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Angaben zur Veröffentlichung / Publication details:

Carstensen, Carsten, and Ronald H. W. Hoppe. 2009. "Unified framework for an a posteriori error analysis of non-standard finite element approximations of H(curl)-elliptic problems." Augsburg: Universität Augsburg.

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Preprint Nr. 09/2009 — 22. April 2009

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<http://www.math.uni-augsburg.de/>

Impressum:

Herausgeber:

Institut für Mathematik

Universität Augsburg

86135 Augsburg

<http://www.math.uni-augsburg.de/forschung/preprint/>

ViSdP:

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Unified Framework for an A Posteriori Error Analysis of Non-Standard Finite Element Approximations of $H(\text{curl})$ -Elliptic Problems

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22nd April 2009

Abstract — A unified framework for a residual-based a posteriori error analysis of standard conforming finite element methods as well as non-standard techniques such as nonconforming and mixed methods has been developed in [20]–[24]. This paper provides such a framework for an a posteriori error control of nonconforming finite element discretizations of $H(\text{curl})$ -elliptic problems as they arise from low-frequency electromagnetics. These nonconforming approximations include the interior penalty discontinuous Galerkin (IPDG) approach considered in [33,34], and mortar edge element approximations studied in [10], [28]–[31], [41,48].

Keywords: a posteriori error analysis, unified framework, non-standard finite element methods, $H(\text{curl})$ -elliptic problems

Dedicated to the Sixtieth Anniversary of Rolf Rannacher

1. INTRODUCTION

The a posteriori error control and the design of adaptive mesh-refining algorithms is key to the actual scientific computing with any standard or nonstandard finite element method. The unifying theory of a posteriori error analysis [20]–[24] illustrates that *all* finite element methods allow for some a posteriori error control in energy norms for the Laplace, the Stokes, or the Lamé equations. This paper concerns the particular case of an $H(\text{curl})$ -elliptic problem

$$\text{curl } \mu^{-1} \text{curl } \mathbf{u} + \sigma \mathbf{u} = \mathbf{f}$$

in a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$ as it arises from a semi-discretization in time of the eddy current equations [35]. The idea is to rewrite the second-order PDE

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The first author has been supported by the DFG Research Center MATHEON, Project C13. The second author acknowledges support by the NSF under Grants No. DMS-0707602, DMS-0810156, and DMS-0811153

as a system of two first-order PDEs in weak form

$$\mathcal{A}(\mathbf{u}, \mathbf{p}) = \ell_1 + \ell_2 .$$

Here, the operator \mathcal{A} is given by

$$(\mathcal{A}(\mathbf{u}, \mathbf{p}))(\mathbf{v}, \mathbf{q}) := \mathbf{a}(\mathbf{p}, \mathbf{q}) - \mathbf{b}(\mathbf{u}, \mathbf{q}) + \mathbf{b}(\mathbf{v}, \mathbf{p}) + \mathbf{c}(\mathbf{u}, \mathbf{v})$$

in terms of bilinear forms $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and the linear functionals ℓ_1, ℓ_2 associated with the data of the problem (see Section 3 for details).

We prove in Proposition 3.1 that \mathcal{A} is linear, bounded and bijective with bounded inverse. Therefore, the natural norms of any error is equivalent to the respective dual norms of the residuals.

Given some approximations $\tilde{\mathbf{u}}_h$ of \mathbf{u} and $\tilde{\mathbf{p}}_h$ of \mathbf{p} , in the general analysis of residuals

$$\begin{aligned} \mathbf{Res}_1(\mathbf{q}) &:= \ell_1(\mathbf{q}) - \mathbf{a}(\tilde{\mathbf{p}}_h, \mathbf{q}) + \mathbf{b}(\tilde{\mathbf{u}}_h, \mathbf{q}) , \\ \mathbf{Res}_2(\mathbf{v}) &:= \ell_2(\mathbf{v}) - \mathbf{b}(\mathbf{v}, \tilde{\mathbf{p}}_h) - \mathbf{c}(\tilde{\mathbf{u}}_h, \mathbf{v}) \end{aligned}$$

we rediscover the error estimators of [7,8,32,43] for the curl-conforming edge elements of Nédélec's first family and those of [34] for an interior penalty discontinuous Galerkin method. In comparison with [34], the general framework even results in sharper estimates. In particular, with regard to the existing estimates with mesh-depending norms on the jumps, it is an innovative new feature of this paper (and of [21]) that those terms are obtained as known upper bounds while the consistency errors are actually smaller.

The remaining parts of this paper are organized as follows. Section 2 is devoted to the Sobolev spaces $\mathbf{H}(\mathbf{curl}; \Omega)$ and $\mathbf{H}(\mathbf{div}; \Omega)$ and various trace spaces thereof. The unified framework in Section 3 provides the details for the aforementioned operator \mathcal{A} and the associated errors and residuals. Sections 4 and 5 recast the interior penalty discontinuous Galerkin method and the mortar edge element method in the above format and provide a new proof of the estimates in [34] and [31].

2. $\mathbf{H}(\mathbf{curl}; \Omega)$, $\mathbf{H}(\mathbf{div}; \Omega)$, AND THEIR TRACES

Let $\Omega \subset \mathbb{R}^3$ be a simply connected polyhedral domain with boundary $\Gamma = \partial\Omega$ which can be split into J open faces $\Gamma_1, \dots, \Gamma_J$ with $\Gamma = \cup_{j=1}^J \bar{\Gamma}_j$. We denote by $\mathcal{D}(\Omega)$ the space of all infinitely often differentiable functions with compact support in Ω and by $\mathcal{D}'(\Omega)$ its dual space referring to $\langle \cdot, \cdot \rangle$ as the dual pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. We further adopt standard notation from Lebesgue and Sobolev space theory. We refer to $\mathbf{H}(\mathbf{curl}; \Omega)$ as the linear space

$$\mathbf{H}(\mathbf{curl}; \Omega) := \{ \mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega) \},$$

which is a Hilbert space with respect to the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{curl}, \Omega} := (\mathbf{u}, \mathbf{v})_{0, \Omega} + (\mathbf{curl} \mathbf{u}, \mathbf{curl} \mathbf{v})_{0, \Omega} \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega)$$

and associated norm $\|\cdot\|_{\text{curl},\Omega}$. We further refer to $\mathbf{H}(\mathbf{curl}^0;\Omega)$ as the subspace of irrotational vector fields

$$\mathbf{H}(\mathbf{curl}^0;\Omega) = \{\mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega) \mid \mathbf{curl} \mathbf{u} = \mathbf{0}\},$$

which admits the characterization $\mathbf{H}(\mathbf{curl}^0;\Omega) = \mathbf{grad} H^1(\Omega)$. Its orthogonal complement

$$\mathbf{H}^\perp(\mathbf{curl};\Omega) = \{\mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega) \mid (\mathbf{u}, \mathbf{u}^0)_{0,\Omega} = 0, \mathbf{u}^0 \in \mathbf{H}(\mathbf{curl}^0;\Omega)\}$$

can be interpreted as the subspace of weakly solenoidal vector fields. The Hilbert space $\mathbf{H}(\mathbf{curl};\Omega)$ admits the following Helmholtz decomposition

$$\mathbf{H}(\mathbf{curl};\Omega) = \mathbf{H}(\mathbf{curl}^0;\Omega) \oplus \mathbf{H}^\perp(\mathbf{curl};\Omega). \quad (2.1)$$

