Central limit theorems for empirical product densities of stationary point processes

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Abstract

In the present work we investigate kernel-type estimators for product densities and for the pair correlation function for stationary spatial point processes. In the setting of Brillinger-mixing point processes we present central limit theorems for these estimators and for the integrated squared error of the estimators for the second-order product density and the pair correlation function. Based on these central limit theorems we can construct asymptotic goodness-of-fit tests for the distribution of a stationary point process.

Zusammenfassung

In dieser Arbeit untersuchen wir Kernschätzer für Produktdichten und für die Paarkorrelationsfunktion für stationäre räumliche Punktprozesse. Im Fall von Brillinger-mischenden Punktprozessen leiten wir für diese Schätzer und für den integrierten quadratischen Fehler der empirischen Produktdichte zweiter Ordnung und der empirischen Paarkorrelationsfunktion Zentrale Grenzwertsätze her. Aus diesen Zentralen Grenzwertsätzen lassen sich Anpassungstests zur Prüfung auf die Verteilung eines stationären Punktprozesses konstruieren.

AMS Mathematics Subject Classification: primary: 60G55, 62M30; secondary: 62G20

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Notation

$\oplus$	Minkowski addition (see page 46)
$\ominus$	Minkowski subtraction (see page 117)
$\lvert A\rvert$	d-dimensional Lebesgue measure of the set $A \in \mathfrak{B}(\mathbb{R}^d)$ (see page 7)
$\xrightarrow[n\to\infty]{\mathrm{d}}$	weak convergence as $n \to \infty$ (see page 62)
$\alpha^{(k)}$	kth-order factorial moment measure (see page 6)
$\alpha^{(k)}_{\mathrm{red}}$	kth-order reduced factorial moment measure (see page 8)
b_n	bandwidth (see page 46)
$b(x,r)$	closed ball in $\mathbb{R}^d$ with radius $r > 0$ and midpoint $x \in \mathbb{R}^d$ (see page 10)
$b^o(x,r)$	open ball in $\mathbb{R}^d$ with radius $r > 0$ and midpoint $x \in \mathbb{R}^d$ (see page 14)
$b_{\ell-1}(x,r)$	closed ball in $(\mathbb{R}^d)^{\ell-1}$ with radius $r > 0$ and midpoint $x \in (\mathbb{R}^d)^{\ell-1}$, $\ell \geq 2$ (see page 42)
$b^o_{\ell-1}(x,r)$	open ball in $(\mathbb{R}^d)^{\ell-1}$ with radius $r > 0$ and midpoint $x \in (\mathbb{R}^d)^{\ell-1}$, $\ell \geq 2$ (see page 69)
$\mathfrak{B}(\mathbb{R}^d)$	Borel-σ-algebra on $\mathbb{R}^d$ (see page 5)
$c^{(k)}$	kth-order cumulant density (see page 9)
$\mathrm{Cum}_k(X_1,\dots,X_k)$	mixed cumulant of the random vector $(X_1,\dots,X_k)'$ (see page 32)
χ^2_q	χ^2-distribution with q degrees of freedom (see page 62)
Δ_n	scaled deviation of empirical product densities (see pages 61 and 72)
$\Delta^{(g)}_n$	scaled deviation of the empirical pair correlation function (see page 67)
$F^{-1}_{\chi^2_q}$	quantile function of the χ^2-distribution with q degrees of freedom (see page 114)
g	pair correlation function (see page 9)
$\Gamma_k(X)$	kth cumulant of the random variable X (see page 32)

$\gamma^{(k)}$	kth-order factorial cumulant measure (see page 7)
$\gamma_{\text{red}}^{(k)}$	kth-order reduced factorial cumulant measure (see page 8)
$\lVert\gamma_{\text{red}}^{(k)}\rVert$	total variation of $\gamma_{\text{red}}^{(k)}$ (see page 8)
$\mathcal{H}_{d-1}$	$(d-1)$-dimensional Hausdorff measure (see page 107)
$I_n(K)$	integrated squared error of the estimated second-order product density and the pair correlation function (see pages 78 and 105)
$k,\ k^{(\ell,d)}$	kernel function (see page 42)
$\tilde{k}$	convolution of the kernel function (see page 43)
λ	intensity (see page 7)
$\mu^*_{k_1,\dots,k_j}$	sum of indecomposable integrals (see page 37)
N	set of all locally-finite counting measures on $\mathbb{R}^d$ (see page 5)
$\mathcal{N}$	σ-algebra induced by N (see page 5)
$\mathbb{N}$	set of all positive integers
$\mathrm{N}(\mu,\sigma^2)$	normal distribution with mean μ and variance σ^2 (see page 99)
$\mathrm{N}(\mu,\Sigma_q)$	normal distribution with mean μ and covariance matrix Σ_q (see page 62)
o	origin in $\mathbb{R}^d$ (see page 10)
$o(.),\ O(.)$	Landau notation (see page 52)
ω_d	volume of the unit ball in $\mathbb{R}^d$ (see page 10)
Π_λ	distribution of a stationary Poisson process with intensity λ (see page 11)
Φ^{-1}_{μ,σ^2}	quantile function of the normal distribution with mean μ and variance σ^2 (see page 113)
$\varrho^{(k)}$	kth-order product density (see page 8)
$\varrho,\ \varrho^{(2)}$	second-order product density (see page 8)
$\hat{\varrho}_n^{(\ell)}$	estimator for the ℓth-order product density (see page 46)
$\hat{\varrho}_n,\ \hat{\varrho}_n^{(2)}$	estimator for the second-order product density (see page 46)
$\rho(W)$	inscribed radius of the set $W \subseteq \mathbb{R}^d$ (see page 46)
T_x	translation operator (see page 12)
W_n	observation window (see page 46)

1

Introduction

A point process can be thought of as a set of points randomly scattered in space. Formally, a point process is defined as a random locally-finite counting measure on the d-dimensional Euclidean space $\mathbb{R}^d$. Fields of applications for point processes are material sciences (see Ohser and Mücklich [48] and Torquato [63]), image processing, analysis of the structure of tissues in medical sciences, geological sciences and seismology, forestry and ecology, astronomy, and statistical physics. Most of these applications refer to point processes in $\mathbb{R}^2$. Comprehensive representations of spatial statistics and statistics for random geometrical structures can be found in Stoyan et al. [57] and Cressie [7]. Overviews of statistics for spatial point processes are given in Ripley [51], Karr [42], Diggle [14], and Stoyan [56].

An important task of statistics for point processes is model identification, that is, the formulation of a mathematical model giving a satisfactory description of an observed point pattern. With such mathematical models one can, for instance, draw conclusions about properties of certain materials or tissues. In the last two decades various articles on nonparametric problems in point process theory have been published, mainly concerning so-called summary statistics such as Ripley's K-function, the nearest-neighbor distance distribution function, the empty-space function, and other functions based on these characteristics such as the L- or the J-function. These summary statistics are used for verifying or rejecting hypothetical point process models by graphical investigations or simulation tests, see Stoyan et al. [57], Baddeley and Turner [2], and Baddeley et al. [3]. Often these investigations focus on complete spatial randomness, see Zimmerman [67], Grabarnik and Chiu [18], and Ho and Chiu [36]. Most tests used in applications are based on heuristic considerations rather than on mathematical models. This is mainly due to the latter models' complexity caused by dimension and by stochastic dependencies of neighboring areas. Furthermore one has to cope with rather little information on the point process since in most cases there is only one observation in a given observation window available.

An approach originating from ergodic theory for stationary spatial processes is the idea of considering point processes in a convex observation window expanding in every direction. This allows the derivation of consistency properties of estimators and, under additional mixing assumptions, limiting distributions of these estimators. These limiting distributions can be used for test procedures, where the true distribution of the considered test statistic in finite observation windows is not known. A frequent problem in this context is that such limiting distributions depend on the underlying assumptions in a complicated way. This is not the case for the limiting distributions of the test statistics studied in the present work.

The derivation of limit theorems for estimators of product densities and of the pair correlation function requires mixing properties that guarantee weak dependence between numbers of points in areas that are far apart from each other. In this work we use the concept of Brillinger-mixing point processes. Another mixing property that is suitable for asymptotical investigations for statistics for point processes and random sets is absolute regularity (also called β-mixing). The asymptotic behavior of absolutely regular tessellations has been studied in Heinrich [27]. The concepts developed there have been applied to germ-grain models in Heinrich and Molchanov [31].

An important characteristic for stationary point processes is the second-order product density $\varrho(x)$, $x \in \mathbb{R}^d$, which is defined as the Lebesgue density of the second-order reduced factorial moment measure and contains information about interaction between points. If the stationary point process is isotropic, that is, if its distribution is rotation invariant, one can consider the pair correlation function instead of the second-order product density without loss of information. The pair correlation function is derived from the product density by $g(r) = \varrho(x)/\lambda^2$ for $r = \|x\| \geq 0$, $x \in \mathbb{R}^d$. (Here, λ denotes the intensity of the point process.) Due to its simple interpretation and straightforward graphical representation the pair correlation function is more popular in applications than the second-order product density. Although second-order quantities do not characterize the distribution of the point process (see Baddeley and Silverman [4] for an illustrative example) they still give a rather informative description of the point pattern.

In the present work we will use the second-order product density and its isotropic analogue, the pair correlation function, to construct goodness-of-fit tests for a wide class of stationary point processes. These goodness-of-fit tests are based on central limit theorems for kernel-type estimators of product densities and the pair correlation function under mild mixing conditions. Based on one realization of a point process in a convex observation window expanding in every direction we determine kernel-type estimators for product densities and the pair correlation function and study their deviations from a hypothetical product density or pair correlation function. Such deviations are, for example, the sum of squared differences between estimated and hypothetical product densities evaluated on a given finite set of points, or the integrated squared error of the estimated second-order product density and of the estimated pair correlation function. We will derive limit theorems for these deviations in the setting of Brillinger-mixing point processes. The limiting distribution will solely depend on the underlying hypothetical second-order quantity, the intensity of the point process, and the kernel function. This allows the construction of distribution-free testing procedures. We will also discuss under which assumptions various point process models are Brillinger-mixing.

The normal convergence of empirical product densities has been proved in Jolivet [40] for Brillinger-mixing point processes based on a decomposition of their cumulants (see Leonov and Shiryaev [45] and Jolivet [39]). This method has already been used for proving asymptotic normality of empirical reduced factorial moment measures on fixed sets such as Ripley's K-function (see Jolivet [39]). Jolivet [40] sketches the proof of asymptotic normality of estimated product densities only briefly and does not state all necessary assumptions. The present work will be more rigorous.

In Heinrich et al. [29] and Heinrich and Muche [32] the pair correlation function of the point process of nodes in a stationary Poisson Voronoi tessellation has been determined, and Heinrich et al. [33] derived a corresponding formula for Poisson hyperplane tessellations. The asymptotic behavior of the empirical product density in the setting of absolutely regular point processes has been studied in Heinrich and Liebscher [30]. For Poisson cluster processes corresponding results can be found in Heinrich [26]. For the point process of exposed tangent points of Boolean models the asymptotic investigations in Heinrich and Werner [35] yield a χ^2-goodness-of-fit test for a hypothetical radius distribution of Boolean models with spherical grains.

Heinrich [28] proves a functional central limit theorem for the estimated K-function. David [10] extends this result to a multivariate K-function based on cuboids of varying size. However, the methods used in Heinrich [28] and David [10] for achieving functional limit theorems cannot be used for deriving similar results for product densities.

There is a vast amount of literature on estimators for probability densities (see Silverman [54], Devroye [12], and Wand and Jones [64], for example). Hall [22] proves a central limit theorem for the integrated squared error of nonparametric probability density estimators. Csörgö and Horváth [8] derive central limit theorems for L_p-norms of kernel-type density estimators. Horváth [37] extends this result to multivariate probability densities.

The present work is organized as follows. Chapter 2 gives a brief introduction to the theory of point processes. We introduce product densities and the pair correlation function, and we present some examples of point process models. In Chapter 3 we give an overview of mixing properties for point processes and study assumptions that imply various classes of point processes to be Brillinger-mixing. Chapter 4 presents properties of cumulants that will be useful for the derivation of the central limit theorems in Chapters 6 and 7. In Chapter 5 we introduce estimators for product densities and the pair correlation function. The asymptotic behavior of these estimators is studied in Chapter 6. More precisely we show asymptotic normality for the scaled deviations of these estimators and their respective hypothetical quantities and give asymptotic representations for the mean and the variance. In Chapter 7 we derive central limit theorems for the integrated squared error of the estimated second-order product density and pair correlation function and give asymptotic representations for the mean and the variance. Chapter 8 gives a short description of how the central limit theorems in Chapters 6 and 7 can be used for constructing asymptotic goodness-of-fit tests. The last chapter summarizes the main results of the present work and outlines some open questions. Finally, Appendix A presents some properties of the sequence of observation windows that are needed for the asymptotic results in Chapters 6 and 7.

2

Point processes

This chapter gives an introduction to the theory of point processes. We restrict our presentation primarily to the notions relevant for the following chapters. Comprehensive representations of point process theory can for instance be found in Stoyan et al. [57], König and Schmidt [43], and Daley and Vere-Jones [9].

The first section presents the definition of point processes and basic concepts such as stationarity and isotropy. We continue with notions like moment measures and kth-order stationarity in the second section. The third section focuses on point process characteristics and their interpretation. The chapter concludes with some examples of point process models.

