



Cutoff on Trees is Rare

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Abstract

We study the simple random walk on trees and give estimates on the mixing and relaxation times. Relying on a seminal result by Basu, Hermon and Peres characterizing cutoff on trees, we give geometric criteria that are easy to verify and allow to determine whether the cutoff phenomenon occurs. We provide a general characterization of families of trees with cutoff, and show how our criteria can be used to prove the absence of cutoff for several classes of trees, including spherically symmetric trees, Galton–Watson trees of a fixed height, and sequences of random trees converging to the Brownian continuum random tree.

Keywords Random walk · Spectral gap · Mixing time · Cutoff phenomenon

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1 Introduction and Main Results

In recent breakthroughs, random walks on various families of random graphs were shown to exhibit the cutoff phenomenon, which is a sharp transition in the convergence to equilibrium [8, 9, 24]. Motivated by these results, we address the question, ‘For which families of (random) trees does the simple random walk have/exhibit cutoff?’ First natural candidates to study are the simple random walk on Galton–Watson trees,

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when the offspring distribution μ has mean $m \geq 1$ and finite variance $\sigma^2 \in (0, \infty)$, and the simple random walk on spherically symmetric trees. Our results on cutoff for these families of trees are summarized as follows.

Theorem 1.1 *The simple random walk does (almost surely) not exhibit cutoff if the underlying trees $(G_n)_{n \geq 1}$ come from one of the following three constructions:*

- *A supercritical Galton–Watson tree ($m > 1$) conditioned on survival and truncated at generation n .*
- *A family of Galton–Watson trees conditioned on having n sites.*
- *A spherically symmetric tree of bounded degree truncated at generation n .*

We refer to Sect. 5 for precise formal definitions and statements. Intuitively, Theorem 1.1 is due to the fact that Galton–Watson trees are typically *short and fat*, as shown by Addario–Berry [1], while spherically symmetric trees are *balanced* in the number of sites by construction. Our main contribution is to show that trees are typically *long, thin and imbalanced* when the simple random walk exhibits cutoff; see Theorems 1.3–1.6. Our arguments rely on a characterization of the cutoff phenomenon on trees that was recently proven by Basu, Hermon and Peres [7]. With their characterization at hand, we provide simple geometric criteria for the occurrence of cutoff, which are easy to verify for specific examples of trees. The criteria in Theorems 1.3 and 1.4 are sharp in the sense that we can either exclude the occurrence of cutoff or slightly modify the trees via a retraction such that the simple random walk exhibits cutoff. This allows us to construct a very simple example for a family of trees on which the simple random walk has cutoff; see Corollary 1.5. Previously, a first example of trees with cutoff was obtained in [31].

We now give a brief introduction to the theory of mixing times and refer to Levin et al. [22] for a more comprehensive treatment. Let $G = (V, E, o)$ be a finite, rooted graph. Let $(\eta_t)_{t \geq 0}$ denote the (continuous-time) simple random walk on G in the variable speed model, i.e., the walker moves along every edge of the tree at rate 1. For two probability measures μ, ν on V , let their *total-variation distance* be

$$\|\mu - \nu\|_{\text{TV}} := \max_{A \subseteq V} |\mu(A) - \nu(A)|. \quad (1)$$

The ε -mixing time of $(\eta_t)_{t \geq 0}$ is now given, for $\varepsilon \in (0, 1)$, by

$$t_{\text{mix}}(\varepsilon) := \inf \left\{ t \geq 0 : \max_{x \in V} \|\mathbb{P}(\eta_t \in \cdot | \eta_0 = x) - \pi\|_{\text{TV}} \leq \varepsilon \right\} \quad (2)$$

where π is the uniform distribution on V . For a sequence of graphs $(G_n)_{n \geq 1}$, let $(t_{\text{mix}}^n(\varepsilon))_{n \geq 1}$ denote the collection of mixing times of the random walks on (G_n) . We say that the family of random walks on (G_n) exhibits *cutoff* if for any $\varepsilon \in (0, 1)$

$$\lim_{n \rightarrow \infty} \frac{t_{\text{mix}}^n(1 - \varepsilon)}{t_{\text{mix}}^n(\varepsilon)} = 1. \quad (3)$$

The cutoff phenomenon was first verified in [15], and obtained its name in the seminal paper of Aldous and Diaconis [6]. Ever since there has been a lot of activity toward

showing that specific examples of Markov chains exhibit cutoff; see, for example, [8, 9, 24]. The ultimate goal is to produce a necessary and sufficient condition for cutoff. It turns out that cutoff has deep connections to the spectral properties of the underlying random walk; see the results [17, 23, 27, 28] of Miclo and others for a discussion of the spectrum of random walks on trees, and the work of Jonasson [20] for a collection of results on the spectrum of random walks on random graphs.

It was conjectured by Peres that a necessary and sufficient condition for cutoff is that for some $\varepsilon \in (0, 1)$ (or, equivalently, for all $\varepsilon \in (0, 1)$)

$$\lim_{n \rightarrow \infty} t_{\text{mix}}^n(\varepsilon) \lambda_n = \infty \quad (4)$$

where λ_n denotes the spectral gap of the random walk on G_n ; see also Sect. 2 for a formal definition. However, while this product criterion is indeed necessary, Aldous showed that it is not sufficient; see Chapter 18 of Levin et al. [22]. In a recent breakthrough, Salez provides a sufficient entropy criterion for cutoff which is for example satisfied by nonnegatively curved Markov chains [32]. Nevertheless, it is believed that the product criterion in (4) is sufficient for cutoff for a wide range of families of Markov chains.

For birth-and-death chains, Ding et al. showed that the product criterion is sharp [16]. Moreover, Chen and Saloff-Coste provided simple conditions such that (4) holds [12, 13]. Using their results, we will be able to exclude cutoff on spherically symmetric trees in Sect. 5.1, following a projection argument due to Nestoridi and Nguyen for d -regular trees [29]. It was verified for the simple random walk on trees by Basu et al. [7] that the product criterion is sufficient to see cutoff; see also Theorem 3.3 for a formal statement of their result. A first example of a family of trees with cutoff was found by Peres and Sousi [31]. Note that although their results are stated for a discrete-time model of the simple random walk, they can be converted to the continuous-time setup using Chen and Saloff-Coste [11]. We will re-obtain the result of Peres and Sousi using Theorem 1.6; see Corollaries 4.5 and 1.5 for a simplified example of a family of trees with cutoff.

In the following, we will focus on random walks on finite trees and assume that the underlying graphs will always be a sequence of rooted trees. Before we come to the main theorems, we introduce some notations. We let $\ell(v)$ denote the set of edges in the shortest path of a vertex $v \in V$ from the root o and write $|v| := |\ell(v)|$. For every edge $e \in E$, we let $e_-, e_+ \in V$ with $|e_-| < |e_+|$ denote the endpoints of the edge and set $|e| := |e_+|$. Further, let T_v be the largest subtree of G rooted at v which consists only of sites with distance at least $|v|$ from o , and we use the conventions that $T_e := T_{e_+}$ and $|T_e| := |V(T_{e_+})|$ for all $e \in E$. We say that a vertex x is a δ -center of mass if there are two trees T and \tilde{T} of G with $V(T) \cap V(\tilde{T}) = \{x\}$ and $V(T) \cup V(\tilde{T}) = V(G)$ such that

$$|V(T)| \geq \delta |V(G)|, \quad |V(\tilde{T})| \geq \delta |V(G)|. \quad (5)$$

The existence of a δ -center of mass is guaranteed for all $\delta \in [0, \frac{1}{3}]$ by the following result. For its proof, we refer to the appendix.

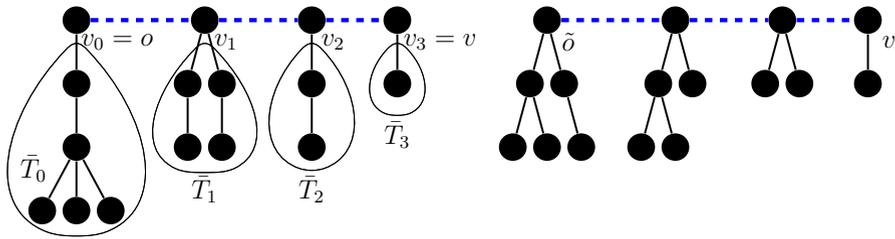


Fig. 1 Example of a tree G on the left-hand side and its corresponding v -retraction \tilde{G} on the right-hand side. The edges in the segment $\ell(v)$ connecting the root and the site v are depicted in dashed blue

Proposition 1.2 *For all trees G , there must be at least one vertex which is a $\frac{1}{3}$ -center of mass.*

From now on, for a given family of trees (G_n) , we fix some $\varepsilon > 0$ and assume without loss of generality that the root o_n is chosen to be a δ -center of mass for some $\delta > 0$ which does not depend on n . For a pair of real-valued, nonnegative sequences $(f(n))_{n \geq 0}$ and $(g(n))_{n \geq 0}$, we write $f \asymp g$ if f and g have the same order of magnitude, and we write $f \ll g$ if $f(n)/g(n)$ converges to 0 for n going to infinity. We start with a criterion on trees which allows us to determine when cutoff does not occur for the simple random walk.

Theorem 1.3 *For a sequence of trees (G_n) suppose that*

$$\max_{e \in E(G_n)} |e||T_e| \asymp \max_{v \in V(G_n)} \sum_{e \in \ell(v)} |T_e|. \tag{6}$$

Then, the simple random walk on (G_n) does not exhibit cutoff.