Likewise, the space $\mathbf{H}(\text{div};\Omega)$ is defined by

$$\mathbf{H}(\text{div};\Omega) := \{\mathbf{q} \in \mathbf{L}^2(\Omega) \mid \text{div} \mathbf{q} \in L^2(\Omega)\}$$

which is a Hilbert space with respect to the inner product

$$(\mathbf{u}, \mathbf{v})_{\text{div},\Omega} := (\mathbf{u}, \mathbf{v})_{0,\Omega} + (\text{div} \mathbf{u}, \text{div} \mathbf{v})_{0,\Omega} \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{H}(\text{div};\Omega)$$

and associated norm $\|\cdot\|_{\text{div},\Omega}$. For vector fields $\mathbf{u} \in \mathcal{D}(\bar{\Omega})^3 := \{\varphi|_\Omega \mid \varphi \in \mathcal{D}(\mathbb{R}^3)\}$, the normal component trace reads

$$\eta_{\mathbf{n}}(\mathbf{u})|_{\Gamma_j} := \mathbf{n}_j \cdot \mathbf{u}|_{\Gamma_j} \quad \text{for } j = 1, \dots, J$$

with the exterior unit normal vector \mathbf{n}_j on Γ_j . The normal component trace mapping can be extended by continuity to a surjective, continuous linear mapping (cf. [26]; Thm. 2.2)

$$\eta_{\mathbf{n}} : \mathbf{H}(\text{div};\Omega) \rightarrow \mathbf{H}^{-1/2}(\Gamma).$$

We define $\mathbf{H}_0(\text{div};\Omega)$ as the subspace of vector fields with vanishing normal components on Γ

$$\mathbf{H}_0(\text{div};\Omega) := \{\mathbf{u} \in \mathbf{H}(\text{div};\Omega) \mid \eta_{\mathbf{n}}(\mathbf{u}) = 0\}.$$

In order to study the traces of vector fields $\mathbf{q} \in \mathbf{H}(\mathbf{curl};\Omega)$, following [16,17,18], we introduce the spaces

$$\begin{aligned} \mathbf{L}_{\mathbf{t}}^2(\Gamma) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \eta_{\mathbf{n}}(\mathbf{u}) = 0\}, \\ \mathbf{H}_{-}^{1/2}(\Gamma) &:= \{\mathbf{u} \in \mathbf{L}_{\mathbf{t}}^2(\Gamma) \mid \mathbf{u}|_{\Gamma_j} \in \mathbf{H}^{1/2}(\Gamma_j) \text{ for all } j = 1, \dots, J\}. \end{aligned}$$

For $\Gamma_j, \Gamma_k \subset \Gamma$ with $j \neq k$ and $E_{jk} := \bar{\Gamma}_j \cap \bar{\Gamma}_k \in \mathcal{E}_h$, the set of edges, we denote by \mathbf{t}_j and \mathbf{t}_k the tangential unit vectors along Γ_j and Γ_k and by \mathbf{t}_{jk} the unit vector parallel to E_{jk} such that Γ_j is spanned by $\mathbf{t}_j, \mathbf{t}_{jk}$ and Γ_k by $\mathbf{t}_k, \mathbf{t}_{jk}$. Let

$$\mathcal{J}_k := \{j \in \{1, \dots, N\} \mid \bar{\Gamma}_j \cap \bar{\Gamma}_k = E_{jk} \in \mathcal{E}_h\}$$

and define

$$\mathbf{H}_{||}^{1/2}(\Gamma) := \{\mathbf{u} \in \mathbf{H}_{-}^{1/2}(\Gamma) \mid (\mathbf{t}_{jk} \cdot \mathbf{u}_j)|_{E_{jk}} = (\mathbf{t}_{jk} \cdot \mathbf{u}_k)|_{E_{jk}} \text{ for } k = 1, \dots, N \text{ and } j \in \mathcal{J}_k\},$$

$$\mathbf{H}_{\perp}^{1/2}(\Gamma) := \{\mathbf{u} \in \mathbf{H}_{-}^{1/2}(\Gamma) \mid (\mathbf{t}_j \cdot \mathbf{u}_j)|_{E_{jk}} = (\mathbf{t}_k \cdot \mathbf{u}_k)|_{E_{jk}} \text{ for } k = 1, \dots, N \text{ and } j \in \mathcal{J}_k\}.$$

We refer to $\mathbf{H}_{||}^{-1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\Gamma)$ as the dual spaces of $\mathbf{H}_{||}^{1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ with $\mathbf{L}_{\mathbf{t}}^2(\Gamma)$ as the pivot space. For $\mathbf{u} \in \mathcal{D}(\bar{\Omega})^3$ we further define the tangential trace mapping

$$\gamma_{\mathbf{t}}|_{\Gamma_j} := \mathbf{u} \wedge \mathbf{n}_j|_{\Gamma_j} \text{ for } j = 1, \dots, n$$

and the tangential components trace

$$\pi_{\mathbf{t}}|_{\Gamma_j} := \mathbf{n}_j \wedge (\mathbf{u} \wedge \mathbf{n}_j)|_{\Gamma_j} \text{ for } j = 1, \dots, n.$$

Moreover, for a smooth function $u \in \mathcal{D}(\bar{\Omega})$ we define the tangential gradient operator $\nabla_{\Gamma} = \mathbf{grad}|_{\Gamma}$ as the tangential components trace of the gradient operator ∇

$$\nabla_{\Gamma} u|_{\Gamma_j} := \nabla_{\Gamma_j} u = \pi_{\mathbf{t},j}(\nabla u) = \mathbf{n}_j \wedge (\nabla u \wedge \mathbf{n}_j) \text{ for } j = 1, \dots, n,$$

which leads to a continuous linear mapping $\nabla_{\Gamma} : H^{3/2}(\Gamma) \rightarrow \mathbf{H}_{||}^{1/2}(\Gamma)$. The tangential divergence operator

$$\operatorname{div}|_{\tau} : \mathbf{H}_{||}^{-1/2}(\Gamma) \rightarrow H^{-3/2}(\Gamma)$$

is defined, with the respective dual pairings $\langle \cdot, \cdot \rangle$, as the adjoint operator of $-\nabla_{\Gamma}$

$$\langle \operatorname{div}|_{\tau} \mathbf{u}, v \rangle = - \langle \mathbf{u}, \nabla_{\Gamma} v \rangle \quad \text{for all } v \in H^{3/2}(\Gamma) \text{ and } \mathbf{u} \in \mathbf{H}_{||}^{-1/2}(\Gamma).$$

Finally, for $u \in \mathcal{D}(\Omega)$ we define the tangential curl operator $\mathbf{curl}|_{\tau}$ as the tangential trace of the gradient operator

$$\mathbf{curl}_{\tau} u|_{\Gamma_j} = \mathbf{curl}|_{\Gamma_j} u = \gamma_{\mathbf{t},j}(\nabla u) = \nabla u \wedge \mathbf{n}_j \text{ for } j = 1, \dots, n. \quad (2.2)$$

The vectorial tangential curl operator is a linear continuous mapping

$$\mathbf{curl}_{\tau} : H^{3/2}(\Gamma) \rightarrow \mathbf{H}_{\perp}^{1/2}(\Gamma).$$

The scalar tangential curl operator

$$\text{curl}_\tau : \mathbf{H}_\perp^{-1/2}(\Gamma) \rightarrow H^{-3/2}(\Gamma)$$

is defined as the adjoint of the vectorial tangential curl operator via $\mathbf{curl}|_\tau$, i.e.,

$$\langle \text{curl}|_\tau \mathbf{u}, v \rangle = \langle \mathbf{u}, \mathbf{curl}|_\Gamma v \rangle \quad \text{for all } v \in H^{3/2}(\Gamma) \text{ and } \mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\Gamma) .$$

The range spaces of the tangential trace mapping γ_t and the tangential components trace mapping π_t on $H(\mathbf{curl}; \Omega)$ can be characterized by means of the spaces

$$\begin{aligned} \mathbf{H}^{-1/2}(\text{div}|_\Gamma, \Gamma) &:= \{ \lambda \in \mathbf{H}_\parallel^{-1/2}(\Gamma) \mid \text{div}|_\Gamma \lambda \in H^{-1/2}(\Gamma) \} , \\ \mathbf{H}^{-1/2}(\text{curl}|_\Gamma, \Gamma) &:= \{ \lambda \in \mathbf{H}_\perp^{-1/2}(\Gamma) \mid \text{curl}|_\Gamma \lambda \in H^{-1/2}(\Gamma) \} , \end{aligned}$$

which are dual to each other with respect to the pivot space $\mathbf{L}_t^2(\Gamma)$. We refer to $\|\cdot\|_{-1/2, \text{div}|_\Gamma, \Gamma}$ and $\|\cdot\|_{-1/2, \text{curl}|_\Gamma, \Gamma}$ as the respective norms and denote by $\langle \cdot, \cdot \rangle_{-1/2, \Gamma}$ the dual pairing (see, e.g., [18] for details).

