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The Doubly Nonlocal Hele-Shaw–Cahn–Hilliard System with Singular Potential and Nonconstant Mobility

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Abstract
We present the rigorous asymptotic analysis in thin domains of a diffuse interface model of two-component Hele-Shaw flow based on an advective nonlocal Cahn–Hilliard equation with singular potential and nonconstant nondegenerate mobility for the relative concentration. The velocity is determined by a Stokes system in which the inhomogeneous viscosity is highly oscillating and dependent on the relative concentration. Using the notion of sigma-convergence for thin heterogeneous media, we obtain in the homogenization limit a new doubly nonlocal Hele-Shaw–Cahn–Hilliard-type model system containing an additional term arising from the dependence of the viscosity on the relative concentration. In the case when both the viscosity and the mobility coefficients do not depend on the relative concentration, we additionally prove that the new model is well posed and we establish the existence of global strong solutions.

Keywords Nonlocal Cahn–Hilliard–Stokes system with singular potential · Sigma-convergence · Thin domains · Doubly nonlocal Cahn–Hilliard–Hele-Shaw system · Deterministic homogenization

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1 Introduction and Main Results

In fluid dynamics, topological transitions of interfaces between macroscopically immiscible fluids are prominent phenomena and play an important role in many applications. Two-phase flow involving the slow flow at low Reynolds number of a fluid between two parallel flat plates separated by a small distance can lead to surprising phenomena like the Saffmann–Taylor fingering instability (Saffmann and Taylor 1985). Thus, careful derivation and analysis of such models are required.

In this work, we are interested in the mathematically rigorous derivation by homogenization and analysis of a new doubly nonlocal Hele-Shaw–Cahn–Hilliard model. We base our derivation of this new model on the nonlocal Cahn–Hilliard–Navier–Stokes model at low Reynolds number (so that the convective term in the Navier–Stokes equation is neglected) stated on the microscale, which we introduce first.

1.1 The ε-Model

On the microscopic scale, the fluid domain is of Hele-Shaw type, that is, we consider a region confined in between two rigid parallel plates described as follows: let Ω be a bounded open Lipschitz domain in $\mathbb{R}^{d-1}$ ($d = 2, 3$) and let $\varepsilon > 0$ be a fixed small parameter. The domain Ω_ε is given by $\Omega_\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$. Any $x \in \Omega_\varepsilon$ is written $x = (\overline{x}, x_d)$ where $\overline{x} \in \Omega$ and $-\varepsilon < x_d < \varepsilon$. The heterogeneity of the domain Ω_ε is implicit and arises from the fact that the fluids are mixed at length scale ε. This is reflected in both the viscosity and the mobility terms which oscillate at scale ε, the distribution function of the microstructures being represented by an assumption made on the fast spatial variable $\overline{y} = \overline{x}/\varepsilon$ covering several concrete behaviours such as the periodic (uniform) distribution, the almost periodic distribution and many more beside, specified in the context of sigma-convergence below. The domain Ω_ε is a thin layer shrinking to $\Omega \times \{0\} \equiv \Omega$ as part of passing to the homogenization limit $\varepsilon \to 0$. For a given $T > 0$, we set $Q_\varepsilon = (0, T) \times \Omega_\varepsilon$ and $Q = (0, T) \times \Omega$. In Ω_ε, we consider the ε-model problem given by:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}_\varepsilon}{\partial t} - \varepsilon^2 \mathrm{div}(\eta^\varepsilon(\cdot, \varphi_\varepsilon)\nabla \boldsymbol{u}_\varepsilon) + \nabla p_\varepsilon - \mu_\varepsilon \nabla\varphi_\varepsilon = \boldsymbol{h} \text{ in } Q_\varepsilon, \\ \mathrm{div}\boldsymbol{u}_\varepsilon = 0 \text{ in } Q_\varepsilon, \\ \dfrac{\partial \varphi_\varepsilon}{\partial t} + \boldsymbol{u}_\varepsilon \cdot \nabla\varphi_\varepsilon - \mathrm{div}(m^\varepsilon(\cdot, \varphi_\varepsilon)\nabla\mu_\varepsilon) = 0 \text{ in } Q_\varepsilon, \\ \mu_\varepsilon = \varepsilon^{-1}(a_\varepsilon\varphi_\varepsilon - J * \varphi_\varepsilon) + F'(\varphi_\varepsilon) \text{ in } Q_\varepsilon, \\ \dfrac{\partial \mu_\varepsilon}{\partial \nu} = 0 \text{ and } \boldsymbol{u}_\varepsilon = 0 \text{ on } (0, T) \times \partial\Omega_\varepsilon, \\ \boldsymbol{u}_\varepsilon(0, x) = \boldsymbol{u}_0^\varepsilon(x) \text{ and } \varphi_\varepsilon(0, x) = \varphi_0^\varepsilon(x) \text{ in } \Omega_\varepsilon. \end{cases} \tag{1.1}$$

In (1.1), $\boldsymbol{u}_\varepsilon$ represents the velocity of the fluid mixture, φ_ε is the order parameter representing the relative concentration of the mixture (relative difference of the two concentrations), μ_ε is the chemical potential, p_ε is the pressure, J is a suitable interaction kernel, a_ε is a coefficient (depending on J, see below), η^ε is the inhomogeneous oscillating viscosity, m^ε is the oscillating mobility, $\boldsymbol{h}$ stands for an external force density acting on the fluid mixture, F is the configuration potential accounting for the presence of two phases, while ν is the outward unit normal on $\partial\Omega_\varepsilon$. In (1.1), the data are constrained as follows:

(A1) the viscosity $\eta^\varepsilon(\,\cdot\,,\varphi_\varepsilon)(t,x) = \eta(x/\varepsilon, \varphi_\varepsilon(t,x))$, where $\eta : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, lies in $L^\infty(\mathbb{R}^d; \mathcal{C}^{0,1}_{\mathrm{loc}}(\mathbb{R}))$ and there exist $\eta_1, \eta_2 > 0$ such that

$$\eta_1 \leq \eta(y,r) \leq \eta_2 \text{ for a.e. } y \in \mathbb{R}^d \text{ and for all } r \in \mathbb{R}; \tag{1.2}$$

(A2) the interaction kernel $J \in W^{1,1}(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^{d-1} \setminus \{0\})$ satisfies $J(y) = J(-y)$ and is related to the coefficient a_ε via $a_\varepsilon(x) = \int_{\Omega_\varepsilon} J(x-z)\,\mathrm{d}z \geq 0$, $x \in \Omega_\varepsilon$, which satisfies $\varepsilon^{-1} a_\varepsilon \in L^\infty(\Omega_\varepsilon)$ and there is $\beta > 0$ such that

$$\beta \leq \varepsilon^{-1} a_\varepsilon \leq \beta^{-1} \text{ for all } x \in \Omega_\varepsilon;$$

(A3) the potential F can be written in the form $F = F_1 + F_2$ with a singular part $F_1 \in \mathcal{C}([-1,1]) \cap \mathcal{C}^2(-1,1)$ and a regular component $F_2 \in \mathcal{C}^2([-1,1])$, which satisfy the following assumptions: there exist $a_2 > 4(\beta^{-1} - \beta - \alpha)$ (where $\alpha := \min_{[-1,1]} F_2''$), $0 < \sigma_0 < 1$ and $c_0 > 0$ such that

(i) $F_1''(s) \geq a_2$ for all $s \in (-1, -1+\sigma_0] \cup [1-\sigma_0, 1)$,
(ii) $F_1''(s) + \beta \geq c_0$ for all $s \in (-1,1)$,
(iii) F_1'' is nondecreasing in $[1-\sigma_0, 1)$ and nonincreasing in $(-1, -1+\sigma_0]$,
(iv) $\lim_{x\to -1^+} F_1' = +\infty$ and $\lim_{x \to 1^-} F_1' = -\infty$;

(A4) the mobility $m^\varepsilon(\,\cdot\,,\varphi_\varepsilon)(t,x) = m(x/\varepsilon, \varphi_\varepsilon(t,x))$, where $m : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, lies in $L^\infty(\mathbb{R}^d; \mathcal{C}^{0,1}_{\mathrm{loc}}(\mathbb{R}))$ and there exist $m_1, m_2 > 0$ such that

$$m_1 \leq m(y,r) \leq m_2 \text{ for a.e. } y \in \mathbb{R}^d \text{ and for all } r \in \mathbb{R}; \tag{1.3}$$

(A5) the initial values $\varphi_0^\varepsilon \in L^\infty(\Omega_\varepsilon)$ and $\boldsymbol{u}_0^\varepsilon \in L^2(\Omega_\varepsilon)^d$ satisfy

$$\begin{aligned} &\|\boldsymbol{u}_0^\varepsilon\|_{L^2(\Omega_\varepsilon)^d} + \|\varphi_0^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c_1 \varepsilon^{\frac{1}{2}}, \quad \|\varphi_0^\varepsilon\|_{L^1(\Omega_\varepsilon)} \leq c_2 \varepsilon; \\ &\varepsilon^{-\frac{1}{2}} \|\boldsymbol{u}_0^\varepsilon - \boldsymbol{u}^0\|_{L^2(\Omega_\varepsilon)^d} + \varepsilon^{-\frac{1}{2}} \|\varphi_0^\varepsilon - \varphi^0\|_{L^2(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0, \end{aligned} \tag{1.4}$$

for some positive constants c_1 and c_2, where $\boldsymbol{u}^0 \in L^2(\Omega)^d$ and $\varphi^0 \in L^\infty(\Omega)$; finally, the forcing $\boldsymbol{h}$ has the form

$$\boldsymbol{h}(t,x) = (\boldsymbol{h}_1(t,\overline{x}), 0) \text{ for a.e. } (t, x = (\overline{x}, x_d)) \in (0,T) \times \Omega \times (-1,1) =: Q_1, \tag{1.5}$$

where $\boldsymbol{h}_1 \in L^2(Q)^{d-1}$.

Remark 1.1 Part (iv) of assumption (A3) on F' shows that there exists $s_0 \in (-1, 1)$ such that $F'(s_0) = 0$. We assume without loss of generality that $s_0 = 0$, and since the potential F can be chosen up to a constant, we assume henceforth that

$$F(0) = F'(0) = 0. \tag{1.6}$$

The physically most relevant examples of functions F and J are the logarithmic potential

$$F(s) = \frac{\theta}{2}\left((1+s)\log(1+s) + (1-s)\log(1-s)\right) - \frac{\theta_c}{2}s^2, \ s \in (-1, 1) \tag{1.7}$$

and, for $d = 3$, $J(x) = \beta_1 |x|^{-1}$ or, for $d = 2$, $J(x) = -\beta_2 \log |x|$, respectively. Here, θ and θ_c are the absolute temperature and the critical temperature, respectively. Below the critical temperature, i.e. when $0 < \theta < \theta_c$, phase separation occurs; otherwise, the mixed phase is stable. The parameters β_1 and β_2 are positive constants while $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. In (1.7), the splitting in singular and regular parts is realized by taking $F_1(s) = \frac{\theta}{2}\left((1+s)\log(1+s) + (1-s)\log(1-s)\right)$ and $F_2(s) = -\frac{\theta_c}{2}s^2$.

Now, assuming that each fluid has a viscosity $\eta_i(x/\varepsilon)$ ($i = 1, 2$), then the viscosity of the mixture is modelled by the concentration-dependent quantity $\eta^\varepsilon(x, \varphi) = \eta(x/\varepsilon, \varphi)$. In the unmatched viscosity case ($\eta_1 \neq \eta_2$), a typical form of η is the interpolation between η_1 and η_2 given by

$$\eta(y, r) = \eta_1(y)\frac{1+r}{2} + \eta_2(y)\frac{1-r}{2}, \ r \in [-1, 1], \text{ for a.e. } y \in \mathbb{R}^d.$$

The special case $\eta_1 = \eta_2$ is called the matched inhomogeneous viscosity; $\eta_1 = \eta_2$ can also be chosen to be a positive constant. In any case, the bounds (1.2) are assumed and we refer to Abels (2009) for more information on the unmatched case arising from different densities of the fluids.

It is important to note that the coefficient $1/\varepsilon$ in front of $(a_\varepsilon\varphi_\varepsilon - J * \varphi_\varepsilon)$ is to preserve the relative size of Ω_ε for small ε. It is easy to see that

$$\varepsilon^{-1}(a_\varepsilon\varphi_\varepsilon - J * \varphi_\varepsilon)(t, x) = \int_{\Omega_1} (J_\varepsilon(x - \zeta)(\widetilde{\varphi}_\varepsilon(t, x) - \widetilde{\varphi}_\varepsilon(t, \zeta))\,\mathrm{d}\zeta, \tag{1.8}$$

where $J_\varepsilon(x) = J(\overline{x}, \varepsilon x_d)$ for $x = (\overline{x}, x_d) \in \Omega_1$ and $\widetilde{\varphi}_\varepsilon(t, x) = \varphi_\varepsilon(t, \overline{x}, \varepsilon x_d)$ for $(t; x) \in Q_1$.

The system (1.1) is a nonlocal Cahn–Hilliard–Stokes system arising from a diffuse-interface model, which describes the evolution of an incompressible mixture of two immiscible fluids in Hele-Shaw cells (see, for example, Della Porta et al. 2018; Frigeri and Grasselli 2012 and the references therein). The local version of (1.1) was considered in Cheng and Feng (2017) (see also Gurtin et al. 1996). The system (1.1) is a good approximation of the Cahn–Hilliard–Navier–Stokes system in the context of low Reynolds number, see, for example, Han et al. (2013) in which a number of applications are discussed as well.

Our model (1.1) is a more general one as, in $(1.1)_1$, we consider a concentration-dependent inhomogeneous viscosity, thereby generalizing all the existing models. Another new aspect is the dependence of the nonconstant nondegenerate mobility coefficient upon both the spatial variable and the order parameter. These functions are highly oscillating with respect to the fast spatial scale $\overline{y} = \overline{x}/\varepsilon$. We will make a structured hypothesis on the behaviour of $\eta(\cdot, y_d, r)$ and $m(\cdot, y_d, r)$ which covers a wide range of concrete behaviours such as the periodicity, the almost periodicity and many more beside in the context of sigma-convergence.

Our goal in this work is twofold: (i) investigate the limiting behaviour of (1.1) when the thickness ε of the domain as well as the spatial oscillation of the coefficients simultaneously approaches zero in order to derive the upscaled model; and (ii) analyze the homogenized model by addressing first its well-posedness and, second, the regularity of its solutions. The above first goal will be achieved through deterministic homogenization theory by means of the sigma-convergence concept for thin heterogeneous media introduced in Jäger and Woukeng (2022) as a generalization of the two-scale convergence for thin periodic domains (Neuss-Radu and Jäger 2007). It is worth noting that the two-scale convergence for thin periodic domains is now well known while its general deterministic counterpart is at its early stage. Indeed, the only works using the latter concept appear to be Cardone et al. (2024), Jäger and Woukeng (2022), Peter and Woukeng (2024).

More generally, very few results on homogenization of multiphase flow are available in the literature; see, for example, Auriault et al. (1989), Banas and Mahato (2017), Cardone et al. (2024), Peter and Woukeng (2024), Sharmin et al. (2022). We also note the related recent works on upscaling of the Cahn–Hilliard equation coupled with the equations of linear elasticity, the so-called Cahn–Larché system, Reischmann and Peter (2020), Reischmann and Peter (2022).

1.2 The Main Results

In order to state our main results in a compact form, we introduce shorthand notations for certain mean integrals. For any fixed $\varepsilon > 0$ and function ψ defined on Q_ε, we define the partial mean integral $M_\varepsilon \psi$ of ψ on $Q = (0, T) \times \Omega$ as follows

$$M_\varepsilon \psi(t, \overline{x}) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \psi(t, \overline{x}, x_d)\, \mathrm{d}x_d, \quad (t, \overline{x}) \in Q. \tag{1.9}$$

The usual spatial average is denoted by an overbar, i.e. $\overline{\psi} = |\Omega|^{-1} \int_\Omega \psi$.

Assuming that the viscosity $\eta(y, r)$ and the mobility $m(y, r)$ satisfy a structure hypothesis in the context of sigma-convergence (see assumption (A6) in "Appendix A") with respect to $\overline{y}$, we obtain from (1.1) in the limit as $\varepsilon \to 0$ a new Hele-Shaw–Cahn–Hilliard-type model stated in our first main result as follows.

Theorem 1.1 (Upscaled model) *For any* $\varepsilon > 0$, *there exists a weak solution* $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, p_\varepsilon)$ *of* (1.1) *in the sense of Definition* 2.1. *Moreover, up to a subsequence* *of ε not relabelled, there exist functions $\boldsymbol{u} \in L^2(0, T; \mathbb{H})$, $\varphi \in \mathcal{C}([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, $\mu \in L^2(0, T; H^1(\Omega))$, $p \in L^2(0, T; L_0^2(\Omega))$ and $H(\varphi, \boldsymbol{u}) \in L^1(0, T; L^1(\Omega)^{d-1})$ such that, when $\varepsilon \to 0$,*

$$
\begin{aligned}
&M_\varepsilon \boldsymbol{u}_\varepsilon \to (\boldsymbol{u}, 0) \text{ in } L^2(Q)^d\text{-weak,}\\
&M_\varepsilon \varphi_\varepsilon \to \varphi \text{ in } L^2(Q)\text{-strong and in } L^2(0, T; H^1(\Omega))\text{-weak,}\\
&M_\varepsilon \mu_\varepsilon \to \mu \text{ in } L^2(0, T; H^1(\Omega))\text{-weak and } M_\varepsilon p_\varepsilon \to p \text{ in } L^2(Q)\text{-weak,}
\end{aligned}
\tag{1.10}
$$

where the quintuple $(\boldsymbol{u}, \varphi, \mu, p, H(\varphi, \boldsymbol{u}))$ solves the effective system

$$
\begin{cases}
\boldsymbol{u} + H(\varphi, \boldsymbol{u}) = G\boldsymbol{u}^0 + G * (\boldsymbol{h}_1 + \mu \nabla_{\overline{x}}\varphi - \nabla_{\overline{x}} p) \text{ in } Q,\\
\operatorname{div}_{\overline{x}} \boldsymbol{u} = 0 \text{ in } Q \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } (0, T) \times \partial\Omega,\\
\dfrac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla_{\overline{x}}\varphi - \operatorname{div}_{\overline{x}}(\widehat{m}(\varphi)\nabla\mu) = 0 \text{ in } Q,\\
\mu = \widehat{a}\varphi - \widehat{J} * \varphi + F'(\varphi) \text{ in } Q,\\
\dfrac{\partial \mu}{\partial \boldsymbol{n}} = 0 \text{ on } (0, T) \times \partial\Omega,\\
\varphi(0) = \varphi^0 \text{ in } \Omega,
\end{cases}
\tag{1.11}
$$

where the convolution operator in $(1.11)_1$ is with respect to time while the one in $(1.11)_4$ is with respect to space, G is a symmetric positive definite $(d-1) \times (d-1)$ matrix-valued function bounded a.e. in space and continuous in time defined in (3.2) and $H(\varphi, \boldsymbol{u})$ is defined in (3.47), (3.48). Furthermore, if the function η is φ-independent, that is, $\eta(y, r) = \eta(y)$, then the function $H(\varphi, \boldsymbol{u})$ vanishes, and $\boldsymbol{u} \in \mathcal{C}([0, T]; \mathbb{H})$ and $p \in L^2(0, T; H^1(\Omega) \cap L_0^2(\Omega))$.

Further notation and concepts involved in the statement of Theorem 1.1 are provided in Sect. 3 and “Appendix A”.

Equation $(1.11)_1$ is a nonlocal Hele-Shaw-type equation involving an extra term $H(\varphi, \boldsymbol{u})$ arising from the contribution of the viscosity coefficient η, especially from its dependence upon the relative concentration φ; see Sect. 3. The convective Cahn–Hilliard equation $(1.11)_3$ also has a special form as the macroscopic mobility term $\widehat{m}(\varphi)$ is a symmetric and positive definite matrix, in contrast to the well-known situation where $\widehat{m}(\varphi)$ is a scalar function of φ. We note that if η does not depend on φ, then $H(\varphi, \boldsymbol{u})$ is identically zero so that we recover the nonlocal (in time) Hele-Shaw equation obtained in Peter and Woukeng (2024). In this case, though $(1.11)_1$ has recently been derived in Peter and Woukeng (2024), the global model (1.11) is still new in the literature in that the Cahn–Hilliard equation has a new form (the mobility is a matrix) and the potential is singular. So our model (1.11) is a doubly nonlocal Hele-Shaw–Cahn–Hilliard system with nonconstant nondegenerate mobility and with a singular potential. Therefore, the analysis of this model is more involved compared to the one in Peter and Woukeng (2024) . To the best of our knowledge, this is the first time that such a model is considered in the literature.

For the simplified case that η and m do not depend on φ_ε (but may depend on y), we present some further analysis of the models. To start out with, it turns out that the ε-model and the upscaled one are both well posed. The well-posedness of the limit model is our second main result.

Theorem 1.2 (Continuous dependence on the data) *Assume $d = 3$ as well as the assumptions of Theorem 1.1. Assume moreover that the functions η and m are φ -independent, that is, $\eta(y,r) = \eta(y)$ and $m(y,r) = m(y)$. Let $(\boldsymbol{u}, \varphi, \mu, p)$ be as in Theorem 1.1. Then, problem (1.1) possesses a unique solution. Furthermore, the function $H(\varphi, \boldsymbol{u})$ in $(1.11)_1$ vanishes and the mobility coefficient $\widehat{m}$ is a matrix independent of φ and defined by*

$$\widehat{m} = \frac{1}{2}\int_{-1}^{1} M(m(\cdot, y_3)(I_2 + \nabla_{\overline{y}}\omega(\cdot, y_3)))\,\mathrm{d}y_3,$$

where the function $\omega = (\omega_j)_{j=1,2} \in [B^{1,2}_{\#A}(\mathbb{R}^2; H^1_0(I))]^2$ is the unique solution of the corrector problem

$$-\mathrm{div}_y(m(y)(\nabla_y\omega_j + e_j)) = 0 \text{ in } \mathbb{R}^2 \times I, \quad \omega_j \in B^{1,2}_{\#A}(\mathbb{R}^2; H^1_0(I)),$$

I_2 is the 2×2 identity matrix and e_j is the jth vector of the canonical basis in $\mathbb{R}^3$. The effective system (1.11) becomes

$$\begin{cases} \boldsymbol{u} = G\boldsymbol{u}^0 + G * (\boldsymbol{h}_1 + \mu\nabla_{\overline{x}}\varphi - \nabla_{\overline{x}}p) \text{ in } Q, \\ \mathrm{div}_{\overline{x}}\boldsymbol{u} = 0 \text{ in } Q \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } (0,T)\times\partial\Omega, \\ \dfrac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla_{\overline{x}}\varphi - \mathrm{div}_{\overline{x}}(\widehat{m}\nabla\mu) = 0 \text{ in } Q, \\ \mu = \widehat{a}\varphi - \widehat{J} * \varphi + F'(\varphi) \text{ in } Q, \\ \dfrac{\partial\mu}{\partial\boldsymbol{n}} = 0 \text{ on } (0,T)\times\partial\Omega, \\ \varphi(0) = \varphi^0 \text{ in } \Omega. \end{cases} \tag{1.12}$$

Assume in addition that $\boldsymbol{h}_1 \in L^\infty(0,T; L^4(\Omega)^2)$. If $(\boldsymbol{u}_1, \varphi_1)$ and $(\boldsymbol{u}_2, \varphi_2)$ are two weak solutions of (1.12) corresponding to the initial data $(\boldsymbol{u}^0_1, \varphi^0_1)$ and $(\boldsymbol{u}^0_2, \varphi^0_2)$ with source terms $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$, and if further $\left|\overline{\varphi^0_i}\right| < 1$, $i = 1, 2$, then there is a positive constant C depending on the norms of the solutions such that, for almost all $t \in [0,T]$,

$$\begin{aligned} &\|\varphi_1(t) - \varphi_2(t)\|^2_\# + \int_0^t \left(\|\varphi_1(\tau) - \varphi_2(\tau)\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_1(\tau) - \boldsymbol{u}_2(\tau)\|^2_{L^2(\Omega)}\right)\mathrm{d}\tau \\ &\le C\left(\left\|\varphi^0_1 - \varphi^0_2\right\|^2_\# + \left|\overline{\varphi^0_1} - \overline{\varphi^0_2}\right| + \left\|\boldsymbol{u}^0_1 - \boldsymbol{u}^0_2\right\|^2_{L^2(\Omega)} + \|\boldsymbol{h}_1(t) - \boldsymbol{h}_2(t)\|^2_{L^2(\Omega)}\right). \end{aligned} \tag{1.13}$$

In particular, the weak solution of (1.12) *is unique so that the whole sequence* $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ *converges in the sense of* (1.10).

System (1.12) is actually very close to the doubly nonlocal Hele-Shaw–Cahn–Hilliard model derived in Peter and Woukeng (2024) although it generalizes the latter. Indeed, in Peter and Woukeng (2024), we obtained a system like (1.12) but with $\widehat{m} = I_2$ (the identity matrix stemming from the fact that the mobility coefficient was constant and equal to 1). It is also very important to note that in Peter and Woukeng (2024) the system (1.12) involved a regular potential, while the function F' is singular at its endpoints in the current situation. This model is also new because of the previous facts and will be investigated. Noticing that the model in (1.12) is a two-dimensional one, it is useful to point out that the Hele-Shaw–Cahn–Hilliard system was originally a two-dimensional model (Dedè et al. 2018). It is also worth noting that a simplified version of (1.12) has already been investigated in Giorgini et al. (2018), Della Porta et al. (2018), Della Porta and Grasselli (2016). Indeed, in Della Porta et al. (2018) the following model was considered:

$$\begin{cases} \boldsymbol{u} = -\nabla p + \mu\nabla\varphi, \quad \mathrm{div}\boldsymbol{u} = 0, \\ \frac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi - \Delta\varphi = 0, \qquad\qquad \text{in } \Omega\times(0,T), \\ \mu = F'(\varphi) - J*\varphi, \end{cases} \tag{1.14}$$

with F being the convex logarithmic potential given by the singular part in (1.7), that is,

$$F(s) = \frac{\theta}{2}\left((1+s)\log(1+s) + (1-s)\log(1-s)\right), \ s\in(-1,1).$$

In Giorgini et al. (2018), the local version of (1.14) was considered while, in Della Porta and Grasselli (2016), the following local version of (1.14),

$$\begin{cases} \boldsymbol{u} = -\nabla p + \mu\nabla\varphi + \boldsymbol{h}, \quad \mathrm{div}\boldsymbol{u} = 0, \\ \frac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi - \Delta\varphi = 0, \qquad\qquad \text{in } \Omega\times(0,T), \\ \mu = a\varphi - J*\varphi + F'(\varphi), \end{cases}$$

was derived in a asymptotic procedure from a Cahn–Hilliard–Brinkman model by letting the constant viscosity therein tend to zero.

The next main result deals with the regularity properties of the unique weak solution of (1.12) provided that some further assumptions on the initial and source terms are made.

Theorem 1.3 (Regularity of the solutions) *Assume* $d = 3$ *as well as the assumptions of Theorem* 1.2. *Assume further that* $\boldsymbol{u}^0 = 0$ *and* $\nabla F'(\varphi^0) \in L^2(\Omega)^2$. *In addition, suppose that* $\boldsymbol{h}_1 \in W^{1,\infty}(0,T;L^2(\Omega)^2)$. *Let* $(\boldsymbol{u}, \varphi, \mu, p)$ *be as in Theorem* 1.2. *Then, the (unique) weak solution of* (1.12) *is a strong solution and satisfies*

$$\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^4(0,T;W^{1,4}(\Omega)) \cap L^2(0,T;W^{1,r}(\Omega)), \tag{1.15}$$

$$\frac{\partial \varphi}{\partial t} \in L^\infty(0,T;H^1(\Omega)') \cap L^2(0,T;L^2(\Omega)), \tag{1.16}$$

$$\mu \in L^\infty(0,T;H^1(\Omega)) \cap L^4(0,T;W^{1,4}(\Omega)) \cap L^2(0,T;H^2(\Omega)), \tag{1.17}$$

$$p \in L^\infty(0,T;H^1(\Omega) \cap L^2_0(\Omega)), \tag{1.18}$$

$$F'(\varphi) \in L^\infty(0,T;H^1(\Omega)), \tag{1.19}$$

where $2 \le r < \infty$.
Moreover, if $\operatorname{curl} \boldsymbol{h} \in L^\infty(0,T;L^r(\Omega))$ *for some* $2 \le r < \infty$, *then*

$$\boldsymbol{u} \in \begin{cases} L^2(0,T;W^{1,r}(\Omega)^2) \text{ for the same } r \text{ as } \operatorname{curl} \boldsymbol{h}, \\ L^4(0,T;W^{1,4}(\Omega)^2) \text{ if } r = 4, \\ L^\infty(0,T;H^1(\Omega)^2) \text{ if } r = 2. \end{cases} \tag{1.20}$$

The results in Theorem 1.3 have been proved in Della Porta et al. (2018) for the simplified model (1.14). They are new for our model (1.12).

The last main result deals with the two-dimensional ε-model posed in $\Omega_\varepsilon = (a,b) \times (-\varepsilon, \varepsilon)$ and it is stated as follows.

Theorem 1.4 *Assume* $d = 2$ *and* $\boldsymbol{u}^0 = 0$. *Assume further that the function* η *is constant (equal to* 1*). For each* $\varepsilon > 0$, *let* $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, p_\varepsilon)$ *be the solution to* (1.1). *Then, the sequence* $(M_\varepsilon \boldsymbol{u}_\varepsilon, M_\varepsilon \mu_\varepsilon, M_\varepsilon p_\varepsilon)_{\varepsilon>0}$ *weakly converges (as* $\varepsilon \to 0$*) in* $L^2((0,T)\times(a,b))^2 \times L^2((0,T)\times(a,b)) \times L^2((0,T)\times(a,b))$ *towards* $(0,\mu,p)$ *and the sequence* $(M_\varepsilon \varphi_\varepsilon)_{\varepsilon>0}$ *strongly converges in* $L^2((0,T)\times(a,b))$ *towards* φ *with* $\varphi \in L^\infty(0,T;H^1(a,b))$, $\mu \in L^2(0,T;H^1(a,b))$ *and* $p \in L^2(0,T;L^2_0(a,b))$. *Moreover, the pair* (φ,μ) *is the unique solution to the* 1*D nonlocal Cahn–Hilliard equation*

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} - \dfrac{\partial}{\partial x_1}\left(\widehat{m}(\varphi)\dfrac{\partial \mu}{\partial x_1}\right) = 0 \text{ in } (0,T)\times(a,b), \\ \mu = \widehat{a}\varphi - \widehat{J} * \varphi + F'(\varphi) \text{ in } (0,T)\times(a,b), \\ \dfrac{\partial \mu}{\partial x_1}(t,a) = \dfrac{\partial \mu}{\partial x_1}(t,b) = 0 \text{ in } (0,T), \\ \varphi(0) = \varphi^0 \text{ in } (a,b). \end{cases} \tag{1.21}$$

Furthermore, the pressure p *is the unique solution of*

$$\frac{\partial p}{\partial x_1} = \boldsymbol{h}_1 + \mu \frac{\partial \varphi}{\partial x_1}, \quad \int_a^b p \, \mathrm{d}x_1 = 0. \tag{1.22}$$

1.3 Outline of the Paper

Existence and the proof of the uniform estimates for the sequence of solutions to (1.1) are addressed in Sect. 2. Section 3 is concerned with the passage to the limit in (1.1) when the thickness of the domain shrinks to zero using the notion of sigma-convergence in thin heterogeneous media. We also derive therein the upscaled model and complete the proof of Theorem 1.1 with the limit passage. Section 4 deals with the continuous dependence of the solutions upon the initial data and some regularity results of the solution, proving the remaining main theorems in particular. Finally, in Sect. 5, we work out some concrete problems by relying on specific types of heterogeneities. Standard results on sigma-convergence in thin heterogeneous media are gathered in "Appendix A", while an additional uniqueness result (for the microscopic problem) is proved in "Appendix B".