Using Theorem 1.3, one can directly check that, for example, the simple random walk on the segment of size n or on the regular tree truncated at level n does not exhibit cutoff. In order to state a converse theorem, which guarantees the existence of cutoff, we require the following definition. Fix a tree G and a vertex $v \neq o$. Let $v_0 = o, v_1, \dots, v_k = v$ denote the sites in $\ell(v)$ for $k = |v|$. For each v_i , let \tilde{T}_i be the largest subtree which is attached to v_i and does not intersect with $\ell(v)$ in any site other than v_i . A v -retraction of G is a tree \tilde{G} rooted at some site \tilde{o} , which consists of a segment of size $|\ell(v)|$ with \tilde{o} at one of its endpoints. In addition, we let a binary tree of size $|\tilde{T}_i|$ emerge from each site in the segment at distance i from the root \tilde{o} (if $|\tilde{T}_i|$ is not a power of two, consider the binary tree with leaves missing at the last level). Intuitively, the v -retraction replaces ‘long and thin’ subtrees by ‘short and fat’ binary trees (Fig. 1).

Theorem 1.4 *For every G_n in $(G_n)_{n \geq 1}$, suppose that we find some $v_n^* \in V(G_n)$ with*

$$\sum_{e \in \ell(v_n^*)} |T_e| \asymp \max_{v \in V(G_n)} \sum_{e \in \ell(v)} |T_e| \gg |V(G_n)|. \tag{7}$$

If we have that

$$\max_{e \in \ell(v_n^*)} |e| |T_e| \ll \max_{v \in V(G_n)} \sum_{e \in \ell(v)} |T_e| \tag{8}$$

then for any sequence (\tilde{G}_n) of v_n^* -retractions of (G_n) , we have that the simple random walk on (\tilde{G}_n) exhibits cutoff.

Note that taking retractions is necessary for cutoff in Theorem 1.4 as we will see in Sect. 5.1 where we study random walks on spherically symmetric trees; see Remark 5.3. Further, note that instead of attaching binary trees, we may allow also for attaching other classes of graphs with sufficiently fast growth; see Remark 4.3 for precise conditions on the attached trees. We give now a simple example of a family of trees on which the simple random walk exhibits cutoff. This follows as a consequence of Theorem 1.4.

Corollary 1.5 *Consider the family of trees $(G_n)_{n \geq 1}$, where G_n consists of a segment of size $\lfloor \sqrt{n} \rfloor$, rooted at one of its endpoints, and binary trees of size $\lfloor n/(i+1)^2 \rfloor$ attached at distance i from the root for all $i \in \{0, 1, \dots, \lfloor \sqrt{n} \rfloor\}$. Then, the root is a δ -center of mass for $\delta = \frac{1}{6}$ and the simple random walk on (G_n) exhibits cutoff.*

Our last main result is a criterion which is particularly suited to verify cutoff for *thin* and *long* trees; see also the trees constructed in [31]. For all $k \geq 0$, we set

$$V_k = V_k(G) := \bigcup_{v: |v|=k} V(T_v) = \bigcup_{v: |v| \geq k} \{v\}. \tag{9}$$

Theorem 1.6 *Recall (9), i.e., the set of all vertices at distance at least k from the root. Suppose that a family of trees (G_n) with maximum degrees (Δ_n) satisfies*

$$\max_{k \geq 1} k |V_k(G_n)| \ll \frac{1}{\Delta_n} \max_{v \in V(G_n)} \sum_{e \in \ell(v)} |T_e|. \tag{10}$$

Then, the simple random walk on (G_n) exhibits cutoff.

This paper will be organized as follows. In Sect. 2, we discuss preliminary estimates on the spectral gap of the simple random walk on trees. We present two bounds on the spectral gap, a first bound using a characterization of the spectral gap via a discrete Hardy inequality, and a second bound using weighted paths which follows directly from the first characterization. In Sect. 3, we present preliminary facts on upper and lower bounds on the mixing time using a representation as hitting times of large sets. Building on these results, we prove our main criteria for the occurrence of cutoff in Sect. 4. Section 5 is dedicated to applying these criteria to the families of trees in Theorem 1.1, and showing that the simple random walk does not exhibit cutoff. In particular, we verify the absence of cutoff for the simple random walk on spherically symmetric trees, supercritical Galton–Watson trees conditioned on survival, and families of trees which converge to the Brownian continuum random tree.

The latter includes Galton–Watson trees conditioned to contain a certain number of sites. We conclude with an outlook on open problems.

2 Some Facts About the Spectral Gap on Trees

In order to study cutoff for the simple random walk on trees, our key tool will be to give bounds on the spectral gap of the random walk $(\eta_t)_{t \geq 0}$. Let \mathcal{L} denote the generator of the random walk on a tree $G = (V, E)$. We study pairs of eigenvalues and eigenfunctions for the random walk, i.e., we want to find (κ, f) for $\kappa \in \mathbb{C}$ and $f: V \rightarrow \mathbb{C}$ such that

$$(\mathcal{L}f)(x) = \sum_{y: \{x,y\} \in E} (f(y) - f(x)) = \kappa f(x) \quad (11)$$

holds for all $x \in V$. Note that since $(\eta_t)_{t \geq 0}$ is reversible, all eigenvalues of \mathcal{L} are real-valued and non-positive. Moreover, the function $f \equiv 1$ is always an eigenfunction with respect to the eigenvalue $\lambda = 0$. Our goal is to investigate the *spectral gap* λ of the process, i.e., the absolute value of the second largest eigenvalue of \mathcal{L} , respectively, the *relaxation time*

$$t_{\text{rel}} := \frac{1}{\lambda}. \quad (12)$$

Recall the following variational characterization of λ ; see Definition 2.1.3 in [33].

Lemma 2.1 *Let λ be the spectral gap of simple random walk $(\eta_t)_{t \geq 0}$ on the tree G . Then, we have that*

$$\lambda = \min_{f: V \rightarrow \mathbb{R}, \text{Var}(f) \neq 0} \frac{\mathcal{E}(f)}{\text{Var}(f)} \quad (13)$$

where we set

$$\mathcal{E}(f) := \frac{1}{|V|} \sum_{e \in E} (f(e_+) - f(e_-))^2, \quad \text{Var}(f) := \frac{1}{|V|} \sum_{v \in V} f(v)^2 - \frac{1}{|V|^2} \left(\sum_{v \in V} f(v) \right)^2.$$

The quantities $\mathcal{E}(f)$ and $\text{Var}(f)$ are the Dirichlet form and the variance of the function f ; see Chapter 13 of Levin et al. [22] for a more general introduction.

2.1 A Discrete Hardy Inequality on Trees

Using Lemma 2.1, we obtain the following characterization of the spectral gap in terms of a (discrete) Hardy inequality on trees; see also [17].

Lemma 2.2 *Recall that o is a δ -center of mass for some $\delta > 0$, and let T and \tilde{T} be two trees which satisfy (5). Let A be the smallest constant such that we have*

$$\sum_{v \in V(T)} \left(\sum_{e \in \ell(v)} g(e) \right)^2 \leq A \sum_{e \in E(T)} g(e)^2 \tag{14}$$

for all functions $g : E(T) \rightarrow \mathbb{R}$ with $g \not\equiv 0$ as well as

$$\sum_{v \in V(\tilde{T})} \left(\sum_{e \in \ell(v)} \tilde{g}(e) \right)^2 \leq A \sum_{e \in E(\tilde{T})} \tilde{g}(e)^2 \tag{15}$$

for all functions $\tilde{g} : E(\tilde{T}) \rightarrow \mathbb{R}$ with $\tilde{g} \not\equiv 0$. Then, we have that the spectral gap λ of the simple random walk on G satisfies

$$\lambda \in \left[\frac{1}{A}, \frac{1}{\delta A} \right]. \tag{16}$$

Proof For any function $f : V(G) \rightarrow \mathbb{R}$, we set

$$f_T(v) := (f(v) - f(o))1_{v \in V(T)}, \quad f_{\tilde{T}}(v) := (f(v) - f(o))1_{v \in V(\tilde{T})} \tag{17}$$

for all $v \in V(G)$. Using the definition of $\text{Var}(f)$ and that for positive a, b, c, d

$$\frac{a+b}{c+d} \geq \min \left\{ \frac{a}{c}, \frac{b}{d} \right\} \tag{18}$$

holds, we see that

$$\frac{\mathcal{E}(f)}{\text{Var}(f)} \geq \frac{\mathcal{E}(f_T) + \mathcal{E}(f_{\tilde{T}})}{\bar{f}_T + \bar{f}_{\tilde{T}}} \geq \min \left\{ \frac{\mathcal{E}(f_T)}{\bar{f}_T}, \frac{\mathcal{E}(f_{\tilde{T}})}{\bar{f}_{\tilde{T}}} \right\} \tag{19}$$

where we set

$$\bar{f}_T := \frac{1}{|V(G)|} \sum_{v \in V(T)} (f_T(v))^2, \quad \bar{f}_{\tilde{T}} := \frac{1}{|V(G)|} \sum_{v \in V(\tilde{T})} (f_{\tilde{T}}(v))^2. \tag{20}$$

We set $g_T(e) := f_T(e_+) - f_T(e_-)$ for $e \in E(T)$ and $g_{\tilde{T}}(e) := f_{\tilde{T}}(e_+) - f_{\tilde{T}}(e_-)$ for $e \in E(\tilde{T})$. The inequalities (14) and (15) applied to g_T and $g_{\tilde{T}}$ yield $\mathcal{E}(f_T) \geq \frac{1}{A} \bar{f}_T$ and $\mathcal{E}(f_{\tilde{T}}) \geq \frac{1}{A} \bar{f}_{\tilde{T}}$. Now, we use Lemma 2.1 and (19) to conclude the lower bound on λ claimed in (16).