2.1 Definition of point processes, stationarity, and isotropy

Let N be the set of all locally-finite counting measures on $\mathbb{R}^d$, $d \geq 1$, and let $\mathcal{N}$ be the σ-algebra induced by the sets $\{\psi \in N : \psi(B) = k\}$, where $k \in \mathbb{N}_0$ and $B \in \mathfrak{B}(\mathbb{R}^d)$ is bounded. Here $\mathfrak{B}(\mathbb{R}^d)$ denotes the Borel-σ-algebra on $\mathbb{R}^d$. A point process in $\mathbb{R}^d$ is defined as a measurable mapping Ψ from a probability space $[\Omega, \mathcal{A}, \mathbb{P}]$ into $[N, \mathcal{N}]$. Let $P = \mathbb{P} \circ \Psi^{-1}$ be the probability measure on $[N, \mathcal{N}]$ induced by Ψ. We call P the *distribution of the point process* Ψ and write $\Psi \sim P$.

In the present work we will only consider *simple point processes in* $\mathbb{R}^d$, that is, $\mathbb{P}(\Psi(\{x\}) \leq 1$ for all $x \in \mathbb{R}^d\}) = 1$.

Two important classes of statistically tractable point processes are given by the concepts of *stationarity* and *isotropy*.

Definition 2.1.1 *A point process $\Psi \sim P$ in $\mathbb{R}^d$ is said to be* stationary *if*

$$(\Psi(B_1 + x), \ldots, \Psi(B_k + x)) \quad \stackrel{\mathrm{d}}{=} \quad (\Psi(B_1) \ldots, \Psi(B_k))$$

for all $x \in \mathbb{R}^d$, $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$ and $k \geq 1$. Here $\stackrel{\mathrm{d}}{=}$ denotes identity of distributions.

A point process $\Psi \sim P$ in $\mathbb{R}^d$ is said to be isotropic *if*

$$(\Psi(U^{-1}B_1), \ldots, \Psi(U^{-1}B_k)) \quad \stackrel{\mathrm{d}}{=} \quad (\Psi(B_1), \ldots, \Psi(B_k))$$

for all $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$, $k \geq 1$, and matrices $U \in \mathrm{SO}(d)$, where $\mathrm{SO}(d)$ is the special orthogonal group over $\mathbb{R}^d$. □

2.2 Moment measures and *k*th-order stationarity

Let $\mathrm{supp}(\psi)$ be the support of the counting measure $\psi \in N$. In the following we will use the abbreviated notation $x \in \Psi$ for $x \in \mathrm{supp}(\Psi)$. Further, $\sum^*$ denotes summation over summands with index tuples having pairwise distinct components.

Definition 2.2.1 *The* kth-order factorial moment measure $\alpha^{(k)}$ *of the point process $\Psi \sim P$ in $\mathbb{R}^d$ is defined by*

$$\alpha^{(k)}(B_1 \times \cdots \times B_k) \quad := \quad \mathbb{E} \sum_{x_1,\ldots,x_k \in \Psi}^{*} \mathbb{1}_{B_1}(x_1) \cdots \mathbb{1}_{B_k}(x_k),$$

where $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$ and $k \geq 1$. □

The first-order factorial moment measure $\alpha^{(1)}$ is called *intensity measure* of the point process Ψ due to $\alpha^{(1)}(B) = \mathbb{E}\Psi(B)$ being the mean number of points of Ψ in a Borel set B.

A popular approach in the analysis of point processes is the use of characteristics based on factorial moment measures, especially the second-order factorial moment measure, although in general the distribution of Ψ is not uniquely determined by the factorial moment measures. Baddeley and Silverman [4] give examples of non-Poissonian point processes with the same first- and second-order factorial moment measures as the Poisson process.

Definition 2.2.2 *The* kth-order factorial cumulant measure $\gamma^{(k)}$ *of the point process* $\Psi \sim P$ *in* $\mathbb{R}^d$ *is defined by*

$$\gamma^{(k)}(B_1 \times \cdots \times B_k) \;\; := \;\; \sum_{\ell=1}^{k}(-1)^{\ell-1}(\ell-1)! \sum_{\substack{K_1 \cup \ldots \cup K_\ell \\ =\{1,\ldots,k\}}} \prod_{j=1}^{\ell} \alpha^{(|K_j|)}\Big(\mathop{\times}_{k_j \in K_j} B_{k_j} \Big),$$

where $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$ *and* $k \geq 1$*. Here,* $\sum_{\substack{K_1 \cup \ldots \cup K_\ell \\ =\{1,\ldots,k\}}}$ *denotes summation over all* ℓ*-partitions of* $\{1, \ldots, k\}$*, where an* ℓ*-partition of* $\{1, \ldots, k\}$ *is a family of nonempty pairwise disjoint subsets* $K_1, \ldots, K_\ell \subseteq \{1, \ldots, k\}$ *with* $K_1 \cup \ldots \cup K_\ell = \{1, \ldots, k\}$*.* □

Note that $\gamma^{(k)}$ is a signed measure on $[(\mathbb{R}^d)^k, \mathfrak{B}\big((\mathbb{R}^d)^k\big)]$.

Due to Definition 2.2.2 the kth-order factorial moment measure $\alpha^{(k)}$ can be expressed by the factorial cumulant measures up to order k by

$$\alpha^{(k)}\Big(\mathop{\times}_{j=1}^{k} B_j \Big) \;\; = \;\; \sum_{j=1}^{k} \sum_{\substack{K_1 \cup \ldots \cup K_j \\ =\{1,\ldots,k\}}} \prod_{i=1}^{j} \gamma^{(|K_i|)}\Big(\mathop{\times}_{k_i \in K_i} B_{k_i} \Big) \tag{2.1}$$

with $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$ and $k \geq 1$.

Definition 2.2.3 *Let* $k \geq 1$*. A point process* $\Psi \sim P$ *in* $\mathbb{R}^d$ *is said to be* kth-order stationary *(henceforth abbreviated as* k-stationary*) if* $\mathbb{E}\Psi^k([0,1]^d) < \infty$ *and*

$$\alpha^{(j)}((B_1 + x) \times \cdots \times (B_j + x)) \;\; = \;\; \alpha^{(j)}(B_1 \times \cdots \times B_j)$$

for all $B_1, \ldots, B_j \in \mathfrak{B}(\mathbb{R}^d)$*,* $j = 1, \ldots, k$*, and* $x \in \mathbb{R}^d$*.*

A 2*-stationary point process* Ψ *is called* weakly stationary*.* □

Clearly, stationarity implies weak stationarity.

If the point process Ψ is at least 1-stationary, then its intensity measure is translation invariant and thus a multiple of the Lebesgue measure $|\,.\,|$ on $[\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d)]$, that is, there exists $\lambda \in (0, \infty)$ with

$$\alpha^{(1)}(B) \; = \; \lambda |B|$$

for all $B \in \mathfrak{B}(\mathbb{R}^d)$. The constant $\lambda = \mathbb{E}\Psi([0,1]^d)$ is called the *intensity* of Ψ and is the mean number of points of Ψ in the unit cube $[0,1]^d$.

Definition 2.2.4 *Let the point process $\Psi \sim P$ in $\mathbb{R}^d$ be k-stationary with intensity $\lambda > 0$ and $k \geq 2$. The measure $\alpha_{\text{red}}^{(k)}$ on $[(\mathbb{R}^d)^{k-1}, \mathfrak{B}((\mathbb{R}^d)^{k-1})]$ that is uniquely determined by*

$$\alpha^{(k)}(B_1 \times \cdots \times B_k) \quad = \quad \lambda \int_{B_k} \alpha_{\text{red}}^{(k)}((B_1 - x) \times \cdots \times (B_{k-1} - x))\,\mathrm{d}x$$

for all $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$ is called kth-order reduced factorial moment measure*. The measure $\gamma_{\text{red}}^{(k)}$ on $[(\mathbb{R}^d)^{k-1}, \mathfrak{B}((\mathbb{R}^d)^{k-1})]$ that is uniquely determined by*

$$\gamma^{(k)}(B_1 \times \cdots \times B_k) \quad = \quad \lambda \int_{B_k} \gamma_{\text{red}}^{(k)}((B_1 - x) \times \cdots \times (B_{k-1} - x))\,\mathrm{d}x$$

for all $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$ is called kth-order reduced factorial cumulant measure*.*

The total variation measure $|\gamma_{\text{red}}^{(k)}|$ *is defined by*

$$|\gamma_{\text{red}}^{(k)}|(\,.\,) = (\gamma_{\text{red}}^{(k)})^{+}(\,.\,) + (\gamma_{\text{red}}^{(k)})^{-}(\,.\,),$$

where the measures $(\gamma_{\text{red}}^{(k)})^{+}$ and $(\gamma_{\text{red}}^{(k)})^{-}$ are given by the Jordan decomposition

$$\gamma_{\text{red}}^{(k)}(\,.\,) = (\gamma_{\text{red}}^{(k)})^{+}(\,.\,) - (\gamma_{\text{red}}^{(k)})^{-}(\,.\,).$$

The total variation *of $\gamma_{\text{red}}^{(k)}$ is defined by*

$$\|\gamma_{\text{red}}^{(k)}\| \quad := \quad |\gamma_{\text{red}}^{(k)}|\big((\mathbb{R}^d)^{k-1}\big).$$

□

The fact that $\alpha_{\text{red}}^{(k)}$ coincides with the $(k-1)$th-order moment measure of the reduced Palm distribution will be used for the interpretation of the quantities introduced in the following section. For the definition of the reduced Palm distribution see, for instance, Stoyan et al. [57], page 121.

If the kth-order reduced factorial moment measure $\alpha_{\text{red}}^{(k)}$ is absolutely continuous with respect to the Lebesgue measure on $[(\mathbb{R}^d)^{k-1}, \mathfrak{B}((\mathbb{R}^d)^{k-1})]$, then its Lebesgue density $\varrho^{(k)}$ is given by

$$\alpha_{\text{red}}^{(k)}(B_1 \times \cdots \times B_{k-1}) \quad = \quad \int_{B_1} \cdots \int_{B_{k-1}} \varrho^{(k)}(x_1, \ldots, x_{k-1})\,\mathrm{d}x_1 \cdots \mathrm{d}x_{k-1},$$

where $B_1, \ldots, B_{k-1} \in \mathfrak{B}(\mathbb{R}^d)$, and is called the *kth-order reduced product density*, henceforth abbreviated as *kth-order product density*.

If the kth-order reduced factorial cumulant measure $\gamma_{\text{red}}^{(k)}$ is absolutely continuous with respect to the Lebesgue measure on $[(\mathbb{R}^d)^{k-1}, \mathfrak{B}\big((\mathbb{R}^d)^{k-1}\big)]$, then its Lebesgue density $c^{(k)}$ is given by

$$\gamma_{\text{red}}^{(k)}(B_1 \times \cdots \times B_{k-1}) \;=\; \int\limits_{B_1} \cdots \int\limits_{B_{k-1}} c^{(k)}(x_1, \ldots, x_{k-1})\, \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1},$$

where $B_1, \ldots, B_{k-1} \in \mathfrak{B}(\mathbb{R}^d)$, and is called the *kth-order reduced cumulant density*, henceforth abbreviated as *kth-order cumulant density.*

Remark 2.2.5 If the kth-order cumulant density exists, then the assumption

$$\|\gamma_{\text{red}}^{(k)}\| \;=\; |\gamma_{\text{red}}^{(k)}|\big((\mathbb{R}^d)^{k-1}\big) = \int\limits_{(\mathbb{R}^d)^{k-1}} |c^{(k)}(x)| \mathrm{d}x < \infty$$

on the total variation implies the kth-order cumulant density $c^{(k)}(x)$ to be finite for Lebesgue-almost all $x \in (\mathbb{R}^d)^{k-1}$ and to satisfy

$$\int\limits_{(\mathbb{R}^d)^{\# S}} |c^{(k)}(x_1, \ldots, x_{k-1})| \mathrm{d}\mathbf{x}_S < \infty$$

for Lebesgue-almost all $\mathbf{x}_{\{1,\ldots,k-1\}\setminus S} \in (\mathbb{R}^d)^{k-1-\# S}$ for all $S \subseteq \{1, \ldots, k-1\}$ with $S \neq \emptyset$. Here $\# S$ denotes the cardinality of S and $\mathbf{x}_S = (x_i)_{i \in S}$. □

2.3 Point process characteristics

In the present work we will focus on product densities as introduced in the previous section and, in particular, on the second-order product density $\varrho^{(2)}$ and the *pair correlation function* defined by

$$g(r) := \frac{\varrho^{(2)}(x)}{\lambda},$$

where $r = \|x\|$, $x \in \mathbb{R}^d$, and λ is the intensity of the 2-stationary point process. For 2-stationary and isotropic point processes with intensity λ the pair correlation function g contains the same information as the second-order product density $\varrho^{(2)}$ normalized with the intensity λ. The standardization with the intensity λ implies the pair correlation function to be the constant 1 for the stationary Poisson process, see Section 2.4. The pair correlation function is more popular for data analysis than the second-order product density due to its simple interpretation and straightforward graphical representation. An alternative second-order characteristic is *Ripley's*