1.4 Notation

Unless otherwise specified, the vector spaces throughout are assumed to be real vector spaces, and the scalar functions are assumed to take real values. If X and F denote a locally compact space and a Banach space, respectively, then we write $\mathcal{C}(X; F)$ and $\mathrm{BUC}(X; F)$ for continuous mappings of X into F and bounded uniformly continuous mappings of X into F, respectively. We shall always assume that $\mathrm{BUC}(X; F)$ is equipped with the supremum norm $\|u\|_\infty = \sup_{x\in X} \|u(x)\|$, where $\|\cdot\|$ denotes the norm in F. For brevity, we will write $\mathcal{C}(X) = \mathcal{C}(X; \mathbb{R})$ and $\mathrm{BUC}(X) = \mathrm{BUC}(X; \mathbb{R})$. Likewise, the usual space $L^p(X; F)$ and $L^p_{\mathrm{loc}}(X; F)$ (X provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{\mathrm{loc}}(X)$, respectively, in the case when $F = \mathbb{R}$. Finally, it will always be assumed that the numerical spaces $\mathbb{R}^m$ ($m \geq 1$) and their open sets are each equipped with Lebesgue measure $\mathrm{d}y = \mathrm{d}y_1 \dots \mathrm{d}y_m$. The space $\mathbb{R}^m_\xi$ will denote the numerical space $\mathbb{R}^m$ of generic variable ξ.

Throughout the work, C will denote a generic constant independent of $\varepsilon > 0$ which may change from line to line.

2 Existence Result and Uniform Estimates

2.1 Existence Result

We begin with the functional-analytic setup. If X is a Banach space, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X and its topological dual X'. We set $\mathbb{X} = X \times \cdots \times X$, d times, and we equip $\mathbb{X}$ with the product topology. In case X is a real Hilbert space with inner product $(\cdot,\cdot)_X$, we shall denote by $\|\cdot\|_X$ the induced norm. We therefore introduce the classical Hilbert spaces for the Navier–Stokes systems with no-slip boundary condition (see, for example, Temam 2001) $\mathbb{H}_\varepsilon$ and $\mathbb{V}_\varepsilon$ defined by $\mathbb{V}_\varepsilon = \{\boldsymbol{u} \in \mathbb{H}^1_0(\Omega_\varepsilon) : \mathrm{div}\boldsymbol{u} = 0 \text{ in } \Omega_\varepsilon\}$ and $\mathbb{H}_\varepsilon = \{\boldsymbol{u} \in \mathbb{L}^2(\Omega_\varepsilon) : \mathrm{div}\boldsymbol{u} = 0 \text{ in } \Omega_\varepsilon$ and $\boldsymbol{u}\cdot\nu = 0 \text{ on } \partial\Omega_\varepsilon\}$, where ν is the outward unit normal to $\partial\Omega_\varepsilon$. The space $\mathbb{H}_\varepsilon$ is endowed with the scalar product denoted by $(\cdot,\cdot)$, the associated norm of which is denoted by $\|\cdot\|_{\mathbb{H}_\varepsilon}$. The space $\mathbb{V}_\varepsilon$ is equipped with the inner product

$$((\boldsymbol{u}, \boldsymbol{v})) = (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}), \quad (\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}_\varepsilon)$$

whose associated norm is the norm of the gradient. Owing to the Poincaré inequality, the norm in $\mathbb{V}_\varepsilon$ is equivalent to the $\mathbb{H}^1(\Omega_\varepsilon)$-norm. We also define the space $L^2_0(\Omega_\varepsilon) = \{v \in L^2(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} v\,\mathrm{d}x = 0\}$. We denote by $\mathbb{V}$ (resp. $\mathbb{H}$) the space defined as $\mathbb{V}_\varepsilon$ (resp. $\mathbb{H}_\varepsilon$) when replacing Ω_ε by Ω. Finally, for any $f \in H^1(D)'$ (D being any open bounded domain in $\mathbb{R}^d$), $\overline{f}$ will stand for the average of f over D, i.e. $\overline{f} = |D|^{-1}\langle f, 1\rangle$ where $|D|$ denotes the Lebesgue measure of D.

The following is the notion of weak solutions that will be considered in this work.

Definition 2.1 Let $\boldsymbol{u}_0^\varepsilon \in \mathbb{H}_\varepsilon$ and $\varphi_0^\varepsilon \in L^\infty(\Omega_\varepsilon)$ with $F(\varphi^0) \in L^1(\Omega_\varepsilon)$, $|\varphi_0^\varepsilon| \leq 1$ and $0 < T < \infty$ be given. A couple $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon)$ is a weak solution of (1.1) on $[0, T]$ corresponding to $(\boldsymbol{u}_0^\varepsilon, \varphi^0)$ if

- $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon)$ and μ_ε satisfy
  - (i) $\boldsymbol{u}_\varepsilon \in \mathcal{C}([0, T]; \mathbb{H}_\varepsilon) \cap L^2(0, T; \mathbb{V}_\varepsilon)$ with $\partial \boldsymbol{u}_\varepsilon/\partial t \in L^2(0, T; \mathbb{V}'_\varepsilon)$,
  - (ii) $\varphi_\varepsilon \in \mathcal{C}([0, T]; L^2(\Omega_\varepsilon)) \cap L^2(0, T; H^1(\Omega_\varepsilon))$ with $\partial\varphi_\varepsilon/\partial t \in L^2(0, T; H^1(\Omega_\varepsilon)')$,
  - (iii) $\varphi_\varepsilon \in L^\infty(Q_\varepsilon)$, $|\varphi_\varepsilon(t, x)| < 1$ a.e. $(t, x) \in Q_\varepsilon$;
- Setting $\rho_\varepsilon(x, \varphi_\varepsilon) = \varepsilon^{-1}a_\varepsilon(x)\varphi_\varepsilon + F'(\varphi_\varepsilon)$, we have for every $\psi \in H^1(\Omega_\varepsilon)$, $v \in \mathbb{V}_\varepsilon$ and for a.e. $t \in (0, T)$,

$$\left\langle \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}, v\right\rangle + \varepsilon^2\left(\eta^\varepsilon(\,\cdot\,, \varphi_\varepsilon)\nabla \boldsymbol{u}_\varepsilon, \nabla v\right) = -\int_{\Omega_\varepsilon}(v\cdot\nabla\mu_\varepsilon)\varphi_\varepsilon\,\mathrm{d}x + \int_{\Omega_\varepsilon}\boldsymbol{h}(t)v\,\mathrm{d}x,$$
$$\left\langle \frac{\partial \varphi_\varepsilon}{\partial t}, \psi\right\rangle + (\nabla\rho_\varepsilon, \nabla\psi) = \int_{\Omega_\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla\psi)\varphi_\varepsilon\,\mathrm{d}x + \int_{\Omega_\varepsilon}\varepsilon^{-1}(\nabla J * \varphi_\varepsilon)\cdot\nabla\psi\,\mathrm{d}x;$$

- $\boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0^\varepsilon$ and $\varphi_\varepsilon(0) = \varphi_0^\varepsilon$.

The following existence result holds.

Theorem 2.1 *Let the assumptions* (A1)–(A5) *be satisfied. Let $\varepsilon > 0$ be fixed and assume further that $\left|\overline{\varphi_0^\varepsilon}\right| < 1$. Then, there exists a solution $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon)$ of* (1.1) *in the sense of Definition* 2.1. *Moreover, to each solution $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon)$ is associated a unique $p_\varepsilon \in L^2(0, T; L^2_0(\Omega_\varepsilon))$ such that* $(1.1)_1$ *holds in the sense of distributions.*

We can also prove a uniqueness results under additional assumptions. Namely, if in addition the functions $\eta(y, r)$ and $m(y, r)$ are independent of r and further if $\boldsymbol{h}_1 \in L^\infty(0, T; L^4(\Omega)^{d-1})$ and $\boldsymbol{u}_0^\varepsilon \in L^4(\Omega_\varepsilon)^d$, then the solution $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon)$ is unique. As this result is not relevant for what follows, the proof is relegated to "Appendix B".

2.2 Proof of Theorem 2.1

The proof is divided in three steps developed in the following subsections.

2.2.1 Approximate Solutions

We follow the same way of reasoning as in (Frigeri and Grasselli 2012, Proof of Theorem 1) (see also Frigeri et al. 2015, Proof of Theorem 2) by regularizing the singular potential F and defining the approximate problem. To this end, we fix $0 < \sigma < 1$ and define the smooth potential as follows: $F_\sigma = F_{1\sigma} + \overline{F}_2$ where

$$F_{1\sigma}(s) = \begin{cases} F_1'(1-\sigma) \text{ for } s \geq 1-\sigma \\ F_1'(s) \text{ for } |s| \leq 1-\sigma \\ F_1'(-1+\sigma) \text{ for } s \leq -1+\sigma \end{cases} \tag{2.1}$$

with $F_{1\sigma}(0) = F_1(0)$, $F_{1\sigma}'(0) = F_1'(0)$, and $\overline{F}_2$ is a $\mathcal{C}^2$-extension of F_2 on $\mathbb{R}$ with quadratic growth satisfying

$$\overline{F}_2(s) \geq \min_{[-1,1]} F_2 - 1 \text{ and } \overline{F}_2''(s) \geq \min_{[-1,1]} F_2'' \text{ for all } s \in \mathbb{R}. \tag{2.2}$$

Then, as it can be seen in (Frigeri et al. 2015, p. 1271), assumption (A3) infers that

$$F_{1\sigma}(s) \leq F_1(s) \text{ for all } s \in (-1, 1) \text{ and } \sigma \in (0, \sigma_0]. \tag{2.3}$$

Putting together (2.3) and the fact that $\overline{F}_2$ has quadratic growth, and invoking the assumption $F(\varphi_0^\varepsilon) \in L^1(\Omega_\varepsilon)$, yields

$$\int_{\Omega_\varepsilon} F_\sigma(\varphi_0^\varepsilon)\,\mathrm{d}x \leq \int_{\Omega_\varepsilon} F_1(\varphi_0^\varepsilon)\,\mathrm{d}x + C < \infty \text{ for all } \sigma \in (0, \sigma_0], \tag{2.4}$$

as it can easily be shown that $F_1(\varphi_0^\varepsilon) \in L^1(\Omega_\varepsilon)$ (recall that $F(\varphi_0^\varepsilon) \in L^1(\Omega_\varepsilon)$). We also get from (A3) as in Frigeri et al. (2015) that there exist $0 < \delta \in \mathbb{R}$ and $\delta_0, C_1, C_2 \in \mathbb{R}$ such that, for all $s \in \mathbb{R}$,

$$\begin{cases} F_\sigma(s) \geq \delta s^2 - \delta_0, \\ F_\sigma''(s) + \beta \geq c_0 \\ \left|F_\sigma'(s)\right|^2 \leq C_1 \left|F_\sigma(s)\right| + C_2, \end{cases} \quad \text{for all } \sigma \in (0, \sigma_0]\,. \tag{2.5}$$

This being so, we consider the following approximate problem

$$\begin{cases} \dfrac{\partial \boldsymbol{u}_{\varepsilon\sigma}}{\partial t} - \varepsilon^2 \mathrm{div}(\eta^\varepsilon(\,\cdot, \varphi_{\varepsilon\sigma})\nabla \boldsymbol{u}_{\varepsilon\sigma}) + \nabla p_{\varepsilon\sigma} - \mu_{\varepsilon\sigma}\nabla\varphi_{\varepsilon\sigma} = \boldsymbol{h} \text{ in } Q_\varepsilon, \\ \mathrm{div}\boldsymbol{u}_{\varepsilon\sigma} = 0 \text{ in } Q_\varepsilon, \\ \dfrac{\partial \varphi_{\varepsilon\sigma}}{\partial t} + \boldsymbol{u}_{\varepsilon\sigma} \cdot \nabla\varphi_{\varepsilon\sigma} - \mathrm{div}(m^\varepsilon(\,\cdot, \varphi_{\varepsilon\sigma})\nabla\mu_{\varepsilon\sigma}) = 0 \text{ in } Q_\varepsilon, \\ \mu_{\varepsilon\sigma} = \varepsilon^{-1}(a_\varepsilon\varphi_{\varepsilon\sigma} - J * \varphi_{\varepsilon\sigma}) + F_\sigma'(\varphi_{\varepsilon\sigma}) \text{ in } Q_\varepsilon, \\ \dfrac{\partial \mu_{\varepsilon\sigma}}{\partial \nu} = 0 \text{ and } \boldsymbol{u}_{\varepsilon\sigma} = 0 \text{ on } (0, T) \times \partial\Omega_\varepsilon, \\ \boldsymbol{u}_{\varepsilon\sigma}(0, x) = \boldsymbol{u}_0^\varepsilon(x) \text{ and } \varphi_{\varepsilon\sigma}(0, x) = \varphi_0^\varepsilon(x) \text{ in } \Omega_\varepsilon, \end{cases} \tag{2.6}$$

where σ lies in $(0, \sigma_0]$. Putting properties (2.4)–(2.5) together, we see that the function F_σ satisfies the same hypotheses like in Frigeri et al. (2015) (see assumptions (H3)–(H5) therein), so that taking into account the above-mentioned properties in conjunction with (A1), (A2) and (A4), we may appeal to (Frigeri et al. 2015, Theorem 1) to derive the existence of a vector function $(\boldsymbol{u}_{\varepsilon\sigma}, \varphi_{\varepsilon\sigma})$ satisfying

$$\boldsymbol{u}_{\varepsilon\sigma} \in \mathcal{C}([0,T];\mathbb{H}_\varepsilon) \cap L^2(0,T;\mathbb{V}_\varepsilon) \text{ with } \frac{\partial \boldsymbol{u}_{\varepsilon\sigma}}{\partial t} \in L^2(0,T;\mathbb{V}'_\varepsilon), \tag{2.7}$$

$$\varphi_{\varepsilon\sigma} \in \mathcal{C}([0,T];L^2(\Omega_\varepsilon)) \cap L^2(0,T;H^1(\Omega_\varepsilon)) \text{ with } \frac{\partial \varphi_{\varepsilon\sigma}}{\partial t} \in L^2(0,T;H^1(\Omega_\varepsilon)'), \tag{2.8}$$

$$\mu_{\varepsilon\sigma} = \varepsilon^{-1}(a_\varepsilon \varphi_{\varepsilon\sigma} - J * \varphi_{\varepsilon\sigma}) + F'_\sigma(\varphi_{\varepsilon\sigma}) \in L^2(0,T;H^1(\Omega_\varepsilon)) \tag{2.9}$$

and solving system (2.6). It is important to note that, as we assumed low Reynolds number, the convective term $(\boldsymbol{u}_{\varepsilon\sigma} \cdot \nabla)\boldsymbol{u}_{\varepsilon\sigma}$ which appears in Frigeri et al. (2015) is of no effect in our model, so that the estimates of the velocity here do not depend on the dimension d as in Frigeri et al. (2015). Also, the fact that the diffusion term depends on x/ε does not tamper the proof since the operator $-\mathrm{div}(\eta^\varepsilon(\cdot,\varphi_{\varepsilon\sigma})\nabla)$ has the same properties as the one in Frigeri et al. (2015). We also obtain as in Peter and Woukeng (2024) the existence of a pressure $p_{\varepsilon\sigma} \in L^2(0,T;L^2_0(\Omega_\varepsilon))$ satisfying $(2.6)_1$.

2.2.2 A Priori Bounds Based on the Energy Estimate

We need to derive uniform estimates which will be useful in the limit passage first in σ and then in ε. (The passage to the limit in ε will be done through the homogenization process.) To begin with, we first use the dilatation in the vertical variable x_d, i.e. $y_d = x_d/\varepsilon$, and therefore define the new functions and coefficients as follows: $\widetilde{\varphi}_{\varepsilon\sigma}(t,x) = \varphi_{\varepsilon\sigma}(t,\overline{x},\varepsilon x_d)$ (for $(t,x) \in Q_1$) and the same definition for $\widetilde{\boldsymbol{u}}_{\varepsilon\sigma}, \widetilde{\mu}_{\varepsilon\sigma}, \widetilde{p}_{\varepsilon\sigma}, \widetilde{\eta^\varepsilon}(\cdot,\widetilde{\varphi}_{\varepsilon\sigma})$. We also set $J_\varepsilon(x) = J(\overline{x},\varepsilon x_d)$ and $\widetilde{a}_\varepsilon(x) = \varepsilon^{-1}a_\varepsilon(x) \equiv (J_\varepsilon * 1)(x)$ for $x \in \Omega_1$. With this definition, we recall that we have (1.8).

This being so, we set as in Peter and Woukeng (2024)

$$\nabla_\varepsilon = \left(\nabla_{\overline{x}}, \varepsilon^{-1}\frac{\partial}{\partial x_d}\right) \text{ and } \mathrm{div}_\varepsilon = \mathrm{div}_{\overline{x}} + \varepsilon^{-1}\frac{\partial}{\partial x_d} \equiv \nabla_\varepsilon \cdot .$$

We denote by $H^1_\varepsilon(\Omega_1)$ the space $H^1(\Omega_1)$ equipped with the H^1-norm, the usual gradient operator ∇ being replaced by ∇_ε, i.e.

$$\|u\|_{H^1_\varepsilon(\Omega_1)} = \left(\|u\|^2_{L^2(\Omega_1)} + \|\nabla_\varepsilon u\|^2_{L^2(\Omega_1)}\right)^{\frac{1}{2}} \text{ for } u \in H^1(\Omega_1).$$

In the new notation, the approximate problem reads as follows:

$$\begin{cases} \dfrac{\partial \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}}{\partial t} - \varepsilon^2 \mathrm{div}(\widetilde{\eta}^{\varepsilon}(\cdot, \widetilde{\varphi}_{\varepsilon\sigma})\nabla_\varepsilon \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}) + \nabla_\varepsilon \widetilde{p}_{\varepsilon\sigma} - \widetilde{\mu}_{\varepsilon\sigma}\nabla_\varepsilon \widetilde{\varphi}_{\varepsilon\sigma} = \boldsymbol{h} \text{ in } Q_1, \\ \mathrm{div}\widetilde{\boldsymbol{u}}_{\varepsilon\sigma} = 0 \text{ in } Q_1, \\ \dfrac{\partial \widetilde{\varphi}_{\varepsilon\sigma}}{\partial t} + \widetilde{\boldsymbol{u}}_{\varepsilon\sigma} \cdot \nabla_\varepsilon \widetilde{\varphi}_{\varepsilon\sigma} - \mathrm{div}(\widetilde{m}^{\varepsilon}(\cdot, \widetilde{\varphi}_{\varepsilon\sigma})\nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma}) = 0 \text{ in } Q_1, \\ \widetilde{\mu}_{\varepsilon\sigma} = \widetilde{a}_\varepsilon \widetilde{\varphi}_{\varepsilon\sigma} - J_\varepsilon * \widetilde{\varphi}_{\varepsilon\sigma} + F'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma}) \text{ in } Q_1, \\ \dfrac{\partial \widetilde{\mu}_{\varepsilon\sigma}}{\partial \nu} \equiv \nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma} \cdot \nu = 0 \text{ and } \widetilde{\boldsymbol{u}}_{\varepsilon\sigma} = 0 \text{ on } (0,T) \times \partial\Omega_1, \\ \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(0,x) = \widetilde{\boldsymbol{u}}_0^{\varepsilon}(x) \text{ and } \widetilde{\varphi}_{\varepsilon\sigma}(0,x) = \widetilde{\varphi}_0^{\varepsilon}(x) \text{ in } \Omega_1. \end{cases} \tag{2.10}$$

Then, setting

$$\begin{aligned} \mathcal{E}(\widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(t), \widetilde{\varphi}_{\varepsilon\sigma}(t)) = \frac{1}{2} \left\| \widetilde{\boldsymbol{u}}_{\varepsilon\sigma} \right\|_{L^2}^2 \\ + \frac{1}{4} \iint_{\Omega_1 \times \Omega_1} J_\varepsilon(x-\xi)(\widetilde{\varphi}_{\varepsilon\sigma}(t,x) - \widetilde{\varphi}_{\varepsilon\sigma}(t,\xi))^2 \, \mathrm{d}x \, \mathrm{d}\xi \\ + \int_{\Omega_1} F_\sigma(\widetilde{\varphi}_{\varepsilon\delta}(t)) \, \mathrm{d}x, \end{aligned} \tag{2.11}$$

and proceeding exactly as in the proof of (Peter and Woukeng 2024, Lemma 2.2), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\widetilde{\boldsymbol{u}}_{\varepsilon\sigma}, \widetilde{\varphi}_{\varepsilon\sigma}) + \varepsilon^2 \left(\widetilde{\eta}^{\varepsilon}(\cdot, \widetilde{\varphi}_{\varepsilon\sigma})\nabla_\varepsilon \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}, \nabla_\varepsilon \widetilde{\boldsymbol{u}}_{\varepsilon\sigma} \right) + \left(\widetilde{m}^{\varepsilon}(\cdot, \widetilde{\varphi}_{\varepsilon\sigma})\nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma}, \nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma} \right) = (\boldsymbol{h}, \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}). \tag{2.12}$$

Hence, integrating (2.12) over $(0, t)$ ($t > 0$ freely fixed) gives

$$\begin{aligned} & \mathcal{E}(\widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(t), \widetilde{\varphi}_{\varepsilon\sigma}(t)) + \int_0^t \left(\varepsilon^2 \left\| \sqrt{\widetilde{\eta}^{\varepsilon}(\cdot, \widetilde{\varphi}_{\varepsilon\sigma})}\nabla_\varepsilon \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(\tau) \right\|_{L^2}^2 + \left\| \sqrt{\widetilde{m}^{\varepsilon}(\cdot, \widetilde{\varphi}_{\varepsilon\sigma})}\nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma}(\tau) \right\|_{L^2}^2 \right) \mathrm{d}\tau \\ & \leq \mathcal{E}(\widetilde{\boldsymbol{u}}_0^{\varepsilon}, \widetilde{\varphi}_0^{\varepsilon}) + \int_0^t \boldsymbol{h}(\tau) \cdot \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(\tau) \, \mathrm{d}\tau, \end{aligned} \tag{2.13}$$

or, taking into account (A1) and (A4),

$$\begin{aligned} & \mathcal{E}(\widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(t), \widetilde{\varphi}_{\varepsilon\sigma}(t)) + \int_0^t \left(\eta_1 \varepsilon^2 \left\| \nabla_\varepsilon \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(\tau) \right\|_{L^2}^2 + m_2 \left\| \nabla_\varepsilon \widetilde{\mu}_{\varepsilon}(\tau) \right\|_{L^2}^2 \right) \mathrm{d}\tau \\ & \leq \mathcal{E}(\widetilde{\boldsymbol{u}}_0^{\varepsilon}, \widetilde{\varphi}_0^{\varepsilon}) + \int_0^t \boldsymbol{h}(\tau) \cdot \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(\tau) \, \mathrm{d}\tau, \end{aligned} \tag{2.14}$$

for all $t \in [0, T]$.

Before proceeding to the a priori estimates, let us state the following result found in (Marusić and Marusić-Paloka 2000, Lemmas 8, 11 and Remark 5), see also (Peter and Woukeng 2024, Lemma 2.1 and Remark 2.1).

Lemma 2.1 *There exists a positive constant C independent of ε such that*

$$\left\| \widetilde{v} \right\|_{L^2(\Omega_1)} \leq C\varepsilon \left\| \nabla_\varepsilon \widetilde{v} \right\|_{L^2(\Omega_1)^d}, \tag{2.15}$$

and

$$\|\widetilde{v}\|_{L^4(\Omega_1)} \leq C\varepsilon^{\frac{3}{4}} \|\nabla_\varepsilon \widetilde{v}\|_{L^2(\Omega_1)^d} , \tag{2.16}$$

for all $v \in H_0^1(\Omega_\varepsilon)$.

By a simple change of variable $y_d = \varepsilon x_d$ in the above lemma, we get for the above constant C,

$$\|v\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla v\|_{L^2(\Omega_\varepsilon)^d} , \tag{2.17}$$

and

$$\|v\|_{L^4(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{3}{4}} \|\nabla v\|_{L^2(\Omega_\varepsilon)^d} \tag{2.18}$$

for any $u \in H_0^1(\Omega_\varepsilon)$.

Bearing this in mind, the following result holds.

Proposition 2.1 *For any* $\sigma \in (0, \sigma_0]$ *and* $\varepsilon > 0$ *fixed, let* $(\boldsymbol{u}_{\varepsilon\sigma}, \varphi_{\varepsilon\sigma}, \mu_{\varepsilon\sigma})$ *be a weak solution of* (2.6) *in the sense of* (Peter and Woukeng (2024), Definition 2.1) *such that (2.7)–(2.9) are satisfied. Then, there exists a positive constant C independent of both* ε *and* σ *such that*

$$\|\widetilde{\boldsymbol{u}}_{\varepsilon\sigma}\|_{L^\infty(0,T;L^2(\Omega_1)^d)} \leq C, \tag{2.19}$$

$$\varepsilon \|\nabla_\varepsilon \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}\|_{L^2(Q_1)^{d\times d}} \leq C, \tag{2.20}$$

$$\|\widetilde{\varphi}_{\varepsilon\sigma}\|_{L^2(0,T;H^1_\varepsilon(\Omega_1))} \leq C, \tag{2.21}$$

$$\|\widetilde{\mu}_{\varepsilon\sigma}\|_{L^2(0,T;H^1_\varepsilon(\Omega_1))} \leq C, \tag{2.22}$$

and

$$\left\|F'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma})\right\|_{L^2(0,T;L^1(\Omega_1))} \leq C. \tag{2.23}$$

Proof A review of the proof of (Peter and Woukeng 2024, Proposition 2.1) reveals that, if we follow the same line of reasoning, we get

$$\begin{aligned} &\|\widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(t)\|^2_{L^2} + \alpha \|\widetilde{\varphi}_{\varepsilon\sigma}(t)\|^2_{L^2} + \int_0^t \left(\eta_1\varepsilon^2 \|\nabla_\varepsilon \widetilde{\boldsymbol{u}}_{\varepsilon\sigma}(\tau)\|^2_{L^2} + 2m_2 \|\nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma}(\tau)\|^2_{L^2}\right) \mathrm{d}\tau \\ &\leq \|\widetilde{\boldsymbol{u}}_0^\varepsilon\|^2_{L^2} + \frac{1}{2}\iint_{\Omega_1\times\Omega_1} J_\varepsilon(x-\zeta)(\widetilde{\varphi}_0^\varepsilon(x) - \widetilde{\varphi}_0^\varepsilon(\zeta))^2 \,\mathrm{d}x\,\mathrm{d}\zeta + 2\int_{\Omega_1} F_\sigma(\widetilde{\varphi}_0^\varepsilon)\,\mathrm{d}x + C, \end{aligned} \tag{2.24}$$

where C is a positive constant depending on Ω_1 and on the given constants of the assumptions, and $\alpha > 0$ depends on $\|J\|_{L^1}$. The assumptions on a_ε and J lead us to

$$\iint_{\Omega_1\times\Omega_1} J_\varepsilon(x-\zeta)(\widetilde{\varphi}_0^\varepsilon(x) - \widetilde{\varphi}_0^\varepsilon(\zeta))^2 \,\mathrm{d}x\,\mathrm{d}\zeta \leq C \left\|\widetilde{\varphi}_0^\varepsilon\right\|^2_{L^2(\Omega_1)}.$$

This, together with (2.4), shows that the right-hand side of (2.24) is bounded by a positive constant C independent of both ε and $\sigma \in (0, \sigma_0]$. Hence, we infer from (2.24) that (2.19) and (2.20) hold, and moreover, we have

$$\|\nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma}\|_{L^2(Q_1)} \le C \tag{2.25}$$

and

$$\|\widetilde{\varphi}_{\varepsilon\sigma}\|_{L^\infty(0,T;L^2(\Omega_1))} \le C. \tag{2.26}$$

Now, we take the gradient ∇_ε of $(2.10)_4$ and then take the scalar product in $L^2(\Omega_1)$ of the resulting equality with $\nabla_\varepsilon \widetilde{\varphi}_{\varepsilon\sigma}$. Then, proceeding as in the proof of (Peter and Woukeng 2024, Proposition 2.1) , we obtain (see (2.30) in Peter and Woukeng (2024))

$$\|\nabla_\varepsilon \widetilde{\varphi}_{\varepsilon\sigma}\|^2_{L^2(Q_1)} \le C\left(\|\nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma}\|^2_{L^2(Q_1)} + \|\widetilde{\varphi}_{\varepsilon\sigma}\|^2_{L^2(Q_1)}\right) \le C,$$

which, together with (2.26), yields (2.21). It remains to check (2.22) and (2.23). Let us start with (2.23). It is worth recalling that we have assumed that $F(0) = F'(0) = 0$ (see (1.6)). This being so, we follow the steps of (Frigeri and Grasselli 2012, Proof of Theorem 1) and define the function

$$H_\sigma(s) = F_\sigma(s) + \frac{\beta}{2}s^2, \quad s \in \mathbb{R},$$

where β is given in assumption (A3) so that $(2.5)_2$ is satisfied. Then, from $(2.5)_2$, we notice that H'_σ is monotone. Therefore, relying on the fact that $\left|\overline{\varphi_0^\varepsilon}\right| < 1$, we infer from the proof of (Frigeri and Grasselli 2012, (3.36)) that there exists a positive constant $\tilde{C}$ depending on $\overline{\varphi_0^\varepsilon}$ but neither on ε otherwise nor on σ such that

$$\left\|H'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma})\right\|_{L^1(\Omega_1)} \le \tilde{C}\int_{\Omega_1}(\widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon})H'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma})\,\mathrm{d}x + \tilde{C}. \tag{2.27}$$

Indeed, if we have a look at the proof of (3.36) in (Frigeri and Grasselli 2012, Section 3) , we see that

$$\tilde{C} = \frac{\kappa_2 - \kappa_1}{\delta}\,|\Omega_1| \max_{[\kappa_1,\kappa_2]}\left(\left|F'_1\right| + \left|F'_2\right| + \beta\delta_2\right),$$

where $\kappa_1, \kappa_2 \in (-1, 1)$ are fixed such that $\kappa_1 \le 0 \le \kappa_2$ and $\kappa_1 < \overline{\varphi_0^\varepsilon} < \kappa_2$, $\delta = \min\left\{\overline{\widetilde{\varphi}_0^\varepsilon} - \kappa_1, \kappa_2 - \overline{\widetilde{\varphi}_0^\varepsilon}\right\}$ and $\delta_2 = \max\{-\kappa_1, \kappa_2\}$ so that we may assume that (2.27) holds true with a constant completely independent of ε in the place of $\tilde{C}$. Indeed, in view of assumption (1.4) on φ_0^ε, we infer that $\left\|\widetilde{\varphi}_0^\varepsilon - \varphi^0\right\|_{L^2(\Omega_1)} \to 0$ as $\varepsilon \to 0$. This yields at once

$$\begin{aligned}\left|\overline{\widetilde{\varphi}_0^{\varepsilon}} - \overline{\varphi^0}\right| &= \left||\Omega_1|^{-1} \int_{\Omega_1} (\widetilde{\varphi}_0^{\varepsilon} - \varphi^0) dx\right| \\ &\leq C(\Omega_1) \left\|\widetilde{\varphi}_0^{\varepsilon} - \varphi^0\right\|_{L^2(\Omega_1)} \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

Now, assume without loss of generality that $\kappa_i \neq 0$ for $i = 1, 2$. We distinguish two different cases.

