Stochastic Processes and their Applications 189 (2025) 104706

Contents lists available at ScienceDirect

Stochastic Processes and their Applications

journal homepage: www.elsevier.com/locate/spa

Planar reinforced k-out percolation

Gideon Amir [a], Markus Heydenreich [b], Christian Hirsch [c,d,*]

[a] Department of Mathematics, Bar-Ilan University, Ramat-Gan, 5290002, Israel
[b] Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany
[c] Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark
[d] DIGIT Center, Aarhus University, Finlandsgade 22, 8200 Aarhus N, Denmark

ABSTRACT

We investigate the percolation properties of a planar reinforced network model. In this model, at every time step, every vertex chooses $k \geqslant 1$ incident edges, whose weight is then increased by 1. The choice of this k-tuple occurs proportionally to the product of the corresponding edge weights raised to some power $\alpha > 0$.

Our investigations are guided by the conjecture that the set of infinitely reinforced edges percolates for $k = 2$ and $\alpha \gg 1$. First, we study the case $\alpha = \infty$, where we show the percolation for $k = 2$ after adding arbitrarily sparse independent sprinkling and also allowing dual connectivities. We also derive a finite-size criterion for percolation without sprinkling. Then, we extend this finite-size criterion to the $\alpha < \infty$ case. Finally, we verify these conditions numerically.

1. Introduction

We target the interesting though challenging question which mechanism gives networks the shape and property they have. This has been addressed from various perspectives. A challenge for mathematicians is the rigorous derivation of global network properties from local interactions. The topic of this paper is to propose a network model giving rise to global connectivities based on a mechanism of local self-enhancement. More precisely, we consider a model where vertices are given a priori, and it is the edges that are forming according to preferential attachment.

The motivation for our network layout comes from neural networks. The vertices indicate neural cells whilst edges represent neural connections through axons. Neuroplasticity postulates that neural connections are strengthened through usage, and a neural connection that has been used frequently is thus more likely to be used in the future. A strong form of neuroplasticiy has been suggested by Markram et al. as tabula-rasa hypothesis [1]. There they start from a theoretical all-to-all connectivity, and then enhance the links between cells upon usage to ultimately become functional connections (synapses), whereas rarely used links deteriorate. The new research field of graph-based reinforcement models aims to understand the mathematical feasibility of such scenarios.

In this work, we propose a percolating model of graph-based reinforcement, which is based on the idea of *k-out networks*. The evolution of the network takes place in rounds, i.e., discrete time steps. In each round, each vertex $z \in \mathbb{Z}^d$ reinforces precisely k of its incident edges. The selection of such a k-tuple is proportional to the product of the current edge weights raised to some reinforcement parameter $\alpha > 0$, see Section 2.2 below for a precise model definition. We also consider the setting where only the maximal-weight edges incident to a node are eligible for reinforcement. As we will explain in more detail below, this can be considered as the case $\alpha = \infty$, where even a small weight difference makes an edge infinitely stronger than another one with lower weight.

* Corresponding author.
E-mail addresses: gideon.amir@biu.ac.il (G. Amir), markus.heydenreich@uni-a.de (M. Heydenreich), hirsch@math.au.dk (C. Hirsch).

https://doi.org/10.1016/j.spa.2025.104706
Received 29 July 2024; Received in revised form 27 March 2025; Accepted 19 May 2025
Available online 4 June 2025

Our main focus is the question of percolation of the set of infinitely-reinforced edges, which we denote by $\mathcal{E}_\infty$. More specifically, we concentrate on the case $d = k = 2$. Here, our overarching conjecture is that percolation occurs once α is sufficiently large. Our main result, Theorem 3 below, is a major step towards this conjecture in the case $\alpha = \infty$. More precisely, we show that there is percolation after an arbitrarily sparse sprinkling of independent open edges when using a modified notion of connectivity that will be specified precisely below. Here, the sprinkling means that we consider the union of $\mathcal{E}_\infty$ with an i.i.d. ε-bond percolation on $\mathbb{Z}^d$. Moreover, we provide several rigorously proven finite-range criteria for percolation, for which we then obtain overwhelming numerical evidence by Monte Carlo simulation. All of these results are stated in full detail in Section 2 below.

We conclude the present introduction by elaborating on the connections of our work to two vibrant fields of research, namely (i) graph-based reinforcement, and (ii) dependent models for percolation, especially those violating the FKG condition.

Graph-based reinforcement. Pioneering works in the area of graph-based reinforcement are [2,3] where the reinforcement is set on the vertices of the graph. In the context of neuroscience, it is more natural to consider edge-based reinforcement, and those were introduced in the seminal work [4]. One of the major drawbacks of this model is that even for $\alpha \gg 1$, it does not give rise to percolating structures [5]. Percolation could be achieved when relying on tree-based models [6,7]. However, this imposes already very strong a priori structures. An interesting alternative form of graph reinforcement comes from ant-based walks [8,9].

Percolation without FKG. In recent years, the investigation of dependent percolation models has attracted considerable attention. We refer the reader to [10] and references therein. However, although a broad range of models has been considered, the vast majority satisfies the *FKG inequality*, i.e., positive correlation of increasing events. When considering questions beyond the validity of the FKG inequality, only a few isolated results are available. The results treated in [11,12] fundamentally rely on the Gaussian field and Poisson point process setting, respectively, and therefore do not apply in our setting. To solve this problem in our setting, we develop a completely novel stochastic domination property.

While our investigation concerns a reinforced model for k-out percolation, we stress here that already the question of *independent* k-out percolation has been the topic of vigorous research. More precisely, to put our investigation into perspective, we first review the independent k-out model discussed in [13]. Here, at each site $z \in \mathbb{Z}^d$, we select independently and uniformly at random a k-element subset $E_z^{(k)}$ of all bonds incident at z. We let $G_k^d = \bigcup_{z\in\mathbb{Z}^d} E_z^{(k)}$ denote the random graph obtained as the union of all these bonds. Then, we let

$$k_c(d) := \min\{k \geqslant 1 : \mathbb{P}(G_k^d \text{ percolates}) > 0\}$$

denote the smallest value of $k \geqslant 1$ such that G_k^d percolates, noting that $k_c(d) \leqslant d+1$. A key property of this model is the negative correlation of the vacant edges.

Proposition 1 (*Percolation of Independent d-out Percolation on $\mathbb{Z}^d$; [13]*)**.** *For every $d \geqslant 3$, we have that $k_c(d) \leqslant 3$. Moreover, $k_c(2) = 2$.*

The rest of the manuscript is organized as follows. In Section 2, we give a detailed description of our k-out percolation model and state all our main results. Section 3 contains the proof of a crucial domination property. In Section 4, we prove a sprinkling result by establishing a sharp phase transition and combining it with the domination result. Subsequently, in Section 5, we derive certain finite-size criteria for percolation, for which we then present overwhelming numerical evidence in Section 6.

2. Model and main results

In this work, we consider a reinforced model for k-out percolation, which depends on a reinforcement parameter $\alpha > 0$. First, in Section 2.1, we focus on the case $\alpha = \infty$. Then, in Section 2.2, we consider the setting of finite α.

2.1. The case $\alpha = \infty$

In this model, we assign weights (W_e) to the edges E of the hyper-cubic lattice $\mathbb{Z}^d = (V, E)$, which evolve in discrete time steps. Initially, all edges have weight $W_e(0) = 1$. Then, at every discrete time $t \geqslant 1$ the weights are updated as follows.

