semigroup $\{S_i(t)\}_{t\geq 0}$ and the continuum version of the corresponding discrete semigroup, applied to the high-frequency part of the data. Both of the semigroups will be bounded using the evolution estimate, while we use a further estimate on the discrete partial difference quotient operators eigenvalues, to ensure the constant bounding the sum of both semigroups norms is independent of N. As our cutoff projector P_δ^a converges pointwise to the identity, the norm of the high-frequency terms vanishes in the limit. While for $s < r$ it becomes even easier, as we can squeeze a rate of convergence out of the norm itself, by returning to the $H^r(\mathbb{T}^q)$ norm, and then bound the norm of the high-frequency part by the norm of the data.

Lemma 5.2.4
Under the assumptions of Theorem 3.3.1 for every $i \in \{1,2,3\}$, every $T > 0$ and every $\varepsilon > 0$, there exists a $N(\varepsilon) \in \mathbb{N}^*$ such that for every $N \geq N(\varepsilon)$:

$$\sup_{t\in[0,T]} \left\| S(t) - \tilde{S}_N^d(t) \right\|_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}^2 \left\| (P_\delta^a)^\perp \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix} \right\|_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}^2 < \varepsilon.$$

Proof
By using Lemma 1.1.8 for both semigroups, we compute

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\| S(t) - \tilde{S}_N^d(t) \right\|_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}^2 \\
&\leq C \sup_{t\in[0,T]} \left(\|S(t)\|_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}^2 + \left\| \tilde{S}_N^d(t) \right\|_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))}^2 \right) \\
&\leq C \sup_{t\in[0,T]} \left(Me^{\omega t} + M_d e^{\omega_d t} \right) \\
&\leq C \max\{M, M_d\} \max\{1, e^{\omega T}, e^{\omega_d T}\} =: C(T).
\end{aligned}$$

Note that the constants M_d and ω_d could be chosen independent of N (as the semigroups are either contractions or their growth rate can be bound (As the highest degree term of the polynomial $p(A,\cdot)$, in every direction, is even and negative $p(A,\cdot)$ can be bounded from above independent of N. Hence the real part of the growth rate ω_d can be bounded independent of N) using Lemma 1.2.55), therefore we have for $\eta = s - r$

$$\begin{aligned}
&\left\| (P_\delta^a)^\perp \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix} \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
\leq &\left\| \frac{\langle \lfloor N^\delta \rfloor \rangle^\eta}{\langle \lfloor N^\delta \rfloor \rangle^\eta} (P_\delta^a)^\perp \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix} \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
\leq &\langle \lfloor N^\delta \rfloor \rangle^{2\eta} \left\| (P_\delta^a)^\perp \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix} \right\|^2_{H^{r+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q)} .
\end{aligned}$$

Now $\langle \lfloor N^\delta \rfloor \rangle^{2\eta}$ vanishes for $s < r$ and $\langle \lfloor N^\delta \rfloor \rangle^{2\eta} = 1$ for $s = r$. Furthermore

$$\left\| (P_\delta^a)^\perp \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix} \right\|^2_{H^{r+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q)} \overset{N\to\infty}{\longrightarrow} 0,$$

as $(P_\delta^a)^\perp v \overset{N\to\infty}{\longrightarrow} 0$ for every $v \in H^{r+m}(\mathbb{T}^q) \times H^r(\mathbb{T}^q)$ Thus there exists $N(\varepsilon) \in \mathbb{N}^*$ such that

$$\langle \lfloor N^\delta \rfloor \rangle^{2\eta} \left\| (P_\delta^a)^\perp \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix} \right\|^2_{H^{r+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q)} < \varepsilon,$$

for all $N \geq N(\varepsilon)$. Thus proving the claim. ■

Lemma 5.2.5
Under the assumptions of Theorem 3.3.1 for every $T > 0$ and every $\varepsilon > 0$, there exists a $N(\varepsilon) \in \mathbb{N}^*$ such that for every $N \geq N(\varepsilon)$:

$$\int_0^T \sup_{t\in[0,T]} \left\| (S_i(t) - \tilde{S}^d_{N,i}(t)) \right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \left\| (P_\delta^a)^\perp \tilde{f}_N(t',\cdot) \right\|^2_{H^s(\mathbb{T}^q)} \, ds < \varepsilon.$$

Proof
Again using the estimate

$$\sup_{t\in[0,T]} \left\| (S_i(t) - \tilde{S}^d_{N,i}(t)) \right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \leq C_i(T),$$

derived from Lemma 1.1.8 and that for $\eta = s - r$ we have

$$\begin{aligned}
&\int_0^T \sup_{t\in[0,T]} \left\| (P_\delta^a)^\perp \tilde{f}_N(t',\cdot) \right\|^2_{H^s(\mathbb{T}^q)} \, ds \\
\leq &\int_0^T \sup_{t\in[0,T]} \left\| \frac{\langle \lfloor N^\delta \rfloor \rangle^\eta}{\langle \lfloor N^\delta \rfloor \rangle^\eta} (P_\delta^a)^\perp \tilde{f}_N(s,x) \right\|^2_{H^s(\mathbb{T}^q)} \, ds \\
\leq &\int_0^T \langle \lfloor N^\delta \rfloor \rangle^\eta \sup_{t\in[0,T]} \left\| (P_\delta^a)^\perp \tilde{f}_N(t',\cdot) \right\|^2_{H^r(\mathbb{T}^q)} \, ds \\
\leq &\langle \lfloor N^\delta \rfloor \rangle^{2\eta} C(T) \left\| (P_\delta^a)^\perp \tilde{f}_N \right\|^2_{C([0,T],H^r(\mathbb{T}^q)}.
\end{aligned}$$

Now $\langle \lfloor N^\delta \rfloor \rangle^{2\eta}$ vanishes for $s < r$ and $\langle \lfloor N^\delta \rfloor \rangle^{2\eta} = 1$ for $s = r$. Furthermore

$$\left\| (P_\delta^a)^\perp \tilde{f}_N(t',\cdot) \right\|^2_{H^r(\mathbb{T}^q)} \overset{N\to\infty}{\longrightarrow} 0,$$

for every $t' \in [0,T)$, as $(P_\delta^a)^\perp v \overset{N\to\infty}{\longrightarrow} 0$ for every $v \in H^r(\mathbb{T}^q)$ Thus there exists $N(\varepsilon) \in \mathbb{N}^*$ such that

$$\langle \lfloor N^\delta \rfloor \rangle^{2\eta} C(T) \left\| (P_\delta^a)^\perp \tilde{f}_N \right\|^2_{C([0,T],H^r(\mathbb{T}^q)} < \varepsilon$$

for all $N \geq N(\varepsilon)$. And we obtain the assertion. ■

Again these deterministic estimates can be recycled via Lemma 5.1.2 and we only need an additional estimate for the difference of the stochastic integrals.

Lemma 5.2.6
Under the assumptions of Theorem 3.3.1 for every $i \in \{1,2,3\}$, depending on the damping, every $T > 0$, every $\delta \in (0,1)$ and every $p > 1$, there exists a $N(\varepsilon) \in \mathbb{N}^*$ such that:

$$\int_0^T \mathbb{E}\left[\left\|\int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))(P^a_\delta)^\perp \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} \, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q)}\right] dt < \varepsilon$$

for every $N \geq N(\varepsilon)$.

Proof
In a first step we compute for $\eta = s - r$

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[\left\|\int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))(P^a_\delta)^\perp \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} \, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q)}\right] d\tau \\
\leq & C\int_0^T \left\|(S_i(t) - \tilde{S}^d_{N,i}(t))\right\|^{2p}_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \|\sigma(t)\|_\infty \, \mathbb{E}\left[\left\|\int_0^t (P^a_\delta)^\perp \, d\tilde{B}_N(t',\cdot)\right\|^2_{H^s(\mathbb{T}^q)} 2p\right] dt \\
\leq & C\int_0^T \left\|(S_i(t) - \tilde{S}^d_{N,i}(t))\right\|^{2}_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \|\sigma(t)\|_\infty \, \mathbb{E}\left[\left\|\frac{\langle \lfloor N^\delta \rfloor\rangle^\eta}{\langle \lfloor N^\delta \rfloor\rangle^\eta}\int_0^t (P^a_\delta)^\perp \, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^s(\mathbb{T}^q)}\right] dt \\
\leq & C\int_0^T \left\|(S_i(t) - \tilde{S}^d_{N,i}(t))\right\|^{2}_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \|\sigma(t)\|_\infty \langle \lfloor N^\delta \rfloor\rangle^{2\eta} \mathbb{E}\left[\left\|\int_0^t (P^a_\delta)^\perp \, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^r(\mathbb{T}^q)}\right] dt.
\end{aligned}$$

Using once more the estimate, by Lemma 1.1.8,

$$\sup_{t\in[0,T]} \left\|(S_i(t) - \tilde{S}^d_{N,i}(t))\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \leq C_i(T),$$

and

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[\left\|\int_0^t (P^a_\delta)^\perp \, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^r(\mathbb{T}^q)}\right] dt \\
\overset{\text{Itō-Isom.}}{\leq} & C\int_0^T t^p \left(\sum_{k\in\mathbb{Z}^q\setminus[-\lfloor N^\delta\rfloor, \lfloor N^\delta\rfloor-1]^q} \langle k\rangle^{2r} \prod_{j=1}^q \left(\frac{e^{i\frac{\pi k_j}{N}} - 1}{i\frac{k_j\pi}{N}}\right)^2\right)^p \\
\leq & CT^p \left(\sum_{k\in\mathbb{Z}^q\setminus[-\lfloor N^\delta\rfloor, \lfloor N^\delta\rfloor-1]^q} \langle k\rangle^{2r}\right)^p.
\end{aligned}$$

Once more $\langle \lfloor N^\delta \rfloor\rangle^{2\eta}$ vanishes for $s < r$ and $\langle \lfloor N^\delta \rfloor\rangle^{2\eta} = 1$ for $s = r$. Furthermore

$$\sum_{k\in\mathbb{Z}^q\setminus[-\lfloor N^\delta\rfloor, \lfloor N^\delta\rfloor-1]^q} \langle k\rangle^{2r} \overset{N\to\infty}{\longrightarrow} 0,$$

as the remainder of a convergent series. Thus there exists $N(\varepsilon) \in \mathbb{N}^*$ such that

$$\int_0^T \mathbb{E}\left[\left\|\int_0^t (S_i(t-t') - \tilde{S}^d_{N,i}(t-t'))(P^a_\delta)^\perp \begin{pmatrix} 0 \\ \sigma(t') \end{pmatrix} \, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q)}\right] dt < \varepsilon$$

for every $N \geq N(\varepsilon)$. And we obtain the assertion. ∎

5.3 Semigroup Error Estimates

Finally we have to control the error between the continuous semigroup S and the continuation $\tilde{S}_{N,d}$, of the corresponding discrete semigroup $S_{N,d}$, acting on the low-frequency terms. In a first step we will use that the Fourier transform is an isometric isomorphism, with respect to the $\|\cdot\|_{H^s(\mathbb{T}^q)}$, for all $s \in \mathbb{R}$, to get it into the norm. Now we have access to the Fourier semigroup representation developed in section 4.3 and can use Taylor expansions of the semigroups coefficient functions to obtain the desired error estimates.

Naturally the Taylor estimates will be outsourced from the proof this sections main result and build up individually for each of the coefficient functions, where we use the Landau Big-$\mathcal{O}$ notation, which we will recall now, to give a qualitative description of the error.

Notation 5.3.1
Let $f : E \to F$ and $g : E \to [0, \infty)$, such that $g(x) > 0$ on an environment U of 0_E.
We say $f(x) = \mathcal{O}(\|g(x)\|_E)$ for $x \to 0_E$ if

$$\overline{\lim_{x \to a}} \frac{\|f(x)\|_E}{g(x)} < \infty.$$

Now we will begin the bottom up construction of our error estimates, recalling the square root functions R and $\tilde{R}_N^d$ [2)], introduced at the beginning of Section 4.3 and encoding the eigenvalues of the respective operator matrix, we will rewrite the discrete eigenvalues as the continuous eigenvalues plus an error term, for this we will use the Taylor expansion[3)] of a (shifted) square root function, with the symbol $p(A, \cdot)$ as its center.

Lemma 5.3.2
Under the assumptions of Theorem 3.3.1 every $N \in \mathbb{N}^*$, every $\delta \in \left(0, \frac{1}{m+1}\right)$ and $k \in \hat{Q}^q_{\lfloor N^\delta \rfloor} \cap supp(R_i(\cdot))$ one has

$$\tilde{R}_N^d(k) - R(k) = \mathcal{O}\left(\left| \frac{|k|^2 R(k)}{N^2} \right| \right).$$

Proof

$$\begin{aligned}
\tilde{R}_N^d(k) &= \sqrt{\left(\frac{\rho_2 p_{N,d}(\Delta_{N,d}, k) + \rho_1}{2} \right)^2 + p(A, \tilde{g}(k))} \\
&\overset{LB.1}{=} \sqrt{\left(\frac{\rho_2 p(\Delta, k) + \rho_1}{2} \right)^2 + p(A, k)} + \mathcal{O}\left(\left| \frac{p(A,k) + \left(\frac{\rho_2 p_{N,d}(\Delta_{N,d}, k) + \rho_1}{2} \right)^2 - p(A,k) - \left(\frac{\rho_2 p(\Delta, k) + \rho_1}{2} \right)^2}{\sqrt{\left(\frac{\rho_2 p(\Delta, k) + \rho_1}{2} \right)^2 + p(A,k)}} \right| \right) \\
&= \sqrt{\left(\frac{\rho_2 p(\Delta, k) + \rho_1}{2} \right)^2 + p(A, k)} + \mathcal{O}\left(\left| \frac{|k|^2 R(k)}{N^2} \right| \right).
\end{aligned}$$

■

On the same „level “we get the first order Taylor expansion of exponential term, in the strongly damped setting, as it only relies on the estimates for the (discrete) Laplacian.

2) At this point we should remind ourself, that the symbols and these square roots may very well be complex valued and we need to be consistent in our choice of branch for each pair of evaluations.

3) As the spectra of our operators are discrete and the roots don't appear in the semigroup representation, when they are zero, we are always far enough away from zero, to use Taylor expansions.

Lemma 5.3.3
Under the assumptions of Theorem 3.3.1 respectively Theorem 3.3.1, for $i = 3$, every $N \in \mathbb{N}^*$, every $k \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$ and $t > 0$ one has

$$e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},k)t}{2}} - e^{\frac{\rho_2 p(\Delta,k)t}{2}} = \mathcal{O}\left(\left|\frac{|k|^4 t^2}{N^2}\right|\right).$$

Proof
Using the series representation of the exponentialfunction and the relation between the continuous and discrete symbol we compute:

$$\begin{aligned} e^{\frac{\rho_2 p_{N,d}(\Delta_{N,d},k)t}{2}} &\overset{L\ B.5}{=} e^{\frac{\rho_2 p(\Delta,k)t}{2}} + \mathcal{O}\left(\left|e^{\frac{\rho_2 p(\Delta,k)t}{2}} \frac{\rho_2 t}{2}(p_{N,d}(\Delta_{N,d},k) - p(\Delta,k))\right|\right) \\ &= e^{\frac{\rho_2 p(\Delta,k)t}{2}} + \mathcal{O}\left(\left|\frac{p(\Delta,k)t^2|k|^2}{N^2}\right|\right) = e^{\frac{\rho_2 p(\Delta,k)\tau}{2}} + \mathcal{O}\left(\left|\frac{t^2|k|^4}{N^2}\right|\right). \end{aligned}$$

■

Building upon Lemma 5.3.2 we can now compute the error estimates for all the functions appearing in our semigroups coefficients, this time using tR_1, tR_2 respectively tR_3 as center of the Taylor expansion.

Lemma 5.3.4
Under the assumptions of Theorem 3.3.1 respectively Theorem 3.3.1, for every $i \in \{1,2,3\}$, every $N \in \mathbb{N}^*$, every $k \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$ and $t > 0$ one has

$$\cosh(\tilde{R}^d_{N,i}(k)t) - \cosh(R_i(k)t) = \mathcal{O}\left(\left|\frac{|k|^{2m+2}t^2}{N^2}\right|\right).$$

Proof
We linearize $cosh(x)$ and use the series expression of $sinh(x)$ and Lemma 5.3.2 to express the error in terms of k/N:

$$\begin{aligned} \cosh(\tilde{R}^d_{N,i}(k)t) &\overset{L\ B.3}{=} \cosh(R_i(k)t) + \sinh(R_i(k)t)\left(\tilde{R}^d_{N,i}(k)t - R_i(k)t\right) \\ &\overset{L\ 5.3.2}{=} \cosh(R_i(k)t) + \sinh(R_i(k)t)\mathcal{O}\left(\frac{R_i(k)|k|^2 t}{N^2}\right) \\ &\overset{L\ B.2}{=} \cosh(R_i(k)t) + \mathcal{O}\left(\left|\frac{(R_i(k))^2|k|^2t^2}{N^2}\right|\right) = \cosh(R_i(k)t) + \mathcal{O}\left(\left|\frac{p(A,k)|k|^2t^2}{N^2}\right|\right) \\ &= \cosh(R_i(k)t) + \mathcal{O}\left(\left|\frac{(R_i(k))^2|k|^2t^2}{N^2}\right|\right) = \cosh(R_i(k)t) + \mathcal{O}\left(\left|\frac{|k|^{2m+2}\tau^2}{N^2}\right|\right). \end{aligned}$$

■

Lemma 5.3.5
Under the assumptions of Theorem 3.3.1 respectively Theorem 3.3.1, for every $i \in \{1,2,3\}$, every $N \in \mathbb{N}^*$, every $k \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$ and $t > 0$ one has

$$\frac{\sinh(\tilde{R}^d_{N,i}(k)t)}{\tilde{R}^d_{N,i}(k)} - \frac{\sinh(R_i(k)t)}{R_i(k)} = \mathcal{O}\left(\left|\frac{t^2|k|^{2m+2}}{N^2}\right|\right).$$

Proof
Multiplying by $\frac{t}{t}$ we then can use the linearization of $\sinh(x)/x$ to compute:

$$\frac{\sinh(\tilde{R}^d_{N,i}(k)t)}{\tilde{R}^d_{N,i}(k)} = \frac{t\sinh(\tilde{R}^d_{N,i}(k)t)}{\tilde{R}^d_{N,i}(k)t}$$

$$\overset{\text{Lemma}B.4}{=} \frac{t\sinh(R_i(k)t)}{R_i(k)t} + \frac{R_i(k)t\cosh(R_i(k)t) - \sinh(R_i(k)t)}{(R_i(k)t)^2}(\tilde{R}^d_{N,i}(k)t - R_i(k)t),$$

and using the series expansions of $sinh(x)$ and $xcosh(x)$ we obtain

$$\begin{aligned}\frac{\sinh(\tilde{R}^d_{N,i}(k)t)}{\tilde{R}^d_{N,i}(k)} &= \frac{\sinh(R_1(k)t)}{R_i(k)t} + \mathcal{O}(R_i(k)t)(\tilde{R}^d_{N,i}(k)t - R_i(k)t) \overset{L\ 5.3.2\ (i)}{=} \frac{\sinh(R_1(k)t)}{R_i(k)t} + \mathcal{O}\left(\left|\frac{p(A,k)t^2|k|^2}{N^2}\right|\right)\\ &= \frac{\sinh(R_1(k)t)}{R_i(k)t} + \mathcal{O}\left(\left|\frac{t^2|k|^{2m+2}}{N^2}\right|\right).\end{aligned}$$

■

Lemma 5.3.6
Under the assumptions of Theorem 3.3.1 respectively Theorem 3.3.1, for every $i \in \{1,2,3\}$, every $N \in \mathbb{N}^*$, every $k \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$ and $t > 0$ one has

$$\tilde{R}^d_{N,i}(k)\sinh(\tilde{R}^d_{N,i}(k)t) - R_i(k)\sinh(R_i(k)t) = \mathcal{O}\left(\left|\frac{t^2|k|^{2m+2}}{N^2}\right|\right).$$