Case 1) We assume that κ_1, κ_2 are such that $\frac{\kappa_1}{2} < \overline{\varphi^0} < \frac{\kappa_2}{2}$. Let $\varepsilon_0 > 0$ be such that $\left|\overline{\widetilde{\varphi}_0^{\varepsilon}} - \overline{\varphi^0}\right| \leq -\frac{\kappa_1}{2}$ for all $0 < \varepsilon \leq \varepsilon_0$. This yields

$$\overline{\varphi^0} - \frac{\kappa_1}{2} \leq \overline{\widetilde{\varphi}_0^{\varepsilon}} - \kappa_1 \;\; \forall 0 < \varepsilon \leq \varepsilon_0. \tag{2.28}$$

We also choose $\varepsilon_1 > 0$ such that $\left|\overline{\widetilde{\varphi}_0^{\varepsilon}} - \overline{\varphi^0}\right| \leq \frac{\kappa_2}{2}$ for all $0 < \varepsilon \leq \varepsilon_1$ to get

$$\frac{\kappa_2}{2} - \overline{\varphi^0} \leq \kappa_2 - \overline{\widetilde{\varphi}_0^{\varepsilon}} \;\; \forall 0 < \varepsilon \leq \varepsilon_1. \tag{2.29}$$

(2.28) and (2.29) lead to

$$\delta_1 := \min\left(\overline{\varphi^0} - \frac{\kappa_1}{2}, \frac{\kappa_2}{2} - \overline{\varphi^0}\right) \leq \delta = \min\left\{\overline{\widetilde{\varphi}_0^{\varepsilon}} - \kappa_1, \kappa_2 - \overline{\widetilde{\varphi}_0^{\varepsilon}}\right\},$$

for all $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_1)$.

Case 2) We assume that $\frac{\kappa_2}{2} \leq \overline{\varphi^0} < \kappa_2$. (The case $\kappa_1 < \overline{\varphi^0} \leq \frac{\kappa_1}{2}$ is treated similarly as the case $\frac{\kappa_2}{2} \leq \overline{\varphi^0} < \kappa_2$.) Let n_0 be a positive integer such that $\frac{\kappa_2}{2} \leq \overline{\varphi^0} < (1 - 2^{-n_0})\kappa_2$. Such an integer exists. Let $\varepsilon_2 > 0$ be such that $\left|\overline{\widetilde{\varphi}_0^{\varepsilon}} - \overline{\varphi^0}\right| \leq 2^{-n_0}\kappa_2$ for all $0 < \varepsilon \leq \varepsilon_2$. Then we have

$$(1 - 2^{-n_0})\kappa_2 - \overline{\varphi^0} \leq \kappa_2 - \overline{\widetilde{\varphi}_0^{\varepsilon}} \quad \forall 0 < \varepsilon \leq \varepsilon_2.$$

Since $\frac{\kappa_2}{2} \leq \overline{\varphi^0}$, it follows that $\frac{\kappa_1}{2} < \overline{\varphi^0}$, so that, arguing as in the previous case, we have $\overline{\varphi^0} - \frac{\kappa_1}{2} \leq \overline{\widetilde{\varphi}_0^{\varepsilon}} - \kappa_1$ for all $0 < \varepsilon \leq \varepsilon_0$. Therefore, taking

$$\delta_1 := \min\left(\overline{\varphi^0} - \frac{\kappa_1}{2}, (1 - 2^{-n_0})\kappa_2 - \overline{\varphi^0}\right),$$

we have $\delta_1 \leq \delta$ for all $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_2)$.

Now, letting

$$C = \frac{\kappa_2 - \kappa_1}{\delta_1} |\Omega_1| \max_{[\kappa_1, \kappa_2]} (\left|F_1'\right| + \left|F_2'\right| + \beta\delta_2),$$

we get immediately $\tilde{C} \leq C$ for all $0 < \varepsilon \leq \min(\varepsilon_0, \rho)$ ($\rho = \varepsilon_1$ or ε_2), and so, it holds that, for all $\sigma \in (0, \sigma_0]$ and all $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_1)$,

$$\left\| H'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma}) \right\|_{L^1(\Omega_1)} \leq C \int_{\Omega_1} (\widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon}) H'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma})\, \mathrm{d}x + C, \tag{2.30}$$

where in (2.30) the positive constant C is independent of both σ and ε.

We assume in the sequel that ε satisfies the above requirement. This being so, testing $\widetilde{\mu}_{\varepsilon\sigma}$ by $\widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon}$ and using the fact that $\overline{\widetilde{\varphi}}_{\varepsilon\sigma}(t) = \overline{\varphi_0^\varepsilon}$ for all $t \geq 0$ (the mass conservation property) yields

$$\begin{aligned} \int_{\Omega_1} (\widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon}) F'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma})\, \mathrm{d}x &= \int_{\Omega_1} (\widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon})(\widetilde{\mu}_{\varepsilon\sigma} - \overline{\widetilde{\mu}}_{\varepsilon\sigma})\, \mathrm{d}x \\ &\quad - \int_{\Omega_1} (\widetilde{a}_\varepsilon \widetilde{\varphi}_{\varepsilon\sigma} - J_\varepsilon * \widetilde{\varphi}_{\varepsilon\sigma})(\widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon})\, \mathrm{d}x, \end{aligned}$$

so that

$$\int_{\Omega_1} (\widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon}) F'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma})\, \mathrm{d}x \leq C \left(\|\nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma}\|_{L^2} + \|\widetilde{\varphi}_{\varepsilon\sigma}\|_{L^2} \right) \left\| \widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon} \right\|_{L^2}, \tag{2.31}$$

where we have used the Poincaré–Wirtinger inequality for $\widetilde{\mu}_{\varepsilon\sigma} - \overline{\widetilde{\mu}}_{\varepsilon\sigma}$. Going back to (2.30) and using therein the definition of F_σ (in terms of H_σ) and appealing to (2.31), we get

$$\begin{aligned} \left\| F'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma}) \right\|_{L^1(\Omega_1)} &\leq \left\| H'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma}) \right\|_{L^1(\Omega_1)} + \beta \|\widetilde{\varphi}_{\varepsilon\sigma}\|_{L^1(\Omega_1)} \\ &\leq C \int_{\Omega_1} (\widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon})(F'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma}) + \beta \widetilde{\varphi}_{\varepsilon\sigma})\, \mathrm{d}x + C + \beta \|\widetilde{\varphi}_{\varepsilon\sigma}\|_{L^1(\Omega_1)} \\ &\leq C \left(\|\nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma}\|_{L^2} + \|\widetilde{\varphi}_{\varepsilon\sigma}\|_{L^2} \right) \left\| \widetilde{\varphi}_{\varepsilon\sigma} - \overline{\varphi_0^\varepsilon} \right\|_{L^2} \\ &\quad + C \|\widetilde{\varphi}_{\varepsilon\sigma}\|_{L^2(\Omega_1)} + C. \end{aligned} \tag{2.32}$$

Hence, using inequalities (2.25) and (2.26), we infer from the last series of above inequalities that

$$\left\| F'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma}) \right\|_{L^1(\Omega_1)} \leq C,$$

that is, (2.23). Now, since $\int_{\Omega_1} \widetilde{\mu}_{\varepsilon\sigma}\, \mathrm{d}x = \int_{\Omega_1} F'_\sigma(\widetilde{\varphi}_{\varepsilon\sigma})\, \mathrm{d}x$, we readily get $\left\| \overline{\widetilde{\mu}}_{\varepsilon\sigma} \right\|_{L^2(0,T)} \leq C$, so that, by the Poincaré–Wirtinger inequality and (2.25), it follows at once that

$$\begin{aligned} \int_{Q_1} |\widetilde{\mu}_{\varepsilon\sigma}|^2\, \mathrm{d}x\, \mathrm{d}t &\leq C \left[\int_{Q_1} |\nabla_\varepsilon \widetilde{\mu}_{\varepsilon\sigma}|^2\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \left| \int_{\Omega_1} \widetilde{\mu}_{\varepsilon\sigma}\, \mathrm{d}x \right|^2 \mathrm{d}t \right] \\ &\leq C. \end{aligned} \tag{2.33}$$

Therefore, (2.22) stems from (2.25) and (2.33). □

Based on the results contained in Proposition 2.1, we deduce the following estimates for the solution of (2.6).

Corollary 2.1 *Let* $(\boldsymbol{u}_{\varepsilon\sigma}, \varphi_{\varepsilon\sigma}, \mu_{\varepsilon\sigma})$ *be the solution of* (2.6) *given by* (2.7), (2.8) *and* (2.9). *Then, it holds that*

$$\|\boldsymbol{u}_{\varepsilon\sigma}\|_{L^\infty(0,T;L^2(\Omega_\varepsilon)^d)} \leq C\varepsilon^{\frac{1}{2}}, \tag{2.34}$$

$$\|\boldsymbol{u}_{\varepsilon\sigma}\|_{L^2(0,T;L^4(\Omega_\varepsilon)^d)} \leq C\varepsilon^{\frac{1}{4}}, \tag{2.35}$$

$$\varepsilon\,\|\nabla\boldsymbol{u}_{\varepsilon\sigma}\|_{L^2(Q_\varepsilon)^{d\times d}} \leq C\varepsilon^{\frac{1}{2}}, \tag{2.36}$$

$$\|\varphi_{\varepsilon\sigma}\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\varphi_{\varepsilon\sigma}\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{1}{2}}, \tag{2.37}$$

$$\|\mu_{\varepsilon\sigma}\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{1}{2}}, \tag{2.38}$$

$$\left\|\frac{\partial\boldsymbol{u}_{\varepsilon\sigma}}{\partial t}\right\|_{L^2(0,T;\mathbb{V}_\varepsilon')} \leq C\varepsilon^{\frac{3}{2}}, \tag{2.39}$$

$$\left\|\frac{\partial\varphi_{\varepsilon\sigma}}{\partial t}\right\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} \leq C\varepsilon^{\frac{1}{2}}, \tag{2.40}$$

$$\left\|F_\sigma'(\varphi_{\varepsilon\sigma})\right\|_{L^2(0,T;L^1(\Omega_\varepsilon))} \leq C\varepsilon, \tag{2.41}$$

and

$$\|p_{\varepsilon\sigma}\|_{L^2(Q_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}, \tag{2.42}$$

where C *is a positive constant independent of both* $\varepsilon > 0$ *and* $\sigma \in (0, \sigma_0]$.

Proof The estimates (2.34), (2.36), (2.37), (2.38) and (2.41) are easy consequences of the straightforward identities $\|\phi\|_{L^2(\Omega_\varepsilon)} = \varepsilon^{\frac{1}{2}}\left\|\widetilde{\phi}\right\|_{L^2(\Omega_1)}$, $\|\nabla\phi\|_{L^2(\Omega_\varepsilon)} = \varepsilon^{\frac{1}{2}}\left\|\nabla_\varepsilon\widetilde{\phi}\right\|_{L^2(\Omega_1)}$ and $\|\phi\|_{L^1(\Omega_\varepsilon)} = \varepsilon\left\|\widetilde{\phi}\right\|_{L^1(\Omega_1)}$. Estimate (2.39) can be obtained exactly as we did it in the proof of (Peter and Woukeng 2024, Corollary 2.1). With this in mind, let us check (2.35) and (2.40). To this end, let $v \in H^1(\Omega_\varepsilon)$; then,

$$\begin{aligned}\left|\left\langle\frac{\partial\varphi_{\varepsilon\sigma}}{\partial t}(t), v\right\rangle\right| &= \left|-\left\langle\operatorname{div}(\boldsymbol{u}_{\varepsilon\sigma}\varphi_{\varepsilon\sigma}), v\right\rangle - \left\langle\operatorname{div}(m^\varepsilon(\cdot,\varphi_{\varepsilon\sigma})\nabla\mu_{\varepsilon\sigma}), v\right\rangle\right| \\ &\leq \left|\int_{\Omega_\varepsilon}\varphi_{\varepsilon\sigma}\boldsymbol{u}_{\varepsilon\sigma}\cdot\nabla v\,\mathrm{d}x\right| + \left|\int_{\Omega_\varepsilon}m^\varepsilon(\cdot,\varphi_{\varepsilon\sigma})\nabla\mu_{\varepsilon\sigma}\cdot\nabla v\,\mathrm{d}x\right| \\ &\leq C\,\|\boldsymbol{u}_{\varepsilon\sigma}\|_{L^4(\Omega_\varepsilon)}\,\|\varphi_{\varepsilon\sigma}\|_{L^4(\Omega_\varepsilon)}\,\|\nabla v\|_{L^2(\Omega_\varepsilon)} \\ &\quad + m_2\,\|\nabla\mu_{\varepsilon\sigma}\|_{L^2(\Omega_\varepsilon)}\,\|\nabla v\|_{L^2(\Omega_\varepsilon)}\,.\end{aligned}$$

Thus,

$$\sup_{v\in H^1(\Omega_\varepsilon),\|v\|_{H^1}\leq 1}\left|\left\langle\frac{\partial\varphi_{\varepsilon\sigma}}{\partial t}(t), v\right\rangle\right| \leq C(\|\boldsymbol{u}_{\varepsilon\sigma}\|_{L^4(\Omega_\varepsilon)}\,\|\varphi_{\varepsilon\sigma}\|_{L^4(\Omega_\varepsilon)} + \|\nabla\mu_{\varepsilon\sigma}\|_{L^2(\Omega_\varepsilon)}).$$

Integrating the square of both members of the above last inequality over $(0, T)$, we obtain

$$\left\| \frac{\partial \varphi_{\varepsilon\sigma}}{\partial t} \right\|_{L^2(0,T;H^1(\Omega_\varepsilon)')} \leq C(\|\boldsymbol{u}_{\varepsilon\sigma}\|_{L^2(0,T;L^4(\Omega_\varepsilon))} \|\varphi_{\varepsilon\sigma}\|_{L^2(0,T;L^4(\Omega_\varepsilon))} + \|\nabla\mu_{\varepsilon\sigma}\|_{L^2(Q_\varepsilon)}). \tag{2.43}$$

We use the Gagliardo–Nirenberg inequality to get

$$\|\boldsymbol{u}_{\varepsilon\sigma}\|_{L^2(0,T;L^4(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{1}{4}} \text{ and } \|\varphi_{\varepsilon\sigma}\|_{L^2(0,T;L^4(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{5}{8}}. \tag{2.44}$$

Indeed, if $d = 3$, we have from (2.18)

$$\|\boldsymbol{u}_{\varepsilon\sigma}\|_{L^4(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{3}{4}} \|\nabla\boldsymbol{u}_{\varepsilon\sigma}\|_{L^2(\Omega_\varepsilon)},$$

so that using (2.36) we get

$$\begin{aligned}\left(\int_0^T \|\boldsymbol{u}_{\varepsilon\sigma}\|^2_{L^4(\Omega_\varepsilon)}\, \mathrm{d}t\right)^{\frac{1}{2}} &\leq C\varepsilon^{\frac{3}{4}} \left(\int_0^T \|\nabla\boldsymbol{u}_{\varepsilon\sigma}\|^2_{L^2(\Omega_\varepsilon)}\, \mathrm{d}t\right)^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{1}{4}}.\end{aligned}$$

As for $\varphi_{\varepsilon\sigma}$, one has, using the $3D$ Gagliardo–Nirenberg inequality,

$$\|\varphi_{\varepsilon\sigma}\|_{L^4(\Omega_\varepsilon)} \leq C\, \|\varphi_{\varepsilon\sigma}\|^{\frac{1}{4}}_{L^4(\Omega_\varepsilon)} \|\nabla\varphi_{\varepsilon\sigma}\|^{\frac{3}{4}}_{L^4(\Omega_\varepsilon)}.$$

Hence,

$$\begin{aligned}\left(\int_0^T \|\varphi_{\varepsilon\sigma}\|^2_{L^4(\Omega_\varepsilon)}\, \mathrm{d}t\right)^{\frac{1}{2}} &\leq C\left(\int_0^T \|\varphi_{\varepsilon\sigma}\|^4_{L^2(\Omega_\varepsilon)}\, \mathrm{d}t\right)^{\frac{1}{8}} \left(\int_0^T \|\nabla\varphi_{\varepsilon\sigma}\|^2_{L^2(\Omega_\varepsilon)}\, \mathrm{d}t\right)^{\frac{3}{8}} \\ &\leq C\varepsilon^{\frac{5}{8}}.\end{aligned}$$

This yields (2.40) for the case $d = 3$. Now, if $d = 2$, the Gagliardo–Nirenberg inequality gives

$$\begin{aligned}\int_0^T \|\boldsymbol{u}_{\varepsilon\sigma}\|^4_{L^4(\Omega_\varepsilon)}\, \mathrm{d}t &\leq C\, \|\boldsymbol{u}_{\varepsilon\sigma}\|^2_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \int_0^T \|\nabla\boldsymbol{u}_{\varepsilon\sigma}\|^2_{L^2(\Omega_\varepsilon)}\, \mathrm{d}t \\ &\leq C; \text{ see (2.34) and (2.36)},\end{aligned}$$

and

$$\begin{aligned}\int_0^T \|\varphi_{\varepsilon\sigma}\|^4_{L^4(\Omega_\varepsilon)}\, \mathrm{d}t &\leq C\, \|\varphi_{\varepsilon\sigma}\|^2_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \int_0^T \|\nabla\varphi_{\varepsilon\sigma}\|^2_{L^2(\Omega_\varepsilon)}\, \mathrm{d}t \\ &\leq C\varepsilon^2; \text{ see (2.37) and (2.26)}.\end{aligned}$$

This also leads to (2.40) for $d = 2$. Altogether, this gives (2.35) and (2.40). Finally, (2.42) is obtained by arguing exactly as in the proof of (Peter and Woukeng 2024, Proposition 2.2). This concludes the proof. □

2.2.3 The Cahn–Hilliard–Stokes System: the Limit $\sigma \to 0$

Consider the sequence $(\boldsymbol{u}_{\varepsilon\sigma}, \varphi_{\varepsilon\sigma}, \mu_{\varepsilon\sigma}, p_{\varepsilon\sigma})_{0<\sigma\leq\sigma_0}$. Thanks to the uniform controls (2.34)–(2.42) and to well-known compactness results, we obtain the existence of a subsequence of the above-mentioned sequence (not relabelled) and of functions $\boldsymbol{u}_\varepsilon \in L^\infty(0, T; \mathbb{H}_\varepsilon) \cap L^2(0, T; \mathbb{V}_\varepsilon) \cap H^1(0, T; \mathbb{V}'_\varepsilon)$, $\varphi_\varepsilon \in L^\infty(0, T; L^2(\Omega_\varepsilon)) \cap L^2(0, T; H^1(\Omega_\varepsilon)) \cap H^1(0, T; H^1(\Omega_\varepsilon)')$, $\mu_\varepsilon \in L^2(0, T; H^1(\Omega_\varepsilon))$ and $p_\varepsilon \in L^2(0, T; L^2_0(\Omega_\varepsilon))$ such that, as $\sigma \to 0$,

$$\boldsymbol{u}_{\varepsilon\sigma} \to \boldsymbol{u}_\varepsilon \text{ in } L^\infty(0, T; \mathbb{H}_\varepsilon)\text{-weak} * \text{ and in } L^2(0, T; \mathbb{V}_\varepsilon)\text{-weak}, \tag{2.45}$$

$$\boldsymbol{u}_{\varepsilon\sigma} \to \boldsymbol{u}_\varepsilon \text{ in } L^2(0, T; \mathbb{H}_\varepsilon)\text{-strong and a.e. in } Q_\varepsilon, \tag{2.46}$$

$$\boldsymbol{u}_{\varepsilon\sigma} \to \boldsymbol{u}_\varepsilon \text{ in } L^2(0, T; L^4(\Omega_\varepsilon)^d)\text{-weak}, \tag{2.47}$$

$$\frac{\partial \boldsymbol{u}_{\varepsilon\sigma}}{\partial t} \to \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \text{ in } L^2(0, T; \mathbb{V}'_\varepsilon)\text{-weak}, \tag{2.48}$$

$$\varphi_{\varepsilon\sigma} \to \varphi_\varepsilon \text{ in } L^\infty(0, T; L^2(\Omega_\varepsilon))\text{-weak} * \text{ and in } L^2(0, T; H^1(\Omega_\varepsilon))\text{-weak}, \tag{2.49}$$

$$\varphi_{\varepsilon\sigma} \to \varphi_\varepsilon \text{ in } L^2(0, T; L^2(\Omega_\varepsilon))\text{-strong and a.e. in } Q_\varepsilon, \tag{2.50}$$

$$\frac{\partial \varphi_{\varepsilon\sigma}}{\partial t} \to \frac{\partial \varphi_\varepsilon}{\partial t} \text{ in } L^2(0, T; H^1(\Omega_\varepsilon)')\text{-weak}, \tag{2.51}$$

$$\mu_{\varepsilon\sigma} \to \mu_\varepsilon \text{ in } L^2(0, T; H^1(\Omega_\varepsilon))\text{-weak}, \tag{2.52}$$

$$p_{\varepsilon\sigma} \to p_\varepsilon \text{ in } L^2(Q_\varepsilon)\text{-weak}. \tag{2.53}$$

With these convergence results at hand, we need to pass to the limit in the variational formulation of (2.6) in order to prove that the quadruple $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, p_\varepsilon)$ solves (1.1). Prior to that, we need to show that $|\varphi_\varepsilon| < 1$ a.e. in Q_ε . Proceeding as in (Frigeri and Grasselli 2012, Proof of Theorem 1), we rely on the monotonicity of H'_σ to reach our goal, that is, to show that

$$\varphi_\varepsilon \in L^\infty(Q_\varepsilon) \text{ with } |\varphi_\varepsilon(t, x)| < 1 \text{ a.e. in } Q_\varepsilon. \tag{2.54}$$

Thus, it follows from (2.54) noting the pointwise convergence (2.50) and the uniform convergence of F'_σ to F' on every compact subinterval of $(-1, 1)$ (recall that F' and F'_σ are continuous on $(-1, 1)$) that

$$F'_\sigma(\varphi_{\varepsilon\sigma}) \to F'(\varphi_\varepsilon) \text{ a.e. in } Q_\varepsilon. \tag{2.55}$$

This being so, by a mere routine, we may use the convergence results (2.45)–(2.53) and (2.55) to pass to the limit in the variational form of (2.6) and, therefore, solve (1.1) in the sense of Definition 2.1.

2.3 Uniform Estimates

Owing to the above convergence results, we may pass to the limit in the energy inequality (2.14) and obtain

$$\begin{aligned}&\mathcal{E}(\widetilde{\boldsymbol{u}}_{\varepsilon}(t),\widetilde{\varphi}_{\varepsilon}(t))+\int_0^t\left(\eta_1\varepsilon^2\left\|\nabla_{\varepsilon}\widetilde{\boldsymbol{u}}_{\sigma}(\tau)\right\|_{L^2}^2+m_2\left\|\nabla_{\varepsilon}\widetilde{\mu}_{\varepsilon}(\tau)\right\|_{L^2}^2\right)\mathrm{d}\tau\\&\quad\leq\mathcal{E}(\widetilde{\boldsymbol{u}}_0^{\varepsilon},\widetilde{\varphi}_0^{\varepsilon})+\int_0^t\boldsymbol{h}(\tau)\cdot\widetilde{\boldsymbol{u}}_{\varepsilon}(\tau)\,\mathrm{d}\tau\text{ for all }t\in[0,T].\end{aligned}\tag{2.56}$$

Thus, proceeding as in Sect. 2.2.2 and taking into account (2.54), we obtain the following uniform bounds (constants $C>0$ independent of ε):

$$\|\boldsymbol{u}_{\varepsilon}\|_{L^{\infty}(0,T;L^2(\Omega_{\varepsilon})^d)}\leq C\varepsilon^{\frac{1}{2}},\tag{2.57}$$

$$\|\boldsymbol{u}_{\varepsilon}\|_{L^{2}(0,T;L^4(\Omega_{\varepsilon})^d)}\leq C\varepsilon^{\frac{1}{4}},\tag{2.58}$$

$$\varepsilon\left\|\nabla\boldsymbol{u}_{\varepsilon}\right\|_{L^2(Q_{\varepsilon})^{d\times d}}\leq C\varepsilon^{\frac{1}{2}},\tag{2.59}$$

$$\|\varphi_{\varepsilon\sigma}\|_{L^{\infty}(0,T;L^2(\Omega_{\varepsilon}))}+\|\varphi_{\varepsilon\sigma}\|_{L^{2}(0,T;H^1(\Omega_{\varepsilon}))}\leq C\varepsilon^{\frac{1}{2}},\tag{2.60}$$

$$\|\varphi_{\varepsilon}\|_{L^{\infty}(Q_{\varepsilon})}\leq 1,\tag{2.61}$$

$$\|\mu_{\varepsilon}\|_{L^{2}(0,T;H^1(\Omega_{\varepsilon}))}\leq C\varepsilon^{\frac{1}{2}},\tag{2.62}$$

$$\left\|\frac{\partial\boldsymbol{u}_{\varepsilon}}{\partial t}\right\|_{L^2(0,T;\mathbb{V}_{\varepsilon}')}\leq C\varepsilon^{\frac{3}{2}},\tag{2.63}$$

$$\left\|F'(\varphi_{\varepsilon})\right\|_{L^2(0,T;L^1(\Omega_{\varepsilon}))}\leq C\varepsilon,\tag{2.64}$$

and

$$\|p_{\varepsilon}\|_{L^2(Q_{\varepsilon})}\leq C\varepsilon^{\frac{1}{2}}.\tag{2.65}$$

We need a further estimate which will be useful in the forthcoming sections. Using the notation $M_{\varepsilon}\varphi_{\varepsilon}$ for the partial mean integral of the order parameter, see (1.9), we have the following result.

Proposition 2.2 *We have $M_{\varepsilon}\varphi_{\varepsilon}\in L^2(0,T;H^1(\Omega))$ with $\frac{\partial M_{\varepsilon}\varphi_{\varepsilon}}{\partial t}\in L^2(0,T;H^1(\Omega)')$ and it holds*

$$\sup_{\varepsilon>0}\left[\|M_{\varepsilon}\varphi_{\varepsilon}\|_{L^2(0,T;H^1(\Omega))}+\left\|\frac{\partial M_{\varepsilon}\varphi_{\varepsilon}}{\partial t}\right\|_{L^2(0,T;H^1(\Omega)')}\right]\leq C,$$

where the positive constant C is independent of ε.

Proof The proof is very similar to the one of (Peter and Woukeng 2024, Proposition 2.3) and is therefore omitted. □

3 Passage to the Homogenization Limit

In this section, we aim to pass to the limit in system (1.1) for $\varepsilon \to 0$. This will be achieved provided that we make a structure hypothesis on the oscillating viscosity and mobility terms. To proceed with, let A be an ergodic algebra with mean value on $\mathbb{R}^{d-1}$, and let $B_A^2(\mathbb{R}^{d-1}; L^2(I))$ be the associated vector-valued generalized Besicovitch space defined in "Appendix A". We assume throughout the current section that the functions η and m satisfy:

(A6) $\eta(\cdot, r), m(\cdot, r) \in B_A^2(\mathbb{R}^{d-1}; L^2(I))$ for all $r \in \mathbb{R}$, where $I = (-1, 1)$.

If for instance $A = C_{\text{per}}(Y)$, the algebra of Y-continuous periodic function in $\mathbb{R}^{d-1}$, then $B_A^2(\mathbb{R}^{d-1}; L^2(I)) = L^2_{\text{per}}(Y; L^2(I))$, the subspace of $L^2_{\text{loc}}(\mathbb{R}^{d-1}; L^2(I))$ consisting of Y-periodic functions, $Y = (0, 1)^{d-1}$. In this case, assumption (A6) will amount to the functions $\eta(y, r)$ and $m(y, r)$ being periodic with respect to $\overline{y}$, where $y = (\overline{y}, y_d)$. So assumption (A6) is of capital interest in the limit passage when the coefficients are oscillating like in (1.1).

3.1 Preliminaries

Let us first define some function spaces. Let

$$\mathcal{V} = \{u \in (A^\infty(\mathbb{R}^{d-1}; \mathcal{C}_0^1(\overline{I})))^d : \text{div} u = 0\},$$

and let V (resp. H) be the closure of $\mathcal{V}$ in $\mathcal{B}_A^{1,2}(\mathbb{R}^{d-1}; H_0^1(I))^d$ (resp. $\mathcal{B}_A^2(\mathbb{R}^{d-1}; L^2(I))^d$). For $u \in \mathcal{B}_A^2(\mathbb{R}^{d-1}; L^2(I))^d$, we set

$$\|u\|_2 = \left(\int_I M\left(|u(\cdot, y_d)|^2 \right) \mathrm{d}y_d \right)^{\frac{1}{2}}.$$

Then, endowed with the norm $\|\cdot\|_2$, $\mathcal{B}_A^2(\mathbb{R}^{d-1}; L^2(I))^d$ is a Hilbert space. Now, set $\overline{\nabla} = (\frac{\overline{\partial}}{\partial y_1}, \dots, \frac{\overline{\partial}}{\partial y_{d-1}}, \frac{\partial}{\partial y_d})$, where $\frac{\overline{\partial}}{\partial y_i}$ is defined in "Appendix A". For $u \in \mathcal{B}_A^{1,2}(\mathbb{R}^{d-1}; H_0^1(I))^d$, we set

$$\left\| \overline{\nabla} u \right\|_2 = \left(\int_I M\left(\left| \overline{\nabla} \otimes u(\cdot, y_d) \right|^2 \right) \mathrm{d}y_d \right)^{1/2},$$

where $\overline{\nabla} \otimes u = \left(\frac{\overline{\partial} u_i}{\partial y_j} \right)_{1 \le i, j \le d}$ with $\frac{\overline{\partial}}{\partial y_d} := \frac{\partial}{\partial y_d}$ (the usual partial derivative in the distributional sense). As shown in (Cardone et al. 2024, Lemma 4.1), $\mathcal{B}_A^{1,2}(\mathbb{R}^{d-1}; H_0^1(I))$ is a Hilbert space under the norm $\left\| \overline{\nabla} . \right\|_2$. With this in mind, it is a fact that $V = \{u \in \mathcal{B}_A^{1,2}(\mathbb{R}^{d-1}; H_0^1(I))^d : \overline{\text{div}} u = 0\}$ where $\overline{\text{div}} = \overline{\nabla} \cdot$. We equip V and H with the relative topologies. In practice, we shall rather consider the subspace V_d

of V defined below:

$$V_d = \left\{ \boldsymbol{u} = (u_i)_{1\leq i\leq d} \in V : \int_I M(u_d(\cdot, y_d))\, \mathrm{d}y_d = 0 \right\},$$

a closed subspace of V endowed with the relative norm.