K-function (see Ripley [50]) defined by

$$K(r) := \frac{1}{\lambda}\,\alpha^{(2)}_{\mathrm{red}}\big(b(o,r)\big), \quad r \geq 0,$$

where $o = (0,\ldots,0)' \in \mathbb{R}^d$ denotes the origin and $b(x,r) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ denotes the closed ball in $\mathbb{R}^d$ with radius $r > 0$ and midpoint $x \in \mathbb{R}^d$. The K-function provides an alternative definition of the pair correlation function given by

$$g(r) = \left.\frac{\frac{\mathrm{d}}{\mathrm{d}s}K(s)}{d\,\omega_d\,s^{d-1}}\right|_{s=r}, \quad r \geq 0,$$

where $\omega_d = \frac{\sqrt{\pi^d}}{\Gamma(1+d/2)}$ denotes the volume of the unit ball in $\mathbb{R}^d$. The interpretations of Ripley's K-function,

$$\lambda K(r) \;\; \hat{=} \;\; \begin{bmatrix}\text{mean number of points in a ball of radius } r \text{ centered at a}\\ \text{typical point}\end{bmatrix}, \tag{2.2}$$

the pair correlation function,

$$\lambda g(r) \;\; \hat{=} \;\; \frac{\begin{bmatrix}\text{mean number of points in an “infinitesimal annulus” with}\\ \text{distance } r \text{ to a typical point}\end{bmatrix}}{[\text{volume of the “infinitesimal annulus”}]}, \tag{2.3}$$

and the second-order product density,

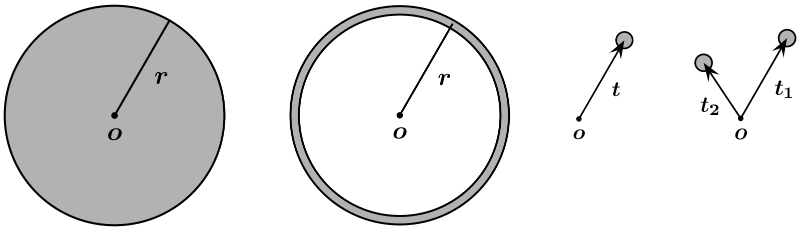
$$\varrho^{(2)}(t) \;\; \hat{=} \;\; \begin{bmatrix}\text{mean number of points in an “infinitesimal neighbor-}\\ \text{hood” of the vector } t \text{ attached to a typical point}\end{bmatrix}, \tag{2.4}$$

are illustrated in Figure 2.1. For the kth-order product density we have the interpretation

$$\varrho^{(k)}(t_1,\ldots,t_{k-1}) \;\; \hat{=} \;\; \begin{bmatrix}\text{mean number of points in “infinitesimal neighborhoods”}\\ \text{of the vectors } t_1,\ldots,t_{k-1} \text{ attached to a typical point}\end{bmatrix} \tag{2.5}$$

which is also illustrated in Figure 2.1 for $k = 3$.

Figure 2.1: Interpretation of the K-function, the pair correlation function, and product densities for point processes in $\mathbb{R}^2$



The quantity $\lambda K(r)$ is the mean number of points in a ball of radius r centered at the origin (far left). Similarly, $\lambda g(r)$ is the mean number of points in an infinitesimal annulus with radius r about the origin, relative to this annulus' volume (left). Product densities are based on "directed distances" rather than on radii. The second-order product density $\varrho^{(2)}(t)$ is the mean number of points in an infinitesimal neighborhood of the vector t (right) while the third-order product density $\varrho^{(3)}(t_1, t_2)$ is the mean number of points in infinitesimal neighborhoods of the vectors t_1 and t_2 (far right).

2.4 Examples of point processes

We will now give some examples of point processes in $\mathbb{R}^d$.

Example 2.4.1 *Poisson processes.*
A point process $\Psi \sim P$ in $\mathbb{R}^d$ is called a *Poisson process with intensity measure* Λ if

(i) $\Psi(B_1), \ldots, \Psi(B_k)$ are independent for disjoint $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$ and all $k \geq 1$, and

(ii) $\Psi(B) \sim \text{Poi}(\Lambda(B))$ for all bounded $B \in \mathfrak{B}(\mathbb{R}^d)$,

where Λ is a locally-finite measure on $[\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d)]$. The measure Λ coincides with the intensity measure. The Poisson process is stationary if it is 1-stationary, that is, $\Lambda(B) = \lambda|B|$ for all $B \in \mathfrak{B}(\mathbb{R}^d)$. We use the notation Π_λ for the distribution of a stationary Poisson process with intensity λ. Figure 2.2 shows a simulated realization of a stationary Poisson process in $\mathbb{R}^2$. The kth-order product density of a stationary Poisson process with intensity λ satisfies

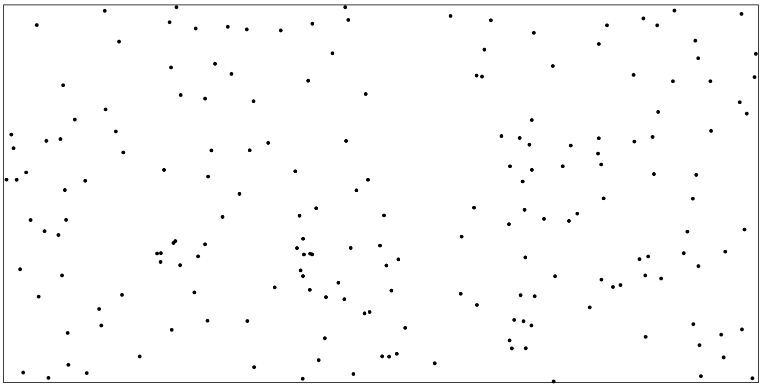
$$\varrho^{(k)}(t) = \lambda^{k-1} \qquad \text{for all } t \in (\mathbb{R}^d)^{k-1},$$

see Stoyan et al. [57], page 39. This implies the pair correlation function to take the form

$$g(r) = 1 \qquad \text{for all } r \geq 0.$$

□

Figure 2.2: Simulated realization of a stationary Poisson process in $\mathbb{R}^2$



A simulated realization of a stationary Poisson process in $\mathbb{R}^2$ with intensity 1 in a window with height 10 and width 20.

Example 2.4.2 *Cluster processes.*
A *cluster process* $\Psi \sim P$ in $\mathbb{R}^d$ consists of the *primary process* $\Psi_\mathrm{p} \sim P_\mathrm{p}$ and the *secondary process* $\Psi_\mathrm{c} \sim P_\mathrm{c}$ (also called *typical cluster*). Each point $x \in \operatorname{supp}(\Psi_\mathrm{p})$ triggers a point process $\Psi_\mathrm{c}^{[x]} \sim P_\mathrm{c}^{[x]}$ (a *cluster*) which is independent of Ψ_p and $\Psi_\mathrm{c}^{[y]}$, $y \neq x$, and has the same distribution as the translated process $T_x\Psi_\mathrm{c}$, that is, $P_\mathrm{c}^{[x]}(Y) = P_\mathrm{c}(T_xY)$ for all $Y \in \mathcal{N}$. Here, T_x denotes the translation operator defined by $(T_x\varphi)(B) = \varphi(B + x)$, where $B \in \mathfrak{B}(\mathbb{R}^d)$, $\varphi \in N$, and $x \in \mathbb{R}^d$. The cluster process Ψ is then given by

$$\Psi = \sum_{x\in\Psi_\mathrm{p}} \Psi_\mathrm{c}^{[x]}.$$

The condition $\mathbb{E}\Psi_\mathrm{c}(\mathbb{R}^d) < \infty$ guarantees the existence of the cluster process Ψ. The *cluster radius* denotes the radius of the smallest ball containing the support of the typical cluster Ψ_c.

If the primary process is stationary, then Ψ is stationary, too. If the primary and the secondary process are isotropic so is Ψ. Further, the cluster process inherits k-stationarity from the primary process. The intensity λ of a cluster process with 1-stationary primary process with intensity λ_p is given by $\lambda = \lambda_\mathrm{p}\ \mathbb{E}\Psi_\mathrm{c}(\mathbb{R}^d)$.

If the primary process is a Poisson process, then Ψ is called a *Poisson cluster process*, and if the primary process is a stationary Poisson process, then Ψ is called a *stationary Poisson cluster process*. Figure 2.3 shows a simulated realization of a stationary Poisson cluster process.

Neyman-Scott processes are examples of Poisson cluster processes. Here, the points of the typical cluster are random in number and are scattered independently with identical distribution about the origin.

Let $\Psi \sim P$ be a Neyman-Scott process in $\mathbb{R}^d$ with intensity λ and let λ_{p} be the intensity of the underlying Poisson process. Let F be the distribution function of the difference of two independent random points $X = (X_1, \ldots, X_d)'$ and $Y = (Y_1, \ldots, Y_d)'$ of the typical cluster $\Psi_{\mathrm{c}} \sim P_{\mathrm{c}}$, that is, $F(t) = P(X_1 - Y_1 \leq t_1, \ldots, X_d - Y_d \leq t_d)$ with $t = (t_1, \ldots, t_d)' \in \mathbb{R}^d$. If F has a Lebesgue density f, $f(t) = \frac{\partial^d}{\partial t_1 \cdots \partial t_d} F(t)$, then the second-order product density of the stationary Poisson cluster process Ψ is given by

$$\lambda \varrho^{(2)}(t) = \lambda^2 + \lambda_{\mathrm{p}} \sum_{n=2}^{\infty} p_n n(n-1) f(t) \tag{2.6}$$

for all $t \in \mathbb{R}^d$, where $p_n = \mathbb{P}\left(\Psi_{\mathrm{c}}(\mathbb{R}^d) = n\right)$. In order to find a representation for the pair correlation function, let F now be the distribution function of the distance between two independent random points of the typical cluster $\Psi_{\mathrm{c}} \sim P_{\mathrm{c}}$, that is, $F(r) = P(\|X - Y\| \leq r)$ with $r \in \mathbb{R}$. If F has a Lebesgue density $f(r) = \frac{\mathrm{d}}{\mathrm{d}r} F(r)$ then the pair correlation function of the stationary Poisson cluster process Ψ is given by

$$\lambda^2 g(r) = \lambda^2 + \lambda_{\mathrm{p}} \sum_{n=2}^{\infty} p_n n(n-1) \frac{f(r)}{d\omega_d r^{d-1}} \tag{2.7}$$

for all $r \geq 0$, see Stoyan et al. [57], page 159.

Matérn's cluster process is a stationary Poisson cluster process for which the number of points of the typical cluster is Poisson distributed with intensity μ and the points of the typical cluster are independently uniformly scattered in the ball $b(o, R)$ with radius $R > 0$. Here the density f occurring in equation (2.7) is of the form

$$f(r) = \begin{cases} \left(1 - \dfrac{r}{2R}\right)/R, & d = 1, \\ 4r/(\pi R^2)\left(\arccos\dfrac{r}{2R} - \dfrac{r}{2R}\sqrt{1 - \dfrac{r^2}{4R^2}}\right), & d = 2, \\ \dfrac{3}{2}\dfrac{r^2}{R^6}(R - r/2)^2(2R + r/2), & d = 3, \end{cases}$$

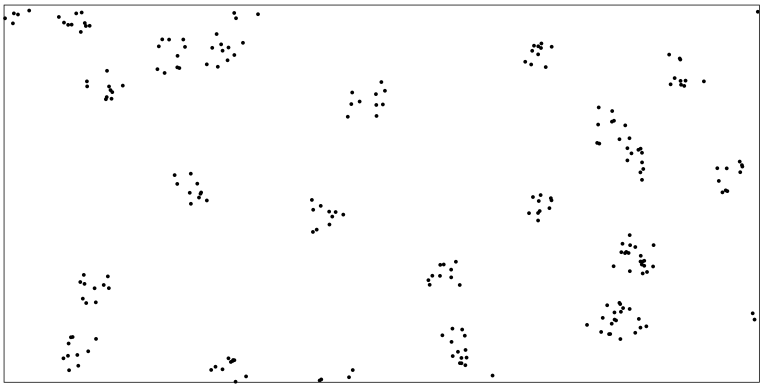
for $0 < r < 2R$ and $f(r) = 0$ otherwise, see Santaló [52], page 212.

The *modified Thomas process* is a stationary Poisson cluster process for which the number of points of the typical cluster is Poisson distributed with intensity μ and the points of the typical cluster are independently distributed according to a normal distribution with mean vector o and covariance matrix $\sigma^2 I_d$. Here, I_d denotes the identity matrix of size d. For $d = 2$ the pair correlation function takes the form

$$\lambda^2 g(r) = \lambda^2 + \frac{\lambda_{\mathrm{p}} \mu^2}{4\pi\sigma^2} \exp\left(-\frac{r^2}{4\sigma^2}\right)$$

for $r \geq 0$, see Cox and Isham [6], page 148. □

Figure 2.3: Simulated realization of a stationary Poisson cluster process in $\mathbb{R}^2$



A simulated realization of a stationary Poisson cluster process in $\mathbb{R}^2$ observed in a window of height 10 and width 20. The primary process has intensity $\frac{1}{10}$. The points in the typical cluster are independently uniformly scattered in a square with side length 1 centered at o and there are 10 points in each cluster.

Example 2.4.3 *Hard-core processes.*
A *hard-core process* is a point process where there is a certain minimum distance $R > 0$ between the points. Matérn [46] introduced two such models that both emerge from a stationary Poisson process $\Psi_{\mathrm{p}} \sim \Pi_{\lambda_{\mathrm{p}}}$ by dependent thinning (for a definition of thinning see Stoyan et al. [57], Chapter 5.1). For *Matérn's hard-core process I* all points $x \in \Psi_{\mathrm{p}}$ for which there is another point $y \in \Psi_{\mathrm{p}}$, $y \neq x$, with $\|x - y\| < R$ are deleted. For *Matérn's hard-core process II* the points of Ψ_{p} are marked independently by random numbers uniformly distributed over $(0, 1)$. A point $x \in \Psi_{\mathrm{p}}$ with mark $m(x)$ is deleted if and only if there exists another point $y \in \Psi_{\mathrm{p}}$ with $m(y) < m(x)$ and $y \in b^o(x, R)$, where $b^o(x, r) := \{y \in \mathbb{R}^d : \|y - x\| < r\}$ denotes the open ball in $\mathbb{R}^d$ with radius $r > 0$ and midpoint $x \in \mathbb{R}^d$. □

Example 2.4.4 *Soft-core processes.*
Stoyan and Stoyan [58] introduced soft-core processes as a generalization of Matérn's hard-core process II where the minimum distance is random. For this model the points of a stationary Poisson process $\Psi_\mathrm{p} \sim \Pi_{\lambda_\mathrm{p}}$ are marked independently with independent marks $m \sim U(0,1)$ and $r \sim G$, where $U(0,1)$ denotes the uniform distribution on $(0,1)$ and the *distribution of the radius* G has support contained in $[0,\infty)$. A point $x \in \Psi_\mathrm{p}$ is deleted if and only if there is another point $y \in \Psi_\mathrm{p}$, $y \neq x$, with $y \in b^o(x, r(x))$ and $m(y) < m(x)$.

