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Global controllability properties of linear control systems

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ABSTRACT

For linear control systems with bounded control range, the state space is compactified using the Poincaré sphere. The linearization of the induced control flow allows the construction of invariant manifolds on the sphere and of corresponding manifolds in the state space of the linear control system.

1. Introduction

We study global controllability properties of linear control systems on $\mathbb{R}^n$ with control restrictions of the form

$$\dot{x}(t) = Ax(t) + Bu(t),\ u \in \mathcal{U}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and the set of control functions is defined by

$$\mathcal{U} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \,|\, u(t) \in U \text{ for almost all } t \in \mathbb{R}\}; \tag{2}$$

here the control range U is a compact and convex neighborhood of $0 \in \mathbb{R}^m$. We denote the solution for initial condition $x(0) = x_0 \in \mathbb{R}^n$ and control $u \in \mathcal{U}$ by $\varphi(t, x_0, u)$, $t \in \mathbb{R}$.

Note that the convexity assumption for U is not a restriction since for a compact control range U the trajectories for controls taking values in the convex hull of U can, uniformly on bounded intervals, be approximated by trajectories for controls with values in U (cf. Lee and Markus [1, Theorem 1 A, p. 164] for this classical result).

If the Kalman rank condition $\operatorname{rank}[B\ \ AB \cdots A^{n-1}B] = n$ holds, there is a unique maximal set D, where complete controllability holds, and D contains the origin in the interior. The present paper analyzes global controllability properties based on a compactification of $\mathbb{R}^n$ by a variant of the classical Poincaré compactification from the theory of polynomial differential equations.

The basic geometric idea is quite simple. A copy of $\mathbb{R}^n$ is attached to the sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ at the north pole. Then one takes the central projection in $\mathbb{R}^{n+1}$ to the northern hemisphere $\mathbb{S}^{n,+}$ of $\mathbb{S}^n$ called the Poincaré sphere. The equator $\mathbb{S}^{n,0}$ of $\mathbb{S}^n$ represents infinity since for points in $\mathbb{R}^n$ with $\|x_k\| \to \infty$ the images in $\mathbb{S}^n$ approach $\mathbb{S}^{n,0}$. The local analysis of points on $\mathbb{S}^{n,0}$ allows us to arrive at conclusions about the behavior of the original system "near infinity". In order to work this out, the machinery of control flows is helpful: The (open loop) behavior of control systems is described by a continuous flow Φ on $\mathcal{U} \times \mathbb{R}^n$ and tools from dynamical systems theory, in particular, the Selgrade decomposition can be invoked.

The approach of the present paper is mainly motivated by techniques from Poincaré compactification in the theory of polynomial differential equations due to Poincaré [2]; cf. Cima and Llibre [3], Perko [4], Dumortier, Llibre, and Artes [5], Llibre and Teruel [6]. We use the smooth structure and not only the topological properties of the associated flows as in the earlier paper Colonius, Santana, and Viscovini [7]. We expect that some of these techniques will also be useful when applied to affine control systems and to polynomial control systems.

The theory of control flows, control sets, and chain control sets is developed in Colonius and Kliemann [8] and Kawan [9]. For further contributions we refer to Ayala, da Silva, and Mamani [10], da Silva [11], Boarotto and Sigalotti [12], Tao, Huang, and Chen [13]. Cannarsa and Sigalotti [14] show that approximate controllability for bilinear control systems is equivalent to exact controllability. The Selgrade decomposition for linear flows on vector bundles is due to Selgrade [15]; cf. Salamon and Zehnder [16] and Colonius and Kliemann [8,17]. For linear flows with chain transitive base space, Selgrade's theorem provides a Whitney decomposition of the vector bundle into subbundles

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such that the projections of the subbundles to the projective bundle yield the maximal chain transitive sets of the induced projective flow; cf. e.g. [17, Theorem 9.2.5]. We will lift linear control systems of the form (1) to bilinear control systems on $\mathbb{R}^{n+1}$ and derive Selgrade decompositions for the lifted linear control flow Φ^1 on $\mathcal{U}\times\mathbb{R}^{n+1}$ as well as for the linearization of the projected control flow $\pi\Phi^1$ on $\mathcal{U}\times\mathbb{S}^n$. Furthermore, corresponding invariant manifolds in $\mathbb{S}^n$ and $\mathbb{R}^n$ are constructed. For some pertinent references concerning invariant manifolds, see the introduction of Section 5.

The main results of this paper are Corollary 1 clarifying the limit behavior for time tending to infinity of trajectories on the Poincaré sphere $\mathbb{S}^n$. Theorem 7 determines the Lyapunov exponents for the induced control flow and Corollary 3 describes stable manifolds on $\mathbb{S}^n$. Consequences for the original linear control system in $\mathbb{R}^n$ are drawn in Theorem 9.

The contents of this paper are as follows. Section 2 contains preliminary results on control sets, chain control sets, and control flows for control-affine systems on manifolds. Results from Colonius, Santana, and Viscovini [7] for linear control systems are recalled. Section 3 uses the Selgrade decomposition of the lifted control flow Φ^1 on $\mathcal{U}\times\mathbb{R}^{n+1}$ to characterize the limit behavior of trajectories on the Poincaré sphere $\mathbb{S}^n$ by the chain control sets. There are one or two chain control sets which are not subsets of the equator $\mathbb{S}^{n,0}$. Each Lyapunov space $L(\lambda_i)$ of the matrix A yields one or two chain control sets ${}_{\mathbb{S}}L(\lambda_i)^\infty$ contained in the equator $\mathbb{S}^{n,0}$. In Section 4 the induced control flow $\pi\Phi^1$ on $\mathcal{U}\times\mathbb{S}^n$ is linearized. When the base space of the linearized flow $T\pi\Phi^1$ is restricted to $\mathcal{U}\times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty, \lambda_{i_0}\neq 0$, the corresponding Selgrade decomposition and the Lyapunov exponents are determined. Section 5 contains results on stable manifolds and Section 6 presents several examples.

Notation. The closure of a set A in a metric space is denoted by $\overline{A}$. The origin in $\mathbb{R}^n$ is 0_n and 0_1 is abbreviated by 0. The projection from $\mathbb{R}^n_0=\mathbb{R}^n\setminus\{0_n\}$ to the sphere $\mathbb{S}^{n-1}$ is $\pi x=\frac{x}{\|x\|}$ for $x\in\mathbb{R}^n_0$.

2. Preliminaries

For general nonlinear control systems, control sets, chain control sets, and control flows are defined and some of their properties are recalled. Then, for linear control systems, control sets and chain control sets are characterized.

2.1. Control sets, chain control sets, and control flows

Consider control-affine systems of the form

$$\dot{x}(t)=X_0(x(t))+\sum_{i=1}^m u_i(t)X_i(x(t)),\ u\in\mathcal{U}, \tag{3}$$

where $X_0,X_1,\dots,X_m$ are smooth (C^∞-)vector fields on a smooth manifold M and

$$\mathcal{U}=\{u\in L^\infty(\mathbb{R},\mathbb{R}^m)\,|u(t)\in U \text{ for almost all } t\in\mathbb{R}\};$$

here U is a compact convex neighborhood of $0\in\mathbb{R}^m$. We assume that for every control $u\in\mathcal{U}$ and every initial state $x_0\in M$ there exists a unique (Carathéodory) solution $\varphi(t,x_0,u), t\in\mathbb{R}$, with $\varphi(0,x_0,u)=x_0$. Background on linear and nonlinear control systems is provided by the monographs Sontag [18], Jurdjevic [19].

For $x\in M$ the controllable set $\mathbf{C}(x)$ and the reachable set $\mathbf{R}(x)$ are defined as

$$\mathbf{C}(x)=\{y\in M\,|\exists u\in\mathcal{U}\ \exists T>0: \varphi(T,y,u)=x\},$$
$$\mathbf{R}(x)=\{y\in M\,|\exists u\in\mathcal{U}\ \exists T>0: y=\varphi(T,x,u)\}.$$

The following definition introduces sets of complete approximate controllability.

Definition 1. A nonvoid set $D\subset M$ is called a control set of system (3) if it has the following properties: (i) for all $x\in D$ there is a control $u\in\mathcal{U}$ such that $\varphi(t,x,u)\in D$ for all $t\geq 0$, (ii) for all $x\in D$ one has $D\subset\overline{\mathbf{R}(x)}$, and (iii) D is maximal with these properties, that is, if $D'\supset D$ satisfies conditions (i) and (ii), then $D'=D$.

Next we introduce a notion of controllability in infinite time allowing for (small) jumps between pieces of trajectories. We fix a metric d compatible with the topology of M.

Definition 2. Let $x,y\in M$. For $\varepsilon,\tau>0$ a controlled (ε,τ)-*chain* ζ from x to y is given by $k\in\mathbb{N},\ x_0=x,x_1,\dots,x_k=y\in M,\ u_0,\dots,u_{k-1}\in\mathcal{U}$, and $T_0,\dots,T_{k-1}\geq\tau$ with

$$d(\varphi(T_j,x_j,u_j),x_{j+1})<\varepsilon \text{ for all } j=0,\dots,k-1.$$

If for every $\varepsilon,\tau>0$ there is a controlled (ε,τ)-chain from x to y, the point x is chain controllable to y.

The chain reachable set is

$$\mathbf{R}^c(x)=\{y\in M\,|x \text{ is chain controllable to } y\}. \tag{4}$$

In analogy to control sets, we define chain control sets as maximal chain controllable sets.

Definition 3. A nonvoid set $E\subset M$ is called a *chain control set* of system (3) if for all $x,y\in E$ and $\varepsilon,\tau>0$ there is a controlled (ε,τ)-chain from x to y, and E is maximal with this property.

The control flow associated with control system (3) is the flow on $\mathcal{U}\times M$ defined by

$$\Phi:\mathbb{R}\times\mathcal{U}\times M\to\mathcal{U}\times M, \Phi_t(u,x)=(\theta_t u,\varphi(t,x,u)), \tag{5}$$

where $\theta_t u=u(t+\cdot)$ is the right shift on $\mathcal{U}$. Note that $\Phi_{t+\tau}=\Phi_t\circ\Phi_\tau$ for $t,\tau\in\mathbb{R}$, due to the cocycle property $\varphi(t+\tau,x,u)=\varphi(t,\varphi(\tau,x,u),u(\tau+\cdot))$. The space $\mathcal{U}$ is a compact metrizable space with respect to the weak* topology of $L^\infty(\mathbb{R},\mathbb{R}^m)$ (we fix such a metric) and the shift flow θ is continuous; cf. Kawan [9, Proposition 1.15]. Furthermore, the flow Φ is continuous; cf. [9, Proposition 1.17].

A chain transitive set of the control flow is a subset $\mathcal{E}$ of $\mathcal{U}\times M$ such that for all $(u,x),(v,y)\in\mathcal{E}$ and all $\varepsilon,\tau>0$ there is an (ε,τ)-chain ζ for Φ from (u,x) to (v,y) given by $k\in\mathbb{N}$, $T_0,\dots,T_{k-1}\geq\tau$, and $(u_0,x_0)=(u,x),(u_1,x_1),\dots,(u_k,x_k)=(v,y)\in\mathcal{U}\times M$ with $d(\Phi(T_i,u_i,x_i),(u_{i+1},x_{i+1}))<\varepsilon$ for $i=0,\dots,k-1$. The relation between chain control sets and the control flow Φ defined in (5) has been proved in Colonius and Kliemann [8, Theorem 4.3.11]. In Colonius, Santana, and Viscovini [7, Theorem 2.15] it is explained why the assumptions on invariance and compactness are not necessary.

Theorem 1. *Let $\mathcal{E}\subset\mathcal{U}\times M$ be a maximal chain transitive set for the control flow Φ. Then it follows that $\{x\in M\,|\exists u\in\mathcal{U}:(u,x)\in\mathcal{E}\}$ is a chain control set. Conversely, if $E\subset M$ is a chain control set, then*

$$\mathcal{E}:=\{(u,x)\in\mathcal{U}\times M\,|\varphi(t,x,u)\in E \text{ for all } t\in\mathbb{R}\} \tag{6}$$

is a maximal chain transitive set.

2.2. The control sets and chain control sets of linear control systems

First we consider the uncontrolled equation $\dot{x}=Ax$. Let $\psi(t,x_0)=e^{At}x_0, t\in\mathbb{R}$, be the generated linear flow on $\mathbb{R}^n$. The Lyapunov exponents $\lambda(x_0):=\lim_{t\to\pm\infty}\frac{1}{t}\log\|\psi(t,x_0)\|$ are equal to the real parts of the eigenvalues of A and the Lyapunov spaces $L(\lambda_i)$ are the sums of the (real) generalized eigenspaces for eigenvalues with real parts λ_i. The following theorem characterizes the Lyapunov spaces of $\dot{x}=Ax$ by the maximal chain transitive sets of the induced flow on projective space.

Theorem 2. *Let* $\mathbb{P}\psi$ *be the projection onto* $\mathbb{P}^{n-1}$ *of the linear flow* $\psi(t,x) = e^{At}x$ *on* $\mathbb{R}^n$ *and denote the Lyapunov exponents by* $\lambda_1 > \cdots > \lambda_\ell, 1 \le \ell \le n$.

(i) Then the state space $\mathbb{R}^n$ *can be decomposed into the Lyapunov spaces* $L(\lambda_i)$,

$$\mathbb{R}^n = L(\lambda_1) \oplus \cdots \oplus L(\lambda_\ell). \tag{7}$$

(ii) The projections $\mathbb{P}L(\lambda_i)$ *to* $\mathbb{P}^{n-1}$ *of the Lyapunov spaces* $L(\lambda_i)$ *are the maximal chain transitive sets of the induced flow* $\mathbb{P}\psi$ *on* $\mathbb{P}^{n-1}$.

Assertion (i) is clear. A proof of assertion (ii) is contained in Colonius and Kliemann [17, Section 4.1]. A simpler proof can be given, when one uses that the center Lyapunov space $L(0)$ of $\dot{x} = Ax$ is a maximal chain transitive set if 0 is a Lyapunov exponent; cf. Colonius, Santana, and Viscovini [7, Theorem 3.2]. The other Lyapunov spaces $L(\lambda_i)$ are maximal chain transitive sets for the shifted systems $\dot{x} = (A - \lambda_i I)x$. Then also the projections $\mathbb{P}L(\lambda_j)$ to $\mathbb{P}^{n-1}$ are chain transitive.

It is convenient to introduce the following notation. Define L^0 as the Lyapunov space $L(0)$ if 0 is a Lyapunov exponent and $L^0 := \{0\}$ otherwise. The space $\mathbb{R}^n$ can be decomposed into the sum of the unstable, center, and stable subspaces of A given by

$$\mathbb{R}^n = L^+ \oplus L^0 \oplus L^-, \tag{8}$$

where $L^+ = \bigoplus_{\lambda_i > 0} L(\lambda_i)$ and $L^- = \bigoplus_{\lambda_i < 0} L(\lambda_i)$. Denote the associated projections by $\pi^0 : \mathbb{R}^n \to L^0$ and $\pi^h : \mathbb{R}^n \to L^+ \oplus L^-$.