It can be shown that the tangential trace mapping is a continuous linear mapping

$$\gamma_t : \mathbf{H}(\mathbf{curl}; \Omega) \rightarrow \mathbf{H}^{-1/2}(\text{div}|_\Gamma, \Gamma) ,$$

whereas the tangential components trace mapping is a continuous linear mapping

$$\pi_t : \mathbf{H}(\mathbf{curl}; \Omega) \rightarrow \mathbf{H}^{-1/2}(\text{curl}|_\Gamma, \Gamma) .$$

The previous results imply that the tangential divergence of the tangential trace and the scalar tangential curl of the tangential components trace coincide: For $\mathbf{u} \in \mathbf{H}(\mathbf{curl}; \Omega)$ it holds

$$\text{div}|_\Gamma (\mathbf{u} \wedge \mathbf{n}) = \text{curl}|_\Gamma (\mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n})) = \mathbf{n} \cdot \mathbf{curl} \mathbf{u} .$$

We define $\mathbf{H}_0(\mathbf{curl}; \Omega)$ as the subspace of $\mathbf{H}(\mathbf{curl}; \Omega)$ with vanishing tangential traces on Γ

$$\mathbf{V} := \mathbf{H}_0(\mathbf{curl}; \Omega) := \{ \mathbf{u} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid \gamma_t(\mathbf{u}) = 0 \} .$$

3. THE UNIFIED FRAMEWORK

As a model problem, for given $\mathbf{f} \in \mathbf{H}(\text{div}; \Omega)$ and $\mu > 0, \sigma > 0$, we consider the following elliptic boundary-value problem (BVP)

$$\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u} + \sigma \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \tag{3.1a}$$

$$\gamma_t(\mathbf{u}) = 0 \quad \text{on } \Gamma. \tag{3.1b}$$

This BVP can be interpreted as the stationary form of the 3D eddy currents equations with μ, σ being related to the magnetic permeability and electric conductivity, respectively, and \mathbf{f} standing for a current density. The weak formulation of (3.1a)-(3.1b) amounts to the computation of $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$ such that

$$\int_{\Omega} \left(\mu^{-1} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} + \sigma \mathbf{u} \cdot \mathbf{v} \right) dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \text{ for all } \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega). \quad (3.2)$$

With $\mathbf{p} := \mu^{-1} \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega)$, (3.1a) can be recast as the first-order system

$$\mu \mathbf{p} - \mathbf{curl} \mathbf{u} = 0, \quad (3.3a)$$

$$\mathbf{curl} \mathbf{p} + \sigma \mathbf{u} = \mathbf{f}. \quad (3.3b)$$

The fundamental Hilbert spaces

$$\mathbf{V} := \mathbf{H}_0(\mathbf{curl}; \Omega) \quad \text{and} \quad \mathbf{Q} := \mathbf{L}^2(\Omega)$$

allow for the definition of the bilinear forms

$$\mathbf{a}(\cdot, \cdot) : \mathbf{Q} \times \mathbf{Q} \rightarrow \mathbb{R}, \mathbf{b}(\cdot, \cdot) : \mathbf{V} \times \mathbf{Q} \rightarrow \mathbb{R}, \text{ and } \mathbf{c}(\cdot, \cdot) : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$$

as well as functionals $\ell_1 \in \mathbf{Q}^*$ and $\ell_2 \in \mathbf{V}^*$ according to

$$\mathbf{a}(\mathbf{p}, \mathbf{q}) := \int_{\Omega} \mu \mathbf{p} \cdot \mathbf{q} dx \quad \text{for all } \mathbf{p}, \mathbf{q} \in \mathbf{Q}, \quad (3.4a)$$

$$\mathbf{b}(\mathbf{u}, \mathbf{q}) := \int_{\Omega} \mathbf{curl}_h \mathbf{u} \cdot \mathbf{q} dx \quad \text{for all } \mathbf{u} \in \mathbf{V}, \mathbf{q} \in \mathbf{Q}, \quad (3.4b)$$

$$\mathbf{c}(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \sigma \mathbf{u} \cdot \mathbf{v} dx \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}, \quad (3.4c)$$

$$\ell_1(\mathbf{q}) := 0 \quad \text{for all } \mathbf{q} \in \mathbf{Q}, \quad (3.4d)$$

$$\ell_2(\mathbf{v}) := \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \quad \text{for all } \mathbf{v} \in \mathbf{V}. \quad (3.4e)$$

Here and throughout the paper, \mathbf{curl}_h refers to the piecewise action of the \mathbf{curl} -operator used later for discrete vector-valued functions (note that $\mathbf{curl}_h \mathbf{u} = \mathbf{curl} \mathbf{u}$ for $\mathbf{u} \in \mathbf{V}$) and $\ell_1 \in \mathbf{Q}^*$ has been formally introduced for later purposes as well.

The weak formulation of (3.3a)-(3.3b) is to find $(\mathbf{u}, \mathbf{p}) \in \mathbf{V} \times \mathbf{Q}$ such that

$$\mathbf{a}(\mathbf{p}, \mathbf{q}) - \mathbf{b}(\mathbf{u}, \mathbf{q}) = \ell_1(\mathbf{q}) \quad \text{for all } \mathbf{q} \in \mathbf{Q}, \quad (3.5a)$$

$$\mathbf{b}(\mathbf{v}, \mathbf{p}) + \mathbf{c}(\mathbf{u}, \mathbf{v}) = \ell_2(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{V}. \quad (3.5b)$$

The operator-theoretic framework involves the operator $\mathcal{A} : (\mathbf{V} \times \mathbf{Q}) \rightarrow (\mathbf{V} \times \mathbf{Q})^*$ defined, for all $(\mathbf{u}, \mathbf{p}), (\mathbf{v}, \mathbf{q}) \in \mathbf{V} \times \mathbf{Q}$, by

$$(\mathcal{A}(\mathbf{u}, \mathbf{p}))(\mathbf{v}, \mathbf{q}) := \mathbf{a}(\mathbf{p}, \mathbf{q}) - \mathbf{b}(\mathbf{u}, \mathbf{q}) + \mathbf{b}(\mathbf{v}, \mathbf{p}) + \mathbf{c}(\mathbf{u}, \mathbf{v}). \quad (3.6)$$

Then, the system (3.5a)-(3.5b) is recast in compact form as

$$\mathcal{A}(\mathbf{u}, \mathbf{p}) = \ell_1 + \ell_2. \quad (3.7)$$

Proposition 3.1. *For positive μ, σ , the operator \mathcal{A} is a continuous, linear, and bijective and, hence, \mathcal{A} has a bounded inverse.*