The intensity of such a soft-core process satisfies

$$\lambda = \int_0^\infty \frac{1-\exp(-\lambda_\mathrm{p}\omega_d r^d)}{\omega_d r^d}\,\mathrm{d}G(r).$$

The pair correlation function takes the form

$$g(r) = \frac{\lambda_\mathrm{p}^2 h(r)}{\lambda^2}$$

with

$$\begin{aligned}
h(r) &= \int\int_{\substack{\{s+t\geq r,\\ r>s, r>t\}}} [A(s,t,r) + A(t,s,r)]\,\mathrm{d}G(s)\,\mathrm{d}G(t)\\
&\quad + \int\int_{\substack{\{s+t\geq r,\\ r>s, r\leq t\}}} A(s,t,r)\,\mathrm{d}G(s)\,\mathrm{d}G(t) + \int\int_{\substack{\{s+t\geq r,\\ r\leq s, r>t\}}} A(t,s,r)\,\mathrm{d}G(s)\,\mathrm{d}G(t)\\
&\quad + \int\int_{\{s+t<r\}} \frac{1-\exp(-a(s))}{a(s)} \cdot \frac{1-\exp(-a(t))}{a(t)}\,\mathrm{d}G(s)\,\mathrm{d}G(t),
\end{aligned}$$

where

$$\begin{aligned}
A(x,y,r) &= \left(\frac{1}{a(x)+b(x,y,r)} - \frac{1}{b(x,y,r)\exp(a(x))}\right)\Big/ a(x)\\
&\quad + \frac{1}{b(x,y,r)(a(x)+b(x,y,r))\exp(a(x)+b(x,y,r))}
\end{aligned}$$

and

$$a(x) = \lambda_\mathrm{p}\omega_d x^d, \qquad b(x,y,r) = \lambda_\mathrm{p}|b(o,y) \setminus b(\mathbf{r},x)|,$$

where $\mathbf{r} = (r,\ldots,r)' \in \mathbb{R}^d$, and $|\,.\,|$ denotes the d-dimensional Lebesgue measure (see also Stoyan and Stoyan [58]). For $d = 2$ we have

$$\begin{aligned}\frac{1}{\lambda_{\mathrm{p}}}b(R,r,t) &= b(0,r) - b(0,r)\cap b(t,R)\\ &= r^2\pi\\ &\quad - r^2\left(\arccos\frac{t^2+r^2-R^2}{2tr} - \frac{t^2+r^2-R^2}{4t^2r^2}(4t^2r^2-(t^2+r^2-R^2)^2)^{1/2}\right)\\ &\quad - R^2\left(\arccos\frac{t^2+R^2-r^2}{2tR} - \frac{t^2+R^2-r^2}{4t^2R^2}(4t^2R^2-(t^2+R^2-r^2)^2)^{1/2}\right)\end{aligned}$$

see Stoyan and Stoyan [59], Appendix K.

A further generalization of this model is obtained by using a random set $\Xi \sim Q$ with $o \in \Xi$ instead of the ball $b^o(o,r)$ with random radius r for the thinning procedure, that is, by marking independently the points of a stationary Poisson process $\Psi_{\mathrm{p}} \sim \Pi_{\lambda_{\mathrm{p}}}$ with independent marks $m \sim U(0,1)$ and a random set $\Xi \sim Q$ with $o \in \Xi$. For the definition of random sets see for example Stoyan et al. [57], Chapter 6. Then a point $x \in \Psi_{\mathrm{p}}$ is deleted if and only if there is another point $y \in \Psi_{\mathrm{p}}$, $y \neq x$, with $y \in T_x\Xi$ and $m(y) < m(x)$. As a result the intensity λ satisfies

$$\lambda = \lambda_{\mathrm{p}} \int_{\mathcal{K}}\int_{N} \frac{1}{\varphi(K)+1}\Pi_{\lambda_{\mathrm{p}}}(\mathrm{d}\varphi)Q(\mathrm{d}K) = \int_{\mathcal{K}} \frac{1-\exp(-\lambda_{\mathrm{p}}|K|)}{|K|}Q(\mathrm{d}K).$$

□

Example 2.4.5 *Cox processes.*
A Cox process is a Poisson process with random intensity measure, that is, a point process being Poisson conditional on the realization of the intensity measure. For a formal definition let Q be a distribution on the measurable space $[M, \mathcal{M}]$ of non-negative locally-finite measures on $\mathbb{R}^d$. Let Π_Γ be the distribution of a Poisson process with intensity measure Γ and let Λ be a random measure with distribution Q. Then the *Cox process* Ψ_Λ *with driving random measure* Λ has distribution

$$P_{\Psi_\Lambda}(Y) = \int_N P_\Gamma(Y)Q(\mathrm{d}\Gamma)$$

for $Y \in \mathcal{N}$ (see Stoyan et al. [57], page 154).

The kth-order factorial moment measure of Ψ_Λ and the kth-order moment measure (defined for instance in Stoyan et al. [57], page 110) of the driving random measure coincide, that is, assuming that the following moments exist, we have

$$\alpha^{(k)}(B_1 \times \cdots \times B_k) = \mathbb{E}\Lambda(B_1)\cdots\Lambda(B_k)$$

for $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$ and $k \geq 1$. Hence, Ψ_Λ is k-stationary if and only if Λ is k-stationary. Plugging this into Definition 2.2.2 of the kth-order factorial cumulant measure we obtain

$$\begin{aligned} \gamma^{(k)}(B_1 \times \cdots \times B_k) &= \sum_{\ell=1}^{k} (-1)^{\ell-1} (\ell-1)! \sum_{\substack{K_1 \cup \ldots \cup K_\ell \\ = \{1, \ldots, k\}}} \prod_{j=1}^{\ell} \mathbb{E} \prod_{k_j \in K_j} \Lambda(B_{k_j}) \\ &= \mathrm{Cum}_k(\Lambda(B_1), \ldots, \Lambda(B_k)), \end{aligned}$$

where $\mathrm{Cum}_k(X_1, \ldots, X_k)$ is the kth mixed cumulant of the random vector $(X_1, \ldots, X_k)'$. (The definition of mixed cumulants will follow in Chapter 4.) □

3

Mixing properties

After giving an overview of several types of mixing properties and relations between these we will focus on Brillinger-mixing. Heinrich [26] and Heinrich and Schmidt [34] state conditions on some classes of point processes for being Brillinger-mixing. We will sum up these results and examine some more classes of point processes. An overview of mixing properties can be found in Doukhan [15], for instance.

3.1 Definitions

The following definitions of mixing properties—mixing, α-mixing, φ-mixing, and Brillinger-mixing—can be found in Ivanoff [38]. For the definition of strong Brillinger-mixing see König and Schmidt [43], page 333.

Mixing

A stationary point process $\Psi \sim P$ is said to be *mixing* (see Daley and Vere-Jones [9], page 341) if

$$\lim_{\|x\|\to\infty} \big(P(T_x V \cap W) - P(V)P(W)\big) = 0$$

for all $V, W \in \mathcal{N}$, where T_x is the translation operator, see page 12.

α-mixing (strong mixing)

Let $\Psi \sim P$ be a stationary point process and let $A \in \mathfrak{B}(\mathbb{R}^d)$. Let $\mathcal{F}(A)$ be the σ-algebra generated by all random variables $\Psi(A')$ with $A' \in \mathfrak{B}(\mathbb{R}^d)$ satisfying $A' \subseteq A$. Let $\mathrm{diam}(A)$ denote the diameter of A, that is, $\mathrm{diam}(A) = \sup_{x,y \in A} \rho(x,y)$ where $\rho(x,y) = \max_{1 \le i \le d} |x_i - y_i|$. Furthermore, let $\rho(B,C) = \min_{x \in B, y \in C} \rho(x,y)$, and define $\alpha(s,t)$ by

$$\alpha(s,t) = \sup_{\substack{\rho(A_1,A_2) \ge s, \\ \mathrm{diam}(A_1) \le t, \mathrm{diam}(A_2) \le t}} \sup_{\substack{U_1 \in \mathcal{F}(A_1), \\ U_2 \in \mathcal{F}(A_2)}} |P(U_1 \cap U_2) - P(U_1)P(U_2)|$$

for all $s, t \in [0, \infty)$. The point process $\Psi \sim P$ is said to be *α-mixing* or *strongly mixing*, if $\lim_{x \to \infty} \alpha(xs, xt) = 0$.

φ-mixing (uniform strong mixing)

Define $\varphi(s)$ by

$$\varphi(s) = \sup_{\rho(A_1,A_2) \ge s} \sup_{\substack{U_1 \in \mathcal{F}(A_1), U_2 \in \mathcal{F}(A_2), \\ P(U_1) > 0}} |P(U_2 | U_1) - P(U_2)|$$

for all $s \in [0, \infty)$. Then the point process $\Psi \sim P$ is said to be *φ-mixing* or *uniformly strongly mixing* if $\lim_{s \to \infty} \varphi(s) = 0$.

Brillinger-mixing

A stationary point process $\Psi \sim P$ is said to be *Brillinger-mixing* if $\mathbb{E}\Psi^k([0,1]^d) < \infty$ and

$$\|\gamma_{\mathrm{red}}^{(k)}\| = \int_{(\mathbb{R}^d)^{k-1}} |\gamma_{\mathrm{red}}^{(k)}(\mathrm{d}(x_1, \ldots, x_{k-1}))| < \infty \tag{3.1}$$

for all $k \ge 2$.

In the present work, all results involving the assumption of a Brillinger-mixing point process remain true when stationarity is replaced by the condition that k-stationarity holds for all $k \ge 2$.

Strong Brillinger-mixing

A stationary point process $\Psi \sim P$ is said to be *strongly Brillinger-mixing* if $\mathbb{E}\Psi^k([0,1]^d) < \infty$ and if there exist constants $a, b \in [0, \infty)$ such that

$$\|\gamma_{\text{red}}^{(k)}\| = \int_{(\mathbb{R}^d)^{k-1}} |\gamma_{\text{red}}^{(k)}(\mathrm{d}(x_1, \ldots, x_{k-1}))| \leq ab^k k! \tag{3.2}$$

for all $k \geq 2$.

3.2 Implications of mixing properties on second-order characteristics

Now we will study the behavior of the K-function, the second-order product density and the pair correlation function under some mixing conditions.

The finiteness of the second total variation $\left\|\gamma_{\text{red}}^{(2)}\right\|$ of a 2-stationary point process Ψ implies

$$\sup_{r \geq 0} \left|\alpha_{\text{red}}^{(2)}\left(b(o,r)\right) - \lambda \omega_d r^d\right| < \infty,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$, see page 10. Hence we have

$$\lim_{r \to \infty} \frac{K(r)}{\omega_d r^d} = 1,$$

see König and Schmidt [43], page 332 ff. This means that the K-function of a 2-stationary point process Ψ with $\left\|\gamma_{\text{red}}^{(2)}\right\| < \infty$ converges to the K-function of the Poisson process. König and Schmidt [43], page 333, state that absolute continuity of the K-function and the property

$$\int_0^\infty \left|\frac{\mathrm{d}K(r)}{\mathrm{d}r} - d\omega_d r^{d-1}\right| \mathrm{d}r = \|\gamma_{\text{red}}^{(2)}\| < \infty$$

entail the convergence $\lim_{r \to \infty} \left|\frac{\mathrm{d}K(r)}{\mathrm{d}r} - d\omega_d r^{d-1}\right| = 0$ and hence the convergence $\lim_{x \to \infty} g(x) = 1$. However, in general the finiteness of $\int_0^\infty |f(x)| \mathrm{d}x$ does not imply $\lim_{x \to \infty} f(x) = 0$ for a function f. For instance, the function

$$f(x) = \sum_{n=1}^{\infty} \mathbb{1}_{[n,\, n+1/n^2)}(x)$$

satisfies $\int_0^\infty |f(x)| \mathrm{d}x = \sum_{n=1}^\infty \frac{1}{n^2} < \infty$ and $\limsup_{x\to\infty} f(x) = 1 \neq 0 = \liminf_{x\to\infty} f(x)$. The discontinuity of the function f can be removed by smoothing the edges. As a result one may find examples of infinitely often differentiable functions f for which the above-mentioned conclusion from König and Schmidt [43]) does not hold. Thus the assumption $\|\gamma_{\mathrm{red}}^{(2)}\| < \infty$ does not ensure the convergence of the pair correlation $\lim_{x\to\infty} g(x) = 1$. The same argument holds for the second-order product density ϱ.

Theorem 12.4.V in Daley and Vere-Jones [9], page 488, a result attributed by Delasnerie [11] to Neveu, implies the second-order product density ϱ of a stationary mixing point process to satisfy $\lim_{\|x\|\to\infty} \varrho(x) = \lambda$.