Proof
Using the series expansions of $x\sinh(x)$ to linearize:

$$\begin{aligned}\tilde{R}^d_{N,i}(k)\sinh(\tilde{R}^d_{N,i}(k)t) &= \frac{\tilde{R}^d_{N,i}(k)t\sinh(\tilde{R}^d_{N,i}(k)t)}{t}\\ &\overset{L\ B.7}{=} R_i(k)\sinh(R_i(k)t) + \frac{\sinh(R_i(k)t) + R_i(k)t\cosh(R_i(k)t)}{t}(\tilde{R}^d_{N,i}(k)t - R_i(k)t)\end{aligned}$$

and using the series expansions of $sinh(x)$ and $xcosh(x)$ we obtain

$$\begin{aligned}\tilde{R}^d_{N,i}(k)\sinh(\tilde{R}^d_{N,i}(k)t) =& R_i(k)\sinh(R_i(k)t) + \mathcal{O}(|R_i(k)t|)(\tilde{R}^d_{N,i}(k)t - R_i(k)t)\\ \overset{L\ 5.3.2\ (i)}{=}& R_i(k)\sinh(R_i(k)t) + \mathcal{O}\left(\left|\frac{p(A,k)t^2|k|^2}{N^2}\right|\right)\\ =& R_i(k)\sinh(R_i(k)t) + \mathcal{O}\left(\left|\frac{t^2|k|^{2m+2}}{N^2}\right|\right).\end{aligned}$$

■

Corollary 5.3.7
Under the assumptions of Theorem 3.3.1 respectively Theorem 3.3.1, for every $i \in \{1,2,3\}$, every $N \in \mathbb{N}^*$, every $k \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$ and $t > 0$ one has

$$\frac{p(A,\tilde{g}(k))\sinh(\tilde{R}^d_{N,i}(k)t)}{\tilde{R}^d_{N,i}(k)} - \frac{p(A,j)\sinh(R_i(k)t)}{R_i(k)} = \mathcal{O}\left(\frac{|k|^{2m+2}t^2}{N^2}\right).$$

Proof
We rewrite

$$\frac{p(A,\tilde{g}(k))\sinh(\tilde{R}^d_{N,i}(k)t)}{\tilde{R}^d_{N,i}(k)} = \tilde{R}^d_{N,i}(k)\sinh(\tilde{R}^d_{N,i}(k)t) - \frac{\rho_1^2\sinh(\tilde{R}^d_{N,i}(k)t)}{4\tilde{R}^d_{N,i}(k)}.$$

By Lemma 5.3.6 we have for the first term on the right-hand side

$$\tilde{R}^d_{N,i}(k)\sinh(\tilde{R}^d_{N,i}(k)t) - R_i(k)\sinh(R_i(k)t) = \mathcal{O}\left(\frac{t^2|k|^{2m+2}}{N^2}\right)$$

and by and by Lemma 5.3.5, also

$$\frac{\sinh(\tilde{R}^d_{N,i}(k)t)}{\tilde{R}^d_{N,i}(k)} - \frac{\sinh(R_i(k)t)}{R_i(k)} = \mathcal{O}\left(\frac{t^2|k|^{2m+2}}{N^2}\right).$$

As $\mathcal{O}\left(\frac{|k|^{2m+2}t^2}{N^2}\right)$ is closed under addition, we obtain the claim. ∎

Corollary 5.3.8
Under the assumptions of Theorem 3.3.1 respectively Theorem 3.3.1, for every $i \in \{2,3\}$, every $N \in \mathbb{N}^*$, every $k \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$ and $t > 0$ one has

$$\left(\cosh(\tilde{R}^d_{N,i}(j)t) \pm \frac{\rho_{i-1}\sinh(\tilde{R}^d_{N,i}(j)t)}{2\tilde{R}^d_{N,i}(j)}\right) - \cosh(R_i(j)t) \mp \frac{\rho_{i-1}\sinh(R_i(j)t)}{2R_i(j)} = \mathcal{O}\left(\frac{|k|^{2m+2}t^2}{N^2}\right).$$

Proof
By Lemma 5.3.4 we have

$$\cosh(\tilde{R}^d_{N,i}(j)t) - \cosh(R_i(j)t) = \mathcal{O}\left(\frac{|k|^{2m+2}t^2}{N^2}\right)$$

and by Lemma 5.3.5, also

$$\pm\frac{\rho_{i-1}\sinh(\tilde{R}^d_{N,i}(j)t)}{2\tilde{R}^d_{N,i}(j)} \mp \frac{\rho_{i-1}\sinh(R_i(j)t)}{2R_i(j)} = \mathcal{O}\left(\frac{|k|^{2m+2}t^2}{N^2}\right).$$

As $\mathcal{O}\left(\frac{|k|^{2m+2}t^2}{N^2}\right)$ is closed under addition, we obtain the claim. ∎

As already announced, in the introduction to this section, we will now bring in the Fourier transform, so we can use all the previously computed estimates. Furthermore we use that, through the projection onto the low-frequency Fourier modes, we are working on in a finite dimensional setting to further simplify the norm.

Lemma 5.3.9
Under the assumptions of Theorem 3.3.1 respectively Theorem 3.3.1, for every $i \in \{1,2,3\}$, every $N \in \mathbb{N}^*$ and $t > 0$ one has

$$\begin{aligned}
&\left\|(S_i(t) - \tilde{S}^d_{N,i}(t))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q))} \\
\leq &C \sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left|\left\langle \mathcal{F}(S_i(t) - \tilde{S}^d_{N,i}(t))\begin{pmatrix}\varphi_j\\0\end{pmatrix}, \begin{pmatrix}\varphi_j\\0\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right| \\
&+ C \sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left|\left\langle \mathcal{F}(S_i(t) - \tilde{S}^d_{N,i}(t))\begin{pmatrix}0\\\varphi_j\end{pmatrix}, \begin{pmatrix}0\\\varphi_j\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|.
\end{aligned}$$

Proof
By the definition of the operator norm, as $dim(ran(P^a_\delta)) < \infty$ and $H^r(\mathbb{T}^q) \supset H^{s+m}(\mathbb{T}^q)$

$$\left\|(S_i(t) - \tilde{S}^d_{N,i}(t))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \leq \frac{C}{\sqrt{2}} \sup_{k,j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left\|(S_i(t) - \tilde{S}^d_{N,i}(t))\begin{pmatrix}\langle j\rangle^{-r}\varphi_j\\\langle k\rangle^{-r}\varphi_k\end{pmatrix}\right\|^2_{H^r(\mathbb{T}^q)\times H^r(\mathbb{T}^q)}.$$

And by Theorem 1.2.9

$$\sup_{k,j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left\|(S_i(t) - \tilde{S}^d_{N,i}(t))\begin{pmatrix}\langle j\rangle^{-r}\varphi_j\\\langle k\rangle^{-r}\varphi_k\end{pmatrix}\right\|^2_{H^r(\mathbb{T}^q)\times H^r(\mathbb{T}^q)}$$

$$
\begin{aligned}
&\leq \sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left\| (S_i(t)-\tilde{S}^d_{N,i}(t)) \begin{pmatrix} \langle j\rangle^{-r}\varphi_j \\ 0 \end{pmatrix} \right\|^2_{H^r(\mathbb{T}^q)\times H^r(\mathbb{T}^q)} \\
&\quad + \sup_{k\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left\| (S_i(t)-\tilde{S}^d_{N,i}(t)) \begin{pmatrix} 0 \\ \langle k\rangle^{-r}\varphi_k \end{pmatrix} \right\|^2_{H^r(\mathbb{T}^q)\times H^r(\mathbb{T}^q)} \\
&\overset{\text{Theorem 1.2.9}}{\leq} C \sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left| \left\langle \mathcal{F}(S_i(t)-\tilde{S}^d_{N,i}(t)) \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix}, \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix} \right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)} \right| \\
&\quad + C \sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left| \left\langle \mathcal{F}(S_i(t)-\tilde{S}^d_{N,i}(t)) \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix} \right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)} \right|.
\end{aligned}
$$

■

Now that we have collected all the needed components we will combine them into the main result of this section, that is the error estimates for the semigroups, on the low-frequency Fourier modes.

Theorem 5.3.10
Under the assumptions of Theorem 3.3.1 respectively Theorem 3.3.1, for every $i \in \{1,2,3\}$ and $t > 0$ one has: For every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}^*$, such that

$$
\left\| (S_i(t)-\tilde{S}^d_{N,i}(t))P^q_\delta \right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} < \varepsilon,
$$

for all $N \geq N(\varepsilon)$.

Proof

(i) By Lemma 5.3.9 it suffices to consider

$$
\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left| \left\langle \mathcal{F}(S_1(t)-\tilde{S}^d_{N,1}(t)) \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix}, \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix} \right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)} \right|
$$

and

$$
\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left| \left\langle \mathcal{F}(S_1(t)-\tilde{S}^d_{N,1}(t)) \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix} \right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)} \right|
$$

for every $j \in \hat{Q}^q_{\lfloor N^\delta\rfloor}$.
Now using the representations in Lemma 4.3.2, Lemma 4.3.3, and Corollary 4.3.4 we have

$$
\begin{aligned}
&\mathcal{F}(S_1(t)-\tilde{S}^d_{N,1}(t)) \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix} \\
&= \begin{cases}
\begin{pmatrix}
\left(\cosh(R_1(j)t) - \left(\prod_{h=1}^q \left(\dfrac{e^{i\frac{\pi j_h}{N}}-1}{i\frac{j_h\pi}{N}} \right) \right) \cosh(\tilde{R}^d_{N,1}(j)t) \right) \varphi_j \\
\left(\dfrac{p(A,j)\sinh(R_1(j)t)}{R_1(j)} - \dfrac{p_{N,d}(A_{N,d},j)\sinh(\tilde{R}^d_{N,1}(j)t)}{\tilde{R}^d_{N,1}(j)} \left(\prod_{h=1}^q \left(\dfrac{e^{i\frac{\pi k_h}{N}}-1}{i\frac{k_h\pi}{N}} \right) \right) \right) \varphi_j
\end{pmatrix}, & R_1(j) = 0 \\
\left(1 - \prod_{h=1}^q \left(\dfrac{e^{i\frac{\pi j_h}{N}}-1}{i\frac{j_h\pi}{N}} \right) \right) \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix}, & R_1(j) = 0.
\end{cases}
\end{aligned}
$$

and

$$
\mathcal{F}(S_1(t)-\tilde{S}^d_{N,1}(t)) \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix}
$$

$$=\begin{cases}\left(\begin{pmatrix}\dfrac{\sinh(R_1(j)t)}{R_1(j)} - \left(\prod_{h=1}^{q}\left(\dfrac{e^{i\frac{\pi j_h}{N}}-1}{i\frac{j_h\pi}{N}}\right)\right)\dfrac{\sinh(\tilde{R}^d_{N,1}(j)t)}{\tilde{R}^d_{N,1}(j)}\end{pmatrix}\varphi_j \\ \left(\cosh(R_1(k)t) - \cosh(\tilde{R}^d_{N,1}(j)t)\left(\prod_{h=1}^{q}\left(\dfrac{e^{i\frac{\pi j_h}{N}}-1}{i\frac{j_h\pi}{N}}\right)\right)\right)\varphi_j\end{pmatrix}, & R_1(j)\neq 0,\\ \left(1-\prod_{h=1}^{q}\left(\dfrac{e^{i\frac{\pi j_h}{N}}-1}{i\frac{j_h\pi}{N}}\right)\right)\begin{pmatrix} t\varphi_j \\ \varphi_j\end{pmatrix}, & R_1(j)=0.\end{cases}$$

Then by the Lemmata 5.3.4, 5.3.5, B.5 and Corollary 5.3.7

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_1(t)-\tilde{S}^d_{N,1}(t))\begin{pmatrix}\varphi_j\\0\end{pmatrix},\begin{pmatrix}\varphi_j\\0\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right| = \begin{cases}\mathcal{O}\left(\frac{|j|^{2m+2}t^2}{N^2}\right), & R_1(j)\neq 0,\\ \mathcal{O}\left(\frac{|j|^q}{N^q}\right), & R_1(j)=0.\end{cases}$$

and

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_1(t)-\tilde{S}^d_{N,1}(t))\begin{pmatrix}0\\\varphi_j\end{pmatrix},\begin{pmatrix}0\\\varphi_j\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right| = \begin{cases}\mathcal{O}\left(\frac{|j|^{2m+2}t^2}{N^2}\right), & R_1(j)\neq 0,\\ \mathcal{O}\left(\frac{|j|^q|t|}{N^q}\right), & R_1(j)=0.\end{cases}$$

Thus for every $j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}$

$$\begin{aligned}&\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_1(t)-\tilde{S}^d_{N,1}(t))\begin{pmatrix}\varphi_j\\0\end{pmatrix},\begin{pmatrix}\varphi_j\\0\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|\\ \leq& C(T)\max\left\{\frac{N^{(4m+4)\delta}}{N^4},\frac{N^{q\delta}}{N^q}\right\}\\ \leq& C(T)\underbrace{\max\left\{N^{4((m+1)\delta-1)},N^{q(\delta-1}\right\}}_{\overset{N\to\infty}{\longrightarrow}0}\overset{N\to\infty}{\longrightarrow}0,\end{aligned}$$

and

$$\begin{aligned}&\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_1(t)-\tilde{S}^d_{N,1}(t))\begin{pmatrix}0\\\varphi_j\end{pmatrix},\begin{pmatrix}0\\\varphi_j\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|\\ \leq& C(T)\max\left\{\frac{N^{(4m+4)\delta}}{N^4},\frac{N^{q\delta}}{N^q}\right\}\\ \leq& C(T)\underbrace{\max\left\{N^{4((m+1)\delta-1)},N^{q(\delta-1}\right\}}_{\overset{N\to\infty}{\longrightarrow}0}\overset{N\to\infty}{\longrightarrow}0,\end{aligned}$$

which gives the assertion for $i=1$.

(ii) By Lemma 5.3.9 it suffices to consider

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_2(t)-\tilde{S}^d_{N,2}(t))\begin{pmatrix}\varphi_j\\0\end{pmatrix},\begin{pmatrix}\varphi_j\\0\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|$$

and

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_2(t)-\tilde{S}^d_{N,2}(t))\begin{pmatrix}0\\\varphi_j\end{pmatrix},\begin{pmatrix}0\\\varphi_j\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|$$

for every $j \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$.
Again using the representation results Lemma 4.3.2, Lemma 4.3.3 and Corollary 4.3.4 we obtain a representation of the differences

$$\mathcal{F}(S_2(\tau) - \tilde{S}^d_{N,2}(\tau))\begin{pmatrix}\varphi_j\\0\end{pmatrix} \text{ and } \mathcal{F}(S_2(\tau) - \tilde{S}^d_{N,2}(\tau))\begin{pmatrix}0\\\varphi_j\end{pmatrix}$$

and by Lemma 5.3.4, 5.3.5,B.5 and the Corollary 5.3.7 and 5.3.8 we obtain

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_2(t) - \tilde{S}^d_{N,2}(t))\begin{pmatrix}\varphi_j\\0\end{pmatrix}, \begin{pmatrix}\varphi_j\\0\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right| = \begin{cases}\mathcal{O}\left(\frac{|j|^{2m+2}t^2}{N^2}\right), & R_2(j)\neq 0,\\ \mathcal{O}\left(\frac{|j|^q}{N^q}\right), & R_2(j) = 0.\end{cases}$$

and

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_2(t) - \tilde{S}^d_{N,2}(t))\begin{pmatrix}0\\\varphi_j\end{pmatrix}, \begin{pmatrix}0\\\varphi_j\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right| = \begin{cases}\mathcal{O}\left(\frac{|j|^{2m+2}t^2}{N^2}\right), & R_2(j)\neq 0,\\ \mathcal{O}\left(\frac{|j|^q|t|}{N^q}\right), & R_2(j) = 0.\end{cases}$$

Thus for every $j \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$

$$\begin{aligned}&\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_2(t) - \tilde{S}^d_{N,2}(t))\begin{pmatrix}\varphi_j\\0\end{pmatrix}, \begin{pmatrix}\varphi_j\\0\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|\\ \leq& C(T)\max\left\{\frac{N^{(4m+4)\delta}}{N^4}, \frac{N^{q\delta}}{N^q}\right\}\\ \leq& C(T)\underbrace{\max\left\{N^{4((m+1)\delta-1)}, N^{q(\delta-1}\right\}}_{\stackrel{N\to\infty}{\longrightarrow}0} \stackrel{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

and

$$\begin{aligned}&\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_2(t) - \tilde{S}^d_{N,2}(t))\begin{pmatrix}0\\\varphi_j\end{pmatrix}, \begin{pmatrix}0\\\varphi_j\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|\\ \leq& C(T)\max\left\{\frac{N^{(4m+4)\delta}}{N^4}, \frac{N^{q\delta}}{N^q}\right\}\\ \leq& C(T)\underbrace{\max\left\{N^{4((m+1)\delta-1)}, N^{q(\delta-1}\right\}}_{\stackrel{N\to\infty}{\longrightarrow}0} \stackrel{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

which gives the assertion for $i = 2$.

(iii) By Lemma 5.3.9 it suffices to consider

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_3(t) - \tilde{S}^d_{N,3}(t))\begin{pmatrix}\varphi_j\\0\end{pmatrix}, \begin{pmatrix}\varphi_j\\0\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|$$

and

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}}\left|\left\langle \mathcal{F}(S_3(t) - \tilde{S}^d_{N,3}(t))\begin{pmatrix}0\\\varphi_j\end{pmatrix}, \begin{pmatrix}0\\\varphi_j\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|$$

for every $j \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$.
Now using the representation results Lemma 4.3.2, Lemma 4.3.3 and Corollary 4.3.4 we find a representation of the differences

$$\mathcal{F}(S_3(\tau) - \tilde{S}^d_{N,3}(\tau))\begin{pmatrix}\varphi_j\\0\end{pmatrix} \text{ and } \mathcal{F}(S_3(\tau) - \tilde{S}^d_{N,3m}(\tau))\begin{pmatrix}0\\\varphi_j\end{pmatrix}$$

and by the Lemmata 5.3.4, 5.3.5,B.5 and Corollary 5.3.7

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left|\left\langle \mathcal{F}(S_3(t) - \tilde{S}^d_{N,3}(t))\begin{pmatrix}\varphi_j\\0\end{pmatrix}, \begin{pmatrix}\varphi_j\\0\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right| = \begin{cases}\mathcal{O}\left(\frac{|j|^{2m+2}t^2}{N^2}\right), & R_3(j)\neq 0,\\ \max\left\{\mathcal{O}\left(\frac{|j|^q|t|}{N^q}\right), \mathcal{O}\left(\frac{k^4t^2}{N^2}\right)\right\}, & R_3(j) = 0.\end{cases}$$

and

$$\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left|\left\langle \mathcal{F}(S_3(t) - \tilde{S}^d_{N,3}(t))\begin{pmatrix}0\\\varphi_j\end{pmatrix}, \begin{pmatrix}0\\\varphi_j\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right| = \begin{cases}\mathcal{O}\left(\frac{|j|^{2m+2}t^2}{N^2}\right), & R_3(j)\neq 0,\\ \max\left\{\mathcal{O}\left(\frac{|j|^q|t|}{N^q}\right), \mathcal{O}\left(\frac{k^4t^2}{N^2}\right)\right\}, & R_3(j) = 0.\end{cases}$$

With the additional error terms of order $\mathcal{O}\left(\frac{k^4t^2}{N^2}\right)$, being the result of Lemma 5.3.3 applied to the strong damping terms. While they don't affect the order of the already existing ones as they only appear in terms of lower order in those entries. Thus for every $j \in \hat{Q}^q_{\lfloor N^\delta \rfloor}$

$$\begin{aligned}&\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left|\left\langle \mathcal{F}(S_3(t) - \tilde{S}^d_{N,3}(t))\begin{pmatrix}\varphi_j\\0\end{pmatrix}, \begin{pmatrix}\varphi_j\\0\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|\\ \leq & C(T)\max\left\{\frac{N^{(4m+4)\delta}}{N^4}, \frac{N^{q\delta}}{N^q}, \frac{N^{4\delta}}{N^2}\right\}\\ \leq & C(T)\underbrace{\max\left\{N^{4((m+1)\delta-1)}, N^{q(\delta-1)}, N^{4\delta-2}\right\}}_{\overset{N\to\infty}{\longrightarrow}0} \overset{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

and

$$\begin{aligned}&\sup_{j\in\hat{Q}^q_{\lfloor N^\delta\rfloor}} \left|\left\langle \mathcal{F}(S_3(t) - \tilde{S}^d_{N,3}(t))\begin{pmatrix}0\\\varphi_j\end{pmatrix}, \begin{pmatrix}0\\\varphi_j\end{pmatrix}\right\rangle_{L_2(\mathbb{T}^q)\times L_2(\mathbb{T}^q)}\right|\\ \leq & C(T)\max\left\{\frac{N^{(4m+4)\delta}}{N^4}, \frac{N^{q\delta}}{N^q}, \frac{N^{4\delta}}{N^2}\right\}\\ \leq & C(T)\underbrace{\max\left\{N^{4((m+1)\delta-1)}, N^{q(\delta-1)}, N^{4\delta-2}\right\}}_{\overset{N\to\infty}{\longrightarrow}0} \overset{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

which gives the assertion for $i = 3$.