(1) At every site $z \in V$, we select one tuple $\sigma \in E_z^{(k)}$, where $E_z^{(k)}$ denote the family of k-tuples of edges incident to z: we order all adjacent edges $e_1, \dots, e_{2d}$ according to their momentary weights $W_{e_1}(t), \dots, W_{e_{2d}}(t)$ in decreasing order (breaking ties by independent coin flips). We set σ to be the k-tuple consisting of first k of these edges in this decreasing order.

(2) The weight W_e of each edge $e \in \sigma$ is then increased by the number of times it was selected. This selection is done simultaneously for all vertices $z \in V$ based on the weights $\{W_e(t)\}_{e\in E}$. As it can happen that an edge is selected by each of its two endpoints, one has $W_e(t+1) - W_e(t) \in \{0, 1, 2\}$.

This results in the "final" configuration

$$\mathcal{E}_\infty := \left\{e \in E : \liminf_{t\uparrow\infty}(W_e(t) - W_e(t-1)) \geqslant 1\right\} \tag{1}$$

of edges that are eventually reinforced in every round. Our arguments in Section 3 show that this set coincides with the set of edges that are reinforced infinitely often. The same set of arguments shows that our percolation model is stochastically dominated by the independent k-out model mentioned in Proposition 1. This is explained in the proof of Proposition 7 below.

To simplify the terminology, we call an edge $e \in E$ *open* if $e \in \mathcal{E}_\infty$ and *closed* otherwise. As elucidated in the introduction, we hypothesize that $\mathcal{E}_\infty$ percolates for $d = k = 2$. That is, we make the following conjecture.

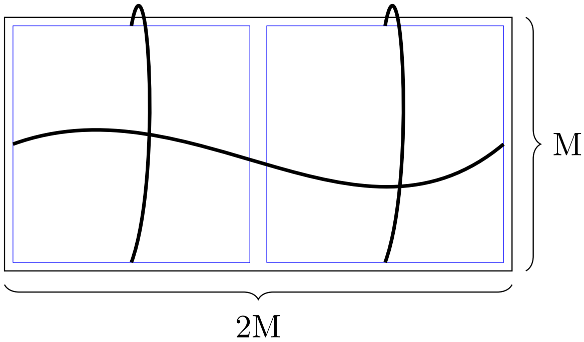

Fig. 1. Illustration of the crossings in an (M,n)-open rectangle.

Conjecture 2 (*Percolation for* $d = k = 2$). *Assume that* $d = k = 2$. *Then,* $\mathbb{P}(\mathcal{E}_\infty$ *percolates*$) = 1$.

The aim of this work is to present a series of results supporting this core conjecture. First, superpose the edges of interest it with some *sprinkling*, i.e., by adding an independent Bernoulli percolation process with some small parameter $\varepsilon > 0$. We write $\mathcal{E}_{\varepsilon,\mathsf{spr}}$ for the sprinkled set of edges and put $\mathcal{E}_{\infty,\varepsilon} := \mathcal{E}_\infty \cup \mathcal{E}_{\varepsilon,\mathsf{spr}}$. Adding the sprinkled edges makes it substantially easier to find an infinite connected component. One of the striking properties of planar Bernoulli bond percolation is its self-duality. While our dependent model is still planar, it is no longer self-dual. This means that when applying classical techniques from percolation theory, the implications are often different than in Bernoulli bond percolation. However, for our argument to work, we still need to allow the alternative that there is an infinite component where the adjacency notation from the original lattice is replaced by the adjacencies from the dual lattice. That is, two edges of $\mathbb{Z}^2$ are dually adjacent if there exists $z \in \mathbb{Z}^2$ such that $\partial(z + [0,1]^2)$ contains both edges.

We say that the model percolates if there exists an infinite sequence of distinct edges $(e_0, e_1, e_2, \dots)$ in $\mathcal{E}_\infty$ such that for all $i \in \mathbb{N}$, we have that e_{i-1} and e_i are adjacent. We further say that the model *percolates dually* if there exists an infinite sequence of distinct edges $(e_0, e_1, e_2, \dots)$ in $\mathcal{E}_\infty$ such that for all $i \in \mathbb{N}$, we have that e_{i-1} and e_i are *dually adjacent*.

Theorem 3 (*Percolation of the Model with Sprinkling*). *Let* $d = k = 2$ *and* $\varepsilon > 0$. *Then,*

$$\mathbb{P}(\mathcal{E}_{\infty,\varepsilon} \textit{ percolates}) = 1 \quad \textit{or} \quad \mathbb{P}(\mathcal{E}_{\infty,\varepsilon} \textit{ percolates dually}) = 1.$$

The proof of Theorem 3 relies fundamentally on the OSSS inequality from [14]. In order to apply it, we need to derive bounds on influences and revealment probabilities. Here, we note that the influences are intimately related to probabilities of pivotal events, which have been studied for geometrically challenging models such as Voronoi percolation [15]. However, the arguments of [15] heavily rely on the FKG inequality, which is not available for our model. While recently, there has been progress in applying the OSSS inequality in models without FKG inequality [11,16], these arguments are highly model dependent and do not extend to the present setting. We will address this challenge through a completely novel stochastic domination property in our setting, see Proposition 7 below. It is when extending the techniques of [11,16] to our model that the finite range of dependence will be crucial.

Conjecture 2 is a percolation result for a model with parameter k, which is similar to the continuous model considered in [17]. Here points of a Poisson point process are connected to their $k \geqslant 1$ nearest neighbors. To study this percolation process, the authors develop a rigorous finite-size criterion, which is then verified “with high confidence” through a simulation.

We next aim to transfer this finite-size approach to our situation, similar adaptations were considered in [18,19]. The key idea is to identify edges e, where it is already possible to say after a finite number of rounds whether $e \in \mathcal{E}_\infty$. More precisely, an edge is after $n \geqslant 1$ rounds

(1) *certainly vacant* if it is reinforced at most $n - 1$ times;
(2) *potentially occupied* if it is reinforced precisely n times;
(3) *certainly occupied* if it is reinforced at least $n + 1$ times. Moreover, if a node is incident to two certainly vacant edges, then the remaining two edges are also n-*certainly occupied*.

The reason for this terminology is that as shown in the proof of Lemma 8, if an edge is certainly occupied, then it is eventually reinforced in every round. Similarly, if an edge is certainly vacant, it is only reinforced a finite number of times.

Given $M > 2n$ we say that a $2M \times M$ rectangle is (M,n)-*open* if there exist a horizontal crossing of the central $((2M-2n)\times(M-2n))$-rectangle and vertical crossings of the left and right $((M-2n)\times(M-2n))$-squares in the central rectangle, all consisting only of edges that are *certainly occupied* after n rounds.

Fig. 1 shows the crossings of the central rectangle. Henceforth, we often think of the $(2M \times M)$-rectangle as being a horizontal edge in a coarse-grained lattice.

Having introduced the notion of (M,n)-openness, $n < M/2$, we define $E_{\mathsf{fs},\infty}(M,n)$ as the event that the coarse-grained edge $(0,0) \to (1,0)$ is (M,n)-open. Then, we show that if $\mathbb{P}(E_{\mathsf{fs},\infty}(M,n)) > 0.8457$, then $\mathbb{P}(\mathcal{E}_\infty$ percolates$) = 1$. The deeper reason behind the value 0.8457 is the recent paper [20, Theorem 1], which shows that any 1-dependent 2D bond percolation model percolates if the marginal probability exceeds this value.