Proof. The mapping properties are straightforward and the proof here focuses on the bijectivity which essentially follows from the inf-sup condition. In fact, given any $(\mathbf{u}, \mathbf{p}) \in \mathbf{V} \times \mathbf{Q}$ one calculates

$$\begin{aligned} (\mathcal{A}(\mathbf{u}, \mathbf{p}))(3\mathbf{u}, 2\mathbf{p} - \mu^{-1} \text{curl}_h \mathbf{u}) &= (\mathcal{A}(3\mathbf{u}, 2\mathbf{p} + \mu^{-1} \text{curl}_h \mathbf{u}))(\mathbf{u}, \mathbf{p}) \\ &= 2\mu \|\mathbf{p}\|_{L^2(\Omega)}^2 + 3\sigma \|\mathbf{u}\|_{L^2(\Omega)}^2 + \mu^{-1} \|\text{curl}_h \mathbf{u}\|_{L^2(\Omega)}^2. \end{aligned}$$

This implies the inf-sup condition and the remaining degeneracy condition which leads to bijectivity. \square

As an immediate consequence, given any $\ell_1 \in \mathbf{Q}^*, \ell_2 \in \mathbf{V}^*$, there exists a unique solution $(\mathbf{u}, \mathbf{p}) \in \mathbf{V} \times \mathbf{Q}$ of (3.7). Moreover, given any $(\tilde{\mathbf{u}}_h, \tilde{\mathbf{p}}_h) \in \mathbf{V} \times \mathbf{Q}$, it holds

$$\|(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{p} - \tilde{\mathbf{p}}_h)\|_{\mathbf{V} \times \mathbf{Q}} \approx \|\mathbf{Res}_1\|_{\mathbf{Q}^*} + \|\mathbf{Res}_2\|_{\mathbf{V}^*} \quad (3.8)$$

with residuals $\mathbf{Res}_1 \in \mathbf{Q}^*$ and $\mathbf{Res}_2 \in \mathbf{V}^*$,

$$\mathbf{Res}_1(\mathbf{q}) := \ell_1(\mathbf{q}) - \mathbf{a}(\tilde{\mathbf{p}}_h, \mathbf{q}) + \mathbf{b}(\tilde{\mathbf{u}}_h, \mathbf{q}) \quad \text{for all } \mathbf{q} \in \mathbf{Q}, \quad (3.9a)$$

$$\mathbf{Res}_2(\mathbf{v}) := \ell_2(\mathbf{v}) - \mathbf{b}(\mathbf{v}, \tilde{\mathbf{p}}_h) - \mathbf{c}(\tilde{\mathbf{u}}_h, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{V}. \quad (3.9b)$$

The first residual $\mathbf{Res}_1(\mathbf{q})$ equals the function $\tilde{\mathbf{p}}_h - \mu^{-1} \text{curl}_h \tilde{\mathbf{u}}_h$ times the test function \mathbf{q} in the scalar product of $\mathbf{L}^2(\Omega)$. The corresponding dual norm is therefore the $\mathbf{L}^2(\Omega)$ norm of $\tilde{\mathbf{p}}_h - \mu^{-1} \text{curl}_h \tilde{\mathbf{u}}_h$, i.e.,

$$\|\mathbf{Res}_1\|_{\mathbf{Q}^*} = \|\tilde{\mathbf{p}}_h - \mu^{-1} \text{curl}_h \tilde{\mathbf{u}}_h\|_{0,\Omega}.$$

The analysis of the second residual \mathbf{Res}_2 involves an integration by parts and some dual norm with test functions in \mathbf{V} . Therefore, the analysis of $\|\mathbf{Res}_2\|_{\mathbf{V}^*}$ is more involved and requires additional properties from the weak form and the discrete solutions.

We assume \mathcal{T}_h to be a regular simplicial triangulation with $\mathcal{E}_h(D)$ and $\mathcal{F}_h(D)$ denoting the sets of edges and faces of \mathcal{T}_h in $D \subset \overline{\Omega}$. The curl-conforming edge elements of Nédélec's first family with respect to $T \in \mathcal{T}_h$ read

$$\mathbf{Nd}_1(T) := \{\mathbf{v} | \exists \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \forall \mathbf{x} \in T, \mathbf{v}(\mathbf{x}) := \mathbf{a} + \mathbf{b} \wedge \mathbf{x}\} \quad (3.10)$$

with degrees of freedom given by the zero-order moments of the tangential components along the edges $E \in \mathcal{E}_h(T)$ and

$$\mathbf{Nd}_1(\Omega; \mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{V} | \forall T \in \mathcal{T}_h, \mathbf{v}_h|_T \in \mathbf{Nd}_1(T)\}.$$

Under the condition

$$\mathbf{Nd}_1(\Omega; \mathcal{T}_h) \subset \text{Ker } \mathbf{Res}_2, \quad (3.11)$$

reliability holds for the explicit residual-based error estimator which, for each $T \in \mathcal{T}_h$ and with tangential and normal jumps across interior faces $F \in \mathcal{F}_h(\Omega)$, reads

$$\eta_T := h_T \|\mathbf{f} - \sigma \tilde{\mathbf{u}}_h - \mathbf{curl}_h \tilde{\mathbf{p}}_h\|_{0,T} + h_T \|\text{div}(\mathbf{f} - \sigma \tilde{\mathbf{u}}_h)\|_{0,T}, \quad (3.12a)$$

$$\eta_F := h_F^{1/2} \|[\pi_t(\tilde{\mathbf{p}}_h)]\|_{0,F} + h_F^{1/2} \|\mathbf{n}_F \cdot [\sigma \tilde{\mathbf{u}}_h]\|_{0,F}. \quad (3.12b)$$

Proposition 3.2 [32,43]. *Using the notation before and under the condition (3.11) it holds*

$$\|\mathbf{Res}_2\|_{\mathbf{V}^*}^2 \lesssim \eta^2 := \sum_{T \in \mathcal{T}_h} \eta_T^2 + \sum_{F \in \mathcal{F}_h(\Omega)} \eta_F^2. \quad (3.13)$$

Proof. Given any $\mathbf{v} \in \mathbf{V}$, Theorem 1 of [43] shows that there exist $\mathbf{v}_h \in \mathbf{Nd}_1(\Omega; \mathcal{T}_h)$, $\varphi \in H_0^1(\Omega)$, and $\mathbf{z} \in H_0^1(\Omega)^3$ with

$$\mathbf{v} - \mathbf{v}_h = \nabla \varphi + \mathbf{z}$$

plus approximation and stability properties. The proof then follows that of Corollary 2 of [43] for

$$\mathbf{Res}_2(\mathbf{v}) = \mathbf{Res}_2(\mathbf{v} - \mathbf{v}_h) = \mathbf{Res}_2(\nabla \varphi + \mathbf{z})$$

and employs integration by parts followed by trace inequalities and approximation estimates of $\nabla \varphi$ and \mathbf{z} . Since the proof in [43] is quite explicit, details are dropped here. \square

The converse estimate holds up to data oscillations [8,32].