■

Remark 5.3.11
Now that we have done all the error estimates, we wish to point that for s strictly smaller than r, it is possible to obtain a rate of convergence, that would depend on s, the chosen cutoff range δ, as well as the rate of convergence of the discrete initial data/forcing and noise damping towards their continuum counterparts.

6 Proofs of the Main Results

In this chapter we are finally combining all the existence results, representations, decomposition results and estimates, done in the previous two chapters, into the lemmata proving our main results.

6.1 Deterministic Proofs

Lemma 6.1.1
Under the assumptions of Theorem 3.3.1with $D = \rho_2\Delta$, for some constant $\rho_2 > 0$, for every $s < r$ (EQ1) has a unique mild solution $u \in H^{s+m}(\mathbb{T}^q)$, (DEQ1) has a unique mild solution $X_N \in h^{s+m}(Q_N^q)$ for every $N \in \mathbb{N}$ and for every $T > 0$ we have:

$$\left\|u - \tilde{X}_N\right\|^2_{C([0,T],H^{s+m}(\mathbb{T}^q))} \overset{N\to\infty}{\longrightarrow} 0.$$

and

$$\left\|\partial_t u - \partial_t \tilde{X}_N t\right\|^2_{C([0,T],H^{s}(\mathbb{T}^q))} \overset{N\to\infty}{\longrightarrow} 0.$$

Proof
The existence of unique mild solutions follows by Corollary 4.2.4 respectively Corollary 4.2.7.

Consider

$$\left\|u - \tilde{X}_N\right\|^2_{C([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_t u - \partial_t \tilde{X}_N\right\|^2_{C([0,T],H^{s}(\mathbb{T}^q))}.$$

Let $\varepsilon > 0$. By Lemma 5.1.1, Lemma 5.1.3, Lemma 5.1.4, Lemma 5.1.6 and Lemma 5.1.7 we get

$$\begin{aligned}
&\left\|u - \tilde{X}_N\right\|^2_{C([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_t u - \partial_t \tilde{X}_N\right\|^2_{C([0,T],H^{s}(\mathbb{T}^q))} \\
&\leq \sup_{t\in[0,T]} \left\| S_3(t)\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix}\right)\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
&\quad + \sup_{t\in[0,T]} \left\| \int_0^t S_3(t-t')\begin{pmatrix} 0 \\ f(t',\cdot) - \tilde{f}_N(t',\cdot) \end{pmatrix} dt' \right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
&\quad + C \sup_{t\in[0,T]} \left\|(S_3(t) - \tilde{S}^d_{N,3}(t))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \left\|\begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,1} \end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
&\quad + C \sup_{t\in[0,T]} \left\|S_3(t) - \tilde{S}^d_{N,3}(t)\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \left\|(P^a_\delta)^\perp \begin{pmatrix} \tilde{X}_{N,0} \\ \tilde{X}_{N,3} \end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \\
&\quad + C \int_0^T \sup_{t\in[0,T]} \left\|(S_3(t-t') - \tilde{S}^d_{N,3}(t-t'))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \left\|\tilde{f}_N(t',\cdot)\right\|^2_{H^s(\mathbb{T}^q)} \, ds \\
&\quad + C \int_0^T \sup_{t\in[0,T]} \left\|(S_3(t) - \tilde{S}^d_{N,3}(t))\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \left\|(P^a_\delta)^\perp \tilde{f}_N(s,x)\right\|^2_{H^s(\mathbb{T}^q)} \, ds.
\end{aligned}$$

By Lemma 5.3.10(iii) we find $N_1 \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \left\|(S_3(t) - \tilde{S}^d_{N,3}(t))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \left\|\begin{pmatrix}\tilde{X}_{N,0}\\ \tilde{X}_{N,1}\end{pmatrix}\right\|^2_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)} \leq \frac{\varepsilon}{6}$$

for all $N \geq N_1$ and $N_2 \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \left\|(S_3(t) - \tilde{S}^d_{N,3}(t))P^a_\delta\right\|^2_{\mathcal{L}(H^{s+m}(\mathbb{T}^q)\times H^r(\mathbb{T}^q))} \sup_{t\in[0,T]} \left\|\tilde{f}_N(t,\cdot)\right\|^2_{H^s(\mathbb{T}^q)} \leq \frac{\varepsilon}{6}$$

for all $N \geq N_2$. Therefore we find $N_3 \in \mathbb{N}$ such that the total of the remaining terms is bounded by $\frac{2}{3}\varepsilon$, for all $N \geq N_3$, and setting $N_0 := \max\{N_1, N_2, N_3, N_4, N_5, N_6\}$ we obtain:

$$\left\|u(\tau,x) - \tilde{X}_N(\tau,x)\right\|^2_{C([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_\tau u(\tau,x) - \partial_\tau \tilde{X}_N(\tau,x)\right\|^2_{C([0,T],H^s(\mathbb{T}^q))} < \varepsilon,$$

for all $N \geq N_0$, thus proving the claim. ∎

6.2 Stochastic Proofs

Before we proof our stochastic results we will proof a Lemma that will allow us to reuse the deterministic estimates for the deterministic part of the stochastic solutions, that is everything but the difference of the respective stochastic convolutions.

Lemma 6.2.1
Under the assumptions of Theorem 3.3.1, let $u \in H^{s+m}(\mathbb{T}^q)$ the unique, stochastic strong, mild solution of (EQ1) and for every $N \in \mathbb{N}^*$ let $X_N \in h^{s+m}(Q^q_N)$ the unique stochastic strong, mild solution of (DEQ1). Then for every $i \in \{1,2,3\}$, depending on the damping, $T > 0$ and every $p > 1$ we have and every $\varepsilon > 0$, there exists a $N_0 \in \mathbb{N}$ and a constant $C(T,p) > 0$, such that for every $N \geq N_0$:

$$\begin{aligned}
&\mathbb{E}\left[\left\|u\tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_t u - \partial_t \tilde{X}_N\right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))}\right] \\
&\leq \int_0^T \frac{\varepsilon}{2} + C(T,p)\mathbb{E}\left[\left\|\int_0^t S_i(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\, dW(t',\cdot) - \int_0^t \tilde{S}^d_{N,i}(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^s(\mathbb{T}^q)}\right] dt.
\end{aligned}$$

Proof
Let $v \in H^{s+m}(\mathbb{T}^q)$ the unique mild solution of (EQ1) with $\sigma \equiv 0$ and for every $N \in \mathbb{N}$ let $Y_N \in h^{s+m}(Q^q_N)$ the unique mild solution of (DEQ1) with $\sigma \equiv 0$, then we have by triangular inequality and monotony and linearity of the expectation:

$$\begin{aligned}
&\mathbb{E}\left[\left\|u - \tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_t u - \partial_t \tilde{X}_N\right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))}\right] \\
\leq &C(p)\mathbb{E}\left[\left\|v - \tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_t v(\tau,x) - \partial_t \tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))}\right] \\
&+ C(p)\mathbb{E}\left[\left\|\int_0^t S_i(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\, dW(t',\cdot) - \int_0^t \tilde{S}^d_{N,i}(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))}\right].
\end{aligned}$$

By then Stochastic Fubini Theorem we can exchange expectation and the L_p-norm and obtain:

$$\begin{aligned}
&C(p)\mathbb{E}\left[\left\|\int_0^t S_i(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\, dW(t',\cdot) - \int_0^t \tilde{S}^d_{N,i}(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))}\right] \\
\leq &C(T,p)\int_0^T \mathbb{E}\left[\left\|\int_0^t S_i(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\, dW(t',\cdot) - \int_0^t \tilde{S}^d_{N,i}(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix}\, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^s(\mathbb{T}^q)}\right] d\tau.
\end{aligned}$$

Furthermore as v and $(Y_N)_{N\in\mathbb{N}}$ are deterministic we can drop the expectation and estimate

$$\begin{aligned}
&\mathbb{E}\left[\left\|v-\tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))}+\left\|\partial_t v-\partial_t\tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^{s}(\mathbb{T}^q))}\right]\\
=&\left\|v-\tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))}+\left\|\partial_t v-\partial_t\tilde{Y}_N\right\|^{2p}_{L_p([0,T],H^{s}(\mathbb{T}^q))}\\
\leq& C(T,p)\left[\left\|v-\tilde{Y}_N\right\|^{2}_{C([0,T],H^{s+m}(\mathbb{T}^q))}+\left\|\partial_t v-\partial_t\tilde{Y}_N\right\|^{2}_{C([0,T],H^{s}(\mathbb{T}^q))}\right].
\end{aligned}$$

By Theorem 3.3.1 we now find $N_0\in\mathbb{N}$ such that for every $N\geq N_0$

$$C(T,p)\left[\left\|v-\tilde{Y}_N\right\|^{2}_{C([0,T],H^{s+m}(\mathbb{T}^q))}+\left\|\partial_t v-\partial_t\tilde{Y}_N\right\|^{2}_{C([0,T],H^{s}(\mathbb{T}^q))}\right]<\frac{\varepsilon}{2},$$

thus proving the claim. ∎

Now once again we proceed to prove our stochastic continuum limits in order of increasing damping strength.

Lemma 6.2.2
Under the assumptions of Theorem 3.3.1 with $D=\rho\Delta_{N,d}$, for some constant $\rho_2>0$, for every $s<r$ (EQ1) has a unique stochastic strong, mild solution $u\in H^{s+m}(\mathbb{T}^q)$, (DEQ1) has a unique stochastic strong, mild solution $X_N\in h^{s+m}(Q_N^q)$ for every $N\in\mathbb{N}$ and for every $T>0$ and every $p>1$ we have:

$$\mathbb{E}\left[\left\|u-\tilde{X}_N\right\|^{2}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))}\right]\overset{N\to\infty}{\longrightarrow}0.$$

and

$$\mathbb{E}\left[\left\|\partial_t u-\partial_t\tilde{X}_N\right\|^{2}_{L_p([0,T],H^{s}(\mathbb{T}^q))}\right]\overset{N\to\infty}{\longrightarrow}0.$$

Proof
The existence of unique stochastic strong, mild solutions follows by Corollary 4.2.3 respectively Corollary 4.2.6.

Now we consider

$$\mathbb{E}\left[\left\|u-\tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))}+\left\|\partial_t u-\partial_t\tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s}(\mathbb{T}^q))}\right].$$

Let $\varepsilon>0$. By Lemma 6.2.1 there exist a $C(T,p)>0$ and $N_0\in\mathbb{N}$ such that

$$\begin{aligned}
&\mathbb{E}\left[\left\|u-\tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))}+\left\|\partial_t u-\partial_t\tilde{X}_N p\right\|_{L_{H^s(\mathbb{T}^q)}([0,T],)}2p\right]\\
\leq&\frac{\varepsilon}{2}+C(T,p)\int_0^T\mathbb{E}\left[\left\|\int_0^t S_3(t-t')\begin{pmatrix}0\\\sigma(t')\end{pmatrix}dW(t',\cdot)-\int_0^t\tilde{S}^d_{N,3}(t-t')\begin{pmatrix}0\\\sigma(t')\end{pmatrix}d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^s(\mathbb{T}^q)}\right]dt.
\end{aligned}$$

Now by Lemma 5.1.5 and Lemma 5.1.8 we have

$$\begin{aligned}
&\int_0^T\mathbb{E}\left[\left\|\int_0^t S_3(t-t')\begin{pmatrix}0\\\sigma(t')\end{pmatrix}dW(t',\cdot)-\int_0^t\tilde{S}^d_{N,3}(t-t')\begin{pmatrix}0\\\sigma(t')\end{pmatrix}d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^s(\mathbb{T}^q)}\right]dt\\
\leq&\int_0^T\mathbb{E}\left[\left\|\int_0^t(S_3(t-t')-\tilde{S}^d_{N,3}(t-t'))P^a_\delta\sigma(t')\,d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\right]dt\\
&+\int_0^T\mathbb{E}\left[\left\|\int_0^t(S_3(t-t')-\tilde{S}^d_{N,3}(t-t'))(P^a_\delta)^\perp\sigma(t')\,d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\right]d\\
&+\int_0^T\mathbb{E}\left[\left\|\int_0^t S_3(t-t')\begin{pmatrix}0\\\sigma(t')\end{pmatrix}d(W(t',\cdot)-\tilde{B}_N(t',\cdot))\right\|^{2p}_{H^s(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\right]dt
\end{aligned}$$

by Itō-Isometry and Lemma 5.3.10 we find $N_1 \in \mathbb{N}$ such that for ever $N \geq N_1$:

$$\int_0^T \mathbb{E}\left[\left\|\int_0^t (S_3(t-t') - \tilde{S}^d_{N,3}(t-t'))P^a_\delta \sigma(t')\, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\right] dt < \frac{\varepsilon}{6},$$

by Lemma 5.2.6 we find $N_2 \in \mathbb{N}$ such that for ever $N \geq N_2$:

$$\int_0^T \mathbb{E}\left[\left\|\int_0^t (S_3(t-t') - \tilde{S}^d_{N,3}(t-t'))(P^a_\delta)^\perp \sigma(t')\, d\tilde{B}_N(t',\cdot)\right\|^{2p}_{H^{s+m}(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\right] dt < \frac{\varepsilon}{6}$$

and by Lemma 5.2.3 we find $N_3 \in \mathbb{N}$ such that for ever $N \geq N_3$:

$$\int_0^T \mathbb{E}\left[\left\|\int_0^t S_3(t-t')\begin{pmatrix}0\\ \sigma(t')\end{pmatrix} d(W(t',\cdot) - \tilde{B}_N(t',\cdot))\right\|^{2p}_{H^s(\mathbb{T}^q)\times H^s(\mathbb{T}^q)}\right] dt < \frac{\varepsilon}{6}.$$

Thus for every $N \geq \max\{N_0, N_1, N_2, N_3\}$ we get:

$$\mathbb{E}\left[\left\|u - \tilde{X}_N\right\|^{2p}_{L_p([0,T],H^{s+m}(\mathbb{T}^q))} + \left\|\partial_t u - \partial_t \tilde{X}_N\right\|^{2p}_{L_p([0,T],H^s(\mathbb{T}^q))}\right] < \varepsilon.$$

■

Part II

Nonlinear to Linear Continuum Limit

7 Scaling of the Non-linear Term

In the second part of this thesis we want to study the limiting behaviour of semilinear discrete equations, that is we will add a non-linear function in X_N to the right-hand side of (DEQ1).

$$\begin{cases} \partial_{tt} X_N(t,l) = A_{d,N} X_N(t,l) + \partial_t D_N X_N(t,l) + b_N(X_N, N) + f_N(t,l) + \sigma(t) B_N(t,l), & (t,l) \in \mathbb{R}_+ \times Q_N^q, \\ X_N(0,\cdot) = X_{N,0}(\cdot), & \text{in } Q_N^q, \\ \partial_t X_N(0,\cdot) = X_{N,1}(\cdot), & \text{in } Q_N^q, \end{cases} \tag{NLEQ1}$$

where the new part $b_N(\cdot)$ is a scaled non-linear function, that is the combination of a N-dependent scaling function and a polynomial, that does not directly depend on τ.

($\tilde{A}1$) $u_0, u_1 \in L_2(\mathbb{T}^q)$, and $f \in C(\mathbb{R}_{\geq 0}, L_2(\mathbb{T}^q))$.

($\tilde{A}2$) For every $N \in \mathbb{N}^*$ we have $X_{N,0}, X_{N,1} \in l_2(Q_N^q)$, $f_N \in C(\mathbb{R}_{\geq 0}, l_2(Q_N^q)$ such that

$$\begin{aligned} \tilde{X}_{N,0} &\overset{N\to\infty}{\longrightarrow} u_0 \text{ in } L_2(\mathbb{T}^q), \\ \tilde{X}_{N,1} &\overset{N\to\infty}{\longrightarrow} u_1 \text{ in } L_2(\mathbb{T}^q) \text{ and} \\ \tilde{f}_N &\overset{N\to\infty}{\longrightarrow} f \text{ in } C([0,T], L_2(\mathbb{T}^q)) \ \forall T > 0. \end{aligned}$$

To treat these equations we will employ a technique similar to the renomalization scheme developed for the continous stochastic wave equation (see for example [GKO18a] or [GKO18b]), but instead of adding a divergent (time-dependent) constant, we reign in the non-linearity by dividing through an appropriate scaling function (in N), such that the non-linearity collapses into a linear expression in the limiting equation. As in the continuous case the ability to implement this renormalization scheme on an algebraic level limits the choice of admissible non-linearitys to polynomials and trigonometric, hyperbolic and exponential power series, though the later won't be covered here as their treatment requires some more control on the dependency between the monomial powers and their renormalization constants.
As we lack any a priori knowledge on the solutions $(Z_N)_{N\in\mathbb{N}}$ of our non-linear equations NLEQ1, in Section 7.2 we will turn back to the linear solution $(X_N)_{N\in\mathbb{N}}$ of DEQ1 and the scaling of $(b(X_N))_{N\in\mathbb{N}}$[1)], that is the linear solution under the N-independent non-liner part of b_N, as a reference point. Finally in Section 7.3 we will investigate the difference between the linear solution, that by the previous part converges to our expected linear continuous limiting equation, and the non-linear equation, under a stronger scaling than the reference value.

7.1 Moment Bounds for the Linear Solutions

In this section we will collect some useful estimates on the pseudo-covariance of the linear continuous solution, that we will frequently use when determining the scaling of $(b([P_N u])))_{N\in\mathbb{N}^*}$.

1) Though, as they are interchangeable, by the theory for linear equations we will actually use $(b([P_N u]))_{N\in\mathbb{N}}$ in our computations, as its symbol immediately gives us the scaling in terms of a hyperharmonic series.

Where u is a solution of the continuous stochastic equation

$$\begin{cases} \partial_{tt} u(t,x) = Au(t,x) + Du(t,x) + f(t,x) + \sigma(t)W(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{T}^q, \\ u(0,\cdot) = u_0(\cdot), & \text{in } \mathbb{T}^q, \\ \partial_t u(0,\cdot) = u_1(\cdot), & \text{in } \mathbb{T}^q, \end{cases} \tag{LEQ1}$$

Remark 7.1.1
At this point it is good to remember that the polynomial $p(A,\cdot)$ and its square root can, and in general will, take values in the complex numbers.