Consider the differential equation induced on the hyperbolic subspace $L^+ \oplus L^-$,

$$\dot{y}(t) = A\pi^h y(t) + \pi^h Bu(t). \tag{9}$$

It is well known that, for every $u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$, this inhomogeneous hyperbolic differential equation has a unique bounded solution $e(u,t), t \in \mathbb{R}$. (For example, the unique bounded solution of an equation on L^- is $e(u,t) = \int_{-\infty}^t e^{A(t-s)}\pi^- Bu(s)ds, t \in \mathbb{R}$, where π^- is the projection to L^-.) Note that $e(u,t) = e(u(t+\cdot),0)$ for $t \in \mathbb{R}$ and $u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$.

The following theorem characterizes the control set containing $0 \in \mathbb{R}^n$ and the unique chain control set of a linear control system.

Theorem 3. *Consider the linear control system* (1).

(i) There is a control set D_0 *with* $0 \in D_0$. *It is convex and satisfies*

$$D_0 = \left(\mathbf{C}(0) \cap L^+\right) \oplus \left(L^0 \cap \mathrm{Im}[B\ AB \cdots A^{n-1}B]\right) \oplus \left(\overline{\mathbf{R}(0)} \cap L^-\right).$$

The sets $\mathbf{C}(0) \cap L^+$ *and* $\overline{\mathbf{R}(0)} \cap L^-$ *are bounded. If* $\mathrm{Im}[B\ AB \cdots A^{n-1}B] = \mathbb{R}^n$, *then there is a unique control set* D *with nonvoid interior, and* $D = D_0$.

(ii) There exists a unique chain control set E. *It is given by*

$$E = \overline{D_0} + L^0 = (\overline{\mathbf{C}(0)} \cap L^+) \oplus L^0 \oplus (\overline{\mathbf{R}(0)} \cap L^-)$$
$$= \left\{ e(u,t) + y \,\middle|\, u \in \mathcal{U}, t \in \mathbb{R}, \text{ and } y \in L^0 \right\}.$$

Proof. Assertion (i) is Colonius, Santana, and Viscovini [7, Corollary 2.17]. The first two equalities in (ii) hold by [7, Theorem 4.8]. Finally, [7, Lemma 2.18(iii)] shows that $(\overline{\mathbf{C}(0)} \cap L^+) \oplus (\overline{\mathbf{R}(0)} \cap L^-)$ is the chain control set of system (9) with $u \in \mathcal{U}$ and hence bounded. Thus it consists of the points $e(u,t)$ with $t \in \mathbb{R}$ and $u \in \mathcal{U}$. □

3. Selgrade decomposition for the lifted control flow

In this section, we embed linear control systems of the form (1) into bilinear control systems on $\mathbb{R}^n \times \mathbb{R}$, which can be projected to the Poincaré sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. The associated control flow Φ^1 is a linear flow on the vector bundle $\mathcal{U} \times \mathbb{R}^{n+1}$. It admits a Selgrade decomposition into invariant subbundles and we determine the exponential growth rates of solutions. The subbundles yield the chain control sets of the projected control system on $\mathbb{S}^n$, which are the possible limit sets for time tending to infinity.

Recall from Colonius, Santana, and Viscovini [7] the following construction. Linear control systems of the form (1) on $\mathbb{R}^n$ can be lifted to bilinear control systems with states $(x(t), x_{n+1}(t))$ in $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ by

$$\dot{x}(t) = Ax(t) + x_{n+1}(t)Bu(t),\ \dot{x}_{n+1}(t) = 0,\ u \in \mathcal{U}. \tag{10}$$

The solutions for initial condition $\left(x(0), x_{n+1}(0)\right) = \left(x_0, r\right) \in \mathbb{R}^n \times \mathbb{R}$, may be written as

$$\varphi^1(t, x_0, r, u) = \left(e^{At}x_0 + r\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, r\right),\ t \in \mathbb{R}. \tag{11}$$

Observe that, for $r = 0$, one has $\varphi^1(t,x,0,u) = (\psi(t,x),0)$ and, for $r = 1$, one has $\varphi^1(t,x,1,u) = (\varphi(t,x,u),1)$ for all $t \in \mathbb{R}, x \in \mathbb{R}^n, u \in \mathcal{U}$. Hence, on the hyperplane $\mathbb{R}^n \times \{0\}$ one obtains a copy of the differential equation $\dot{x} = Ax$ and on the affine hyperplane $\mathbb{R}^n \times \{1\}$ one obtains a copy of control system (1).

Control system (10) is a bilinear control system on $\mathbb{R}^{n+1}$ which, with $b_i, i = 1, \ldots, m$, denoting the ith column of B, may be written as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}_{n+1}(t) \end{pmatrix} = \left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=1}^m u_i(t)\begin{pmatrix} 0 & b_i \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} x(t) \\ x_{n+1}(t) \end{pmatrix}. \tag{12}$$

Define subsets of projective space $\mathbb{P}^n$ and of the unit sphere $\mathbb{S}^n$ by

$$\mathbb{P}^{n,0} = \{\mathbb{P}(x,0) \,|\, x \in \mathbb{R}^n_0\},\ \mathbb{P}^{n,1} = \{\mathbb{P}(x,r) \,|\, x \in \mathbb{R}^n, r \neq 0\}, \tag{13}$$
$$\mathbb{S}^{n,+} := \{(x,r) \in \mathbb{S}^n \,|\, x \in \mathbb{R}^n, r > 0\},\ \mathbb{S}^{n,0} = \{(x,0) \in \mathbb{S}^n \,|\, x \in \mathbb{R}^n_0\},$$

respectively. The northern hemisphere $\mathbb{S}^{n,+}$ can be identified with $\mathbb{P}^{n,1}$ via $\frac{(x,1)}{\|(x,1)\|} \sim \mathbb{P}(x,1)$. Furthermore, we call $\mathbb{P}^{n,0}$ the projective equator since it is the projection of the equator $\mathbb{S}^{n,0}$. Define the map

$$h^1 : \mathbb{R}^n \to \mathbb{P}^{n,1}, h^1(x) = \mathbb{P}(x,1), x \in \mathbb{R}^n. \tag{14}$$

Control system (10) induces a control flow Φ^1 on $\mathcal{U} \times \mathbb{R}^{n+1}$ defined by

$$\Phi^1_t(u,x,r) = (u(t+\cdot), \varphi^1(t,x,r,u)), t \in \mathbb{R}, (x,r) \in \mathbb{R}^{n+1},\ u \in \mathcal{U}. \tag{15}$$

The maps between the fibers $\{u\} \times \mathbb{R}^{n+1}$ are linear, hence Φ^1 is a linear flow. The projection of system (10) to projective space yields a projective control flow $\mathbb{P}\Phi^1$ on the projective Poincaré bundle $\mathcal{U} \times \mathbb{P}^n$. The subsets $\mathcal{U} \times \mathbb{P}^{n,0}$ and $\mathcal{U} \times \mathbb{P}^{n,1}$ are invariant under the flow $\mathbb{P}\Phi^1$. The following proposition (cf. [7, Proposition 4.2]) shows some properties of the flows on the projective Poincaré bundle.

Proposition 1. *(i) The projectivized flow* $\mathbb{P}\psi(t,p), p \in \mathbb{P}^{n-1}$, *of the uncontrolled system* $\dot{x} = Ax$ *and the flow* $\mathbb{P}\varphi^1(t,p,0), p \in \mathbb{P}^n$, *restricted to the projective equator* $\mathbb{P}^{n,0} \subset \mathbb{P}^n$ *are conjugate by the analytical isometry* $e_{\mathbb{P}}(p) = \mathbb{P}(x,0)$ *for* $p = \mathbb{P}x \in \mathbb{P}^{n-1}$.

(ii) The map

$$\left(\mathrm{id}_{\mathcal{U}}, h^1\right) : \mathcal{U} \times \mathbb{R}^n \to \mathcal{U} \times \mathbb{P}^{n,1}, (u,x) \mapsto (u, \mathbb{P}(x,1)), \tag{16}$$

is a conjugacy of the flows Φ *on* $\mathcal{U} \times \mathbb{R}^n$ *and* $\mathbb{P}\Phi^1$ *restricted to* $\mathcal{U} \times \mathbb{P}^{n,1}$.

Recall that, for every $u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$, the induced differential Eq. (9) on $L^+ \oplus L^-$ has a unique bounded solution $e(u,t), t \in \mathbb{R}$. Note the following lemma.

Lemma 1. *For* $u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ *and* $y \in L^0$, *one obtains the exponential growth rate*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \left\| e^{At}(e(u,0) + y) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma \right\| = 0. \tag{17}$$

Proof. Since $e(u,t), t \in \mathbb{R}$, is a solution of (9) it satisfies

$$e(u,t) = e^{A\pi^h t}e(u,0) + \int_0^t e^{A(t-\sigma)}\pi^h Bu(\sigma)d\sigma \in L^+ \oplus L^-.$$

We compute

$$e^{At}(e(u,0) + y) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma$$

$$= e^{A\pi^h t}e(u,0) + e^{A\pi^0 t}y + \int_0^t e^{A(t-\sigma)}\pi^h Bu(\sigma)d\sigma + \int_0^t e^{A(t-\sigma)}\pi^0 Bu(\sigma)d\sigma$$
$$= e(u,t) + e^{A\pi^0 t}y + \int_0^t e^{A\pi^0\sigma}Bu(t-\sigma)d\sigma.$$

Here $e(u,t)$ is bounded and the components of $e^{A\pi^0 t}$ are polynomials in t. Since $u \in {}^{\vee\infty}(\mathbb{R},\mathbb{R}^m)$ also the components of $\int_0^t e^{A\pi^0\sigma}Bu(t-\sigma)d\sigma$ can be estimated by polynomials. This implies the claim. □

The following theorem determines the Selgrade decomposition of the associated linear control flow Φ^1 on $\mathcal{U}\times\mathbb{R}^{n+1}$. Denote $L(\lambda_i)^\infty := L(\lambda_i)\times\{0\}\subset\mathbb{R}^{n+1}$.

Theorem 4. *Consider the linear control flow Φ^1 on $\mathcal{U}\times\mathbb{R}^{n+1}$ associated to the lift (12) of a linear control system of the form (1).*

(i) Then Φ^1 has the Selgrade decomposition

$$\mathcal{U}\times\mathbb{R}^{n+1} = \bigoplus_{\lambda_i>0}\left(\mathcal{U}\times L(\lambda_i)^\infty\right)\oplus\mathcal{V}_c\oplus\bigoplus_{\lambda_i<0}\left(\mathcal{U}\times L(\lambda_i)^\infty\right), \tag{18}$$

where $\mathcal{V}_c$ is the central Selgrade bundle

$$\mathcal{V}_c = \left\{(u, re(u,0)+y, r)\,\middle|\, u\in\mathcal{U}, y\in L^0, r\in\mathbb{R}\right\}. \tag{19}$$

(ii) The dimensions of the subbundles of $\mathcal{U}\times\mathbb{R}^{n+1}$ are $\dim(\mathcal{U}\times L(\lambda_i)^\infty) = \dim L(\lambda_i)$ for all i and $\dim\mathcal{V}_c = 1+\dim L^0$.

(iii) For all $\lambda_i\neq 0$ and every $(u,x,0)\in\mathcal{U}\times L(\lambda_i)^\infty$ the Lyapunov exponent is λ_i and for every $(u,x,r)\in\mathcal{V}_c$ the Lyapunov exponent is 0.

Proof. (i), (ii) By Colonius, Santana, and Viscovini [7, Theorem 4.3], the lifted flow Φ^1 has the Selgrade decomposition

$$\mathcal{U}\times\mathbb{R}^{n+1} = \left(\mathcal{U}\times L(\lambda_1)^\infty\right)\oplus\cdots\oplus\left(\mathcal{U}\times L(\lambda_{\ell^+})^\infty\right)\oplus\mathcal{V}_c\oplus$$
$$\oplus\left(\mathcal{U}\times L(\lambda_{\ell^+ +\ell^0+1})^\infty\right)\oplus\cdots\oplus\left(\mathcal{U}\times L(\lambda_\ell)^\infty\right)$$

with $\lambda_{\ell^+}>0$ and $\lambda_{\ell^+ +\ell^0+1}<0$ (we use a different numbering of the Lyapunov exponents). The assertion on the dimension of $\mathcal{U}\times L(\lambda_i)^\infty$ is obvious. For general affine flows on vector bundles and their lifts to linear flows, the central Selgrade bundle $\mathcal{V}_c$ has been described in Colonius and Santana [20, Theorem 25]. This is specialized in [7, Theorem 5.3] for the case of linear control systems providing also the formula for the dimension. A sign mistake in [20, Theorem 25] is corrected in Colonius and Santana [21]. The result is formula (19). It follows that $\ell^0=0$ and L^0 is trivial, if 0 is not a Lyapunov exponent, and $\ell^0=1$ otherwise. This proves assertions (i) and (ii). In (iii), the assertion for $\mathcal{U}\times L(\lambda_i)^\infty$ is clear. For $(u, re(u,0)+y, r)\in\mathcal{V}_c$ we may replace y by ry. Eq. (11) yields

$$\frac{1}{t}\log\left\|\varphi^1(t, re(u,0)+ry, r, u)\right\|$$
$$= \frac{1}{t}\log|r| + \frac{1}{t}\log\left\|\left(e^{At}(e(u,0)+y) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, 1\right)\right\|.$$

Now the assertion follows by Lemma 1. □

The chain control sets in the projective Poincaré sphere are determined by [7, Corollary 4.4 and Corollary 5.4] and Theorem 2 . Recall from (14) that $h^1(x) = \mathbb{P}(x,1), x\in\mathbb{R}^n$.

Theorem 5. *(i) For the induced system on the projective Poincaré sphere $\mathbb{P}^n$ there is a unique chain control set ${}_{\mathbb{P}}E_c$ with ${}_{\mathbb{P}}E_c\cap\mathbb{P}^{n,1}\neq\emptyset$. It is given by $h^1(E)$, where E is the unique chain control set of (1).*

(iii) The other chain control sets on $\mathbb{P}^n$ are contained in the projective equator $\mathbb{P}^{n,0}$ and are given by $\mathbb{P}(L(\lambda_i)^\infty)$ for $\lambda_i\neq 0$.

Any bilinear control system can be projected to the unit sphere. For system (12) on $\mathbb{R}^{n+1}$ abbreviate

$$A_0 := \begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix} \text{ and } A_i = \begin{pmatrix} 0 & b_i\\ 0 & 0\end{pmatrix} \text{ for } i=1,\dots,m.$$

Then the induced system on $\mathbb{S}^n$ is described by

$$\dot{s}(t) = h_0(s(t)) + \sum_{i=1}^m u_i(t)h_i(s(t)), u\in\mathcal{U}, \tag{20}$$

where $h_i(s) = \left[A_i - s^\top A_i s\cdot I_n\right]s$ for $i=0,1,\dots,m$.