Theorem 4 (*Finite-Size Criterion*). *Let $d = k = 2$ and $1 \leqslant n < M/2$ be such that $\mathbb{P}(E_{\mathsf{fs},\infty}(n)) > 0.8457$. Then,*

$$\mathbb{P}(\mathcal{E}_\infty \text{ percolates}) = 1.$$

Using Monte Carlo simulation with $M = 40$ and $n = 4$, we verify numerically that with a certainty exceeding $1 - 10^{-300}$, the finite-size crossing probability in Theorem 4 indeed exceeds the threshold 0.8457 of 1-dependent percolation, this is explained in Section 6.

2.2. The case $\alpha < \infty$

Next, we deal with the case $\alpha < \infty$. We go through the algorithm for the weight evolution similarly as in the case where $\alpha = \infty$ and again initialize the weights with $W_e(0) = 1$ for every $e \in E$. Then, for each $t \geqslant 1$, we carry out the following updates

(1) At every site $z \in \mathbb{Z}^d$, we select one tuple $\sigma \in E_z^{(k)}$ whose edge weights are increased by 1.
(2) These edges are selected with probability proportional to $\prod_{e\in\sigma} W_e(t-1)^\alpha$.

In the limiting case $\alpha = \infty$, we recover the simplified reinforcement mechanism discussed above. Therefore, it is plausible to extend Conjecture 2 from $\alpha = \infty$ to the case of large but finite α.

Conjecture 5 (*Percolation of Reinforced 2-Out Percolation on $\mathbb{Z}^2$; Finite α*). *Assume that $d = k = 2$. Then, there exists $\alpha_0 > 0$ such that $\mathbb{P}(\mathcal{E}_\infty$ percolates$) = 1$ whenever $\alpha > \alpha_0$.*

As before, we add independent sprinkling on the edges with a sprinkling probability $p_{\mathsf{spr}} > 0$. As in the case of $\alpha = \infty$, we consider then the union $\mathcal{E}_{\infty,\varepsilon} := \mathcal{E}_\infty \cup \mathcal{E}_{\varepsilon,\mathsf{spr}}$ of edges that are reinforced infinitely often together with the sprinkled edges.

Ideally, as a first step, we would like to present a finite-size criterion as in Theorem 4. The key difficulty is that now, it is more complicated to identify edges where we know that they will be reinforced either finitely or infinitely often. For instance, if an edge has weight 1 after 10 rounds, it could still happen (although extremely unlikely) that it will still be reinforced in each of the following 100 rounds.

Hence, we need to introduce a process of corrupted vertices to take into account such effects. More precisely, let $p_* \in [0,1]$ be a fixed (small) corruption probability. Then, each vertex incident to at least one edge of weight at most n after n rounds is declared *n-corrupted* independently with probability p_*. Having introduced the corrupted vertices, we can now extend the concept of certainly occupied edges. More precisely, after $n \geqslant 1$ rounds, we say that an edge that is not incident to a corrupted vertex is

- *certainly vacant* if it is reinforced at most $n - 1$ times;
- *certainly occupied* if it is not certainly vacant and is incident to a node which itself is incident to two certainly vacant edges.

To introduce the finite-size criterion, we now proceed in the same way as in $\alpha = \infty$. Again, to avoid boundary effects, we consider a central rectangle $(2M - 2n) \times (M - 2n)$ rectangle with edges that are certainly occupied after n rounds. Then, we say again that it is *(M,n)-open* if there exist the three types of crossings considered in the case $\alpha = \infty$. Now, we say that a horizontal edge between (x,y) and $(x+1,y)$ is *(M,n)-open* if the rectangle $[Mx - M, Mx + M) \times [My - M/2, My + M/2)$ has this property in the above sense. Similarly, we define the openness of vertical edges. Now, we let $E_{\mathsf{fs},\alpha} = E_{\mathsf{fs},\alpha}(n)$ denote the event that the coarse-grained edges $(0,0) \to (1,0)$ is (M,n)-open with corruption probability p_* set as

$$p_*(\alpha, n) := 1 \wedge \Big((n-1)^\alpha \sum_{j\geqslant n} j^{-\alpha}\Big). \tag{2}$$

We note that the infinite sum cannot be evaluated in closed forms. However, it would be possible to present closed-form bounds through suitable integral bounds. For fixed n, we can (and will) choose α so that p_* is sufficiently small.

Theorem 6 (*Percolation of Reinforced 2-Out Percolation on $\mathbb{Z}^2; \alpha \gg 0$*). *Let $d = k = 2$. Furthermore, assume that $\alpha > 0$ and n are such that $\mathbb{P}(E_{\mathsf{fs},\alpha}(n)) > 0.8457$. Then, $\mathbb{P}(|\mathcal{E}_\infty| = \infty) = 1$.*

We note that the bound on p_* is the result of a comparison with a strongly-reinforced Pólya urn process, which we elaborate on in Lemma 15 below. Again, in Section 6, we provide overwhelming evidence from Monte Carlo simulations that $\mathbb{P}(E_{\mathsf{fs},\alpha,p_*}(n)) > 0.8457$.

Finally, we stress again that the difficulty of analyzing the reinforced model is that it exhibits long-range dependencies and does not satisfy the FKG inequality. This means that the vast majority of the standard percolation arguments break down for this model. Nevertheless, in the case $\alpha = \infty$, the model enjoys the following intriguing domination property.

Proposition 7 (*Domination of $\mathcal{E}_\infty$ When $\alpha = \infty$*). *If $\alpha = \infty$, then the random set $\mathcal{E}_\infty$ stochastically dominates $E \setminus \mathcal{E}_\infty$.*

It is unclear if the stochastic-domination property holds for $\alpha < \infty$. However, we believe that an analog of Proposition 7 could be possible for a clever choice of the finite reinforcement function.

3. Proof of Proposition 7

We will describe a new colored process, equivalent to our original reinforcement process for $\alpha = \infty$, from which the derivation of Proposition 7 will be straightforward. In this colored process, each edge $e \in E(\mathbb{Z}^d)$ has a blue counter $b_t(e)$ and a red counter $r_t(e)$ that increase with time. At the beginning we set $r_0(e) = b_0(e) = 1$ for all edges.

At each step, every vertex chooses d of its adjacent edges, which increase their blue counter by 1, and the other d increase their red counter by 1. The choice is made according to the same reinforcement mechanism as in our regular process — we choose the d edges incident to v with the highest $b(e)$ values and increase their blue counter, breaking ties by choosing uniformly at random, while all the other d incident edges increase their red counter. Set

$$B_n := \{e : b_n(e) > n+1\},\ R_n(e) := \{e : r_n(e) > n+1\},\ \text{ and } U_n := \{e : b_n(e) = n+1\}.$$

Finally set

$$B_\infty := \bigcup_n B_n,\ R_\infty := \bigcup_n R_n \text{ and } U_\infty = \bigcap_n U_n.$$

We make the following simple observations:

Lemma 8.