Lemma 7.1.2
Let u a solution of (LEQ1), $N \in \mathbb{N}^*$ and $k \in \hat{Q}_N^q \cap supp(p(A,k))$, then

$$\mathbb{E}\left[|u_k(t)|^2\right] = \int_0^t \frac{|\sinh((t-s)\sqrt{p(A,k)})|^2}{|p(A,k)|}\, ds$$

and for $k \notin suppp(A,k)$ we have

$$\mathbb{E}\left[|u_k(t)|^2\right] = t^3.$$

Proof
First we note that for $k \notin \text{supp}(p(A,k))$:

$$\mathbb{E}\left[|u_0(t)|^2\right] = \mathbb{E}\left[\left|\int_0^t (t-s)\, d\beta_0(s)\right|^2\right] \overset{Thm 2.2.10}{=} \int_0^t (t-s)^2\, ds = t^3.$$

And $k \in \hat{Q}_N^q \cap \text{supp}p(A.k)$ we have:

$$\begin{aligned} \mathbb{E}\left[|u_k(t)|^2\right] =& \mathbb{E}\left[\left|\int_0^t \frac{\sinh((t-s)\sqrt{p(A,k)})}{\sqrt{p(A,k)}}\, d\beta_k(s)\right|^2\right] \\ \overset{Thm 2.2.10}{=}& \int_0^t \frac{|\sinh((t-s)\sqrt{p(A,k)})|^2}{|p(A,k)|}\, ds \end{aligned}$$

∎

To further estimate this the following bound on the real part of $\sqrt{p(A,k)}$ is needed, as its boundedness ensures that the hyperbolic functions appearing in our solution stay bounded.

Lemma 7.1.3
Under our general assumption (A3) there exist a $C > 0$ such that for all $k \in \mathbb{Z}^q$:

$$0 < |\text{Re}\sqrt{p(A,k)}| < C.$$

Proof
Let $p(A,k) = a_k + ib_k \in \mathbb{C}$ then the real part of its square root is given by

$$\text{Re}\sqrt{p(A,k)} = \pm\sqrt{\frac{\sqrt{a_k^2 + b_k^2} + a_k}{2}},$$

depending on our choice of square root. But now for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}^*$ such that for ever k outside $[-N(\varepsilon), N(\varepsilon)]^q$ we have

$$\sqrt{\frac{\sqrt{a_k^2 + b_k^2} + a_k}{2}} < \varepsilon,$$

as by our general assumption (A3), there exists an $N_0 \in \mathbb{N}^*$ such that $a_k < 0$, for all $k > N_0$, and

$$\frac{b_k}{a_k} \overset{k\to\infty}{\longrightarrow} 0.$$

∎

Now we can derive the following bounds.

Lemma 7.1.4
Let u a solution of (LEQ1), $N \in \mathbb{N}^*$ and $k \in \hat{Q}_N^q \cap supp(p(A,k))$, then there exist $C_1(T), C_2(T) > 0$ such that

$$C_1(T)\int_0^t \frac{\sinh^2((t-s)\operatorname{Re}\sqrt{p(A,k)}) + \sin^2((t-s)\operatorname{Im}\sqrt{p(A,k)}}{|p(A,k)|}\,ds \le \mathbb{E}\left[|u_k(t)|^2\right]$$

and

$$C_2(T)\int_0^t \frac{\sinh^2((t-s)\operatorname{Re}\sqrt{p(A,k)}) + \sin^2((t-s)\operatorname{Im}\sqrt{p(A,k)}}{|p(A,k)|}\,ds.$$

Proof
Expanding the complex sinh we get

$$\begin{aligned}&|\sinh((t-s)\sqrt{p(A,k)})|^2\\ =&\sinh^2((t-s)\operatorname{Re}\sqrt{p(A,k)})\cos^2((t-s)\operatorname{Im}\sqrt{p(A,k)}) + \cosh^2((t-s)\operatorname{Re}\sqrt{p(A,k)})\sin^2((t-s)\operatorname{Im}\sqrt{p(A,k)}).\end{aligned}$$

As $0 < \cos^2(x) < 1$ for all $x \in \mathbb{R}$ and $\cosh^2((t-s)\operatorname{Re}\sqrt{p(A,k)})$ stays bounded on $[0,T]$, for every $T > 0$, by Lemma 7.1.3 we get the claim. ∎

Lemma 7.1.5
For $N \in \mathbb{N}^*$, $k \in (\hat{Q}_N^q \cap supp(p(A,k))) \setminus [-N_0, N_0+1]^q$ there exist constants $C_1, C_2 > 0$ such that

$$C_1\left(\frac{t}{|p(A,k)|^m} + \frac{t^3}{6}\right) \le \left(\frac{t}{2|p(A,k)|} - \frac{\sin(2t\operatorname{Im}\sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2\operatorname{Im}\sqrt{p(A,k)}}\right) \le C_2\left(\frac{t}{|p(A,k)|^m} + \frac{t^3}{6}\right).$$

Where N_0 denotes the cutoff from (A3.3).

Proof
For $k \in \hat{Q}_N^2$ such that $|\sqrt{p(A,k)}| \le \frac{1}{2t}$ we get from the power series of $\sin(x)$ the existence of constants $\tilde{C}_1, \tilde{C}_2$:

$$\tilde{C}_1\frac{t^3}{6} \le \left(\frac{t}{2|p(A,k)|} - \frac{\sin(2t\operatorname{Im}\sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2\operatorname{Im}\sqrt{p(A,k)}}\right) \le \tilde{C}_2\frac{t^3}{6},$$

while for $k \in \hat{Q}_N^2$ such that $|\operatorname{Im}\sqrt{p(A,k)}| > \frac{1}{2t}$ using $|\sin(x)| \le 1$ we find constants $\hat{C}_1, \hat{C}_2$:

$$\hat{C}_1\left(\frac{t}{|p(A,k)|}\right) \le \left(\frac{t}{2|p(A,k)|} - \frac{\sin(2t\operatorname{Im}\sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2\operatorname{Im}\sqrt{p(A,k)}}\right) \le \hat{C}_2\left(\frac{t}{|p(A,k)|}\right).$$

Now by (A3.3) we get with possibly different constants $\hat{C}_1, \hat{C}_2$:

$$\hat{C}_1\left(\frac{t}{|k|^{2m}}\right) \le \hat{C}_1\left(\frac{t}{|p(A,k)|}\right) \le \left(\frac{t}{2|p(A,k)|} - \frac{\sin(2t\operatorname{Im}\sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2\operatorname{Im}\sqrt{p(A,k)}}\right) \le \hat{C}_2\left(\frac{t}{|p(A,k)|}\right) \le \hat{C}_2\left(\frac{t}{|k|^{2m}}\right).$$

The claim follows by choosing $C_1 := \min\{\hat{C}_1, \tilde{C}_1\}$ and $C_2 := \max\{\hat{C}_2, \tilde{C}_2\}$. ∎

Lemma 7.1.6
For $N \in \mathbb{N}^*$, $k \in (\hat{Q}_N^q \cap supp(p(A,k))) \setminus [-N_0, N_0+1]^q$ there exist constants $C_1, C_2 > 0$ such that

$$C_1\left(\frac{t}{|p(A,k)|^m} + \frac{t^3}{6}\right) \le \left(\frac{t}{2|p(A,k)|} + \frac{\sinh(2t\operatorname{Re}\sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2(\operatorname{Re}\sqrt{p(A,k)})}\right) \le C_2\left(\frac{t}{|p(A,k)|^m} + \frac{t^3}{6}\right).$$

Where N_0 denotes the cutoff from (A3.3).

Proof
For $k \in \hat{Q}_N^2$ such that $\sqrt{p(A,k)}| \leq \frac{1}{2t}$ we get from the power series of $\sinh(x)$ the existence of constants $\tilde{C}_1, \tilde{C}_2$:

$$\tilde{C}_1 \frac{t^3}{6} \leq \left(\frac{t}{2|p(A,k)|} + \frac{\sinh(2t \operatorname{Re} \sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2 (\operatorname{Re} \sqrt{p(A,k)})} \right) \leq \tilde{C}_2 \frac{t^3}{6},$$

while for $k \in \hat{Q}_N^2$ such that $|\operatorname{Re} \sqrt{p(A,k)}| > \frac{1}{2t}$, using that by Lemma 7.1.3 the real part of $\sqrt{(p(A,k)}$ and therefore $|\sinh(2t \operatorname{Re} \sqrt{p(A,k)}/(\operatorname{Re} \sqrt{p(A,k)})$ is bounded by a constant $C(T)$ on $[0,T]$, we find constants $\hat{C}_1(T), \hat{C}_2(T)$:

$$\hat{C}_1(T) \left(\frac{1}{|p(A,k)|} \right) \leq \left(\frac{t}{2|p(A,k)|} + \frac{\sinh(2t \operatorname{Re} \sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2 (\operatorname{Re} \sqrt{p(A,k)})} \right) \leq \hat{C}_2(T) \left(\frac{1}{|p(A,k)|} \right).$$

Now by (A3.3) we get with possibly different constants $\hat{C}_1, \hat{C}_2$:

$$\hat{C}_1 \left(\frac{t}{|k|^{2m}} \right) \leq \hat{C}_1 \left(\frac{t}{|p(A,k)|} \right) \leq \left(\frac{t}{2|p(A,k)|} + \frac{\sinh(2t \operatorname{Re} \sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2 (\operatorname{Re} \sqrt{p(A,k)})} \right) \leq \hat{C}_2 \left(\frac{t}{|p(A,k)|} \right) \leq \hat{C}_2 \left(\frac{t}{|k|^{2m}} \right).$$

The claim follows by choosing $C_1(T) := \min\{\hat{C}_1(T), \tilde{C}_1\}$ and $C_2(T) := \max\{\hat{C}_2(T), \tilde{C}_2\}$. ∎

Now we combine the previous results into their final form.

Corollary 7.1.7
Let u a solution of (sSWEQ) and $N \in \mathbb{N}^*$, then there exists constants $C_1(T), C_2(T) > 0$:

$$C_1(T) \left(t \sum_{k \in \hat{Q}_{N/2}^q} \langle k \rangle^{-2} + t^3 + 1 \right) \leq \sum_{k \in \hat{Q}_N^q} \mathbb{E}\left[|u_k(t)|^2 \right] \leq C_2(T) \left(t \sum_{k \in \hat{Q}_N^q} \langle k \rangle^{-2} + t^3 + 1 \right).$$

Proof
Let N_0 be the cutoff from (A3.3), then there exist constants $C_1(T) \geq 0$ and $C_2(T) > 0$ such that

$$C_1(T) \leq \sum_{k \in \hat{Q}_N^q} \mathbb{E}\left[|u_k(t)|^2 \right] \leq C_2(T).$$

Now using Lemma 7.1.2 and further expanding using Lemma 7.1.4

$$\int_0^t \frac{\sinh^2((t-s) \operatorname{Re} \sqrt{p(A,k)})}{|p(A,k)|} \, ds = \left(\frac{t}{2|p(A,k)|} + \frac{\sinh(2t \operatorname{Re} \sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2 (\operatorname{Re} \sqrt{p(A,k)})} \right)$$

and

$$\int_0^t \frac{+\sin^2((t-s) \operatorname{Im} \sqrt{p(A,k)}}{|p(A,k)|} \, ds = \left(\frac{t}{2|p(A,k)|} - \frac{\sin(2t \operatorname{Im} \sqrt{p(A,k)})}{4|\sqrt{p(A,k)}|^2 \operatorname{Im} \sqrt{p(A,k)}} \right).$$

Bounding everything using Lemma 7.1.5 and Lemma 7.1.6 we obtain the claim, by noting

$$\sum_{k \in \hat{Q}_{N/2}^q} \mathbb{E}\left[|u_k(t)|^2 \right] \leq \sum_{k \in \hat{Q}_N^q} \mathbb{E}\left[|u_k(t)|^2 \right]$$

and applying all the previous estimates together. ∎

Finally we will give a moment bound, that uses a worst case estimate, to create a shift back to the Sobolev norms in the odd power cases.

Lemma 7.1.8
Let u a solution of (LEQ1), $N \in \mathbb{N}$ and $\alpha \in \hat{Q}_N^q$ then there exists a $C(t) > 0$:

$$\sum_{k\in\hat{Q}_N^q} \langle k\rangle^{-2r}\mathbb{E}\Big[|[P_N u(t)]_{N,q}(k-\alpha)|^2\Big] \le C(t) \sum_{k\in\hat{Q}_N^q} \langle k\rangle^{-2r}\mathbb{E}\Big[|u_k(t)|^2\Big].$$

Proof
Firt we note that by Lemma 1.2.55 there exists a $\tilde{\alpha}_k \in \mathbb{Z}^q$ such that $k - \tilde{\alpha}_k \in \hat{Q}_N^q$ and

$$[P_N u(t)]_{N,q}(k-\alpha) = [P_N u(t)]_{N,q}(k-\tilde{\alpha}_k).$$

Now we decompose the left hand side of the claim as follows:

$$\begin{aligned}
&\sum_{k\in\hat{Q}_N^q} \langle k\rangle^{-2r}\mathbb{E}\Big[|[P_N u(t)]_{N,q}(k-\alpha)|^2\Big] \\
&= \sum_{k\in\hat{Q}_N^q} \mathbb{1}_{\{|k|\ge|k-\tilde{\alpha}_k|\}}(k)\langle k\rangle^{-2r}\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big] + \sum_{k\in\hat{Q}_N^q} \mathbb{1}_{\{|k|<|k-\tilde{\alpha}_k|\}}(k)\langle k\rangle^{-2r}\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big].
\end{aligned}$$

For $k : |k| \ge |k-\tilde{\alpha}_k|$ by

$$\langle k\rangle^{-2r}\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big] \le \langle k-\tilde{\alpha}_k\rangle^{-2r}\mathbb{E}\Big[|u_{(k-\tilde{\alpha}_k}(t)|^2\Big],$$

we obtain

$$\sum_{k\in\hat{Q}_N^q} \mathbb{1}_{\{|k|\ge|k-\tilde{\alpha}_k|\}}(k)\langle k\rangle^{-2r}\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big] \le \sum_{k\in\hat{Q}_N^q} \mathbb{1}_{\{|k|\ge|k-\tilde{\alpha}_k|\}}(k)\langle k-\tilde{\alpha}_k\rangle^{-2r}\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big].$$

By setting $k' := k - \tilde{\alpha}_k$ we estimate:

$$\begin{aligned}
&\sum_{k\in\hat{Q}_N^q} \mathbb{1}_{\{|k|\ge|k-\tilde{\alpha}_k|\}}(k)\langle k\rangle^{-2r}\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big] \\
&\le \sum_{k'\in\hat{Q}_N^q} \mathbb{1}_{\{|(k'+\tilde{\alpha}_k)|\ge|k'|\}}((k'+\tilde{\alpha}_k)\langle k'\rangle^{-2r}\mathbb{E}\Big[|u_{k'}(t)|^2\Big] \\
&\le \sum_{k'\in\hat{Q}_N^q} \langle k'\rangle^{-2r}\mathbb{E}\Big[|u_{k'}(t)|^2\Big].
\end{aligned} \tag{7.1.1}$$

For $k : |k| < |k-\tilde{\alpha}_k|$ we use Lemma 7.1.5 and the explicit form of the Fourier coefficients.

(i) For $|k-\tilde{\alpha}_k| \in [0, \frac{1}{2t}]$, by Lemma 7.1.5, we find constants $C, \tilde{C} > 0$ such that

$$\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big] \le Ct^3 \le \tilde{C}\mathbb{E}\Big[|u_k(t)|^2\Big].$$

(ii) For $|k-\tilde{\alpha}_k| \in (\frac{1}{2t}, \infty)$, we find $C > 0$ and $k(t) \in \hat{Q}_N^q$, such that $|k(t)| = C\left\lceil \frac{1}{2t}\right\rceil$, this yields:

$$t^3 = Ct^2\left\lceil \frac{1}{2t}\right\rceil^2 \frac{t}{C\left\lceil \frac{1}{2t}\right\rceil^2} \le Ct\left\lceil \frac{1}{2t}\right\rceil \mathbb{E}\Big[|u_{k(t)}(t)|^2\Big], \tag{7.1.2}$$

and by Lemma 7.1.5, we find new constants $C, \tilde{C}(t) > 0$ such that

$$\begin{aligned}
&\sum_{k\in\hat{Q}_N^q} \mathbb{1}_{\{|k|<|k-\tilde{\alpha}_k|>\frac{1}{2t}\}}(k)\langle k\rangle^{-2r}\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big] \\
\le &C \sum_{k\in\hat{Q}_N^q} \mathbb{1}_{\{|k|<|k-\tilde{\alpha}_k|\}>\frac{1}{2t}}(k)\langle k\rangle^{-2r}\frac{t}{|k-\alpha|^2} \le C\sum_{k\in\hat{Q}_N^q} \mathbb{1}_{\{|k|<|(k-\alpha)mod_{\hat{Q}_N^q}|>\frac{1}{2t}\}}(k)\langle k\rangle^{-2r}t^3 \\
\le &C\sum_{k\in\hat{Q}_N^q} \langle k\rangle^{-2r}t^3 \le C't^3 \overset{L\ 7.1.2}{\le} \tilde{C}t\left\lceil \frac{1}{2t}\right\rceil \mathbb{E}\Big[|u_{k(t)}(t)|^2\Big] \le \tilde{C}t^2\left\lceil \frac{1}{2t}\right\rceil^2 \sum_{k\in\hat{Q}_N^q} \langle k\rangle^{-2r}\mathbb{E}\Big[|u_k(t)|^2\Big].
\end{aligned}$$

Thus

$$\sum_{k\in\hat{Q}_N^q} \mathbb{1}_{\{|k|<|(k-\tilde{\alpha}_k|\}}(k)\langle k\rangle^{-2r}\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big] \leq C\left(t\left\lceil\frac{1}{2t}\right\rceil+1\right)\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{-2r}\mathbb{E}\Big[|u_k(t)|^2\Big].$$

And together with Lemma 7.1.1

$$\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{-2r}\mathbb{E}\Big[|u_{k-\tilde{\alpha}_k}(t)|^2\Big] \leq C\left(t^2\left\lceil\frac{1}{2t}\right\rceil^2+2\right)\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{-2r}\mathbb{E}\Big[|u_k(t)|^2\Big].$$

■

7.2 Scaling of linear solutions under nonlinearitys

By considering the moments of our linear solutions $\{X_N\}_{N\in\mathbb{N}^*}$ we determine their scaling behaviour as N tends to infinity. First we consider the toy case of a stochastic wave equation with a simple, monomial nonlinearity, that is either quadratic or cubic. From the cubic case upwards, all degrees of the monomial nonlinearity, require some combinatorics, that will be developed before we prove the scaling results for a general partial differential operator A and all monomial nonlinearitys. Finally we consider three different renormalization approaches for polynomial nonlinearities.

7.2.1 Scaling of a Simple Stochastic Wave Equation

In a first step we will consider the following stochastic wave equation without forcing and zero initial data:

$$\begin{cases}\partial_t u = \Delta u + W & (t,x)\in\mathbb{R}_+\times\mathbb{T}^q\\ u(0,\cdot)=0=\partial_t u(0,\cdot), & \end{cases} \tag{sSWEQ}$$

to better understand the interaction between the non-linearity and the stochastic convolution

$$\int_0^t S(t-s)\,dW(s)$$

and how it is different from the interaction of the non-linearity with a deterministic forcing of the same spatial regularity. We begin by introducing the following notation, that is motivated by the multiplicative properties of multi-indices, that will help to keep the notation lucid, especially for higher degree non-linearities, the iterated discrete convolutions lead to longer and longer products of Fourier coefficients.