3.3 Relations between mixing properties

In this section we give a slightly informal presentation of relations between the mixing properties defined in Section 3.1.

Ivanoff [38] derived the following relations between mixing conditions:

$$\varphi\text{-mixing} \quad \Rightarrow \quad \alpha\text{-mixing}. \tag{3.3}$$

Furthermore, we obviously have

$$\text{strong Brillinger-mixing} \quad \Rightarrow \quad \text{Brillinger-mixing}. \tag{3.4}$$

Note that the mixing properties in (3.3) are conditions on the distribution of the point process while those in (3.4) are conditions on associated cumulant measures. As a result there is no obvious relation between the mixing properties in (3.3) and (3.4).

If the distribution P of a stationary point process $\Psi \sim P$ is uniquely determined by its moment measures, then Brillinger-mixing implies mixing. This is due to the fact that Brillinger-mixing implies

$$\alpha^{(k)}(A_1 \times \ldots \times A_{k-1} \times (A_k + t)) \xrightarrow[t\to\infty]{} \alpha^{(k-1)}(A_1 \times \ldots \times A_{k-1})\alpha^{(1)}(A_k)$$

for all $k \geq 2$ and all $A_1, \ldots, A_k \in \mathfrak{B}(\mathbb{R}^d)$ with $|A_i| < \infty$, $i = 1, \ldots, k$. If the distribution of the Brillinger-mixing point process Ψ is uniquely determined by its moment measures, then Ψ is mixing.

However, there exist mixing point processes that are not Brillinger-mixing. An example is given in König and Schmidt [43], page 333: Let $\Psi \sim P$ be a Cox process with a stationary and isotropic Poisson hyperplane process with intensity λ_1 as its driving random measure. This process has intensity $\lambda = \lambda_1$, and its K-function satisfies

$$\lambda K(r) = \alpha_{\text{red}}^{(2)}(b(o,r)) = \omega_{d-1} r^{d-1} + \lambda \omega_d r^d.$$

This implies

$$\sup_{r \geq 0} \left|\alpha_{\text{red}}^{(2)}(b(o,r)) - \lambda\, \omega_d r^d\right| = \sup_{r \geq 0} \left|\omega_{d-1} r^{d-1}\right| = \infty$$

and hence $\left\|\gamma_{\text{red}}^{(2)}\right\| = \infty$ (see Section 3.2).

Ivanoff [38] claims that the condition that for all $\varepsilon > 0$ there is a compact set $C \in \mathfrak{B}(\mathbb{R}^d)$ such that

$$\int\limits_{(\mathbb{R}^d)^{k-2}} \int\limits_{D \setminus C} |\gamma_{\text{red}}^{(k)}(\mathrm{d}(x_1, \ldots, x_{k-1}))| < \varepsilon$$

is sufficient for

$$\text{Brillinger-mixing} \quad \Rightarrow \quad \text{mixing}.$$

Examining the proof for this claim one can observe that Ivanoff [38] tacitly assumed the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int\limits_{(\mathbb{R}^d)^k} (\xi_1 + S_t \xi_2)(x_1) \cdot \ldots \cdot (\xi_1 + S_t \xi_2)(x_k) \gamma_{\text{red}}^{(k+1)}(\mathrm{d}(x_1, \ldots, x_k)) \tag{3.5}$$

to be absolutely convergent. The condition

$$\sum_{k=1}^{\infty} \frac{2^k}{k!} \int\limits_{(\mathbb{R}^d)^k} |\gamma_{\text{red}}^{(k+1)}(\mathrm{d}(x_1, \ldots, x_k))| < \infty.$$

ensures the absolute convergence of the series (3.5). However, this condition implies the total variation $\left\|\gamma_{\text{red}}^{(k)}\right\|$ to be of order $o(k!/2^k)$ as $k \to \infty$. For large k this assumption is stricter than the assumption on $\left\|\gamma_{\text{red}}^{(k)}\right\|$ for strong Brillinger-mixing.

3.4 Examples for Brillinger-mixing point processes

In the following chapters we will show central limit theorems for estimators of product densities and the pair correlation function as well as central limit theorems for the integrated squared errors of the estimated second-order product density and pair correlation function. These results will be based on the assumption of Brillinger-mixing. Therefore we will now investigate conditions on some of the point process classes from Section 2.4 that are sufficient for Brillinger-mixing or even strong Brillinger-mixing.

Stationary Poisson processes

The factorial cumulant measures of the stationary Poisson process $\Psi \sim \Pi_\lambda$ with intensity λ satisfy $\gamma^{(k)} = 0$ for all $k \geq 2$, see König and Schmidt [43], page 173. Hence Ψ is strongly Brillinger-mixing, see König and Schmidt [43], page 333.

Stationary cluster processes

A stationary cluster process $\Psi \sim P$ generated by the stationary primary process $\Psi_\mathrm{p} \sim P_\mathrm{p}$ and secondary process $\Psi_\mathrm{c} \sim P_\mathrm{c}$ is Brillinger-mixing if the primary process is Brillinger-mixing and the secondary process has finite moments, that is $\mathbb{E}\Psi_\mathrm{c}^k(\mathbb{R}^d) < \infty$ for all $k \geq 1$, see Heinrich [26]. This can be seen by the representation

$$\gamma^{(k)}(B_1 \times \ldots \times B_k) = \sum_{\ell=1}^{k} \sum_{\substack{K_1 \cup \ldots \cup K_\ell \\ = \{1,\ldots,k\}}} \int_{(\mathbb{R}^d)^\ell} \prod_{j=1}^{\ell} \alpha_{P_\mathrm{c}}^{(|K_j|)}(B_{k_j} - x_j;\ k_j \in K_j) \gamma_{P_\mathrm{p}}^{(\ell)}(\mathrm{d}(x_1, \ldots, x_\ell)) \quad (3.6)$$

of the factorial cumulant measure $\gamma^{(k)}$ of Ψ with $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$, see Heinrich and Schmidt [34].

Stationary Poisson cluster processes

As a consequence of the above results on stationary Poisson and stationary cluster processes, a stationary Poisson cluster process with secondary process $\Psi_\mathrm{c} \sim P_\mathrm{c}$ is Brillinger-mixing if $\mathbb{E}\Psi_\mathrm{c}^k(\mathbb{R}^d) < \infty$ for all $k \geq 1$, see Heinrich [26]. Note that this is a condition on the number of points in the typical cluster, not on the cluster radius. In particular, assuming a bounded cluster radius of the secondary process is not sufficient for Ψ being Brillinger-mixing. Examples for Brillinger-mixing Poisson cluster processes are Matérn's cluster process, the modified Thomas process, and the Gauss-Poisson process, see König and Schmidt [43], page 332.

Matérn's cluster process is even strongly Brillinger-mixing, see König and Schmidt [43], page 333.

Cox processes

Example 2.4.5 already showed the kth-order factorial cumulant measure of a Cox process $\Psi_\Lambda \sim P$ with stationary driving random measure Λ to satisfy

$$\gamma^{(k)}(B_1 \times \ldots \times B_k) = \mathrm{Cum}_k(\Lambda(B_1), \ldots \Lambda(B_k)) = \int\limits_{B_k} C_\Lambda(B_1 - x, \ldots, B_{k-1} - x)\mathrm{d}x,$$

where C_Λ is a well-defined measure which is equal to $\gamma^{(k)}_{\mathrm{red}}$ by Definition 2.2.4 of the kth-order reduced factorial cumulant measure. Therefore the Cox process Ψ_Λ satisfies condition (3.1) if and only if $\int\limits_{(\mathbb{R}^d)^{k-1}} |C_\Lambda(\mathrm{d}(x_1, \ldots, x_{k-1}))| < \infty$, see Heinrich [26] and Heinrich and Schmidt [34]. Likewise, the Cox process Ψ_Λ satisfies condition (3.2) if and only if there exist constants a, b such that

$$\int\limits_{(\mathbb{R}^d)^{k-1}} |C_\Lambda(\mathrm{d}(x_1, \ldots, x_{k-1}))| \leq ab^k k!$$

for all $k \geq 2$.

Soft-core processes and Matérn's hard-core processes

A soft-core process $\Psi \sim P$ with stationary Poisson process Π_{λ_p} with intensity $\lambda_\mathrm{p} > 0$ as primary process and random set $\Xi \sim Q$ is Brillinger-mixing if all moments of the diameter of the random set Ξ are finite. Hence Matérn's hard-core process II is Brillinger-mixing. To prove $\left\|\gamma^{(k)}_{\mathrm{red}}\right\| < \infty$ for all $k \geq 2$ we first determine the factorial moment measures $\alpha^{(k)}(.)$, $k \geq 1$. Using a generalization of the refined Campbell theorem for the n-fold Palm distribution $P^!_{x_1,\ldots,x_k}$ due to Hanisch [23] we find

$$\begin{aligned}
&\alpha^{(k)}(B_1 \times \ldots \times B_k) \\
&\quad = \int\limits_{\mathcal{K}^k} \int\limits_{N} \int\limits_0^1 \cdots \int\limits_0^1 \sum_{x_1,\ldots,x_k \in \varphi}^{*} \mathbb{1}_{B_1}(x_1) \cdot \ldots \cdot \mathbb{1}_{B_k}(x_k) \\
&\qquad\qquad \times \prod_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i,\ x_j \in K_i + x_i}}^{k} \mathbb{1}_{(u_i,1]}(u_j)\mathrm{d}u_1 \ldots \mathrm{d}u_k P(\mathrm{d}\varphi)Q(\mathrm{d}K_1) \ldots Q(\mathrm{d}K_k)
\end{aligned}$$

$$= \int_{\mathcal{K}^k} \int_N \sum\nolimits^*_{x_1,\ldots,x_k \in \varphi} \mathbb{1}_{B_1}(x_1) \cdot \ldots \cdot \mathbb{1}_{B_k}(x_k)$$

$$\times \sum_{\pi \in \mathcal{S}(\{1,\ldots,k\})} \frac{\prod_{j=2}^k \mathbb{1}_{(\bigcup_{i=1}^{j-1}(K_{\pi(i)}+x_{\pi(i)}))^C}(x_{\pi(j)})}{\prod_{j=1}^k \varphi(\bigcup_{i=1}^j (K_{\pi(i)} + x_{\pi(i)}))} P(\mathrm{d}\varphi) Q(\mathrm{d}K_1) \ldots Q(\mathrm{d}K_k)$$

$$= \lambda_{\mathrm{p}}^k \int_{\mathcal{K}^k} \int_{(\mathbb{R}^d)^k} \mathbb{1}_{B_1}(x_1) \cdot \ldots \cdot \mathbb{1}_{B_k}(x_k) \int_N \sum_{\pi \in \mathcal{S}(\{1,\ldots,k\})} \frac{\prod_{j=2}^k \prod_{i=1}^{j-1} \mathbb{1}_{K^C_{\pi(i)}}(x_{\pi(j)} - x_{\pi(i)})}{\prod_{j=1}^k [\varphi(\bigcup_{i=1}^j (K_{\pi(i)} + x_{\pi(i)})) + j]}$$

$$P^!_{x_1,\ldots,x_k}(\mathrm{d}\varphi) \mathrm{d}(x_1,\ldots,x_k) Q(\mathrm{d}K_1) \ldots Q(\mathrm{d}K_k)$$

for Borel sets $B_1, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$. For the Poisson process we have $P^!_{x_1,\ldots,x_k} = \Pi_{\lambda_{\mathrm{p}}}$ (see Hanisch [23]) as a generalization of Slivnyak's theorem. This implies

$$\alpha^{(k)}(B_1 \times \ldots \times B_k) = \lambda_{\mathrm{p}}^k \int_{\mathcal{K}^k} \int_{(\mathbb{R}^d)^k} \mathbb{1}_{B_1}(x_1) \mathbb{1}_{B_2 - x_1}(y_2) \cdot \ldots \cdot \mathbb{1}_{B_k - x_1}(y_k)$$

$$\times \int_N \sum_{\substack{\pi \in \mathcal{S}(\{1,\ldots,k\}) \\ y_1 := 0}} \frac{\prod_{j=2}^k \prod_{i=1}^{j-1} \mathbb{1}_{K^C_{\pi(i)}}(y_{\pi(j)} - y_{\pi(i)})}{\prod_{j=1}^k [\varphi(\bigcup_{i=1}^j (K_{\pi(i)} + y_{\pi(i)})) + j]} \Pi_{\lambda_{\mathrm{p}}}(\mathrm{d}\varphi)$$

$$\mathrm{d}(x_1, y_2, \ldots, y_k) Q(\mathrm{d}K_1) \ldots Q(\mathrm{d}K_k).$$

Furthermore we get

$$\alpha^{(k)}_{\mathrm{red}}(B_2 \times \ldots \times B_k) = \frac{\lambda_{\mathrm{p}}^k}{\lambda} \int_{\mathcal{K}^k} \int_{(\mathbb{R}^d)^{k-1}} \mathbb{1}_{B_2}(y_2) \cdot \ldots \cdot \mathbb{1}_{B_k}(y_k)$$

$$\times \int_N \sum_{\substack{\pi \in \mathcal{S}(\{1,\ldots,k\}) \\ y_1 := 0}} \frac{\prod_{j=2}^k \prod_{i=1}^{j-1} \mathbb{1}_{K^C_{\pi(i)}}(y_{\pi(j)} - y_{\pi(i)})}{\prod_{j=1}^k [\varphi(\bigcup_{i=1}^j (K_{\pi(i)} + y_{\pi(i)})) + j]} \Pi_{\lambda_{\mathrm{p}}}(\mathrm{d}\varphi)$$