For the corresponding upper bound on λ , we note that the minimizer g^* for A is given by (14) with equality and satisfies $g^*(e) \geq 0$ for all $e \in E(T)$ (consider $|g^*|$ otherwise to see that this has to be the case). Define a function $h : V(G) \rightarrow \mathbb{R}$ by

$$h(v) := 1_{v \in V(T)} \sum_{e \in \ell(v)} g^*(e). \quad (21)$$

Recall that π is the uniform measure on $V(G)$ and use the Paley–Zygmund inequality to see that

$$\text{Var}(h) \geq (1 - \pi(h > 0)) \frac{1}{|V(G)|} \sum_{v \in V(G)} h(v)^2$$

and hence, we must have

$$\text{Var}(h) \geq \pi(h = 0) \mathcal{E}(h) A \geq \delta A \mathcal{E}(h).$$

Using the characterization of λ in Lemma 2.1, this concludes the proof. \square

We will use Lemma 2.2 in Sects. 4.1 and 4.3 in order to obtain upper and lower bounds on the spectral gap, respectively.

2.2 A Bound on the Relaxation Time by Weighted Paths

Next, we give an estimate which allows us to achieve upper bounds on the relaxation time for the simple random walk on trees. Note that this bound was obtained already in [23] in a more general setup, including a corresponding lower bound of the same form within a factor of 2. For the convenience of the reader, we provide a short proof for our special case of the simple random walk. Recall (12).

Proposition 2.3 *Let $(a_e)_{e \in E(G)}$ be any family of positive edge weights for the tree G . For any choice of the (a_e) 's, we have that*

$$t_{\text{rel}} \leq \max_{e \in E(G)} a_e^{-1} \sum_{v \in T_e} \sum_{\tilde{e} \in \ell(v)} a_{\tilde{e}}. \quad (22)$$

Proof Recall the trees T, \tilde{T} for G from (5). By Lemmas 2.1 and 2.2, if we have for some constant C that

$$\sum_{v \in V(T)} \left(\sum_{e \in \ell(v)} g(e) \right)^2 \leq C \sum_{e \in E(T)} g(e)^2 \quad (23)$$

for all functions $g : E(T) \rightarrow \mathbb{R}$, and a similar statement with respect to \tilde{T} , we conclude $t_{\text{rel}} \leq C$. In the following, we only show (23) with respect to the tree T . Fix positive

edge weights a_e and note that by the Cauchy-Schwarz inequality,

$$\sum_{v \in V(T)} \left(\sum_{e \in \ell(v)} g(e) \right)^2 \leq \sum_{v \in V(T)} \left(\sum_{\tilde{e} \in \ell(v)} a_{\tilde{e}} \right) \sum_{e \in \ell(v)} a_e^{-1} g(e)^2.$$

Rearranging the sum according to the edges and assuming without loss of generality that g is nonnegative, we get

$$\sum_{v \in V(T)} \left(\sum_{e \in \ell(v)} g(e) \right)^2 \leq \sum_{e \in E(T)} \left(\sum_{v \in T_e} \sum_{\tilde{e} \in \ell(v)} a_{\tilde{e}} \right) a_e^{-1} g(e)^2.$$

Hence, for any choice of (a_e) , we can take the right-hand side of (22) as C in (23). □

Note that we can choose positive weights (a_e) in any particular way to obtain an upper bound. In the following, we present three special cases of (a_e) 's and the respective bounds on the relaxation time. We start with the case where we set $a_e = \frac{1}{|e|}$ in Proposition 2.3 to obtain the following bound.

Corollary 2.4 *We have that*

$$t_{\text{rel}} \leq (\log(\text{diam}(G)) + 1) \max_{e \in E} |e||T_e| \tag{24}$$

where $\text{diam}(G)$ denotes the diameter of the tree G .

In Proposition 4.1, we give a corresponding lower bound on the relaxation time, but without the additional factor of $\log(\text{diam}(G))$. Next, consider $a_e = \frac{1}{f(|e|)}$ for some function $f : \mathbb{N} \rightarrow \mathbb{R}$, whose reciprocals are summable. We obtain the following bound as a consequence of Proposition 2.3.

Corollary 2.5 *Suppose that we have some function f on the integers with*

$$\sum_{n \in \mathbb{N}} f(n)^{-1} = C < \infty$$

Then we have that

$$t_{\text{rel}} \leq C \max_{e \in E} f(|e|)|T_e|. \tag{25}$$

Next, we have the following bound by choosing the weights $a_e = |T_e|$ for all $e \in E(T)$ proportional to size of the tree T_e attached tree to e .

Corollary 2.6 *It holds that*

$$t_{\text{rel}} \leq \max_{v \in V} \sum_{e \in \ell(v)} |T_e|. \tag{26}$$

This follows directly from Proposition 2.3 noting that

$$t_{\text{rel}} \leq \max_{e \in E} \frac{1}{|T_e|} \sum_{v \in T_e} \sum_{\tilde{e} \in \ell(v)} |T_{\tilde{e}}| \leq \max_{e \in E} \frac{1}{|T_e|} \sum_{v \in T_e} \max_{\tilde{v} \in V} \sum_{\tilde{e} \in \ell(\tilde{v})} |T_{\tilde{e}}| = \max_{v \in V} \sum_{e \in \ell(v)} |T_e|.$$

We will see in the upcoming section that the right-hand side in (26) gives also an upper bound on the mixing time for the simple random walk.

3 Some Facts About the Mixing Time on Trees

In this section, we discuss mixing time estimates for the simple random walk $(\eta_t)_{t \geq 0}$ on a tree $G = (V, E, o)$. The main result presented in this section is that the ε -mixing time can be bounded in terms of the hitting time of the root o . For sites $x, y \in V$, let

$$\tau_{\text{hit}}(x) := \inf \{t \geq 0: \eta_t = x\} \tag{27}$$

be the *hitting time* of x and let $\mathbb{E}_y[\tau_{\text{hit}}(x)]$ denote the expected hitting time of x when starting the random walk from y . The following proposition gives an upper bound on the ε -mixing time.

Proposition 3.1 *Let $t_{\text{mix}}(\varepsilon)$ be the ε -mixing time of $(\eta_t)_{t \geq 0}$. There is a universal constant $C > 0$ such that we have for all $\varepsilon \in (0, \frac{1}{2})$*

$$t_{\text{mix}}(\varepsilon) \leq C \log(\varepsilon^{-1}) \left(1 + \max_{v \in V} \mathbb{E}_v[\tau_{\text{hit}}(o)] \right) \leq 2C \log(\varepsilon^{-1}) \max_{v \in V} \sum_{e \in \ell(v)} |T_e|. \tag{28}$$

In particular,

$$t_{\text{mix}}(\varepsilon) \leq 2C \log(\varepsilon^{-1}) |V| \text{diam}(G). \tag{29}$$

Proof Note that the first inequality in (28) is immediate from Theorem 5 in [2] together with Theorem 20.3 in Levin et al. [22] and Lemma 5.2 of Peres and Sousi [30]. To see that the second inequality holds, we claim that for two adjacent sites $v, w \in V$ with $|v| < |w|$, we have

$$\mathbb{E}_w[\tau_{\text{hit}}(v)] \leq \sum_{x \in T_w} \text{deg}(x) - 1. \tag{30}$$

To see this, consider the embedded discrete-time simple random walk, where we move in each time step to a neighbor chosen uniformly at random. The claim follows from the well-known fact—see equation (4) in [10]—that this discrete-time simple random walk satisfies

$$\mathbb{E}_w[\tau_{\text{hit}}(v)] = \sum_{x \in T_w} \text{deg}(x) - 1, \tag{31}$$

and we can conclude with a time-change argument. Since for any tree \tilde{G} , we have

$$\sum_{x \in V(\tilde{G})} \deg(x) = 2|V(\tilde{G})| - 2 \tag{32}$$

this yields

$$\mathbb{E}_v[\tau_{\text{hit}}(o)] \leq 2 \sum_{e \in \ell(v)} |T_e|,$$

and the second inequality in (28) follows. □

Note that the bound in Proposition 3.1 does not use the assumption that the root is a δ -center of mass. In fact, we see that (28) holds for an arbitrarily chosen root. Next, we state a lower bound on the mixing time, following the ideas of Lemma 5.4 in [7], which does indeed require that the root is a δ -center of mass.