Denote by $\pi:\mathbb{R}_0^{n+1}=\mathbb{R}^{n+1}\setminus\{0\}\to\mathbb{S}^n, y\mapsto y/\|y\|$, the canonical projection. Note that the cocycle φ^1 defined in (15) satisfies

$$\pi\varphi^1(t,\pi(x,r),u) = \frac{\varphi^1(t,x,r,u)}{\left\|\varphi^1(t,x,r,u)\right\|}. \tag{21}$$

The relations between the chain control sets on projective space and on the sphere are described by Colonius and Santana [22, Theorem 8]. This yields the following.

Theorem 6. *Consider the bilinear control system (12) on $\mathbb{R}^{n+1}$ and the chain control set ${}_{\mathbb{P}}E_c$ of the induced control system on $\mathbb{P}^n$.*

(i) The set ${}_{\mathbb{S}}E_c^0 := \{s\in\mathbb{S}^n\,|\mathbb{P}s\in{}_{\mathbb{P}}E_c\}$ is the unique chain control set in $\mathbb{S}^n$ which projects onto ${}_{\mathbb{P}}E_c$ if and only if there is $s_0\in\mathbb{S}^n$ with $\mathbb{P}s_0\in{}_{\mathbb{P}}E_c$ and $-s_0\in\mathbf{R}^c(s_0)$.

(ii) There are two chain control sets ${}_{\mathbb{S}}E_c^1 = -{}_{\mathbb{S}}E_c^2$ projecting onto ${}_{\mathbb{P}}E_c$ with

$${}_{\mathbb{S}}E_c^1\cup{}_{\mathbb{S}}E_c^2 = \{s\in\mathbb{S}^n\,|\mathbb{P}s\in{}_{\mathbb{P}}E_c\},$$

if and only if for all $s_0\in\mathbb{S}^n$ with $\mathbb{P}s_0\in{}_{\mathbb{P}}E_c$ it holds that $-s_0\notin\mathbf{R}^c(s_0)$.

(iii) Define for $j\in\{0,1,2\}$ and $u\in\mathcal{U}$

$$F^j(u) := \left\{x\in\mathbb{R}^{n+1}\,\middle|\, x=0 \text{ or } \pi\varphi(t,x,u)\in{}_{\mathbb{S}}E_c^j \text{ for all } t\in\mathbb{R}\right\}.$$

In case (i) it follows that $F^0(u)$ is a linear subspace. In case (ii) it follows for $j=1,2$ that $F^j(u)$ is a convex cone.

Thus there are one or two chain control sets ${}_{\mathbb{S}}E_c^j$, $j=0$ or $j=1,2$, on the Poincaré sphere $\mathbb{S}^n$ projecting onto the central chain control set ${}_{\mathbb{P}}E_c$ in $\mathbb{P}^n$. They are not subsets of the equator $\mathbb{S}^{n,0}$ and can also be obtained by

$$\mathbb{S}^n\cap\left\{(x,r)\in\mathbb{R}^{n+1}\,\middle|\,\exists u\in\mathcal{U}:(u,x,r)\in\mathcal{V}_c\right\}. \tag{22}$$

Furthermore, the maximal chain transitive sets $\mathbb{P}(L(\lambda_i)), \lambda_i\neq 0$, of the flow $\mathbb{P}(\psi,0)$ yield the chain transitive sets $\mathcal{U}\times\mathbb{P}(L(\lambda_i)^\infty)$ of the flow $\mathbb{P}\Phi^1$ restricted to the projective equator $\mathbb{P}^{n,0}$. By [22, Theorem 8], each of them gives one or two maximal chain transitive sets $\mathcal{U}\times{}_{\mathbb{S}}L(\lambda_i)_j^\infty\subset\mathcal{U}\times\mathbb{S}^{n,0}$, $j=0$ or $j=1,2$. Here ${}_{\mathbb{S}}L(\lambda_i)_0^\infty$ is the unique subset of $\mathbb{S}^{n,0}$ that projects to $\mathbb{P}(L(\lambda_i)^\infty)$ or else ${}_{\mathbb{S}}L(\lambda_i)_1^\infty$ and ${}_{\mathbb{S}}L(\lambda_i)_2^\infty$ are two subsets of $\mathbb{S}^{n,0}$ that project to $\mathbb{P}(L(\lambda_j)^\infty)$. If, for example, λ_i is a simple real eigenvalue, the first case occurs, if λ_i is the real part of a complex conjugate pair of eigenvalues, the second case occurs.

We note the following corollary which clarifies the limit behavior of trajectories on the Poincaré sphere. The α- and ω-limit sets for a point $s_0\in\mathbb{S}^n$ and a control $u\in\mathcal{U}$ are

$$\alpha(s_0,u) := \left\{s\in\mathbb{S}^n\,\middle|\,\exists t_k\to-\infty:\pi\varphi^1(t_k,s_0,u)\to s \text{ for } k\to\infty\right\}, \tag{23}$$
$$\omega(s_0,u) := \left\{s\in\mathbb{S}^n\,\middle|\,\exists t_k\to\infty:\pi\varphi^1(t_k,s_0,u)\to s \text{ for } k\to\infty\right\}.$$

Corollary 1. *Consider the control system induced on the Poincaré sphere $\mathbb{S}^n$ by a linear control system of the form (1). For all $u\in\mathcal{U}$ and $s_0=\frac{(x_0,1)}{\|(x_0,1)\|}\in\mathbb{S}^{n,+}$, $x_0\in\mathbb{R}^n$, the limit sets $\alpha(s_0,u)$ and $\omega(s_0,u)$ of the trajectory $\pi\varphi^1(t,s_0,u)=\frac{(\varphi(t,x_0,u),1)}{\|(\varphi(t,x_0,u),1)\|}$ are contained in one of the central chain control sets ${}_{\mathbb{S}}E_c^j$, $j=0$ or $j=1,2$, or in one of the sets ${}_{\mathbb{S}}L(\lambda_i)_j^\infty\subset\mathbb{S}^{n,0}$, $j=0$ or $j=1,2$, $\lambda_i\neq 0$.*

Proof. For any flow on a compact metric space, it is well known that the α- and ω-limit sets are contained in the chain recurrent set defined as the set of points x such that for all $\varepsilon,\tau>0$ there is an (ε,τ)-chain from x to x; cf. e.g. Alongi and Nelson [23, Corollary 2.7.15] or Colonius and Kliemann [17, Proposition 3.1.12]. For the control flow associated with the control system on $\mathbb{P}^n$ the chain recurrent set is the union of the sets $\mathcal{U}\times\mathbb{P}L(\lambda_i)^\infty, \lambda_i\neq 0$, with the projected central subbundle $\mathbb{P}\mathcal{V}_c$, which coincides with the lift of the central chain control set ${}_{\mathbb{P}}E_c$ to the chain transitive set ${}_{\mathbb{P}}\mathcal{E}_c = \left\{(u,p) \in \mathcal{U} \times \mathbb{P}^n \,\middle|\, \mathbb{P}\varphi^1(t,p,u) \in {}_{\mathbb{P}}E_c \text{ for } t \in \mathbb{R}\right\}$. Define

$$
{}_{\mathbb{S}}\mathcal{E}_c^j := \left\{(u,s) \in \mathcal{U}\times\mathbb{S}^n \,\middle|\, \pi\varphi^1(t,s,u) \in {}_{\mathbb{S}}E_c^j \text{ for all } t \in \mathbb{R}\right\}, \quad (24)
$$

$$
\mathcal{U} \times {}_{\mathbb{S}}L(\lambda_i)_j^\infty := \left\{(u,s) \in \mathcal{U}\times\mathbb{S}^n \,\middle|\, \pi\varphi^1(t,s,u) \in {}_{\mathbb{S}}L(\lambda_i)_j^\infty \text{ for all } t \in \mathbb{R}\right\}. \quad (25)
$$

Thus, for the control flow associated with the control system on $\mathbb{S}^n$, the chain recurrent set is the union of the sets $\mathcal{U} \times {}_{\mathbb{S}}L(\lambda_i)_j^\infty, j = 0$ or $j = 1,2, \lambda_i \neq 0$, with ${}_{\mathbb{S}}\mathcal{E}_c^j, j = 0$ or $j = 1,2$. Since the limit sets (23) are the projections of the limit sets of the control flow the assertion follows. □

Corollary 1 shows that the possible limit sets of controlled trajectories are the central chain control sets ${}_{\mathbb{S}}E_c^j$ and the chain control sets ${}_{\mathbb{S}}L(\lambda_i)_j^\infty$ on the equator. On the level of control flows, the possible limit sets are the lifts of these chain control sets to $\mathcal{U}\times\mathbb{S}^n$. In the next section, we will determine the linearization about the sets (25).

4. Selgrade decomposition on the sphere

In this section we determine the Selgrade decomposition and the corresponding exponential growth rates for the linearization of the control flow induced by a linear control system on the Poincaré sphere. Since this is a Riemannian manifold, we have to be more careful concerning the tangent spaces. Some arguments are taken from Crauel [24, Section 3], who considers random differential systems.

It is convenient to endow $\mathbb{R}^n$ with a scalar product, which makes the Lyapunov spaces pairwise orthogonal, i.e., $\langle x, y\rangle = 0$ for $x \in L(\lambda_i), y \in L(\lambda_j)$ with $\lambda_i \neq \lambda_j$. On $\mathbb{R}^{n+1}$ we use the scalar product

$$
\left\langle (x, x_{n+1}), (y, y_{n+1})\right\rangle' = \langle x, y\rangle + x_{n+1}\cdot y_{n+1}, \text{ for } x, y \in \mathbb{R}^n \text{ and } x_{n+1}, y_{n+1} \in \mathbb{R}.
$$

This does not change the Lyapunov exponents since they are independent of the norm. In the following, we write also the scalar product in $\mathbb{R}^{n+1}$ as $\langle\cdot,\cdot\rangle$ since it will be clear from the context what is meant. The tangent bundle $T\mathbb{R}_0^{n+1}$ of $\mathbb{R}_0^{n+1}$ is trivial and is identified with $\mathbb{R}_0^{n+1} \times \mathbb{R}^{n+1}$. Similarly, we identify the projective bundle $P\mathbb{R}_0^{n+1}$ with $\mathbb{R}_0^{n+1} \times \mathbb{P}^n$. We consider the sphere $\mathbb{S}^n$ as an embedded compact n-dimensional submanifold of $\mathbb{R}_0^{n+1}$ and identify the tangent bundle $T\mathbb{S}^n$ with the subset of $T\mathbb{R}_0^{n+1}$ given by

$$
T\mathbb{S}^n = \left\{(s,v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \,|\, s \in \mathbb{S}^n \text{ and } \langle v, s\rangle = 0\right\}. \quad (26)
$$

Fix the Riemannian metric on $\mathbb{S}^n$ induced by this identification. Note that $T\mathbb{S}^n$ can be written as a set of pairs (s, v_s) where $s \in \mathbb{S}^n$ and $v_s \in T_s\mathbb{S}^n$. With $\pi : \mathbb{R}_0^{n+1} \to \mathbb{S}^n, \pi(x, x_{n+1}) := \frac{(x,x_{n+1})}{\|(x,x_{n+1})\|}$, the points $s \in \mathbb{S}^n$ may be written as $s = \pi(x, x_{n+1}), \|(x, x_{n+1})\| = 1$.

The induced control system on the Poincaré sphere $\mathbb{S}^n$ generates the control flow $\pi\Phi^1 : \mathbb{R}\times\mathcal{U}\times\mathbb{S}^n \to \mathcal{U}\times\mathbb{S}^n$ given by

$$
\pi\Phi_t^1(u,s) = (u(t+\cdot), \pi\varphi^1(t,s,u)), t \in \mathbb{R}, s = \pi(x, x_{n+1}), u \in \mathcal{U}. \quad (27)
$$

This flow can be linearized with respect to the component in $\mathbb{S}^n$, which yields the following continuous flow $T\pi\Phi^1 : \mathbb{R}\times\mathcal{U}\times T\mathbb{S}^n \to \mathcal{U}\times T\mathbb{S}^n$ given by

$$
T\,\big|_s\, \pi\Phi_t^1(u,s)(v, v_{n+1}) = \left(u(t+\cdot), \pi\varphi^1(t,s,u), D_s\pi\varphi^1(t,s,u)(v, v_{n+1})\right), \quad (28)
$$

where $u \in \mathcal{U}, s \in \mathbb{S}^n$, and $D_s\pi\varphi^1(t,s,u)(v, v_{n+1})$ means the derivative of $\pi\varphi^1(t,s,u)$ with respect to the second variable at the point s applied to $(v, v_{n+1}) \in T_s\mathbb{S}^n$ as a linear map.

This is a linear flow on the vector bundle $\mathcal{U}\times T\mathbb{S}^n$ with base space $\mathcal{U}\times\mathbb{S}^n$. For a Lyapunov space $L(\lambda_{i_0}), \lambda_{i_0} \neq 0$, of the matrix A, recall that $L(\lambda_{i_0})^\infty = L(\lambda_{i_0})\times\{0\} \subset \mathbb{R}^{n+1}$. By the remarks following Theorem 6, the set $L(\lambda_{i_0})^\infty \cap \mathbb{S}^n$ consists of one or two chain transitive sets denoted by $\mathcal{U}\times {}_{\mathbb{S}}L(\lambda_{i_0})_j^\infty \subset \mathcal{U}\times\mathbb{S}^{n,0}, j = 0$ or $j = 1,2$. We choose one of these sets and denote it, for some notational simplification, by ${}_{\mathbb{S}}L(\lambda_{i_0})^\infty$. Define

$$
T_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}\mathbb{S}^n := \bigcup_{s\in {}_{\mathbb{S}}L(\lambda_{i_0})^\infty} T_s\mathbb{S}^n \text{ and } P_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}\mathbb{S}^n := \bigcup_{s\in {}_{\mathbb{S}}L(\lambda_{i_0})^\infty} \mathbb{P}_s\mathbb{S}^n, \quad (29)
$$

where $\mathbb{P}_s\mathbb{S}^n := \{s\} \times \left\{\mathbb{P}(v, v_{n+1}) \,|\, (v, v_{n+1}) \in T_s\mathbb{S}^n\right\}$.

Proposition 2. *For the linearized flow (28), the base space can be restricted to the compact invariant set $\mathcal{U}\times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$. This results in the following flow defined on $\mathcal{U}\times T_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}\mathbb{S}^n$ given by*

$$
T\,\big|_{\pi(x,0)}\, \pi\Phi_t^1(u,s)(v, v_{n+1}) = \left(u(t+\cdot), \pi\varphi^1(t,x,0,u), D_s\pi\varphi^1(t,x,0,u)(v, v_{n+1})\right),
$$

for $t \in \mathbb{R}, u \in \mathcal{U}, s = \pi(x,0) = (x,0) \in {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$, and $(v, v_{n+1}) \in T_s\mathbb{S}^n$. This is a linear flow again denoted by $T\pi\Phi^1$ on a vector bundle with chain transitive base space $\mathcal{U}\times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ and Selgrade decomposition in the form

$$
\mathcal{V}_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty} := \mathcal{U}\times T_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}\mathbb{S}^n = {}_{\mathbb{S}}\mathcal{V}_1 \oplus \cdots \oplus {}_{\mathbb{S}}\mathcal{V}_k. \quad (30)
$$

Here the ${}_{\mathbb{S}}\mathcal{V}_i$ are invariant subbundles and their projections to $\mathcal{U}\times P_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}\mathbb{S}^n$ are the maximal chain transitive sets of the flow induced by $T\pi\Phi^1$.