Notation 7.2.1
Let $p\in\mathbb{N}^*$, $k\in\mathbb{Z}^q$, $I\subset\mathbb{Z}^q$, $\alpha,\in(\mathbb{Z}^q)^p$ and $\{u_k\}_{k\in I}\subset\mathbb{C}$. From now on the following notations will be used:

- $\alpha(k) := \left(\alpha, k-\sum_{i=1}^p \alpha_i\right),$
- $u_\alpha = \prod_{k=1}^p u_k.$

Lemma 7.2.2
Let u a solution of (LEQ1) $u_0\equiv u_1\equiv f(t,\cdot)\equiv 0 \quad \forall t\geq 0$, $w(N,j,q)$ the weight function from Definition 1.2.51 and $\alpha\in(\mathbb{Z}^q)^p$. Then

(i) $\overline{u_\alpha} = u_{-\alpha}$,

(ii) $\overline{w_\alpha(N,q)} = w_{-\alpha}(N,q)$.

Proof

(i) This follows as by the Definition 2.2 $\overline{u_k} = u_{-k} \quad \forall\, k \in \mathbb{Z}^q$.

(ii) This follows as $\overline{\exp(x)} = \exp(\overline{x}) \quad \forall\, x \in \mathbb{C}$.

Definition 7.2.3
We define the dimension dependent scaling function $\beta_q : \mathbb{N}^* \to \mathbb{R}$ via

$$\beta_q(N) := \begin{cases} 1, & q = 1, \\ \log(N), & q = 2, \\ N^{q-2}, & q \geq 3, \end{cases}$$

for all $N \in \mathbb{N}^*$.

Theorem 7.2.4 (Critical Scaling for quadratic nonlinearities)
Let $b(a) = a^2$, $N \in \mathbb{N}^*$, $T > 0$ and u a solution of (sSWEQ) then, for every $t \in [0,T]$, there exist constants $C_1(q), C_2(q) > 0$ such that

$$C_1(q)(t\beta_q(N) + t^3 + 1)^2 \leq \mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q^q_N)}\right] \leq C_2(q)(t\beta_q(N) + t^3 + 1)^2.$$

Proof
We begin with the lower bound, where we use the lower bound for the norm weights to switch to classical Fourier weights. Then we drop everything but the $k = 0$ term. Next we use that the discrete Fourier transform behaves analogous to the continuous, i.e. it transforms multiplication into a discrete convolution, to express everything in terms of the discrete Fourier coefficients. Now we can replace the discrete Fourier coefficients by the continuous ones, at the price of the weight function w.

$$\begin{aligned}
&\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q^q_N)}\right] \\
\overset{(1.2.6)}{\geq}\ &\mathbb{E}\left[\sum_{k\in\hat{Q}^q_N} \langle k\rangle^{2r} |\widehat{b([P_N u]_{N,q}(t,k))}^d|^2\right] \\
\geq\ &\mathbb{E}\left[|\widehat{b([P_N u]_{N,q}(t,0))}^d|^2\right] \\
\overset{Thm 1.2.46}{=}\ &\mathbb{E}\left[\left|\sum_{\alpha\in\hat{Q}^q_N} \widehat{[P_N u]_{N,q}(t,\alpha)}^d\, \widehat{[P_N u]_{N,q}(t,-\alpha)}^d\right|^2\right] \\
\overset{Lemma 1.2.53}{=}\ &\mathbb{E}\left[\left|\sum_{\alpha\in\hat{Q}^q_N} w_{\alpha(0)}(N,q) u_{\alpha(0)}(t)\right|^2\right]
\end{aligned}$$

Using the identities

$$\begin{aligned}
w_{\alpha(0)}(N,q) &= w_\alpha(N,q) w_{-\alpha}(N,q) = |w_\alpha(N,q)|^2, \\
u_{\alpha(0)}(t) &= u_\alpha(t) u_{-\alpha}(t) = |u_\alpha(t)|^2
\end{aligned}$$

we obtain

$$\mathbb{E}\left[\left|\sum_{\alpha\in\hat{Q}^q_N} w_{\alpha(0)}(N,q) u_{\alpha(0)}(t)\right|^2\right] = \mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}^q_N} |w_\alpha(N,q)|^2 |u_\alpha(t)|^2\right)^2\right] \geq \mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}^q_{N/2}} |w_\alpha(N,q)|^2 |u_\alpha(t)|^2\right)^2\right].$$

The right-hand side of the last inequality enables us to give a nonzero lower bound for the absolute values of the weight functions and using the complex Gaussian distribution of the Fourier coefficients and previously established covariance bounds we compute:

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] \geq \mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}^q_{N/2}} |w_\alpha(N,q)|^2|u_\alpha(t)|^2\right)^2\right]$$

$$\overset{(1.2.5)}{\geq} C_{\frac{1}{2}}\mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}^q_{N/2}} |u_\alpha(t)|^2\right)^2\right] \overset{Cor.2.2.12}{=} C\mathbb{E}\left[\sum_{\alpha\in\hat{Q}^q_{N/2}} |u_\alpha(t)|^2\right]^2 \overset{Cor.7.1.7}{\geq} \tilde{C}\left(t\sum_{\alpha\in\hat{Q}^q_{N/2}}\langle\alpha\rangle^{-2}+t^3+1\right)^2$$

$$\overset{LA.4}{\geq} C_1(q)(t\beta_q(N)+t^3+1)^2.$$

Now for the upper bound we start with the upper bounds on the norm weights to switch to classical Fourier weights though in this direction we get an additional prefactor. Next we use that the discrete Fourier transform behaves analogous to the continuous, i.e. it transforms multiplication into a discrete convolution, to express everything in terms of the discrete Fourier coefficients. Now we can replace the discrete Fourier coefficients by the continuous ones, at the price of the weight function w (note that by the periodicity of the discrete Fourier coefficients we can assume w.l.o.g. that $(k-\alpha)\in\hat{Q}^q_N$). And compute similar to the lower bound

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]$$

$$\overset{(1.2.6)}{\leq} 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}^q_N}\langle k\rangle^{2r}|\widehat{b([P_N u]_{N,q}(t,k))}^d|^2\right]$$

$$\overset{Thm1.2.46}{=} 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}^q_N}\langle k\rangle^{2r}\left|\sum_{\alpha\in\hat{Q}^q_N}\widehat{[P_N u]_{N,q}(t,\alpha)}^d\,\widehat{[P_N u]_{N,q}(t,k-\alpha)}^d\right|^2\right]$$

$$\overset{L1.2.53}{=} 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}^q_N}\langle k\rangle^{2r}\left|\sum_{\alpha\in\hat{Q}^q_N} w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]$$

$$\leq 10^{-2r}\mathbb{E}\left[\left|\sum_{\alpha\in\hat{Q}^q_N} w_{\alpha(0)}(N,q)u_{\alpha(0)}(t)\right|^2\right] + 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}^q_N\setminus\{0\}}\langle k\rangle^{2r}\left|\sum_{\alpha\in\hat{Q}^q_N} w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right].$$

While we treat the first term as we did for the lower bound we can't simply drop the second term. Instead we have to bound it by the first. In a first step we use the cancelations to get the absolute value inside the sum, before bounding the weight functions

$$10^{-2r}\mathbb{E}\left[\left|\sum_{\alpha\in\hat{Q}^q_N} w_{\alpha(0)}(N,q)u_{\alpha(0)}(t)\right|^2\right] + 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}^q_N\setminus\{0\}}\langle k\rangle^{2r}\left|\sum_{\alpha\in\hat{Q}^q_N} w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]$$

$$\overset{Lemma7.2.6}{\leq} 10^{-2r}\mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}^q_N}|w_\alpha(N,q)|^2|u_\alpha(t)|^2\right)^2\right]$$

$$+2(10^{-2r})\mathbb{E}\left[\sum_{k\in\hat{Q}^q_N\setminus\{0\}}\langle k\rangle^{2r}\sum_{\alpha\in\hat{Q}^q_N}|w_{\alpha(k)}(N,q)|^2|u_\alpha(t)|^2|u_{(k-\alpha)}(t)|^2\right]$$

$$\overset{(1.2.5)}{\leq} 10^{-2r}\mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}^q_N}|u_\alpha(t)|^2\right)^2\right] + 2(10^{-2r})\mathbb{E}\left[\sum_{k\in\hat{Q}^q_N\setminus\{0\}}\langle k\rangle^{2r}\sum_{\alpha\in\hat{Q}^q_N}|u_\alpha(t)|^2|u_{(k-\alpha)}(t)|^2\right].$$

Now we bound $|u_{(k-\alpha)}(t)|^2$ by the sum over $\hat{Q}_N^q$ and use that r is chosen negative enough for the series of Fourier weights to converge, and therefore can be bounded independent of N.

$$\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q\setminus\{0\}} \langle k\rangle^{2r} \sum_{\alpha\in\hat{Q}_N^q} |u_\alpha(t)|^2 |u_{k-\alpha}(t)|^2\right] \leq \mathbb{E}\left[\sum_{k\in\hat{Q}_N^q\setminus\{0\}} \langle k\rangle^{2r} \sum_{\alpha\in\hat{Q}_N^q} |u_\alpha(t)|^2 \sum_{\tilde{\alpha}\in\hat{Q}_N^q} |u_{\tilde{\alpha}}(t)|^2\right]$$
$$\leq \sum_{k\in\hat{Q}_N^q\setminus\{0\}} \langle k\rangle^{2r} \mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}_N^q} |u_\alpha(t)|^2\right)^2\right] = C_q \mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}_N^q} |u_\alpha(t)|^2\right)^2\right].$$

Inserting this into the previous estimate we obtain

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]$$
$$\leq C_q \mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}_N^q} |u_\alpha(t)|^2\right)^2\right] \overset{Cor2.2.12}{=} C_q \mathbb{E}\left[\sum_{\alpha\in\hat{Q}_{N/2}^q} |u_\alpha(t)|^2\right]^2 \overset{Cor7.1.7}{\leq} C_q\left(t\sum_{\alpha\in\hat{Q}_{N/2}^q} \langle\alpha\rangle^{-2} + t^3 + 1\right)^2$$
$$\overset{L.A.4}{\leq} C_2(q)(t\beta_q(N) + t^3 + 1)^2.$$

■

While we could also go for a straightforward computation for cubic non-linearities it is much more convenient to already introduce a the tool we will use in the general case.

Definition 7.2.5
Let $p \in \mathbb{N}$ and $\tilde{p} \in \mathbb{N}_0$.

(i) We define the set of all p−tuples with $\tilde{p}$ conjugate pairs as

$$\mathcal{M}(\mathbb{Z}^q, p, \tilde{p}) :=$$
$$\left\{\alpha \in (\mathbb{Z}^q)^p \;:\; \exists\, \sigma \in S_p : \alpha_{\sigma^{-1}(i)} \begin{cases} = -\alpha_{\sigma^{-1}(\tilde{p}+i)}, & \text{for } i \in \{1,\ldots,\tilde{p}\}, \\ \neq -\alpha_{\sigma^{-1}(j)} & \forall j \in \{2\tilde{p}+1,\ldots,p\}\setminus\{i\}, \text{ for } i \in \{2\tilde{p}+1,\ldots,p\}. \end{cases}\right\}$$

*(ii) We define the **matchfunction** $M^p_{\tilde{p}} : (\mathbb{Z}^q)^p \to \{0,1\}$ via*

$$M^p_{\tilde{p}}(\alpha) := \mathbb{1}_{\mathcal{M}(\mathbb{Z}^q,p,\tilde{p})}(\alpha).$$

*(iii) The **matchnumber** $\tilde{m}(\alpha)$ of $\alpha \in (\mathbb{Z}^q)^p$ is defined as the unique number $\tilde{m}(\alpha) \in \mathbb{N}_0$, such that*

$$\tilde{m}(\alpha) := [\tilde{p} \mapsto M^p_{\tilde{p}}(\alpha)]^{-1}(\{1\}).$$

With this we obtain the following computational properties that will allow us to compute or atleast greatly simplify the expectations of product of Fourier coefficients.

Lemma 7.2.6
Let $I \subset \mathbb{Z}^q$ and $\{u_k\}_{k\in I}$ a family of centered complex Gaussian random variables, such that $\mathbb{E}\left[u_k^2\right] = 0$ and u_k and u_j are independent, unless $j \in \{k, -k\}$, with $u_{-k} = \overline{u_k}$. Then we have for $p \in \mathbb{N}^*$ and $\alpha_1, \alpha_2 \in I^p$:

(i) If $\overline{m}(\alpha_1) \neq \overline{m}(\alpha_2)$ then $\mathbb{E}\left[u_{\alpha_1} u_{\alpha_2}\right] = 0$.

(ii) If $\overline{m}(\alpha_1) = \overline{m}(\alpha_2) = 0$ then

$$\mathbb{E}\left[u_{\alpha_1} u_{\alpha_2}\right] = \begin{cases} \mathbb{E}\left[|u_{\alpha_1}|^2\right], & \exists \sigma \in S_p :\; \alpha_2 = -\sigma(\alpha_1), \\ 0, & \text{else.} \end{cases}$$

(iii)

$$\mathbb{E}\left[\left(\sum_{\alpha_1\in I^p} u_{\alpha_1}\right)\overline{\left(\sum_{\alpha_2\in I^p} u_{\alpha_2}\right)}\right] = \sum_{\alpha_1\in I^p}\mathbb{E}\left[|u_{\alpha_1}|^2\right].$$

Proof

(i) W.l.o.g. we can assume $\overline{m}(\alpha_1) > \overline{m}(\alpha_2)$, we find $\alpha_{1,1}\in I^{\overline{m}(\alpha_1)}, \alpha_{1,2}\in I^{p-2\overline{m}(\alpha_1)}, \alpha_{2,1}\in I^{\overline{m}(\alpha_2)}$ and $\alpha_{2,2}\in I^{p-2\overline{m}(\alpha_2)}$ such that $\overline{m}(\alpha_{1,2}) = 0 = \overline{m}(\alpha_{2,2})$, $\overline{m}(\alpha_{1,1}) = \overline{m}(\alpha_1)$, $\overline{m}(\alpha_{2,1}) = \overline{m}(\alpha_2)$ and

$$\begin{aligned}\mathbb{E}\left[u_{\alpha_1}u_{\alpha_2}\right] =& \mathbb{E}\left[u_{\alpha_{1,1}}\overline{u_{\alpha_{1,1}}}u_{\alpha_{1,2}}u_{\alpha_{2,1}}\overline{u_{\alpha_{2,1}}}u_{\alpha_{2,2}}\right] = \mathbb{E}\left[|u_{\alpha_{1,1}}|^2u_{\alpha_{1,2}}|u_{\alpha_{2,1}}|^2u_{\alpha_{2,2}}\right]\\ \overset{Coro 2.2.12}{=}& C(p,\alpha_1,\alpha_2)\mathbb{E}\left[|u_{\alpha_{1,1}}|^2|u_{\alpha_{2,1}}|^2\right]\mathbb{E}\left[u_{\alpha_{1,2}}u_{\alpha_{2,2}}\right].\end{aligned}$$

Now $\overline{m}((\alpha_{1,2},\alpha_{2,2})) \le \min\{p-2\overline{m}(\alpha_1), p-2\overline{m}(\alpha_1)\} = p-2\overline{m}(\alpha_2)$. Thus there exist $i\in\{1,\dots,p-2\overline{m}(\alpha_2)\}$, $r(i)\in\mathbb{N}$, $\alpha(i)\in I$ and $\tilde{\alpha}(i)\in I^{2p-\overline{m}(\alpha_2)-\overline{m}(\alpha)-r(i)}$ such that $u_{\alpha(i)}$ and $u_{\tilde{\alpha}(i)}$ are independent and

$$u_{\alpha_{1,2}}u_{\alpha_{2,2}} = u_{\alpha(i)}^{r(i)}u_{\tilde{\alpha}(i)},$$

with this and Corollary 2.2.12 we compute

$$\begin{aligned}&C(p,\alpha_1,\alpha_2)\mathbb{E}\left[|u_{\alpha_{1,1}}|^2|u_{\alpha_{2,1}}|^2\right]\mathbb{E}\left[u_{\alpha_{1,2}}u_{\alpha_{2,2}}\right]\\ =&\tilde{C}(p,\alpha_1,\alpha_2)\mathbb{E}\left[|u_{\alpha_{1,1}}|^2|u_{\alpha_{2,1}}|^2\right]\mathbb{E}\left[u_{\tilde{\alpha}(i)}\right]\underbrace{\mathbb{E}\left[u_{\alpha(i)}^{r(i)}\right]}_{=0} = 0,\end{aligned}$$

where $\mathbb{E}\left[u_{\alpha(i)}^{r(i)}\right] = 0$, as it is complex Gaussian with pseudo-covariance 0.

(ii) If there is a $\sigma\in S_p$ such that $\alpha_2 = -\sigma(\alpha_1)$ then

$$\mathbb{E}\left[u_{\alpha_1}u_{\alpha_2}\right] = \mathbb{E}\left[u_{\alpha_1}u_{-\sigma(\alpha_1)}\right] = \mathbb{E}\left[u_{\alpha_1}\overline{u_{\sigma(\alpha_1)}}\right] = \mathbb{E}\left[u_{\alpha_1}\overline{u_{\alpha_1}}\right] = \mathbb{E}\left[|u_{\alpha_1}|^2\right].$$

If there exist no such permutation, we find a $\alpha\in I$ such that

$$A_1 := \mathcal{H}_0(\{i\ \ (\alpha_1)_i\}) \ne \mathcal{H}_0(\{i\ \ (\alpha_2)_i\}) =: A_2.$$

Let $\sigma_1,\sigma_2\in S_p : \sigma_i(\alpha_i) = (\underbrace{\alpha,\dots,\alpha}_{A_i},*,\dots,*)$ and define $\tilde{\alpha}_i$ such that $\sigma_i(\alpha_i) = (\underbrace{\alpha,\dots,\alpha}_{A_i},\tilde{\alpha}_i)$ for $i=1,2$.

Then assuming w.l.o.g. $A_1 > A_2$ we compute, using Corollary 2.2.12 respectively independence:

$$\begin{aligned}\mathbb{E}\left[u_{\alpha_1}u_{\alpha_2}\right] =& \mathbb{E}\left[u_{\tilde{\alpha}_1}u_{\tilde{\alpha}_2}(|u_\alpha|^2)^{A_2}(u_\alpha)^{A_1-A_2}\right]\\ =& C(p,\alpha_1,\alpha_2)\mathbb{E}\left[u_{\tilde{\alpha}_1}u_{\tilde{\alpha}_2}\right]\mathbb{E}\left[(|u_\alpha|^2)^{A_2}\right]\underbrace{\mathbb{E}\left[(u_\alpha)^{A_1-A_2}\right]}_{=0} = 0.\end{aligned}$$

Again $\mathbb{E}\left[(u_\alpha)^{A_1-A_2}\right] = 0$, as it is complex Gaussian with pseudo-covariance 0.

(iii) Follows by (ii) and linearity of the expectation operator.