4. INTERIOR PENALTY DISCONTINUOUS GALERKIN METHODS

Let \mathcal{T}_h be a geometrically conforming, shape-regular simplicial triangulation of Ω . The discrete spaces \mathbf{V}_h and \mathbf{Q}_h are chosen as elementwise polynomials of degree $\leq p$,

$$\mathbf{V}_h := \Pi_p(\mathcal{T}_h; \mathbb{R}^3) \quad \text{and} \quad \mathbf{Q}_h := \Pi_p(\mathcal{T}_h; \mathbb{R}^3).$$

For this choice and some penalty parameter $\alpha \geq \alpha_{\min} > 0$, set

$$\begin{aligned} \mathbf{J}_1(\mathbf{v}_h, \mathbf{q}_h) &:= \sum_{F \in \mathcal{F}_h(\Omega)} \int_F \{\pi_t(\mathbf{q}_h)\} \cdot [\gamma_t(\mathbf{v}_h)] ds, \\ \mathbf{J}_2(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_{F \in \mathcal{F}_h(\Omega)} \int_F (\{\pi_t(\mathbf{curl} \mathbf{u}_h)\} - \alpha [\gamma_t(\mathbf{u}_h)]) \cdot ([\gamma_t(\mathbf{v}_h)]) ds. \end{aligned}$$

The first formulation of the *Interior Penalty Discontinuous Galerkin Method* reads: Find $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ such that

$$\mathbf{a}(\mathbf{p}_h, \mathbf{q}_h) - \mathbf{b}(\mathbf{u}_h, \mathbf{q}_h) = \ell_1(\mathbf{q}_h) + \mathbf{J}_1(\mathbf{u}_h, \mathbf{q}_h) \quad \text{for all } \mathbf{q}_h \in \mathbf{Q}_h, \quad (4.1a)$$

$$\mathbf{b}(\mathbf{v}_h, \mathbf{p}_h) + \mathbf{c}(\mathbf{u}_h, \mathbf{v}_h) = \ell_2(\mathbf{v}_h) + \mathbf{J}_2(\mathbf{u}_h, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h. \quad (4.1b)$$

The second formulation in the primal variable reads: Find $\mathbf{u}_h \in \mathbf{V}_h$ such that, for all $\mathbf{v}_h \in \mathbf{V}_h$, it holds

$$\begin{aligned} & \mathbf{c}(\mathbf{u}_h, \mathbf{v}_h) + \sum_{T \in \mathcal{T}_h} (\mu^{-1} \mathbf{curl} \mathbf{u}_h, \mathbf{curl} \mathbf{v}_h)_{0,T} \\ &= \ell_1(\mu^{-1} \mathbf{curl} \mathbf{v}_h) + \ell_2(\mathbf{v}_h) + \mathbf{J}_1(\mathbf{u}_h, \mu^{-1} \mathbf{curl} \mathbf{v}_h) + \mathbf{J}_2(\mathbf{u}_h, \mathbf{v}_h). \end{aligned} \quad (4.2)$$

Theorem 4.1. *The formulations (4.1a)-(4.1b) and (4.2) are formally equivalent in the following sense. If $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ solves (4.1a)-(4.1b), then $\mathbf{u}_h \in \mathbf{V}_h$ solves (4.2). Conversely, if $\mathbf{u}_h \in \mathbf{V}_h$ solves (4.2), then there exists some $\mathbf{p}_h \in \mathbf{Q}_h$ such that $(\mathbf{u}_h, \mathbf{p}_h)$ solves (4.1a)-(4.1b).*

Proof. Suppose that $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ solves (4.1a)-(4.1b). Since μ is constant on each element $T \in \mathcal{T}_h$, $\mathbf{q}_h := \mu^{-1} \mathbf{curl} \mathbf{v}_h$ is a proper test function in (4.1a) for any $\mathbf{v}_h \in \mathbf{V}_h$. The resulting identity involves

$$\mathbf{a}(\mathbf{p}_h, \mu^{-1} \mathbf{curl} \mathbf{v}_h) = \mathbf{b}(\mathbf{v}_h, \mathbf{p}_h).$$

This and (4.1b) imply (4.2).

Conversely, let $\mathbf{u}_h \in \mathbf{V}_h$ solve (4.2). Then, the expression

$$\mathbf{b}(\mathbf{u}_h, \mathbf{q}_h) + \ell_1(\mathbf{q}_h) + \mathbf{J}_1(\mathbf{u}_h, \mathbf{q}_h)$$

is a linear and bounded functional as a function of $\mathbf{q}_h \in \mathbf{Q}_h$. Since \mathbf{a} is a scalar product on \mathbf{Q}_h , there exists a unique Riesz representation $\mathbf{a}(\mathbf{p}_h, \cdot)$ of this linear functional. Then, $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ solves (4.1a). Again, $\mathbf{q}_h := \mu^{-1} \mathbf{curl} \mathbf{v}_h$ is a proper test function in (4.1a). The resulting expression combined with (4.2) allows the proof of (4.1b). \square

Given the solution $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ of (4.1a)-(4.1b), consider the *consistency error*

$$\xi := \min_{\tilde{\mathbf{v}}_h \in \mathbf{V}} (\|\mathbf{u}_h - \tilde{\mathbf{v}}_h\|_{L^2(\Omega)}^2 + \|\mathbf{curl}_h \mathbf{u}_h - \mathbf{curl} \tilde{\mathbf{v}}_h\|_{L^2(\Omega)}^2)^{1/2} \quad (4.3)$$

and notice that the minimum is attained with a minimiser $\tilde{\mathbf{u}}_h \in \mathbf{V}$, i.e.,

$$\xi^2 = \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{L^2(\Omega)}^2 + \|\mathbf{curl}_h \mathbf{u}_h - \mathbf{curl} \tilde{\mathbf{u}}_h\|_{L^2(\Omega)}^2.$$

Since there exist computable upper bounds for ξ , it is not necessary to compute the minimiser $\tilde{\mathbf{u}}_h \in \mathbf{V}$ for error control. For instance, in Proposition 4.1 of [34], it is shown that

$$\xi^2 \lesssim \alpha \sum_{F \in \mathcal{F}_h(\Omega)} h_F^{-1} \|\gamma_t(\mathbf{u}_h)\|_{0,F}^2 =: \bar{\xi}^2.$$

Since, the jumps are also error terms, e.g.,

$$h_F^{-1} \|[\gamma_t(\mathbf{u}_h)]\|_{0,F}^2 = h_F^{-1} \|[\gamma_t(\mathbf{u} - \mathbf{u}_h)]\|_{0,F}^2,$$

they are seen as a contribution to the DG error norm and, at the same time, are computable a posteriori and so arise in the upper bounds in [34]. However, in this paper, we consider those jump contributions $\bar{\xi}$ as one known upper bound of ξ whose efficiency is less clear to us.

Given the aforementioned minimiser $\tilde{\mathbf{u}}_h \in \mathbf{V}$ in the definition of ξ , we let

$$\tilde{\mathbf{p}}_h := \mu^{-1} \mathbf{curl} \tilde{\mathbf{u}}_h \in \mathbf{Q}.$$

Then, the unified approach leads to (3.8) with the residuals (3.9a)-(3.9b). Here,

$$\mathbf{Res}_1(\mathbf{q}) = 0 \quad \text{for all } \mathbf{q} \in \mathbf{Q}$$

and, for all $\mathbf{v} \in \mathbf{V}$,

$$\mathbf{Res}_2(\mathbf{v}) := \int_{\Omega} (\mathbf{f} \cdot \mathbf{v} - \mu^{-1} \mathbf{curl}_h \tilde{\mathbf{u}}_h \cdot \mathbf{curl} \mathbf{v} - \sigma \tilde{\mathbf{u}}_h \cdot \mathbf{v}) dx.$$

Lemma 4.1. *For any $\mathbf{v}_h \in \mathbf{Nd}_1(\Omega; \mathcal{T}_h)$, it holds*

$$\mathbf{Res}_2(\mathbf{v}_h) = \mathbf{c}(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \mathbf{v}_h).$$