Proof. The flows on the compact metric spaces $\mathcal{U}$ and ${}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ are chain transitive. By Alongi and Nelson [23, Theorem 2.7.18], for both flows, it suffices to consider chains with all jump times equal to 1. This implies that the product flow on $\mathcal{U}\times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ is chain transitive noting that the flow on ${}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ does not depend on the element in $\mathcal{U}$. (Alternatively, this also holds by the remarks following Theorem 6.) By Selgrade's theorem (cf. e.g. Colonius and Kliemann [17, Theorem 9.2.5]), every linear flow with chain transitive base space admits a Selgrade decomposition. □

In the following, we will determine the Selgrade bundles in (30).

Remark 1. For the linear flow $T\pi\Phi^1$ on $\mathcal{U}\times T\mathbb{S}^n$ the base space can also be restricted to ${}_{\mathbb{S}}\mathcal{E}_c^j \subset \mathcal{U}\times\mathbb{S}^n$ instead of $\mathcal{U}\times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$. Here also a Selgrade decomposition is valid since the flow on the compact metric space ${}_{\mathbb{S}}\mathcal{E}_c^j$ is chain transitive. However, the Selgrade bundles will have a more complicated structure and it would be more difficult to determine their stability properties; cf. Theorem 7.

Since π is a submersion, $T\pi : T\mathbb{R}_0^{n+1} \to T\mathbb{S}^n$ acts as a projection along $\ker(T\pi)$. By (26) and, with the particular choice of the Riemannian metric, $T\pi$ becomes an orthogonal projection, and hence, for $y \in \mathbb{R}_0^{n+1}$ and $v \in T_y\mathbb{R}_0^{n+1}$,

$$
(T_y\pi)(y, v) = \left(\pi y, \|y\|^{-1}\left(v - \|y\|^{-2}\langle v, y\rangle\, y\right)\right) \in T_{\pi y}\mathbb{S}^n. \quad (31)
$$

For $\|y\| = 1$, this simplifies to $(T_y\pi)(y, v) = (y, v - \langle v, y\rangle\, y)$. The geometric interpretation of formula (31) is the following: Project $y \in \mathbb{R}_0^{n+1}$ to $s = \pi y = \frac{y}{\|y\|} \in \mathbb{S}^n$. Then the tangent vector $v \in T_y\mathbb{R}_0^{n+1}$ is mapped to $\|y\|^{-1} v$ minus $\|y\|^{-1}$ times the projection $\left\langle v, \frac{y}{\|y\|}\right\rangle \frac{y}{\|y\|}$ to the radial component.

Note that $\ker(T_y\pi) = \operatorname{span}(y)$. In fact, $(T_y\pi)v = 0$ is equivalent to $v = \left\langle v, \frac{y}{\|y\|}\right\rangle \frac{y}{\|y\|}$ and hence $v \in \operatorname{span}(y)$. The converse follows by taking in $v = \alpha y, \alpha \in \mathbb{R}$ the scalar product with y.

The spaces $L(\lambda_i)^\infty = L(\lambda_i)\times\{0\}$ are subspaces of $\mathbb{R}^{n+1}$ and $\{y\}\times L(\lambda_i)^\infty$ are subspaces of the tangent spaces $T_y\mathbb{R}_0^{n+1} \cong \{y\}\times\mathbb{R}^{n+1}$. Consider the map $(\mathrm{id}_{\mathcal{U}}, T_{(x,0)}\pi) : \mathcal{U}\times T_{(x,0)}\mathbb{R}^{n+1} \to \mathcal{U}\times T_s\mathbb{S}^n$. In the following, we write the points in $\mathbb{S}^{n,0}$ as $s = \pi(x,0)$ with $x \in \mathbb{R}^n$ satisfying $\|x\| = 1$.

Lemma 2. *Let $s = \pi(x,0) \in {}_{\mathbb{S}}L(\lambda_{i_0})^\infty \subset \mathbb{S}^{n,0}$ with $(x,0) \in L(\lambda_{i_0})^\infty \subset \mathbb{R}_0^{n+1}$ and $\|x\| = 1$. Then one obtains*

$$
(\mathrm{id}_{\mathcal{U}}, T_{(x,0)}\pi)(\mathcal{U}\times\{(x,0)\}\times L(\lambda_i)^\infty) = \mathcal{U}\times\left\{(s, v, 0) \,|\, (v,0) \in L(\lambda_i)^\infty\right\}, \lambda_i \neq \lambda_{i_0},
$$

$$
(\mathrm{id}_{\mathcal{U}}, T_{(x,0)}\pi)(\mathcal{U}\times\{(x,0)\}\times L(\lambda_{i_0})^\infty) = \mathcal{U}\times\left\{(s, (v - \langle v, x\rangle\, x, 0)) \,|\, (v,0) \in L(\lambda_{i_0})^\infty\right\},
$$

$$(\mathrm{id}_{\mathcal{U}}, T_{(x,0)}\pi)(\{(x,0)\}\times\mathcal{V}_c) = \left\{ (u,s,(w-\langle w,x\rangle x, r)) | (u,w,r)\in\mathcal{V}_c \right\}.$$

The kernel of $T_{(x,0)}\pi$ restricted to $\{(x,0)\}\times L(\lambda_{i_0})$ equals the span of $(x,0)$.

Proof. For $(x,0)\in L(\lambda_{i_0})^\infty$, $\|x\|=1$, and $(v,v_{n+1})\in\mathbb{R}^n\times\mathbb{R}\cong T_{(x,0)}\mathbb{R}_0^{n+1}$, formula (31) implies

$$\begin{aligned}(T_{(x,0)}\pi)(x,0)(v,v_{n+1}) &= \left(\pi(x,0),((v,v_{n+1})-\left\langle (v,v_{n+1}),(x,0)\right\rangle (x,0))\right)\\ &= \left(s,\left((v,v_{n+1})-\langle v,x\rangle (x,0)\right)\right)\\ &= \left(s,\left(v-\langle v,x\rangle x, v_{n+1}\right)\right).\end{aligned}$$

This proves the formula above for $i=i_0$. For $i\neq i_0$ and $(v,0)\in L(\lambda_i)^\infty$, orthogonality of $L(\lambda_{i_0})$ and $L(\lambda_i)$ implies that $\langle v,x\rangle=0$, and hence

$$\begin{aligned}(T_{(x,0)}\pi)\left((x,0),L(\lambda_i)^\infty\right) &= \left\{ (s,(v-\langle v,x\rangle x,0)) | (v,0)\in L(\lambda_i)^\infty \right\}\\ &= \left\{ (s,v,0) \,|(v,0)\in L(\lambda_i)^\infty \right\}.\end{aligned}$$

Concerning the central Selgrade bundle, recall that $\mathcal{V}_c$ is given by (19) and again by (31)

$$(T_{(x,0)}\pi)\left((x,0),(w,r)\right) = (s, w-\langle w,x\rangle x, r). \quad \square$$

Next we determine the derivative of the cocycle $\pi\varphi^1(t,x,0,u)$ in a point $s_0\in\mathbb{S}^{n,0}$ in direction (v,v_{n+1}).

Lemma 3. *The derivative of the map $\pi\varphi^1(t,\cdot,u):\mathbb{S}^n\to\mathbb{S}^n$ in the point $s=\pi(x,0)\in\mathbb{S}^n$, $\|x\|=1$, in direction $(v,v_{n+1})\in T_s\mathbb{S}^n$ is the element of $T_{\pi\varphi^1(t,s,u)}\mathbb{S}^n$ given by*

$$\begin{aligned}&D_s\pi\varphi^1(t,x,0,u)(v,v_{n+1})\\ &= \left\|e^{At}x\right\|^{-1}\left(e^{At}v+v_{n+1}\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, v_{n+1}\right)\\ &\quad-\left\|e^{At}x\right\|^{-3}\left(e^{At}x,0\right)\left\langle e^{At}v+v_{n+1}\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, e^{At}x\right\rangle.\end{aligned} \tag{32}$$

Proof. By (21), the cocycle on $\mathbb{S}^n$ satisfies

$$\pi\varphi^1(t,\pi(x,0),u) = \frac{\varphi^1(t,x,0,u)}{\left\langle \varphi^1(t,x,0,u),\varphi^1(t,x,0,u)\right\rangle^{1/2}}.$$

We compute for the derivatives in $(x,0)$ in direction $(v,v_{n+1})\in T_s\mathbb{S}^n$

$$D_s\varphi^1(t,x,0,u)(v,v_{n+1}) = \left(e^{At}v+v_{n+1}\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, v_{n+1}\right)$$

and

$$\begin{aligned}&D_s\left\langle \varphi^1(t,x,0,u),\varphi^1(t,x,0,u)\right\rangle^{1/2}(v,v_{n+1})\\ &= \frac{\left\langle (e^{At}v+v_{n+1}\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, v_{n+1}),\varphi^1(t,x,0,u)\right\rangle}{\left\langle \varphi^1(t,x,0,u),\varphi^1(t,x,0,u)\right\rangle^{1/2}}\\ &= \left\|e^{At}x\right\|^{-1}\left\langle e^{At}v+v_{n+1}\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, e^{At}x\right\rangle.\end{aligned}$$

This yields

$$\begin{aligned}&D_s\pi\varphi^1(t,x,0,u)(v,v_{n+1})\\ &= \left\|\varphi^1(t,x,0,u)\right\|^{-2}\Big[D_s\varphi^1(t,x,0,u)(v,v_{n+1})\left\langle \varphi^1(t,x,0,u),\varphi^1(t,x,0,u)\right\rangle^{1/2}\\ &\qquad-\varphi^1(t,x,0,u)D_s\left\langle \varphi^1(t,x,0,u),\varphi^1(t,x,0,u)\right\rangle^{1/2}(v,v_{n+1})\Big]\\ &= \left\|e^{At}x\right\|^{-2}\Big[\left(e^{At}v+v_{n+1}\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, v_{n+1}\right)\left\|\left(e^{At}x,0\right)\right\|\\ &\qquad-\left(e^{At}x,0\right)\left\|e^{At}x\right\|^{-1}\left\langle e^{At}v+v_{n+1}\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, e^{At}x\right\rangle\Big]\\ &= \left\|e^{At}x\right\|^{-1}\left(e^{At}v+v_{n+1}\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, v_{n+1}\right)\\ &\quad-\left\|e^{At}x\right\|^{-3}\left(e^{At}x,0\right)\left\langle e^{At}v+v_{n+1}\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, e^{At}x\right\rangle. \quad \square\end{aligned}$$

For the control flow $\pi\Phi^1$ defined in (27) with linearization $T\pi\Phi$ defined in (28), the Lyapunov exponent of a point $s=(x,x_{n+1})\in\mathbb{S}^n\subset\mathbb{R}^{n+1}$ and control $u\in\mathcal{U}$ in direction $(0,0)\neq(v,v_{n+1})\in T_s\mathbb{S}^n\subset\mathbb{R}^n\times\mathbb{R}$ is given by

$$\lambda(s,u;v,v_{n+1}) := \lim_{|t|\to\infty}\frac{1}{t}\log\left\|D_{(x,x_{n+1})}\pi\varphi^1(t,x,x_{n+1},u)(v,v_{n+1})\right\|,$$

where $(u,s,v,v_{n+1})\in\mathcal{U}\times T_s\mathbb{S}^n$. The following theorem describes the Selgrade bundles and their Lyapunov exponents.

Theorem 7. *Consider the projected linear control flow $\pi\Phi^1$ on $\mathcal{U}\times\mathbb{S}^n$ associated with the lift (12) of a linear control system of the form (1) and the linearized flow $T\pi\Phi$ with base space restricted to $\mathcal{U}\times{}_{\mathbb{S}}L(\lambda_{i_0})^\infty$, as described in Proposition 2.*

(i) Then the Selgrade bundles in (30) have the following form:

$$\begin{aligned}&{}_{\mathbb{S}}\mathcal{V}_c = (\mathrm{id}_{\mathcal{U}},T\pi)\mathcal{V}_c \text{ with } \dim{}_{\mathbb{S}}\mathcal{V}_c = \dim L^0+1;\\ &{}_{\mathbb{S}}\mathcal{V}_i = (\mathrm{id}_{\mathcal{U}},T\pi)\left(\mathcal{U}\times L(\lambda_i)^\infty\right) = \mathcal{U}\times{}_{\mathbb{S}}L(\lambda_{i_0})^\infty\times L(\lambda_i)^\infty\\ &\qquad\text{with } \dim{}_{\mathbb{S}}\mathcal{V}_i = \dim L(\lambda_i),\end{aligned}$$

for $\lambda_i\neq 0,\lambda_{i_0}$. If $\dim L(\lambda_{i_0})>1$ there is an additional Selgrade bundle given by

$${}_{\mathbb{S}}\mathcal{V}_{i_0} = (\mathrm{id}_{\mathcal{U}},T\pi)(\mathcal{U}\times L(\lambda_{i_0})^\infty) \text{ with } \dim{}_{\mathbb{S}}\mathcal{V}_{i_0} = \dim L(\lambda_{i_0})-1.$$

(ii) The Lyapunov exponents are

$$\begin{aligned}&\lambda(s,u;v,0) = \lambda_i-\lambda_{i_0} \text{ for all } (u,s,v,0)\in{}_{\mathbb{S}}\mathcal{V}_i \text{ with } \lambda_i\neq 0,\lambda_{i_0},\\ &\lambda(s,u;v,v_{n+1}) = -\lambda_{i_0} \text{ for } (u,s,v,v_{n+1})\in{}_{\mathbb{S}}\mathcal{V}_c.\end{aligned}$$

For $\dim L(\lambda_{i_0})>1$

$$\lambda(s,u;v,0) = 0 \text{ for all } (u,s,v,0)\in{}_{\mathbb{S}}\mathcal{V}_{i_0}. \tag{33}$$

Proof. (i) Recall that $T_s\mathbb{S}^n=\{s\}\times\left\{v\in\mathbb{R}^{n+1}\,|\langle v,s\rangle=0\right\}$ and $T\left(\mathbb{R}_0^n\times\mathbb{R}\right)\subset T\mathbb{R}_0^{n+1}$. The Selgrade decomposition for the linear control flow Φ^1 associated with the lifted control system (12) is given in (18). Let $s=\pi(x,0)\in{}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ with $\|x\|=1$. By Lemma 2, the surjection

$$(\mathrm{id}_{\mathcal{U}},T\pi):\mathcal{U}\times T_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}\mathbb{R}_0^{n+1}\to\mathcal{U}\times T_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}\mathbb{S}^n,$$

maps $\mathcal{U}\times L(\lambda_{i_0})^\infty\times L(\lambda_i)^\infty\subset\mathcal{U}\times T_{L(\lambda_{i_0})^\infty}\mathbb{R}_0^{n+1}, i\neq 0,i_0$, onto $\mathcal{U}\times{}_{\mathbb{S}}L(\lambda_{i_0})^\infty\times L(\lambda_i)^\infty$. Furthermore, $\mathcal{U}\times{}_{\mathbb{S}}L(\lambda_{i_0})^\infty\times\mathcal{V}_c$ is mapped onto

$${}_{\mathbb{S}}\mathcal{V}_c = \left\{(u,s,w-\langle w,x\rangle x,r)\,\middle|\, s\in{}_{\mathbb{S}}L(\lambda_{i_0})^\infty \text{ and } (u,w,r)\in\mathcal{V}_c\right\}.$$