Proposition 3.2 *Assume that the root o is a δ -center of mass for some $\delta > 0$. Let Δ be the maximum degree in G . Then for all $\varepsilon \leq \delta$,*

$$t_{\text{mix}}\left(\frac{\varepsilon}{2}\right) \geq \frac{\varepsilon}{2} \max_{v \in V} \mathbb{E}_v[\tau_{\text{hit}}(o)] \geq \frac{\varepsilon}{2\Delta} \max_{v \in V} \sum_{e \in \ell(v)} |T_e|. \tag{33}$$

Proof Let $v \in V$ with $v \neq o$ be fixed and recall that π denotes the uniform distribution on $V(G)$. Moreover, recall the trees T and \tilde{T} in (5) from the definition of the δ -center of mass and assume that $v \in V(T)$. We now claim that

$$\mathbb{P}_v(\tau_o \leq t_{\text{mix}}(\varepsilon/2)) \geq \mathbb{P}_v\left(\eta_{t_{\text{mix}}(\varepsilon/2)} \in V(\tilde{T})\right) \geq \pi(V(\tilde{T})) - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}. \tag{34}$$

The first inequality in (34) follows since $\eta_{t_{\text{mix}}(\varepsilon/2)} \in V(\tilde{T})$ implies that the root was visited before $t_{\text{mix}}(\varepsilon/2)$, the second inequality uses the definition of the mixing time and the third inequality follows since $\pi(V(\tilde{T})) \geq \delta \geq \varepsilon$. In words, (34) says that with probability at least $\varepsilon/2$, the random walk hits the root by time $t_{\text{mix}}(\varepsilon/2)$. Using the Markov property of $(\eta_t)_{t \geq 0}$, since (34) holds for any $v \in V(T)$, we can iterate (34) to get that the probability of hitting the root for the first time by time $k \cdot t_{\text{mix}}(\varepsilon/2)$ is at least the probability that a Geometric- $(\varepsilon/2)$ -random variable takes a value $\leq k$. This yields

$$\mathbb{E}_v[\tau_{\text{hit}}(o)] \leq \frac{2}{\varepsilon} t_{\text{mix}}\left(\frac{\varepsilon}{2}\right).$$

Since $v \in T$ was arbitrary, we obtain the first inequality in (33). For the second inequality, recall (31) and (32), and use a comparison with a discrete-time simple random walk, where we choose in each step one neighbor uniformly at random, and speed the walk up by a factor of Δ . □

So far, we gathered techniques in order to give upper and lower bounds on the mixing and relaxation time. The next seminal result by Basu et al. gives a characterization of cutoff for the simple random walks on trees; see [7].

Theorem 3.3 (c.f. Theorem 1 in [7]) *Fix $\varepsilon \in (0, 1)$. Let $(G_n)_{n \geq 1}$ be a family of trees with ε -mixing times (t_{mix}^n) and relaxation times (t_{rel}^n) for the simple random walk on (G_n) , respectively. The simple random walk on (G_n) exhibits cutoff if and only if*

$$t_{\text{rel}}^n \ll t_{\text{mix}}^n. \quad (35)$$

In other words, we see cutoff whenever then product criterion (4) holds.

Remark 3.4 In view of Corollary 2.6, Proposition 3.2 and Theorem 3.3, we aim at verifying cutoff by giving weights (a_e) in Proposition 2.3, which lead to a strictly improved bound over the weights $a_e = |T_e|$.

4 Proof of the Main Criteria for Cutoff

We now use the preliminary bounds from the previous two sections in order to establish our main criteria on cutoff for the simple random walk on trees, i.e., we prove Theorems 1.3, 1.4 and 1.6.

4.1 A General Lower Bound on the Relaxation Time

We start with a lower bound on the relaxation time. Recall that for a family of graphs (G_n) , the root is chosen to be a δ -center of mass for some fixed $\delta > 0$.

Proposition 4.1 *The constant A from Lemma 2.2 satisfies*

$$A \geq \max_{e \in E(G)} |e||T_e|. \quad (36)$$

In particular, Lemma 2.2 implies that

$$t_{\text{rel}} \geq \delta \max_{e \in E(G)} |e||T_e|. \quad (37)$$

Proof We show the lower bound on A by considering an explicit function g_{e^*} in the definition of A in (14) and (15). Fix some $e^* \in E(G)$ which maximizes the right-hand side of (36). Recall that $e_-^*, e_+^* \in V$ with $|e_-^*| < |e_+^*|$ denote the endpoints of the edge e^* . We define $g_{e^*} : E(G) \rightarrow \mathbb{R}$ to be

$$g_{e^*}(e) := \begin{cases} \frac{1}{|e_+^*|} & \text{if } e \in \ell(e_+^*) \\ 0 & \text{if } e \notin \ell(e_+^*) \end{cases}$$

and note that g_{e^*} satisfies

$$\sum_{v \in V(G)} \left(\sum_{e \in \ell(v)} g_{e^*}(e) \right)^2 \geq \sum_{v \in T_{e^*}} \left(\sum_{e \in \ell(v)} g_{e^*}(e) \right)^2 = |T_{e^*}|$$

as well as

$$\sum_{e \in E(T)} g_{e^*}(e)^2 = \frac{1}{|e_+^*|} = \frac{1}{|e^*|}.$$

Using the definition of A in Lemma 2.2, we see that (36) holds. □

Proof of Theorem 1.3 Recall that by Theorem 3.3, there is no cutoff when the mixing time and the relaxation time are of the same order. Note that Proposition 3.1 yields an upper bound on the mixing time, while Proposition 4.1 establishes a lower bound on the relaxation time. Both bounds are of the same order, due to assumption (6), which finishes the proof.

4.2 Cutoff for v -Retractions of Trees

In this section, we prove Theorem 1.4. We use Proposition 2.3 with a specific choice of edge weights (a_e) improving the choice of weights leading to Corollary 2.6.

Lemma 4.2 *Suppose that assumptions (7) and (8) of Theorem 1.4 hold and recall that $(\tilde{G}_n)_{n \geq 1}$ denotes the sequence of v_n^* -retractions of $(G_n)_{n \geq 1}$. Then, we have*

$$\tilde{t}_{\text{rel}}^n \ll \max_{v \in V(\tilde{G}_n)} \sum_{e \in \ell(v)} |T_e| \tag{38}$$

where \tilde{t}_{rel}^n is the relaxation time corresponding to \tilde{G}_n .

Proof Let $k := |v_n^*|$ and let $(\tilde{T}_i)_{i \in \{0, \dots, k\}}$ denote the trees in \tilde{G}_n attached along the segment $\ell(v_n^*)$, ordered according to their distances from the root, which are used in the construction of the v_n^* -retraction. We consider the following choice of the weights $(a_e)_{e \in E(G_n)}$. For $e \in \ell(v_n^*)$, we let $a_e := |e|^{-1/2}$. For $e \in E(\tilde{T}_i)$, we set

$$a_e := \frac{1}{\sqrt{\max\{i, 1\}(|e| - i)^2}}. \tag{39}$$

In order to apply Proposition 2.3, we will now give an upper bound for

$$A_e := a_e^{-1} \sum_{v \in T_e} \sum_{\tilde{e} \in \ell(v)} a_{\tilde{e}} \tag{40}$$

for any possible choice of the edge $e \in E(\tilde{G}_n)$. We claim that without loss of generality, it suffices to consider $e \in \ell(v_n^*)$. To see this, consider $e \in \tilde{T}_i$ for $i \geq 1$ and let e_i be

the corresponding edge in $\ell(v_n^*)$ with $|e_i| = i$. For every $v \in \bar{T}_i$, we have

$$\sum_{\tilde{e} \in \ell(v)} a_{\tilde{e}} = \sum_{j=1}^i j^{-\frac{1}{2}} + i^{-\frac{1}{2}} \sum_{l=i+1}^{|e|} \frac{1}{(l-i)^2} \in [\sqrt{i}, 4\sqrt{i}]. \tag{41}$$

Together with the fact that \bar{T}_i has a binary structure, we get that

$$A_e \leq 4i(|e| - i)^2 |T_e| \leq 4i(|e| - i)^2 2^{-(|e|-i)} |T_{e_i}| \leq 5i |T_{e_i}| \leq 5A_{e_i}$$

holds. Similarly, for $e \in \bar{T}_0$, we see that $A_e \leq 5|\bar{T}_0| \leq 5|V(G_n)|$, which is of smaller order than the right-hand side in (38) by assumption (7). Hence, it suffices to bound A_e for all edges e within $\ell(v_n^*)$ to conclude.

From (41), we see that the right-hand side of (40) is bounded from above by

$$A_e = \sqrt{|e|} \sum_{i=|e|}^k \sum_{v \in \bar{T}_i} \sum_{\tilde{e} \in \ell(v)} a_{\tilde{e}} \leq 4\sqrt{|e|} \sum_{i=|e|}^k \sqrt{i} |\bar{T}_i|.$$

It remains to show that

$$\sqrt{|e|} \sum_{i=|e|}^k \sqrt{i} |\bar{T}_i| \ll \sum_{e \in \ell(v_n^*)} |T_e|. \tag{42}$$

We see that for all $e \in \ell(v_n^*)$,

$$\left(\sqrt{|e|} \sum_{i=|e|}^k \sqrt{i} |\bar{T}_i| \right)^2 \leq \left(|e| \sum_{i=|e|}^k |\bar{T}_i| \right) \left(\sum_{i=|e|}^k i |\bar{T}_i| \right) \ll \left(\sum_{i=1}^k i |\bar{T}_i| \right)^2 \tag{43}$$

using the Cauchy–Schwarz inequality for the first step, as well as the fact that by assumptions (8) and (7)

$$|e| \sum_{i=|e|}^k |\bar{T}_i| \leq \max_{e' \in \ell(v_n^*)} |e'| |T_{e'}| \ll \max_{v \in V(G_n)} \sum_{e' \in \ell(v)} |T_{e'}| \asymp \sum_{i=1}^k i |\bar{T}_i|$$

for the second step. Taking square roots of both sides of (43) finishes the proof.