Bearing this in mind, we consider the following auxiliary Stokes system, which corresponds to local cell problems in the periodic case: for a.e. $\overline{x} \in \Omega$, find $\omega^j(\overline{x}, \cdot, \cdot) \equiv \omega^j(\overline{x})$ solving

$$\begin{cases} \dfrac{\partial \omega^j(\overline{x})}{\partial t} - \overline{\mathrm{div}}_y(\eta(\cdot, \varphi(\cdot, \overline{x}))\overline{\nabla}_y \omega^j(\overline{x})) + \overline{\nabla}_y \pi^j = 0 \text{ in } (0,\infty) \times \mathbb{R}^{d-1} \times I, \\ \overline{\mathrm{div}}_y \omega^j(\overline{x}) = 0 \text{ in } (0,\infty) \times \mathbb{R}^{d-1} \times I, \\ \omega^j(\overline{x}) = 0 \text{ on } (0,\infty) \times \mathbb{R}^{d-1} \times \{-1, 1\}, \\ \omega^j(\overline{x}, 0, \cdot) = e_j \text{ in } \mathbb{R}^{d-1} \times I, \end{cases} \tag{3.1}$$

where e_j is the jth vector of the canonical basis in $\mathbb{R}^d$ and where $\varphi \in \mathcal{C}([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$. The following result holds.

Proposition 3.1 *Let the assumptions* (A2) *and* (A6) *be satisfied. Then, there exists a unique solution* $\omega^j(\overline{x}) = (\omega_i^j(\overline{x}))_{1\leq i\leq d} \in \mathcal{C}([0, T]; H) \cap L^2(0, T; V_d)$ *for* $1 \leq j \leq d-1$ *(resp.* $\mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ *for* $j = d$*) of (3.1) verifying* $\frac{\partial \omega^j(\overline{x})}{\partial t} \in L^2(0, T; V')$*. Moreover, it holds that* $\omega^j \in \mathcal{C}([0, T]; L^2(\Omega; H)) \cap L^2(Q; V_d)$ *for* $1 \leq j \leq d-1$ *(resp.* $\mathcal{C}([0, T]; L^2(\Omega; H)) \cap L^2(Q; V)$ *for* $j = d$*). If, in addition, we set*

$$\begin{aligned} G_{ij}(t, \overline{x}) &= \frac{1}{2} \int_{-1}^{1} M(\omega^i(\overline{x}, t, \cdot, y_d)) e_j \, \mathrm{d}y_d \text{ for } (t, \overline{x}) \in \overline{Q} \text{ and } 1 \leq i, j \leq d \\ &\equiv \frac{1}{2} \int_{-1}^{1} M(\omega_j^i(\overline{x}, t, \cdot, y_d))\, \mathrm{d}y_d, \end{aligned} \tag{3.2}$$

then the G_{ij} *are uniformly bounded a.e. in* Ω *by a function continuous in time and decreasing exponentially as* t *increases and* $G_{jd} = G_{dj} = 0$ *for all* $1 \leq j \leq d-1$*. Moreover, the matrix* $G = (G_{ij})_{1\leq i,j\leq d-1}$ *is symmetric and positive definite.*

Proof Although the coefficient $\eta(\cdot, \varphi)$ depends on the order parameter φ, the proof of (Cardone et al. 2024, Section 4.1) (see also Peter and Woukeng 2024, Section 4.1) carries over mutatis mutandis to the present setting. Indeed, the well-posedness of (3.1) follows a classical setting (recall that $\eta(y, r) \geq \eta_1$ a.e. in (y, r)). Also, it is a fact that the matrix G is symmetric, positive definite and its coefficients decrease exponentially in time. The bound on the entries stems from the inequality (see Cardone et al. 2024, Propositions 4.1 and 4.2)

$$\left\|\omega^j(\overline{x},t)\right\|_2 \leq \sqrt{2}\exp\left(-\frac{\eta_1}{4}t\right) \text{ for all } t \in [0,T], \text{ a.e. } \overline{x} \in \Omega,$$

where η_1 is given by (A1), so that

$$\left|G_{ij}(t,\overline{x})\right| \leq \exp\left(-\frac{\eta_1}{4}t\right), \text{ all } t \in [0,T] \text{ and a.e. } \overline{x} \in \Omega,$$

which completes the proof. □

3.2 The Limit Procedure

Owing to bounds (2.57)–(2.65) and to Proposition 2.2, we are in position to apply the compactness results stated in "Appendix A" as follows: given any ordinary sequence $E = (\varepsilon_n)_{n\in\mathbb{N}^*}$, there exist a subsequence E' of E and functions $\boldsymbol{u}_0 \in L^2(Q;\mathcal{B}_A^{1,2}(\mathbb{R}^{d-1};H_0^1(I)))^d$, $(\varphi,\varphi_1),(\mu,\mu_1) \in L^2(0,T;H^1(\Omega)) \times L^2(Q;B_{\#A}^{1,2}(\mathbb{R}^{d-1};H^1(I)))$ and $p \in L^2(Q;\mathcal{B}_A^2(\mathbb{R}^{d-1};L^2(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0 \text{ in } L^2(Q_\varepsilon)^d\text{-weak } \Sigma_A, \tag{3.3}$$
$$\varepsilon\nabla\boldsymbol{u}_\varepsilon \to \overline{\nabla}_y\boldsymbol{u}_0 \text{ in } L^2(Q_\varepsilon)^{d\times d} \text{ -weak } \Sigma_A, \tag{3.4}$$
$$\varphi_\varepsilon \to \varphi \text{ in } L^2(Q_\varepsilon)\text{-strong } \Sigma_A, \tag{3.5}$$
$$M_\varepsilon\varphi_\varepsilon \to \varphi \text{ in } L^\infty(Q)\text{-weak}*, \tag{3.6}$$
$$\nabla\varphi_\varepsilon \to \nabla_{\overline{x}}\varphi + \nabla_y\varphi_1 \text{ in } L^2(Q_\varepsilon)^d\text{-weak } \Sigma_A, \tag{3.7}$$
$$\mu_\varepsilon \to \mu \text{ in } L^2(Q_\varepsilon) \text{ -weak } \Sigma_A, \tag{3.8}$$
$$\nabla\mu_\varepsilon \to \nabla_{\overline{x}}\mu + \nabla_y\mu_1 \text{ in } L^2(Q_\varepsilon)^d\text{-weak } \Sigma_A, \tag{3.9}$$

and

$$p_\varepsilon \to p \text{ in } L^2(Q_\varepsilon)\text{-weak } \Sigma_A. \tag{3.10}$$

We recall that $\nabla_{\overline{x}}\phi = (\frac{\partial\phi}{\partial x_1},\ldots,\frac{\partial\phi}{\partial x_{d-1}},0)$ for $\phi = \varphi,\mu$. One can easily see that in view of the equation $\mathrm{div}\boldsymbol{u}_\varepsilon = 0$ in Q_ε, we get $\overline{\mathrm{div}}_y\boldsymbol{u}_0 = 0$ in $Q \times \mathbb{R}^{d-1} \times I$. Next, let

$$\boldsymbol{u}(t,\overline{x}) = \frac{1}{2}\int_{-1}^{1} M(\boldsymbol{u}_0(t,\overline{x},\cdot,x_d))\,\mathrm{d}x_d,\ (t,\overline{x}) \in Q, \tag{3.11}$$

and set

$$\overline{\boldsymbol{u}} = (u_i)_{1\leq i\leq d-1}, \text{ where } \boldsymbol{u} = (u_i(t,\overline{x}))_{1\leq i\leq d}. \tag{3.12}$$

Then we have

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \text{ in } L^2(Q)^d\text{-weak,} \tag{3.13}$$

and we deduce from the bound (2.58) that $\boldsymbol{u} \in L^2(0,T; L^4(\Omega)^d)$. Also, as shown in (Cardone et al. 2024, Section 4), we have $u_d = 0$ so that $\boldsymbol{u} = (\overline{\boldsymbol{u}}, 0)$, where $\overline{\boldsymbol{u}} \in L^2(Q)^{d-1}$ with

$$\mathrm{div}_{\overline{x}}\overline{\boldsymbol{u}} = 0 \text{ in } Q \text{ and } \overline{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \text{ on } (0,T) \times \partial\Omega, \tag{3.14}$$

where $\boldsymbol{n}$ stands for the outward unit normal to $\partial\Omega$. One also has

$$\int_\Omega \int_I M(p(t, \overline{x}, \cdot, \zeta))\, \mathrm{d}\zeta\, \mathrm{d}\overline{x} = 0,$$

which stems from the identity $\int_{\Omega_\varepsilon} p_\varepsilon\, \mathrm{d}x = 0$.

This being so, let

$$\widehat{a}(\overline{x}) = (\widehat{J} * 1)(\overline{x}) = \int_\Omega J(\overline{x} - \overline{z}, 0)\, \mathrm{d}\overline{z} \text{ for } \overline{x} \in \Omega. \tag{3.15}$$

We shall need the following auxiliary lemma.

Lemma 3.1 *Let $(v_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^2(Q_\varepsilon)$ satisfying $v_\varepsilon \to v_0$ in $L^2(Q_\varepsilon)$-strong Σ_A as $E \ni \varepsilon \to 0$, where $v_0 \in L^2(0,T; L^2(\Omega))$. Let $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying $f(\cdot, r) \in L^\infty(\mathbb{R}^d)$ (for any fixed $r \in \mathbb{R}$), and let there exist a positive constant κ such that*

$$|f(y,r) - f(y,s)| \le \kappa\, |r - s| \text{ for a.e. } y \in \mathbb{R}^d \text{ and all } r, s \in \mathbb{R}.$$

Assume in addition that $f(\cdot, r) \in B^2_A(\mathbb{R}^{d-1}; L^2(I))$ for all $r \in \mathbb{R}$. Then, as $E \ni \varepsilon \to 0$,

$$f^\varepsilon(\cdot, v_\varepsilon) \to f(\cdot, v_0) \text{ in } L^2(Q_\varepsilon)\text{-strong } \Sigma_A, \tag{3.16}$$

where $f^\varepsilon(\cdot, v_\varepsilon)(t,x) = f(x/\varepsilon, v_\varepsilon(t,x))$ for $(t,x) \in Q_\varepsilon$.

Proof It is sufficient to show that, as $E \ni \varepsilon \to 0$,

$$\varepsilon^{-\frac{1}{2}} \left\| f^\varepsilon(\cdot, v_\varepsilon) \right\|_{L^2(Q_\varepsilon)} \to \| f(\cdot, v_0) \|_{L^2(Q; \mathcal{B}^2_A(\mathbb{R}^{d-1}; L^2(I)))}. \tag{3.17}$$

First of all, set

$$g(t, \overline{x}, y) = f(y, v_0(t, \overline{x})), \quad (t, \overline{x}, y) \in Q \times \mathbb{R}^{d-1} \times I.$$

Then, $g(t,\overline{x},\,\cdot\,)\in B_A^{2,\infty}(\mathbb{R}^{d-1};L^2(I))$ for a.e. $(t,\overline{x})\in Q$, where $B_A^{2,\infty}=B_A^2\cap L^\infty$, such that $\|g(t,\overline{x},\,\cdot\,)\|_{B_A^{2,\infty}}\leq B(t,\overline{x})$ for a $B\in L^2(Q)$. It is a well-known fact that for any function $g_0\in\mathcal{C}(\overline{Q};B_A^{2,\infty}(\mathbb{R}^{d-1};L^2(I)))$, one has

$$g_0^\varepsilon\to g_0 \text{ in } L^2(Q_\varepsilon)\text{ -strong } \Sigma_A, \tag{3.18}$$

where $g_0^\varepsilon(t,x)=g_0(t,\overline{x},x/\varepsilon)$. Owing to the density of $\mathcal{C}(\overline{Q};B_A^{2,\infty}(\mathbb{R}^{d-1};L^2(I)))$ in the corresponding space which is only square-integrable over Q, we may show that (3.18) still holds for g_0 in the latter space. This shows that (3.18) is true for g defined above. Therefore, proving (3.17) amounts to checking

$$\varepsilon^{-\frac{1}{2}}\left\|f^\varepsilon(\,\cdot\,,v_\varepsilon)-g^\varepsilon\right\|_{L^2(Q_\varepsilon)}\to 0 \text{ when } E\ni\varepsilon\to 0.$$

But in view of the properties of f, we have

$$\varepsilon^{-\frac{1}{2}}\left\|f^\varepsilon(\,\cdot\,,v_\varepsilon)-g^\varepsilon\right\|_{L^2(Q_\varepsilon)}\leq\kappa\varepsilon^{-\frac{1}{2}}\|v_\varepsilon-v_0\|_{L^2(Q_\varepsilon)}\to 0$$

as $E\ni\varepsilon\to 0$ (recall that $v_\varepsilon\to v_0$ in $L^2(Q_\varepsilon)$-strong Σ_A and that v_0 does not depend on y). This concludes the proof. □

Now, we have all the ingredients to pass to the limit in the variational formulation of (1.1). To this end, let $\Psi\in(\mathcal{C}_0^\infty(Q)\otimes A^\infty(\mathbb{R}^{d-1};\mathcal{C}_0^\infty(I)))^d$, $(\phi_0,\phi_1)\in\mathcal{C}_0^\infty(Q)\times(\mathcal{C}_0^\infty(Q)\otimes A^\infty(\mathbb{R}^{d-1};\mathcal{C}_0^\infty(I)))$ and $\chi_0\in\mathcal{C}_0^\infty(Q)$. Define the functions Ψ^ε and ϕ_ε on Q_ε as follows:

$$\Psi^\varepsilon(t,x)=\Psi\left(t,\overline{x},\frac{x}{\varepsilon}\right),\quad \phi_\varepsilon(t,x)=\phi_0(t,\overline{x})+\varepsilon\phi_1\left(t,\overline{x},\frac{x}{\varepsilon}\right),$$

for $(t,x)\in Q_\varepsilon$. We take $(\Psi^\varepsilon,\phi_\varepsilon,\chi_0)\in\mathcal{C}_0^\infty(Q_\varepsilon)^d\times\mathcal{C}_0^\infty(Q_\varepsilon)\times\mathcal{C}_0^\infty(Q)$ as test function in the variational formulation of (1.1) to get

$$\begin{cases}-\int_{Q_\varepsilon}\boldsymbol{u}_\varepsilon\left(\frac{\partial\Psi}{\partial t}\right)^\varepsilon\mathrm{d}x\,\mathrm{d}t+\varepsilon^2\int_{Q_\varepsilon}\eta^\varepsilon(\varphi_\varepsilon)\nabla\boldsymbol{u}_\varepsilon\cdot\left((\nabla_{\overline{x}}\Psi)^\varepsilon+\frac{1}{\varepsilon}(\nabla_y\Psi)^\varepsilon\right)\mathrm{d}x\,\mathrm{d}t\\ \quad-\int_{Q_\varepsilon}p_\varepsilon\left((\mathrm{div}_{\overline{x}}\Psi)^\varepsilon+\frac{1}{\varepsilon}(\mathrm{div}_y\Psi)^\varepsilon\right)\mathrm{d}x\,\mathrm{d}t-\int_{Q_\varepsilon}\mu_\varepsilon\nabla\varphi_\varepsilon\Psi^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\ \qquad=\int_{Q_\varepsilon}\boldsymbol{h}\Psi^\varepsilon\,\mathrm{d}x\,\mathrm{d}t,\end{cases} \tag{3.19}$$

$$\begin{cases}-\int_{Q_\varepsilon}\varphi_\varepsilon\frac{\partial\phi_\varepsilon}{\partial t}\mathrm{d}x\,\mathrm{d}t+\int_{Q_\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla\varphi_\varepsilon)\phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\ \quad+\int_{Q_\varepsilon}m^\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon\cdot(\nabla_{\overline{x}}\phi_0+\varepsilon(\nabla_{\overline{x}}\phi_1)^\varepsilon+(\nabla_y\phi_1)^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t=0,\end{cases} \tag{3.20}$$

$$\int_{Q_\varepsilon}\mu_\varepsilon\chi_0\,\mathrm{d}x\,\mathrm{d}t=\int_{Q_\varepsilon}\varepsilon^{-1}a_\varepsilon\varphi_\varepsilon\chi_0\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_\varepsilon}\varepsilon^{-1}(J*\varphi_\varepsilon)\chi_0\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_\varepsilon}F'(\varphi_\varepsilon)\chi_0\,\mathrm{d}x\,\mathrm{d}t. \tag{3.21}$$

We deal with each equation separately. First, we consider (3.19). Letting $E' \ni \varepsilon \to 0$, we obtain

$$\int_Q \int_I M(p\,\mathrm{div}_y \Psi)\,\mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t = 0,$$

showing that p is independent of y, that is, $p(t, \overline{x}, y) = p(t, \overline{x})$. As a result, the identity $\int_{\Omega_0} \int_I M(p(t, \overline{x}, \cdot, y_d))\,\mathrm{d}y_d\,\mathrm{d}\overline{x} = 0$ obtained above amounts to $\int_{\Omega_0} p(t, \overline{x})\,\mathrm{d}\overline{x} = 0$, meaning that $p \in L^2(0, T; L_0^2(\Omega))$. Next, we restrict to Ψ satisfying $\mathrm{div}_y \Psi = 0$ and divide (3.19) by ε to obtain

$$\begin{cases} -\frac{1}{\varepsilon} \int_{Q_\varepsilon} \boldsymbol{u}_\varepsilon \left(\frac{\partial \Psi}{\partial t}\right)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon} \int_{Q_\varepsilon} \varepsilon^2 \eta^\varepsilon(\varphi_\varepsilon) \nabla \boldsymbol{u}_\varepsilon \cdot \left((\nabla_{\overline{x}} \Psi)^\varepsilon + \frac{1}{\varepsilon} (\nabla_y \Psi)^\varepsilon \right) \mathrm{d}x\,\mathrm{d}t \\ \quad -\frac{1}{\varepsilon} \int_{Q_\varepsilon} p_\varepsilon (\mathrm{div}_{\overline{x}} \Psi)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\varepsilon} \int_{Q_\varepsilon} \mu_\varepsilon \nabla \varphi_\varepsilon \Psi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\varepsilon} \int_{Q_\varepsilon} \boldsymbol{h} \Psi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \end{cases} \tag{3.22}$$

Our aim is to pass to the limit in (3.22) when $E' \ni \varepsilon \to 0$. A quick look at (3.22) reveals that only the second and the last terms on its left-hand side require attention. So, as for the second term, we first use the strong sigma-convergence of $(\varphi_\varepsilon)_{\varepsilon \in E'}$ to get the strong sigma-convergence of $\eta^\varepsilon(\varphi_\varepsilon)$ towards $\eta(\,\cdot\,, \varphi)$ in $L^2(Q_\varepsilon)$, see Lemma 3.1. Next, we use (3.4) and appeal to Theorem A.4 to obtain, as $E' \ni \varepsilon \to 0$,

$$\frac{1}{\varepsilon} \int_{Q_\varepsilon} \varepsilon \eta^\varepsilon(\varphi_\varepsilon) \nabla \boldsymbol{u}_\varepsilon \cdot (\nabla_y \Psi)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \to \int_Q \int_I M(\eta(\,\cdot\,, \varphi) \overline{\nabla}_y \boldsymbol{u}_0 \cdot \nabla_y \Psi)\,\mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t. \tag{3.23}$$

Concerning the last term on the left-hand side of (3.22), we have

$$\begin{aligned} \frac{1}{\varepsilon} \int_{Q_\varepsilon} \mu_\varepsilon \nabla \varphi_\varepsilon \Psi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t &= -\frac{1}{\varepsilon} \int_{Q_\varepsilon} \varphi_\varepsilon (\Psi^\varepsilon \nabla \mu_\varepsilon + \mu_\varepsilon (\mathrm{div}_{\overline{x}} \Psi)^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &\to \int_Q \int_I M\left(\varphi \left[(\nabla_{\overline{x}} \mu + \nabla_y \mu_1) \Psi + \mu \mathrm{div}_{\overline{x}} \Psi \right]\right) \mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t. \end{aligned}$$

Hence, dividing both sides of (3.22) by two and taking the limit of the resulting equality when $E' \ni \varepsilon \to 0$ yield

$$\begin{cases} -\frac{1}{2} \int_Q \int_I M\left(\boldsymbol{u}_0(t, \overline{x}, \,\cdot\,, y_d) \frac{\partial \Psi}{\partial t}(t, \overline{x}, \cdot, y_d) \right) \mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t \\ +\frac{1}{2} \int_Q \int_I M(\eta(\,\cdot\,, \varphi) \overline{\nabla}_y \boldsymbol{u}_0 \cdot \nabla_y \Psi)\,\mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t \\ \quad -\frac{1}{2} \int_Q \int_I M\left(\varphi \left[(\nabla_{\overline{x}} \mu + \nabla_y \mu_1) \Psi + \mu \mathrm{div}_{\overline{x}} \Psi \right]\right) \mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t \\ \quad\quad -\frac{1}{2} \int_Q \int_I M(p \mathrm{div}_{\overline{x}} \Psi)\,\mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t = \frac{1}{2} \int_Q \int_I M(\boldsymbol{h} \Psi)\,\mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t \end{cases} \tag{3.24}$$

for all $\Psi \in (\mathcal{C}_0^\infty(Q) \otimes A^\infty(\mathbb{R}^{d-1}; \mathcal{C}_0^\infty(I)))^d$ satisfying $\mathrm{div}_y \Psi = 0$.

Now, we turn to (3.20) and divide it by 2ε and proceed as we did in obtaining (3.24) (see especially the proof of (3.23)). Using the equality

$$\int_{Q_\varepsilon} (\boldsymbol{u}_\varepsilon \nabla \varphi_\varepsilon)\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = -\int_{Q_\varepsilon} \varphi_\varepsilon \boldsymbol{u}_\varepsilon \nabla \phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t,$$

and passing to the limit $\varepsilon \to 0$ yields

$$\begin{cases} -\frac{1}{2}\int_Q \int_I M\left(\varphi \frac{\partial \phi_0}{\partial t}\right) \mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t - \frac{1}{2}\int_Q \int_I M(\varphi \boldsymbol{u}_0(\nabla_{\overline{x}}\phi_0 + \nabla_y \phi_1))\,\mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t \\ +\frac{1}{2}\int_Q \int_I M\left(m(\,\cdot\,,\varphi)(\nabla_{\overline{x}}\mu + \nabla_y \mu_1)(\nabla_{\overline{x}}\phi_0 + \nabla_y \phi_1)\right) \mathrm{d}y_d\,\mathrm{d}\overline{x}\,\mathrm{d}t = 0. \end{cases} \tag{3.25}$$

Let us now have a look at (3.21). First and foremost, we need to show that

$$|\varphi(t, \overline{x})| < 1 \quad \text{for a.e. } (t, \overline{x}) \in Q. \tag{3.26}$$

This would clarify the limit passage in the term involving $F'(\varphi_\varepsilon)$. First of all, we know from (3.5) that

$$\varepsilon^{-1} \int_{Q_\varepsilon} |\varphi_\varepsilon(t, x) - \varphi(t, \overline{x})|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_1} |\varphi_\varepsilon(t, \overline{x}, \varepsilon x_d) - \varphi(t, \overline{x})|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\to 0 \text{ when } E' \ni \varepsilon \to 0.$$

So, setting $\widetilde{\varphi}_\varepsilon(t, x) = \varphi_\varepsilon(t, \overline{x}, \varepsilon x_d)$, $(t, x) \in Q_1$, we see that

$$\widetilde{\varphi}_\varepsilon \to \varphi \text{ in } L^2(Q_1)\text{-strong and a.e. in } Q_1.$$

With this in mind, let us introduce the sets

$$E_\delta^\varepsilon = \{(t, x) \in Q_1 : |\widetilde{\varphi}_\varepsilon(t, x)| > 1 - \delta\},$$
$$E_\delta = \{(t, x) \in Q_1 : |\widetilde{\varphi}_0(t, x)| > 1 - \delta\},$$

where $0 < \delta < 1$ is arbitrarily given and where $\widetilde{\varphi}(t, x) = \varphi(t, \overline{x})$ for $(t, x) \in Q_1$. Then, from the pointwise convergence of $\widetilde{\varphi}_\varepsilon$ and the Fatou lemma, we get that, for any $\delta > 0$,

$$\mathrm{meas}(E_\delta) \leq \liminf_{\varepsilon \to 0} \mathrm{meas}(E_\delta^\varepsilon),$$

where meas stands for the Lebesgue measure in Q_1.

Now define the function H by $H(s) = F(s) + \frac{\beta}{2}s^2$, $s \in (-1, 1)$. Since H' is nondecreasing, we have that $H'(s) \geq 0$ for $s \in [0, 1)$ and $H'(s) \leq 0$ for $s \in (-1, 0]$.

Thus, we can write

$$\min\left(H'(1-\delta), -H'(-1+\delta)\right)\operatorname{meas}(E_\delta^\varepsilon) \leq \left\| H'(\widetilde{\varphi}_\varepsilon)\right\|_{L^1(Q_1)}.$$

However,

$$\begin{aligned}\left\| H'(\varphi_\varepsilon)\right\|_{L^1(Q_\varepsilon)} &\leq \left\| F'(\varphi_\varepsilon)\right\|_{L^1(Q_\varepsilon)} + \beta \left\|\varphi_\varepsilon\right\|_{L^1(Q_\varepsilon)} \\ &\leq T^{\frac{1}{2}} \left\| F'(\varphi_\varepsilon)\right\|_{L^2(0,T;L^1(\Omega_\varepsilon))} + \beta \left|\Omega_\varepsilon\right|^{\frac{1}{2}} \left\|\varphi_\varepsilon\right\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \\ &\leq C\varepsilon,\end{aligned}$$

so that $\left\| H'(\widetilde{\varphi}_\varepsilon)\right\|_{L^1(Q_1)} \leq C$, C being independent of both ε and δ. Thus, we have

$$\operatorname{meas}(E_\delta) \leq \frac{C}{\min\left(H'(1-\delta), -H'(-1+\delta)\right)}.$$

Letting $\delta \to 0$ in the last inequality yields at once

$$\operatorname{meas}(\{(t,x) \in Q_1 : |\widetilde{\varphi}_0(t,x)| \geq 1\}) = 0.$$

This infers

$$|\varphi(t,\overline{x})| < 1 \ \text{ a.e. } (t,\overline{x}) \in Q,$$

that is, (3.26). We are now in a position to proceed with (3.21). We first deal with the term $\frac{1}{\varepsilon}\int_{Q_\varepsilon} F'(\varphi_\varepsilon)\chi_0 \, dx \, dt$. We have

$$\frac{1}{\varepsilon}\int_{Q_\varepsilon} F'(\varphi_\varepsilon)\chi_0 \, dx \, dt = \int_{Q_1} F'(\widetilde{\varphi}_\varepsilon)\chi_0 \, dx \, dt.$$

By virtue of the pointwise convergence $\widetilde{\varphi}_\varepsilon \to \varphi$ a.e. in Q_1, we deduce from the continuity of F' in $(-1, 1)$ that $F'(\widetilde{\varphi}_\varepsilon) \to F'(\varphi)$ a.e. in Q_1. Next, it holds that $\left\| F'(\widetilde{\varphi}_\varepsilon)\right\|_{L^1(Q_1)} \leq C$ (recall that $\left\| F'(\varphi_\varepsilon)\right\|_{L^1(Q_\varepsilon)} \leq C\varepsilon$). The Lebesgue dominated convergence theorem gives

$$\frac{1}{\varepsilon}\int_{Q_\varepsilon} F'(\varphi_\varepsilon)\chi_0 \, dx \, dt \to \int_Q \int_I F'(\varphi)\chi_0 \, dy_d \, d\overline{x} \, dt. \tag{3.27}$$

As for the other terms in (3.21), we rely on (Peter and Woukeng 2024, Proposition 4.1) to get, as $E' \ni \varepsilon \to 0$,

$$\frac{1}{\varepsilon}\int_{Q_\varepsilon} \varepsilon^{-1}(a_\varepsilon\varphi_\varepsilon - J * \varphi_\varepsilon)\chi_0 \, dx \, dt \to \int_Q \int_I (\widehat{a}\varphi - \widehat{J} * \varphi)\chi_0 \, dy_d \, d\overline{x} \, dt. \tag{3.28}$$

Collecting the convergence results (3.27)–(3.28), we get from (3.21) (which has been divided by 2ε) the following equation after the limit passage

$$\begin{cases} \frac{1}{2}\int_Q \int_I \mu\chi_0 \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t = \frac{1}{2}\int_Q \int_I \widehat{a}\varphi\chi_0 \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t - \frac{1}{2}\int_Q \int_I (\widehat{J} * \varphi)\chi_0 \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t \\ \quad +\frac{1}{2}\int_Q \int_I F'(\varphi)\chi_0 \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t. \end{cases} \tag{3.29}$$

Finally, since $\boldsymbol{u}_0^\varepsilon \to \boldsymbol{u}^0$ in $L^2(\Omega_\varepsilon)^d$-strong Σ_A and $\varphi_0^\varepsilon \to \varphi^0$ in $L^2(\Omega_\varepsilon)^d$-strong Σ_A, we conclude by integration by parts that $\boldsymbol{u}_0(0) = \boldsymbol{u}^0$ and $\varphi_0(0) = \varphi^0$. Recalling the definition (3.12), we infer from the equality $u_d = 0$ that $\boldsymbol{u}^0 = (\overline{\boldsymbol{u}^0}, 0)$, that is, the last component of $\boldsymbol{u}^0$ is zero.

In summary, we have just proved the following result.

Proposition 3.2 *The function tuple $(\boldsymbol{u}_0, \varphi, \varphi_1, \mu, \mu_1, p)$ solves the following system:*

$$\begin{cases} -\frac{1}{2}\int_Q \int_I M\left(\boldsymbol{u}_0(t,\overline{x},\,\cdot\,,y_d)\frac{\partial \Psi}{\partial t}(t,\overline{x},\,\cdot\,,y_d)\right) \mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t \\ +\frac{1}{2}\int_Q \int_I M(\eta(\cdot,\varphi)\overline{\nabla}_y \boldsymbol{u}_0 \cdot \nabla_y \Psi) \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t \\ -\frac{1}{2}\int_Q \int_I M\left(\varphi\left[(\nabla_{\overline{x}}\mu + \nabla_y \mu_1)\Psi + \mu \mathrm{div}_{\overline{x}}\Psi\right]\right) \mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t \\ -\frac{1}{2}\int_Q \int_I M(p\mathrm{div}_{\overline{x}}\Psi) \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t = \frac{1}{2}\int_Q \int_I M(\boldsymbol{h}\Psi) \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t, \end{cases} \tag{3.30}$$

$$\begin{cases} -\frac{1}{2}\int_Q \int_I M\left(\varphi\frac{\partial \phi_0}{\partial t}\right) \mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t - \frac{1}{2}\int_Q \int_I M(\varphi \boldsymbol{u}_0(\nabla_{\overline{x}}\phi_0 + \nabla_y \phi_1)) \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t \\ +\frac{1}{2}\int_Q \int_I M\left(m(\cdot,\varphi)(\nabla_{\overline{x}}\mu + \nabla_y \mu_1)(\nabla_{\overline{x}}\phi_0 + \nabla_y \phi_1)\right) \mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t = 0, \end{cases} \tag{3.31}$$

$$\begin{cases} \frac{1}{2}\int_Q \int_I \mu\chi_0 \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t = \frac{1}{2}\int_Q \int_I \widehat{a}\varphi\chi_0 \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t - \frac{1}{2}\int_Q \int_I (\widehat{J} * \varphi)\chi_0 \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t \\ \quad +\frac{1}{2}\int_Q \int_I F'(\varphi)\chi_0 \,\mathrm{d}y_d \,\mathrm{d}\overline{x} \,\mathrm{d}t, \end{cases} \tag{3.32}$$

$$|\varphi(t,\overline{x})| < 1 \text{ a.e. } (t,\overline{x}) \in Q, \tag{3.33}$$

$$\boldsymbol{u}_0(0,\overline{x},y) = \boldsymbol{u}^0(\overline{x}) \text{ and } \varphi(0,\overline{x}) = \varphi^0(\overline{x}) \text{ for a.e. } \overline{x} \in \Omega \text{ and } y \in \mathbb{R}^{d-1} \times I, \tag{3.34}$$

for all $\Psi \in (\mathcal{C}_0^\infty(Q) \otimes A^\infty(\mathbb{R}^{d-1}; \mathcal{C}_0^\infty(I)))^d$ with $\mathrm{div}_y\Psi = 0$, $(\phi_0, \phi_1) \in \mathcal{C}_0^\infty(Q) \times (\mathcal{C}_0^\infty(Q) \otimes A^\infty(\mathbb{R}^{d-1}; \mathcal{C}_0^\infty(I)))$ and $\chi_0 \in \mathcal{C}_0^\infty(Q)$.