Finally, $\mathcal{U}\times L(\lambda_{i_0})^\infty\times L(\lambda_{i_0})^\infty$ is mapped onto

$$\begin{aligned}&(\mathrm{id}_{\mathcal{U}},T\pi)\left(\mathcal{U}\times L(\lambda_{i_0})^\infty\times L(\lambda_{i_0})^\infty\right)\\ &= \mathcal{U}\times\left\{(s,v-\langle v,x\rangle x,0)\,\middle|\, s\in{}_{\mathbb{S}}L(\lambda_{i_0})^\infty \text{ and } (v,0)\in L(\lambda_{i_0})^\infty\right\}.\end{aligned}$$

Let

$$P_{L(\lambda_{i_0})^\infty}\mathbb{R}_0^{n+1} := \bigcup_{(x,0)\in L(\lambda_{i_0})^\infty} P_{(x,0)}\mathbb{R}_0^{n+1} = \bigcup_{(x,0)\in L(\lambda_{i_0})^\infty}\{(x,0)\}\times\mathbb{P}^n.$$

In the projective bundle $\mathcal{U}\times P_{L(\lambda_{i_0})^\infty}\mathbb{R}_0^{n+1}$, the images of the Selgrade bundles $\mathcal{V}_c$ and $\mathcal{U}\times L(\lambda_i)^\infty$ are the maximal chain transitive sets. Their images in $\mathcal{U}\times P_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}\mathbb{S}^n$ (cf. (29) are also chain transitive since they are the continuous images of chain transitive sets in a compact metric space. The same arguments show that the image of the Selgrade bundle $\mathcal{U}\times{}_{\mathbb{S}}L(\lambda_{i_0})^\infty\times L(\lambda_{i_0})^\infty$ is chain transitive. It follows that the subbundles defined in assertion (i) are the Selgrade bundles. The kernel of $T_{(x,0)}\pi$ restricted to $L(\lambda_{i_0})^\infty$ equals the span of $(x,0)$. Hence

$$\dim{}_{\mathbb{S}}\mathcal{V}_{i_0} = \dim\left(T_{(x,0)}\pi\right)L(\lambda_{i_0})^\infty = \dim L(\lambda_{i_0})^\infty-1 = \dim L(\lambda_{i_0})-1.$$

The dimensions of the subbundles ${}_{\mathbb{S}}\mathcal{V}_c$ and ${}_{\mathbb{S}}\mathcal{V}_i, i \neq i_0$ are preserved under $(\mathrm{id}_{\mathcal{U}}, T\pi)$ and

$$\sum_{\lambda_i \neq 0} \dim {}_{\mathbb{S}}\mathcal{V}_i + \dim {}_{\mathbb{S}}\mathcal{V}_c = \sum_{\lambda_i \neq 0, \lambda_{i_0}} \dim L(\lambda_i) + \dim L(\lambda_{i_0}) - 1 + \dim L^0 + 1$$
$$= n = \dim T_s \mathbb{S}^n.$$

(ii) By Lemma 2, any direction in $(T_{(x,0)}\pi)L(\lambda_i)^\infty$ satisfies $(v, v_{n+1}) = (v, 0) \in L(\lambda_i)^\infty$. Thus Lemma 3 implies that

$$D_s \pi\varphi^1(t, x, 0, u)(v, 0) \tag{34}$$
$$= \left\|e^{At}x\right\|^{-1} \left(e^{At}v, 0\right) - \left\|e^{At}x\right\|^{-3} \left(e^{At}x, 0\right) \left\langle e^{At}v, e^{At}x \right\rangle.$$

For $i \neq i_0$, the A-invariant subspaces $L(\lambda_i)$ and $L(\lambda_{i_0})$ are orthogonal, and hence $\langle e^{At}v, e^{At}x\rangle = 0$ implying

$$D_s \pi\varphi^1(t, x, 0, u)(v, 0) = \left\|e^{At}x\right\|^{-1} \left(e^{At}v, 0\right).$$

For the Lyapunov exponent in direction $(v, 0) \in L(\lambda_i)^\infty, i \neq i_0$, this implies

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \left\|D_s \pi\varphi^1(t, x, 0, u)(v, 0)\right\| = \lim_{t\to\pm\infty} \frac{1}{t} \log \frac{\|(e^{At}v, 0)\|}{\|e^{At}x\|}$$
$$= \lim_{t\to\pm\infty} \frac{1}{t} \log \left\|e^{At}v\right\| - \lim_{t\to\pm\infty} \frac{1}{t} \log \left\|e^{At}x\right\| = \lambda_i - \lambda_{i_0}.$$

Next consider the Lyapunov exponents for $(u, s, v, 0) \in (\mathrm{id}_{\mathcal{U}}, T_{(x,0)}\pi)\mathcal{V}_{i_0}$. (34) yields

$$D_s \pi\varphi^1(t, x, 0, u)(v, 0) = \left(\frac{e^{At}v}{\|e^{At}x\|}v, 0\right) - \left(\frac{e^{At}x}{\|e^{At}x\|}, 0\right) \left\langle \frac{e^{At}v}{\|e^{At}x\|}, \frac{e^{At}x}{\|e^{At}x\|} \right\rangle.$$

Since $x, v \in L_{i_0}$ it follows that this term has exponential growth 0 for $t \to \pm\infty$. This proves (33). Finally, we compute the Lyapunov exponents for $(u, s, v, v_{n+1}) \in {}_{\mathbb{S}}\mathcal{V}_c = (\mathrm{id}_{\mathcal{U}}, T_{(x,0)}\pi)\mathcal{V}_c$. Lemma 2 and Eq. (19) imply

$$(v, v_{n+1}) = (w - \langle w, x\rangle x, r) \text{ for } (u, w, r) = (u, re(u, 0) + ry, r) \in \mathcal{V}_c.$$

Inserting this into formula (32), we obtain

$$D_s \pi\varphi^1(t, \pi(x, 0), u)(v, v_{n+1})$$
$$= \left\|e^{At}x\right\|^{-1} \left(e^{At}w + r\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, r\right) - \langle w, x\rangle \left(\frac{e^{At}x}{\|e^{At}x\|}, 0\right)$$
$$+ \left\|e^{At}x\right\|^{-1} \left\|e^{At}x\right\|^{-2} \left(e^{At}x, 0\right) \left\langle e^{At}w + r\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, e^{At}x \right\rangle$$
$$- \left(\frac{e^{At}x}{\|e^{At}x\|}, 0\right) \langle w, x\rangle \left\langle \frac{e^{At}x}{\|e^{At}x\|}, \frac{e^{At}x}{\|e^{At}x\|} \right\rangle.$$

First note that we may omit the summands with bounded norms which do not contribute to the exponential growth rate. Then the remaining terms may be written as $f(t)g(t)$, where $f(t) := \|e^{At}x\|^{-1}$ and

$$g(t) := r\left(e^{At}(e(u, 0) + y) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, 1\right)$$
$$+ r\left(\frac{e^{At}x}{\|e^{At}x\|}, 0\right) \left\langle e^{At}(e(u, t) + y) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, \frac{e^{At}x}{\|e^{At}x\|} \right\rangle.$$

We have $f(t)^{-1} = \|e^{At}x\|$, and hence

$$\lim_{t\to\pm\infty} \frac{1}{t} \log f(t) = -\lambda_{i_0}, \quad \lim_{t\to\pm\infty} \frac{1}{t} \log f(t)^{-1} = \lambda_{i_0}.$$

By Lemma 2, the exponential growth rate of g is $\lim_{t\to\pm\infty} \frac{1}{t} \log \|g(t)\| = 0$. By Cesari [25, (3.12.v)] it follows that

$$\lim_{t\to\pm\infty} \frac{1}{t} \log [f(t)\|g(t)\|] = -\lambda_{i_0}.$$

This concludes the proof. □

For $s \in \mathbb{S}^n$ and $u \in \mathcal{U}$ a direction is called stable, if the corresponding Lyapunov exponents for $t \to \pm\infty$ are negative and unstable if they are positive. An immediate corollary of Theorem 7 is the following.

Corollary 2. *In the situation of Theorem 7, for $\lambda_{i_0} \neq 0$, the subbundles ${}_{\mathbb{S}}\mathcal{V}_i, i > i_0$, and ${}_{\mathbb{S}}\mathcal{V}_i, i < i_0$, consist for all $u \in \mathcal{U}$ of stable and unstable directions, respectively. The directions in ${}_{\mathbb{S}}\mathcal{V}_c$ are stable for $\lambda_{i_0} > 0$ and unstable for $\lambda_{i_0} < 0$.*

(i) Define

$$\text{for } \lambda_{i_0} > 0, \ {}_{\mathbb{S}}\mathcal{V}^- := \bigoplus_{i > i_0, \lambda_i \neq 0} {}_{\mathbb{S}}\mathcal{V}_i \oplus {}_{\mathbb{S}}\mathcal{V}_c \text{ and } {}_{\mathbb{S}}\mathcal{V}^+ := \bigoplus_{i < i_0} {}_{\mathbb{S}}\mathcal{V}_i, \tag{35}$$

$$\text{for } \lambda_{i_0} < 0, \ {}_{\mathbb{S}}\mathcal{V}^- := \bigoplus_{i > i_0} {}_{\mathbb{S}}\mathcal{V}_i \text{ and } {}_{\mathbb{S}}\mathcal{V}^+ := {}_{\mathbb{S}}\mathcal{V}_c \oplus \bigoplus_{i < i_0, \lambda_i \neq 0} {}_{\mathbb{S}}\mathcal{V}_i. \tag{36}$$

Then

$$\mathcal{V}_{\mathbb{S}L(\lambda_{i_0})^\infty} = \mathcal{U} \times T_{\mathbb{S}L(\lambda_{i_0})^\infty}\mathbb{S}^n = {}_{\mathbb{S}}\mathcal{V}^- \oplus {}_{\mathbb{S}}\mathcal{V}_{i_0} \oplus {}_{\mathbb{S}}\mathcal{V}^+$$

is a decomposition into the stable, center, and unstable subbundles.

(ii) For the stable subbundle ${}_{\mathbb{S}}\mathcal{V}^-$, the supremal exponential growth rate $\kappa\left({}_{\mathbb{S}}\mathcal{V}^-\right) := \sup\{\lambda(u, x, v^-) | (u, x, v^-) \in {}_{\mathbb{S}}\mathcal{V}^-\}$ satisfies

for $\lambda_{i_0} > 0$, $\kappa({}_{\mathbb{S}}\mathcal{V}^-) = \lambda_{i_1} - \lambda_{i_0}$ where $\lambda_{i_1} = \max\{\lambda_i \,|\lambda_i > 0 \text{ and } i \geq i_0\}$,

for $\lambda_{i_0} < 0$, $\kappa\left({}_{\mathbb{S}}\mathcal{V}^-\right) = \lambda_\ell - \lambda_{i_0}$ if $i_0 < \ell$ and if $i_0 = \ell$ then ${}_{\mathbb{S}}\mathcal{V}^-$ is trivial.

Proof. Recall that $\lambda_1 > \cdots > \lambda_\ell$. Theorem 7 (iii) shows that

$$\kappa({}_{\mathbb{S}}\mathcal{V}_i) = \lambda_i - \lambda_{i_0} < 0 \text{ for } 0 \neq \lambda_i < \lambda_{i_0}, \text{ i.e., } i > i_0. \tag{37}$$

For $\lambda_{i_0} > 0$, it follows that $\kappa({}_{\mathbb{S}}\mathcal{V}_c) = -\lambda_{i_0} < 0$. This implies that ${}_{\mathbb{S}}\mathcal{V}^-$ defined in (35) satisfies $\kappa({}_{\mathbb{S}}\mathcal{V}^-) = \lambda_{i_1} - \lambda_{i_0} < 0$, where $\lambda_{i_1} := \max\{\lambda_i \,|\lambda_i > 0 \text{ and } i \geq i_0\}$. If $\lambda_i < 0$ for all $i > i_0$ then $\kappa({}_{\mathbb{S}}\mathcal{V}^-) = -\lambda_{i_0}$. For $\lambda_{i_0} < 0$ it holds that $\kappa({}_{\mathbb{S}}\mathcal{V}_c) = -\lambda_{i_0} > 0$ and (37) implies that ${}_{\mathbb{S}}\mathcal{V}^-$ defined in (36) satisfies $\kappa({}_{\mathbb{S}}\mathcal{V}^-) = \lambda_\ell - \lambda_{i_0} < 0$ if $i_0 < \ell$. Hence the assertions follow. □

Remark 2. Note that $\lambda_i - \lambda_{i_0} \neq 0$ for $i \neq i_0$ and $\lambda_{i_0} \neq 0$. It follows that the linear flow $T\pi\Phi^1$ is uniformly hyperbolic, if $\dim L(\lambda_{i_0}) = 1$, i.e., if λ_{i_0} is a simple real eigenvalue of A.

Remark 3. The Selgrade bundles ${}_{\mathbb{S}}\mathcal{V}_i$, ${}_{\mathbb{S}}\mathcal{V}_c$, and ${}_{\mathbb{S}}\mathcal{V}_{i_0}$ actually are Sacker-Sell bundles since their spectral intervals (which degenerate to points) do not overlap; cf. Johnson, Palmer, and Sell [26].

5. Invariant manifolds

In this section we determine invariant manifolds on the Poincaré sphere $\mathbb{S}^n$ and on $\mathbb{R}^n$. In order to determine the behavior of linear control systems of the form (1) "near infinity" we analyze invariant manifolds for points on the equator $\mathbb{S}^{n,0}$ of $\mathbb{S}^n$. For the linearization $T\pi\Phi^1$ about the invariant sets ${}_{\mathbb{S}}L(\lambda_{i_0})^\infty \subset \mathbb{S}^{n,0}$, Theorem 7 determines the Selgrade bundles and their spectra.

We need stable manifolds for the nonlinear flow $\pi\Phi^1$ on $\mathcal{U} \times \mathbb{S}^n$. From the vast literature on invariant manifolds, we refer to the following related versions of stable manifold theorems. Johnson [27] proved results on invariant manifolds tangential to Sacker-Sell bundles. His main result [27, Theorem 2.25] concerns differential equations in $\mathbb{R}^n$ which are embedded in a flow where the base space is of Bebutov type. Chow and Yi [28] consider differential equations in $\mathbb{R}^n$ with a base space which is a compact and connected manifold. Alternatively, also invariant manifold theorems for (single) Carathéodory differential equations are presented in Aulbach and Wanner [29] .