(1) For every edge e and every $n \geqslant 0$, $b_n(e) + r_n(e) = 2n+2$.
(2) For every $n \geqslant 0$, it holds that $B_n \sqcup R_n \sqcup U_n = \mathbb{Z}^d$, where $\sqcup$ means that the union is disjoint.
(3) B_n, R_n are increasing sequences, U_n is decreasing.
(4) $B_\infty \cup R_\infty \cup U_\infty = \mathbb{Z}^d$.
(5) B_n and R_n have the same distribution, B_∞ and R_∞ have the same distribution.
(6) $B_\infty \cup U_\infty = \{e : b_n(e) \to \infty\}$, $R_\infty \cup U_\infty = \{e : r_n(e) \to \infty\}$.

Before proving Lemma 8, we explain how to deduce Proposition 7, where the main step will be to identify the models in such a way that $\mathcal{E}_\infty = B_\infty \cup U_\infty$. This identification also gives that $R_1 \cap \mathcal{E}_\infty = \emptyset$, which implies that our model is stochastically dominated by the independent k-out model with $k = d$.

Proof of Proposition 7. The connection to our reinforcement model comes by looking only at the blue counters. Under this identification we get that B_n is the set of all certainly occupied edges and R_n is the set of certainly vacant edges at time n. Also $\mathcal{E}_\infty = B_\infty \cup U_\infty$. The claim now follows directly from clause (4) of Lemma 8. □

Now, we give the proof of Lemma 8.

Proof of Lemma 8.

(1) The assertion follows from the fact that each edge gets either a red counter or a blue counter from each of its endpoints.
(2) Clause (1) implies that R_n, B_n, U_n are a disjoint partition of $\mathbb{E}(\mathbb{Z}^d)$.
(3) We will show this for B_n, and R_n follows by symmetry. Take some edge $e = (v_+, v_-) \in B_n$ and consider the first time $m < n$ for which $e \in B_{m+1}$. Then at time m, the edge e got a blue counter from both its endpoints. Therefore, $b_m(e)$ was greater or equal to the median of the blue counters of the edges incident to v_+ (at time m). Since the blue counter of any edge can increase by at most 2 at every step, it follows that $b_{m+1}(e)$ is strictly larger than the d lowest blue counters of edges incident to v_+ (and similarly for v_-). As a consequence, the edge e will keep being chosen by both v_+ and v_- for all rounds $k \geqslant m+1$. This implies that $e \in B_k$ for all $k \geqslant m+1$. Finally, the fact that U_n is decreasing now follows from clause (2).
(4) Clause (3) implies that the unions defining B_∞ and R_∞ are increasing, and that the intersection defining U_∞ is decreasing. Therefore, the statement follows from clause (2).
(5) By clause (1), choosing the edges with the highest blue counter is the same as not choosing those with the highest red counter. This makes the definition of the process symmetric w.r.t. flipping the colors.
(6) This follows from a similar argument to the monotonicity clause (2). □

4. Proof of Theorem 3

In this section, we prove Theorem 3. That is, we show that percolation occurs in the dual or original lattice after adding arbitrarily sparse independent sprinkling. In (1), we defined the set $\mathcal{E}_\infty$ of edges that are eventually reinforced in every round. One challenge of dealing with this set is that it has an unbounded range of dependencies. To overcome this difficulty, we now introduce an approximated version $\mathcal{E}_N$ of the set $\mathcal{E}_\infty$ after a finite number of $N \geqslant 1$ rounds. We say that an edge e is *N-potentially occupied* (and write $e \in \mathcal{E}_N$) given the configuration after N rounds if it is still possible with positive probability that $e \in \mathcal{E}_\infty$. Further, we call an edge e *strictly N-potentially occupied* if $e \in \mathcal{E}_N \setminus \mathcal{E}_\infty$. The main idea is to show percolation for each of the approximated sets $\mathcal{E}_N$ and then to use a stochastic domination result to control the difference $\mathcal{E}_N \setminus \mathcal{E}_\infty$.

This strategy is similar to that used in [15], where the authors prove a similar result for Voronoi percolation. The key similarity is that both Voronoi percolation and our model are spatial percolation models with exponential decay of correlations.

To summarize, the proof of Theorem 3 relies on two central results, namely Propositions 9 and 10 below.

Proposition 9 (*Sharp Threshold for Approximation*)**.** *Fix $N \geqslant 1$ and let $p_{\mathsf{spr}}(N)$ denote the critical sprinkling probability in the N-approximated model. Then, for $p < p_{\mathsf{spr}}(N)$, the diameter of the clusters has exponentially decaying tails.*

Let S_N denote the collection of sites that are incident to at least one edge that is strictly N-potentially occupied.

Proposition 10 (*Stochastic Domination*)**.** *Let $N \geqslant 2$. Then, S_N is dominated by a Bernoulli site percolation process with marginal probability $\delta_N = 6^{-N/2}$.*

We first discuss how to prove Theorem 3 subject to the validity of Propositions 9 and 10. Then, we prove these propositions in the subsequent subsections.

Proof of Theorem 3. Proposition 10 gives a stochastic domination of S_N by a Bernoulli site percolation process with marginal probability δ_N. We first claim that from this result, we can conclude that the edges incident to sites in S_N are stochastically dominated by a Bernoulli bond percolation process with probability $\sqrt{\delta_N}$. Indeed, consider an arbitrary finite edge set $F \subseteq E$. Let V_F denote the set of left or lower endpoints of the edges in F. Then, by Proposition 10,

$$\mathbb{P}(F \subseteq S_N) \leqslant \delta_N^{|V_F|} \leqslant \delta_N^{|F|/2},$$

thereby proving the claimed stochastic domination. A consequence is that $\mathcal{E}_{N(\varepsilon)} \subseteq \mathcal{E}_\infty \cup \mathcal{E}_{\varepsilon,\mathsf{spr}}$ provided that $N(\varepsilon)$ is so large that $\delta_N^{1/2} \leqslant \varepsilon$. We also note that the independent superposition of two Bernoulli bond percolation processes with parameter $\varepsilon > 0$ is again a Bernoulli bond percolation process, whose parameter is then $1 - (1-\varepsilon)(1-\varepsilon) = 2\varepsilon - \varepsilon^2$.

Now, to derive a contradiction, assume that for some $\varepsilon > 0$

$$\mathbb{P}(\mathcal{E}_\infty \cup \mathcal{E}_{2\varepsilon-\varepsilon^2,\mathsf{spr}} \text{ percolates}) = \mathbb{P}(\mathcal{E}_\infty \cup \mathcal{E}_{2\varepsilon-\varepsilon^2,\mathsf{spr}} \text{ percolates dually}) = 0.$$

Then, Proposition 10 implies that,

$$\mathbb{P}(\mathcal{E}_{N(\varepsilon)} \cup \mathcal{E}_{\varepsilon,\mathsf{spr}} \text{ percolates}) = \mathbb{P}(\mathcal{E}_{N(\varepsilon)} \cup \mathcal{E}_{\varepsilon,\mathsf{spr}} \text{ percolates dually}) = 0.$$

In other words, ε is a subcritical sprinkling probability both with respect to original and dual connectivities of the $N(\varepsilon)$-approximated model. In particular, $\mathcal{E}_{N(\varepsilon)} = \mathcal{E}_{N(\varepsilon)} \cup \mathcal{E}_{0,\mathsf{spr}}$ is in the strictly subcritical regime, where Proposition 9 guarantees the exponential tail decay of the cluster sizes. In particular, by Proposition 9, both the sizes of the clusters with respect to the original connectivities and with respect to the dual connectivities have exponentially decaying tails. Consequently, the probability to have left–right vacant crossings of large squares tends to 1 as the side length of the square tends to infinity.