Proof of Theorem 1.4 For the lower bound on the mixing time from Proposition 3.2, note that the leading order does not decrease when we replace (G_n) by (\tilde{G}_n) for v_n^* from (7), and that the root is still a $\tilde{\delta}$ -center of mass for some $\tilde{\delta} > 0$. Theorem 1.4 follows together with the product criterion from Theorem 3.3, and Lemma 4.2.

Remark 4.3 Note that instead of binary trees, we may consider other trees \tilde{T}_i rooted at v_i and attached to the segment at distance i from \tilde{o} for which

$$\max_{v \in V(T_e)} \sum_{\tilde{e} \in \ell(v)} |T_{\tilde{e}}| \leq \alpha |T_e| \tag{44}$$

holds for some constant α . The above choice of binary trees satisfies (44) with $\alpha = 2$. Depending on (8), we may also allow α to go to infinity with n sufficiently slow.

4.3 A Sufficient Condition for Cutoff on Trees

In this section, we give an upper bound on the relaxation time, which will allow us to give a sufficient condition for cutoff on trees. For a finite rooted tree $G = (V, E, o)$, we let $\mathcal{S} := \{B \subseteq V : B \neq \emptyset, o \notin B\}$, and define for all $B \in \mathcal{S}$

$$v_B := \inf \left\{ \sum_{e \in E} f(e)^2 : \sum_{e \in \ell(v)} f(e) \geq 1 \text{ for all } v \in B \right\} > 0, \tag{45}$$

where the infimum is taken over functions $f : E \rightarrow \mathbb{R}$. For the following bound on the relaxation time, we use the ideas of Evans et al. for proving a Hardy inequality on continuous weighted trees [17]. A corresponding Hardy inequality for discrete weighted trees was obtained by Miclo; see Proposition 16 in [27]. We will now provide a similar result on the relaxation time of the simple random walk.

Proposition 4.4 *Recall (9). For any finite tree G , we have that*

$$t_{\text{rel}} \leq 32 \max_{B \in \mathcal{S}} \frac{|B|}{v_B} \leq 32 \max_{k \in \mathbb{N}} k |V_k|. \tag{46}$$

Proof Recall the trees T, \tilde{T} for G from (5). For the first inequality in (46), it suffices by Lemma 2.2 to show that for every function $g : E(T) \rightarrow \mathbb{R}$ with $g \not\equiv 0$, we have

$$\sum_{v \in V(T)} \left(\sum_{e \in \ell(v)} g(e) \right)^2 \leq 32 \left(\max_{B \in \mathcal{S}} \frac{|B|}{v_B} \right) \sum_{e \in E(T)} g(e)^2, \tag{47}$$

and a similar statement with respect to \tilde{T} . We will now show (47) only for T . Fix a nonnegative function g (otherwise consider $|g|$), and define for all $i \in \mathbb{Z}$ the set

$$B_i := \left\{ v \in V(T) : 2^i \leq \sum_{e \in \ell(v)} g(e) < 2^{i+1} \right\}.$$

We let $\mathcal{I} := \{i \in \mathbb{Z} : B_i \neq \emptyset\}$ and note that $B_i \in \mathcal{S}$ when $i \in \mathcal{I}$. Observe that

$$\sum_{v \in V(T)} \left(\sum_{e \in \ell(v)} g(e) \right)^2 \leq \sum_{i \in \mathcal{I}} 2^{2i+2} |B_i| \leq \left(\max_{B \in \mathcal{S}} \frac{|B|}{v_B} \right) \sum_{i \in \mathcal{I}} 2^{2i+2} v_{B_i}$$

holds. Hence, it suffices now to prove that for all $i \in \mathcal{I}$

$$\nu_{B_i} \leq 2^{-2i+2} \sum_{e \in E(T): e_+ \in B_{i-1} \cup B_i} g(e)^2 \quad (48)$$

is satisfied in order to conclude (47). To see that (48) indeed holds, consider

$$g_i(e) := 2^{-i+1} g(e) \mathbb{1}_{\{e_+ \in B_{i-1} \cup B_i\}}$$

for all $e \in E(T)$. Using the definition of the sets $(B_i)_{i \in \mathbb{Z}}$ and the fact that g is nonnegative, we have for all $v \in B_i$

$$\sum_{e \in \ell(v)} g_i(e) = 2^{-i+1} \left(\sum_{e \in \ell(v)} g(e) - \sum_{\tilde{e} \in \ell(v): \tilde{e}_+ \notin B_{i-1} \cup B_i} g(\tilde{e}) \right) \geq 2^{-i+1} (2^i - 2^{i-1}) = 1.$$

Plugging g_i into the definition (45) of ν_{B_i} yields (48), and hence the first inequality in (46). For the second inequality in (46), observe that for all $B \in \mathcal{S}$

$$\frac{\nu_B}{|B|} \geq \frac{1}{|B|} \max_{v \in B} \inf \left\{ \sum_{e \in E(G)} f(e)^2 : \sum_{e \in \ell(v)} f(e) = 1 \right\} = \frac{1}{|B|} \max_{v \in B} \frac{1}{|v|} \quad (49)$$

holds, where the infimum is attained when $f(e) = |v|^{-1} \mathbb{1}_{\{e \in \ell(v)\}}$. We then optimize in (49) over the choice of $B \in \mathcal{S}$ to conclude.

Proof of Theorem 1.6 Similar to the proof of Theorem 1.3, recall that by Theorem 3.3, we have cutoff if and only if the relaxation time is of strictly smaller order than the mixing time. Note that by Proposition 3.2 we obtain a lower bound on the mixing time, while Proposition 4.4 gives us an upper bound on the relaxation time. We conclude taking into account assumption (10).

As a direct consequence of Theorem 1.6, we see that the simple random walk on the following family of trees constructed by Peres and Sousi [31] exhibits cutoff: Fix k and consider the tree G_N for $N = N(k)$ which is constructed as follows: Set $n_i = 2^{2^i}$ and start with a segment of length n_k , rooted at one of the endpoints. At the root, we attach a binary tree of size $N := n_k^3$. Then at distance n_i from the root, attach a binary tree of size N/n_i for all $i \in \{k/2, \dots, k\}$. We reindex the sequence of trees (G_N) to obtain a family of trees $(G_n)_{n \geq 1}$ for all $n \geq 1$. A visualization of these trees can be found in [31]. Note that in this construction, the root is always a δ -center of mass for some $\delta > 0$ which does not depend on n .

Corollary 4.5 *The family of trees described by Peres and Sousi has a mixing time of order Nk and a relaxation time of order N , and hence exhibits cutoff provided that $k \rightarrow \infty$ with $N \rightarrow \infty$.*

Proof We first consider the ε -mixing time for some $\varepsilon > 0$. In Proposition 3.2, we choose v to be a leaf in the tree attached at distance n_k from the root and get

$$t_{\text{mix}}(\varepsilon) \geq c \sum_{i=k/2}^{k-1} (n_{i+1} - n_i) \sum_{j=i}^{k-1} \frac{N}{n_j} \geq \frac{c}{4} kN$$

for some constant $c = c(\varepsilon) > 0$, when summing only over the sites in the attached binary trees for (33). Since $N/n_i \gg n_i$ for all $i \leq k$, a corresponding upper bound follows from Proposition 3.1. While a lower bound on t_{rel} of order N is immediate from Proposition 4.1, we obtain a corresponding upper bound of the same order by Proposition 4.4, by a straight-forward count of the sites of distance at least m from the root. We conclude that cutoff occurs by Theorem 3.3. \square

5 Cutoff for SRW on Trees is Atypical

We now use the previous estimates and ask about cutoff for the simple random walk on the following three families of trees from Theorem 1.1: infinite spherically symmetric trees truncated at level n , supercritical Galton–Watson trees conditioned on survival and truncated at height n , and families of combinatorial trees converging to the Brownian continuum random tree (CRT). The latter includes critical Galton–Watson trees conditioned to have exactly n sites. In all three cases, we will in the following verify that cutoff does not occur.

5.1 Spherically Symmetric Trees

Let $G = (V, E, o)$ be a rooted tree. We say that G is *spherically symmetric* if $\deg(v) = \deg(v')$ holds for all $v, v' \in V$ with $|v| = |v'|$. Examples for such trees are regular trees. We write \deg_k for the degree of the vertices at generation k provided that $V_k \neq \emptyset$ holds (recall (9)).