Proof. Since $\mathbf{v}_h \in \mathbf{Nd}_1(\Omega; \mathcal{T}_h) \subset \Pi_p(\mathcal{T}_h; \mathbb{R}^3)$ is an admissible test function for \mathbf{Res}_2 , the jump contribution

$$\mathbf{J}_2(\mathbf{u}_h, \mathbf{v}_h) = 0$$

vanishes. A comparison with (4.2) shows, for $\mathbf{v}_h \in \mathbf{Nd}_1(\Omega; \mathcal{T}_h)$, that

$$\mathbf{Res}_2(\mathbf{v}_h) = \mathbf{c}(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \mathbf{v}_h) + (\mu^{-1} \mathbf{curl}_h(\mathbf{u}_h - \tilde{\mathbf{u}}_h), \mathbf{curl}_h \mathbf{v}_h)_{0,\Omega} - \mathbf{J}_1(\mathbf{u}_h, \mu^{-1} \mathbf{curl}_h \mathbf{v}_h).$$

Since $\mathbf{curl}_h \mathbf{curl}_h \mathbf{v}_h = 0$ and $[\gamma_t(\tilde{\mathbf{u}}_h)] = 0$, Stokes theorem yields

$$\begin{aligned} (\mu^{-1} \mathbf{curl}_h(\mathbf{u}_h - \tilde{\mathbf{u}}_h), \mathbf{curl}_h \mathbf{v}_h)_{0,\Omega} &= \sum_{T \in \mathcal{T}_h} \int_T \mu^{-1} \mathbf{curl}_h(\mathbf{u}_h - \tilde{\mathbf{u}}_h) \cdot \mathbf{curl}_h \mathbf{v}_h dx = \\ &= \sum_{F \in \mathcal{F}_h(\Omega)} \pi_t(\mu^{-1} \mathbf{curl}_h \mathbf{v}_h) \cdot [\gamma_t(\mathbf{u}_h)] d\sigma = \mathbf{J}_1(\mathbf{u}_h, \mu^{-1} \mathbf{curl}_h \mathbf{v}_h). \end{aligned}$$

This implies the assertion of the lemma. \square

The unified theory leads to the following result which is stronger than the estimate of [34]. In fact, it implies the estimate [34] if one employs $\xi \lesssim \bar{\xi}$.

Proposition 4.1. *With volume and face contributions for some new*

$$\eta^2 := \sum_{T \in \mathcal{T}_h} \eta_T^2 + \sum_{F \in \mathcal{F}_h(\Omega)} \eta_F^2$$

defined, for $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h(\Omega)$, by

$$\begin{aligned} \eta_T &:= h_T \|\mathbf{f} - \boldsymbol{\sigma} \mathbf{u}_h - \mathbf{curl}_h \mu^{-1} \mathbf{curl}_h \mathbf{u}_h\|_{0,T} + h_T \|\text{div}(\mathbf{f} - \boldsymbol{\sigma} \mathbf{u}_h)\|_{0,T}, \\ \eta_F &:= h_F^{1/2} \|[\pi_t(\mu^{-1} \mathbf{curl}_h) \mathbf{u}_h]\|_{0,F} + h_F^{1/2} \|\mathbf{n}_F \cdot [\boldsymbol{\sigma} \mathbf{u}_h]\|_{0,F} \end{aligned}$$

it holds that

$$\|(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{p} - \tilde{\mathbf{p}}_h)\|_{\mathbf{V} \times \mathbf{Q}} \approx \|\mathbf{Res}_1\|_{\mathbf{Q}^*} + \|\mathbf{Res}_2\|_{\mathbf{V}^*} \lesssim \eta + \xi.$$

Proof. Lemma 4.1 suggests to consider the new functional

$$\mathbf{Res}_3 := \mathbf{Res}_2 - \mathbf{c}(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \cdot) = \ell_2 - \mathbf{b}(\cdot, \mu^{-1} \mathbf{curl} \tilde{\mathbf{u}}_h) - \mathbf{c}(\mathbf{u}_h, \cdot),$$

which is the form of the functional \mathbf{Res}_2 in Proposition 3.2 and indeed satisfies

$$\mathbf{Nd}_1(\Omega; \mathcal{T}_h) \subset \text{Ker}(\mathbf{Res}_3).$$

This is (3.11) when \mathbf{Res}_2 there is replaced by \mathbf{Res}_3 from this proof. Consequently, with the new estimators defined in the proposition,

$$\|\mathbf{Res}_3\|_{\mathbf{V}^*}^2 \lesssim \eta^2 := \sum_{T \in \mathcal{T}_h} \eta_T^2 + \sum_{F \in \mathcal{F}_h(\Omega)} \eta_F^2.$$

We thus obtain

$$\|\mathbf{Res}_2\|_{\mathbf{V}^*} \lesssim \eta + \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{0,\Omega} \leq \eta + \xi,$$

which concludes the proof. \square

5. MORTAR EDGE ELEMENT APPROXIMATIONS

We consider the so-called macrohybrid formulation of (3.1) in case $\mathbf{f} \in \mathbf{H}_0(\text{div}; \Omega)$ with respect to a non overlapping decomposition of the computational domain Ω into N mutually disjoint subdomains

$$\bar{\Omega} = \bigcup_{j=1}^N \bar{\Omega}_j \quad \text{with} \quad \Omega_j \cap \Omega_k \neq \emptyset \quad \text{for all } 1 \leq j < k \leq N. \quad (5.1)$$

We assume the decomposition to be geometrically conforming, i.e., two adjacent subdomains either share a face, an edge, or a vertex. The skeleton S of the decomposition

$$S = \bigcup_{m=1}^M \bar{\gamma}_m \quad (5.2)$$

consists of the interfaces $\gamma_1, \dots, \gamma_M$ between all adjacent subdomains Ω_j and Ω_k . We refer to $\gamma_{m(j)}$ as the mortar associated with subdomain Ω_j , while the other face, which geometrically occupies the same place, is denoted by $\delta_{m(j)}$ and is called the nonmortar. Based on (5.1) we introduce the product space

$$\mathbf{X} := \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \forall j = 1, \dots, N, \mathbf{u}|_{\Omega_j} \in \mathbf{H}(\mathbf{curl}; \Omega_j) \text{ and } \gamma_t(\mathbf{u})|_{\partial\Omega_j \cap \partial\Omega} = 0\} \quad (5.3)$$

equipped with the norm

$$\|\mathbf{u}\|_{\mathbf{X}} := \left(\sum_{j=1}^N \|\mathbf{u}\|_{\mathbf{curl}, \Omega_j^2} \right)^{1/2}. \quad (5.4)$$

A subdomainwise application of Stokes' theorem shows that vanishing jumps

$$\gamma_t(\mathbf{u})_{\gamma_m} = 0 \quad \text{for all } 1 \leq m \leq M$$

of some $\mathbf{u} \in \mathbf{X}$ imply

$$\mathbf{u} \in \mathbf{V} := \mathbf{H}_0(\mathbf{curl}; \Omega). \quad (5.5)$$

In general, we cannot expect (5.5) to hold true and need to enforce weak continuity of the tangential traces across γ_m by means of Lagrange multipliers in the space

$$\mathbf{M}(S) := \prod_{m=1}^M \mathbf{H}^{-1/2}(\text{div}_\tau; \gamma_m) \quad (5.6)$$

equipped with the norm

$$\|\mu\|_{\mathbf{M}(S)} := \left(\sum_{m=1}^M \|\mu|_{\gamma_m}\|_{-1/2, \text{div}_\tau, \gamma_m}^2 \right)^{1/2}. \quad (5.7)$$

We introduce the bilinear form $\mathbf{A}(\cdot, \cdot) : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}$ as the sum of the bilinear forms associated with the subdomain problems according to

$$\mathbf{A}(\mathbf{u}, \mathbf{v}) := \sum_{j=1}^N a_{\Omega_j}(\mathbf{u}|_{\Omega_j}, \mathbf{v}|_{\Omega_j}) = \sum_{j=1}^N \int_{\Omega_j} (\mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} + \sigma \mathbf{u} \cdot \mathbf{v}) \, dx. \quad (5.8)$$