■

Theorem 7.2.7 (Critical Scaling for cubic nonlinearities)
Let $b(a) = a^3$, $N\in\mathbb{N}^*$, $T>0$ and u a solution of (sSWEQ) then there, for every $t\in[0,T]$, exist constants $C_1(q), C_2(q) > 0$ such that

$$C_1(q)(t\beta_q(N)+t^3+1)^2\mathbb{E}\left[\left\|P_{N/2}u(t,\cdot)\right\|^2_{H^r(\mathbb{T}^q)}\right] \le \mathbb{E}\left[\left\|b([P_Nu]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]$$

and

$$\mathbb{E}\left[\left\|b([P_Nu]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] \le C_2(q)(t\beta_q(N)+t^3+1)^2\mathbb{E}\left[\|P_Nu(t,\cdot)\|^2_{H^r(\mathbb{T}^q)}\right].$$

Proof

As usual by now we begin with the lower bounds. First we switch to classical the Fourier weights. As for the higher even powers iterating Theorem 1.2.46 and using Lemma 1.2.53 we express the discrete Fourier coefficients by their continuous counterparts (again we can w.l.o.g. assume that $(k-\alpha_1-\alpha_2) \in \hat{Q}_N^q$ by periodicity of the discrete Fourier coefficients).

$$
\begin{aligned}
&\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] \\
\overset{(1.2.6)}{\geq}\,&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}|\widehat{b([P_N u]_{N,q}(t,k))}^d|^2\right] \\
\overset{Thm1.2.46}{=}&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2}\widehat{[P_N u]_{N,q}(t,\alpha_1)}^d\,\widehat{[P_N u]_{N,q}(t,\alpha_2)}^d\,\widehat{[P_N u]_{N,q}(t,k-\alpha_1-\alpha_2)}^d\right|^2\right] \\
\overset{L1.2.53}{=}&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2} w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right].
\end{aligned}
$$

Using our match function we compute:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2} w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
=&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2}\sum_{\nu=0}^{1} M_\nu^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]. \\
\overset{Lemma7.2.6}{=}&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2} M_1^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
&\underbrace{+\,\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2} M_0^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]}_{\geq 0} \\
\geq&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2} M_1^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]
\end{aligned}
$$

Now we use that for $\alpha\in\mathcal{M}(\mathbb{Z}^q,3,1)$ we have (with some $\alpha'\in\mathbb{Z}^q$)

$$
\begin{aligned}
w_{\alpha(k)}(N,q) =& w_{\alpha'}(N,q)w_{-\alpha'}(N,q)w_{k+\alpha'-\alpha'}(N,q) = |w_{\alpha'}(N,q)|^2 w_k(N,q), \\
u_{\alpha(k)}(t) =& u_{\alpha'}(t)u_{-\alpha'}(t)u_{k+\alpha'-\alpha'}(t) = |u_\alpha(t)|^2 u_k(t),
\end{aligned}
$$

therefore we get, factorising the double sum using stochastic independence respectively Coro2.2.12 and the lower bound on the weight function (omitting all non-negative terms for which the weight drops below the threshold chosen by us) we compute:

$$
\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2} M_1^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]
$$

$$
\begin{aligned}
&= C(3,1)\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha'\in\hat{Q}_N^q}|w_{\alpha'}(N,q)|^2 w_k(N,q)|u_\alpha(t)|^2 u_k(t)\right|^2\right]\\
&\overset{Cor 2.2.12}{=} C\mathbb{E}\left[\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right]^2 \mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}|w(N,k,q)u_k(t)|^2\right]\\
&\overset{(1.2.5)}{\geq} C\mathbb{E}\left[\sum_{m\in\hat{Q}_{N/2}^q}|u_m(t)|^2\right]^2 \mathbb{E}\left[\left\|P_{N/2}u\right\|_{H^r(\mathbb{T}^q)}\right] \qquad . \qquad (7.2.1)
\end{aligned}
$$

By Corollary 7.1.7 and the lower bound from Lemma A.4 we get

$$
\mathbb{E}\left[\sum_{m\in\hat{Q}_{N/2}^q}|u_m(t)|^2\right]^2 \overset{Cor 7.1.7}{\geq} \tilde{C}\left(t\sum_{\alpha\in\hat{Q}_{N/2}^q}\langle\alpha\rangle^{-2}+t^3+1\right)^2 \overset{Lemma A.4}{\geq} C_1(q)(t\beta_q(N)+t^3+1)^2,
$$

which gives the claimed lower bound when inserted in (7.2.1).

Now for the upper bound we begin by switch to classical the Fourier weights. As for the higher even powers iterating Theorem 1.2.46 and using Lemma 1.2.53 we express the discrete Fourier coefficients by their continuous counterparts (Again we can assume $(k-\alpha_1-\alpha_2)\in\hat{Q}_N^q$).

$$
\begin{aligned}
\mathbb{E}\left[\left\|b([P_Nu]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] &\overset{(1.2.6)}{\leq} 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}|b(\widehat{[P_Nu]_{N,q}(t,\cdot))}^d(k)|^2\right]\\
&\overset{Thm 1.2.46}{=} 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2}\widehat{[P_Nu]_{N,q}(t,\cdot)}^d(\alpha_1)\widehat{[P_Nu]_{N,q}(t,\cdot)}^d(\alpha_2)\widehat{[P_Nu]_{N,q}(t,\cdot)}^d(k-\alpha_1-\alpha_2)\right|^2\right]\\
&\overset{L1.2.53}{=} 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2}w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]
\end{aligned}
$$

Next we use our match function to sort out the highest match number terms that give us the scaling and everything else:

$$
\begin{aligned}
&10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2}w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]\\
&= 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2}\sum_{\nu=0}^{1}M_\nu^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right].\\
&\overset{Lemma 7.2.6}{=} 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2}M_1^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]\\
&\quad + 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2}M_0^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right].
\end{aligned}
$$

By (7.2.1) but with the upper bound on the weights we obtain

$$\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2} M_1^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \overset{(1.2.5)}{\leq} C\mathbb{E}\left[\sum_{m\in\hat{Q}_N^q}|u_m(t)|^2\right]^2 \mathbb{E}\left[\|P_N u\|^2_{H^r(\mathbb{T}^q)}\right].$$

For the second term we show that it indeed is controlled by the highest match number term:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^2} M_0^3(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]\\
\overset{L7.2.6}{=}&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\alpha_1\in\hat{Q}_N^q\setminus\{k\}}\sum_{\alpha_2\in\hat{Q}_N^q\setminus\{k,\alpha_1\}}|u_{\alpha_1}(t)|^2|u_{\alpha_2}(t)|^2|u_{k-\alpha_1-\alpha_2}(t)|^2\right]\\
\overset{Cor2.2.12}{=}&C\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\alpha_1\in\hat{Q}_N^q\setminus\{k\}}\sum_{\alpha_2\in\hat{Q}_N^q\setminus\{k,\alpha_1\}}\mathbb{E}\left[|u_{\alpha_1}(t)|^2\right]\mathbb{E}\left[|u_{\alpha_2}(t)|^2\right]\mathbb{E}\left[|u_{k-\alpha_1-\alpha_2}(t)|^2\right]\\
\overset{L7.1.8}{\leq}&C\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\alpha_1\in\hat{Q}_N^q\setminus\{k\}}\sum_{\alpha_2\in\hat{Q}_N^q\setminus\{k,\alpha_1\}}\mathbb{E}\left[|u_{\alpha_1}(t)|^2\right]\mathbb{E}\left[|u_{\alpha_2}(t)|^2\right]\mathbb{E}\left[|u_{k}(t)|^2\right]\\
\leq&C\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\alpha_1\in\hat{Q}_N^q}\sum_{\alpha_2\in\hat{Q}_N^q}\mathbb{E}\left[|u_{\alpha_1}(t)|^2\right]\mathbb{E}\left[|u_{\alpha_2}(t)|^2\right]\mathbb{E}\left[|u_{k}(t)|^2\right]\\
\overset{Cor2.2.12}{=}&C\left(\mathbb{E}\left[\sum_{m\in\hat{Q}_N^q}|u_m(t)|^2\right]\right)^2\mathbb{E}\left[\|P_N u\|_{H^r(\mathbb{T}^q)}\right].
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]\leq C\left(\mathbb{E}\left[\sum_{m\in\hat{Q}_N^q}|u_m(t)|^2\right]\right)^2\mathbb{E}\left[\|P_N u\|_{H^r(\mathbb{T}^q)}\right]. \tag{7.2.2}$$

Now by Coro 7.1.7 and the upper bound from Lemma A.4 we get

$$\left(\mathbb{E}\left[\sum_{m\in\hat{Q}_N^q}|u_m(t)|^2\right]\right)^2 \overset{Cor7.1.7}{\leq} \tilde{C}\left(t\sum_{\alpha\in\hat{Q}_N^q}\langle\alpha\rangle^{-2}+t^3+1\right)^2 \overset{LA.4}{\leq} C_1(q)(t\beta_q(N)+t^3+1)^2,$$

which gives the claimed upper bound when inserted in (7.2.2). ■

7.2.2 Renormalization Scaling for General Non-linearities

As we now return to the general case we will see the full usefulness of our match function in dealing with the combinatorics arising from higher pseudo-covariance moments of our linear solution. While the computation of the linear solutions quadratic and cubic powers, pseudo covariance, could be performed directly[2)], as we were in a binary situation of there either being a pre-existing[3)] complex conjugate pair, of Fourier coefficients, in the summand or not, for higher monomial terms both the potential number of conjugate pairs, as well as the combinations of Fourier coefficients that can produce them, increase drastically necessitating a slightly more sophisticated bookkeeping.We also need to adjust our scaling function β to account for the more general order of our differential operator A.

2) though we already committed to use matching for a more convenient and lucid notation

3) That is without the complex conjugation of the pseudo covariance

Definition 7.2.8
We define the dimension and parameter dependent scaling function $\beta_{q,m} : \mathbb{N} \to \mathbb{R}$ via

$$\beta_{q,m}(N) := \begin{cases} 1, & q < 2m, \\ \log(N), & q = 2m, \\ N^{q-2m}, & q \geq 2m+1, \end{cases}$$

for all $m, N \in \mathbb{N}$.

Monomial Non-linearities

We begin with with the monomial non-linearities as they are the building blocks of all further non-linearities for which we can implement our renormalization scheme.

Theorem 7.2.9 (Critical Scaling for higher even powered nonlinearities)
Let $b(a) = a^{2p}$, $p, N \in \mathbb{N}$, $T > 0$ and u a solution of (LEQ1), with $0 = u_0 = u_1 \equiv f$, then, for every $t \in [0, T]$, there exist constants $C_1(q), C_2(q) > 0$ such that

$$C_1(q)(t\beta_{q,m}(N) + t^3 + 1)^{2p} \leq \mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] \leq C_2(q)(t\beta_{q,m}(N) + t^3 + 1)^{2p}.$$

Proof
Again we begin with the lower bound. Iterating Theorem 1.2.46 and using Lemma 1.2.53 we express the discrete Fourier coefficients by their continuous counterparts.

$$\begin{aligned}
&\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] \\
\overset{L1.2.52}{\geq}\ &\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q} \langle k\rangle^{2r} |\widehat{b([P_N u]_{N,q}(\tau,\cdot))}^d(k)|^2\right] \\
\underset{L1.2.53}{\overset{Thm1.2.46}{=}}\ &\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q} \langle k\rangle^{2r} \left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} w_{\alpha(k)}(N,q) u_{\alpha(k)}(t)\right|^2\right]
\end{aligned}$$

In a first step we again drop all terms except the one with $k = 0$, as those can't contain the maximum number of matches, that is p matches.

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q} \langle k\rangle^{2r} \left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} w_{\alpha(k)}(N,q) u_{\alpha(k)}(t)\right|^2\right] \\
\geq &\mathbb{E}\left[\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} w_{\alpha(0)}(N,q) u_{\alpha(0)}(t)\right|^2\right]
\end{aligned}$$

In the next step we use our Match function to further filter out everything that doesn't contain p matches:

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} w_{\alpha(0)}(N,q) u_{\alpha(0)}(t)\right|^2\right] \\
= &\mathbb{E}\left[\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} \sum_{\nu=0}^{p} M_\nu^{2p}(\alpha(0)) w_{\alpha(k)}(N,q) u_{\alpha(0)}(t)\right|^2\right]
\end{aligned}$$

$$\overset{Lemma 7.2.6}{=}\mathbb{E}\left[\sum_{\nu=0}^{p}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_\nu^{2p}(\alpha(k))w_{\alpha(0)}(N,q)u_{\alpha(0)}(t)\right|^2\right]$$
$$\geq\mathbb{E}\left[\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_p^{2p}(\alpha(0))w_{\alpha(k)}(N,q)u_{\alpha(0)}(t)\right|^2\right].$$

Finally combining the conjugate pairs into squares of the absolute value we obtain

$$\mathbb{E}\left[\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_p^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]\geq\mathbb{E}\left[\left|\left(\sum_{\alpha\in\hat{Q}_{N/2}^q}|w_\alpha(N,q)|^2|u_\alpha(t)|^2\right)^p\right|^2\right].$$

With the inequality being only due to the fact that we dropped terms such that we now can use this bound to continue our previous estimate and get rid of the weight functions by using a nonzero lower bound on their absolute values to proceed as in the quadratic case, with the constants now depending on p due to the moment change with Corollary 2.2.12:

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]\geq\mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}_{N/2}^q}|w_\alpha(N,q)|^2|u_\alpha(t)|^2\right)^{2p}\right]\overset{(1.2.5)}{\geq}C_{\frac{1}{2}}\mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}_{N/2}^q}|u_\alpha(t)|^2\right)^{2p}\right]$$
$$\overset{Cor 2.2.12}{=}C(p)\mathbb{E}\left[\sum_{\alpha\in\hat{Q}_{N/2}^q}|u_\alpha(t)|^2\right]^{2p}\overset{Cor 7.1.7}{\geq}\tilde{C}(p)\left(t\sum_{\alpha\in\hat{Q}_{N/2}^q}\langle\alpha\rangle^{-2}+t^3+1\right)^{2p}$$
$$\overset{LA.4}{\geq}C_1(p,q)(t\beta_q(N)+t^3+1)^{2p}.$$

For the upper bound we start as for the lower bound only with the inverted inequalities and corresponding changed prefactor

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]$$
$$\overset{Corollary 1.2.57}{\leq}10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}|\widehat{b([P_N u]_{N,q}(t,\cdot))}^d(k)|^2\right]$$
$$\overset{Theorem 1.2.46, Lemma 1.2.53}{=}10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right].$$

Now we use our Match function to split of the terms with p- matches that give us our scaling and a remainder term we have to control by the p match terms.

$$\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]$$
$$=\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}}\sum_{\nu=0}^{p} M_\nu^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]$$
$$\overset{Lemma 7.2.6}{=}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_\nu^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_p^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
&\quad + \mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_\nu^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right].
\end{aligned}
$$

First note, again because p- Matches are only possible if $k=0$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_p^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
&= \mathbb{E}\left[\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_p^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
&= \mathbb{E}\left[\left|\left(\sum_{\alpha\in\hat{Q}_N^q}|w_\alpha(N,q)|^2|u_\alpha(t)|^2\right)^p\right|^2\right] \\
&\overset{Cor2.2.12}{=} \mathbb{E}\left[\left(\sum_{\alpha\in\hat{Q}_N^q}|w_\alpha(N,q)|^2|u_\alpha(t)|^2\right)\right]^{2p}.
\end{aligned}
$$

Now we only need to bound the terms with less conjugate pairs to complete the proof in the same way we did for the quadratic case.

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_\nu^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
&= \mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}C(p,\nu)\left(\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right)^{2\nu}\left|\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-1-2\nu}} w_{\alpha_\nu(k)}(N,q)u_{\alpha_\nu(k)}(t)\right|^2\right] \\
&\overset{C2.2.12}{=} \sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}C(p,\nu)\mathbb{E}\left[\left(\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right)\right]^{2\nu}\mathbb{E}\left[\left|\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-1-2\nu}} w_{\alpha_\nu(k)}(N,q)u_{\alpha_\nu(k)}(t)\right|^2\right] \\
&\overset{L7.2.6}{=} \sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}C(p,\nu)\mathbb{E}\left[\left(\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right)\right]^{2\nu}\mathbb{E}\left[\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-1-2\nu}} |w_{\alpha_\nu(k)}(N,q)|^2|u_{\alpha_\nu(k)}(t)|^2\right].
\end{aligned}
$$

Now we use (again bounding $|u_{(k-\sum_{j=0}^{2p-2\nu-1}\alpha_j)}(t)|^2$ by the entire sum, as $k-\sum_{j=0}^{2p-2\nu-1}\alpha_j$ can again be assumed in $\hat{Q}_N^q$ by periodicity)

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-1-2\nu}} |w_{\alpha_\nu(k)}(N,q)|^2|u_{\alpha_\nu(k)}(t)|^2\right] \\
&\leq \mathbb{E}\left[\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-1-2\nu}} |w_{\alpha_\nu}(N,q)|^2|u_{\alpha_\nu}(t)|^2\left(\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right)\right]
\end{aligned}
$$

to compute (i.e bounding all lower match expressions by the highest possible number of matches)

$$\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p-1}} M_\nu^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]$$

$$\overset{Cor 2.2.12}{\leq}\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}C(p,\nu)\mathbb{E}\left[\left(\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t|^2\right)\right]^{2p}$$

$$\leq C(p)\mathbb{E}\left[\left(\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right)\right]^{2p}$$

$$\overset{(1.2.5)}{\leq} C(p)\mathbb{E}\left[\sum_{\alpha\in\hat{Q}_N^q}|u_\alpha(t)|^2\right]^{2p}.$$

Plugging this into our original estimate we can compute

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]$$

$$\leq C(p)\mathbb{E}\left[\sum_{\alpha\in\hat{Q}_N^q}|u_\alpha(t)|^2\right]^{2p}\overset{Cor 7.1.7}{\leq}C(p,q)\left(t\sum_{\alpha\in\hat{Q}_N^q}\langle\alpha\rangle^{-2}+t^3+1\right)^{2p}\overset{LA.4}{\leq}C_2(p,q)(t\beta_q(N)+t^3+1)^{2p}.$$

■

Theorem 7.2.10 (Critical Scaling for higher odd power nonlinearities)
Let $b(a)=a^{2p+1}$, $p,N\in\mathbb{N}$, $T>0$ and u a solution of (LEQ1), with $0=u_0=u_1\equiv f$, then, for every $t\in[0,T]$, there exist constants $C_1(q),C_2(q)>0$ such that

$$C_1(q)(t\beta_{q,m}(N)+t^3+1)^{2p}\mathbb{E}\left[\left\|P_{N/2}u(t,\cdot)\right\|^2_{H^r(\mathbb{T}^q)}\right]\leq\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]$$

and

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]\leq C_2(q)(t\beta_{q,m}(N)+t^3+1)^{2p}\mathbb{E}\left[\|P_N u(t,\cdot)\|^2_{H^r(\mathbb{T}^q)}\right].$$

Proof
Again we begin with the lower bound.After the usual switch to the classical Fourier weights we iterate Theorem 1.2.46 and using Lemma 1.2.53 we express the discrete Fourier coefficients (after replacing them by their copy in $\hat{Q}_N^q$ by periodicity of the discrete Fourier coefficients) by their continuous counterparts:

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]$$

$$\overset{(1.2.6)}{\geq}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}|\widehat{b([P_N u]_{N,q}(t,\cdot))}^d(k)|^2\right]$$

$$\underset{L1.2.53}{\overset{Thm 1.2.46}{=}}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}}w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]$$

Now we drop all the terms with less than p conjugate pairs, using our Match function:

$$\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}}w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}}\sum_{\nu=0}^{p} M_\nu^{2p+1}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
&\overset{L7.2.6}{=} \mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}} M_\nu^{2p+1}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
&\geq \mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}} M_p^{2p+1}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right].
\end{aligned}
$$

Analogous to (7.2.1) we factorize the multi sum and use a lower bound on the weight functions (dropping the terms where the bound doesn't hold)

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}} M_p^{2p+1}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
&= C(2p+1,p)\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\left(\sum_{\alpha'\in\hat{Q}_N^q}|w_{\alpha'}(N,q)|^2|u_\alpha(t)|^2\right)^p w_k(N,q)u_k(t)\right|^2\right] \\
&\overset{Cor2.2.12}{=} C\mathbb{E}\left[\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right]^{2p}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}|w(N,k,q)u_k(t)|^2\right] \\
&\overset{(1.2.5)}{\geq} C\mathbb{E}\left[\sum_{m\in\hat{Q}_{N/2}^q}|u_m(t)|^2\right]^{2p}\mathbb{E}\left[\left\|P_{N/2}u\right\|_{H^r(\mathbb{T}^q)}^2\right].
\end{aligned}
\tag{7.2.3}
$$

Putting everything together we get the claimed lower bound.