Now, by Proposition 7, the occupied edges dominate the vacant ones in $\mathcal{E}$, thereby contradicting the exponential decay of the cluster sizes. □

We will prove Propositions 9 and 10 in Sections 4.1 and 4.2 below. For both results, it is convenient to describe more precisely how our model can be constructed as a factor of a model with independent inputs. First, we introduce an iid sequence $\{U_e^{p_{\mathsf{spr}}}\}_{e\in E}$ that encodes the sprinkling of edges. More precisely, each $U_e^{p_{\mathsf{spr}}}$ is uniformly distributed in $[0,1]$ and the set of sprinkled edges is then given by

$$X := \{e \in E : U_e^{\mathsf{spr}} \leqslant p_{\mathsf{spr}}\},$$

Second, we attach to each site $z \in \mathbb{Z}^2$ a sequence of iid decision variables $U_z^{(1)}, U_z^{(2)}, \ldots$. All variables are independent for different values of z and also independent of the sprinkling variables X.

Given the configuration of edge reinforcements up to iteration n, the variable $U_z^{(n)}$ is used to decide which of the edges are reinforced by z in iteration n. We write

$$Y_z := \{U_z^{(k)}\}_{k\geqslant 1}$$

for the sequence of all decision variables at z. Then, we write $Y := \{Y_z\}_{z\in\mathbb{Z}^2}$ for the collection of all Y_z. Hence, the entire randomness of the model can be encoded in the pair

$$Z := (X, Y).$$

4.1. Proof of sharp thresholds for approximation – Proposition 9

We will deduce Proposition 9 from a differential inequality in the spirit of [10, Lemma 1.7]. We also write $\theta_n(p_{\mathsf{spr}}(N))$ for the probability that in the sprinkled model the connected percolation component of the origin has ℓ_∞-diameter exceeding n, which we denote as $o \leftrightsquigarrow \partial B_n$.

To make the argumentation self-contained, we first give a general introduction to the OSSS inequality (after O'Donnell, Saks, Schramm and Servedio, [14]). In the presentation, we follow [21]. Let $X = (X_i)_{i\in I}$ be a finite collection of independent random variables defined on some Borel spaces Ω_i and let $f : \prod_{i\in I} \Omega_i \to \{0,1\}$ be measurable. Then,

$$\mathsf{Inf}_i(f) := \mathbb{P}(f(W) \neq f(\tilde{W})),$$

denotes the *influence* of the ith coordinate, where $\tilde{W}$ results from W by resampling the ith coordinate. The second ingredient to the OSSS inequality is an algorithm T revealing the entries of W. This is done sequentially where the index i of the next variable W_i to be revealed is chosen based on the already revealed values. The algorithm stops once the value of $f(W)$ is determined by the already revealed values. Now,

$$\delta_i(T) := \mathbb{P}(T \text{ reveals } W_i)$$

denotes the probability that the algorithm T reveals W_i. Then, the OSSS inequality states that

$$\mathsf{Var}(f(W)) \leqslant \sum_{i\in I} \delta_i(T)\mathsf{Inf}_i(f). \tag{3}$$

Proposition 11 (*Differential Inequality*). *There exists $c_{\mathsf{Diff}} > 0$ such that for every $n \geqslant 1$ and $p_{\mathsf{spr}} > 0$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}p_{\mathsf{spr}}}\theta_n(p_{\mathsf{spr}}) \geqslant c_{\mathsf{Diff}} \frac{n}{\sum_{s\leqslant n}\theta_s(p_{\mathsf{spr}})}\theta_n(p_{\mathsf{spr}})(1-\theta_n(p_{\mathsf{spr}})).$$

We note that once Proposition 11 is established, the derivation of the sharp-threshold property asserted in Proposition 9 is standard. However, to make our presentation self-contained, we repeat the derivation here. First, we reproduce a key technical lemma from [22, Lemma 3.7].

Lemma 12 (*[22]*). *Consider a converging sequence of differentiable functions $f_n : [0,p_0] \longrightarrow [0,M]$ which are increasing in x and satisfy*

$$f_n' \geqslant \frac{n}{\Sigma_n} f_n$$

for all $n \geqslant 1$, where $\Sigma_n = \sum_{k=0}^{n-1} f_k$. Then, there exists $p_1 \in [0,p_0]$ such that

P1 *For any $p < p_1$, there exists $c_p > 0$ such that for any n large enough, $f_n(p) \leqslant \exp(-c_p n)$.*
P2 *For any $p > p_1$, $f = \lim_{n\to\infty} f_n$ satisfies $f(p) \geqslant p - p_1$.*

Next, we follow the arguments in [22] to deduce Proposition 9 from Proposition 11 and Lemma 12.

Proof of Proposition 9. We fix first $p_0 \in (p_{\mathsf{spr}}(N))$ and note that $\inf_{p\leqslant p_0} 1-\theta_1(p) > 0$. Then, Proposition 11 shows that we can apply Lemma 12 to $f_n = (c_{\mathsf{Diff}}(1-\theta_1(p_0)))^{-1}\theta_n$. Hence, we deduce that there exists $p_{c,1} \in [0,p_0]$ such that

(1) For any $p < p_{c,1}$, we have $\limsup_{n\to\infty} \log\theta_n(p) < 0$.
(2) For any $p > p_{c,1}$, we have $\theta(p) \geqslant p - p_{c,1}$.

Combining these two items shows that $p_{c,1} = p_{\mathsf{spr}}(N)$ and therefore implies the asserted sharp threshold property. □

We apply the OSSS inequality to $Z = (X,Y)$, which is indexed by $z \in \mathbb{Z}^d$ and $e \in E(\mathbb{Z}^d)$. Hence, inequality (3) applied to an algorithm T determining $f_n := \mathbb{1}\{o \leftrightsquigarrow \partial B_n\}$ gives that

$$\theta_n(p_{\mathsf{spr}})(1-\theta_n(p_{\mathsf{spr}})) = \mathsf{Var}(f_n(Z)) \leqslant \sum_{z\in\mathbb{Z}^d} \delta_z(T)\mathsf{Inf}_z^Y + \sum_{e\in E(\mathbb{Z}^d)} \delta_e(T)\mathsf{Inf}_e^X. \tag{4}$$

Henceforth, we rely on a specific algorithm T from [10], which involves an additional randomization through a uniform integer $U \in \{1,\dots,n\}$. For the convenience of the reader, and to make the manuscript self-contained, we briefly recall the idea behind this algorithm. The idea is to explore the clusters by starting from ∂B_U. More precisely, we proceed as follows:

(1) Reveal the value of $U_e^{p_{\mathsf{spr}}}$ for all edges incident to ∂B_U. Also reveal Y_z for all $z \in \mathbb{Z}^2$ at ℓ_∞-distance at most N from a point in ∂B_U.
(2) Suppose that the values of $X_{e_1},\dots,X_{e_k}$ and of $Y_{z_1},\dots,Y_{z_\ell}$ have already been revealed at the start of iteration t of the algorithm. Let C_t be the union of all connected components that are revealed by this iteration. Then, according to some arbitrary rule, we pick out some unrevealed site z_0 that is incident to an edge in C_t. We then reveal the values of X_e for any edge adjacent to z_0. We also reveal the values of Y_z for all $z \in \mathbb{Z}^2$ at ℓ_∞-distance at most N from z_0.