While the methods developed in these (and other) papers presumably can be adapted so that they apply to our situation, the results are not immediately applicable. Instead we will use a local stable manifold theorem in Colonius and Kliemann [8, Theorem 6.4.3], which is based on an abstract stable manifold theorem due to Bronstein and Chernii [30] (cf. [8, Theorem 5.6.1] for a detailed proof and the monograph by Bronstein and Kopanskii [31] for abstract invariant manifold theory). We need the following notational preliminaries.

Consider a control-affine system on a Riemannian manifold M of the form

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)), u \in \mathcal{U}, \tag{38}$$

satisfying the assumptions on system (3). The linearized system on the tangent bundle TM has the form

$$\frac{d}{dt} Tx(t) = TX_0(Tx(t)) + \sum_{i=1}^{m} u_i(t) TX_i(Tx(t)), \tag{39}$$

where for a vector field X on M its linearization is denoted by TX. The control flow $\Phi_t(u,x) = (u(t+\cdot), \varphi(t,x,u))$ on $\mathcal{U} \times M$ for (38) can be linearized and yields the control flow $T\Phi$ on $\mathcal{U} \times TM$ for (39).

The following result is a minor modification of a local stable manifold theorem on Riemannian manifolds presented in [8, Theorem 6.4.9] (a mistake in the formulation is corrected); cf. also the local stable manifold theorem [8, Theorem 6.4.3] on $\mathbb{R}^n$. Instead of the compact closure of a control set D in M, we consider an arbitrary compact set $K \subset M$ and define the lift of K to $\mathcal{U} \times M$ by

$$\mathcal{K} := \{(u,x) \in \mathcal{U} \times M \,|\varphi(t,x,u) \in K \text{ for all } t \in \mathbb{R}\}.$$

Assume that $\mathcal{K}$ is nonvoid. Thus $\mathcal{K}$ is a compact invariant set for the control flow Φ, and we assume that the linearized flow $T\Phi$ restricted to the vector bundle

$$\mathcal{V}_{\mathcal{K}} = \{(u,x,v) \in \mathcal{U} \times TM \,|(u,x) \in \mathcal{K}\} \to \mathcal{K}$$

admits the following decomposition into invariant subbundles,

$$\mathcal{V}_{\mathcal{K}} = \mathcal{V}^- \bigoplus \mathcal{V}^+. \tag{40}$$

Here $\mathcal{V}^-$ and $\mathcal{V}^+$ are exponentially separated meaning that there are constants $c_0 > 0$ and $\varepsilon_0 > 0$ with

$$\left\| T\Phi_t(u,x,v^-) \right\| \le c_0 e^{-\varepsilon_0 t} \left\| T\Phi_t(u,x,v^+) \right\| \text{ for } t \ge 0, \tag{41}$$

for all $(u,x,v^-) \in \mathcal{V}^-$, $(u,x,v^+) \in \mathcal{V}^+$ with $\|v^-\| = \|v^+\| = 1$, and the subbundle $\mathcal{V}^-$ is stable, i.e., it satisfies

$$\kappa(\mathcal{V}^-) := \sup \left\{ \limsup_{t\to\infty} \frac{1}{t} \log \left\| D_x \varphi(t,x,u) v^- \right\| |(u,x,v^-) \in \mathcal{V}^- \right\} < 0. \tag{42}$$

Note that we do not assume hyperbolicity; that is, we allow that $\mathcal{V}^+$ includes points with nonpositive Lyapunov exponents. For the Riemannian manifold M, the exponential map exp yields a map $(x,v) \mapsto (x, \exp(x,v))$ from a neighborhood of the zero section in TM to $M \times M$. Recall the following definitions (cf. e.g. Husemoller [32, p. 15]). A bundle is a continuous map $\pi : E \to B$, where E and B are metric spaces. Let $\pi : E \to B$ and $\pi' : E' \to B'$ be two bundles. A bundle morphism $F = (g,h) : E \to E'$ is a pair of continuous maps $g : E \to E'$ and $h : B \to B'$ such that $\pi' \circ g = h \circ \pi$. A bundle morphism is a bundle isomorphism if there exists an inverse bundle morphism.

Theorem 8. *Consider control system (38) on M with associated control flow Φ on $\mathcal{U} \times M$ and let $K \subset M$ be compact with lift $\mathcal{K}$ to $\mathcal{U} \times M$. Suppose for the linearized control system (39) that the associated linearized control flow $T\Phi$ restricted to the vector bundle $\mathcal{V}_{\mathcal{K}}$ admits the decomposition (40), (41) into subbundles and the subbundle $\mathcal{V}^-$ satisfies stability condition (42). Then there are $\delta > 0$ and a map*

$$S^- : \{(u,x,v^-) \in \mathcal{V}^- \,|\|v^-\| < \delta\} \to \mathcal{K} \times M,$$

which is a bundle isomorphism onto its image $\mathcal{W}^-_{loc} := \operatorname{im} S^-$ with the following properties:

(i) Every $(u,x_0,x) \in \mathcal{W}^-_{loc}$ satisfies

$$\lim_{t\to\infty} e^{-\alpha t} d(\varphi(t,x,u), \varphi(t,x_0,u)) = 0 \text{ for every } \alpha \in \left(\kappa({}_{\mathbb{S}}\mathcal{V}^-), 0\right),$$

where d denotes the Riemannian distance on M. The set $\mathcal{W}^-_{loc} \subset \mathcal{K} \times M$ is called a local stable manifold corresponding to the stable subbundle $\mathcal{V}^-$.

(ii) For every $(u,x_0) \in \mathcal{K}$, define the local stable manifold for (u,x_0) by

$$\mathcal{W}^-_{loc}(u,x_0) := \left\{ x \in M \,\middle|(u,x_0,x) \in \mathcal{W}^-_{loc} \right\} \subset M.$$

Then the topological dimension of $\mathcal{W}^-_{loc}(u,x_0)$ equals the dimension of $\mathcal{V}^-$.

(iii) The local stable manifold $\mathcal{W}^-$ is positively invariant under the control flow Φ, i.e., for $x \in \mathcal{W}^-_{loc}(u,x_0)$ it follows that $\varphi(t,x,u) \in \mathcal{W}^-_{loc}(u(t+\cdot), \varphi(t,x_0,u))$ for all $t \ge 0$.

(iv) The distance of the subbundle $\mathcal{W}^-_{loc}$ to $\mathcal{V}^-$ can be made arbitrarily small in the following Lipschitz sense by choosing $\delta > 0$ small: For all $h > 0$ there is $\delta > 0$ such that $\mathcal{W}^-$ is contained in the set $C(\mathcal{V}^-, h)$ of angle h around $\mathcal{V}^-$ given by

$$C(\mathcal{V}^-, h) := \left\{(u,x,\exp(x,v^+ + v^-)) \in \mathcal{K} \times M \,\middle|(u,x,v^\pm) \in \mathcal{V}^\pm, \left\|v^+\right\| \le h \left\|v^-\right\| \right\}.$$

Remark 4. Using time reversal, one can also obtain a result on local unstable manifolds, cf. [8, Remark 6.4.5].

We will apply Theorem 8 to control system (20) on the Poincaré sphere $M = \mathbb{S}^n$, the associated control flow $\pi\Phi^1$ on $\mathcal{U} \times \mathbb{S}^n$, and its linearization $T\pi\Phi^1$ on $\mathcal{U} \times T\mathbb{S}^n$. The set $K := {}_{\mathbb{S}}L(\lambda_{i_0})^\infty \subset \mathbb{S}^{n,0}$ is compact and its lift to $\mathcal{U} \times \mathbb{S}^n$ is given by $\mathcal{U} \times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ since on the equator $\mathbb{S}^{n,0}$ the controls u act trivially. Observe that this also implies that $\pi\varphi^1(t,s,u) = \pi\varphi^1(t,s,0)$ for $s \in \mathbb{S}^{n,0}$. Recall that Corollary 2 presents decompositions with a stable subbundle ${}_{\mathbb{S}}\mathcal{V}^-$ in the form

$$\mathcal{V}_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty} = \mathcal{U} \times T_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}\mathbb{S}^n = {}_{\mathbb{S}}\mathcal{V}^- \oplus ({}_{\mathbb{S}}\mathcal{V}_{i_0} \oplus {}_{\mathbb{S}}\mathcal{V}^+). \tag{43}$$

Here ${}_{\mathbb{S}}\mathcal{V}_{i_0}$ is nontrivial if and only if $\dim L(\lambda_{i_0}) > 0$. We obtain the following corollary to Theorem 8.

Corollary 3. *Consider control system (20) on the Poincaré sphere $\mathbb{S}^n$ and the associated control flow $\pi\Phi^1$ on $\mathcal{U} \times \mathbb{S}^n$. For the compact set ${}_{\mathbb{S}}L(\lambda_{i_0})^\infty \subset \mathbb{S}^{n,0}$, $\lambda_{i_0} \neq 0$, and the stable subbundle ${}_{\mathbb{S}}\mathcal{V}^-$ in (43), there are $\delta > 0$ and a map*

$$S^- : \left\{(u,s,v,v_{n+1}) \in {}_{\mathbb{S}}\mathcal{V}^- \,\middle|\left\|(v,v_{n+1})\right\| < \delta \right\} \to \mathcal{U} \times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty \times \mathbb{S}^n,$$

which is a bundle isomorphism onto its image $\mathcal{W}^-_{loc} := \operatorname{im} S^-$, with the following properties:

(i) Every $(u,s_0,s) \in \mathcal{W}^-_{loc}$ satisfies

$$\lim_{t\to\infty} e^{-\alpha t} d(\pi\varphi^1(t,s,u), \pi\varphi^1(t,s_0,0)) = 0 \text{ for every } \alpha \in \left(\kappa({}_{\mathbb{S}}\mathcal{V}^-), 0\right), \tag{44}$$

where d denotes the Riemannian distance on $\mathbb{S}^n$. The set $\mathcal{W}^-_{loc} \subset \mathcal{U} \times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty \times \mathbb{S}^n$ is a local stable manifold corresponding to the stable subbundle ${}_{\mathbb{S}}\mathcal{V}^-$.

(ii) For every $(u,s_0) \in {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ the local stable manifold for (u,s_0) defined by

$$\mathcal{W}^-_{loc}(u,s_0) := \left\{ s \in \mathbb{S}^n \,\middle|(u,s_0,s) \in \mathcal{W}^-_{loc} \right\} \subset \mathbb{S}^n$$

has topological dimension equal to the dimension of ${}_{\mathbb{S}}\mathcal{V}^-$.

(iii) The local stable manifold $\mathcal{W}^-_{loc}$ is positively invariant under the control flow $\pi\Phi^1$, i.e., for $s \in \mathcal{W}^-_{loc}(u,s_0)$ it holds that $\pi\varphi^1(t,s,u) \in \mathcal{W}^-_{loc}(u(t+\cdot), \pi\varphi^1(t,s_0,0))$ for all $t \ge 0$.

(iv) The distance of the subbundle $\mathcal{W}^-_{loc}$ to ${}_{\mathbb{S}}\mathcal{V}^-$ can be made arbitrarily small by choosing $\delta > 0$ small: For all $h > 0$ there is $\delta > 0$ such that $\mathcal{W}^-_{loc}$ is contained in the set $C({}_{\mathbb{S}}\mathcal{V}^-, h)$ of angle h around ${}_{\mathbb{S}}\mathcal{V}^-$ given by

$$C({}_{\mathbb{S}}\mathcal{V}^-, h) = \Big\{(u,s,\exp(s,v^+ + v^-)) \in \mathcal{U} \times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty \times \mathbb{S}^n \,\Big|(u,s,v^\pm) \in \mathcal{V}^\pm, \left\|v^+\right\| \le h \left\|v^-\right\| \Big\}.$$

Note that for the sphere $\mathbb{S}^n$ the exponential map is $\exp : T\mathbb{S}^n \to \mathbb{S}^n \times \mathbb{S}^n : (s,v) \mapsto (s, \exp(s,v)) = \left(s, \frac{s+v}{\|s+v\|}\right)$.

Remark 5. We can apply Theorem 8 also to the flow on $\mathbb{S}^{n,0}$, which has the form $(\pi\psi(t,s),0), t \in \mathbb{R}, s \in \mathbb{S}^{n,0} \simeq \mathbb{S}^{n-1} \times \{0\}$; here π denotes the projection $\pi : \mathbb{R}^n_0 \to \mathbb{S}^{n-1}$. When the base space of the linearized flow is restricted to ${}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ and $\lambda_{i_0} < 0$, the stable subbundle is

$$\bigoplus_{i>i_0} T_{{}_{\mathbb{S}}L(\lambda_{i_0})^\infty}(T\pi)\, L(\lambda_i)^\infty = {}_{\mathbb{S}}L(\lambda_{i_0})^\infty \times \bigoplus_{i>i_0} L(\lambda_i)^\infty.$$

One obtains a corresponding stable manifold in $\mathbb{S}^{n,0}$. If $\lambda_{i_0} > 0$ one has to add the subbundle $(T\pi)\, L(\lambda_{i_0})^\infty$, which has dimension $\dim L(\lambda_{i_0}) - 1$.

Consider $(u, s_0, s) \in \mathcal{W}^-_{loc} \subset \mathcal{U} \times \mathbb{S}^{n,0} \times \mathbb{S}^n$. Here only the values $u(t), t \geq 0$, are relevant. We can extend $\mathcal{W}^-_{loc}$ to a global stable manifold $\mathcal{W}^-$defined by

$$\mathcal{W}^- := \Big\{(u, s_0, s) \in \mathcal{U} \times \mathbb{S}^{n,0} \times \mathbb{S}^n \,\Big|\, \exists T \geq 0 : (u(T+\cdot), \pi\varphi^1(T, s_0, 0), \pi\varphi^1(T, s, u)) \in \mathcal{W}^-_{loc} \Big\}.$$

Note that, for $t \to -\infty$, the α-limit set of $\pi\varphi^1(t, s, u)$, $s \in \mathbb{S}^{n,+}$, is contained in one of the central chain control sets ${}_{\mathbb{S}}E^j_c, j = 0$ or $j = 1, 2$, or in one of the sets ${}_{\mathbb{S}}L(\lambda_i)^\infty_j \subset \mathbb{S}^{n,0}, j = 0$ or $j = 1, 2, \lambda_i \neq 0$; cf. Corollary 1.

The following corollary is immediate.