$$\mathrm{d}(y_2, \ldots, y_k) Q(\mathrm{d}K_1) \ldots Q(\mathrm{d}K_k)$$

for $B_2, \ldots, B_k \in \mathfrak{B}(\mathbb{R}^d)$. By Jensen's inequality we obtain

$$\|\gamma^{(k)}_{\mathrm{red}}\| \leq \lambda_{\mathrm{p}}^k \int_{\mathcal{K}^k} \int_{(\mathbb{R}^d)^{k-1}} \int_N \Big| \sum_{l=1}^k (-1)^{l-1}(l-1)! \sum_{\substack{K_1 \cup \ldots \cup K_l \\ = \{1,\ldots,k\},\ k \in K_1}} f_{\mathrm{red}}(\mathbf{x}_{K_1 \setminus \{k\}}) \prod_{j=2}^l f(\mathbf{x}_{K_j}) \Big|$$

$$\Pi_{\lambda_{\mathrm{p}}}(\mathrm{d}\varphi) \mathrm{d}(x_1, \ldots, x_{k-1}) Q(\mathrm{d}K_1) \ldots Q(\mathrm{d}K_k), \tag{3.7}$$

where $\mathcal{S}(T)$ denotes the set of all permutations $\pi : \{1,\dots,|T|\} \to T$ and

$$\begin{aligned} f(\mathbf{x}_T) &= \sum_{\pi\in\mathcal{S}(T)} \frac{\prod_{j=2}^{|T|}\prod_{i=1}^{j-1}\mathbb{1}_{K^C_{\pi(i)}}(x_{\pi(j)}-x_{\pi(i)})}{\prod_{j=1}^{|T|}[\varphi(\bigcup_{i=1}^{j}(K_{\pi(i)}+x_{\pi(i)}))+j]}, \qquad \mathbf{x}_T=(x_i)_{i\in T},\\ f_{\mathrm{red}}(\mathbf{y}_T) &= \frac{1}{\lambda}\sum_{\substack{\pi\in\mathcal{S}(T\cup\{k\})\\ y_k:=0}} \frac{\prod_{j=2}^{|T|+1}\prod_{i=1}^{j-1}\mathbb{1}_{K^C_{\pi(i)}}(y_{\pi(j)}-y_{\pi(i)})}{\prod_{j=1}^{|T|+1}[\varphi(\bigcup_{i=1}^{j}(K_{\pi(i)}+y_{\pi(i)}))+j]}, \qquad \mathbf{y}_T=(y_i)_{i\in T}, \end{aligned}$$

for subsets $T \subseteq \{1,\dots,k-1\}$, where $f_{\mathrm{red}}(\mathbf{y}_\emptyset) := 1$. Note that the functions f and f_{red} are bounded from above by the number of permutations of T and $T\cup\{k\}$ and hence by $(k-1)!$ and $k!$, respectively. Now we dissect the region of integration $(\mathbb{R}^d)^k$ in $A\cup A^C = (\mathbb{R}^d)^{k-1}$ with

$$A := \big\{(x_2,\dots,x_k)\in(\mathbb{R}^d)^{k-1}\,\big|\,\forall j=1,\dots,k\ \exists i\neq j : (K_j+x_j)\cap(K_i+x_i)\neq\emptyset\big\}.$$

Using the implication

$$(K_j+x_j)\cap(K_i+x_i)=\emptyset \quad\Rightarrow\quad x_i\in K_j^C+x_j\ \vee x_j\in K_i^C+x_i$$

and the representation of the intensity

$$\lambda = \lambda_{\mathrm{p}}\int_{\mathcal{K}}\int_{N}\frac{1}{\varphi(K)+1}\Pi_{\lambda_{\mathrm{p}}}(\mathrm{d}\varphi)Q(\mathrm{d}K),$$

see Example 2.4.4, it can be shown that the integral on the right-hand side of the inequality (3.7) vanishes on A^C. Together with the triangle inequality we obtain

$$\begin{aligned} \|\gamma^{(k)}_{\mathrm{red}}\| \;&\le\; \lambda_{\mathrm{p}}^k\int_{\mathcal{K}^k}\int_A\int_N\Big|\sum_{l=1}^{k}(-1)^{l-1}(l-1)!\sum_{\substack{K_1\cup\dots\cup K_l\\=\{1,\dots,k\},\ k\in K_1}} f_{\mathrm{red}}(\mathbf{x}_{K_1\setminus\{k\}})\prod_{j=2}^{l}f(\mathbf{x}_{K_j})\Big|\\ &\qquad \Pi_{\lambda_{\mathrm{p}}}(\mathrm{d}\varphi)\mathrm{d}(x_1,\dots,x_{k-1})Q(\mathrm{d}K_1)\dots Q(\mathrm{d}K_k)\\ &\quad+\lambda_{\mathrm{p}}^k\int_{\mathcal{K}^k}\int_{A^C}\int_N\Big|\sum_{l=1}^{k}(-1)^{l-1}(l-1)!\sum_{\substack{K_1\cup\dots\cup K_l\\=\{1,\dots,k\},\ k\in K_1}} f_{\mathrm{red}}(\mathbf{x}_{K_1\setminus\{k\}})\prod_{j=2}^{l}f(\mathbf{x}_{K_j})\Big|\\ &\qquad \Pi_{\lambda_{\mathrm{p}}}(\mathrm{d}\varphi)\mathrm{d}(x_1,\dots,x_{k-1})Q(\mathrm{d}K_1)\dots Q(\mathrm{d}K_k)\\ &= \lambda_{\mathrm{p}}^k\int_{\mathcal{K}^k}\int_A\int_N\Big|\sum_{l=1}^{k}(-1)^{l-1}(l-1)!\sum_{\substack{K_1\cup\dots\cup K_l\\=\{1,\dots,k\},\ k\in K_1}} f_{\mathrm{red}}(\mathbf{x}_{K_1\setminus\{k\}})\prod_{j=2}^{l}f(\mathbf{x}_{K_j})\Big|\\ &\qquad \Pi_{\lambda_{\mathrm{p}}}(\mathrm{d}\varphi)\mathrm{d}(x_1,\dots,x_{k-1})Q(\mathrm{d}K_1)\dots Q(\mathrm{d}K_k). \end{aligned}$$

Applying the triangle inequality and the inequality

$$\int_N |f_{\mathrm{red}}(.) \prod f(.)\Big| \Pi_{\lambda_{\mathrm{p}}}(\mathrm{d}\varphi) \le (k-1)!k!$$

we obtain

$$\|\gamma_{\mathrm{red}}^{(k)}\| \le (k-1)!(k!)^2 \frac{\lambda_{\mathrm{p}}^k}{\lambda} \int_{\mathcal{K}^k} |K_1 \oplus \check{K}_2| \cdot \ldots \cdot |K_1 \oplus \check{K}_k| Q(\mathrm{d}K_1) \ldots Q(\mathrm{d}K_k).$$

This leads to the upper bound

$$\|\gamma_{\mathrm{red}}^{(k)}\| \le C \frac{\lambda_{\mathrm{p}}^k}{\lambda} \omega_d^{k-1} \mathbb{E}(\mathrm{diam}(\Xi))^{(k-1)d},$$

where $\mathrm{diam}(\Xi)$ denotes the diameter of Ξ. We see that the condition $\mathbb{E}(\mathrm{diam}(\Xi))^{(k-1)d} < \infty$ is sufficient for the total variation $\|\gamma_{\mathrm{red}}^{(k)}\|$ to be finite. Hence Ψ is Brillinger-mixing if all moments of the diameter of Ξ exist.

Independent $\pi(x)$-thinning

A point process Ψ with intensity λ resulting from an independent $\pi(x)$-thinning of a Brillinger-mixing point process $\Psi_{\mathrm{p}} \sim P_{\mathrm{p}}$ with intensity λ_{p} with stationary random field $\pi = \{\pi(x) : x \in \mathbb{R}^d\} \sim P_\pi$ on $\mathbb{R}^d$ is Brillinger-mixing. This is due to the kth-order reduced factorial cumulant measure of Φ taking the form

$$\begin{aligned}&\gamma_{\mathrm{red}}^{(k)}(B_1 \times \ldots \times B_{k-1})\\ &= \frac{\lambda_{\mathrm{p}}}{\lambda} \int_{(\mathbb{R}^d)^{k-1}} \mathbb{1}_{B_1}(x_1) \cdot \ldots \cdot \mathbb{1}_{B_{k-1}}(x_{k-1}) \, \mathbb{E}[\pi(0)\pi(x_1) \cdot \ldots \cdot \pi(x_{k-1})] \gamma_{\mathrm{p,red}}^{(k)}(\mathrm{d}(x_1, \ldots, x_k)).\end{aligned}$$

Superposition of two independent point processes

A point process $\Psi \sim P$ resulting from a superposition of two independent point processes Ψ_1 and Ψ_2, that is, $\Psi = \Psi_1 + \Psi_2$, is Brillinger-mixing if Ψ_1 and Ψ_2 are Brillinger-mixing. This is due to $G(.) = G_1(.)G_2(.)$, where G and G_i, $i = 1, 2$, are the probability generating functionals of Ψ and Ψ_i, $i = 1, 2$, respectively. (For the definition of the probability generating functional see, for instance, Stoyan et al. [57], page 115.)

Stationary infinitely divisible point process

A stationary infinitely divisible point process $\Psi \sim P$ governed by the KLM measure $\tilde{P}$ (for a definition see Daley and Vere-Jones [9], page 258) is Brillinger-mixing if and only if the kth-order reduced factorial moment measure of the KLM measure is finite, that is, $\alpha^{(k)}_{\mathrm{red},\tilde{P}}((\mathbb{R}^d)^{k-1}) < \infty$, for all $k \geq 2$. This is due to $\gamma^{(k)}(\cdot) = \alpha^{(k)}_{\mathrm{red},\tilde{P}}(\cdot)$ for all $k \geq 2$, which can be seen by the shape of the probability generating functional

$$G(h) = \exp\left\{ \int \left[\prod_{x \in s(\varphi)} h(x) - 1 \right] \tilde{P}(\mathrm{d}\varphi) \right\},$$

see Heinrich [26], with $h = 1 - u$ for some measurable function $u : \mathbb{R}^d \to [0, 1]$ with bounded support. However, the above-mentioned condition $\alpha^{(k)}_{\mathrm{red},\tilde{P}}((\mathbb{R}^d)^{k-1}) < \infty$ for all $k \geq 2$ implies that the KLM measure is supported by the set $\{\varphi \in N : \varphi(\mathbb{R}^d) < \infty\}$ and hence Ψ is regular, see Definition 8.4.VI in Daley and Vere-Jones [9], page 259. But Ψ is regular if and only if it can be represented as a Poisson cluster process with almost surely finite clusters, see Daley and Vere-Jones [9], Proposition 8.4.VIII(ii), page 260. Thus a Brillinger-mixing infinitely divisible point process can be represented as stationary Poisson cluster process satisfying the conditions mentioned on page 24.

Stationary renewal process

The stationary renewal process $\Psi \sim P$ generated by the distribution law $F(\cdot)$ of the distance between two consecutive points is Brillinger-mixing if, for some $n \in \mathbb{N}$, the convolution $F^{*n}(\cdot)$ has a non-trivial absolutely continuous component and $\int_0^\infty x^k F(\mathrm{d}x) < \infty$ for every $k \geq 1$, see Heinrich [26].

4

Cumulants

Cumulants, also called semi-invariants, were first introduced in 1889 by Thiele [61], page 19, by a recursion formula involving the moments. In 1899 Thiele [62] gave the modern definition of cumulants as coefficients in the power series representation of the logarithm of the moment generating function. Despite the fact that the first k cumulants contain the same information as the first k moments, $k \geq 1$, cumulants are more convenient to work with in certain settings. This is due to properties such as the cumulant of a sum of independent random variables being the sum of the cumulants of these random variables (see Thiele [61]), and the cumulants of order three and higher of normal distributions being zero (see Shiryaev [53], page 293).

An overview of Thiele's contributions on cumulants can be found in Hald [20]. Recursive formulas for obtaining moments from cumulants and vice versa are given in Smith [55].

In this chapter we present some well-known properties of cumulants and joint cumulants. Lemma 4.2.1 gives a representation of the cumulants of certain random variables as a sum of indecomposable integrals and will be the key tool in the proofs of the central limit theorems in the following chapters.

4.1 Definition

The definitions of the kth cumulant of a random variable and the mixed cumulant of a random vector given in this section are taken from Shiryaev [53], page 289 ff.

Definition 4.1.1 *Let $X \in \mathbb{R}^\ell$, $\ell \geq 1$, be a random vector. The function h defined by*

$$h : \mathbb{R}^\ell \to \mathbb{R}, \qquad t \mapsto \log \mathbb{E} e^{\mathrm{i}t'X},$$

is the cumulant generating function of X. □

The following definition of the kth cumulant of a real-valued random variable originally goes back to Thiele [62] (a translation to English can be found in Hald [21]).

Definition 4.1.2 *Let X be a real-valued random variable with cumulant generating function h, let $k \geq 1$, and let $\mathbb{E}|X|^k < \infty$. The* kth cumulant $\Gamma_k(X)$ of X *is defined by*

$$\Gamma_k(X) = \frac{1}{\mathrm{i}^k} \frac{\mathrm{d}^k}{\mathrm{d}t^k} h(t)|_{t=0}.$$

□

The mixed cumulant (also called joint cumulant or simple semi-invariant) of a real-valued random vector is given by the following definition, to be found in Leonov and Shiryaev [45].