The proof of Proposition 11 relies on the following two central auxiliary results (Lemmas 13 and 14 below) concerning a comparison of the two influences, and a bound on the revealment probabilities, respectively. Let $B_N(z)$ denote the box around $z \in \mathbb{Z}^2$ with ℓ_∞-radius $N \geqslant 1$.

Lemma 13 (*Comparison of Influences*). *Let $N \geqslant 1$. Then, there is $c_1 = c_1(d,N) > 0$ such that for every $z \in \mathbb{Z}^2$, $\mathsf{Inf}_z^Y \leqslant c_1 \sum_{e\subseteq B_N(z)} \mathsf{Inf}_e^X$.*

Lemma 14 (*Bound on Revealment Probabilities*). *Let $N \geqslant 1$. Then, there is $c_2 = c_2(d,N) > 0$ with the following property. Let T be the randomized algorithm starting the exploration from ∂B_U, with U uniform in $\{1,\dots,n\}$. Then,*

$$\Big(\sup_{e\in E}\delta_e(T)\Big) \vee \Big(\sup_{z\in\mathbb{Z}^2}\delta_z(T)\Big) \leqslant \frac{c_2\sum_{s\leqslant n}\theta_s(p_{\mathsf{spr}})}{n}$$

Before proving Lemmas 13 and 14, we explain how to conclude the proof of Proposition 11.

Proof of Proposition 11. First, by the OSSS inequality (4) and Lemmas 13 and 14, we obtain that

$$\theta_n(p_{\mathsf{spr}})(1-\theta_n(p_{\mathsf{spr}})) \leqslant c_1 c_2 |B_N(o)| n^{-1} \sum_{s\leqslant n} \theta_s(p_{\mathsf{spr}}) \sum_{e\in E(\mathbb{Z}^d)} \mathsf{Inf}_e^X.$$

Now, Russo's formula gives that

$$\sum_{e\in E(\mathbb{Z}^d)} \mathsf{Inf}_e^X = \frac{\mathrm{d}}{\mathrm{d}p_{\mathsf{spr}}} \theta_n(p_{\mathsf{spr}}),$$

thereby concluding the proof. □

Hence, we now prove the auxiliary results Lemmas 13 and 14, starting with Lemma 13. The idea is similar to [15], where a comparison between different forms of pivotality also plays a crucial role. However, our situation is a bit simpler in the sense that N-approximated model has bounded dependence range N.

Proof of Lemma 13. For $\sigma \in \{0,1\}$, we introduce the event $E_{\mathsf{coarse},\sigma} := \{f_n(Z_\sigma) = \sigma\}$, where Z_σ denotes the configuration obtained from Z by setting $1 - U_e^{\mathsf{spr}} := \sigma$ for every $e \in E \cap B_N(z)$.

Now, using that the N-approximated model has dependence range N, we see that if the resampling of Y at z changes the value of f_n, then $E_{\mathsf{coarse}} := E_{\mathsf{coarse},0} \cap E_{\mathsf{coarse},1}$ must occur. In other words, $\mathsf{Inf}_z^Y \leqslant \mathbb{P}(E_{\mathsf{coarse}})$. Hence, it suffices to relate the probability of E_{coarse} with the sum of the influences Inf_e^X for $e \in B_N(z)$.

To achieve this goal, we write X^* for the configuration of the model with the sprinkled edges X_e replaced by independent copies for every $e \in B_N(z)$. Then, we let $E_{\mathsf{fine},-}$ denote the event that in X none of the edges in $B_N(z)$ is sprinkled. We also let $E_{\mathsf{fine},+}$ denote the event that in the resampled configuration all edges in $B_N(z)$ are sprinkled. Then, by independence and the definition of E_{coarse},

$$c\mathbb{P}(E_{\mathsf{coarse}}) \leqslant \mathbb{P}\big(E_{\mathsf{coarse}} \cap E_{\mathsf{fine},-} \cap E_{\mathsf{fine},+}\big) \leqslant \mathbb{P}\big(f_n(X,Y) \neq f_n(X^*,Y)\big)$$

for some $c = c(N) > 0$. Noting that the right-hand side is bounded above by $\sum_{e\subseteq B_N(z)} \mathsf{Inf}_e^X$ concludes the proof. □

Hence, it remains to establish the revealment bounds asserted in Lemma 14. Here, we proceed similarly as in [10], noting that again the finite-range property of our model simplifies the argument.

Proof of Lemma 14. We only bound the revealment probability $\delta_z(T)$, as the bound on $\delta_e(T)$ is easier. The key observation is that if the randomized algorithm starts from exploring ∂B_U with $U = m$ for some $m \leqslant n$, then the following holds. If T reveals the state of Y_z for some site $z \in \mathbb{Z}^2$, then $\partial B_m \leftrightsquigarrow B_N(z)$. Therefore,

$$\mathbb{P}(T \text{ reveals } z) \leqslant \mathbb{P}\big(\partial B_m \leftrightsquigarrow B_N(z)\big) \leqslant \theta_{|m-|z|_\infty|-N},$$

where we use the convention $\theta_k = 1$ for $k \leqslant 0$. Thus, picking $m \in \{1,\ldots,n\}$ uniformly at random, we obtain that

$$\delta_z(T) \leqslant \frac{1}{n}\Big(2N + \sum_{m\leqslant n} \theta_m\Big) \leqslant \frac{c}{n} \sum_{m\leqslant n} \theta_m,$$

for a suitable $c = c(N) > 0$. This concludes the proof. □

4.2. *Proof of stochastic domination of approximation – Proposition 10*

To prove Proposition 10, we need to describe more precisely how the decision variables $\{U_z^{(k)}\}_{k\geqslant 1}$ determine the state of the edges.

Proof of Proposition 10. Our construction depends on the parity of k, and we first discuss the case of odd k. Henceforth, we implicitly consider $\mathbb{Z}^2$ to be endowed with a checkerboard pattern so that we can speak of black and white sites.

For a black site $z \in \mathbb{Z}^2$, we let $U_z^{(k)}$ encode in some arbitrary way a random selection of two of the highest-weight edges incident to z. Next, to determine the reinforcements at a white site $z' \in \mathbb{Z}^2$, we work conditioned on the edge weights up to iteration k and also on the decision variables $U_z^{(k)}$ for all black sites $z \in \mathbb{Z}^2$. We note that the probability space Ω of selecting two highest-weight edges incident to z' consists of $\binom{4}{2} = 6$ elements.

First, consider the case where, in step k, at least 2 of the black neighbors of z' reinforced their corresponding edge to the white vertex z'. Then, there is a probability of at least $\delta = 1/6$ that, in step k, the vertex z' selects the same two edges, which are therefore reinforced infinitely often, see clause (3) in Lemma 8.

Second, consider the case when there are at least 2 neighboring black vertices of z' that did not reinforce their corresponding edge to the white vertex z'. Then, again with probability of at least $\delta = 1/6$, in step k, the vertex z' selects the other two edges. Then, again by Lemma 8, these two edges are never reinforced again. Therefore, the vertex z' is no longer incident to any strictly N-potentially occupied edges.