Furthermore, we define the bilinear form $\mathbf{B}(\cdot, \cdot) : \mathbf{X} \times \mathbf{M}(S) \rightarrow \mathbb{R}$ by means of

$$\mathbf{B}(\mathbf{u}, \mu) := \langle \mu, [\gamma_t(\mathbf{u})] \rangle_{-1/2, S} \quad (5.9)$$

with the abbreviation

$$\langle \cdot, \cdot \rangle_{-1/2, S} := \sum_{m=1}^M \langle \cdot, \cdot \rangle_{-1/2, \gamma_m}. \quad (5.10)$$

The macro-hybrid variational formulation of (3.1a),(3.1b) reads: Find $(\mathbf{u}, \lambda) \in \mathbf{X} \times \mathbf{M}(S)$ such that

$$\begin{aligned} \mathbf{A}(\mathbf{u}, \mathbf{v}) + \mathbf{B}(\mathbf{u}, \lambda) &= \ell(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{X}, \\ \mathbf{B}(\mathbf{u}, \mu) &= 0 \quad \text{for all } \mu \in \mathbf{M}(S). \end{aligned} \quad (5.11)$$

The bilinear form $\mathbf{A}(\cdot, \cdot)$ is elliptic on the kernel of the operator associated with the bilinear form $\mathbf{B}(\cdot, \cdot)$ and $\mathbf{B}(\cdot, \cdot)$ satisfies the inf-sup condition

$$0 < \beta \leq \inf_{\mu \in \mathbf{M}(S)} \sup_{\mathbf{v} \in \mathbf{X}} \frac{\mathbf{B}(\mathbf{v}, \mu)}{\|\mathbf{v}\|_{\mathbf{X}} \|\mu\|_{\mathbf{M}(S)}}.$$

The macro-hybrid variational formulation (5.11) has a unique solution (\mathbf{u}, λ) .

The mortar edge element approximation of (3.2) mimics the macro-hybrid formulation (5.11) in the discrete regime and is based on individual shape-regular simplicial triangulations $\mathcal{T}_1, \dots, \mathcal{T}_N$ of the subdomains $\Omega_1, \dots, \Omega_N$ regardless the situation on the skeleton S of the decomposition. In particular, the interfaces inherit two different non-matching triangulations. The discretization of

$$\mathbf{H}_{\mathbf{0}, \partial\Omega_i \cap \partial\Omega}(\text{curl}; \Omega_j) := \{\mathbf{u} \in \mathbf{H}(\text{curl}; \Omega_j) \mid \gamma_{\mathbf{t}}(\mathbf{u})_{\partial\Omega_j \cap \partial\Omega} = 0\}$$

with curl-conforming edge elements of Nédélec's first family [36] considers the edge element spaces $\mathbf{Nd}_{1,\Gamma}(\Omega_j; \mathcal{T}_j)$ of vector fields with vanishing tangential trace on $\Gamma \cap \partial\Omega_j$. For a triangle $T \in \mathcal{T}_{\delta_{m(k)}}$ of diameter h_T with the surface $\delta_{m(k)} \subset S$, let $\mathbf{RT}_0(T)$ be the lowest order Raviart-Thomas element (cf., e.g., [15]). We denote by $\mathbf{RT}_0(\delta_{m(k)}; \mathcal{T}_{\delta_{m(k)}})$ the associated mixed finite element space, and we refer to $\mathbf{RT}_{0,0}(\delta_{m(k)}; \mathcal{T}_{\delta_{m(k)}})$ as the subspace of vector fields with vanishing normal components on $\delta_{m(k)}$. Based on these definitions, the product space

$$\mathbf{X}_h := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) \mid \forall j = 1, \dots, N, \mathbf{v}_h|_{\Omega_j} \in \mathbf{Nd}_{1,\Gamma}(\Omega_j; \mathcal{T}_j)\} \quad (5.12)$$

is equipped with the norm

$$\|\mathbf{v}_h\|_{\mathbf{X}_h} := \left(\|\mathbf{v}_h\|_{\mathbf{X}}^2 + \|\gamma_{\mathbf{t}}(\mathbf{v}_h)\|_{S, +1/2, h, S}^2 \right)^{1/2} \quad \text{for all } \mathbf{v}_h \in \mathbf{X}_h; \quad (5.13)$$

where $\|\cdot\|_{+1/2, h, S}$ is given by

$$\|\gamma_{\mathbf{t}}(\mathbf{v}_h)\|_{S, +1/2, h, S} := \left(\sum_{m=1}^M \|\gamma_{\mathbf{t}}(\mathbf{v}_h)\|_{\gamma_m, +1/2, h, \gamma_m} \right)^{1/2} \quad (5.14)$$

and $\|\cdot\|_{+1/2, h, \gamma_m}$ stands for the mesh-dependent norm

$$\|\gamma_{\mathbf{t}}(\mathbf{v}_h)\|_{\gamma_m, +1/2, h, \gamma_m} := h^{-1/2} \|\gamma_{\mathbf{t}}(\mathbf{v}_h)\|_{\gamma_m, 0, \gamma_m}. \quad (5.15)$$

Due to the occurrence of nonconforming edges on the interfaces between adjacent subdomains, there is a lack of continuity across the interfaces: neither the tangential traces $\gamma_{\mathbf{t}}(\mathbf{v}_h)$ nor the tangential trace components $\pi_{\mathbf{t}}(\mathbf{v}_h)$ can be expected to be continuous. We note that $\gamma_{\mathbf{t}}(\mathbf{v}_h)|_{\delta_{m(j)}} \in \mathbf{RT}_0(\delta_{m(j)}; \mathcal{T}_{\delta_{m(j)}})$ and $\pi_{\mathbf{t}}(\mathbf{v}_h)|_{\delta_{m(j)}} \in \mathbf{Nd}_1(\delta_{m(j)}; \mathcal{T}_{\delta_{m(j)}})$. Therefore, continuity can be enforced either in terms of the tangential traces or the tangential trace components. If we choose the tangential traces, the multiplier space $\mathbf{M}_h(S)$ can be constructed according to

$$\mathbf{M}_h(S) := \prod_{m=1}^M \mathbf{M}_h(\delta_{m(j)}) \quad (5.16)$$

with $\mathbf{M}_h(\delta_{m(j)})$ chosen such that

$$\mathbf{RT}_{0,0}(\delta_{m(j)}; \mathcal{T}_{\delta_{m(j)}}) \subset \mathbf{M}_h(\delta_{m(j)}) , \quad (5.17)$$

$$\dim \mathbf{M}_h(\delta_{m(j)}) = \dim \mathbf{RT}_{0,0}(\delta_{m(j)}; \delta_{m(j)}) . \quad (5.18)$$

We refer to [48] for the explicit construction. The multiplier space $\mathbf{M}_h(S)$ will be equipped with the mesh-dependent norm

$$\|\mu_h\|_{\mathbf{M}_h(S)} := \left(\sum_{m=1}^M \|\mu_h|_{\delta_{m(j)}}\|_{-1/2,h,\delta_{m(j)}} \right)^{1/2} , \quad (5.19)$$

where

$$\|\mu_h|_{\delta_{m(j)}}\|_{-1/2,h,\delta_{m(j)}} := h^{1/2} \|\mu_h|_{\delta_{m(j)}}\|_{0,\delta_{m(j)}} . \quad (5.20)$$

The mortar edge element approximation of (3.1a),(3.1b) then requires the solution of the saddle point problem: Find $(\mathbf{u}_h, \lambda_h) \in \mathbf{X}_h \times \mathbf{M}_h(S)$ such that

$$\begin{aligned} \mathbf{A}_h(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{B}_h(\mathbf{v}_h, \lambda_h) &= \ell(\mathbf{v}_h) \quad \text{for } \mathbf{v}_h \in \mathbf{X}_h, \\ \mathbf{B}_h(\mathbf{u}_h, \mu_h) &= 0 \quad \text{for } \mu_h \in \mathbf{M}_h(S), \end{aligned} \quad (5.21)$$

where the bilinear forms $\mathbf{A}_h(\cdot, \cdot) : \mathbf{X}_h \times \mathbf{X}_h \rightarrow \mathbb{R}$ and $\mathbf{B}_h(\cdot, \cdot) : \mathbf{X}_h \times \mathbf{M}_h(S) \rightarrow \mathbb{R}$ are given by the restriction of $\mathbf{A}(\cdot, \cdot)$ and $\mathbf{B}(\cdot, \cdot)$ to $\mathbf{X}_h \times \mathbf{X}_h$ and $\mathbf{X}_h \times \mathbf{M}_h(S)$, respectively.