Definition 4.1.3 *Let $X = (X_1, \ldots, X_k)' \in \mathbb{R}^k$, $k \geq 1$, be a random vector satisfying $\mathbb{E}|X_j|^k < \infty$, $j = 1, \ldots, k$. The* mixed cumulant $\mathrm{Cum}_k(X_1, \ldots, X_k)$ *of X is defined by*

$$\mathrm{Cum}_k(X_1, \ldots, X_k) = \frac{1}{\mathrm{i}^k} \frac{\partial^k}{\partial t_1 \ldots \partial t_k} h(t_1, \ldots, t_k)|_{t_1 = \ldots t_k = 0}.$$

□

4.2 Properties

In this section we present some properties of cumulants and mixed cumulants. The kth cumulant of a random variable X satisfies

$$\Gamma_k(X) = \mathrm{Cum}_k(X, \ldots, X).$$

The kth cumulant $\Gamma_k(X)$ of a random variable X can be expressed by the moments $M_j(X) := \mathbb{E}X^j$, $j \in I := \{1, \ldots, k\}$, through

$$\Gamma_k(X) = k! \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \sum_{\substack{k_1 + \ldots + k_j = k \\ k_1, \ldots, k_j \geq 1}} \frac{M_{k_1}(X) \cdot \ldots \cdot M_{k_j}(X)}{k_1! \cdot \ldots \cdot k_j!}.$$

From this formula it is easily seen that the first cumulant $\Gamma_1(X)$ equals the mean and the second cumulant $\Gamma_2(X)$ equals the variance.

Conversely, the kth moment $M_k(X)$ can be expressed by the cumulants $\Gamma_j(X)$, $j \in I$, through

$$M_k(X) = k! \sum_{j=1}^{k} \sum_{\substack{k_1+\ldots+k_j=k \\ k_1,\ldots,k_j \geq 1}} \frac{\Gamma_{k_1}(X) \cdot \ldots \cdot \Gamma_{k_j}(X)}{k_1! \cdot \ldots \cdot k_j!}.$$

For the mixed cumulant $\mathrm{Cum}_k(X_1, \ldots, X_k)$ of a random vector $(X_1, \ldots, X_k)'$ we have a representation in terms of the mixed moments $M(X_i : i \in T) := \mathbb{E} \prod_{i \in T} X_i$, $T \subseteq I$, given by

$$\mathrm{Cum}_k(X_1, \ldots, X_k) = \sum_{j=1}^{k} (j-1)!\, (-1)^{j-1} \sum_{I_1 \cup \ldots \cup I_j = I} \prod_{i=1}^{j} M(X_a : a \in I_i).$$

Conversely, the mixed moment $M(X_1, \ldots, X_k)$ can be expressed by the mixed cumulants through

$$M(X_1, \ldots, X_k) = \sum_{j=1}^{k} \sum_{I_1 \cup \ldots \cup I_j = I} \prod_{i=1}^{j} \mathrm{Cum}_{|I_i|}(X_a : a \in I_i), \tag{4.1}$$

see Leonov and Shiryaev [45].

For $k = 1$ we have $\mathrm{Cum}_1(X_1) = \mathbb{E}X_1$, and for $k = 2$ we have $\mathrm{Cum}_2(X_1, X_2) = \mathrm{Cov}(X_1, X_2)$.

Obviously the mixed cumulant $\mathrm{Cum}_k(X_1, \ldots, X_k)$ is invariant under permutation of its arguments $X_1, \ldots, X_k$ and it is homogeneous, that is,

$$\mathrm{Cum}_k(c_1 X_1, \ldots, c_k X_k) = \prod_{j=1}^{k} c_j \cdot \mathrm{Cum}_k(X_1, \ldots, X_k)$$

for constants $c_1, \ldots, c_k \in \mathbb{R}$. Furthermore, the mixed cumulant $\mathrm{Cum}_k(X_1, \ldots, X_k)$ is multilinear: Let $X = (X_1, \ldots, X_k)'$ be a random vector, and Y, Z be random variables with $X_1 = Y + Z$. Then we have

$$\Gamma_1(Y + Z) = \mathbb{E}Y + \mathbb{E}Z = \Gamma_1(Y) + \Gamma_1(Z)$$

for $k = 1$ and

$$\begin{aligned} &\mathrm{Cum}_k(Y + Z, X_2, \ldots, X_k) \\ &= \sum_{j=1}^{k} (j-1)!\, (-1)^{j-1} \sum_{I_1 \cup \ldots \cup I_j = I} \prod_{i=1}^{j} \mathbb{E}\Big[\prod_{a \in I_i} X_a \Big] \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{k} (j-1)!\,(-1)^{j-1} \sum_{\substack{I_1\cup\ldots\cup I_j=I\\ 1\in I_1}} \mathbb{E}\Big[(Y+Z)\cdot \prod_{a\in I_1\setminus\{1\}} X_a\Big] \prod_{i=2}^{j}\mathbb{E}\Big[\prod_{a\in I_i} X_a\Big]\\
&= \sum_{j=1}^{k} (j-1)!\,(-1)^{j-1} \sum_{\substack{I_1\cup\ldots\cup I_j=I\\ 1\in I_1}} \mathbb{E}\Big[Y\cdot \prod_{a\in I_1\setminus\{1\}} X_a\Big] \prod_{i=2}^{j}\mathbb{E}\Big[\prod_{a\in I_i} X_a\Big]\\
&\quad + \sum_{j=1}^{k} (j-1)!\,(-1)^{j-1} \sum_{\substack{I_1\cup\ldots\cup I_j=I\\ 1\in I_1}} \mathbb{E}\Big[Z\cdot \prod_{a\in I_1\setminus\{1\}} X_a\Big] \prod_{i=2}^{j}\mathbb{E}\Big[\prod_{a\in I_i} X_a\Big]\\
&= \mathrm{Cum}_k(Y,X_2,\ldots,X_k) + \mathrm{Cum}_k(Z,X_2,\ldots,X_k)
\end{aligned}$$

for $k \geq 2$. This proves multilinearity due to invariance under permutation of the arguments.

If $Y := (X_i)_{i\in T_1}$ and $Z := (X_i)_{i\in T_2}$ are independent for some $T_1, T_2 \subsetneqq \{1,\ldots,k\}$ with $T_1 \cup T_2 = \{1,\ldots,k\}$, $k \geq 2$, and $T_1 \cap T_2 \neq \emptyset$, then the mixed cumulant $\mathrm{Cum}_k(X_1,\ldots,X_k)$ equals zero. This can be easily seen by differentiating

$$h(t) = \log \mathbb{E}\, e^{\mathrm{i}t'X} = \log \mathbb{E}\, e^{\mathrm{i}s'Y} + \log \mathbb{E}\, e^{\mathrm{i}u'Z}$$

with respect to $t_1,\ldots,t_k$, where $s = (t_i)_{i\in T_1}$ and $u = (t_i)_{i\in T_2}$. This property explains the name "semi-invariant".

Since a constant and a random vector are always independent, the latter property and the multilinearity entail the mixed cumulant to be invariant towards deterministic translation of an arbitrary component, that is,

$$\mathrm{Cum}_k(X+c) = \mathrm{Cum}_k(X)$$

for all random vectors X, constant vectors $c \in \mathbb{R}^d$ and $k \geq 2$.

Let us now consider some properties of cumulants for point process characteristics. For a 4-stationary point process and Borel-measurable functions h_1 and h_2, we have

$$\begin{aligned}
&\mathrm{Cov}\Bigg(\sum_{x,y\in\Psi}^{*} h_1(x,y), \sum_{z,v\in\Psi}^{*} h_2(z,v)\Bigg)\\
&= \int_{(\mathbb{R}^d)^2} h_1(x,y)[h_2(x,y)+h_2(y,x)]\alpha^{(2)}(\mathrm{d}(x,y))\\
&\quad + \int_{(\mathbb{R}^d)^3} h_1(x,y)[h_2(x,z)+h_2(y,z)+h_2(z,x)+h_2(z,y)]\alpha^{(3)}(\mathrm{d}(x,y,z))
\end{aligned}$$

$$+\int\limits_{(\mathbb{R}^d)^4} h_1(x,y)h_2(z,v)[\gamma^{(4)}(\mathrm{d}(x,y,z,v)) + \gamma^{(1)}(\mathrm{d}x)\gamma^{(3)}(\mathrm{d}(y,z,v))$$
$$+ \gamma^{(1)}(\mathrm{d}y)\gamma^{(3)}(\mathrm{d}(x,z,v)) + \gamma^{(1)}(\mathrm{d}z)\gamma^{(3)}(\mathrm{d}(x,y,v)) + \gamma^{(1)}(\mathrm{d}v)\gamma^{(3)}(\mathrm{d}(x,y,z))$$
$$+ \gamma^{(2)}(\mathrm{d}(x,z))\gamma^{(2)}(\mathrm{d}(y,v)) + \gamma^{(2)}(\mathrm{d}(x,v))\gamma^{(2)}(\mathrm{d}(y,z))$$
$$+ \gamma^{(2)}(\mathrm{d}(x,z))\gamma^{(1)}(\mathrm{d}y)\gamma^{(1)}(\mathrm{d}v) + \gamma^{(2)}(\mathrm{d}(x,v))\gamma^{(1)}(\mathrm{d}y)\gamma^{(1)}(\mathrm{d}z)$$
$$+ \gamma^{(2)}(\mathrm{d}(y,z))\gamma^{(1)}(\mathrm{d}x)\gamma^{(1)}(\mathrm{d}v) + \gamma^{(2)}(\mathrm{d}(y,v))\gamma^{(1)}(\mathrm{d}x)\gamma^{(1)}(\mathrm{d}z)], \tag{4.2}$$

provided that the integrals exist, see equation (4.17) in Heinrich [26].

Lemma 4.2.1 will show that the kth cumulant of certain random variables which will be investigated in the following chapters is a sum of integrals that are indecomposable, in the sense that they cannot be represented as a product of two integrals. The rigorous definition of decomposability is as follows.

Let $f_i : (\mathbb{R}^d)^{p_i} \to \mathbb{R}$ be fixed measurable functions, let $k \in \mathbb{N}$ and $p_i \in \mathbb{N}$ with $i \in I = \{1, \dots, k\}$ be fixed and set

$$\Psi^{(p_i)}(f_i) := \sum_{x_1,\dots,x_{p_i} \in \Psi} f_i(x_1, \dots, x_{p_i}).$$

Let $\mathbb{E}\left[|\Psi^{(p_i)}(f_i)|^k\right] < \infty$ for all $i \in I$. We will now find a representation of the mixed moments $M\left(\Psi^{(p_1)}(f_1), \dots, \Psi^{(p_k)}(f_k)\right) = \mathbb{E}\left[\prod_{i=1}^k \Psi^{(p_i)}(f_i)\right]$ as a sum of integrals defined as follows.

For arbitrary $T \subseteq I$, $q \in \{1, \dots, p_T\}$ with $p_T := \sum_{i\in T} p_i$, $r \in \{1, \dots, q\}$, and decompositions $\mathcal{P}_T = \{P_1, \dots, P_q\}$ of $\{1, \dots, p_T\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_r\}$ of $\{1, \dots, q\}$ we define the integral

$$I_{\mathcal{P}_T,\mathcal{Q}}(f_i : \ i \in T)$$
$$:= \int\limits_{(\mathbb{R}^d)^q} \prod_{b=1}^q \prod_{a\in P_b} \mathbb{1}_{\{x_a=z_b\}}\ f_{i_1}(x_1, \dots, x_{p_{i_1}})$$
$$\times\ f_{i_2}(x_{p_{i_1}+1}, \dots, x_{p_{i_1}+p_{i_2}}) \cdot \ldots \cdot f_{i_{|T|}}(x_{\sum_{j=1}^{|T|-1} p_{i_j}+1}, \dots, x_{p_T}) \prod_{c=1}^r \gamma^{(|Q_c|)}(\mathrm{d}\mathbf{z}_{Q_c}),$$

where $\{i_1, \dots, i_{|T|}\} = T$ with $1 \le i_1 < i_2 < \ldots < i_{|T|} \le k$ and $\mathbf{z}_{Q_c} = (z_q)_{q\in Q_c}$. The elements of a set P_b are the indices of the arguments of the functions $f_{i_1}, \dots, f_{i_{|T|}}$ that are identical and distinct from all the arguments in every other set $P_c \neq P_b$. In the above-mentioned integral this is indicated by the term $\prod_{b=1}^q \prod_{a\in P_b} \mathbb{1}_{\{x_a=z_b\}}$.