Writing $Y_-^{(k)}$ for the collection of decision variables of all sites up to step $k-1$ together with the decision variables at the black sites at step k, we conclude that there exists a state $\omega_*(Y_-^{(k)}) \in \Omega$ such that almost surely, (i) $\mathbb{P}(U_{z'}^{(k)} = \omega_*(Y_-^{(k)}) \mid Y_-^{(k)}) \geqslant \delta$ and (ii) if $U_{z'}^{(k)} = \omega_*(Y_-^{(k)})$, then after step k, all potentially reinforced edges incident to z' are in fact reinforced infinitely often.

Taking into account these observations, we now present a two-step construction of the variables $U^{(k)}_{z'}$. For this, let $\{W^{(k)}_z\}_{k\geqslant 1, z\in\mathbb{Z}^2}$ be an iid sequence of Bernoulli random variables with parameter δ. If $W^{(k)}_z = 1$, then we let $U^{(k)}_z$ be the state $\omega_*(Y^{(k)}_-)$. Otherwise, if $W^{(k)}_z = 0$, then we let $U^{(k)}_z$ be the state $\omega_*(Y^{(k)}_-)$ with probability $q_0(Y^{(k)}_-) := \big(\mathbb{P}(U^{(k)}_z = \omega_*(Y^{(k)}_-) \mid Y^{(k)}_-) - \delta\big)/(1-\delta)$, and with probability $1 - q_0(Y^{(k)}_-)$ the random variable $U^{(k)}_z$ is sampled according to the conditional distribution $\mathcal{L}\big(U^{(k)}_z | U^{(k)}_z \neq \omega_*(Y^{(k)}_-), Y^{(k)}_-\big)$.

This describes the construction for odd steps k. For even k, we proceed in precisely the same manner except that the roles of black and white sites are interchanged. In particular, if an edge e incident to a black site z is contained in $\mathcal{E} \setminus \mathcal{E}_N$, then $W^{(1)}_z = W^{(3)}_z = \cdots = W^{(N)}_z = 0$. This proves the asserted domination. □

5. Proofs of Theorems 4 and 6

We deal separately with the cases $\alpha = \infty$ and $\alpha < \infty$. We start with $\alpha = \infty$, where we use a result for 1-dependent percolation with sufficiently high marginal probabilities from [20]. We apply this result to the coarse-grained model. The key observation is that if the coarse-grained edge $(0,0) \to (1,0)$ is n-open, then we are guaranteed three crossings of infinitely-reinforced edges in the rectangle $[0, 2M) \times [0, M)$. Namely,

(1) a horizontal crossing of the $((2M-2n)\times(M-2n))$-rectangle;
(2) vertical crossings of the left and of the right $((M-2n)\times(M-2n))$-squares inside the central rectangle.

Moreover, the existence of such crossings depends only on coarse-grained edges sharing at least one of the end points. Hence, we conclude from planarity that any path of n-open coarse grained edges gives rise to a path of infinitely-reinforced edges. Hence, it suffices to establish the percolation of the coarse-grained model, which we do now.

Proof of Theorem 4; $\alpha = \infty$. The collection of n-open edges defines a 1-dependent family. Now, by our assumption on the finite-size criterion, a coarse-grained edge is open with marginal probability exceeding 0.8457. Now, we can conclude the proof by invoking [20, Theorem 1], which states that any 1-dependent site percolation model with marginal probability exceeding 0.8457 percolates. □

We now turn to the case $\alpha < \infty$. We first argue that percolation of the n-open edges in the coarse-grained model implies percolation of the infinitely-reinforced model in the original model.

Lemma 15 (*Stochastic Domination Property of Certainly Occupied Edges*). *Let $n, \alpha > 1$ and assume*

$$p_* > (n-1)^\alpha \sum_{j\geqslant n} j^{-\alpha}.$$

Then, the process of edges that are reinforced only finitely often in the original model is stochastically dominated by the process of edges that are not certainly occupied.

Proof. The idea of the proof is as follows. If after n rounds an edge e has weight at most n, then for large α, it is highly likely never to be chosen again. Then, loosely speaking, the vertex corruption is used to account for the possibility of this exceptional event.

To make this precise, we let $E_e(n)$ denote the event that the edge e with weight at most $n-1$ in round n is reinforced by at least one of its incident vertices in some round $j \geqslant n$. Then, we claim that $(\mathbb{1}\{E_e(n)\})_e$ is stochastically dominated by a directed Bernoulli bond percolation process with probability p_*. To achieve this goal, we note that at the beginning of round $j \geqslant n$ there are at least 2 high-weight edges, i.e., edges of weight at least j. Hence, the probability that one of the low-weight edges e is reinforced from one of its incident vertices is at most

$$2\frac{(n-1)^\alpha}{2j^\alpha} = \Big(\frac{n-1}{j}\Big)^\alpha.$$

We stress that this upper bound holds irrespective of the weight evolution at any of the other directed edges. Hence, the union bound shows that, as asserted, the edge process $(\mathbb{1}\{E_e(n)\})_{e\in E}$ is dominated by a Bernoulli bond process with probability p_*.

Proof of Theorem 6; $\alpha < \infty$. After the reduction step from Lemma 15, the proof is very similar to that in the case $\alpha = \infty$. Arguing as in the case $\alpha = \infty$, Lemma 15 guarantees that any path of (M, n)-open coarse grained edges gives rise to a path of infinitely-reinforced edges. Moreover, we conclude from the condition (2) that the probability for a coarse-grained edge to be n-open exceeds 0.8457. The theorem then follows, as for $\alpha = \infty$, by invoking [20, Theorem 1]. □

6. Numerical evidence for the finite-size criteria

Now, we give numerical evidence that the finite-size criteria from Theorems 4 and 6 are satisfied.

We start with Theorem 4, i.e., where $\alpha = \infty$. Here, we carried out $N = 10{,}000$ simulations of the process in a (80×40)-rectangle with periodic boundary conditions. In $N_0 = 9{,}553$ of these simulations we found a horizontal crossing of the central (72×36)-rectangle of the nodes that are certainly occupied after $n = 4$ steps. Using that the Monte Carlo variance is $\sqrt{0.0447 \cdot 0.9553}$, this implies that with a certainty exceeding $1 - 10^{-300}$, the actual crossing probability is above the threshold of 0.8457 for 1-dependent percolation

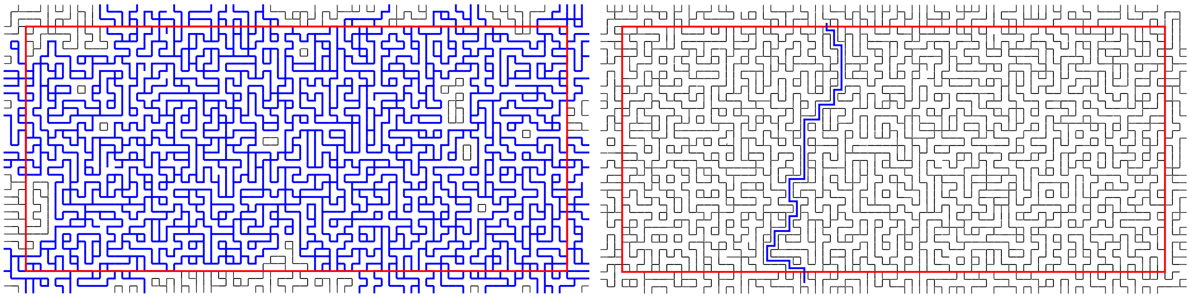

Fig. 2. The left panel shows an example for a crossing with the largest component in color. The right panel shows an example of a non-crossing with a dual vertical crossing in color. In both examples, we have $\alpha = \infty$.