Corollary 4. *In the situation of Corollary 3 the global stable manifold $\mathcal{W}^-$ satisfies (44) for every $(u, s_0, s) \in \mathcal{W}^-$ and it is invariant under the control flow $\pi\Phi^1$, i.e., for $s \in \mathcal{W}^-(u, s_0)$ it holds that $\pi\varphi^1(t, s, u) \in \mathcal{W}^-(u(t+\cdot), \pi\varphi^1(t, s_0, 0))$ for all $t \in \mathbb{R}$.*

Next we analyze the consequences for the original linear control system on $\mathbb{R}^n$. Using the stereographic projection from $\mathbb{R}^{n+1} \to \mathbb{S}^{n,+}$ and $\Delta(x) = \sqrt{x_1^2 + \cdots + x_n^2 + 1}$ we define

$$\phi^+ : \mathbb{R}^n \to \mathbb{S}^{n,+}, \phi^+(x) = \Delta(x)^{-1}(x_1, \ldots, x_n, 1),$$
$$\left(\phi^+\right)^{-1} : \mathbb{S}^{n,+} \to \mathbb{R}^n, \left(\phi^+\right)^{-1}(y_1, \ldots, y_{n+1}) = \left(\frac{y_1}{y_{n+1}}, \ldots, \frac{y_n}{y_{n+1}}\right). \quad (45)$$

For a stable manifold $\mathcal{W}^-_{loc}$ and $(u, s_0) \in \mathcal{U} \times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$, we define

$$W^-_{loc}(u, s_0) := \left(\phi^+\right)^{-1}\left(\mathcal{W}^-(u, s_0) \cap \mathbb{S}^{n,+}\right)$$

and analogously $W^-(u, s_0)$. For $\lambda_{i_0} < 0$ the stable subbundle ${}_{\mathbb{S}}\mathcal{V}^-$ given by (36) does not contain ${}_{\mathbb{S}}\mathcal{V}_c$, and hence $\mathcal{W}^-_{loc}(u, s_0)$ is Lipschitz close to $\mathbb{S}^{n,0}$ (or even contained in $\mathbb{S}^{n,0}$). Since the intersection with $\mathbb{S}^{n,+}$ is relevant, we restrict attention to the case $\lambda_{i_0} > 0$ where, by (35), the subbundle ${}_{\mathbb{S}}\mathcal{V}_c \subset {}_{\mathbb{S}}\mathcal{V}^-$, and hence $\mathcal{W}^-(u, s_0) \cap \mathbb{S}^{n,+}$ is nonvoid.

By Corollary 3(i), it follows for $(u, s_0, s) \in \mathcal{W}^-_{loc}$ that $\pi\varphi^1(t, s, u) = \frac{\varphi^1(t,s,u)}{\|\varphi^1(t,s,u)\|}$ converges to $\pi\varphi^1(t, s_0, u) = \frac{\varphi^1(t,s_0,u)}{\|\varphi^1(t,s_0,u)\|}$. The following example shows what may be expected for the trajectories in $\mathbb{R}^n$.

Example 1. Consider $u = 0$ and let $z_0 \in \mathbb{R}^n$ with $\|z_0\| = 1$ be an eigenvector of A for an eigenvalue $\lambda_{i_0} > 0$. Then $s_0 := (z_0, 0) \in \mathbb{S}^{n,0}$ is an equilibrium of the induced flow on $\mathbb{S}^{n,0}$. We have

$$\varphi(t, x, 0) = e^{At}x, t \geq 0, x \in \mathbb{R}^n, \text{ and } e^{At}z_0 = e^{\lambda_{i_0}t}z_0,$$

and obtain on $\mathbb{S}^n$

$$\pi\varphi^1(t, \pi(z_0, 1), 0) = \phi^+(e^{At}z_0) = \Delta(e^{At}z_0)^{-1}(e^{At}z_0, 1)$$
$$= \Delta(e^{\lambda_{i_0}t}z_0)^{-1}(e^{\lambda_{i_0}t}z_0, 1) \to (z_0, 0) = s_0.$$

This shows that $\pi(z_0, 1) \in \mathcal{W}^-(0, s_0) \cap \mathbb{S}^{n,+}$. Define

$$s_1 = \pi(z_0, 1) \text{ and } s_2 = \pi\varphi^1(\tau, s_1, 0) \text{ for some } \tau > 0.$$

It follows that $s_1, s_2 \in \mathcal{W}^-(0, s_0) \cap \mathbb{S}^{n,+}$ due to invariance of $\mathcal{W}^-(0, s_0)$ and $\mathbb{S}^{n,+}$. Define $z_1 := z_0, z_2 = \left(\phi^+\right)^{-1}(s_1) = \varphi(\tau, z_1, 0)$. The points z_1 and z_2 satisfy

$$\|\varphi(t, z_1, 0) - \varphi(t, z_2, 0)\| = \left\|e^{At}z_0 - e^{A(t+\tau)}z_0\right\| = \left\|e^{\lambda_{i_0}t}z_0 - e^{\lambda_{i_0}(t+\tau)}z_0\right\|$$
$$= e^{\lambda_{i_0}t}\left\|z_0 - e^{\lambda_{i_0}\tau}z_0\right\|.$$

Thus the distance $\|\varphi(t, z_1, 0) - \varphi(t, z_2, 0)\|$ grows with $e^{\lambda_{i_0}t}$. By stability on $\mathbb{S}^n$, it follows for $\alpha \in (\kappa({}_{\mathbb{S}}\mathcal{V}^-), 0) \subset (-\infty, 0)$,

$$d(\pi\varphi^1(t, s_1, 0), \pi\varphi^1(t, s_2, 0))$$
$$\leq d(\pi\varphi^1(t, s_1, 0), \pi\varphi^1(t, s_0, 0)) + d(\pi\varphi^1(t, s_0, 0), \pi\varphi^1(t, s_2, 0))$$
$$\leq e^{-\alpha t}\left[d(s_1, s_0) + d(s_0, s_2)\right] \to 0 \text{ for } t \to \infty.$$

For $t \to \infty$, the points on $\mathcal{W}^-(0, s_0) \cap \mathbb{S}^{n,+}$ converge exponentially, while the points on the preimage $W^-(0, s_0) = \left(\phi^+\right)^{-1}\left(\mathcal{W}^-(0, s_0) \cap \mathbb{S}^{n,+}\right) \subset \mathbb{R}^n$ diverge exponentially with $e^{\lambda_{i_0}t}$. Note that for $t'(t) := t - \tau, t \geq \tau$, one obtains $\|\varphi(t, z_1, 0) - \varphi(t'(t), z_2, 0)\| = 0$.

In the general situation, we obtain the following result.

Theorem 9. *Let the assumptions of Corollary 3 be satisfied and fix $s_0 \in {}_{\mathbb{S}}L(\lambda_{i_0})^\infty \subset \mathbb{S}^{n,0}$ with $\lambda_{i_0} > 0$.*

(i) For every $(u, s_0) \in \mathcal{U} \times {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$, the dimension of $W^-_{loc}(u, s_0) \subset \mathbb{R}^n$ satisfies $\dim W^-_{loc}(u, s_0) \leq \dim {}_{\mathbb{S}}\mathcal{V}^-$. For all $z \in W^-(u, s_0)$ it holds that $\|\varphi(t, z, u)\| \to \infty$ for $t \to \infty$ and the sets $W^-(u, s_0)$ are invariant in the sense that

$$\varphi(t, z, u) \in W^-(u(t+\cdot), \pi\varphi^1(t, s_0, 0)) \text{ for all } t \in \mathbb{R}.$$

(ii) For $j = 1, 2$, let $z_j := \left(\phi^+\right)^{-1}(s_j)$ with $s_j \in \mathcal{W}^-(u, s_0) \cap \mathbb{S}^{n,+}$. Then, for $t > 0$ large enough, there is $t'(t) > 0$ with $t'(t) \to \infty$ for $t \to \infty$ such that

$$\|\varphi(t, z_1, u) - \varphi(t'(t), z_2, u)\| \leq \left\|\varphi^1(t, z_1, 1, u)\right\| d(\pi\varphi^1(t, s_1, u), \pi\varphi^1(t'(t), s_2, u)). \quad (46)$$

Proof. (i) The invariance property follows from invariance of $\mathcal{W}^-(u, s_0)$ and $\mathbb{S}^{n,+}$ and the conjugacy property of ϕ^+. The assertion $\dim W^-_{loc}(u, s_0) \leq \dim {}_{\mathbb{S}}\mathcal{V}^-$ holds since ϕ^+ is a diffeomorphism and $\phi^+(W^-_{loc}(u, s_0)) \subset \mathcal{W}^-_{loc}(u, s_0)$. Furthermore, $\|\varphi(t, z, u)\| \to \infty$ since $\phi^+(\varphi(t, z, u)) \to s_0 \in \mathbb{S}^{n,0}$.
(ii) Corollary 3 and the inequality

$$d(\pi\varphi^1(t, s_1, u), \pi\varphi^1(t, s_2, u))$$
$$\leq d(\pi\varphi^1(t, s_1, u), \pi\varphi^1(t, s_0, u)) + d(\pi\varphi^1(t, s_0, u), \pi\varphi^1(t, s_2, u)),$$

imply that, for every $\alpha \in \left(\kappa({}_{\mathbb{S}}\mathcal{V}^-), 0\right)$,

$$\lim_{t\to\infty} e^{-\alpha t} d(\pi\varphi^1(t, s_1, u), \pi\varphi^1(t, s_2, u)) = 0. \quad (47)$$

For $j = 1, 2$, formula (45) shows that the components of $\pi\varphi^1(t, s_j, u)$ satisfy, for $t \geq 0$,

$$\varphi(t, z_j, u) = \left(\phi^+\right)^{-1}\left(\pi\varphi^1(t, s_j, u)\right) = \left(\pi\varphi^1(t, s_j, u)_{n+1}\right)^{-1}\left(\pi\varphi^1(t, s_j, u)_i\right)^n_{i=1}.$$

With $\pi(z_j, 1) = s_j$, we have

$$\pi\varphi^1(t, s_j, u)_{n+1} = \frac{\varphi^1(t, z_j, 1, u)_{n+1}}{\left\|\varphi^1(t, z_j, 1, u)\right\|} = \frac{1}{\left\|\varphi^1(t, z_j, 1, u)\right\|}$$

implying

$$\varphi(t, z_j, u) = \left\|\varphi^1(t, z_j, 1, u)\right\|\left(\pi\varphi^1(t, \pi(z_j, 1), u)_i\right)^n_{i=1}. \quad (48)$$

We claim that, for $t > 0$ large enough, there is $t'(t) > 0$ with $\left\|\varphi^1(t, z_1, 1, u)\right\| = \left\|\varphi^1(t'(t), z_2, 1, u)\right\|$. In fact, we know that $\left\|\varphi^1(t, z_j, 1, u)\right\| \to \infty$ for $t \to \infty, j = 1, 2$. This implies that $\left\|\varphi^1(t, z_1, 1, u)\right\| > \|(z_2, 1)\|$ for t large enough, and hence $t'(t)$ exists. Thus

$$\|\varphi(t, z_1, u) - \varphi(t'(t), z_2, u)\|$$
$$= \left\|\left\|\varphi^1(t, z_1, 1, u)\right\|\left(\pi\varphi^1(t, s_1, u)\right)^n_{i=1} - \left\|\varphi^1(t'(t), z_2, 1, u)\right\|\left(\pi\varphi^1(t'(t), s_2, u)\right)^n_{i=1}\right\|$$
$$\leq \left\|\varphi^1(t, z_1, 1, u)\right\| d(\pi\varphi^1(t, s_1, u), \pi\varphi^1(t'(t), s_2, u)). \quad \square$$

□

Remark 6. In inequality (46), the first factor converges to ∞ for $t \to \infty$. The second factor satisfies

$$d(\pi\varphi^1(t, s_1, u), \pi\varphi^1(t'(t), s_2, u))$$
$$\leq d(\pi\varphi^1(t, s_1, u), \pi\varphi^1(t, s_2, u)) + d(\pi\varphi^1(t, s_2, u), \pi\varphi^1(t'(t), s_2, u)). \quad (49)$$

We know that $\lim_{t\to\infty} e^{\alpha t} \left\| \pi\varphi^1(t, s_1, u) - \pi\varphi^1(t, s_2, u) \right\| = 0$ for $\alpha \in \left(\kappa({}_{\mathbb{S}}\mathcal{V}^-), 0\right)$. If the exponential growth of $\left\|\varphi^1(t, z_1, 1, u)\right\|$ is smaller than $-\kappa({}_{\mathbb{S}}\mathcal{V}^-)$ one obtains for the first summand in (49) that

$$\lim_{t\to\infty} e^{\alpha t} \left\|\varphi^1(t, z_1, 1, u)\right\| \left\| \pi\varphi^1(t, s_1, u) - \pi\varphi^1(t, s_2, u) \right\| = 0.$$

If $\dim L(\lambda_{i_0} = 1$ then s_0 is an equilibrium on $\mathbb{S}^{n,0}$, and the second summand in (49) converges to 0 for $t \to \infty$. If s_0 is not an equilibrium, it may happen that in the second summand $\pi\varphi^1(t, s_2, u)$ and $\pi\varphi^1(t'(t), s_2, u)$ do not converge for $t \to \infty$. Furthermore, compactness of ${}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ implies that for every sequence $t_k \to \infty$ there are $s_2', s_2'' \in {}_{\mathbb{S}}L(\lambda_{i_0})^\infty$ and a subsequence $t_{k_i}, i \in \mathbb{N}$, such that

$$\pi\varphi^1(t_{k_i}, s_2, u) \to s_2' \text{ and } \pi\varphi^1(t'(t_{k_i}), s_2, u) \to s_2''.$$

Remark 7. Using time reversal, one can show that the results derived in this section for stable manifolds have counterparts for unstable manifolds (cf. Remark 4). Thus one obtains invariant manifolds $W^+(u, s_0)$ in $\mathbb{R}^n$ consisting of points with $\|\varphi(t, z, u)\| \to \infty$ for $t \to -\infty$. We omit the details.

6. Examples

In this section we present several examples illustrating the results in the previous sections.

The following two-dimensional hyperbolic system has been analyzed in Colonius, Santana, Setti [33, Example 2] and Colonius, Santana, Viscovini [7, Example 6.2],

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t) \text{ with } u(t) \in [-1, 1]. \quad (50)$$

For the induced system on the northern hemisphere $\mathbb{S}^{2,+}$ of the Poincaré sphere $\mathbb{S}^2$, one obtains an asymptotically stable equilibrium $s_0 = (1, 0, 0) \subset \mathbb{S}^{2,0} \simeq \mathbb{S}^1$ and an unstable equilibrium $s_0 = (0, 1, 0)$. The phase portrait on $\mathbb{S}^{2,+}$ is sketched in [33, Fig. 2].

We turn to describe higher dimensional examples, where the matrix A is hyperbolic and given in different Jordan normal forms. In the classical treatise Arnol'd [34, § 21] one finds graphical illustrations for the corresponding phase portraits in $\mathbb{R}^3$. For hyperbolic matrix A, Theorem 3 implies that the unique chain control set E in $\mathbb{R}^n$ is bounded and coincides with the closure of the control set D_0 containing the origin. The invariant manifolds depend on the control functions $u \in \mathcal{U}$. For the points at infinity (i.e. on the equator of the Poincaré sphere), the dimensions and the stability properties of the invariant manifolds only depend on the matrix A and not on u. Thus, in the examples below, we do not vary the matrix B in front of $u(t)$.