3.3 The Doubly Nonlocal Homogenized Limit

In this subsection, we aim to recover the homogenized problem for $(\overline{\boldsymbol{u}}, \varphi, \mu, p)$, where $\overline{\boldsymbol{u}}$ is defined by (3.12) and satisfies (3.14). We should consider each of the equations (3.30)–(3.32) separately.

First, we deal with (3.32) to see that it is equivalent to

$$\mu = \widehat{a}\varphi - \widehat{J} * \varphi + F'(\varphi) \text{ in } Q. \tag{3.35}$$

This stems from the fact that none of the terms in (3.32) depends on the variable $y_d \in I$ and, moreover, meas$(I) = 2$.

Now, we proceed with (3.31) to see that it is equivalent to the system

$$\begin{cases} -\int_Q \varphi \frac{\partial \phi_0}{\partial t}\, \mathrm{d}\overline{x}\, \mathrm{d}t - \frac{1}{2}\int_Q \int_I M(\varphi \boldsymbol{u}_0 \cdot \nabla_{\overline{x}}\phi_0)\, \mathrm{d}y_d\, \mathrm{d}\overline{x}\, \mathrm{d}t \\ \quad + \frac{1}{2}\int_Q \int_I M\left(m(\,\cdot\,,\varphi)(\nabla_{\overline{x}}\mu + \nabla_y \mu_1)\cdot \nabla_{\overline{x}}\phi_0\right) \mathrm{d}y_d\, \mathrm{d}\overline{x}\, \mathrm{d}t = 0, \\ \text{for all } \phi_0 \in \mathcal{C}_0^\infty(Q), \end{cases} \tag{3.36}$$

$$\begin{cases} -\frac{1}{2}\int_Q \int_I M(\varphi \boldsymbol{u}_0 \cdot \nabla_y \phi_1)\, \mathrm{d}y_d\, \mathrm{d}\overline{x}\, \mathrm{d}t \\ \quad + \frac{1}{2}\int_Q \int_I M\left(m(\,\cdot\,,\varphi)(\nabla_{\overline{x}}\mu + \nabla_y \mu_1)\nabla_y \phi_1\right) \mathrm{d}y_d\, \mathrm{d}\overline{x}\, \mathrm{d}t = 0, \\ \text{for all } \phi_1 \in \mathcal{C}_0^\infty(Q) \otimes A^\infty(\mathbb{R}^{d-1}; \mathcal{C}_0^\infty(I)). \end{cases} \tag{3.37}$$

We continue with (3.37), where we choose ϕ_1 to be of the form $\phi_1(t, \overline{x}, y) = \phi_1^0(t, \overline{x})\theta(y)$ with $\phi_1^0 \in \mathcal{C}_0^\infty(Q)$ and $\theta \in A^\infty(\mathbb{R}^{d-1}; \mathcal{C}_0^\infty(I))$. Then, we obtain

$$\int_I M(\varphi \boldsymbol{u}_0 \cdot \nabla_y \theta)\, \mathrm{d}y_d + \int_I M\left(m(\,\cdot\,,\varphi)(\nabla_{\overline{x}}\mu + \nabla_y \mu_1)\nabla_y \theta\right) \mathrm{d}y_d = 0,$$

or, taking into account that $\int_I M(\varphi \boldsymbol{u}_0 \nabla_y \theta)\, \mathrm{d}y_d = \int_I M(\varphi \overline{\mathrm{div}}_y(\boldsymbol{u}_0\theta)\, \mathrm{d}y_d = 0$ (recall that φ does not depend on y),

$$\int_I M\left(m(\,\cdot\,,\varphi(t,\overline{x}))(\nabla_{\overline{x}}\mu(t,\overline{x}) + \nabla_y \mu_1(t,\overline{x},\,\cdot\,))\nabla_y \theta\right) \mathrm{d}y_d = 0, \qquad \theta \in A^\infty(\mathbb{R}^{d-1}; \mathcal{C}_0^\infty(I)) \tag{3.38}$$

for a.e. $(t, \overline{x}) \in Q$, which is the weak formulation of the equation

$$-\mathrm{div}_y(m(\,\cdot\,,\varphi(t,\overline{x}))(\nabla_{\overline{x}}\mu(t,\overline{x}) + \nabla_y \mu_1(t,\overline{x},\,\cdot\,)) = 0 \text{ in } \mathbb{R}^{d-1} \times I.$$

So, for $\xi \in \mathbb{R}^{d-1} \times \{0\}$ and $r \in \mathbb{R}$ arbitrarily fixed, we consider the corrector problem

$$\begin{cases} \text{Find } \pi_{\xi,r} \equiv \pi_{\xi,r}(t,\overline{x},\,\cdot\,) \in B_{\#A}^{1,2}(\mathbb{R}^{d-1}; H_0^1(I)) \text{ such that} \\ -\mathrm{div}_y(m(\,\cdot\,,r)(r + \nabla_y \pi_{\xi,r}(t,\overline{x},\,\cdot\,)) = 0 \text{ in } \mathbb{R}^{d-1} \times I. \end{cases} \tag{3.39}$$

Proceeding as in the proof of (Jäger et al. 2023, Theorem 1.2), we may prove that there exists a function $\pi_{\xi,r} \in L^2(Q; B_{\#A}^{1,2}(\mathbb{R}^{d-1}; H_0^1(I)))$ whose gradient $\nabla_y \pi_{\xi,r}$ is unique, and which is such that, for a.e. $(t, \overline{x}) \in Q$, $\pi_{\xi,r}(t, \overline{x}, \cdot) \in B_{\#A}^{1,2}(\mathbb{R}^{d-1}; H_0^1(I))$ solves

(3.39) in the classical sense of distributions in $\mathbb{R}^{d-1} \times I$. This being so, choosing $\xi = \nabla_{\overline{x}}\mu(t, \overline{x})$ and $r = \varphi(t, \overline{x})$ in (3.39) and relying on the uniqueness of the gradient of the solution to (3.39), it emerges that

$$\mu_1(t, \overline{x}, y) = \pi_{\nabla_{\overline{x}}\mu(t,\overline{x}),\varphi(t,\overline{x})}(t, \overline{x}, y) \text{ for a.e. } (t, \overline{x}, y) \in Q \times \mathbb{R}^{d-1} \times I.$$

Now, choosing $\xi = e_j$ (the jth vector of the canonical basis $\mathbb{R}^{d-1}$; remember that we view it as the vector $(e_j, 0) \in \mathbb{R}^d$) and $r = \varphi(t, \overline{x})$, and denoting by $\omega_j(t, \overline{x}, \cdot)$ the corresponding solution of (3.39), we easily get that

$$\mu_1(t, \overline{x}, y) = \nabla_{\overline{x}}\mu(t, \overline{x}) \cdot \omega(t, \overline{x}, y) \text{ with } \omega(t, \overline{x}, \cdot) = (\omega(t, \overline{x}, \cdot))_{1\leq j\leq d-1} \tag{3.40}$$

Bearing this in mind, we define the homogenized mobility term (matrix) as follows

$$\widehat{m}(\varphi)(t, \overline{x}) = \frac{1}{2}\int_I M(m(\cdot, y_d, \varphi(t, \overline{x}))(I_{d-1} + \nabla_{\overline{y}}\omega(t, \overline{x}, \cdot, y_d))\, \mathrm{d}y_d, \ (t, \overline{x}) \in Q, \tag{3.41}$$

where I_{d-1} denotes the $(d-1) \times (d-1)$ identity matrix.

Finally, substituting in (3.36) the expression for μ_1 given by (3.40), we arrive at

$$\frac{\partial\varphi}{\partial t} + \overline{\boldsymbol{u}} \cdot \nabla_{\overline{x}}\varphi - \mathrm{div}_{\overline{x}}(\widehat{m}(\varphi)\nabla_{\overline{x}}\mu) = 0 \text{ in } Q. \tag{3.42}$$

Before proceeding further, it is worth noting that in case the mobility m does not depend on y, we can easily show that $\mu_1 = 0$ and, in that case, the homogenized mobility coefficient is exactly $m(\varphi)$ and not a matrix as in (3.42). This stems from the fact that the function φ does not depend on the microscopic variable y.

Let us consider problem (3.30). By a mere routine using (Jäger and Woukeng 2022, Proposition 3.1), we infer the existence of a function $p_1 \in L^2(Q; \mathcal{B}^2_A(\mathbb{R}^{d-1}; L^2(I)))$ such that

$$\frac{\partial \boldsymbol{u}_0}{\partial t} - \overline{\mathrm{div}}_y(\eta(\cdot, \varphi)\overline{\nabla}_y\boldsymbol{u}_0) + \overline{\nabla}_y p_1 = \boldsymbol{h}_1 - \nabla_{\overline{x}}p + \mu\nabla_{\overline{x}}\varphi \text{ in } Q \times \mathbb{R}^{d-1} \times I. \tag{3.43}$$

In order to deal with (3.43), let us pay attention to the auxiliary Stokes system (3.1) and consider the function $\omega^j = (\omega^j_i)_{1\leq i\leq d}$ defined therein with the associated matrix $G = (G_{ij})_{1\leq i,j\leq d-1}$. Fix $(t, \overline{x}) \in Q$ and choose in the variational formulation of (3.1) the test function $v(\tau, y) = \boldsymbol{u}_0(t - \tau, \overline{x}, y)$ for $(\tau, y) \in (0, t) \times \mathbb{R}^{d-1} \times I$. Then, we have

$$\left\langle \frac{\partial\omega^j}{\partial\tau}(\tau), \boldsymbol{u}_0(t-\tau)\right\rangle + \int_{-1}^1 M(\eta(\cdot, \varphi(\tau))\overline{\nabla}_y\omega^j(\tau) \cdot \overline{\nabla}_y\boldsymbol{u}_0(t-\tau))\, \mathrm{d}y_d = 0,$$

that is,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{-1}^{1} M(\omega^j(\tau)\boldsymbol{u}_0(t-\tau))\,\mathrm{d}y_d + \left\langle \frac{\partial \boldsymbol{u}_0}{\partial \tau}(t-\tau), \omega^j(\tau) \right\rangle \\ \quad + \int_{-1}^{1} M(\eta(\,\cdot\,,\varphi(\tau))\overline{\nabla}_y \omega^j(\tau)\cdot \overline{\nabla}_y \boldsymbol{u}_0(t-\tau))\,\mathrm{d}y_d = 0. \end{cases}$$

Here, the brackets $\langle\, ,\, \rangle$ mean the duality pairing between $\mathcal{B}_A^{1,2}(\mathbb{R}^{d-1}; H_0^1(I))$ and its dual. We integrate the above last equation over $(0, t)$ and divide both sides of the resulting equality by two. Then,

$$\begin{cases} \frac{1}{2}\int_{-1}^{1} M(\omega^j(t)\boldsymbol{u}_0(0))\,\mathrm{d}y_d - \frac{1}{2}\int_{-1}^{1} M(\boldsymbol{u}_0(t)e_j)\,\mathrm{d}y_d + \frac{1}{2}\int_0^t \left\langle \frac{\partial \boldsymbol{u}_0}{\partial \tau}(t-\tau), \omega^j(\tau) \right\rangle \mathrm{d}\tau \\ +\frac{1}{2}\int_0^t \int_{-1}^{1} M(\eta(\,\cdot\,,\varphi(\tau))\overline{\nabla}_y \omega^j(\tau)\cdot \overline{\nabla}_y \boldsymbol{u}_0(t-\tau))\,\mathrm{d}y_d\,\mathrm{d}\tau = 0. \end{cases} \tag{3.44}$$

Now, we test (3.43) by $\Psi(\tau, \overline{x}, y) = \varphi(\overline{x})\omega^j(\overline{x}, t-\tau, y)$ for $(\tau, \overline{x}, y) \in Q \times \mathbb{R}^{d-1} \times I$ with $\varphi \in \mathcal{C}_0^\infty(\Omega)$, where $t \in (0, T]$ is fixed. Then, a simple computation shows that we have, in the sense of distributions in Ω,

$$\begin{aligned} &\frac{1}{2}\int_0^t \left\langle \frac{\partial \boldsymbol{u}_0}{\partial \tau}(\tau), \omega^j(t-\tau) \right\rangle \mathrm{d}\tau + \frac{1}{2}\int_0^t \int_{-1}^{1} M(\eta(\,\cdot\,,\varphi(\tau))\overline{\nabla}_y \boldsymbol{u}_0(\tau)\cdot \overline{\nabla}_y \omega^j(t-\tau))\,\mathrm{d}y_d\,\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^t \int_{-1}^{1} M(\omega^j(t-\tau))\boldsymbol{h}(\tau)\,\mathrm{d}y_d\,\mathrm{d}\tau - \frac{1}{2}\int_0^t \int_{-1}^{1} \nabla_{\overline{x}} p(\tau) M(\omega^j(t-\tau))\,\mathrm{d}y_d\,\mathrm{d}\tau \\ &\quad +\frac{1}{2}\int_0^t \int_{-1}^{1} \mu(\tau)\nabla_{\overline{x}}\varphi(\tau) M(\omega^j(t-\tau))\,\mathrm{d}y_d\,\mathrm{d}\tau. \end{aligned} \tag{3.45}$$

But we have

$$\int_0^t \left\langle \frac{\partial \boldsymbol{u}_0}{\partial \tau}(\tau), \omega^j(t-\tau) \right\rangle \mathrm{d}\tau = \int_0^t \left\langle \frac{\partial \boldsymbol{u}_0}{\partial \tau}(t-\tau), \omega^j(\tau) \right\rangle \mathrm{d}\tau,$$

and

$$\begin{aligned} &\int_0^t \int_{-1}^{1} M(\eta(\,\cdot\,,\varphi(\tau))\overline{\nabla}_y \boldsymbol{u}_0(\tau)\cdot \overline{\nabla}_y \omega^j(t-\tau))\,\mathrm{d}y_d\,\mathrm{d}\tau \\ &\quad = \int_0^t \int_{-1}^{1} M(\eta(\,\cdot\,,\varphi(t-\tau))\overline{\nabla}_y \omega^j(\tau)\cdot \overline{\nabla}_y \boldsymbol{u}_0(t-\tau))\,\mathrm{d}y_d\,\mathrm{d}\tau. \end{aligned}$$

Therefore, comparing (3.44) and (3.45) we are led to

$$\frac{1}{2}\int_{-1}^{1} M(\boldsymbol{u}_0(t))e_j\,\mathrm{d}y_d - \frac{1}{2}\int_{-1}^{1} M(\omega^j(t))\boldsymbol{u}^0\,\mathrm{d}y_d$$
$$+\frac{1}{2}\int_0^t\int_{-1}^{1} M([\eta(\,\cdot\,,\varphi(t-\tau))-\eta(\,\cdot\,,\varphi(\tau))]\overline{\nabla}_y\boldsymbol{u}_0(t-\tau)\cdot\overline{\nabla}_y\omega^j(\tau))\,\mathrm{d}y_d\,\mathrm{d}\tau$$
$$=\frac{1}{2}\int_0^t\int_{-1}^{1} M(\omega^j(t-\tau))\boldsymbol{h}(\tau)\,\mathrm{d}y_d\,\mathrm{d}\tau - \frac{1}{2}\int_0^t\int_{-1}^{1} M(\omega^j(t-\tau))\nabla_{\overline{x}}p(\tau)\,\mathrm{d}\tau\,\mathrm{d}y_d$$
$$+\frac{1}{2}\int_0^t\int_{-1}^{1} M(\omega^j(t-\tau))\mu(\tau)\nabla_{\overline{x}}\varphi(\tau)\,\mathrm{d}y_d\,\mathrm{d}\tau,$$

or, equivalently,

$$\begin{aligned} &u_j(t) - G_j(t)\boldsymbol{u}^0 + H_j(\varphi,\overline{\boldsymbol{u}})(t)\\ &= (G_j * \boldsymbol{h}_1)(t) - (G_j * \nabla_{\overline{x}}p)(t) + (G_j * \mu\nabla_{\overline{x}}\varphi)(t), \end{aligned} \tag{3.46}$$

where the function $H_j(\varphi,\overline{\boldsymbol{u}}) \in \mathcal{C}([0,T];L^1(\Omega))$ is defined by

$$\begin{aligned} H_j(\varphi,\overline{\boldsymbol{u}})(t,\overline{x}) &= \frac{1}{2}\int_0^t\int_{-1}^{1} M([\eta(\,\cdot\,,\varphi(t-\tau))\\ &-\eta(\,\cdot\,,\varphi(\tau))]\overline{\nabla}_y\boldsymbol{u}_0(t-\tau)\cdot\overline{\nabla}_y\omega^j(\tau))\,\mathrm{d}y_d\,\mathrm{d}\tau\\ &\quad\text{for a.e. } (t,\overline{x}) \in Q. \end{aligned} \tag{3.47}$$

Then, setting

$$H(\varphi,\overline{\boldsymbol{u}}) = (H_j(\varphi,\overline{\boldsymbol{u}}))_{1\le j\le d-1}, \tag{3.48}$$

and knowing that $G = (G_j)_{1\le j\le d-1}$, we get at once

$$\overline{\boldsymbol{u}}(t) + H(\varphi,\overline{\boldsymbol{u}})(t) = G(t)\overline{\boldsymbol{u}}^0 + (G*(\boldsymbol{h}_1 - \nabla_{\overline{x}}p + \mu\nabla_{\overline{x}}\varphi))(t) \text{ in } \Omega,\ t\in[0,T]. \tag{3.49}$$

We notice that the functions $\overline{\boldsymbol{u}}$, φ, μ, p and $H(\varphi, \boldsymbol{u})$ solve the following system

$$\begin{cases} \overline{\boldsymbol{u}} + H(\varphi, \overline{\boldsymbol{u}}) = G\overline{\boldsymbol{u}}^0 + G * (\boldsymbol{h}_1 + \mu\nabla_{\overline{x}}\varphi - \nabla_{\overline{x}} p) \text{ in } Q, \\ \mathrm{div}_{\overline{x}}\overline{\boldsymbol{u}} = 0 \text{ in } Q \text{ and } \overline{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \text{ on } (0, T) \times \partial\Omega, \\ \dfrac{\partial \varphi}{\partial t} + \overline{\boldsymbol{u}} \cdot \nabla_{\overline{x}}\varphi - \mathrm{div}_{\overline{x}}(\widehat{m}(\varphi)\nabla_{\overline{x}}\mu) = 0 \text{ in } Q, \\ \mu = \widehat{a}\varphi - \widehat{J} * \varphi + F'(\varphi) \text{ in } Q, \\ \dfrac{\partial \mu}{\partial \boldsymbol{n}} = 0 \text{ on } (0, T) \times \partial\Omega, \\ \varphi(0) = \varphi^0 \text{ in } \Omega. \end{cases} \tag{3.50}$$

We are now in a position to prove the first main result of the work.

Proof of Theorem 1.1 First of all, we note that existence of solutions satisfying uniform a priori estimates was proved in Sect. 2, see Theorem 2.1 and Sect. 2.3. Now, we know from Definition A.1 that if $v_\varepsilon \to v_0$ in $L^r(Q_\varepsilon)$-weak Σ_A, then defining $M_\varepsilon v_\varepsilon$ as in (1.9) we have

$$M_\varepsilon v_\varepsilon \to \frac{1}{2}\int_I M(v_0(\,\cdot\,, y_d))\,\mathrm{d}y_d \text{ in } L^r(Q)\text{-weak.}$$

Bearing this in mind and given an ordinary sequence E of positive real numbers converging to zero, we infer from the above property and from the convergence results (3.3), (3.13), (3.5) and (3.7), (3.8)–(3.9) and (3.10) that, up to a subsequence, we have

$$\begin{aligned} &M_\varepsilon \boldsymbol{u}_\varepsilon \to (\boldsymbol{u}, 0) \text{ in } L^2(Q)^d\text{-weak,} \\ &M_\varepsilon \varphi_\varepsilon \to \varphi \text{ in } L^2(Q)\text{-strong and in } L^2(0, T; H^1(\Omega))\text{-weak,} \\ &M_\varepsilon \mu_\varepsilon \to \mu \text{ in } L^2(0, T; H^1(\Omega))\text{-weak,} \end{aligned}$$

and

$$M_\varepsilon p_\varepsilon \to p \text{ in } L^2(Q) \text{ -weak,}$$

where $\boldsymbol{u} \in L^2(0, T; \mathbb{H})$, $\varphi \in \mathcal{C}([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, $\mu \in L^2(0, T; H^1(\Omega))$ and $p \in L^2(0, T; L^2_0(\Omega))$. Defining $H(\varphi, \boldsymbol{u})$ like in (3.47), (3.48), we have shown that the quintuple $(\boldsymbol{u}, \varphi, \mu, p, H(\varphi, \boldsymbol{u}))$ solves system (1.11). The proof is completed. □

The homogenized mobility $\widehat{m}(\varphi)$ is defined by (3.41). The function $H(\varphi, \boldsymbol{u})$ arises from the contribution of the viscosity coefficient η, especially from its dependence upon the order parameter. If η is φ-independent, then $H(\varphi, \boldsymbol{u})$ vanishes.

Remark 3.1 Assume that the function η does not depend on the second variable, that is, $\eta(y, r) = \eta(y)$ for all $(y, r) \in \mathbb{R}^d \times \mathbb{R}$. Then the function $H(\varphi, \overline{\boldsymbol{u}})$ vanishes. As a by-product, the solution ω^j of the auxiliary Stokes system (3.1) is independent of the macroscopic variable $\overline{x}$. As a result, we recover the nonlocal (in time) Hele-Shaw equation derived in Peter and Woukeng (2024) for smooth potentials F.

For the case that η is φ-independent, we may summarize the results of the analysis made in the current subsection as follows.

Theorem 3.1 *The quintuple $(\overline{\boldsymbol{u}}, \varphi, \mu, p, H(\varphi, \overline{\boldsymbol{u}}))$ defined by (3.12), (3.5), (3.8), (3.10) and (3.50), respectively, is the weak solution to the homogenized system (3.50). If in addition the function η is independent of its second variable, then the function ω^j is independent of the macroscopic variable $\overline{x} \in \Omega$ and we recover in $(3.50)_1$ the nonlocal (in time) Hele-Shaw equation. In this case, the quadruple $(\overline{\boldsymbol{u}}, \varphi, \mu, p)$ solves the Hele-Shaw–Cahn–Hilliard system*

$$\begin{cases} \overline{\boldsymbol{u}} = G\overline{\boldsymbol{u}}^0 + G * (\boldsymbol{h}_1 + \mu\nabla_{\overline{x}}\varphi - \nabla_{\overline{x}} p) \text{ in } Q, \\ \operatorname{div}_{\overline{x}}\overline{\boldsymbol{u}} = 0 \text{ in } Q \text{ and } \overline{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \text{ on } (0, T) \times \partial\Omega, \\ \dfrac{\partial\varphi}{\partial t} + \overline{\boldsymbol{u}} \cdot \nabla_{\overline{x}}\varphi - \operatorname{div}_{\overline{x}}(\widehat{m}(\varphi)\nabla_{\overline{x}}\mu) = 0 \text{ in } Q, \\ \mu = \widehat{a}\varphi - \widehat{J} * \varphi + F'(\varphi) \text{ in } Q, \\ \dfrac{\partial\mu}{\partial\boldsymbol{n}} = 0 \text{ on } (0, T) \times \partial\Omega, \\ \varphi(0) = \varphi^0 \text{ in } \Omega. \end{cases} \tag{3.51}$$

4 Analysis of the Upscaled Doubly Nonlocal Model

In this section, we are concerned with the analysis of the problem (3.51). Our aim is twofold: (i) addressing the well-posedness of (3.51) and (ii) providing some regularity properties of its solution. Throughout this section, we assume that the functions η and m do not depend on the second variable, that is, $\eta(y, r) = \eta(y)$ and $m(y, r) = m(y)$ for all $(y, r) \in \mathbb{R}^d \times \mathbb{R}$. Then, in $(3.50)_1$, the function H vanishes and the homogenized mobility coefficient in $(3.50)_3$ becomes independent of φ and has the form

$$\widehat{m} = \frac{1}{2}\int_I M(m(\,\cdot\,, y_d)(I_{d-1} + \nabla_{\overline{y}}\omega(\,\cdot\,, y_d)))\,\mathrm{d}y_d,$$

where the corrector function $\omega = (\omega_j)_{1\leq j\leq d-1}$ is now independent of $(t, \overline{x})$ and solves the problem

$$-\operatorname{div}_y\left(m(y)(e_j + \nabla_y\omega_j)\right) = 0 \text{ in } \mathbb{R}^{d-1} \times I,\ \omega_j \in B^{1,2}_{\#A}(\mathbb{R}^{d-1}; H^1_0(I)).$$

As a result, $\widehat{m}$ has constant entries. One may easily check that $\widehat{m}$ is symmetric and positive definite.

Now, for $f \in H^1(\Omega)'$, we recall the definition of its average $\overline{f}$ over Ω: $\overline{f} = |\Omega|^{-1} \langle f, 1\rangle$. Here Ω is the open bounded set considered in Sect. 1, which is assumed to be Lipschitz. We recall the definition of the following sets (see (B.1) where this time we replace Ω_1 by Ω):

$$V_0 = \left\{v \in H^1(\Omega) : \overline{v} = 0\right\}, \text{ so that } V_0' = \left\{f \in H^1(\Omega)' : \overline{f} = 0\right\}.$$

We consider the operator $\mathcal{A}\colon H^1(\Omega) \to H^1(\Omega)'$ defined by

$$\langle \mathcal{A}u, v\rangle = \int_\Omega \widehat{m}\nabla u \cdot \nabla v \,\mathrm{d}x \text{ for } u, v \in H^1(\Omega).$$

We observe that $\mathcal{A}$ is a continuous mapping from $H^1(\Omega)$ into V_0'. We also see that its restriction A_0 to V_0 is an isomorphism from V_0 onto V_0'; this stems from the fact that the matrix $\widehat{m}$ is symmetric and positive definite. We denote by A_0^{-1} the inverse of A_0; then,

$$\mathcal{A}A_0^{-1} f = f \;\; \forall f \in V_0', \text{ and } A_0^{-1}\mathcal{A}u = u \;\; \forall u \in V_0.$$

It is also a fact that, for $f \in V_0'$, $u = A_0^{-1} f$ is the unique solution with zero mean value of the Neumann problem

$$-\operatorname{div}(\widehat{m}\nabla u) = f \text{ in } \Omega, \widehat{m}\nabla u \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega, \tag{4.1}$$

$\boldsymbol{n}$ being the unit outward normal to $\partial\Omega$. In addition, it holds that

$$\left\langle \mathcal{A}u, A_0^{-1} f\right\rangle = \langle f, u\rangle \text{ for all } u \in H^1(\Omega) \text{ and all } f \in V_0',$$
$$\left\langle f, A_0^{-1} g\right\rangle = \left\langle g, A_0^{-1} f\right\rangle = \int_\Omega \widehat{m}\nabla(A_0^{-1} f) \cdot \nabla(A_0^{-1} g) \,\mathrm{d}x, \;\; f, g \in V_0'. \tag{4.2}$$

We equip V_0' with the norm $\|f\|_{V_0'} = \left\|\nabla(A_0^{-1} f)\right\|_{L^2(\Omega)}$ for $f \in V_0'$, which makes it a Hilbert space. As a result, one has

$$\left\langle \frac{\partial f}{\partial t}, A_0^{-1} f\right\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|f\|_{V_0'}^2 \text{ for a.e. } t \in (0, T) \text{ and all } f \in H^1(0, T; V_0'). \tag{4.3}$$

It is also known that the following map (defined on $H^1(\Omega)'$)

$$\|f\|_* = \left(\|f - \overline{f}\|_{V_0'}^2 + |\overline{f}|^2\right)^{\frac{1}{2}}, \;\; f \in H^1(\Omega)',$$

is a norm on $H^1(\Omega)'$ equivalent to the usual norm of $H^1(\Omega)'$. Since $\widehat{m}$ has constant coefficients, the following ellipticity estimates for the solutions of (4.1) hold

$$\left\| \nabla A_0^{-1} f \right\|_{H^k(\Omega)} \leq C \, \|f\|_{H^{k-1}(\Omega)} \text{ for all } f \in H^{k-1}(\Omega) \cap L_0^2(\Omega) \text{ and } k = 1, 2. \tag{4.4}$$

Bearing all these preliminaries in mind, we proceed to the first aim of this section.

4.1 Continuous Dependence of the Solutions on the Initial Data

We deal in this subsection with the system (3.51) arising from the upscaling of the $3D$ problem (1.1). It is posed on the $2D$ domain Ω. We rewrite it without using the subscripts, hat, etc. This gives

$$\begin{cases} \boldsymbol{u} = G\boldsymbol{u}^0 + G * (\boldsymbol{h} + \mu\nabla\varphi - \nabla p) \text{ in } Q, \\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } Q \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } (0,T)\times\partial\Omega, \\ \dfrac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi - \operatorname{div}(m(\varphi)\nabla\mu) = 0 \text{ in } Q, \\ \mu = a\varphi - J*\varphi + F'(\varphi) \text{ in } Q, \\ \dfrac{\partial\mu}{\partial\boldsymbol{n}} = 0 \text{ on } (0,T)\times\partial\Omega, \\ \varphi(0) = \varphi^0 \text{ in } \Omega. \end{cases} \tag{4.5}$$

Here the data are constrained as follows:

$(\mathrm{A1})_1$ The matrix $G = G(t)$ is symmetric, positive definite and has entries which decrease exponentially as t increases,

$(\mathrm{A2})_1$ $J \in W^{1,1}(\mathbb{R}^2) \cap \mathcal{C}(\mathbb{R}^2 \setminus \{0\})$, $a(x) = \int_\Omega J(x-\xi)\,\mathrm{d}\xi \geq 0$ $(x \in \Omega)$, and the function F satisfies (A3) (see Sect. 1),

$(\mathrm{A4})_1$ The 2×2 matrix m has constant entries, is symmetric and $(m\xi,\xi) \geq m_1 |\xi|^2$ for all $\xi \in \mathbb{R}^2$, where $m_1 > 0$ is a constant,

$(\mathrm{A5})_1$ $\boldsymbol{u}^0 \in \mathbb{H} = \{\boldsymbol{u} \in L^2(\Omega)^2 : \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$, $\varphi^0 \in L^\infty(\Omega)$ with $F(\varphi^0) \in L^1(\Omega)$ and $\left|\overline{\varphi^0}\right| < 1$, $\boldsymbol{h} \in L^2(Q)^2$.