Proposition 5.1. *The mortar edge element approximation (5.21) admits a unique solution $(\mathbf{u}_h, \lambda_h) \in \mathbf{X}_h \times \mathbf{M}_h(S)$.*

Proof. As has been shown in [48], the bilinear form $\mathbf{A}_h(\cdot, \cdot)$ is elliptic on the kernel of the operator associated with the bilinear form $\mathbf{B}_h(\cdot, \cdot)$ and that $\mathbf{B}_h(\cdot, \cdot)$ satisfies the inf-sup condition

$$0 < \beta \leq \inf_{\mu_h \in \mathbf{M}_h(S)} \sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{\mathbf{B}_h(\mathbf{v}_h, \mu_h)}{\|\mathbf{v}_h\|_{\mathbf{X}_h} \|\mu_h\|_{\mathbf{M}_h(S)}} .$$

This concludes the proof. \square

In the framework of Section 3, with the minimizer $\tilde{\mathbf{u}}_h \in \mathbf{V}$ of the consistency error ξ as given by (4.3) and $\tilde{\mathbf{p}}_h := \mu^{-1} \text{curl } \tilde{\mathbf{u}}_h$ we find

$$\|(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{p} - \tilde{\mathbf{p}}_h)\|_{\mathbf{V} \times \mathbf{Q}} \approx \|\mathbf{Res}_2\|_{\mathbf{V}^*}, \quad (5.22)$$

where

$$\mathbf{Res}_2(\mathbf{v}) = \sum_{i=1}^N \mathbf{Res}_2^{(i)}(\mathbf{v}), \quad (5.23)$$

$$\mathbf{Res}_2^{(i)}(\mathbf{v}) := (\mathbf{f}, \mathbf{v})_{0, \Omega_i} - (\mu^{-1} \text{curl } \tilde{\mathbf{u}}_h, \text{curl } \mathbf{v})_{0, \Omega_i} - (\sigma \tilde{\mathbf{u}}_h, \mathbf{v})_{0, \Omega_i}.$$

Denoting by $\mathbf{Nd}_{1,0}(\Omega_i; \mathcal{T}_{h_i})$ the subspace of $\mathbf{Nd}_1(\Omega_i; \mathcal{T}_{h_i})$ with vanishing tangential trace on $\partial\Omega_i$, a comparison with (5.21) shows that, for $\mathbf{v}_h \in \prod_{i=1}^N \mathbf{Nd}_{1,0}(\Omega_i; \mathcal{T}_{h_i})$, it holds

$$\mathbf{Res}_2(\mathbf{v}_h) = \sum_{i=1}^N \mathbf{Res}_2^{(i)}(\mathbf{v}_h), \quad (5.24)$$

$$\mathbf{Res}_2^{(i)}(\mathbf{v}_h) := (\sigma(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \mathbf{v}_h)_{0, \Omega_i} + (\mu^{-1} \text{curl}_h(\mathbf{u}_h - \tilde{\mathbf{u}}_h), \text{curl } \mathbf{v}_h)_{0, \Omega_i}).$$

Proposition 5.2. *Let η consist of element residuals η_T and face residuals η_F according to*

$$\eta^2 := \sum_{i=1}^N \left(\sum_{T \in \mathcal{T}_i} \eta_T^2 + \sum_{F \in \mathcal{F}_h(\Omega_i)} \eta_F^2 \right), \quad (5.25)$$

where η_T and η_F are given by

$$\begin{aligned} \eta_T &:= h_T \|\mathbf{f} - \text{curl} \mu^{-1} \text{curl} \mathbf{u}_h - \sigma \mathbf{u}_h\|_{0, T} + h_T \|\text{div}(\sigma \mathbf{u}_h)\|_{0, T}, \\ \eta_F &:= h_F^{1/2} \|[\pi_t(\mathbf{p}_h)]\|_{0, F} + h_F^{1/2} \|\mathbf{n}_F \cdot [\sigma \mathbf{u}_h]\|_{0, F}. \end{aligned}$$

Then, it holds

$$\|(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{p} - \tilde{\mathbf{p}}_h)\|_{\mathbf{V} \times \mathbf{Q}} \lesssim \eta + \xi. \quad (5.26)$$

Proof. In view of (5.24) we define

$$\mathbf{Res}_3 := \sum_{i=1}^N \mathbf{Res}_3^{(i)},$$

$$\mathbf{Res}_3^{(i)} := \mathbf{Res}_2^{(i)} - \left((\sigma(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \cdot)_{0, \Omega_i} + (\mu^{-1} (\text{curl}_h(\mathbf{u}_h - \tilde{\mathbf{u}}_h), \text{curl } \cdot)_{0, \Omega_i}) \right).$$

Since $\mathbf{Nd}_{1,0}(\Omega_i; \mathcal{T}_{h_i}) \subset \text{Ker } \mathbf{Res}_3^{(i)}$, a subdomainwise application of Proposition 3.2 yields

$$\|\mathbf{Res}_3\|_{\mathbf{V}^*} \lesssim \eta .$$

Hence, it follows that

$$\|\mathbf{Res}_2\|_{\mathbf{V}^*} \lesssim \eta + \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{0,\Omega} + \|\mathbf{curl}_h \mathbf{u}_h - \mathbf{curl} \tilde{\mathbf{u}}_h\|_{0,\Omega} = \eta + \xi .$$

□

An upper bound $\bar{\xi}$ for the consistency error ξ can be derived using the techniques from [31]. In particular, we obtain

$$\bar{\xi}^2 := \sum_{j=1}^N \sum_{F \in \mathcal{F}_h(\delta_{m(j)})} \left(\eta_F^2 + \hat{\eta}_F^2 \right)$$

with additional face residuals

$$\hat{\eta}_F := h_F^{1/2} \|\lambda_h - \{\pi_t(\mathbf{p}_h)\}\|_{0,F} + h_F^{1/2} \|\lambda_h - \{\mathbf{n}_F \cdot \boldsymbol{\sigma} \mathbf{u}_h\}\|_{0,F} + h_F^{-1/2} \|[\gamma_t(\mathbf{u}_h)]\|_{0,F} .$$

Here, $\lambda_h \in H^{-1/2}(\gamma_m)$ satisfies

$$\langle \lambda_h, \mathbf{curl}_\tau \boldsymbol{\varphi} \rangle_{-1/2, \gamma_m} = - \langle \lambda_h, \boldsymbol{\varphi} \rangle_{-1/2, \gamma_m} \quad \text{for all } \boldsymbol{\varphi} \in H^{1/2}(\gamma_m) . \quad (5.27)$$

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