Proposition 5.1 *Consider an infinite spherically symmetric tree G of maximum degree Δ . For every $n \in \mathbb{N}$, let G_n the tree induced by G by restricted to the sites $V \setminus V_n$. Then, the simple random walk on $(G_n)_{n \geq 1}$ does not exhibit cutoff.*

When $G = \mathbb{N}$ the claim is well known; see [22]. Otherwise, for all $n \in \mathbb{N}$ large enough, we choose the root of G_n to be the first branching point in G , i.e., the vertex closest to the root with degree at least 3. In particular, note that the root of G_n will be a $\frac{1}{4}$ -center of mass for all n sufficiently large. By Propositions 3.1 and 3.2, we see that the $\frac{1}{8}$ -mixing time of G_n satisfies

$$t_{\text{mix}}^n \left(\frac{1}{8} \right) \asymp \sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} \prod_{k=i}^{j-1} (\deg_k - 1). \tag{50}$$

We will now show a lower bound on the relaxation time of the same order by using a comparison to birth-and-death chains. Our strategy will be similar to Evita and Oanh [29], where Nestoridi and Nguyen study eigenvalues and eigenfunctions for the d -regular tree. More precisely, we will exploit that certain eigenfunctions of birth-and-death chains can be converted to eigenfunctions of the simple random walk on spherically symmetric trees. Without loss of generality, we will assume that $\deg(o) > 1$ holds since removing the sites before the first branching point in G changes the relaxation time by at most a constant factor. This can be seen using the characterization of the spectral gap in Lemma 2.1. Let $(X_t)_{t \geq 0}$ be the continuous-time birth-and-death chain on the segment $\{1, \dots, 2n - 1\}$ with nearest neighbor transition rates $(r(x, y))$ given by

$$r(x, y) = \begin{cases} \deg_{x-n} - 1 & \text{if } y = x + 1 > n + 1 \\ \deg_{n-x} - 1 & \text{if } y = x - 1 < n - 1 \\ 1 & \text{otherwise} \end{cases} \quad (51)$$

for all $\{x, y\} \in E(G_n)$. We make the following observation on the spectral gap of $(X_t)_{t \geq 0}$.

Lemma 5.2 *Let $\tilde{\lambda}$ be the spectral gap of $(X_t)_{t \geq 0}$. There exists a corresponding eigenfunction $\tilde{f}: \{1, \dots, 2n - 1\} \rightarrow \mathbb{R}$ which satisfies*

$$\tilde{f}(x) = -\tilde{f}(2n - x) \quad (52)$$

for all $x \in \{1, \dots, 2n - 1\}$.

Proof Note that there exists an eigenfunction g corresponding to the spectral gap of an irreducible birth-and-death chain which is monotone and non-constant; see Lemma 22.17 in [22]. By the symmetry of the transition rates, we see that the function h given by $h(x) = g(2n - x)$ for all $x \in \{1, \dots, 2n - 1\}$ is also an eigenfunction for $(X_t)_{t \geq 0}$ with respect to $\tilde{\lambda}$. Since $g(1) \neq g(2n - 1)$, we see that the function $\tilde{f} := g - h$ is an eigenfunction corresponding to $\tilde{\lambda}$ which has the desired properties. \square

With these observations, we have all tools to give the proof of Proposition 5.1.

Proof of Proposition 5.1 Note that we can extend the eigenfunction \tilde{f} of $(X_t)_{t \geq 0}$ from Lemma 5.2 belonging to the spectral gap to an eigenfunction F with eigenvalue $\tilde{\lambda}$ of the random walk on G_n . To see this, let x_1, x_2 be two sites adjacent to o , and consider the function $F: V \rightarrow \mathbb{R}$ given by

$$F(v) = \begin{cases} f(n - |v|) & \text{if } v \in T_{x_1} \\ f(n + |v|) & \text{if } v \in T_{x_2} \\ 0 & \text{otherwise.} \end{cases} \quad (53)$$

The fact that F is an eigenfunction of the simple random walk on G_n follows by a direct verification using the generator. Hence, in order to give a lower bound on the

relaxation time of the same order as in (50), it suffices to bound the spectral gap $\tilde{\lambda}$ of $(X_t)_{t \geq 0}$. Note that the stationary distribution $\tilde{\pi}$ of $(X_t)_{t \geq 0}$ satisfies

$$\tilde{\pi}(x) = \frac{1}{Z} \sum_{k=\min\{x, 2n-x\}}^{n-2} (\deg_{n-k-1} - 1) \tag{54}$$

for all $x \in \{1, \dots, 2n - 1\}$, where Z is a normalization constant. Recalling that G has maximum degree Δ and using Theorem 4.2 in [12], we see that

$$\frac{1}{\tilde{\lambda}} \geq \frac{1}{16\Delta} \sum_{i=1}^n \frac{1}{\tilde{\pi}(i)} \asymp \sum_{i=1}^n \left(\sum_{j=1}^n \prod_{k=j}^{n-2} (\deg_{n-k-1} - 1) \right) \prod_{k=i}^{n-2} (\deg_{n-k-1} - 1)^{-1}.$$

This gives the desired lower bound on the relaxation time of the tree G_n which is of the same order as the upper bound on the mixing time in (50). Hence, using Theorem 3.3, we see that no cutoff occurs.

Remark 5.3 Using Proposition 5.1, we can see that taking v_n^* -retractions in Theorem 1.4 is necessary for cutoff. More precisely, consider the spherically symmetric tree G with

$$\deg_i = \begin{cases} 3 & \text{if } i = 2^j - 1 \text{ for some } j \in \{0, 1, \dots\} \\ 2 & \text{else} \end{cases} \tag{55}$$

for all $i \in \mathbb{N}$. The corresponding trees $(G_n)_{n \geq 0}$ truncated at level n satisfy (8), but due to Proposition 5.1, the simple random walk on (G_n) does not exhibit cutoff.

5.2 Galton–Watson Trees Conditioned on Survival

In this section, we consider a family of random trees $(G_n)_{n \geq 1}$ which we obtain by truncating a supercritical Galton–Watson tree conditioned on survival. More precisely, let μ be an *offspring distribution* and assume that

$$m := \sum_{j=1}^{\infty} j\mu(j) > 1 \quad \sigma^2 := \sum_{j=1}^{\infty} j^2\mu(j) \in (0, \infty), \tag{56}$$

, i.e., we have a supercritical Galton–Watson process whose offspring distribution has finite variance. In the following, we take the genealogical tree G of a realization of the Galton–Watson process conditioned on survival. We then obtain the trees $(G_n)_{n \geq 1}$ by restricting G onto the sites $V \setminus V_n$ (recalling (9)). We keep the sequence of trees fixed and perform simple random walks on $(G_n)_{n \geq 1}$. We denote the law of G by P (P depends on μ , but we will not write it). We now have the following result on cutoff for the simple random walk on $(G_n)_{n \geq 1}$.

Proposition 5.4 For P -almost all trees G , the family of simple random walks on $(G_n)_{n \geq 1}$ does not exhibit cutoff.

Proof From the proof of Theorem 1.4(b) in [20], we know that the relaxation time is P -almost surely of order m^n for all n sufficiently large. Hence, by Theorem 3.3, it suffices to give an upper bound on the mixing time of order m^n for the random walk on G_n in order to exclude cutoff. Providing an upper bound on the mixing time of order m^n answers a question of Jonasson [20]. Recall that for any supercritical Galton–Watson tree $G = (V, E)$ with offspring distribution μ satisfying (56),

$$\left(\frac{|\{v \in V(G) : |v| = n\}|}{m^n} \right)_{n \geq 1} \tag{57}$$

is a martingale which converges P -almost surely and in L^2 , due to the Kesten–Stigum theorem. For all $v \in V$ and $N \geq 0$ let $k(v, N)$ be the number of sites in the tree T_v with distance at most $|v| + N$ from the root. From (57), we see that for all $v \in V$

$$\sup_{N \in \mathbb{N}} \frac{k(v, N)}{m^N} = Y_v \tag{58}$$

P -almost surely for some random variable Y_v . It is easy to show, applying Doob’s inequality to the martingale in (57), that Y_v has a finite second moment.

Recall that Proposition 3.1 does not require the root to be a δ -center of mass, and hence

$$P \left(t_{\text{mix}}^n \left(\frac{1}{4} \right) \leq T \right) \leq P \left(C \sum_{s=0}^{n-1} m^{n-s} \max_{|v|=s} Y_v \leq T \right) \tag{59}$$

holds P -almost surely for all $T > 0$ and some constant $C > 0$. It remains to give a bound on the random variables Y_v . Note that the random variables Y_v do not depend on the number of sites in the generation $|v|$ in the Galton–Watson tree. Hence, writing E for the expectation corresponding to P , we see that

$$\begin{aligned} P \left(\max_{|v|=s} Y_v > \frac{m^s}{2s^2} \right) &\leq P \left(|\{v : |v| = s\}| > s^2 m^s \right) + s^2 m^n P \left(Y_v > \frac{m^s}{s^2} \right) \\ &\leq \frac{E [|\{v : |v| = s\}|]}{s^2 m^s} + s^2 m^s \frac{E [Y_v^2]}{s^{-4} m^{2s}} \leq cs^{-2} \end{aligned}$$

is satisfied for all $s \in \mathbb{N}$ and some constant $c > 0$. Together with (59) and the first Borel–Cantelli lemma, we see that P -almost surely

$$t_{\text{mix}}^n \left(\frac{1}{4} \right) \leq \tilde{c} m^n$$

holds for some $\tilde{c} > 0$ depending on G and all n sufficiently large. Note that a bound of the same form follows when conditioning the underlying Galton–Watson tree on survival, using its representation as a multi-type Galton–Watson process with one child in every generation having a size-biased offspring distribution; see Chapter 12 of Lyons and Peres [25]. Together with the corresponding bound on the relaxation time for the random walk on (G_n) of order m^n from Johan [20], this finishes the proof. \square

Remark 5.5 For critical or subcritical Galton–Watson trees, i.e., when $m \leq 1$ holds in (56), one can consider the family of random walks on (G_n) which we obtain from the associated Kesten tree truncated at generation n ; see [21] for a formal definition of the tree and [19] for a more comprehensive discussion. Note that the resulting tree consists of a segment of size n with finite trees attached. Using Theorem 1.3, one can show that there is P -almost surely no cutoff.