Assumptions $(\mathrm{A1})_1$, $(\mathrm{A2})_1$, (A3), $(\mathrm{A4})_1$ and $(\mathrm{A5})_1$ being valid, it is known from Sect. 3 that problem (4.5) possesses at least a solution $(\boldsymbol{u}, \varphi, \mu, p)$ satisfying $\boldsymbol{u} \in \mathcal{C}([0,T];\mathbb{H}) \cap L^2(0,T;L^4(\Omega)^2)$, $\varphi \in \mathcal{C}([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^\infty(Q)$ with $|\varphi(t,x)| < 1$ a.e. in Q, $\mu \in L^2(0,T;H^1(\Omega))$ and $p \in L^2(0,T;L_0^2(\Omega))$. That $\boldsymbol{u}$ belongs to $L^2(0,T;L^4(\Omega)^2)$ stems from the convergence result (3.13) and the definition (3.11).

As a further notation, when there is a danger of confusion, we shall write $*_t$ (resp. $*_x$) to denote the convolution operator with respect to time (resp. space).

Theorem 4.1 *Assume $\boldsymbol{h} \in L^\infty(0,T;L^4(\Omega)^2)$ and $\boldsymbol{u}^0 \in L^4(\Omega)^2$. If $(\boldsymbol{u}_1,\varphi_1)$ and $(\boldsymbol{u}_2,\varphi_2)$ are two weak solutions of (4.5) corresponding to the initial conditions $(\boldsymbol{u}_1^0,\varphi_1^0)$ and $(\boldsymbol{u}_2^0,\varphi_2^0)$ with source terms $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ and if further $\left|\overline{\varphi_i^0}\right| < 1$, $i=1,2$, then there exists a positive constant C depending on the norms of the two solutions such that, for all $t \in [0,T]$, we have*

$$\begin{aligned}&\|\varphi_1(t)-\varphi_2(t)\|_*^2 + \int_0^t (\|\boldsymbol{u}_1(\tau)-\boldsymbol{u}_2(\tau)\|_{L^2(\Omega)}^2 + \|\varphi_1(\tau)-\varphi_2(\tau)\|_{L^2(\Omega)}^2)\,\mathrm{d}\tau \\ &\quad \leq C\left(\|\varphi_1^0-\varphi_2^0\|_*^2 + \left|\overline{\varphi_1^0}-\overline{\varphi_2^0}\right| + \|\boldsymbol{u}_1^0-\boldsymbol{u}_2^0\|_{L^2(\Omega)}^2 + \|\boldsymbol{h}_1-\boldsymbol{h}_2\|_{L^\infty(0,T;L^2(\Omega))}^2\right).\end{aligned} \tag{4.6}$$

In particular, the solution of (4.5) is unique.

Note that proving this theorem proves the second main result, Theorem 1.2.

Proof First of all, arguing as in (Peter and Woukeng 2024, Proposition 5.1) we observe that $p \in L^2(0,T;H^1(\Omega)\cap L_0^2(\Omega))$ and $\boldsymbol{u} \in L^\infty(0,T;L^4(\Omega)^2)$. Also, rewriting the term $\mu\nabla\varphi$ in the form

$$\mu\nabla\varphi = \nabla\left(F(\varphi) + a\frac{\varphi^2}{2}\right) - \nabla a\frac{\varphi^2}{2} - (J*\varphi)\nabla\varphi$$

and using $\widetilde{p} = p - (F(\varphi) + a\frac{\varphi^2}{2})$ as the new pressure, Eq. $(4.5)_1$ becomes

$$\boldsymbol{u} = G(t)\boldsymbol{u}^0 + G *_t (\boldsymbol{h} - \frac{\varphi^2}{2}\nabla a + (J *_x \varphi)\nabla\varphi - \nabla\widetilde{p}). \tag{4.7}$$

Let $(\boldsymbol{u}_1,\varphi_1)$ and $(\boldsymbol{u}_2,\varphi_2)$ be two solutions of (4.5) (with $(4.5)_1$ being replaced by (4.7)) corresponding to the initial data and source terms as in the statement of the theorem. Setting $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\varphi = \varphi_1 - \varphi_2$, $\mu = \mu_1 - \mu_2$ and $\widetilde{p} = \widetilde{p}_1 - \widetilde{p}_2$, we have $\varphi(0) = \varphi_1^0 - \varphi_2^0 \equiv \varphi^0$ and $\boldsymbol{u}(0) = \boldsymbol{u}_1^0 - \boldsymbol{u}_2^0 \equiv \boldsymbol{u}^0$ and the quadruple $(\boldsymbol{u},\varphi,\mu,\widetilde{p})$ satisfies the system

$$\begin{cases} \boldsymbol{u} = G(t)\boldsymbol{u}^0 - G *_t (\boldsymbol{h} + \varphi(\varphi_1 \\ \quad +\varphi_2)\frac{\nabla a}{2} + (J *_x \varphi)\nabla\varphi_2 + (J *_x \varphi_1)\nabla\varphi + \nabla\widetilde{p})) \text{ in } Q, \\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } Q \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } (0,T)\times\partial\Omega, \\ \dfrac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi_1 + \boldsymbol{u}_2\cdot\nabla\varphi - \operatorname{div}(m\nabla\mu) = 0 \text{ in } Q, \\ \mu = a\varphi - J*\varphi + F'(\varphi_1) - F'(\varphi_2) \text{ in } Q, \\ \dfrac{\partial\mu}{\partial\boldsymbol{n}} = 0 \text{ on } (0,T)\times\partial\Omega, \\ \varphi(0) = \varphi^0 \text{ in } \Omega. \end{cases} \tag{4.8}$$

The variational formulation of (4.8) is as follows:

$$\left\langle \frac{\partial \varphi}{\partial t}, \psi \right\rangle + (m\nabla\mu, \nabla\psi) = (\boldsymbol{u}\varphi_1, \nabla\psi) + (\boldsymbol{u}_2\varphi, \nabla\psi), \quad \psi \in H^1(\Omega), \tag{4.9}$$

$$\begin{aligned}(\boldsymbol{u}, v) = (G\boldsymbol{u}^0, v) + (G *_t \boldsymbol{h}, v) - \left(G *_t \left(\varphi(\varphi_1 + \varphi_2)\tfrac{\nabla a}{2}\right), v\right) \\ - (G *_t (J *_x \varphi)\nabla\varphi_2, v) + (G *_t (J *_x \varphi_1)\nabla\varphi, v), \quad v \in \mathbb{H}\end{aligned} \tag{4.10}$$

for a.e. $t \in (0, T)$. First of all, taking $\psi = 1$ in (4.9) yields $\overline{\varphi}(t) = \overline{\varphi^0}$ for all $t \in [0, T]$. Still considering (4.9), we take therein $\psi = A_0^{-1}(\varphi - \overline{\varphi})$ to obtain the following identity (see (4.2) and (4.3)):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \overline{\varphi}\|^2_{V_0'} + (\mu, \varphi - \overline{\varphi}) = \left(\boldsymbol{u}\varphi_1, \nabla(A_0^{-1}(\varphi - \overline{\varphi})\right) + \left(\boldsymbol{u}_2\varphi, \nabla(A_0^{-1}(\varphi - \overline{\varphi})\right)$$

or, equivalently,

$$\begin{aligned}\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \overline{\varphi}\|^2_{V_0'} + \left(a\varphi + F'(\varphi_1) - F'(\varphi_2), \varphi\right) = \left(\boldsymbol{u}\varphi_1, \nabla(A_0^{-1}(\varphi - \overline{\varphi})\right) \\ + \left(\boldsymbol{u}_2\varphi, \nabla(A_0^{-1}(\varphi - \overline{\varphi})\right) + (J *_x \varphi, \varphi) + |\Omega|\,\overline{\varphi\mu}.\end{aligned} \tag{4.11}$$

Using part (ii) of Assumption (A3), we see that

$$\left(a\varphi + F'(\varphi_1) - F'(\varphi_2), \varphi\right) \ge c_0 \|\varphi\|^2_{L^2(\Omega)},$$

so that (4.11) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \overline{\varphi}\|^2_{V_0'} + c_0\|\varphi\|^2_{L^2(\Omega)} \le \sum_{k=1}^{4} I_k, \tag{4.12}$$

where

$$\begin{aligned}&I_1 = \left(\boldsymbol{u}\varphi_1, \nabla(A_0^{-1}(\varphi - \overline{\varphi})\right), \quad I_2 = \left(\boldsymbol{u}_2\varphi, \nabla(A_0^{-1}(\varphi - \overline{\varphi})\right), \\ &I_3 = (J *_x \varphi, \varphi) \text{ and } I_4 = |\Omega|\,\overline{\mu\varphi}.\end{aligned}$$

Now, we return to (4.10) and take $v = \boldsymbol{u}$ as the test function. We immediately get from the resulting equality that

$$\begin{aligned}\|\boldsymbol{u}\|_{L^2} \le (\|G *_t ((\varphi_1 + \varphi_2)\nabla a)\|_{L^\infty} + \|G *_t (\nabla J *_x \varphi_1)\|_{L^\infty})\,\|\varphi\|_{L^2} \\ + \left\|G\boldsymbol{u}^0\right\|_{L^2} + \|\boldsymbol{h}(t)\|_{L^2} + \|G *_t (\nabla J *_x \varphi)\|_{L^2},\end{aligned} \tag{4.13}$$

where we have used the fact that $\|\varphi_2\|_{L^\infty} \le 1$. However,

$$\|G *_t (\nabla J *_x \varphi)\|^2_{L^2} = \int_\Omega \left|\int_0^t G(t-\tau)(\nabla J *_x \varphi)(\tau)\,\mathrm{d}\tau\right|^2 \mathrm{d}x$$

$$\leq \|G\|^2_{L^\infty(0,T)} \int_0^t \left(\int_\Omega |\nabla J *_x \varphi|^2 (\tau) \, dx \right) d\tau$$
$$\leq \|G\|^2_{L^\infty(0,T)} \|\nabla J\|^2_{L^1} \int_0^t \|\varphi(\tau)\|^2_{L^2} \, d\tau.$$

Thus,

$$\|G *_t (\nabla J *_x \varphi)\|_{L^2} \leq C \left(\int_0^t \|\varphi(\tau)\|^2_{L^2} \, d\tau \right)^{\frac{1}{2}}, \tag{4.14}$$

where C in (4.14) depends on the norms of G and ∇J. Plugging (4.14) into (4.13), we get after some algebra

$$\|\boldsymbol{u}\|_{L^2} \leq C \left[\left\|\boldsymbol{u}^0\right\|_{L^2} + \|\boldsymbol{h}\|_{L^2} + \|\varphi\|_{L^2} + \left(\int_0^t \|\varphi(\tau)\|^2_{L^2} \, d\tau \right)^{\frac{1}{2}} \right], \tag{4.15}$$

where the constant C depends on the norms of the φ_i, J and a.

With this in mind, we use (4.15) to control the terms in the right-hand side of (4.12) as follows:

$$|I_1| \leq C \|\boldsymbol{u}\|_{L^2} \|\varphi - \overline{\varphi}\|_{V_0'} = C \|\boldsymbol{u}\|_{L^2} \|\varphi - \overline{\varphi}\|_*$$
$$\leq C(\left\|\boldsymbol{u}^0\right\|_{L^2} \|\varphi - \overline{\varphi}\|_* + \|\boldsymbol{h}\|_{L^2} \|\varphi - \overline{\varphi}\|_* + \|\varphi\|_{L^2} \|\varphi - \overline{\varphi}\|_*)$$
$$+ C \|\varphi - \overline{\varphi}\|_* \left(\int_0^t \|\varphi(\tau)\|^2_{L^2} \, d\tau \right)^{\frac{1}{2}}$$

so that

$$|I_1| \leq \frac{c_0}{4} \|\varphi\|^2_{L^2} + C \left(\left\|\boldsymbol{u}^0\right\|^2_{L^2} + \|\boldsymbol{h}\|^2_{L^2} + \|\varphi - \overline{\varphi}\|^2_* + \int_0^t \|\varphi(\tau)\|^2_{L^2} \, d\tau \right). \tag{4.16}$$

Moreover,

$$|I_2| \leq \frac{c_0}{4} \|\varphi\|^2_{L^2} + C |\overline{\varphi}|^2 + C \|\boldsymbol{u}_2\|^4_{L^4} \|\varphi - \overline{\varphi}\|^2_*$$
$$\leq \frac{c_0}{4} \|\varphi\|^2_{L^2} + (C + \|\boldsymbol{u}_2\|^4_{L^4}) \|\varphi - \overline{\varphi}\|^2_*, \tag{4.17}$$

where we have used the $2D$-Ladyzhenskaya inequality ($\|w\|_{L^4(\Omega)} \leq C \|w\|^{\frac{1}{2}}_{L^2(\Omega)} \|w\|^{\frac{1}{2}}_{H^1(\Omega)}$ for all $w \in H^1(\Omega)$) in conjunction with the inequality (4.4) for $k = 1$ both applied to $\nabla(A_0^{-1}(\varphi - \overline{\varphi})$ to obtain the last inequality. For the third term, we estimate

$$|I_3| \leq |(J *_x \varphi, \varphi - \overline{\varphi})| + |(J *_x \varphi, \overline{\varphi})|$$

$$
\begin{aligned}
&= \left|\left(m\nabla(J *_x \varphi), \nabla(A_0^{-1}(\varphi - \overline{\varphi})\right)\right| + |(J *_x \varphi, \overline{\varphi})| \\
&\leq m_2 \|\nabla J\|_{L^1} \|\varphi\|_{L^2} \|\varphi - \overline{\varphi}\|_* + C \|J\|_{L^1} \|\varphi\|_{L^2} |\overline{\varphi}| \\
&\leq \frac{c_0}{4} \|\varphi\|_{L^2}^2 + C \|\varphi - \overline{\varphi}\|_*^2 , \qquad (4.18)
\end{aligned}
$$

while the fourth term is estimated as

$$
\begin{aligned}
|I_4| &= \left|(F'(\varphi_1) - F'(\varphi_2), \overline{\varphi})\right| \\
&\leq \left|\overline{\varphi^0}\right| \left(\left\|F'(\varphi_1)\right\|_{L^1} + \left\|F'(\varphi_2)\right\|_{L^1}\right), \qquad (4.19)
\end{aligned}
$$

where we have used conservation of the total mass ($\overline{\varphi}(t) = \overline{\varphi^0}$, $t \in [0, T]$) in the last inequality.

Gathering (4.12), (4.16), (4.17), (4.18) and (4.19), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\varphi - \overline{\varphi}\|_*^2 + \frac{c_0}{4} \|\varphi\|_{L^2}^2 \leq C\left(\left\|\boldsymbol{u}^0\right\|_{L^2}^2 + \|\boldsymbol{h}\|_{L^2}^2\right) + W_1(t) \|\varphi - \overline{\varphi}\|_*^2 \\
+ \int_0^t \|\varphi(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + C\left|\overline{\varphi^0}\right| W_2(t), \qquad (4.20)
\end{aligned}
$$

where

$$
W_1(t) = C + C\|\boldsymbol{u}_2(t)\|_{L^4}^4 \text{ and } W_2(t) = C(\left\|F'(\varphi_1)\right\|_{L^1} + \left\|F'(\varphi_2)\right\|_{L^1}).
$$

Therefore, owing to the fact that $W_1, W_2 \in L^1(0, T)$, we apply the Gronwall lemma to (4.20) to get at once

$$
\|\varphi(t) - \overline{\varphi}\|_*^2 + \int_0^t \|\varphi(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq C\left(\left\|\boldsymbol{u}^0\right\|_{L^2}^2 + \|\boldsymbol{h}\|_{L^2}^2 + \left|\overline{\varphi^0}\right|\right)
$$

or

$$
\|\varphi(t)\|_*^2 + \int_0^t \|\varphi(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq C\left(\left\|\boldsymbol{u}^0\right\|_{L^2}^2 + \left\|\overline{\varphi^0}\right\|_*^2 + \|\boldsymbol{h}\|_{L^2}^2 + \left|\overline{\varphi^0}\right|\right) \qquad (4.21)
$$

for all $t \in [0, T]$, where we used the equality $\overline{\varphi}(t) = \overline{\varphi^0}$. Now, integrating (4.21), we are led to (4.6), thereby finishing the proof. □

4.2 Further Regularity Results

Our goal now is to prove that the unique weak solution of the doubly nonlocal system (4.5) is actually a strong one provided the initial order parameter satisfies an additional assumption and we assume $\boldsymbol{u}^0 = 0$. More precisely, we aim to prove the following theorem.

Theorem 4.2 *Suppose $\boldsymbol{u}^0 = 0$, $\varphi^0 \in L^\infty(\Omega)$ with $F(\varphi^0) \in L^1(\Omega)$ and $\left|\overline{\varphi^0}\right| < 1$. Assume further that $\nabla F'(\varphi^0) \in L^2(\Omega)^2$ and $\boldsymbol{h} \in W^{1,\infty}(0,T;L^2(\Omega)^2)$. Then, the weak solution of (4.5) is a strong solution and satisfies*

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;L^2(\Omega)^2), \tag{4.22}$$

$$\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^4(0,T;W^{1,4}(\Omega)) \cap L^2(0,T;W^{1,r}(\Omega)), \tag{4.23}$$

$$\frac{\partial \varphi}{\partial t} \in L^\infty(0,T;H^1(\Omega)') \cap L^2(0,T;L^2(\Omega)), \tag{4.24}$$

$$\mu \in L^\infty(0,T;H^1(\Omega)) \cap L^4(0,T;W^{1,4}(\Omega)) \cap L^2(0,T;H^2(\Omega)), \tag{4.25}$$

$$p \in \mathcal{C}([0,T];H^1(\Omega) \cap L^2_0(\Omega)), \tag{4.26}$$

and

$$F'(\varphi) \in L^\infty(0,T;H^1(\Omega)), \tag{4.27}$$

where $2 \le r < \infty$.
If further curl $\boldsymbol{h} \in L^\infty(0,T;L^r(\Omega))$ *for some $2 \le r < \infty$, then*

$$\boldsymbol{u} \in \begin{cases} L^2(0,T;W^{1,r}(\Omega)^2) \text{ for the same } r \text{ as } \operatorname{curl} \boldsymbol{h}, \\ L^4(0,T;W^{1,4}(\Omega)^2) \text{ if } r = 4, \\ L^\infty(0,T;H^1(\Omega)^2) \text{ if } r = 2. \end{cases} \tag{4.28}$$

Note that proving this theorem also proves the third main result, Theorem 1.3.

Let $(\boldsymbol{u}, \varphi, \mu, p)$ be the unique weak solution of (4.5) given by Theorem 4.1. To obtain (4.22)–(4.28), we proceed in three steps detailed in the three following subsections.

4.2.1 First Estimates on Time Derivatives

Let $h > 0$ be fixed and set

$$D^h v(t) = \frac{1}{h}(v(t+h) - v(t)) \text{ for a given function } v.$$

Next, set

$$H(t) = h(t) - \frac{\nabla a}{2}\varphi^2(t) - [(J *_x \varphi)\nabla\varphi](t) - \nabla\widetilde{p}(t), \quad t \in [0,T],$$

so that $\boldsymbol{u}(t) = (G *_t H)(t)$, see $(4.5)_1$ and (4.7). Then, $H \in L^2(0,T;L^2(\Omega)^2)$. An easy computation shows that

$$D^h \boldsymbol{u}(t) = (G *_t D^h H)(t) + (G *_t \alpha_h H(\cdot + h))(t) \text{ for a.e. } t \in (0,T), \tag{4.29}$$

where $\alpha_h(t) = \frac{1}{h}1_{[-h,0]}(t)$ for $t \in \mathbb{R}$ ($1_{[-h,0]}$ being the indicator function of $[-h, 0]$) is an approximation of unity in $\mathbb{R}$. It is easy to see that

$$\alpha_h \to 0 \text{ a.e. in } \mathbb{R} \text{ as } h \to 0. \tag{4.30}$$

Using the fact that $\|H(\cdot + h) - H\|_{L^2(Q)^2} \to 0$ as $h \to 0$ in conjunction with (4.30), one can show that, when $h \to 0$,

$$\alpha_h H(\cdot + h) \to 0 \text{ in } L^2(0, T; L^2(\Omega)^2). \tag{4.31}$$

Bearing this in mind, we consider the variational formulation of (4.5), in which we replace φ in $(4.5)_3$ by $D^h\varphi$ and $\boldsymbol{u}$ in (4.7) by $D^h\boldsymbol{u}$. Then, after setting $\tau_h\varphi = \varphi(\cdot + h)$, we obtain

$$\left\langle \frac{\partial D^h\varphi}{\partial t}, \psi \right\rangle + \left(mD^h\mu, \nabla\psi\right) = \left(\tau_h\varphi D^h\boldsymbol{u}, \nabla\psi\right) + \left(\boldsymbol{u}D^h\varphi, \nabla\psi\right), \quad \psi \in H^1(\Omega), \tag{4.32}$$

$$\begin{aligned} \left(D^h\boldsymbol{u}, v\right) = \left(G *_t D^h\boldsymbol{h}, v\right) - \left(G *_t \tfrac{\nabla a}{2}(\tau_h\varphi + \varphi)D^h\varphi, v\right) \\ - \left(G *_t (\nabla J *_x D^h\varphi)\varphi, v\right) - (G *_t (\nabla J *_x \tau_h\varphi)D^h\varphi, v), \quad v \in \mathbb{H}. \end{aligned} \tag{4.33}$$

We take $v = D^h\boldsymbol{u}$ in (4.33) and we proceed as we did in obtaining (4.15) leading to

$$\left\|D^h\boldsymbol{u}\right\|_{L^2} \le C\left[\left\|D^h\boldsymbol{h}\right\|_{L^2} + \left\|D^h\varphi\right\|_{L^2} + \left(\int_0^t \left\|D^h\varphi(\tau)\right\|_{L^2}^2 \mathrm{d}\tau\right)^{\frac{1}{2}} + \|\alpha_h\tau_h H\|_{L^2}\right]. \tag{4.34}$$

Now, we go back to (4.32) and choose therein $\psi = A_0^{-1}(D^h\varphi)$. Then, recalling that $\overline{D^h\varphi} = 0$ as well as the properties of A_0^{-1}, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^h\varphi\right\|_{V_0'}^2 + \left(D^h\mu, D^h\varphi\right) = \left(\tau_h\varphi D^h\boldsymbol{u}, \nabla(A_0^{-1}(D^h\varphi)\right) + \left(\boldsymbol{u}D^h\varphi, \nabla(A_0^{-1}(D^h\varphi)\right)$$

or, equivalently,

$$\begin{aligned} \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\left\|D^h\varphi\right\|_{V_0'}^2 + \left(aD^h\varphi + \tfrac{F'(\tau_h\varphi)-F'(\varphi)}{h}, D^h\varphi\right) = \left(\tau_h\varphi D^h\boldsymbol{u}, \nabla(A_0^{-1}(D^h\varphi)\right) \\ + \left(\boldsymbol{u}D^h\varphi, \nabla(-A_0^{-1}(D^h\varphi)\right) + (J *_x D^h\varphi, D^h\varphi). \end{aligned} \tag{4.35}$$

Using part (ii) of (A3) combined with (4.35), we are led to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^h\varphi\right\|_{V_0'}^2 + c_0\left\|D^h\varphi\right\|_{L^2(\Omega)}^2 \le \sum_{k=1}^{3} I_k, \tag{4.36}$$

where

$$I_1 = \left(\tau_h \varphi D^h \boldsymbol{u}, \nabla(A_0^{-1}(D^h\varphi)\right), \quad I_2 = \left(\boldsymbol{u} D^h\varphi, \nabla(-A_0^{-1}(D^h\varphi)\right),$$
$$I_3 = (J *_x D^h\varphi, D^h\varphi).$$

By virtue of (4.34), we control I_1 as follows:

$$\begin{aligned}|I_1| &\le C \left\| D^h \boldsymbol{u} \right\|_{L^2} \left\| D^h\varphi \right\|_{V_0'} \\ &\le \frac{c_0}{4} \left\| D^h\varphi \right\|^2_{L^2(\Omega)} + C\left(\left\| D^h \boldsymbol{h} \right\|^2_{L^2} + \left\| D^h\varphi \right\|^2_* + \|\alpha_h \tau_h H\|^2_{L^2} + \int_0^t \left\| D^h\varphi(\tau) \right\|^2_{L^2} \mathrm{d}\tau \right).\end{aligned} \tag{4.37}$$

Next, we control I_2 and I_3 as we did in the proof of Theorem 4.1 to obtain

$$|I_2| \le \frac{c_0}{4} \left\| D^h\varphi \right\|^2_{L^2(\Omega)} + C \|\boldsymbol{u}\|^4_{L^4} \left\| D^h\varphi \right\|^2_* \tag{4.38}$$

and

$$\begin{aligned}|I_3| &= \left| (J *_x D^h\varphi, D^h\varphi) \right| = \left| \left(m\nabla(J *_x D^h\varphi), \nabla(A_0^{-1}(D^h\varphi) \right) \right| \\ &\le m_2 \|\nabla J\|_{L^1} \left\| D^h\varphi \right\|_{L^2} \left\| D^h\varphi \right\|_* \\ &\le \frac{c_0}{4} \left\| D^h\varphi \right\|^2_{L^2} + C \left\| D^h\varphi \right\|^2_*,\end{aligned}$$

that is,

$$|I_3| \le \frac{c_0}{4} \left\| D^h\varphi \right\|^2_{L^2} + C \left\| D^h\varphi \right\|^2_*. \tag{4.39}$$

Putting together (4.36)–(4.39), we are led to

$$\begin{aligned}&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t} \left\| D^h\varphi \right\|^2_* + \tfrac{c_0}{4} \left\| D^h\varphi \right\|^2_{L^2(\Omega)} \\ &\le C\left(\left\| D^h \boldsymbol{h} \right\|^2_{L^2} + \left\| D^h\varphi \right\|^2_* W_3(t) + \|\alpha_h \tau_h H\|^2_{L^2} + \textstyle\int_0^t \left\| D^h\varphi(\tau) \right\|^2_{L^2} \mathrm{d}\tau \right),\end{aligned} \tag{4.40}$$

where $W_3(\cdot) = C + C\|\boldsymbol{u}(\cdot)\|^4_{L^4} \in L^1(0, T)$. An application of Gronwall's lemma gives

$$\begin{aligned}&\left\| D^h\varphi(t) \right\|^2_* + \textstyle\int_0^t \left\| D^h\varphi(\tau) \right\|^2_{L^2} d\tau \\ &\le \left[\left\| D^h\varphi(0) \right\|^2_* + C \textstyle\int_0^T (\left\| D^h \boldsymbol{h}(\tau) \right\|^2_{L^2} + \|\alpha_h(\tau)\tau_h H(\tau)\|^2_{L^2}) \,\mathrm{d}\tau \right] \textstyle\int_0^t x(s)\,\mathrm{d}s,\end{aligned} \tag{4.41}$$

where we have set

$$x(s) = \left\| D^h\varphi(0) \right\|^2_* + C\int_0^s (\left\| D^h \boldsymbol{h}(\tau) \right\|^2_{L^2} + \|\alpha_h(\tau)\tau_h H(\tau)\|^2_{L^2}) \,\mathrm{d}\tau.$$

In order to take full advantage of (4.41), we need to control the term uniformly in h. To do this, we go back to (4.9) to recall that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varphi-\varphi^0\right\|_{V_0'}^2 &= \left\langle\frac{\partial\varphi}{\partial t}, A_0^{-1}(\varphi-\varphi^0)\right\rangle\\ &= (\mu,\varphi-\varphi^0)+\left(\boldsymbol{u}\varphi,\nabla(A_0^{-1}(\varphi-\varphi^0)\right)\\ &= (m\nabla(a\varphi+F'(\varphi)),\nabla(A_0^{-1}(\varphi-\varphi^0)) - (m(\nabla J *_x \varphi),\nabla(A_0^{-1}(\varphi-\varphi^0))\\ &\quad+\left(\boldsymbol{u}\varphi,\nabla(A_0^{-1}(\varphi-\varphi^0)\right).\end{aligned}\tag{4.42}$$

Therefore, in view of (4.42) and using (4.41), we get after mere computations involving Gronwall's lemma,

$$\frac{1}{2}\left\|\varphi-\varphi^0\right\|_{V_0'}^2 \le (C+\left\|\nabla F'(\varphi^0)\right\|_{L^2})t \text{ for all } t\in[0,T].\tag{4.43}$$

Taking $t=h$ in (4.43), we get

$$\left\|D^h\varphi(0)\right\|_* \le C+\left\|\nabla F'(\varphi^0)\right\|_{L^2},\quad h>0.\tag{4.44}$$

Now, combining (4.41) and (4.44), we are led to

$$\begin{aligned}&\left\|D^h\varphi(t)\right\|_*^2+\int_0^t\left\|D^h\varphi(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau\\ &\le\left[C+C\int_0^T(\left\|D^h\boldsymbol{h}(\tau)\right\|_{L^2}^2+\|\alpha_h(\tau)\tau_h H(\tau)\|_{L^2}^2)\,\mathrm{d}\tau\right]\int_0^t x(s)\,\mathrm{d}s.\end{aligned}$$

Now, taking into account (4.34) (which we integrate over $(0,t)$) together with (4.41) and the above last inequality, we end up with

$$\begin{aligned}&\left\|D^h\varphi(t)\right\|_*^2+\int_0^t(\left\|D^h\boldsymbol{u}(\tau)\right\|_{L^2(\Omega)}^2+\left\|D^h\varphi(\tau)\right\|_{L^2(\Omega)}^2)\,\mathrm{d}\tau\\ &\le C+C\left(\|\alpha_h(\tau)\tau_h H(\tau)\|_{L^2}^2+\left\|D^h\boldsymbol{h}\right\|_{L^\infty(0,T;L^2(\Omega))}^2\right).\end{aligned}\tag{4.45}$$

Since $\varphi\in H^1(0,T;H^1(\Omega)')$, we know that $D^h\varphi\to\frac{\partial\varphi}{\partial t}$ in $L^2(0,T;H^1(\Omega)')$ as $h\to0$. We therefore deduce from (4.31) and the assumption on $\boldsymbol{h}$ that we may pass to the limit in (4.45) when $h\to0$ and obtain

$$\begin{aligned}&\left\|\frac{\partial\varphi}{\partial t}(t)\right\|_*^2+\int_0^t(\left\|\frac{\partial\boldsymbol{u}}{\partial t}(\tau)\right\|_{L^2(\Omega)}^2+\left\|\frac{\partial\varphi}{\partial t}(\tau)\right\|_{L^2(\Omega)}^2)\,\mathrm{d}\tau\\ &\le C+C\left\|\frac{\partial\boldsymbol{h}}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))}^2.\end{aligned}\tag{4.46}$$

This yields $\frac{\partial\varphi}{\partial t}\in L^\infty(0,T;H^1(\Omega)')\cap L^2(0,T;L^2(\Omega))$ and $\frac{\partial\boldsymbol{u}}{\partial t}\in L^2(0,T;L^2(\Omega)^2)$.