For the upper bound we once more iterate Theorem 1.2.46 and use Lemma 1.2.53 to express the discrete Fourier coefficients (after replacing them by their copy in $\hat{Q}_N^q$ by periodicity) by their continuous counterparts:

$$
\begin{aligned}
&\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|_{h^r(Q_N^q)}^2\right] \\
&\overset{(1.2.6)}{\leq} 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}|\widehat{b([P_N u]_{N,q}(t,\cdot))}^d(k)|^2\right] \\
&\underset{L1.2.53}{\overset{Thm1.2.46}{=}} 10^{-2r}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}} w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]
\end{aligned}
$$

With our Match functionwe identify the leading order term, and those that need to be bounded by it:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}} w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right] \\
&= \mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}}\sum_{\nu=0}^{p} M_\nu^{2p+1}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&\overset{L7.2.6}{=}\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}}M_\nu^{2p+1}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]\\
&\leq\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}}M_p^{2p+1}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]\\
&\quad+\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}}M_\nu^{2p+1}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right].
\end{aligned}
$$

Analogous to (7.2.3) we compute with the upper bound on the weights

$$
\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}}M_p^{2p+1}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]\overset{(1.2.5)}{\leq}C\mathbb{E}\left[\sum_{m\in\hat{Q}_N^q}|u_m(t)|^2\right]^{2p}\mathbb{E}\left[\|P_Nu\|^2_{H^r(\mathbb{T}^q)}\right].
$$

Now we only need to bound the terms with less conjugate pairs to complete the proof in the same way we did for the quadratic case. In a first step we use the existing matches and Corollary 2.2.12 respectively independence to factorize into a power of the expectation of sums with matching and a sum without any further matches we bound in the following step.

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}\left|\sum_{\alpha\in(\hat{Q}_N^q)^{2p}}M_\nu^{2p}(\alpha(k))w_{\alpha(k)}(N,q)u_{\alpha(k)}(t)\right|^2\right]\\
&=\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}C(p,\nu)\left(\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right)^{2\nu}\left|\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-2\nu}}w_{\alpha_\nu(k)}(N,q)u_{\alpha_\nu(k)}(t)\right|^2\right]\\
&\overset{C2.2.12}{=}\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}C(p,\nu)\mathbb{E}\left[\left(\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right)\right]^{2\nu}\mathbb{E}\left[\left|\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-2\nu}}w_{\alpha_\nu(k)}(N,q)u_{\alpha_\nu(k)}(t))\right|^2\right]\\
&\overset{L7.2.6}{=}\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}C(p,\nu)\mathbb{E}\left[\left(\sum_{m\in\hat{Q}_N^q}|w(N,m,q)|^2|u_m(t)|^2\right)\right]^{2\nu}\mathbb{E}\left[\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-2\nu}}|w_{\alpha_\nu(k)}(N,q)|^2|u_{\alpha_\nu(k)}(t)|^2\right]\\
&\overset{L1.2.53}{\leq}\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\nu=0}^{p-1}C(p,\nu)\mathbb{E}\left[\left(\sum_{m\in\hat{Q}_N^q}|u_m(t)|^2\right)\right]^{2\nu}\mathbb{E}\left[\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-2\nu}}|u_{\alpha_\nu(k)}(t)|^2\right].
\end{aligned}
$$

For $\nu\in[0,p-1]$ we have the bound:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-2\nu}}|u_{\alpha_\nu(k)}(t)|^2\right]\\
&\overset{Cor2.2.12}{=}C(p)\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-2\nu}}\mathbb{E}\left[|u_{\alpha_\nu}(t)|^2\right]\mathbb{E}\left[|u_{k-\sum_{l=1}^{p-\nu}(\alpha_\nu)_l}(t)|^2\right]\\
&\overset{L7.1.8}{\leq}C(p)\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\sum_{\alpha_\nu\in(\hat{Q}_N^q)^{2p-2\nu}}\mathbb{E}\left[|u_{\alpha_\nu}(t)|^2\right]\mathbb{E}\left[|u_k(t)|^2\right]\\
&\leq C(p)\mathbb{E}\left[\sum_{\alpha\in\hat{Q}_N^q}|u_\alpha(t)|^2\right]^{2(p-\nu)}\sum_{k\in\hat{Q}_N^q}\langle k\rangle^{2r}\mathbb{E}\left[|u_k(t)|^2\right]
\end{aligned}
$$

$$=C(p)\mathbb{E}\left[\sum_{\alpha\in\hat{Q}_N^q}|u_\alpha(t)|^2\right]^{2(p-\nu)}\mathbb{E}\left[\|P_N u\|^2_{H^r(\mathbb{T}^q)}\right],$$

on the remaining multi sum without any further matches. Thus putting everything together we get

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]\leq C\left(\mathbb{E}\left[\sum_{m\in\hat{Q}_N^q}|u_m(t)|^2\right]\right)^{2p}\mathbb{E}\left[\|P_N u\|_{H^r(\mathbb{T}^q)}\right]. \tag{7.2.4}$$

Now by Corollary 7.1.7 and the upper bound from Lemma A.4 we get

$$\left(\mathbb{E}\left[\sum_{m\in\hat{Q}_N^q}|u_m(t)|^2\right]\right)^{2p}\overset{Cor7.1.7}{\leq}\tilde{C}\left(t\sum_{\alpha\in\hat{Q}_N^q}\langle\alpha\rangle^{-2}+t^3+1\right)^{2p}\overset{LA.4}{\leq}C_1(q)(t\beta_{q,m}(N)+t^3+1)^{2p},$$

which gives the claimed upper bound when inserted in (7.2.4). ∎

From this we obtain the general results for non-trivial initial data and forcing (as far as admitted by our modified general assumptions) by the following observations:

Corollary 7.2.11
Let $b(a)=a^{2p}$, $p,N\in\mathbb{N}$, $T>0$ and u a solution of (LEQ1), then, for every $t\in[0,T]$, there exist constants $C_1(q),C_2(q)>0$ such that

$$C_1(q)(t\beta_{q,m}(N)+t^3+1)^{2p}\leq\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]\leq C_2(q)(t\beta_{q,m}(N)+t^3+1)^{2p}.$$

Proof
For a constant $C>0, l\in\mathbb{N}_{\leq 2p}$, $g\in C([0,T],L_2(\mathbb{T}^q))$ and v solving (LEQ1) with $0=u_0=u_1\equiv f$ we have

$$\mathbb{E}\left[\left\|([P_N v]_{N,q}(t,\cdot))^{2p-l}[P_N g(t,\cdot)]^l_{N,q]}\right\|^2_{h^r(Q_N^q)}\right]\leq C\|g(t,\cdot)\|^{2l}_{L_2(\mathbb{T}^q)}\mathbb{E}\left[\left\|([P_N v]_{N,q}(t,\cdot))^{2p-1}\right\|^2_{h^r(Q_N^q)}\right]$$

and the claim follows as by Theorem 7.2.10 respectivly Theorem 7.2.9

$$\mathbb{E}\left[\left\|([P_N v]_{N,q}(t,\cdot))^{2p-l}\right\|^2_{h^r(Q_N^q)}\right]\leq C_2(q,t)(t\beta_{q,m}(N)+t^3+1)^{2p-l-1}+C_2(q)(t\beta_{q,m}(N)+t^3+1)^{2p-l}.$$

Here the Expectation in Theorem 7.2.10 is part of the time dependant constant. ∎

Corollary 7.2.12
Let $b(a)=a^{2p+1}$, $p,N\in\mathbb{N}$, $T>0$ and u a solution of (LEQ1), then, for every $t\in[0,T]$, there exist constants $C_1(q),C_2(q)>0$ such that

$$C_1(q)(t\beta_{q,m}(N)+t^3+1)^{2p}\mathbb{E}\left[\left\|P_{N/2}u(t,\cdot)\right\|^2_{H^r(\mathbb{T}^q)}\right]\leq\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]$$

and

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]\leq C_2(q)(t\beta_{q,m}(N)+\tau^3+1)^{2p}\mathbb{E}\left[\|P_N u(t,\cdot)\|^{2p}_{H^r(\mathbb{T}^q)}\right].$$

Proof
Let v solve (LEQ1) with $0=u_0=u_1\equiv f$, then v is precisely the stochastic convolution term of u and we can compute as in the proof of Theorem 7.2.10

$$C_1\left\|P_{N/2}u(t,\cdot)\right\|_{H^r(\mathbb{T}^q)}\mathbb{E}\left[\left\|([P_N v]_{N,q}(t,\cdot))^{2p}\right\|^2_{h^r(Q_N^q)}\right]\leq\mathbb{E}\left[\left\|([P_N v]_{N,q}(t,\cdot))^{2p}[P_N u(t,\cdot)]_{N,q]}\right\|^2_{h^r(Q_N^q)}\right]$$

and

$$\mathbb{E}\left[\left\|([P_N v]_{N,q}(t,\cdot))^{2p}[P_N u(t,\cdot)]_{N,q]}\right\|^2_{h^r(Q_N^q)}\right] \leq C_2 \|P_N u(t,\cdot)\|_{H^s(\mathbb{T}^q)} \mathbb{E}\left[\left\|([P_N v]_{N,q}(t,\cdot))^{2p}\right\|^2_{h^r(Q_N^q)}\right],$$

for some constants $C_1, C_2 > 0$. The remaining terms of $b([P_N u]_{N,q}(t,\cdot))$ are of the form

$$\mathbb{E}\left[\left\|([P_N v]_{N,q}(t,\cdot))^{2p+1-l}[P_N(u-v)(t,\cdot)]^l_{N,q]}\right\|^2_{h^r(Q_N^q)}\right]$$

with $l \in \mathbb{N} \cap [2, 2p+1]$ and $(u-v)(t,\cdot) \in L_2(\mathbb{T}^q)$, a.s.. Thus claim follows as by Theorem 7.2.10 respectivly Theorem 7.2.9

$$\begin{aligned}&\mathbb{E}\left[\left\|([P_N v]_{N,q}(t,\cdot))^{2p+1-l}[P_N(u-v)(t,\cdot)]^l_{N,q]}\right\|^2_{h^r(Q_N^q)}\right]\\ \leq &C_2(q,t)(t\beta_{q,m}(N)+t^3+1)^{2p-l-1} + C_2(q)(t\beta_{q,m}(N)+t^3+1)^{2p-l}.\end{aligned}$$

Here the Expectation in Theorem 7.2.10 is part of the time dependant constant. ■

Polynomial Non-linearities

Now unlike the monomial case there are three possible routes to renormalize a polynomial non-linearity. The first would be simply renormalizing everything with the scaling of the highest order term. This leads to the following two results, as for an even degree polynomial, this renormalization only sees the highest degree monomial.

Corollary 7.2.13

Let $p, N \in \mathbb{N}$, $T > 0$, $b(a) = \sum_{j=0}^{2p} c_j a^j$ a polynomial of even degree and u a solution of (LEQ1) then,for every $t \in [0,T]$ there exist constants $C_1(q), C_2(q) > 0$ such that

$$C_1(q)(t\beta_{q,m}(N)+t^3+1)^{2p} \leq \mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] \leq C_2(q)(t\beta_{q,m}(N)+t^3+1)^{2p}.$$

Proof
First we note the existence of constants $C_1, C_2 > 0$ such that:

$$C_1 b\left(\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]\right) \leq \mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] \leq C_2 b\left(\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]\right),$$

as we simply estimate all the mixed terms in the norm by multiples of the unmixed terms and use linearity of the expectation to pull it inside the polynomial. Now the claim follows by applying Theorem 7.2.9 and Theorem 7.2.10 to the upper and lower bound. ■

While for odd degree polynomials this renormalization sees both the highest degree monomial, as well as the even power directly

Corollary 7.2.14

Let $p, N \in \mathbb{N}$, $T > 0$, $b(a) = \sum_{j=0}^{2p+1} c_j a^j$ a polynomial of odd degree and u a solution of (LEQ1) then, for every $t \in [0,T]$ there exist constants $C_1(q), C_2(q) > 0$ and $C_3(q), C_4(q) \geq 0$ such that

$$C_1(q)(t\beta_{q,m}(N)+t^3+1)^{2p}\mathbb{E}\left[\left\|P_{N/2}u(t,\cdot)\right\|^2_{H^r(\mathbb{T}^q)}\right] + C_3(q)(t\beta_{q,m}(N)+t^3+1)^{2p} \leq \mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right]$$

and

$$\mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] \leq C_2(q)(t\beta_{q,m}(N)+t^3+1)^2\mathbb{E}\left[\|P_N u(t,\cdot)\|^{2p}_{H^r(\mathbb{T}^q)}\right] + C_4(q)(t\beta_{q,m}(N)+t^3+1)^{2p}.$$

Proof
First we note the existence of constants $C_1, C_2 > 0$ such that:

$$C_1 b\left(\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]\right) \leq \mathbb{E}\left[\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)}\right] \leq C_2 b\left(\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]\right),$$

as we simply estimate all the mixed terms in the norm by multiples of the unmixed terms and use linearity of the expectation to pull it inside the polynomial. Now the claim follows by applying Theorem 7.2.9 and Theorem 7.2.10 to the upper and lower bound. ∎

The second renormalization scheme uses the fact that the scaling itself acts as a power of our scaling function, which depends on the power of the monomial, and moves the scaling inside the non-linearity. Though this renormalization will still turn a blind eye to odd powers, it is notable for being the finite version of the power series renormalization.

Corollary 7.2.15

Let $p, N \in \mathbb{N}^*$, $T > 0$, $b(a) = \sum_{j=0}^{2p} c_j a^j$ an even order polynomial, and u a solution of (LEQ1) then, for every $t \in [0,T]$ there exist constants $C_1(q,t), C_2(q,t) \geq 0$ such that

$$C_1(q,t) \leq \mathbb{E}\left[\left\|b\left(\frac{[P_N u]_{N,q}(t,\cdot)}{\beta_{q,m}(N)}\right)\right\|^2_{h^r(Q_N^q)}\right] \leq C_2(q,t).$$

Proof
First we note the existence of constants $C_1, C_2 > 0$ such that:

$$C_1 b\left(\frac{\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]}{\beta_{q,m}(N)}\right) \leq \mathbb{E}\left[\left\|b\left(\frac{[P_N u]_{N,q}(t,\cdot)}{\beta_{q,m}(N)}\right)\right\|^2_{h^r(Q_N^q)}\right] \leq C_2 b\left(\frac{\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]}{\beta_{q,m}(N)}\right),$$

as we simply estimate all the mixed terms in the norm by multiples of the unmixed terms and use linearity of the expectation to pull it inside the polynomial. Now the claim follows by applying Theorem 7.2.9 and Theorem 7.2.10 to the upper and lower bound. ∎

The final possible renormalization scheme circumvents the problem of „blindness for lower order terms "by separating the non-linearity into an odd and an even part, while keeping the scaling inside the non-linearity, and compensating the odd terms by balancing out the surplus power of our scaling function.

Corollary 7.2.16

Let $p, N \in \mathbb{N}^*, T > 0$ $b(a)$ a polynomial, that can be decomposed into the polynomial $b_e(a) = \sum_{j=0}^{p} c_j a^{2j}$ of only even degrees and $b_o(a) = \sum_{j=0}^{p} c_j a^{2j+1}$ a polynomial of purely odd degrees, and u a solution of (LEQ1) then, for every $t \in [0,T]$ there exist constants $C_1(q,t), C_2(q,t) > 0$ and $C_3(q,t), C_4(q,t) \geq 0$ such that

$$C_3(q,t) \leq \mathbb{E}\left[\left\|b_e\left(\frac{[P_N u]_{N,q}(t,\cdot)}{\beta_{q,m}(N)}\right)\right\|^2_{h^r(Q_N^q)}\right] \leq C_4(q,t)$$

as well as

$$C_1(q,t)\mathbb{E}\left[\left\|P_{N/2} u(t,\cdot)\right\|^2_{H^r(\mathbb{T}^q)}\right] \leq \mathbb{E}\left[\left\|\beta_{q,m}(N) b_o\left(\frac{[P_N u]_{N,q}(t,\cdot)}{\beta_{q,m}(N)}\right)\right\|^2_{h^r(Q_N^q)}\right]$$

and

$$\mathbb{E}\left[\left\|\beta_{q,m}(N) b_o\left(\frac{[P_N u]_{N,q}(t,\cdot)}{\beta_{q,m}(N)}\right)\right\|^2_{h^r(Q_N^q)}\right] \leq C_2(q,t)\mathbb{E}\left[\|P_N u(t,\cdot)\|^{2p}_{H^r(\mathbb{T}^q)}\right].$$

Proof
First we note the existence of constants $C_1, C_2 > 0$ such that:

$$C_1 b_e\left(\frac{\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]}{\beta_{q,m}(N)}\right) \leq \mathbb{E}\left[\left\|b_e\left(\frac{[P_N u]_{N,q}(t,\cdot))}{\beta_{q,m}(N)}\right)\right\|^2_{h^r(Q_N^q)}\right] \leq C_2 b\left(\frac{\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]}{\beta_{q,m}(N)}\right),$$

and

$$C_1 b_o\left(\frac{\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]}{\beta_{q,m}(N)}\right) \leq \mathbb{E}\left[\left\|b_o\left(\frac{([P_N u]_{N,q}(t,\cdot))}{\beta_{q,m}(N)}\right)\right\|^2_{h^r(Q_N^q)}\right] \leq C_2 b_o\left(\frac{\mathbb{E}\left[\left\|[P_N u]_{N,q}(t,\cdot)\right\|^2_{h^r(Q_N^q)}\right]}{\beta_{q,m}(N)}\right),$$

as we simply estimate all the mixed terms in the norms by multiples of the unmixed terms and use linearity of the expectation to pull it inside the polynomial. Now the claim follows by applying Theorem 7.2.9 and Theorem 7.2.10 to the upper and lower bound. ■

7.2.3 Deterministic Scaling

As a last point we will discuss what happens in the non-linear case, if instead of a space time white noise we were dealing with a distribution valued forcing, thereby exemplary giving the reason for the change of general assumption (A1). For simplicities sake let us again consider the wave equation this time with a forcing $f \in C([0,T], H^s(\mathbb{T}^q))$, for $\mathbb{R} \ni s < 0$, instead of the space-time white noise

$$\begin{cases} \partial_{tt} u = \Delta u + f & (t,x) \in \mathbb{R}_+ \times \mathbb{T}^q \\ u(0,\cdot) = 0 = \partial_t u(0,\cdot), \end{cases} \tag{DetEQ}$$

and the case of a cubic forcing.

Just as before, just this time without the expectation, using the weight bounds (1.2.6), Lemma 1.2.53 and Theorem 1.2.46 we estimate:

$$\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)} \geq \sum_{k\in\hat{Q}_N^q} \langle k\rangle^{2r} \left|\sum_{\alpha\in(\hat{Q}_N^q)^2} w_{\alpha(k)}(N,q) u_{\alpha(k)}(t)\right|^2$$

and

$$\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)} \leq 10^{-2r} \sum_{k\in\hat{Q}_N^q} \langle k\rangle^{2r} \left|\sum_{\alpha\in(\hat{Q}_N^q)^2} w_{\alpha(k)}(N,q) u_{\alpha(k)}(t)\right|^2.$$

For this lower bound, unlike the stochastic case, where we could use combinatorics to interchange the summation and square of the complex absolute value, here we have no such tool at our disposal. This leaves only zero as available lower bound,as for every $N \in \mathbb{N}$ and $s \in \mathbb{R}$ there exists a $f \in H^s(\mathbb{T}^q) \setminus H^{s+1}(\mathbb{T}^q)$ with $P_N f = 0$ and therefore $P_N u = 0$, as the Fourier coefficients of u are given by

$$u_k(t) = \int_0^t \frac{\sin((t-t')k) f_k(t')}{k}\, dt'.$$

While for the upper bound by Lemma 1.2.52 for every $N \in \mathbb{N}$ we find an $f \in H^s(\mathbb{T}^q)$, for $s < \frac{-q}{2}$, such that

$$f_k(s) = \begin{cases} w^{-1}_{\alpha(k)} \frac{sin((t-s)k)}{k}, & \text{for } k \in \hat{Q}^q_{\lfloor \alpha N \rfloor}, \\ 0, & \text{otherwise,} \end{cases}$$

for some $\alpha \in (0,1)$. Therefore we get

$$\left\|b([P_N u]_{N,q}(t,\cdot))\right\|^2_{h^r(Q_N^q)} \leq 10^{-2r} \sum_{k\in\hat{Q}_N^q} \langle k\rangle^{2r}(\alpha N)^4.$$

Therefore the upper and lower bound don't even show the same scaling behaviour, which leaves no hope for the general critical case. And the scaling needed to be subcritical is twice as large, as it would be in the stochastic setting.