Example 2. Consider the system in $\mathbb{R}^3$ given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(t), \text{ with } u(t) \in [-1, 1]. \quad (51)$$

The Lyapunov spaces are, with $\lambda_1 = 2, \lambda_2 = 1$, and $\lambda_3 = -1$, given by

$$L(2) = \mathbb{R} \times \{0_2\}, L(1) = \{0\} \times \mathbb{R} \times \{0\}, \text{and } L(-1) = \{0_2\} \times \mathbb{R}.$$

For $\lambda_{i_0} = \lambda_2 = 1 > 0$ consider the set ${}_{\mathbb{S}}L(1)^\infty = \{(0, 1, 0, 0)\} \subset \mathbb{S}^{3,0} \simeq \mathbb{S}^2$. The point $s_0 = (0, 1, 0, 0)$ is an equilibrium at infinity. According to Theorem 7, for the linearized flow $T\pi\Phi^1$ with base space restricted to $\mathcal{U} \times {}_{\mathbb{S}}L(1)^\infty$, the one-dimensional subbundle ${}_{\mathbb{S}}\mathcal{V}_c \subset T_{{}_{\mathbb{S}}L(1)^\infty}\mathbb{S}^3 = T_{(0,1,0,0)}\mathbb{S}^3$ is stable with $\kappa({}_{\mathbb{S}}\mathcal{V}_c) = -\lambda_{i_0} = -1$. The central subbundle $\mathcal{V}_c$ is determined by the unique bounded solutions $e(u, \cdot), u \in \mathcal{U}$. The stable subbundle is ${}_{\mathbb{S}}\mathcal{V}^- = {}_{\mathbb{S}}\mathcal{V}_1 \oplus {}_{\mathbb{S}}\mathcal{V}_c$ where

$$\begin{aligned} {}_{\mathbb{S}}\mathcal{V}_1 &= (\mathrm{id}_{\mathcal{U}}, T\pi) \left(\mathcal{U} \times L(\lambda_1)^\infty\right) = \mathcal{U} \times {}_{\mathbb{S}}L(1)^\infty \times L(2)^\infty \\ &= \mathcal{U} \times \{(0, 1, 0, 0)\} \times \left(\mathbb{R} \times \{0_2\}\right) \subset \mathcal{U} \times T\mathbb{S}^{3,0}. \end{aligned}$$

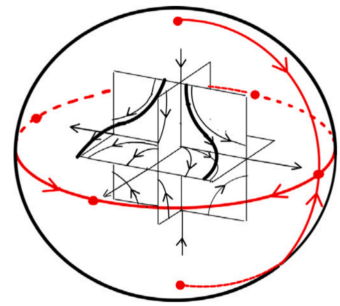

Fig. 1. Phase portraits for $u(t) \equiv 0$ in $\mathbb{R}^3$ and $\mathbb{S}^2 \simeq \mathbb{S}^{3,0}$ in Example 2.

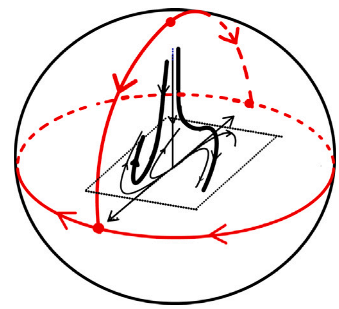

Fig. 2. Phase portraits for $u(t) \equiv 0$ in $\mathbb{R}^3$ and $\mathbb{S}^2 \simeq \mathbb{S}^{3,0}$ in Example 3 .

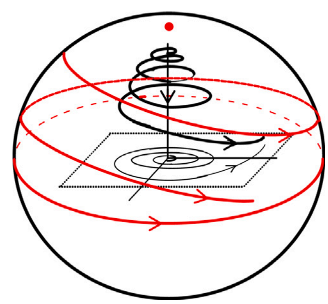

Fig. 3. Phase portraits for $u(t) \equiv 0$ in $\mathbb{R}^3$ and $\mathbb{S}^2 \simeq \mathbb{S}^{3,0}$ in Example 4 .

The unstable subbundle is

$${}_{\mathbb{S}}\mathcal{V}_2 = (\mathrm{id}_{\mathcal{U}}, T\pi) \left(\mathcal{U} \times L(\lambda_3)^\infty\right) = \mathcal{U} \times \{(0, 1, 0, 0)\} \times \left(\{0_2\} \times \mathbb{R}^2\right).$$

By Corollary 3 for $u \in \mathcal{U}$, the local stable manifold $\mathcal{W}^-_{loc}(u, s_0) \subset \mathbb{S}^3$ is two-dimensional and every point $s \in \mathcal{W}^-(u, s_0)$ satisfies for all $\alpha > -1$

$$e^{-\alpha t} d\left(\pi\varphi^1(t, s, u), \pi\varphi^1(t, s_0, u)\right) = e^{-\alpha t} d\left(\pi\varphi^1(t, s, u), s_0\right) \to 0 \text{ for } t \to \infty.$$

For all $u \in \mathcal{U}$ and all $x \in W^-(u, s_0) = \left(\phi^+\right)^{-1} \left(\mathcal{W}^-(u, s_0) \cap \mathbb{S}^{3,+}\right)$ it holds that $\|\varphi(t, x, u)\| \to \infty$ for $t \to \infty$.

The local unstable manifold $\mathcal{W}^+_{loc}(u, s_0) \subset \mathbb{S}^3$ (cf. Remark 7) is one-dimensional, and for all $u \in \mathcal{U}$ and all $x \in W^+(u, s_0) = \left(\phi^+\right)^{-1} \left(\mathcal{W}^+(u, s_0) \cap \mathbb{S}^{3,+}\right)$ it holds that $\|\varphi(t, x, u)\| \to \infty$ for $t \to -\infty$.

For the system with $u(t) \equiv 0$, Fig. 1 sketches the phase portraits on $\mathbb{R}^3$ and on $\mathbb{S}^2$ (we cannot draw the phase portrait on $\mathbb{S}^3$). The sphere $\mathbb{S}^2$ is identified with the equator $\mathbb{S}^{3,0}$ of the Poincaré sphere $\mathbb{S}^3$, hence it represents infinity. There are four equilibria at infinity on the equator $\mathbb{S}^{2,0}$ of $\mathbb{S}^2$ and the poles $(0, 0, \pm 1)$ of $\mathbb{S}^2$ are unstable equilibria at infinity. The equilibrium $s_0 = (0, 1, 0, 0) \in \mathbb{S}^{3,0}$ is identified with $(0, 1, 0) \in \mathbb{S}^{2,0} \subset \mathbb{S}^2$. Its stable and unstable manifolds on $\mathbb{S}^2$ are the half circle between $(0, 0, 1)$ and $(0, 0, -1)$ and the half circle between $(1, 0, 0)$ and $(-1, 0, 0)$, respectively. The set $W^-(0, s_0)$ is contained in the half-space in $\mathbb{R}^3$ spanned by $(0, 1, 0)$ and $(0, 0, 1)$.

In our next example the set ${}_{\mathbb{S}}L(1)^{\infty}$ is not a minimal set for the induced flow on $\mathbb{S}^{3,0}$.

Example 3. Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(t) \text{ with } u(t) \in [-1,1]. \quad (52)$$

The Lyapunov spaces are, with $\lambda_1 = 1$ and $\lambda_2 = -1$ given by

$$L(1) = \mathbb{R}^2 \times \{0\} \text{ and } L(-1) = \{0_2\} \times \mathbb{R}.$$

For $\lambda_{i_0} = \lambda_1 = 1 > 0$ consider the set

$${}_{\mathbb{S}}L(1)^{\infty} = \left\{ (s_1, s_2, 0, 0) \,\middle|\, s_1^2 + s_2^2 = 1 \right\} \subset \mathbb{S}^{3,0}.$$

When we again identify the equator $\mathbb{S}^{3,0}$ with $\mathbb{S}^2$, the set ${}_{\mathbb{S}}L(1)^{\infty}$ is identified with the equator $\mathbb{S}^{2,0} \simeq \mathbb{S}^1$ of $\mathbb{S}^2$. For the linearized flow $T\pi\Phi^1$ with base space restricted to $\mathcal{U} \times {}_{\mathbb{S}}L(1)^{\infty}$, the one-dimensional subbundle ${}_{\mathbb{S}}\mathcal{V}_c \subset T_{{}_{\mathbb{S}}L(1)^{\infty}}\mathbb{S}^3$ is stable with $\kappa({}_{\mathbb{S}}\mathcal{V}_c) = -\lambda_{i_0} = -1$. Furthermore, also the subbundle ${}_{\mathbb{S}}\mathcal{V}_{i_0} = {}_{\mathbb{S}}\mathcal{V}_1$ is one-dimensional and the corresponding Lyapunov exponents are 0. The subbundle ${}_{\mathbb{S}}\mathcal{V}_2$ is stable with Lyapunov exponents equal to $\lambda_2 - \lambda_{i_0} = -1 - 1 = -2$. Hence the set $\mathcal{U} \times {}_{\mathbb{S}}L(1)^{\infty}$ is stable. For the system with $u(t) \equiv 0$, Fig. 2 sketches the phase portraits on $\mathbb{R}^3$ and on $\mathbb{S}^2 \simeq \mathbb{S}^{3,0}$. There are two equilibria $(\pm 1, 0, 0)$ at infinity on the equator $\mathbb{S}^{2,0}$ of $\mathbb{S}^2$ which correspond to the eigenspace $\mathbb{R} \times \{0\}$ of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Furthermore, the poles $(0, 0, \pm 1)$ of $\mathbb{S}^2$ are unstable equilibria at infinity.

In the following example the matrix A has a complex-conjugate pair of eigenvalues. Here a periodic solution at infinity is obtained.

Example 4. Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(t) \text{ with } u(t) \in [-1,1]. \quad (53)$$

The eigenvalues of the matrix A are $\mu_{1,2} = 1 \pm \imath$ and $\mu_3 = -1$, hence the Lyapunov exponents are $\lambda_1 = 1$ and $\lambda_2 = -1$ with Lyapunov spaces

$$L(1) = \mathbb{R}^2 \times \{0\} \text{ and } L(-1) = \{0_2\} \times \mathbb{R}.$$

For $\lambda_{i_0} = \lambda_1 = 1 > 0$ consider the set

$${}_{\mathbb{S}}L(1)^{\infty} = \left\{ (s_1, s_2, 0, 0) \,\middle|\, s_1^2 + s_2^2 = 1 \right\} \subset \mathbb{S}^{3,0}.$$

This set is identified with the equator $\mathbb{S}^{2,0} \simeq \mathbb{S}^1$ of $\mathbb{S}^2 \simeq \mathbb{S}^{3,0}$. Restricted to ${}_{\mathbb{S}}L(1)^{\infty}$ the orbit $\pi\varphi^1(\cdot, s, 0)$ corresponds to

$$\dot{x}_1(t) = x_2(t), \dot{x}_2(t) = -x_1(t), \text{ hence } \left(x_1(t), x_2(t)\right) = (\sin t, \cos t).$$

For the linearized flow $T\pi\Phi^1$ with base space restricted to $\mathcal{U} \times {}_{\mathbb{S}}L(1)^{\infty}$, the subbundle ${}_{\mathbb{S}}\mathcal{V}_c \subset T_{{}_{\mathbb{S}}L(1)^{\infty}}\mathbb{S}^3$ is stable with $\kappa({}_{\mathbb{S}}\mathcal{V}_c) = -\lambda_{i_0} = -1$. The subbundle ${}_{\mathbb{S}}\mathcal{V}_{i_0} = {}_{\mathbb{S}}\mathcal{V}_1$ is one-dimensional and the subbundle ${}_{\mathbb{S}}\mathcal{V}_2$ is stable, hence the set $\mathcal{U} \times {}_{\mathbb{S}}L(1)^{\infty}$, i.e., the periodic solution at infinity, is stable. For the system with $u(t) \equiv 0$, Fig. 3 sketches the phase portraits on $\mathbb{R}^3$ and on $\mathbb{S}^2 \simeq \mathbb{S}^{3,0}$.

In the next example, the periodic solution at infinity has a nontrivial stable manifold.

Example 5. Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} u(t) \text{ with } u(t) \in [-1,1]. \quad (54)$$

The eigenvalues of the matrix A are $\mu_1 = 2, \mu_{2,3} = 1 \pm \imath$, and $\mu_4 = -1$, hence the Lyapunov exponents are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$ with Lyapunov spaces

$$L(2) = \mathbb{R} \times \left\{0_3\right\}, L(1) = \{0\} \times \mathbb{R}^2 \times \{0\} \text{ and } L(-1) = \{0_3\} \times \mathbb{R}.$$

For $\lambda_{i_0} = \lambda_2 = 1 > 0$ consider the set

$${}_{\mathbb{S}}L(1)^{\infty} = \left\{ (0, s_2, s_3, 0, 0) \,\middle|\, s_2^2 + s_3^2 = 1 \right\} \subset \mathbb{S}^{4,0}.$$

This set is identified with the subset $\{0\} \times \mathbb{S}^{2,0} \times \{0\} \simeq \mathbb{S}^1$ of $\mathbb{S}^3 \simeq \mathbb{S}^{4,0}$. Restricted to ${}_{\mathbb{S}}L(1)^{\infty}$ the trajectory $\pi\varphi^1(\cdot, s, 0)$ corresponds to

$$\dot{x}_2(t) = x_3(t), \dot{x}_3(t) = -x_2(t), \text{ hence } \left(x_2(t), x_3(t)\right) = (\sin t, \cos t).$$

For the linearized flow $T\pi\Phi^1$ with base space restricted to $\mathcal{U} \times {}_{\mathbb{S}}L(1)^{\infty}$, the subbundle ${}_{\mathbb{S}}\mathcal{V}_c \subset T_{{}_{\mathbb{S}}L(1)^{\infty}}\mathbb{S}^4$ is stable with $\kappa({}_{\mathbb{S}}\mathcal{V}_c) = -\lambda_{i_0} = -1$. For points in the subbundles ${}_{\mathbb{S}}\mathcal{V}_1$ and ${}_{\mathbb{S}}\mathcal{V}_3$ the Lyapunov exponents are

$$\lambda_1 - \lambda_{i_0} = 2 - 1 = 1 \text{ and } \lambda_3 - \lambda_{i_0} = -1 - 1 = -2,$$

respectively. Hence the stable subbundle is ${}_{\mathbb{S}}\mathcal{V}_c \oplus {}_{\mathbb{S}}\mathcal{V}_3$ and ${}_{\mathbb{S}}\mathcal{V}_1$ is unstable. Let $s_0 \in {}_{\mathbb{S}}L(1)^{\infty}$ and $u \in \mathcal{U}$. The local stable manifold $\mathcal{W}^-_{loc}(u, s_0) \subset \mathbb{S}^4$ is two-dimensional and every point $s \in \mathcal{W}^-(u, s_0)$ satisfies for all $\alpha \in (-1, 0)$

$$e^{-\alpha t} d\left(\pi\varphi^1(t, s, u), \pi\varphi^1(t, s_0, 0)\right) \to 0 \text{ for } t \to \infty.$$

For all $x \in W^{\pm}(u, s_0) = \left(\phi^+\right)^{-1}\left(\mathcal{W}^{\pm}(u, s_0) \cap \mathbb{S}^{4,+}\right)$ it holds that $\|\varphi(t, x, u)\| \to \infty$ for $t \to \pm\infty$.

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We do not use it anywhere.

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