4.2.2 Uniform L^∞-Estimates in Time

First, we take the gradient of μ and test the resulting equality by $\nabla\varphi$ in L^2,

$$\big((a+F''(\varphi))\nabla\varphi,\nabla\varphi\big)=(\nabla\mu,\nabla\varphi)+(\nabla J*\varphi,\nabla\varphi)-(\varphi\nabla a,\nabla\varphi).$$

Applying part (ii) of Assumption (A3), we obtain

$$\begin{aligned}c_0\,\|\nabla\varphi\|_{L^2}^2&\le\|\nabla\mu\|_{L^2}\,\|\nabla\varphi\|_{L^2}+\|\nabla J\|_{L^1}\,\|\varphi\|_{L^2}\,\|\nabla\varphi\|_{L^2}+\|\nabla a\|_{L^\infty}\,\|\varphi\|_{L^2}\,\|\nabla\varphi\|_{L^2}\\&\le C(1+\|\nabla\mu\|_{L^2})\,\|\nabla\varphi\|_{L^2}\,,\end{aligned}$$

where C depends on $\|\varphi\|_{L^2}$, $\|\nabla J\|_{L^1}$ and $\|\nabla a\|_{L^\infty}$. This yields

$$\|\nabla\varphi\|_{L^2}\le C(1+\|\nabla\mu\|_{L^2}).\tag{4.47}$$

Next, defining the function $H(s)=F(s)+\frac{\beta}{2}s^2$, $s\in(-1,1)$, as previously, and using the same reasoning as in (Frigeri and Grasselli 2012, Proof of Theorem 1), we see that there is a positive constant C depending on $\overline{\varphi^0}$ such that

$$\left\|H'(\varphi)\right\|_{L^1(\Omega)}\le C\int_\Omega(\varphi-\overline{\varphi^0})H'(\varphi)\,\mathrm{d}x+C,$$

so that, arguing as previously (see (2.32)), we get

$$\begin{aligned}\left\|F'(\varphi)\right\|_{L^1(\Omega)}&\le C(\|\nabla\mu\|_{L^2}+\|\varphi\|_{L^2})\left\|\varphi-\overline{\varphi^0}\right\|_{L^2}+C\,\|\varphi\|_{L^2}+C\\&\le C(1+\|\nabla\mu\|_{L^2}).\end{aligned}$$

Now, from the equality $\int_\Omega\mu\,\mathrm{d}x=\int_\Omega F'(\varphi)\,\mathrm{d}x$, we deduce from the Poincaré – Wirtinger inequality that

$$\int_\Omega|\mu|^2\,\mathrm{d}x\le C(1+\|\nabla\mu\|_{L^2}^2),$$

and so,

$$\|\mu\|_{H^1(\Omega)}\le C(1+\|\nabla\mu\|_{L^2}).\tag{4.48}$$

This being so, we test $(4.5)_3$ by μ and get

$$(m\nabla\mu,\nabla\mu)=-\left\langle\frac{\partial\varphi}{\partial t},\mu\right\rangle+(\boldsymbol{u}\varphi,\nabla\mu).$$

Using the fact that $\frac{\partial\varphi}{\partial t}\in L^\infty(0,T;H^1(\Omega)')$ and by virtue of $(A4)_1$, we have

$$m_1\,\|\nabla\mu\|_{L^2}^2\le\left\|\frac{\partial\varphi}{\partial t}\right\|_{H^1(\Omega)'}\|\mu\|_{H^1(\Omega)}+\|\boldsymbol{u}\varphi\|_{L^2}\,\|\nabla\mu\|_{L^2}$$

$$\leq C \left\| \frac{\partial \varphi}{\partial t} \right\|_{H^1(\Omega)'} (1 + \|\nabla \mu\|_{L^2}) + \|\boldsymbol{u}\|_{L^2} \|\nabla \mu\|_{L^2}$$
$$\leq C \left\| \frac{\partial \varphi}{\partial t} \right\|^2_{H^1(\Omega)'} + \frac{2m_1}{3} \|\nabla \mu\|^2_{L^2} + C \|\boldsymbol{u}\|^2_{L^2},$$

where we used (4.48) and Young's inequality. This yields, thanks to (4.46),

$$\|\nabla \mu\|_{L^\infty(0,T;L^2(\Omega))} \leq C \left(\left\| \frac{\partial \varphi}{\partial t} \right\|_{L^\infty(0,T;H^1(\Omega)')} + \|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))} \right) \leq C. \tag{4.49}$$

Therefore, in view of (4.48), we have

$$\|\mu\|_{L^\infty(0,T;H^1(\Omega))} \leq C.$$

It also follows from (4.47) in association with (4.49) that

$$\|\varphi\|_{L^\infty(0,T;H^1(\Omega))} \leq C.$$

Now, assuming that $\operatorname{curl} \boldsymbol{h} \in L^\infty(0, T; L^2(\Omega))$ (we recall that for any $\boldsymbol{v} = (v_1, v_2) \in L^1(\Omega)^2$, $\operatorname{curl} \boldsymbol{v} = \nabla \times \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$), we have, for a.e. $t \in (0, T)$,

$$\operatorname{curl} \boldsymbol{u}(t) = G *_t (\operatorname{curl} \boldsymbol{h} - \varphi(\nabla a \times \nabla \varphi) - (\nabla J *_x \varphi) \times \nabla \varphi). \tag{4.50}$$

We know from the equality (3.18) in (Girault and Raviart 1986, Page 44) that there is a positive constant C such that

$$\|\boldsymbol{v}\|_{H^1(\Omega)^2} \leq C(\|\boldsymbol{v}\|_{L^2(\Omega)^2} + \|\nabla \times \boldsymbol{v}\|_{L^2(\Omega)}), \quad \boldsymbol{v} \in H^1(\Omega)^2 \cap \mathbb{H}. \tag{4.51}$$

Rewriting (4.50) in the form

$$\operatorname{curl} \boldsymbol{u}(t) = \int_0^t G(t - \tau)(\operatorname{curl} \boldsymbol{h}(\tau) - \varphi(\nabla a \times \nabla \varphi)(\tau) - (\nabla J *_x \varphi) \times \nabla \varphi)(\tau)) \,\mathrm{d}\tau, \tag{4.52}$$

we have

$$\|\operatorname{curl} \boldsymbol{u}(t)\|_{L^2} \leq \int_0^t |G(t - \tau)| \left(\|\operatorname{curl} \boldsymbol{h}(\tau)\|_{L^2} + C \|\nabla \varphi(\tau)\|_{L^2}\right) \mathrm{d}\tau$$
$$\leq C \left(\|\operatorname{curl} \boldsymbol{h}\|_{L^\infty(0,T;L^2(\Omega))} + C \|\nabla \varphi\|_{L^\infty(0,T;L^2(\Omega))}\right) \tag{4.53}$$

since $\int_0^t |G(t - \tau)| \,\mathrm{d}\tau \leq C$ for all $t \in [0, T]$. We infer by (4.51) and (4.53) that

$$\|\boldsymbol{u}\|_{L^\infty(0,T;H^1(\Omega))} \leq C(\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla \times \boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))})$$

$$\leq C. \tag{4.54}$$

As far as the pressure is concerned, arguing exactly as in Mikelić (1994) (see also Peter and Woukeng 2024) we arrive at $p \in \mathcal{C}([0,T]; H^1(\Omega) \cap L^2_0(\Omega))$. Also, the fact that $F'(\varphi) \in L^\infty(0,T; H^1(\Omega))$ stems from the same property for μ and φ, recalling that $F'(\varphi) = \mu - a\varphi - J * \varphi$.

It remains to check higher-order regularity in space.

4.2.3 Higher-Order Regularity in Space

First and foremost, the fact that $\partial\varphi/\partial t \in L^2(0,T; L^2(\Omega))$, $\mu \in L^\infty(0,T; H^1(\Omega))$ and $\boldsymbol{u}\varphi \in L^\infty(0,T; L^2(\Omega)^2)$ entail that $(4.5)_3$ is satisfied a.e. in Q. As a by-product, we rewrite it as a Neumann problem for μ: for a.e. $t \in (0,T)$,

$$-\operatorname{div}(m\nabla\mu) = -\left(\frac{\partial\varphi}{\partial t} + \operatorname{div}(\boldsymbol{u}\varphi)\right) \text{ in } \Omega, \quad \frac{\partial\mu}{\partial\boldsymbol{n}} = 0 \text{ on } \partial\Omega. \tag{4.55}$$

As the matrix m has constant entries, a classical elliptic regularity result entails that $\mu \in L^2(0,T; H^2(\Omega))$.

Next, observe that assuming Ω to be smooth, we apply the Gagliardo–Nirenberg inequality in dimension 2 to obtain that, for any $1 \leq r < \infty$, there is $C = C(r) > 0$ such that

$$\|\boldsymbol{u}\|_{L^r(\Omega)^2} \leq C \|\boldsymbol{u}\|_{L^1(\Omega)^2}^{\frac{1}{r}} \|\boldsymbol{u}\|_{H^1(\Omega)^2}^{1-\frac{1}{r}} \text{; see [7, Comments on Chapter 9].}$$

This yields readily $\boldsymbol{u} \in L^\infty(0,T; L^r(\Omega)^2)$, where we have taken into account (4.54). This entails that $\boldsymbol{u}\varphi \in L^\infty(0,T; L^r(\Omega)^2)$, i.e.

$$\|\boldsymbol{u}\varphi\|_{L^\infty(0,T;L^r(\Omega)^2)} \leq C \ \ \forall 1 \leq r < \infty. \tag{4.56}$$

Bearing this in mind, we proceed as in (Della Porta et al. 2018, Section 4) to derive the existence of $C > 0$ depending on r such that

$$\begin{aligned}\|\mu\|_{W^{1,r}(\Omega)} &\leq C\left(\left\|\frac{\partial\varphi}{\partial t}\right\|_{L^2(\Omega)} + \|\boldsymbol{u}\varphi\|_{L^r(\Omega)^2} + \|\mu\|_{L^2(\Omega)}\right) \\ &\leq C\left(1 + \left\|\frac{\partial\varphi}{\partial t}\right\|_{L^2(\Omega)}\right)\end{aligned}$$

for any $r > 1$. This leads to

$$\|\mu\|_{L^2(0,T;W^{1,r}(\Omega))} \leq C. \tag{4.57}$$

Let us now check the analogue of (4.57) for φ. To this end, we take the gradient of μ in $(4.5)_4$ and multiply the resulting equality by $|\nabla\varphi|^{r-2}\nabla\varphi$ (we assume here that

$r \geq 2$) and next integrate over Ω. We notice that in light of (4.57) we have

$$\left\|(a + F''(\varphi))\nabla\varphi\right\|_{L^2(0,T;L^r(\Omega))} \leq C \text{ for any } r \geq 2,$$

so that the following resulting equality makes sense:

$$\begin{aligned}\int_\Omega (a + F''(\varphi))\,|\nabla\varphi|^r \,\mathrm{d}x = \int_\Omega \nabla\mu\cdot\nabla\varphi\,|\nabla\varphi|^{r-2}\,\mathrm{d}x - \int_\Omega |\nabla\varphi|^{r-2}\,\varphi\nabla a\cdot\nabla\varphi\,\mathrm{d}x \\ - \int_\Omega (\nabla J * \varphi)\cdot\nabla\varphi\,|\nabla\varphi|^{r-2}\,\mathrm{d}x.\end{aligned}$$

Owing to Assumption (A3), we see that

$$\begin{aligned} c_0\,\|\nabla\varphi\|_{L^r}^r &\leq \|\nabla\mu\|_{L^r}\,\|\nabla\varphi\|_{L^r}^{r-1} + (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1})\,\|\varphi\|_{L^r}\,\|\nabla\varphi\|_{L^r}^{r-1} \\ &\leq \frac{c_0}{2}\,\|\nabla\varphi\|_{L^r}^r + C\,\|\nabla\mu\|_{L^r}^r + C\,\|\varphi\|_{L^r}^r\,. \end{aligned}$$

It follows promptly that

$$\|\nabla\varphi\|_{L^r} \leq C(1 + \|\nabla\mu\|_{L^r}), \tag{4.58}$$

which, by (4.57), yields

$$\|\varphi\|_{L^2(0,T;W^{1,r}(\Omega))} \leq C. \tag{4.59}$$

Now, from the $2D$-Ladyzhenskaya inequality ($\|w\|_{L^4(\Omega)} \leq C\,\|w\|_{L^2(\Omega)}^{\frac{1}{2}}\,\|w\|_{H^1(\Omega)}^{\frac{1}{2}}$ for all $w \in H^1(\Omega)$), we take advantage of (4.57), (4.58) (for $r = 4$) and (4.59) to deduce that

$$\begin{aligned} \|\nabla\mu\|_{L^4(0,T;L^4(\Omega))} &\leq C\,\|\mu\|_{L^\infty(0,T;H^1(\Omega)}^{\frac{1}{2}}\,\|\mu\|_{L^2(0,T;H^2(\Omega))}^{\frac{1}{2}} \\ &\leq C \end{aligned}$$

and

$$\|\nabla\varphi\|_{L^4(0,T;L^4(\Omega))} \leq C(1 + \|\nabla\mu\|_{L^4(0,T;L^4(\Omega))}) \leq C.$$

This shows that $\varphi, \mu \in L^4(0, T; W^{1,4}(\Omega))$.

Now, considering again the equality (4.52), we observe that, for any $r \geq 2$,

$$\|\mathrm{curl}\,\boldsymbol{u}(t)\|_{L^r(\Omega)} \leq C\left(\|\mathrm{curl}\,\boldsymbol{h}\|_{L^\infty(0,T;L^r(\Omega))} + \|\nabla\varphi(t)\|_{L^r(\Omega)}\right).$$

Therefore, proceeding as in (Della Porta et al. 2018, Section 4.3), we find that $\boldsymbol{u} \in L^4(0, T; W^{1,4}(\Omega)^2)$ provided that $\mathrm{curl}\,\boldsymbol{h} \in L^\infty(0, T; L^r(\Omega))$ and that $\boldsymbol{u} \in L^2(0, T; W^{1,r}(\Omega)^2)$ for any $2 \leq r < \infty$ provided that $\mathrm{curl}\,\boldsymbol{h} \in L^\infty(0, T; L^r(\Omega))$. This completes the proof of the theorem.

With these results at hand, we can also prove our final main result.

Proof of Theorem 1.4 It can be proved exactly as in the proof of (Peter and Woukeng 2024, Theorem 1.2). □

5 A Few Concrete Illustrations

The goal of this section is to present some physical situations leading to the use of sigma-convergence with the underlying algebras with mean value leading to the upscaling process in (1.1).

5.1 Equidistribution of Microstructures in Ω

We assume that the heterogeneities are uniformly distributed in Ω. This means that the distribution function of the microstructures is periodic, so that the functions $\overline{y} \mapsto \eta(\overline{y}, y_d, r)$ and $\overline{y} \mapsto m(\overline{y}, y_d, r)$ are 1 -periodic in each of their occurrences. The underlying algebra with mean value here is thus the algebra of Y-periodic continuous functions $A = \mathcal{C}_{\text{per}}(Y)$, $Y = (0, 1)^{d-1}$. The mean value of a function $u \in \mathcal{C}_{\text{per}}(Y)$ is given by

$$M(u) = \int_Y u(y)\, \mathrm{d}y.$$

The function spaces associated with A are as follows: $B_A^p(\mathbb{R}^{d-1}; L^p(I)) = L^p_{\text{per}}(Y; L^p(I))$ (the space of functions in $L^p_{\text{loc}}(\mathbb{R}^{d-1}; L^p(I))$ which are Y -periodic), $B_A^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I)) = W^{1,p}_{\text{per}}(Y; W^{1,p}(I))$ (the subspace of $W^{1,p}_{\text{loc}}(Y; W^{1,p}(I))$ made of Y-periodic functions), and

$$\begin{aligned} &B_{\#A}^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I)) = W_{\#}^{1,p}(Y; W^{1,p}(I)) \\ &:= \left\{ u \in W^{1,p}_{\text{per}}(Y; W^{1,p}(I)) : \int_Z u(y)\, \mathrm{d}y = 0 \right\} \end{aligned}$$

(where $Z = Y \times I$ with $I = (-1, 1)$), which is a Banach space when equipped with the norm

$$\|u\|_{W_{\#}^{1,p}} = \left(\int_Z |\nabla u|^p \, \mathrm{d}y \right)^{1/p}, u \in W_{\#}^{1,p}(Y; W^{1,p}(I)).$$

It is worth noting that $B_A^p(\mathbb{R}^{d-1}; L^p(I)) = \mathcal{B}_A^p(\mathbb{R}^{d-1}; L^p(I))$ since $L^p_{\text{per}}(Y; L^p(I))$ is a Banach space with the corresponding norm, and so, $B_A^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I)) = \mathcal{B}_A^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I))$.

In this case, the sigma-convergence concept is merely the well-known two-scale convergence method for thin heterogeneous domains defined in Neuss-Radu and Jäger

(2007) as follows: A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q_\varepsilon)$ weakly two-scale converges in $L^p(Q_\varepsilon)$ towards $u_0 \in L^p(Q; L^p_{\mathrm{per}}(Y; L^p(I)))$ if, when $\varepsilon \to 0$,

$$\frac{1}{\varepsilon}\int_{Q_\varepsilon} u_\varepsilon(t,x) f\left(t,\overline{x},\frac{x}{\varepsilon}\right)\,\mathrm{d}x\,\mathrm{d}t \to \int_Q\int_Z u_0(t,\overline{x},y) f(t,\overline{x},y)\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t$$

for any $f \in L^{p'}(Q; \mathcal{C}_{\mathrm{per}}(Y; L^{p'}(I)))$ $(1/p' = 1 - 1/p)$.

For the benefit of the reader, we restate the homogenization result in Theorem 1.1 in the periodic setting.

Theorem 5.1 *For any $\varepsilon > 0$, let $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, p_\varepsilon)$ be a weak solution of (1.1) in the sense of Definition 2.1. Then, up to a subsequence of ε not relabelled, there exist functions $\boldsymbol{u} \in L^2(0,T;\mathbb{H})$, $\varphi \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, $\mu \in L^2(0,T;H^1(\Omega))$, $p \in L^2(0,T;L^2_0(\Omega))$ and $H(\varphi,\boldsymbol{u}) \in L^1(0,T;L^1(\Omega)^{d-1})$ such that, when $\varepsilon \to 0$,*

$M_\varepsilon \boldsymbol{u}_\varepsilon \to (\boldsymbol{u}, 0)$ *in* $L^2(Q)^d$*-weak,*
$M_\varepsilon \varphi_\varepsilon \to \varphi$ *in* $L^2(Q)$*-strong and in* $L^2(0,T;H^1(\Omega))$*-weak,*
$M_\varepsilon \mu_\varepsilon \to \mu$ *in* $L^2(0,T;H^1(\Omega))$*-weak and* $M_\varepsilon p_\varepsilon \to p$ *in* $L^2(Q)$*-weak,*

where the quintuple $(\boldsymbol{u}, \varphi, \mu, p, H(\varphi,\boldsymbol{u}))$ solves the effective system

$$\begin{cases} \boldsymbol{u} + H(\varphi,\boldsymbol{u}) = G\boldsymbol{u}^0 + G * (\boldsymbol{h}_1 + \mu\nabla_{\overline{x}}\varphi - \nabla_{\overline{x}} p) \text{ in } Q, \\ \mathrm{div}_{\overline{x}}\boldsymbol{u} = 0 \text{ in } Q \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } (0,T)\times\partial\Omega, \\ \dfrac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla_{\overline{x}}\varphi - \mathrm{div}_{\overline{x}}(\widehat{m}(\varphi)\nabla\mu) = 0 \text{ in } Q, \\ \mu = \widehat{a}\varphi - \widehat{J} * \varphi + F'(\varphi) \text{ in } Q, \\ \dfrac{\partial\mu}{\partial\boldsymbol{n}} = 0 \text{ on } (0,T)\times\partial\Omega, \\ \varphi(0) = \varphi^0 \text{ in } \Omega, \end{cases}$$

where $G = (G_{ij})_{1\le i,j\le d-1}$ is a symmetric positive definite $(d-1)\times(d-1)$ matrix defined by its entries $G_{ij}(t,\overline{x}) = \frac{1}{2}\int_Z \omega^i(\overline{x},t,y)e_j\,\mathrm{d}y$, which are bounded a.e. in space and continuous in time. Here, $\omega^j = (\omega^j_i)_{1\le i\le d}$ is the unique solution in $\mathcal{C}([0,T]; L^2(\Omega; L^2_{\mathrm{per}}(Y; L^2(I))^d)) \cap L^2(Q; W^{1,2}_{\mathrm{per}}(Y; H^1_0(I))^d)$ of the auxiliary Stokes system

$$\begin{cases} \dfrac{\partial\omega^j}{\partial t} - \mathrm{div}_y(\eta(\cdot,\varphi)\nabla_y\omega^j) + \nabla_y\pi^j = 0 \text{ in } (0,T)\times Z, \\ \mathrm{div}_y\omega^j = 0 \text{ in } (0,T)\times Z, \\ \omega^j(0) = e_j \text{ in } Z \text{ and } \int_Z \omega^j_3(\overline{x},t,y)\,\mathrm{d}y = 0, \end{cases}$$

and e_j is the jth vector of the canonical basis in $\mathbb{R}^d$. Furthermore, if the function η is φ-independent, that is, $\eta(y,r)=\eta(y)$, then the function $H(\varphi,\boldsymbol{u})$ vanishes and $\boldsymbol{u}\in\mathcal{C}([0,T];\mathbb{H})$ and $p\in L^2(0,T;H^1(\Omega)\cap L_0^2(\Omega))$.

Proof The result is obtained just by identifying the mean value of a periodic function: for $u\in L^2_{\mathrm{per}}(Y)$, $M(u)=\int_Y u(y)\,\mathrm{d}y$. □

5.2 Almost Periodic Distribution of the Microstructures

Assuming that the heterogeneities are distributed in Ω in an almost periodic way, the functions $\overline{y}\mapsto\eta(\overline{y},y_d,r)$ and $\overline{y}\mapsto m(\overline{y},y_d,r)$ are almost periodic in the Besicovitch sense (Besicovitch 1954; Bohr 1947). The corresponding algebra with mean value in $\mathbb{R}^{d-1}$ is the algebra of Bohr continuous almost periodic functions on $\mathbb{R}^{d-1}$ denoted by $A=\mathrm{AP}(\mathbb{R}^{d-1})$. We recall that $\mathrm{AP}(\mathbb{R}^{d-1})$ (Besicovitch 1954; Bohr 1947) is defined as the algebra of functions on $\mathbb{R}^{d-1}$ which are uniformly approximated by finite linear combinations of functions in the set $\{\cos(k\cdot),\sin(k\cdot):k\in\mathbb{R}^{d-1}\}$ where $\cos(k\cdot)(y)=\cos(2\pi k\cdot y)$ and $\sin(k\cdot)(y)=\sin(2\pi k\cdot y)$, $y\in\mathbb{R}^{d-1}$. It is known that $\mathrm{AP}(\mathbb{R}^{d-1})$ is an algebra wmv called the almost periodic algebra wmv on $\mathbb{R}^{d-1}$. The corresponding generalized Besicovitch space $B_A^p(\mathbb{R}^{d-1})$ is precisely the Besicovitch space $\mathcal{B}^p(\mathbb{R}^{d-1})$ defined in Besicovitch (1954), Bohr (1947).

Under the assumption of an almost periodic distribution of microstructures, the main results of our work are valid with the corresponding function spaces. It is well known from Bohr (1947) that the mean value of a function $u\in\mathrm{AP}(\mathbb{R}^{d-1})$ is the unique constant that belongs to the closed convex hull of the set of translates $\{u(\cdot+a):a\in\mathbb{R}^{d-1}\}$ of u. In any case, it satisfies property (A.1).

5.3 The Asymptotic Periodic/Almost Periodic Setting

Assuming that the distribution of the microstructures inside Ω is almost uniform but with a defect leads to the use of the algebra wmv $A=\mathcal{C}_{\mathrm{per}}(Y)+\mathcal{C}_0(\mathbb{R}^{d-1})$ (Jäger and Woukeng 2021, Section 5.2.3), where $\mathcal{C}_0(\mathbb{R}^{d-1})$ stands for the Banach algebra of continuous functions which vanish at infinity. In this case, hypothesis (A6) holds with $A=\mathcal{C}_{\mathrm{per}}(Y)+\mathcal{C}_0(\mathbb{R}^{d-1})$. We may also deal with the asymptotic almost periodic distribution of heterogeneities with the corresponding algebra wmv $A=\mathrm{AP}(\mathbb{R}^{d-1})+\mathcal{C}_0(\mathbb{R}^{d-1})$ (Jäger and Woukeng 2021, Section 5.2.3).

Appendix A. Sigma-Convergence for Thin Heterogeneous Domains

This section summarizes some well-known results on sigma-convergence for thin heterogeneous domains, which are required for the homogenization process of Section 3. We start with some preliminaries about the algebras with mean value to introduce the generalized Besicovitch spaces.

A.1. Besicovitch Spaces

Let A be an algebra with mean value on $\mathbb{R}^n$ (Zhikov and Krivenko 1983) (integer $n \geq 1$), that is, a closed subalgebra of the $\mathcal{C}^*$-algebra of bounded uniformly continuous real-valued functions on $\mathbb{R}^n$, $\mathrm{BUC}(\mathbb{R}^n)$, which contains the constants, is translation invariant and is such that any of its elements possesses a mean value in the following sense: for every $u \in A$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ ($u^\varepsilon(x) = u(x/\varepsilon)$) weakly$*$-converges in $L^\infty(\mathbb{R}^d)$ to some real number $M(u)$ (called the mean value of u) as $\varepsilon \to 0$. The mean value expresses as

$$M(u) = \lim_{R\to\infty} \fint_{B_R} u(y)\,\mathrm{d}y \text{ for } u \in \mathcal{A}, \tag{A.1}$$

where we have set $\fint_{B_R} = \frac{1}{|B_R|}\int_{B_R}$.

Let A be an algebra wmv on $\mathbb{R}^n$. For any integer $\ell \geq 0$, let $|||u|||_\ell = \sup_{|\alpha|\leq \ell} \left\| D_y^\alpha \psi \right\|_\infty$, where $D_y^\alpha \psi = \frac{\partial^{|\alpha|}\psi}{\partial y_1^{\alpha_1}\ldots\partial y_n^{\alpha_n}}$. Then, the space $A^\infty := \{\psi \in \mathcal{C}^\infty(\mathbb{R}^n) : D_y^\alpha \psi \in A \;\forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ is a Fréchet space under the family of norms $|||\cdot|||_\ell$.

The concept of vector-valued algebra wmv will also be useful in this work. We define it as follows. For $\mathbf{F}$ a Banach space, $\mathrm{BUC}(\mathbb{R}^n; \mathbf{F})$ stands for the Banach space of bounded uniformly continuous functions $u\colon \mathbb{R}^n \to \mathbf{F}$ equipped with the norm

$$\|u\|_\infty = \sup_{y\in\mathbb{R}^n} \|u(y)\|_{\mathbf{F}},$$

where $\|\cdot\|_{\mathbf{F}}$ denotes the norm in $\mathbf{F}$ (cf. Section 1.4). This being so, if A is an algebra with mean value on $\mathbb{R}^n$, we denote by $A \otimes \mathbf{F}$ the usual space of functions of the form

$$\sum_{\text{finite}} u_i \otimes e_i \text{ with } u_i \in A \text{ and } e_i \in \mathbf{F},$$

where $(u_i \otimes e_i)(y) = u_i(y)e_i$ for $y \in \mathbb{R}^n$. Then, we define the vector-valued algebra wmv $A(\mathbb{R}^n; \mathbf{F})$ as the closure of $A \otimes \mathbf{F}$ in $\mathrm{BUC}(\mathbb{R}^n; \mathbf{F})$.

For $1 \leq p < \infty$, we define the Marcinkiewicz space, $\mathfrak{M}^p(\mathbb{R}^n; \mathbf{F})$ as the vector space of functions $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n; \mathbf{F})$ satisfying

$$\|u\|_{p,\mathbf{F}} = \left(\limsup_{R\to\infty} \fint_{B_R} \|u(y)\|_{\mathbf{F}}^p \,\mathrm{d}y\right)^{\frac{1}{p}} < \infty,$$

Then, $\mathfrak{M}^p(\mathbb{R}^n; \mathbf{F})$ is a complete seminormed space when equipped with the seminorm

$$\|u\|_{p,\mathbf{F}} = \left(\limsup_{R\to\infty} \fint_{B_R} \|u(y)\|_{\mathbf{F}}^p \,\mathrm{d}y\right)^{\frac{1}{p}} < \infty.$$

It is a fact that $A(\mathbb{R}^n; \mathbf{F}) \subset \mathfrak{M}^p(\mathbb{R}^n; \mathbf{F})$ since $\|u\|_{p,\mathbf{F}} < \infty$ for any $u \in A(\mathbb{R}^n; \mathbf{F})$. Thus, we define the *generalized Besicovitch space* $B_A^p(\mathbb{R}^n; \mathbf{F})$ to be the closure of $A(\mathbb{R}^n; \mathbf{F})$ in $\mathfrak{M}^p(\mathbb{R}^n; \mathbf{F})$, and it holds that:

(i) $\mathcal{B}_A^p(\mathbb{R}^n; \mathbf{F}) = B_A^p(\mathbb{R}^n; \mathbf{F})/\mathcal{N}$ (where $\mathcal{N} = \{u \in B_A^p(\mathbb{R}^n; \mathbf{F}) : \|u\|_{p,\mathbf{F}} = 0\}$) is a Banach space under the norm $\|u + \mathcal{N}\|_{p,\mathbf{F}} = \|u\|_{p,\mathbf{F}}$ for $u \in B_A^p(\mathbb{R}^n; \mathbf{F})$.

(ii) The mean value $M\colon A(\mathbb{R}^n; \mathbf{F}) \to \mathbf{F}$ extends by continuity to a continuous linear mapping (still denoted by M) on $B_A^p(\mathbb{R}^n; \mathbf{F})$ satisfying

$$L(M(u)) = M(L(u)) \text{ for all } L \in \mathbf{F}' \text{ and } u \in B_A^p(\mathbb{R}^n; \mathbf{F})$$

and we have, for $u \in B_A^p(\mathbb{R}^n; \mathbf{F})$,

$$\|u\|_{p,\mathbf{F}} = \left(M(\|u\|_{\mathbf{F}}^p)\right)^{1/p} \equiv \left(\lim_{R\to\infty} \fint_{B_R} \|u(y)\|_{\mathbf{F}}^p \,\mathrm{d}y\right)^{\frac{1}{p}}.$$

Note that, when $\mathbf{F} = H$ is a Hilbert space, $\mathcal{B}_A^2(\mathbb{R}^n; H)$ is also a Hilbert space with inner product

$$(u, v)_2 = M\left[(u, v)_H\right] \text{ for } u, v \in \mathcal{B}_A^2(\mathbb{R}^n; H), \tag{A.2}$$

where $(\cdot, \cdot)_H$ stands for the inner product in H and $(u, v)_H$ the function $y \mapsto (u(y), v(y))_H$ from $\mathbb{R}^n$ to $\mathbb{R}$, which belongs to $\mathcal{B}_A^1(\mathbb{R}^n)$.