5.3 Combinatorial Trees Converging to the CRT

In the previous two examples, we considered an infinite tree, which we truncated at generation n in order to obtain the family of trees $(G_n)_{n \geq 1}$. We now take a different perspective and study the simple random walk on a sequence of random trees, where we assume that G_n has exactly n sites.

For each tree (G_n) , we assign a labeling to the n sites chosen uniformly at random, and declare the vertex with label 1 to be the root of (G_n) . Let $s : \{1, \dots, 2n - 1\} \rightarrow V_n$ be the *contour function* of G_n , which is given as the walk on V_n when performing depth-first search on G_n , giving priority to sites with a smaller label; see also Sect. 2.6 in [5]. Intuitively, we obtain s by embedding G_n into the plane such that the shortest path distance is preserved and sites with a common ancestor are ordered increasingly according to their labels; see Fig. 2. We will write $|s(\cdot)|$ for the distance of $s(\cdot)$ to the root, and call $|s| : \{1, \dots, 2n - 1\} \rightarrow \mathbb{N}$ again the contour function, with a slight abuse of notation. For $c > 0$, consider the normalized contour function $\tilde{s} : [0, 1] \rightarrow \mathbb{R}$ given by

$$\tilde{s}_n \left(\frac{i}{2n} \right) := cn^{-\frac{1}{2}} |s(i)| \quad (60)$$

for all $i \in \{1, \dots, 2n - 1\}$, $\tilde{s}_n(0) = \tilde{s}_n(1) = 0$ and linear interpolation between these values. We define that a family of random trees $(G_n)_{n \geq 1}$ *converges to the CRT for c* if $(\tilde{s}_n)_{n \geq 1}$ converges in distribution (in the space $\mathcal{C}([0, 1])$) to the Brownian excursion $(e(t))_{t \in [0, 1]}$. Note that the terminology CRT refers to the Brownian continuum random tree, which can be seen as the limit object of $(G_n)_{n \geq 1}$; see [3–5] for an introduction and equivalent definitions, and [18, 26] for examples of trees converging to the CRT.

Arguably the most prominent example of such a sequence of trees are independently chosen critical Galton–Watson trees $(G_n)_{n \geq 1}$ conditioned on having exactly n sites, where we assume that

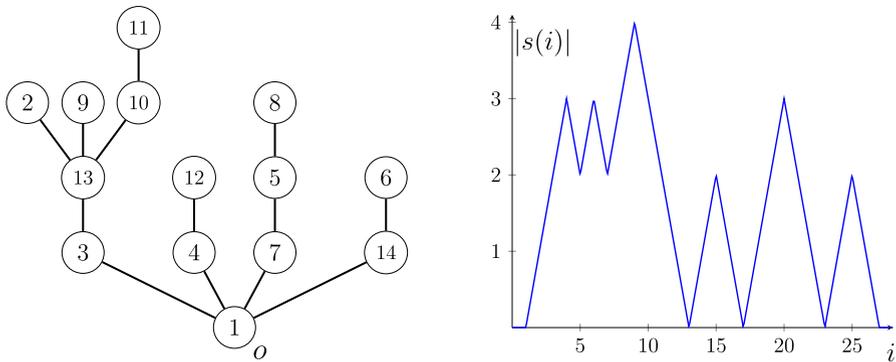


Fig. 2 Visualization of the linear interpolation of the contour function $|s|: \{1, \dots, 27\} \rightarrow \mathbb{N}$ for the graph G given on the left-hand side with $n = 14$ sites

$$\sum_{j=1}^{\infty} j\mu(j) = 1 \quad \sum_{j=1}^{\infty} j^2\mu(j) =: \sigma^2 \in (0, \infty) \quad \gcd(j > 0: \mu(j) > 0) = 1 \tag{61}$$

holds, i.e., the number of offspring has mean 1, finite variance and is not supported on a sublattice of the integers. Note that the assumption of having $m = 1$ is not a restriction as we can transform any given offspring distribution with positive mean into a critical offspring distribution without changing the law of the resulting graphs when conditioning on a given number of sites; see Sect. 2.1 in [4]. The following lemma is a classical result due to Aldous.

Lemma 5.6 (c.f. Theorem 23 in [5]) *Under the assumptions in (61), we have that $(G_n)_{n \geq 1}$ converges to the CRT for $c = \sigma/2$.*

Note that for critical Galton–Watson trees, mixing and relaxation times were studied by Jonasson in [20], relying on estimates of the effective resistance between the root and the leaves. We now present an alternative proof of his results, which extends to more general families of combinatorial trees. In the following, we first choose the trees $(G_n)_{n \geq 1}$, keep them fixed and then perform simple random walks on $(G_n)_{n \geq 1}$. We denote the law of $(G_n)_{n \geq 1}$ by P .

Proposition 5.7 *Let $(G_n)_{n \geq 1}$ be a family of random trees converging to the Brownian CRT for some c . Then, we have that P -almost surely, the family of simple random walks on $(G_n)_{n \geq 1}$ does not exhibit cutoff.*

Intuitively, the absence of cutoff is explained by the fact that the law of the random walk on $(G_n)_{n \geq 1}$ converges to the law of a Brownian motion on the CRT, indicating a smooth decay of the total-variation distance for the random walk at a time of order $N^{3/2}$. In order to prove Proposition 5.7, we will show that P -almost surely, (35) is not satisfied. First, we will prove that the mixing time and the relaxation time are, for all n sufficiently large, with positive probability both of order $n^{3/2}$. We start with the following upper bound on the mixing time.

Lemma 5.8 *Let (G_n) be as in Proposition 5.7. Then, for all $\varepsilon > 0$, there exists a constant C_ε such that for n sufficiently large,*

$$P\left(t_{\text{mix}}^n\left(\frac{1}{4}\right) \leq C_\varepsilon n^{\frac{3}{2}}\right) \geq 1 - \varepsilon. \tag{62}$$

Proof Note that the tree G_n satisfies $\text{diam}(G_n) \leq 2 \max_{i \in \{1, \dots, 2n-1\}} |s(i)|$. From Lemma 5.6, we see that for all $\varepsilon > 0$, there is $c_1 = c_1(\varepsilon) > 0$ such that for n large enough,

$$P\left(\text{diam}(G_n) \geq c_1 n^{\frac{1}{2}}\right) < \varepsilon. \tag{63}$$

Now we use (29) from Proposition 3.1 to conclude. □

It remains to show a corresponding lower bound of order $n^{3/2}$ for the relaxation time of the simple random walk on G_n , which requires a bit of setup. We start by giving some statements for the Brownian excursion. We then carry over these observations to trees using Lemma 5.6. More precisely, we first choose a new root o_* , which will be a δ -center of mass. We then show that there exists a site v at distance of order \sqrt{n} from the new root with order n many sites in the tree T_v . Note that a Brownian excursion $(e(t))_{t \in [0,1]}$ attains almost surely its extrema in every compact subset of $[0, 1]$. We let t_{min} and t_{max} be such that

$$e(t_{\text{min}}) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} e(t), \quad e(t_{\text{max}}) = \max_{t \in [\frac{1}{4}, \frac{3}{4}]} e(t). \tag{64}$$

Consider, for $\theta > 0$ and $\delta > 0$, the events

$$B_1 := \{e(t_{\text{min}}) > \theta\} \cap \left\{ \max_{t \in [0, \delta]} e(t) \leq \frac{\theta}{2} \right\} \tag{65}$$

and

$$B_2 := \{e(t) \geq 2\theta \text{ for all } t \in [t_{\text{max}} - \theta, t_{\text{max}} + \theta]\}, \tag{66}$$

see Fig. 3 for a visualization.

Lemma 5.9 *For every $\varepsilon > 0$, there exists some $\theta > 0$ and some $\delta > 0$ such that $\mathbb{P}(B_1 \cap B_2) \geq 1 - \varepsilon/2$.*

Proof Note that $e(t_{\text{max}}) > e(t_{\text{min}}) > 0$ holds almost surely by the construction of the Brownian excursion. Since $(e(t))_{t \in [0,1]}$ is continuous, and $e(\frac{1}{4})$ and $e(\frac{3}{4})$ have densities with respect to the Lebesgue measure on $[0, \infty)$, the probability of the event

$$\{e(t_{\text{min}}) > \theta\} \cap \{e(t) \geq 2\theta \text{ for all } t \in [t_{\text{max}} - \theta, t_{\text{max}} + \theta]\}$$

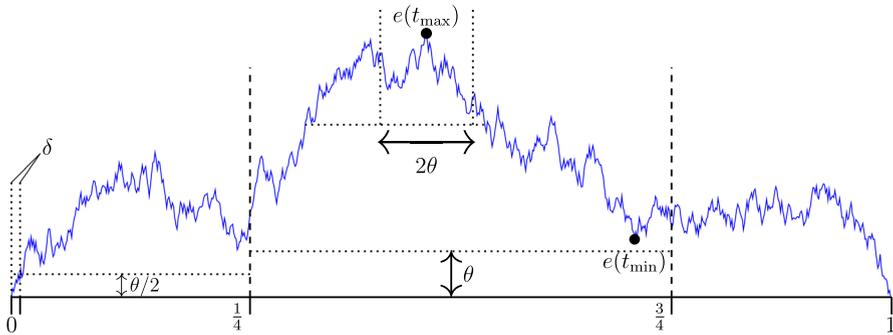


Fig. 3 Visualization of the events B_1 and B_2 for the Brownian excursion

goes to 1 for $\theta \rightarrow 0$. Hence, choose θ such that this probability is at least $1 - \varepsilon/4$. Moreover, since $e(0) = 0$ almost surely and $(e(t))_{t \in [0,1]}$ has continuous paths, we have that for every $\varepsilon, \theta > 0$, there exists some $\delta > 0$ such that the event

$$\left\{ \max_{t \in [0, \delta]} e(t) \leq \frac{\theta}{2} \right\}$$

has probability at least $1 - \varepsilon/4$. This implies that the event $B_1 \cap B_2$ has probability at least $1 - \varepsilon/2$. □

We now use Lemma 5.9 to show the following bound on the relaxation time.