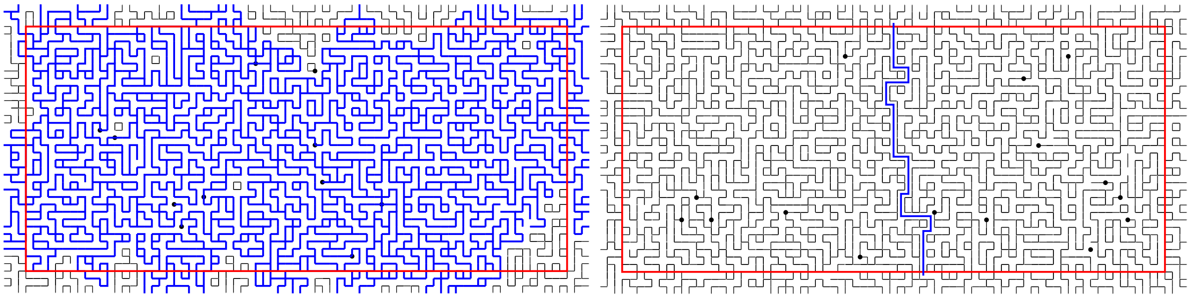

Fig. 3. The left panel shows an example for a crossing with the largest component in color. The right panel shows an example of a non-crossing with a dual vertical crossing in color. In both examples, we have $\alpha = 15$. The circled vertices are the corrupted ones.

The simulations took 22 min 17 s on a 13th Gen Intel Core i5-1345U. Fig. 2 illustrates examples for crossing and non-crossing realizations.

Finally, we discuss Theorem 6, i.e., where $\alpha < \infty$. Our simulations concern $\alpha = 15$. The basic setting is the same as above, namely $N = 10{,}000$ simulations on a (80×40)-rectangle with periodic boundary conditions. In $N_0 = 9,512$ of these simulations we found percolation of the central (72×36)-rectangle of the nodes that are certainly occupied after $n = 4$ steps. Again, with a certainty exceeding $1 - 10^{-300}$, the actual crossing probability is above the threshold of 0.8457 for 1-dependent percolation. The simulations took 1 h 42 min 52 s on a 13th Gen Intel Core i5-1345U. Note that additional time is needed for sampling from a discrete probability distribution and for dealing with the corruption. Fig. 3 illustrates examples for crossing and non-crossing realizations.

Declaration of competing interest

None.

Acknowledgments

We thank the anonymous referees for the careful reading of the manuscript. Their comments helped to improve the quality of the presentation substantially. This work was initiated during the workshop *Adaptive Learning and Opinion Dynamics in Social Networks* at Bar-Ilan University in Tel Aviv. This research was supported by the VILLUM FONDEN under grant agreement no. VIL69126, and by ISF, United States grant #957/20.

References

[1] N. Kalisman, G. Silberberg, H. Markram, The neocortical microcircuit as a tabula rasa, Proc. Natl. Acad. Sci. USA 102 (3) (2005) 880–885.

[2] M. Benaïm, I. Benjamini, J. Chen, Y. Lima, A generalized Pólya's urn with graph based interactions, Random Structures Algorithms 46 (4) (2015) 614–634.

[3] J. Chen, C. Lucas, A generalized Pólya's urn with graph based interactions: convergence at linearity, Electron. Commun. Probab. 19 (67) (2014) 13.

[4] R.v.d. Hofstad, M. Holmes, A. Kuznetsov, W. Ruszel, Strongly reinforced Pólya urns with graph-based competition, Ann. Appl. Probab. 26 (4) (2016) 2494–2539.

[5] C. Hirsch, M. Holmes, V. Kleptsyn, Absence of WARM percolation in the very strong reinforcement regime, Ann. Appl. Probab. 31 (1) (2021) 199–217.

[6] M. Heydenreich, C. Hirsch, A spatial small-world graph arising from activity-based reinforcement, in: K. Avrachenkov, P. Prałat, N. Ye (Eds.), Algorithms and Models for the Web Graph, in: Lecture Notes in Comput. Sci., vol. 8305, Springer, Cham, 2019, pp. 102–114.

[7] M. Heydenreich, C. Hirsch, Extremal linkage networks, Extremes 25 (2022) 229–255.

[8] D. Kious, C. Mailler, B. Schapira, Finding geodesics on graphs using reinforcement learning, Ann. Appl. Probab. 32 (5) (2022) 3889–3929.

[9] D. Kious, C. Mailler, B. Schapira, The trace-reinforced ants process does not find shortest paths, J. Éc. Polytech. Math. 9 (2022) 505–536.
[10] H. Duminil-Copin, A. Raoufi, V. Tassion, Exponential decay of connection probabilities for subcritical Voronoi percolation in $\mathbb{R}^d$, Probab. Theory Related Fields 173 (1–2) (2019) 479–490.
[11] C. Hirsch, B. Jahnel, S. Muirhead, Sharp phase transition for cox percolation, Elect. Comm. Probab. 27 (2022) 1–13.
[12] S. Muirhead, A. Rivera, H. Vanneuville, L. Köhler-Schindler, The phase transition for planar Gaussian percolation models without FKG, Ann. Probab. 51 (5) (2023) 1785–1829.
[13] B. Jahnel, J. Köppl, B. Lodewijks, A. Tóbiás, Percolation in lattice k-neighbor graphs, J. Appl. Probab. (2025, to appear).
[14] R. O'Donnell, M. Saks, O. Schramm, R.A. Servedio, Every decision tree has an influential variable, in: 46th Annual IEEE Symposium on Foundations of Computer Science, FOCS'05, IEEE, 2005, pp. 31–39.
[15] B. Dembin, F. Severo, Supercritical sharpness for Voronoi percolation, Probab. Theory Related Fields (2025).
[16] S. Muirhead, Percolation of strongly correlated Gaussian fields II. Sharpness of the phase transition, Ann. Probab. 52 (3) (2024) 838–881.
[17] P. Balister, B. Bollobás, A. Sarkar, M. Walters, Connectivity of random k-nearest-neighbour graphs, Adv. in Appl. Probab. 37 (1) (2005) 1–24.
[18] P. Balister, B. Bollobás, M. Walters, Continuum percolation with steps in the square or the disc, Random Structures Algorithms 26 (4) (2005) 392–403.
[19] B. Bollabás, A. Stacey, Approximate upper bounds for the critical probability of oriented percolation in two dimensions based on rapidly mixing Markov chains, J. Appl. Probab. 34 (4) (1997) 859–867.
[20] P. Balister, T. Johnston, M. Savery, A. Scott, Improved bounds for 1-independent percolation on $\mathbb{Z}^n$, Elect. J. Probab. 30 (2025) 1–26, paper no. 83.
[21] H. Duminil-Copin, A. Raoufi, V. Tassion, Subcritical phase of d-dimensional Poisson-Boolean percolation and its vacant set, Ann. Henri Lebesgue 3 (2020) 677–700.
[22] H. Duminil-Copin, Sharp threshold phenomena in statistical physics, Jpn. J. Math. 14 (1) (2019) 1–25.