7.3 Subcritical Scaling

In the previous chapter we obtained a candidate for the critical scaling of the non-linearity, by determining the scaling of the continuous linear solution. Now we consider the linear equation

$$\begin{cases} \frac{d^2}{dt^2} X_N(t,\cdot) = \Delta_{N,d} X_N(t,\cdot) + \sigma(t)\dot{B}_N(t,\cdot) \\ X_N(0) \equiv 0 \equiv \frac{d}{dt} X_N(0) \end{cases} \tag{LEQ1}$$

and for $g \in \mathbb{N}_{\geq 2}$ and $\beta > \frac{(q-2)}{2}$ the non-linear equations

$$\begin{cases} \frac{d^2}{dt^2} Z_N(t,\cdot) = \Delta_{N,d} Z_N(t,\cdot) + p\left(\frac{Z_N(t,\cdot)}{N^\beta}\right) + \sigma(t)\dot{B}_N(t,\cdot) \\ Z_N(0) \equiv 0 \equiv \frac{d}{dt} Z_N(0), \end{cases} \tag{NLEQ1}$$

where p is a polynomial of degree $g \in \mathbb{N}$. to show that for scaling β larger than our candidate the non-linear equations (NLEQ1) admit the same limit behaviour as the linear equation (LEQ1).

Theorem 7.3.1
Let $q \geq 2, g \in \mathbb{N}^*, \beta > \frac{(q-2)}{2}$, p a polynomial of degree g and $(X_N)_{N\in\mathbb{N}^*}$, $(Z_N)_{N\in\mathbb{N}^*}$ solution sequences of (LEQ1) respectively (NLEQ1). Define $D_N := Z_N - X_N$ for all $N \in \mathbb{N}^*$, then for all $\eta \in \left(\frac{q-2}{2}, \beta\right)$ and all $\kappa \in (0,1)$ we have

$$\left\|P^a_{\kappa N}\tilde{D}_N\right\|_{L_2([0,T],H^{-\eta}(\mathbb{T}^q))} \xrightarrow{\mathbb{P}} 0.$$

Proof
By definition D_N solves

$$\begin{cases} \frac{d^2}{dt^2} D_N(t,\cdot) = \Delta_{N,d} D_N(t,\cdot) + p\left(\frac{Z_N(t,\cdot)}{N^\beta}\right) \\ D_N(0) \equiv 0 \equiv \frac{d}{dt} D_N(0). \end{cases} \tag{DEQ2}$$

Now the polynomial $p(Z_N)$ can be understood as forcing and the linear theory provides the following mild formulation of D_N

$$D_N(t,\cdot) = \int_0^t \pi_1\left(S_1(t-s)\right)\left(\begin{pmatrix} 0 \\ p\left(\frac{Z_N((s,\cdot)}{N^\beta}\right)\end{pmatrix}\right) ds,$$

where π_1 is the projector mapping onto the first entry. Now for any $\eta \in \left(\frac{q-2}{2}, \frac{\beta}{p}\right)$ there exists a $\nu > 0$ such that $\beta = p\eta + \nu$ and a sequence, $(\tau_N)_{N\in\mathbb{N}}$, of stopping times, defined via

$$\tau_N := \inf_{t\geq 0}\{\|D_N(t)\|_{h^{-\eta}(Q_N^q)} \geq 1\} \wedge \inf_{t\geq 0}\{\|X_N(t)\|_{h^{-\eta}(Q_N^q)} \geq \log(N)\}.$$

Now we estimate the norm of the stopped difference $D_N(\cdot \wedge \tau_N, \cdot)$, by first replacing Z_N by $X_N + D_N$ and using the resulting terms binomial expansion .

$$\|D_N(t\wedge\tau_N)\|_{h^{-\eta}(Q_N^q)} \leq C \int_0^{t\wedge\tau_N} \left\| p\left(\frac{Z_N(s,\cdot)}{N^\beta}\right)\right\|_{h^{-\eta-1}(Q_N^q)} ds$$

$$\leq \int_0^{t\wedge\tau_N} C_p \sum_{j=0}^{g}\sum_{l=0}^{j} \left\| \frac{X_N^l(s,\cdot)D_N^{j-l}(s,\cdot)}{N^\beta} \right\|_{h^{-\eta-1}(Q_N^q)} ds.$$

Now by iterating Theorem 1.2.46 and compressing the interior sum into the constant we obtain the following l_2-bounds:

$$\int_0^{t\wedge\tau_N} C_p \sum_{j=0}^{g}\sum_{l=0}^{j} \left\| \frac{X_N^l(s,\cdot)D_N^{j-l}(s,\cdot)}{N^\beta} \right\|_{h^{-\eta-2}(Q_N^q)} ds$$
$$\leq \int_0^{t\wedge\tau_N} \frac{C_p}{N^\nu}\left(\left\| \frac{X_N(s,\cdot)}{N^\eta}\right\|_{l_2(Q_N^q)}^p + \sum_{l=0}^{p-1} \left\| \frac{X_N(s,\cdot)}{N^\eta}\right\|_{l_2(Q_N^q)}^l \left\| \frac{D_N(s,\cdot)}{N^\eta}\right\|_{l_2(Q_N^q)}^{p-l}\right) ds.$$

As we are working with discrete functions over $\hat{Q}_N^q$, we can now trade powers of N^{-1} for „spatial regularity“, and apply the bounds build into the stopping time τ_N, i.e.

$$\int_0^{t\wedge\tau_N} \frac{C_p}{N^\nu}\left(\left\| \frac{X_N(s,\cdot)}{N^\eta}\right\|_{l_2(Q_N^q)}^g + \sum_{l=0}^{g-1} \left\| \frac{X_N(s,\cdot)}{N^\eta}\right\|_{l_2(Q_N^q)}^l \left\| \frac{D_N(s,\cdot)}{N^\eta}\right\|_{l_2(Q_N^q)}^{g-l}\right) ds$$
$$\leq \int_0^{t\wedge\tau_N} \frac{C_p}{N^\nu}\left(\|X_N(s,\cdot)\|_{h^{-\eta}(Q_N^q)}^p + \sum_{l=0}^{g-1} \|X_N(s,\cdot)\|_{h^{-\eta}(Q_N^q)}^l \left\|P_N^a D_N(s,\cdot)\right\|_{h^{-\eta}(Q_N^q)}^{g-l}\right) ds$$
$$\leq \int_0^{t\wedge\tau_N} \frac{C_p}{N^\nu}\left(\log(N)^g + P_g(\log(N))\|D_N(s,\cdot)\|_{h^{-\eta}(Q_N^q)}\right) ds,$$

where $P_g(\log(N))$ denotes a different polynomial of degree g. By using a redundant stopping we move the stopping time τ_N from the integral into the (time-)argument of D_N.

$$\|D_N(t\wedge\tau_N)\|_{h^{-\eta}(Q_N^q)} \leq \int_0^{t\wedge\tau_N} \frac{C_p}{N^\nu}\left(\log(N)^g + P_g(\log(N))\|D_N(s,\cdot)\|_{h^{-\eta}(Q_N^q)}\right) ds$$
$$\leq \int_0^{t} \frac{C_p}{N^\nu}\left(\log(N)^g + P_g(\log(N))\|D_N(s\wedge\tau_N,\cdot)\|_{h^{-\eta}(Q_N^q)}\right) ds$$

and compute using Gronwall's inequality

$$\|D_N(t\wedge\tau_N)\|_{h^{-\eta}(Q_N^q)} \leq \frac{C_p t\log(N)^g}{N^\nu}\exp\left(\frac{C_p t P_g(\log(N))}{N^\nu}\right).$$

Now use Corollary 1.2.57 we can convert this into an estimate for the continuous norm

$$\left\|P_{\kappa N}\tilde{D}_N(t\wedge\tau_N)\right\|_{H^{-\eta}(\mathbb{T}^q)} \leq \frac{C_p t\log(N)^g}{N^\nu}\exp\left(\frac{C_p t P_g(\log(N))}{N^\nu}\right).$$

And taking the L_2-norm with respect to time we arrive at

$$\left\|P_{\kappa N}^a\tilde{D}_N\right\|_{L_2([0,T\wedge\tau_N],H^{-\eta}(\mathbb{T}^q))} \leq T\left(\frac{C_p T\log(N)^g}{N^\nu}\exp\left(\frac{C_p T P_g(\log(N))}{N^\nu}\right)\right)^2 =: \mathbb{P}_{T,N}.$$

With these „preparations“out of the way we can go on to prove the claimed convergence in probability. First we rewrite the events of interest in such a way, that we can use the previous calculations and standard estimates for probabilities to give a bound, by a term vanishing for $N\to\infty$. For $\delta > 0$ consider

$$\mathbb{P}(\left\|P_{\kappa N}^a\tilde{D}_N\right\|_{L_2([0,T],H^{-\eta}(\mathbb{T}^q))} \geq \delta)$$
$$\leq \mathbb{P}(\tau_N < T \ ,\ \left\|P_{\kappa N}^a\tilde{D}_N\right\|_{L_2([0,T],H^{-\eta}(\mathbb{T}^q))} \geq \delta) + \mathbb{P}(\tau_N \geq T \ ,\ \left\|P_{\kappa N}^a\tilde{D}_N\right\|_{L_2([0,T],H^{-\eta}(\mathbb{T}^q))} \geq \delta)$$

$$\le \mathbb{P}(\tau_N \le T) + \mathbb{P}(\left\| P^a_{\kappa N}\tilde{D}_N \right\|_{L_2([0,T\wedge\tau_N],H^{-\eta}(\mathbb{T}^q))} \ge \delta)$$

$$\le \mathbb{P}(\sup_{t\in[0,T]} \|D_N(t)\|_{h^{-\eta}(Q^q_N)} \ge 1) + \mathbb{P}(\sup_{t\in[0,T]} \|X_N(t)\|_{h^{-\eta}(Q^q_N)} \ge \log(N)) + \mathbb{P}(\left\| P^a_{\kappa N}\tilde{D}_N \right\|_{L_2([0,T\wedge\tau_N],H^{-\eta}(\mathbb{T}^q))} \ge \delta).$$

Using the previous calculations and the Chebyshev Inequality we compute

$$\mathbb{P}(\left\| P^a_{\kappa N}\tilde{D}_N \right\|_{L_2([0,T\wedge\tau_N],H^{-\eta}(\mathbb{T}^q))} \ge \delta) \le \frac{\mathbb{P}_{T,N} \vee \mathbb{P}^2_{T,N}}{\delta^2}$$

and again by the previous computation and this time the Markov Inequality we compute

$$\begin{aligned}\mathbb{P}(\sup_{t\in[0,T]} \|D_N(t)\|_{h^{-\eta}(Q^q_N)} \ge 1) &\le \sup_{t\in[0,T]} \frac{C_p t \log(N)^g}{N^\nu} \exp\left(\frac{C_p t P_g(\log(N))}{N^\nu}\right)\\ &= \frac{C_p T \log(N)^g}{N^\nu} \exp\left(\frac{C_p T P_g(\log(N))}{N^\nu}\right).\end{aligned}$$

For the final probability we use that for N sufficiently large, by Theorem 3.3.2, for u denoting the corresponding continuous solution, and every $t \in [0,T]$ we have

$$\|X_N(t)\|_{h^{-\eta}(Q^q_N)} \le \left\| [P^a_{\kappa N} u]^{\sim}_{N,q}(t) \right\|_{h^{-\eta}(Q^q_N)} + \varepsilon(N) \qquad \mathbb{P}-a.s.$$

with $\varepsilon(N) \overset{N\to\infty}{\longrightarrow} 0$. Now by Corollary 1.2.57 we have, as $u \in C([0,T], H^{-\eta}(\mathbb{T}^q))$,

$$\sup_{t\in[0,T} \left\| [P^a_{\kappa N} u]^{\sim}_{N,q}(t) \right\|_{h^{-\eta}(Q^q_N)} \le \sup_{t\in[0,T\wedge\tau_N]} C \|u(t,\cdot)\|_{H^{-\eta}(\mathbb{T}^q)} \le C(T).$$

plugging this into the Markov Inequality we gain the estimate

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|X_N(t)\|_{h^{-\eta}(Q^q_N)} \ge \log(N)\right) \le \frac{C(T)}{\log(N)}.$$

Combining the last three estimates we get

$$\lim_{N\to\infty} \mathbb{P}\left(\left\| P^a_{\kappa N}\tilde{D}_N \right\|_{L_2([0,T],H^{-\eta}(\mathbb{T}^q))} \ge \delta\right) = 0,$$

for every $\delta > 0$, that is the claimed convergence in probability. ■

Remark 7.3.2
While this covers every β, larger than the critical value for even degree p, for polynomials p, of odd degree, there is a gap of $\frac{q-2}{2}$.

Remark 7.3.3
Aside from a slightly different estimate when iterating Theorem 1.2.46, the exact same result can be obtained for the other two renormalization schemes for polynomials.

Remark 7.3.4
While non-linear potentials weren't considered during the writing of this thesis, at this point it should be pointed out, that by the last theorem we have obtained a (stochastic) continuum limit theory, for discrete equations where the non-linear contribution is such that after the rescaling, in the subcritical case can carry over.

A Miscellaneous Results

In this chapter we collect some standalone results that weren't covered in the Preliminaries.

Theorem A.1 (Chebyshev Inequality)
Let $X \in L_2(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R})$ a square-integrable random variable and $a > 0$, then

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}[X^2]}{a^2}.$$

Proof
Confirm [Tap13a, Satz 6.69].

Theorem A.2 (Markov Inequality)
Let $X \in L_1(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R})$ an integrable random variable and $a > 0$, then

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}[|X|]}{a}.$$

Proof
Confirm [Tap13a, Satz 6.59].

Theorem A.3 (Gronwall-Inequality)
Let $J \subset \mathbb{R}$ an interval, $t_0 \in J$, and $\alpha, \beta, u \in C(J, \mathbb{R}_{\geq 0})$. Then

$$u(t) \leq \alpha(t) + \left| \int_{t_0}^{t} \beta(s) u(s)\, ds \right| \quad \forall\, t \in J,$$

implies

$$u(t) \leq \alpha(t) + \left| \int_{t_0}^{t} \alpha(s)\beta(s) e^{\left| \int_s^t \beta(r)\, dr \right|} \right| \quad \forall\, t \in J. \tag{A.1}$$

And if $\alpha(\cdot) = \alpha_0(|\cdot - t_0|)$ for an increasing function $\alpha_0 \in C(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, then (A.1), becomes

$$u(t) \leq \alpha e^{\left| \int_{t_0}^{t} \beta(s)\, ds \right|} \quad \forall\, t \in J.$$

Proof
Confirm [Ama95, **(6.1)** and **Korollar (6.2)**].

Lemma A.4 (Integral Test for Hyperharmonic Series)
Let $N, m, q \in \mathbb{N}^*$. Then there exist constants $C_1 = C_1(q), C_2 = C_2(q) > 0$ such that

$$\begin{cases} C_1 = \sum_{k \in \hat{Q}_N^q} \langle k \rangle^{-2m}, & q < 2m, \\ C_1 \log(N) \leq \sum_{k \in \hat{Q}_N^q} \langle k \rangle^{-2m} \leq C_2 \log(N), & q = 2m, \\ C_1 N^{q-2m} \leq \sum_{k \in \hat{Q}_N^q} \langle k \rangle^{-2m} \leq C_2 N^{q-2m}, & q \geq 2m+1. \end{cases}$$

Theorem A.5 (Bochner-Minlos)
Let C be a function on $\mathcal{N}$ with the following properties:

(i) C *is continuous on* $\mathcal{N}$,

(ii) C *is positive definite,*

(iii) $C(0) = 1$.

Then there exists a unique probability measure μ_C *on* $\mathcal{N}^*, \mathcal{B}$, *so that for all* $\xi \in \mathcal{N}$:

$$C(\xi) = \int_{\mathcal{N}^*} \exp(i_{\mathcal{N}^*}\langle x, \xi \rangle_{\mathcal{N}})\, d\mu_C(x).$$

Moreover if C *is continuous with respect to* $\|\cdot\|_p$, *for* $p \in \mathbb{N}$, *and if* $n > p$ *is such that the injection* $i_p^n : \mathcal{N}_n \to \mathcal{N}_p$ *is of Hilbert-Schmidt type, then* $\mu_C(\mathcal{N}_{-n}) = 1$.

Proof
[HKPS93, Theorem] ■

B Basic Taylor Approximations

In this chapter we collect the Taylor approximations we use as buildingblocks in our semigroup estimates in chapter 5. As for their validity we refer to the respective chapters on Taylor-Formulas in [AE06a] and [AE06b].

Lemma B.1
For $p > 0$ there are constants $C_1, C_2 > 0$ such that for every $\gamma \in \mathbb{B}_{\mathbb{C}}(0, p)$

$$\sqrt{p} + C_1 \frac{\gamma}{p} \leq \sqrt{p+\gamma} \leq \sqrt{p} + C_2 \frac{\gamma}{p}.$$

Lemma B.2
For $a \in \mathbb{R}$ there are constants $C_1, C_2 > 0$ such that for every $x \in \mathbb{B}_{\mathbb{C}}(a, 1)$

$$\sinh(a) + C_1|x-a| \leq \sinh(x) \leq \sinh(a) + C_2|x-a|.$$

Lemma B.3
For $a \in \mathbb{R}$ there are constants $C_1, C_2 > 0$ such that for every $x \in \mathbb{B}_{\mathbb{C}}(a, 1)$

$$\cosh(a) + C_1|x-a| \leq \cosh(x) \leq \cosh(a) + C_2|x-a|.$$

Lemma B.4
For $a \in \mathbb{R}$ there are constants $C_1, C_2 > 0$ such that for every $x \in \mathbb{B}_{\mathbb{C}}(a, 1)$

$$\frac{\sinh(a)}{a} + C_1|x-a| \leq \sinh(x) \leq \frac{\sinh(a)}{a} + C_2|x-a|.$$

Lemma B.5
For $a \in \mathbb{R}$ there are constants $C_1, C_2 > 0$ such that for every $x \in \mathbb{B}_{\mathbb{C}}(a, 1)$

$$\exp(a) + C_1|x-a| \leq \exp(x) \leq \exp(a) + C_2|x-a|.$$

Lemma B.6
For $a \in \mathbb{R}$ there are constants $C_1, C_2 > 0$ such that for every $x \in \mathbb{B}_{\mathbb{C}}(a, 1)$

$$a\cosh(a) + C_1|x-a| \leq x\cosh(x) \leq a\cosh(a) + C_2|x-a|.$$

Lemma B.7
For $a \in \mathbb{R}$ there are constants $C_1, C_2 > 0$ such that for every $x \in \mathbb{B}_{\mathbb{C}}(a, 1)$

$$a\sinh(a) + C_1|x-a| \leq x\sinh(x) \leq a\sinh(a) + C_2|x-a|.$$

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