We shall also need Sobolev–Besicovitch-type spaces defined as

$$B_A^{1,p}(\mathbb{R}^n; \mathbf{F}) = \{u \in B_A^p(\mathbb{R}^n; \mathbf{F}) : \nabla_y u \in (B_A^p(\mathbb{R}^n; \mathbf{F}))^n\},$$

and endowed with the seminorm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \left\|\nabla_y u\right\|_p^p\right)^{\frac{1}{p}},$$

which makes it a complete seminormed space. Its Banach counterpart is denoted by $\mathcal{B}_A^{1,p}(\mathbb{R}^n; \mathbf{F})$ and is defined by replacing $B_A^p(\mathbb{R}^n; \mathbf{F})$ by $\mathcal{B}_A^p(\mathbb{R}^n; \mathbf{F})$ and $\partial/\partial y_i$ by $\overline{\partial}/\partial y_i$, where $\overline{\partial}/\partial y_i$ is defined by

$$\frac{\overline{\partial}}{\partial y_i}(u + \mathcal{N}) := \frac{\partial u}{\partial y_i} + \mathcal{N} \text{ for } u \in B_A^{1,p}(\mathbb{R}^n; \mathbf{F}). \tag{A.3}$$

Denoting by $\varrho\colon B_A^p(\mathbb{R}^n; \mathbf{F}) \to \mathcal{B}_A^p(\mathbb{R}^n; \mathbf{F}) = B_A^p(\mathbb{R}^n; \mathbf{F})/\mathcal{N}$, $\varrho(u) = u + \mathcal{N}$, the canonical surjection, we see that

$$\frac{\overline{\partial}\varrho(u)}{\partial y_i} = \varrho\left(\frac{\partial u}{\partial y_i}\right) \text{ for } u \in B_A^{1,p}(\mathbb{R}^n; \mathbf{F})$$

as seen above in (A.3).

We define a further notion by restricting ourselves to the case $\mathbf{F} = \mathbb{R}$. We say that the algebra A is ergodic if any $u \in \mathcal{B}_A^1(\mathbb{R}^n; \mathbb{R})$ which is invariant under $(\mathcal{T}(y))_{y\in\mathbb{R}^n}$ is a constant in $\mathcal{B}_A^1(\mathbb{R}^n; \mathbb{R})$. This amounts to the following: if $\mathcal{T}(y)u = u$ in $\mathcal{B}_A^1(\mathbb{R}^n; \mathbb{R})$ for every $y \in \mathbb{R}^n$, then $u = c$ in $\mathcal{B}_A^1(\mathbb{R}^n; \mathbb{R})$ in the sense that $\|u - c\|_1 = 0$, c being a constant.

Now, we assume that $n = d-1$ (integer $d \geq 2$) and we set $I = (-1, 1)$. Any $y \in \mathbb{R}^d$ is written as $y = (\overline{y}, y_d)$. We define the *corrector function* space $B_{\#A}^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I))$ by

$$B_{\#A}^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I)) = \{u \in W_{\text{loc}}^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I)) : \nabla u \in B_A^p(\mathbb{R}^{d-1}; L^p(I))^d \text{ and } \textstyle\int_I M(\nabla_{\overline{y}} u(\cdot, y_d))\, \mathrm{d}y_d = 0\},$$

where, in this case, $\nabla = (\nabla_{\overline{y}}, \frac{\partial}{\partial y_d})$, $\nabla_{\overline{y}}$ being the gradient operator with respect to the variable $\overline{y} \in \mathbb{R}^{d-1}$. We identify two elements of $B_{\#A}^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I))$ by their gradients in the sense that $u = v$ in $B_{\#A}^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I))$ iff $\nabla(u - v) = 0$, i.e. $\int_I \|\nabla(u(\cdot, y_d) - v(\cdot, y_d))\|_p^p \, \mathrm{d}y_d = 0$. The space $B_{\#A}^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I))$ is a Banach space under the norm $\|u\|_{\#,p} = \left(\int_I \|\nabla u(\cdot, y_d)\|_p^p \, \mathrm{d}y_d\right)^{1/p}$.

A.2. Sigma-Convergence for Thin Heterogeneous Domains

We are now in a position to the define the sigma-convergence concept for thin heterogeneous domains. The integer $d \geq 2$ is as above and $\Omega \subset \mathbb{R}^{d-1}$ is the open bounded domain given in Section 1 of this work. We also recall the definition of our thin domain Ω_ε (for a given small $\varepsilon > 0$): $\Omega_\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$ and we set $Q_\varepsilon = (0, T) \times \Omega_\varepsilon$. When $\varepsilon \to 0$, Ω_ε shrinks to the interface $\Omega_0 = \Omega \times \{0\} \equiv \Omega$. We also set and $Q = (0, T) \times \Omega_0 \equiv (0, T) \times \Omega$ as well as $I = (-1, 1)$.

The space $\mathbb{R}_\xi^m$ is the numerical space $\mathbb{R}^m$ of generic variable ξ. In this regard, we set $\mathbb{R}^{d-1} = \mathbb{R}_{\overline{x}}^{d-1}$ or $\mathbb{R}_{\overline{y}}^{d-1}$, where $\overline{x} = (x_1, \ldots, x_{d-1})$, so that $x \in \mathbb{R}^d$ can be written as $(\overline{x}, x_d)$ or $(\overline{x}, \zeta)$. We identify Ω_0 with Ω so that the generic element in Ω_0 is also denoted by $\overline{x}$ instead of $(\overline{x}, 0)$.

Let A be an ergodic algebra with mean value on $\mathbb{R}^{d-1}$. We denote by M the mean value on A as well as its extension on the underlying generalized Besicovitch spaces $B_A^p(\mathbb{R}^{d-1}; L^p(I))$ and $\mathcal{B}_A^p(\mathbb{R}^{d-1}; L^p(I))$, $1 \leq p < \infty$.

Definition A.1 A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q_\varepsilon)$ is

(i) weakly Σ-convergent in $L^p(Q_\varepsilon)$ towards $u_0 \in L^p(Q; \mathcal{B}_A^p(\mathbb{R}^{d-1}; L^p(I)))$ if, as $\varepsilon \to 0$, we have

$$\frac{1}{\varepsilon} \int_{Q_\varepsilon} u_\varepsilon(t, x) f\left(t, \overline{x}, \frac{x}{\varepsilon}\right) \mathrm{d}x\, \mathrm{d}t \to \int_Q \int_I M(u_0(t, \overline{x}, \cdot, y_d) f(t, \overline{x}, \cdot, y_d))\, \mathrm{d}y_d\, \mathrm{d}\overline{x}\, \mathrm{d}t$$

for any $f \in L^{p'}(Q; A(\mathbb{R}^{d-1}; L^{p'}(I)))$ $(1/p' = 1 - 1/p)$; we denote this by "$u_\varepsilon \to u_0$ in $L^p(Q_\varepsilon)$-weak Σ_A";

(ii) strongly Σ-convergent in $L^p(Q_\varepsilon)$ towards $u_0 \in L^p(Q; \mathcal{B}_A^p(\mathbb{R}^{d-1}; L^p(I)))$ if it is weakly sigma-convergent and, additionally,

$$\varepsilon^{-\frac{1}{p}} \|u_\varepsilon\|_{L^p(Q_\varepsilon)} \to \|u_0\|_{L^p(Q;\mathcal{B}_A^p(\mathbb{R}^{d-1};L^p(I)))}; \tag{A.4}$$

we denote this by "$u_\varepsilon \to u_0$ in $L^p(Q_\varepsilon)$-strong Σ_A ".

Remark A.1 (1) If $u_0 \in L^p(Q; A(\mathbb{R}^{d-1}; L^p(I)))$ then (A.4) amounts to

$$\varepsilon^{-\frac{1}{p}} \left\|u_\varepsilon - u_0^\varepsilon\right\|_{L^p(Q_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0, \tag{A.5}$$

where $u_0^\varepsilon(t, x) = u_0(t, \overline{x}, x/\varepsilon)$ for $(t, x) \in Q_\varepsilon$.

(2) In Definition A.1, the test functions in part (i) may also be taken in the space $\mathcal{C}(\overline{Q}; B_A^{'}(\mathbb{R}^{d-1}; L^{p'}(I)) \cap L^\infty(\mathbb{R}^{d-1} \times I))$, see, for example, Woukeng (2015).

In what follows, the letter E denotes any ordinary sequence $(\varepsilon_n)_{n\geq 1}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ when $n \to \infty$. We shall merely denote by ε the generic term of E so that "$\varepsilon \to 0$" shall mean "$\varepsilon_n \to 0$ as $n \to \infty$."

Theorem A.1 *For $1 < p < \infty$, let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(Q_\varepsilon)$ satisfying*

$$\sup_{\varepsilon\in E} \varepsilon^{-1/p} \|u_\varepsilon\|_{L^p(Q_\varepsilon)} \leq C,$$

where $C > 0$ is independent of ε. Then, there exists a subsequence of $(u_\varepsilon)_{\varepsilon\in E}$ which is weakly Σ_A -convergent in $L^p(Q_\varepsilon)$.

The proof of the above theorem is very similar to its homologue stated in Jäger and Woukeng (2022). We also provide some further important results and we refer to Cardone et al. (2024) (see also Jäger and Woukeng 2022) for their proofs.

Theorem A.2 *Let A be an ergodic algebra with mean value on $\mathbb{R}^{d-1}$ and let $1 < p < \infty$. If $(u_\varepsilon)_{\varepsilon\in E}$ is a sequence in $L^p(0, T; W^{1,p}(\Omega_\varepsilon))$ such that*

$$\sup_{\varepsilon\in E} \left(\varepsilon^{-1/p} \|u_\varepsilon\|_{L^p(0,T;W^{1,p}(\Omega_\varepsilon))}\right) \leq C, \tag{A.6}$$

where $C > 0$ is independent of ε, then there exist a subsequence E' of E and a couple (u, u_1) with $u \in L^p(0, T; W^{1,p}(\Omega_0))$ and $u_1 \in L^p(Q; B_{\#A}^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u \textit{ in } L^p(Q_\varepsilon)\textit{-weak } \Sigma_A, \tag{A.7}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \textit{ in } L^p(Q_\varepsilon)\textit{-weak } \Sigma_A \textit{ for } 1 \leq i \leq d-1 \tag{A.8}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_d} \to \frac{\partial u_1}{\partial y_d} \textit{ in } L^p(Q_\varepsilon)\textit{-weak } \Sigma_A. \tag{A.9}$$

Remark A.2 Set

$$\nabla_{\overline{x}}u_0 = \left(\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_{d-1}}, 0\right).$$

Then, (A.8)–(A.9) amount to

$$\nabla u_\varepsilon \to \nabla_{\overline{x}}u_0 + \nabla_y u_1 \text{ in } L^p(Q_\varepsilon)^d\text{-weak } \Sigma_A. \tag{A.10}$$

The following result provides us with sufficient conditions for which the convergence result in (A.7) is strong.

Theorem A.3 *Let the assumptions of Theorem A.2 be satisfied and, moreover, suppose that*

$$\sup_{\varepsilon>0}\left\|\frac{\partial M_\varepsilon u_\varepsilon}{\partial t}\right\|_{L^{p'}(0,T;(W^{1,p}(\Omega))')} \leq C, \tag{A.11}$$

where M_ε is defined by (1.9). Finally assume that Ω is regular enough so that the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. Let (u_0, u_1) and E' be as in Theorem A.2. Then, as $E' \ni \varepsilon \to 0$, the conclusions of Theorem A.2 hold, and in addition, we have

$$u_\varepsilon \to u_0 \textit{ in } L^p(Q_\varepsilon) \textit{ -strong } \Sigma_A. \tag{A.12}$$

The next theorem with its corollary deal with the product of sequences and their proofs are obtained by proceeding as in (Sango and Woukeng 2011, Theorem 6 and Corollary 5) (see also Woukeng 2015).

Theorem A.4 *For $1 < p, q < \infty$, let $r \geq 1$ be such that $1/r = 1/p + 1/q \leq 1$. Suppose that $(u_\varepsilon)_{\varepsilon\in E}$ is a weakly Σ_A-convergent sequence in $L^q(Q_\varepsilon)$ with limit $u_0 \in L^q(Q; \mathcal{B}_A^q(\mathbb{R}^{d-1}; L^q(I)))$ and $(v_\varepsilon)_{\varepsilon\in E}$ is a strongly Σ_A -convergent sequence in $L^p(Q_\varepsilon)$ with limit $v_0 \in L^p(Q; \mathcal{B}_A^p(\mathbb{R}^{d-1}; L^p(I)))$. Then, the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon\in E}$ is weakly Σ_A-convergent in $L^r(Q_\varepsilon)$ towards $u_0 v_0$.*

Corollary A.1 *If $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(Q_\varepsilon)$ and $(v_\varepsilon)_{\varepsilon\in E} \subset L^{p'}(Q_\varepsilon) \cap L^\infty(Q_\varepsilon)$ ($1 < p < \infty$ and $p' = p/(p-1)$) are two sequences such that*

(i) $u_\varepsilon \to u_0$ *in $L^p(Q_\varepsilon)$ -weak Σ_A,*
(ii) $v_\varepsilon \to v_0$ *in $L^{p'}(Q_\varepsilon)$-strong Σ_A,*
(iii) $(v_\varepsilon)_{\varepsilon\in E}$ *is bounded in $L^\infty(Q_\varepsilon)$,*

then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^p(Q_\varepsilon)$-weak Σ_A.

The following proposition is used in finding the limit of the velocity in the homogenization process of (1.1).

Proposition A.1 (Peter and Woukeng 2024, Proposition 3.1) *Let* $(u_\varepsilon)_{\varepsilon\in E}$ *be a sequence in* $L^p(0,T;W^{1,p}(\Omega_\varepsilon))$ *such that*

$$\sup_{\varepsilon\in E}\left(\varepsilon^{-1/p}\left\|u_\varepsilon\right\|_{L^p(Q_\varepsilon)}+\varepsilon^{1-1/p}\left\|\nabla u_\varepsilon\right\|_{L^p(Q_\varepsilon)}\right)\leq C,$$

where $C>0$ *is independent of* ε. *Then, there are a subsequence* E' *of* E *and a function* $u_0\in L^p(Q;\mathcal{B}_A^{1,p}(\mathbb{R}^{d-1};W^{1,p}(I)))$ *such that, as* $E'\ni\varepsilon\to 0$,

$$u_\varepsilon\to u_0 \text{ in } L^p(Q_\varepsilon)\text{-weak } \Sigma_A$$

and

$$\varepsilon\nabla u_\varepsilon\to\overline{\nabla}_y u_0 \text{ in } L^p(Q_\varepsilon)^d\text{-weak } \Sigma_A.$$

Appendix B. Uniqueness of the Solution of (1.1)

Although not relevant for the homogenization result, we can also prove a uniqueness results under additional assumptions. Namely, if the functions $\eta(y,r)$ and $m(y,r)$ are independent of r and, further, if $\boldsymbol{h}_1\in L^\infty(0,T;L^4(\Omega)^{d-1})$ and $\boldsymbol{u}_0^\varepsilon\in L^4(\Omega)^d$, then the microscopic solution $(\boldsymbol{u}_\varepsilon,\varphi_\varepsilon)$ is unique.

Therefore, we assume that the functions η and m are independent of φ_ε, that is, $\eta(y,r)=\eta(y)$ and $m(y,r)=m(y)$ for all $(y,r)\in\mathbb{R}^d\times\mathbb{R}$ in this appendix. This being so, we omit for a while the subscript ε and we replace Ω_ε by Ω_1. For $f\in H^1(\Omega_1)'$, we define its average $\overline{f}$ over Ω_1 by $\overline{f}=|\Omega_1|^{-1}\langle f,1\rangle$. With this in mind, we define the set

$$V_0=\{v\in H^1(\Omega_1):\overline{v}=0\},\text{ so that } V_0'=\{f\in H^1(\Omega_1)':\overline{f}=0\}. \tag{B.1}$$

We consider the operator $B\colon H^1(\Omega_1)\to H^1(\Omega_1)'$ defined by

$$\langle Bu,v\rangle=\int_{\Omega_1}m(x)\nabla u\cdot\nabla v\,\mathrm{d}x \text{ for all } u,v\in H^1(\Omega_1).$$

Then, it is a fact that B maps continuously $H^1(\Omega_1)$ into V_0', and furthermore, the restriction B_0 of B to V_0 is an isomorphism from V_0 onto V_0'. Denoting by B_0^{-1} the inverse of B_0, we have

$$BB_0^{-1}f=f\ \ \forall f\in V_0',\text{ and } B_0^{-1}Bu=u\ \ \forall u\in V_0.$$

One may see that, for $f\in V_0'$, $u=B_0^{-1}f$ is the unique solution with zero mean value of the Neumann problem

$$-\operatorname{div}(m\nabla u)=f \text{ in } \Omega_1,\ m\nabla u\cdot\nu=0 \text{ on } \partial\Omega_1, \tag{B.2}$$

ν being the unit outward normal to $\partial\Omega_1$. In addition, it holds that

$$\left\langle Bu, B_0^{-1} f\right\rangle = \langle f, u\rangle \text{ for all } u \in H^1(\Omega_1) \text{ and all } f \in V_0',$$
$$\left\langle f, B_0^{-1} g\right\rangle = \left\langle g, B_0^{-1} f\right\rangle = \int_{\Omega_1} m\nabla(B_0^{-1} f)\cdot\nabla(B_0^{-1} g)\,\mathrm{d}x \quad \forall f, g \in V_0'. \tag{B.3}$$

With the above definitions and properties, one may endow V_0' with the norm

$$\|f\|_{V_0'} = \left\|\nabla(B_0^{-1} f)\right\|_{L^2(\Omega_1)} \text{ for } f \in V_0'.$$

As a by-product, we have

$$\left\langle \frac{\partial f}{\partial t}, B_0^{-1} f\right\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{V_0'}^2 \text{ for a.e. } t \in (0, T) \text{ and for all } f \in H^1(0, T; V_0'). \tag{B.4}$$

With this in mind, we define the following norm on $H^1(\Omega_1)'$:

$$\|f\|_{\#} = \left(\left\|f - \overline{f}\right\|_{V_0'}^2 + \left|\overline{f}\right|^2\right)^{\frac{1}{2}}, \quad f \in H^1(\Omega_1)'.$$

Then, $\|\cdot\|_{\#}$ is equivalent to the usual norm of $H^1(\Omega_1)'$. The classical regularity theory for elliptic partial differential equations with Neumann boundary conditions entails that the solution of (B.2) satisfies the following estimates:

$$\left\|\nabla B_0^{-1} f\right\|_{H^k(\Omega_1)} \le C\,\|f\|_{H^{k-1}(\Omega_1)} \text{ for all } f \in H^{k-1}(\Omega_1)\cap L_0^2(\Omega_1) \text{ and } k = 1, 2. \tag{B.5}$$

We also recall the following well-known interpolation inequality:

$$\|v\|_{L^4(\Omega_1)} \le C\,\|v\|_{L^2(\Omega_1)}^{1-\frac{d}{4}}\,\|v\|_{H^1(\Omega_1)}^{\frac{d}{4}} \text{ for all } v \in H^1(\Omega_1), d = 2, 3, \tag{B.6}$$

where $C = C(d, \Omega_1) > 0$.

With this in mind, we take advantage of the identity

$$\mu\nabla\varphi = \nabla(F(\varphi) + a\frac{\varphi^2}{2}) - \frac{\varphi^2}{2}\nabla a - (J * \varphi)\nabla\varphi$$

to rewrite $(1.1)_1$ with the extra pressure $\widetilde{p} := p - F(\varphi) - a\frac{\varphi^2}{2}$ in the following form:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \operatorname{div}(\eta(x)\nabla\boldsymbol{u}) + \nabla\widetilde{p} = \boldsymbol{h} - \nabla a\frac{\varphi^2}{2} - (J * \varphi)\nabla\varphi. \tag{B.7}$$

Set $\boldsymbol{H} = \boldsymbol{h} - \nabla a \frac{\varphi^2}{2} - (J * \varphi)\nabla\varphi$. Then, from the equality $((J * \varphi)\nabla\varphi, \boldsymbol{v}) = ((\nabla J * \varphi)\varphi, \boldsymbol{v})$, valid for all $\boldsymbol{v} \in \mathbb{H}$, we see that $\boldsymbol{H} = \boldsymbol{h} - \nabla a \frac{\varphi^2}{2} - (\nabla J * \varphi)\varphi$ in $\mathbb{H}$. Therefore, recalling that $\varphi \in L^\infty(Q_1)$ with $|\varphi| < 1$ a.e. in Q_1, we observe that $\nabla a \frac{\varphi^2}{2} + (\nabla J * \varphi)\varphi \in L^\infty(Q_1)^d$, so that $\boldsymbol{H} \in L^\infty(0, T; L^4(\Omega_1)^d)$. (We have assumed that $\boldsymbol{h}_1 \in L^\infty(0, T; L^4(\Omega)^{d-1})$.) This yields in particular that $\boldsymbol{H} \in L^8(0, T; L^4(\Omega_1)^d)$. This being so, we consider the following Stokes system

$$\begin{cases} \frac{\partial u}{\partial t} - \mathrm{div}(\eta(x)\nabla \boldsymbol{u}) + \nabla\pi = \boldsymbol{H} \text{ in } Q_1, \\ \mathrm{div}\boldsymbol{u} = 0 \text{ in } Q_1, \\ \boldsymbol{u} = 0 \text{ on } (0, T) \times \partial\Omega_1 \text{ and } \boldsymbol{u}(0) = \boldsymbol{u}^0 \text{ in } \Omega_1. \end{cases} \tag{B.8}$$

It is well known from the L^p–L^q estimates of the solution of (B.8) that, since $\boldsymbol{H} \in L^8(0, T; L^4(\Omega_1)^d)$, we have

$$\boldsymbol{u} \in L^8(0, T; L^4(\Omega_1)^d)), \tag{B.9}$$

provided that $\boldsymbol{u}^0 \in (W_0^{1,4}(\Omega_1))^d \cap \mathbf{L}^4_{\mathrm{div}}(\Omega_1)$, where $\mathbf{L}^4_{\mathrm{div}}(\Omega_1)$ is the closure in $L^4(\Omega_1)^d$ of the space $\mathcal{C}^\infty_{0,\sigma}(\Omega_1) = \{\boldsymbol{u} \in \mathcal{C}^\infty_0(\Omega_1)^d : \mathrm{div}\boldsymbol{u} = 0\}$.

Bearing this in mind, let $(\boldsymbol{u}_1, \varphi_1)$ and $(\boldsymbol{u}_2, \varphi_2)$ be two solutions of (1.1) corresponding to the same initial value $(\boldsymbol{u}^0, \varphi^0)$ with $\boldsymbol{u}^0 \in (W_0^{1,4}(\Omega_1))^d \cap \mathbf{L}^4_{\mathrm{div}}(\Omega_1)$ and the same source term $\boldsymbol{h}$. Set $\boldsymbol{u} = \boldsymbol{u}_2 - \boldsymbol{u}_1$, $\varphi = \varphi_2 - \varphi_1$, $\mu = \mu_2 - \mu_1$ and $p = \widetilde{p}_2 - \widetilde{p}_1$. Then, $(\boldsymbol{u}, \varphi, \mu, \widetilde{p})$ satisfies

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \mathrm{div}(\eta(x)\nabla\boldsymbol{u}) + \nabla\widetilde{p} \\ = -\varphi(\varphi_1 + \varphi_2)\frac{\nabla a}{2} - (\nabla J * \varphi)\varphi_2 - (\nabla J * \varphi_1)\varphi) \text{ in } Q_1, \\ \mathrm{div}\boldsymbol{u} = 0 \text{ in } Q_1, \\ \dfrac{\partial\varphi}{\partial t} - \mathrm{div}(m(x)\nabla\mu) = -\mathrm{div}(\boldsymbol{u}\varphi_1) - \mathrm{div}(\boldsymbol{u}_2\varphi) \text{ in } Q_1, \\ \mu = a\varphi - J * \varphi + F'(\varphi_2) - F'(\varphi_1) \text{ in } Q_1, \\ \dfrac{\partial\mu}{\partial\nu} = 0 \text{ and } \boldsymbol{u} = 0 \text{ on } (0, T) \times \partial\Omega_1, \\ \boldsymbol{u}(0) = 0 \text{ and } \varphi(0) = 0 \text{ in } \Omega_1. \end{cases} \tag{B.10}$$

The variational formulation of (B.10) reads, for a.e. $t \in (0, T)$,

$$\left\langle \frac{\partial\varphi}{\partial t}, \psi \right\rangle + (m\nabla\mu, \nabla\psi) = (\boldsymbol{u}\varphi_1, \nabla\psi) + (\boldsymbol{u}_2\varphi, \nabla\psi) \quad \text{for all } \psi \in H^1(\Omega_1), \tag{B.11}$$

$$\left\langle \frac{\partial\boldsymbol{u}}{\partial t}, v \right\rangle + (\eta\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) = -(\varphi(\varphi_1 + \varphi_2)\frac{\nabla a}{2}, \boldsymbol{v}) - ((\nabla J * \varphi)\varphi_2, \boldsymbol{v}) - ((\nabla J * \varphi_1)\varphi), \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathbb{V}_1. \tag{B.12}$$

Note that if we choose $\psi = 1$ in (B.11) then $\overline{\varphi}(t) = \overline{\varphi^0} = 0$ for all $t \in [0, T]$, so that $\varphi \in V_0$. Next, choosing $\psi = B_0^{-1}\varphi$ in (B.11) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2_{V_0'} + (\mu, \varphi) = \left(\boldsymbol{u}\varphi_1, \nabla(B_0^{-1}\varphi)\right) + \left(\boldsymbol{u}_2\varphi, \nabla(B_0^{-1}\varphi)\right),$$

that is,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{V_0'}^2+\Big(a\varphi+F'(\varphi_2)-F'(\varphi_1),\varphi\Big)\\&=\Big(\boldsymbol{u}\varphi_1,\nabla(B_0^{-1}\varphi)\Big)+\Big(\boldsymbol{u}_2\varphi,\nabla(B_0^{-1}\varphi)\Big)+(J*\varphi,\varphi).\end{aligned}\tag{B.13}$$

We infer from (A3) (see (ii) therein) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{V_0'}^2+c_0\|\varphi\|_{L^2(\Omega_1)}^2\le\sum_{k=1}^{3}I_k\tag{B.14}$$

with $I_1=\Big(\boldsymbol{u}\varphi_1,\nabla(B_0^{-1}\varphi)\Big)$, $I_2=\Big(\boldsymbol{u}_2\varphi,\nabla(B_0^{-1}\varphi)\Big)$ and $I_3=(J*\varphi,\varphi)$. Now we take $\boldsymbol{v}=\boldsymbol{u}$ in (B.12) and use (1.2) to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{L^2(\Omega_1)}^2+\eta_1\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)}^2\le\sum_{k=4}^{6}I_k,\tag{B.15}$$

where $I_4=-(\varphi(\varphi_1+\varphi_2)\frac{\nabla a}{2},\boldsymbol{u})$, $I_5=-((\nabla J*\varphi)\varphi_2,\boldsymbol{u})$ and $I_6=-((\nabla J*\varphi_1)\varphi),\boldsymbol{u})$. Addition of (B.14) and (B.15) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\varphi\|_{V_0'}^2+\|\boldsymbol{u}\|_{L^2(\Omega_1)}^2\Big)+c_0\|\varphi\|_{L^2(\Omega_1)}^2+\eta_1\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)}^2\le\sum_{k=1}^{6}I_k.\tag{B.16}$$

We control the right-hand side of (B.16) as follows:

$$\begin{aligned}|I_1|&\le\|\boldsymbol{u}\|_{L^2}\left\|\nabla(B_0^{-1}\varphi)\right\|_{L^2}\|\varphi_1\|_{L^\infty}\\&\le C\|\nabla\boldsymbol{u}\|_{L^2}\left\|\nabla(B_0^{-1}\varphi)\right\|_{L^2}\le\frac{\eta_1}{2}\|\nabla\boldsymbol{u}\|_{L^2}^2+C\|\varphi\|_{\#}^2;\end{aligned}\tag{B.17}$$

$$\begin{aligned}|I_2|&\le\|\boldsymbol{u}_2\|_{L^4}\|\varphi\|_{L^2}\left\|\nabla(B_0^{-1}\varphi)\right\|_{L^4}\\&\le C\|\boldsymbol{u}_2\|_{L^4}\|\varphi\|_{L^2}\left\|\nabla(B_0^{-1}\varphi)\right\|_{L^2}^{1-\frac{d}{4}}\left\|\nabla(B_0^{-1}\varphi)\right\|_{H^1}^{\frac{d}{4}}\\&\le C\|\boldsymbol{u}_2\|_{L^4}\|\varphi\|_{L^2}\|\varphi\|_{\#}^{1-\frac{d}{4}}\|\varphi\|_{L^2}^{\frac{d}{4}}\\&\le\begin{cases}\frac{c_0}{8}\|\varphi\|_{L^2}^2+C\|\boldsymbol{u}_2\|_{L^4}^4\|\varphi\|_{\#}^2 & \text{if } d=2,\\ \frac{c_0}{8}\|\varphi\|_{L^2}^2+C\|\boldsymbol{u}_2\|_{L^4}^8\|\varphi\|_{\#}^2 & \text{if } d=3,\end{cases}\end{aligned}\tag{B.18}$$

where, to obtain the above last inequality, we have used inequality (B.6) associated with inequality (B.5) for $k=1$;

$$|I_3|=\left|(m\nabla(J*\varphi),\nabla(B_0^{-1}\varphi))\right|\le m_2\|\nabla J\|_{L^1}\|\varphi\|_{L^2}\left\|\nabla(B_0^{-1}\varphi)\right\|_{L^2}$$

$$\begin{aligned}
&\leq \frac{c_0}{8} \|\varphi\|_{L^2}^2 + C \|\varphi\|_\#^2 ; && \text{(B.19)}\\
|I_4| &\leq \|\varphi\|_{L^2} \|\varphi_1 + \varphi_2\|_{L^\infty} \|\nabla a\|_{L^\infty} \|\boldsymbol{u}\|_{L^2} \\
&\leq \frac{c_0}{8} \|\varphi\|_{L^2}^2 + C \|\boldsymbol{u}\|_{L^2}^2 ; && \text{(B.20)}\\
|I_5| &\leq \|\nabla J\|_{L^1} \|\varphi\|_{L^2} \|\boldsymbol{u}\|_{L^2} \leq \frac{c_0}{8} \|\varphi\|_{L^2}^2 + C \|\boldsymbol{u}\|_{L^2}^2 ; && \text{(B.21)}
\end{aligned}$$

and

$$|I_6| \leq \|\nabla J\|_{L^1} \|\varphi_1\|_{L^\infty} \|\varphi\|_{L^2} \|\boldsymbol{u}\|_{L^2} \leq \frac{c_0}{8} \|\varphi\|_{L^2}^2 + C \|\boldsymbol{u}\|_{L^2}^2 . \tag{B.22}$$

Using estimates (B.17)–(B.22) in (B.16), we are led to

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|\varphi\|_\#^2 + \|\boldsymbol{u}\|_{L^2(\Omega_1)}^2 \right) + \frac{c_0}{4} \|\varphi\|_{L^2(\Omega_1)}^2 + \frac{7\eta_1}{8} \|\nabla \boldsymbol{u}\|_{L^2(\Omega_1)}^2 \\
&\leq \begin{cases} (C + \|\boldsymbol{u}_2\|_{L^4(\Omega_1)}^4) \|\varphi\|_\#^2 + C \|\boldsymbol{u}\|_{L^2(\Omega_1)}^2 & \text{if } d = 2, \\ (C + \|\boldsymbol{u}_2\|_{L^4(\Omega_1)}^8) \|\varphi\|_\#^2 + C \|\boldsymbol{u}\|_{L^2(\Omega_1)}^2 & \text{if } d = 3. \end{cases}
\end{aligned} \tag{B.23}$$

Each of the functions $t \mapsto C + \|\boldsymbol{u}_2(t)\|_{L^4(\Omega_1)}^4$ and $t \mapsto C + \|\boldsymbol{u}_2(t)\|_{L^4(\Omega_1)}^8$ is integrable over $(0, T)$, see (B.9). Thus, applying the Gronwall lemma to (B.23) yields at once $\varphi = 0$ and $\boldsymbol{u} = 0$. As a result, $\mu = 0$. This concludes the proof of the uniqueness of the microscopic solutions.

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