Lemma 5.10 *For all $\varepsilon > 0$, there exists some constant $c_\varepsilon > 0$ such that for n sufficiently large,*

$$P\left(t_{\text{rel}}^n \geq c_\varepsilon n^{\frac{3}{2}}\right) \geq 1 - \varepsilon. \tag{67}$$

Proof Recall the contour function s and $|s|$. Fix some $\varepsilon > 0$ and let $\theta, \delta > 0$ be the constants from Lemma 5.9. We start the proof by first choosing a new root o_* of G_n as follows: Recall the constant $c > 0$ from the convergence of (G_n) to the CRT. We set

$$o_* := s(i_*) \quad \text{with} \quad i_* = \max \left\{ i \leq \frac{1}{2}n : |s(i)| \leq c^{-1}\theta\sqrt{n} \right\}.$$

We claim that whenever the event B_1^n given by

$$B_1^n := \left\{ |s(i)| > c^{-1}\theta\sqrt{n} \text{ for all } i \in \left[\frac{1}{2}n, \frac{3}{2}n \right] \right\} \cap \left\{ \max_{j \leq 2\delta n} |s(j)| < c^{-1}\frac{\theta}{2}\sqrt{n} \right\}$$

occurs then o_* is a δ -center of mass. To see this, observe that the subtree $T = T_{o_*}$, i.e., the largest subtree containing the new root o_* and only sites of distance at least $|o_*|$ from the old root o , must have at least $n/2$ sites. This is due to the fact that the contour

traverses, between times $n/2$ and $3n/2$, only edges which belong to T and each edge is visited at most twice. To conclude that o_* must be a δ -center of mass, note that the tree \tilde{T} , induced by the vertices $V \setminus V(T) \cup \{o_*\}$, contains at least δn many sites. This follows from the observation that all edges traversed by the contour until time $2\delta n$ belong to \tilde{T} provided that B_1^n occurs. Thus, T and \tilde{T} satisfy (5) and this proves the claim.

Moreover, suppose that in addition the event B_2^n given by

$$B_2^n := \left\{ \exists i \in \left[\left(\frac{1}{2} + 2\theta \right) n, \left(\frac{3}{2} - 2\theta \right) n \right] : \min_{j \in [i - 2\theta n, i + 2\theta n]} |s(j)| \geq 2c^{-1}\theta\sqrt{n} \right\}$$

occurs. We claim that in this case for i from the event B_2^n , the vertex v given by

$$v := s(j_*) \quad \text{with} \quad j_* = \max \left\{ j < i - 2\theta n : |s(j)| < 2c^{-1}\theta\sqrt{n} \right\}$$

has at least distance $c^{-1}\theta\sqrt{n}$ from o_* and it holds that $|T_v| \geq 2\theta n$. Again, this claim follows from the definition of the contour s and the events B_1^n and B_2^n since all edges visited by the contour during $[i - 2\theta n, i + 2\theta n]$ must belong to T_v .

Hence, whenever the events B_1^n and B_2^n both occur, Proposition 4.1 gives a lower bound of $2\delta c^{-1}\theta^2 n^{3/2}$ for the relaxation time. Notice that the events B_1 and B_2 for the Brownian excursion correspond to the events B_1^n and B_2^n . More precisely, we have that

$$P \left(|s(i)| > c^{-1}\theta\sqrt{n} \text{ for all } i \in \left[\frac{1}{2}n, \frac{3}{2}n \right] \right) = P \left(\tilde{s}_n(y) > \theta \text{ for all } y \in \left[\frac{1}{4}, \frac{3}{4} \right] \right)$$

converges to $\mathbb{P}(e(t_{\min}) \geq \theta)$ by Lemma 5.6. A similar statement applies for all other events in the definitions of B_1^n and B_2^n . Thus, for all $\varepsilon > 0$, Lemma 5.6 yields that there exists an $N = N(\varepsilon)$ such that for all $n \geq N$,

$$P(B_1^n \cap B_2^n) \geq \mathbb{P}(B_1 \cap B_2) - \frac{\varepsilon}{2}.$$

We conclude with Lemma 5.9. □

Proof of Proposition 5.7 Recall from Theorem 3.3 that a necessary assumption for the simple random walk to satisfy cutoff is that the product criterion (4) holds. In other words, it is enough to show that

$$Z_n := \frac{t_{\text{mix}}^n \left(\frac{1}{4} \right)}{t_{\text{rel}}^n} \text{ satisfies } P \left(\liminf_{n \rightarrow \infty} Z_n < \infty \right) = 1. \tag{68}$$

Note that Lemmas 5.8 and 5.10 imply (by choosing $K_\varepsilon = \frac{C_\varepsilon}{c_\varepsilon}$) that

$$\forall \varepsilon > 0, \exists K_\varepsilon < \infty \text{ such that } P(Z_n \leq K_\varepsilon) \geq 1 - 2\varepsilon. \tag{69}$$

The following general argument shows that (69) implies (68). Let $\tilde{C} > 0$ and $\tilde{c} \in (0, \tilde{C})$. Then for every $n \in \mathbb{N}$, we have that

$$\begin{aligned} P\left(\liminf_{k \rightarrow \infty} Z_k \geq \tilde{C}\right) &\leq P\left(\exists N_0 = N_0(\omega) < \infty: Z_m \geq \tilde{C} - \tilde{c} \text{ for all } m \geq N_0\right) \\ &\leq P(N_0 \geq n) + P(Z_n \geq \tilde{C} - \tilde{c}). \end{aligned} \quad (70)$$

By choosing n and \tilde{C} large enough, due to (69), both terms on the right-hand side of (70) are $\leq 2\varepsilon$. Since ε was arbitrary, we conclude that (68) holds.

6 Open Questions

In order to exclude cutoff for the simple random walk on spherically symmetric trees, we assumed that the maximum degree is bounded.

Question 6.1 *Can the assumption in Proposition 5.1 on having a bounded maximum degree be relaxed?*

Very recently, this question was answered positively by Chiclana and Peres [14]. In Sect. 5, we showed that we do not see cutoff for Galton–Watson trees, which are typically *short* and *fat*; see also [1].

Question 6.2 *Does a family of trees exist which is short and fat in the sense of [1] for which the simple random walk exhibits cutoff?*

Throughout this article, we used at several points the result of Basu, Hermon and Peres that cutoff occurs for the simple random walk on a family of trees if and only if (4) is satisfied.

Question 6.3 *For which families of graphs is the product criterion in (4) a necessary and sufficient condition such that the simple random walk exhibits cutoff?*

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Declarations

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A Proof of the Existence of a Center of Mass

Proof of Proposition 1.2 Note that each tree must have a vertex separator $x \in V(G)$, that means that when x is removed from the tree, all of the remaining connected components contain at most $|V(G)|/2$ many sites. This is a well-known fact from graph theory. We will now show that each vertex separator of a tree must also be a $\frac{1}{3}$ -center of mass in the above sense.

For a given vertex separator x , let S_1, S_2, \dots, S_d for some $d \geq 2$ denote the connected components of the tree when removing x . Note that when $d = 2$, we take $S_1 \cup \{x\}$ and $S_2 \cup \{x\}$ as vertex sets for the trees T and \tilde{T} . When $d = 3$, associate T with the largest component S_i , which must contain at least $(|V(G)| - 1)/3$, but at most $|V(G)|/2$ many sites. Combine the remaining two components for \tilde{T} .

For $d \geq 4$, we describe now a procedure to reduce the problem of finding the trees T and \tilde{T} for a vertex separator x with d components (S_i), to a problem of finding the trees T and \tilde{T} for $d - 1$ components (\tilde{S}_i) which have all size at most $|V(G)|/2$. Hence, solving the problem of finding the trees T and \tilde{T} for x recursively will allow us to conclude the proof. Let S' and S'' be the smallest and second smallest component in (S_i), respectively. By the pigeonhole principle, we have that

$$|S'| + |S''| \leq \left(\frac{1}{4} + \frac{1}{3}\right) (|V(G)| - 1). \quad (71)$$

We distinguish two cases. If the left-hand side in (71) is at least $(|V(G)| - 1)/3$, then we associate T with $S' \cup S''$ and \tilde{T} with its complement, and we are done. If the left-hand side in (71) is at most $(|V(G)| - 1)/3$, then remove S' and S'' from (S_i) and replace it by $S' \cup S''$ to obtain (\tilde{S}_i).

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