For the special case $T = I$ we have

$$I_{\mathcal{P}_I,\mathcal{Q}}(f_1,\ldots,f_k) = \int\limits_{(\mathbb{R}^d)^q} \prod_{b=1}^{q}\prod_{a\in P_b} \mathbb{1}_{\{x_a=z_b\}}\, f_1(x_1,\ldots,x_{p_1})\cdot\ldots\cdot f_k(x_{\sum_{i=1}^{k-1}p_i+1},\ldots,x_{p_I})\prod_{c=1}^{r}\gamma^{(|Q_c|)}(\mathrm{d}\mathbf{z}_{Q_c}).$$

Now the mixed moment $M\big(\Psi^{(p_1)}(f_1),\ldots,\Psi^{(p_k)}(f_k)\big)$ can be represented as

$$\begin{aligned} &M\big(\Psi^{(p_1)}(f_1),\ldots,\Psi^{(p_k)}(f_k)\big)\\ &= \sum_{q=1}^{p_I}\sum_{\substack{P_1\cup\ldots\cup P_q\\=\{1,\ldots,p_I\}}}\int\limits_{(\mathbb{R}^d)^q}\prod_{b=1}^{q}\prod_{a\in P_b}\mathbb{1}_{\{x_a=z_b\}}\\ &\qquad\times\ f_1(x_1,\ldots,x_{p_1})\cdot\ldots\cdot f_k(x_{\sum_{i=1}^{k-1}p_i},\ldots,x_{p_I})\alpha^{(q)}(\mathrm{d}(z_1,\ldots,z_q))\\ &= \sum_{q=1}^{p_I}\sum_{\substack{P_1\cup\ldots\cup P_q\\=\{1,\ldots,p\}}}\sum_{r=1}^{q}\sum_{\substack{Q_1\cup\ldots\cup Q_r\\=\{1,\ldots,q\}}}\int\limits_{(\mathbb{R}^d)^q}\prod_{b=1}^{q}\prod_{a\in P_b}\mathbb{1}_{\{x_a=z_b\}}\\ &\qquad\times\ f_1(x_1,\ldots,x_{p_1})\cdot\ldots\cdot f_k(x_{\sum_{i=1}^{k-1}p_i+1},\ldots,x_{p_I})\prod_{c=1}^{r}\gamma^{(|Q_c|)}(\mathrm{d}\mathbf{z}_{Q_c}), \end{aligned}$$

see Krickeberg [44]. With the above notation we have

$$M\big(\Psi^{(p_1)}(f_1),\ldots,\Psi^{(p_k)}(f_k)\big) \;=\; \sum_{q=1}^{p_I}\sum_{\substack{P_1\cup\ldots\cup P_q\\=\{1,\ldots,p_I\}}}\sum_{r=1}^{q}\sum_{\substack{Q_1\cup\ldots\cup Q_r\\=\{1,\ldots,q\}}} I_{\mathcal{P}_I,\mathcal{Q}}(f_1,\ldots,f_k).$$

Let $T = \{T_1, T_2\}$ be a decomposition of $I = \{1,\ldots,k\}$. An integral $I_{\mathcal{P}_I,\mathcal{Q}}(f_1,\ldots,f_k)$ is *decomposable with respect to the decomposition* $T = \{T_1,T_2\}$ if there exist a decomposition $\mathcal{P}^{(1)}$ of $\{1,\ldots,p_{T_1}\}$, a decomposition $\mathcal{P}^{(2)}$ of $\{1,\ldots,p_{T_2}\}$, $q_1\in\{1,\ldots,p_{T_1}\}$ and $q_2\in\{1,\ldots,p_{T_2}\}$ with $q_1+q_2=q$, and decompositions $\mathcal{Q}^{(1)}$ of $\{1,\ldots,q_1\}$ and $\mathcal{Q}^{(2)}$ of $\{1,\ldots,q_2\}$ such that

$$I_{\mathcal{P}_I,\mathcal{Q}}(f_1,\ldots,f_k) = I_{\mathcal{P}_{T_1},\mathcal{Q}^{(1)}}(f_i:\ i\in T_1)\cdot I_{\mathcal{P}_{T_1},\mathcal{Q}^{(2)}}(f_i:\ i\in T_2).$$

An integral is called *decomposable* if there exists a nontrivial decomposition of I such that this integral is decomposable with respect to this decomposition. An integral which is not decomposable with respect to any nontrivial decomposition is called *indecomposable*.

The following lemma is the key tool for the proofs of the central limit theorems in Chapters 6 and 7. It gives a representation of the kth cumulant of certain random variables as a sum of indecomposable integrals.

Lemma 4.2.1 *Let $\Psi \sim P$ be a point process in $\mathbb{R}^d$. Let $j, k \in \mathbb{N}$ be fixed, let $C_i \in \mathbb{R}$ be constants for $i = 1, \ldots, j$, and set*

$$\Psi^{(p_i)}(f_i) = \sum_{x_1,\ldots,x_{p_i} \in \Psi} f_i(x_1, \ldots, x_{p_i}),$$

where $f_i : (\mathbb{R}^d)^{p_i} \to \mathbb{R}$ is a fixed measurable function with $p_i \in \mathbb{N}$, for $i = 1, \ldots, j$. Let $\mathbb{E}\left[|\Psi^{(p_i)}(f_i)|^k\right] < \infty$ for all $i = 1, \ldots, j$.

Then we have

$$\Gamma_k\left(\sum_{i=1}^{j} C_i \Psi^{(p_i)}(f_i)\right) = \sum_{\substack{k_1+\ldots+k_j=k \\ k_1,\ldots,k_j \geq 0}} \frac{k!}{k_1! \cdot \ldots \cdot k_j!} C_1^{k_1} \cdot \ldots \cdot C_j^{k_j} \mu^*_{k_1,\ldots,k_j},$$

where

$$\mu^*_{k_1,\ldots,k_j} := \left(\sum_{q=1}^{p_{k_1,\ldots,k_j}} \sum_{\substack{P_1 \cup \ldots \cup P_q \\ = \{1,\ldots,p_{k_1,\ldots,k_j}\}}} \sum_{r=1}^{q} \sum_{\substack{Q_1 \cup \ldots \cup Q_r \\ = \{1,\ldots,q\}}} \right)^* I_{\mathcal{P}_I,\mathcal{Q}}(\underbrace{f_1, \ldots, f_1}_{k_1}, \ldots, \underbrace{f_j, \ldots, f_j}_{k_j}) \tag{4.3}$$

and $p_{k_1,\ldots,k_j} = \sum_{i=1}^{j} p_i k_i$. The summation $(\cdot)^*$ is taken only over the indecomposable integrals.

Proof: Due to multilinearity, symmetry, and homogeneity of the mixed cumulants we have

$$\begin{aligned}
&\Gamma_k\Big(\sum_{i=1}^{j} C_i \Psi^{(p_i)}(f_i)\Big) \\
&= \mathrm{Cum}_k\Big(\sum_{i=1}^{j} C_i \Psi_{f_i}, \ldots, \sum_{i=1}^{j} C_i \Psi^{(p_i)}(f_i)\Big) \\
&= \sum_{\substack{k_1+\ldots+k_j=k \\ k_1,\ldots,k_j \geq 0}} \frac{k!}{k_1! \cdot \ldots \cdot k_j!} C_1^{k_1} \cdot \ldots \cdot C_j^{k_j} \\
&\qquad \times \mathrm{Cum}_k\big(\underbrace{\Psi^{(p_1)}(f_1), \ldots, \Psi^{(p_1)}(f_1)}_{k_1}, \ldots, \underbrace{\Psi^{(p_j)}(f_j), \ldots, \Psi^{(p_j)}(f_j)}_{k_j}\big).
\end{aligned}$$

In order to prove the identity

$$\mu^*_{k_1,\ldots,k_j} = \mathrm{Cum}_k\big(\underbrace{\Psi^{(p_1)}(f_1), \ldots, \Psi^{(p_1)}(f_1)}_{k_1}, \ldots, \underbrace{\Psi^{(p_j)}(f_j), \ldots, \Psi^{(p_j)}(f_j)}_{k_j}\big)$$

for all $k_1, \ldots, k_j \in \{0, \ldots, k\}$ with $\sum_{i=1}^{j} k_i = k$ we will proceed as in Jolivet [39] and Leonov and Shiryaev [45]. Let $k_1, \ldots, k_j \in \{0, \ldots, k\}$ with $\sum_{i=1}^{j} k_i = k$ and set $\Psi_i = \Psi^{(p_i)}(g_i)$, where

$$g_i = \begin{cases} f_1 & \text{for } i \in \{1, \ldots, k_1\}, \\ f_2 & \text{for } i \in \{k_1+1, \ldots, k_1+k_2\}, \\ \vdots & \\ f_j & \text{for } i \in \{k_1+\ldots+k_{j-1}+1, \ldots, k\}. \end{cases}$$

Then we have

$$\begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_{k_1} \\ \Psi_{k_1+1} \\ \vdots \\ \Psi_k \end{pmatrix} = \begin{pmatrix} \Psi^{(p_1)}(f_1) \\ \vdots \\ \Psi^{(p_1)}(f_1) \\ \Psi^{(p_2)}(f_2) \\ \vdots \\ \Psi^{(p_j)}(f_j) \end{pmatrix}.$$

With $M(\Psi_1, \ldots, \Psi_k) = \mathbb{E}\left[\prod_{i=1}^{k} \Psi_i\right]$ and $I = \{1, \ldots, k\}$ we have (see equation (4.1))

$$\begin{aligned} \mathrm{Cum}_k(\Psi_1, \ldots, \Psi_k) &= M(\Psi_1, \ldots, \Psi_k) - \sum_{j=2}^{k} \sum_{I_1 \cup \ldots \cup I_j = I} \prod_{i=1}^{j} \mathrm{Cum}_{|I_i|}(\Psi_a : a \in I_i) \\ &= \Sigma_{\text{indec}} + \Sigma_{\text{dec}} - \mathcal{C}, \end{aligned}$$

where Σ_{dec} is the sum over the decomposable integrals from $M(\Psi_1, \ldots, \Psi_k)$,

$$\mu^*_{k_1, \ldots, k_j} \equiv \Sigma_{\text{indec}} = M(\Psi_1, \ldots, \Psi_k) - \Sigma_{\text{dec}}$$

is the sum over all indecomposable integrals from $M(\Psi_1, \ldots, \Psi_k)$, and

$$\mathcal{C} = \sum_{j=2}^{k} \sum_{I_1 \cup \ldots \cup I_j = I} \prod_{i=1}^{j} \mathrm{Cum}_{|I_i|}(\Psi_a : a \in I_i)$$

denotes the remaining term.

For $j \in \{2, \ldots, k\}$ and a fixed decomposition $\{I_1, \ldots, I_j\}$ of $I = \{1, \ldots, k\}$, a summand $\prod_{i=1}^{j} \mathrm{Cum}_{|I_i|}(\Psi_a : a \in I_i)$ of $\mathcal{C}$ *factorizes with respect to a decomposition* $T = \{T_1, T_2\}$ if for each $i \in \{1, \ldots, j\}$ we have either $I_i \subseteq T_1$ or $I_i \subseteq T_2$, that is, if the summand can be written as

$$\prod_{i=1}^{j} \mathrm{Cum}_{|I_i|}(\Psi_a : a \in I_i) = \prod_{\substack{i=1 \\ I_i \subseteq T_1}}^{j} \mathrm{Cum}_{|I_i|}(\Psi_a : a \in I_i) \cdot \prod_{\substack{i=1 \\ I_i \subseteq T_2}}^{j} \mathrm{Cum}_{|I_i|}(\Psi_a : a \in I_i).$$

Note that due to $j \geq 2$ each summand $\prod_{i=1}^{j} \mathrm{Cum}_{|I_i|}(\Psi_a : \ a \in I_i)$ factorizes with respect to at least one nontrivial decomposition.

Let P_I be the distribution of the vector $(\Psi_1, \ldots, \Psi_k)'$ which is determined by the distribution P of the point process Ψ. For all $S \subseteq I$, let P_S be the distribution of the vector $(\Psi_a)_{a \in S}$. Every term in $\mathcal{C}$ that factorizes with respect to a fixed decomposition $T = \{T_1, T_2\}$ of I is completely determined by the marginals P_{T_1} and P_{T_2}. The same is true for every term in Σ_{dec} that is decomposable with respect to T.

Let $T^{(1)} = \left\{T_1^{(1)}, T_2^{(1)}\right\}$ be an arbitrary fixed decomposition of I. The sum over the terms of Σ_{dec} that are decomposable with respect to $T^{(1)}$ is denoted by $\Sigma_{\mathrm{dec}}^{(1)}$, and the sum over the terms of $\mathcal{C}$ that factorize with respect to $T^{(1)}$ is denoted by $\mathcal{C}_{\mathrm{dec}}^{(1)}$. Let

$$\Sigma^{(1)} = \Sigma_{\mathrm{dec}} - \Sigma_{\mathrm{dec}}^{(1)}$$

and

$$\mathcal{C}^{(1)} = \mathcal{C} - \mathcal{C}_{\mathrm{dec}}^{(1)}.$$

Then we have

$$\mathrm{Cum}_k(\Psi_1, \ldots, \Psi_k) = \Sigma_{\mathrm{indec}} + \Sigma^{(1)} + \Sigma_{\mathrm{dec}}^{(1)} - \mathcal{C}^{(1)} - \mathcal{C}_{\mathrm{dec}}^{(1)}.$$

Now we will show

$$\Sigma_{\mathrm{dec}}^{(1)} = \mathcal{C}_{\mathrm{dec}}^{(1)}. \tag{4.4}$$

To this end we set $\tilde{P}_I := P_{T_1^{(1)}} \otimes P_{T_2^{(1)}}$, where $P_{T_1^{(1)}}$ and $P_{T_2^{(1)}}$ are the distributions of the random vectors $\left(\Psi_a : a \in T_1^{(1)}\right)$ and $\left(\Psi_a : a \in T_2^{(1)}\right)$, respectively. Let $(\tilde{\Psi}_1, \ldots, \tilde{\Psi}_k)'$ be a random vector with distribution $\tilde{P}_I$, and let $\tilde{\Sigma}_{\mathrm{indec}}$, $\tilde{\Sigma}^{(1)}$, $\tilde{\Sigma}_{\mathrm{dec}}^{(1)}$, $\tilde{\mathcal{C}}^{(1)}$, and $\tilde{\mathcal{C}}_{\mathrm{dec}}^{(1)}$ be defined as Σ_{indec}, $\Sigma^{(1)}$, $\Sigma_{\mathrm{dec}}^{(1)}$, $\mathcal{C}^{(1)}$, and $\mathcal{C}_{\mathrm{dec}}^{(1)}$ above, with $(\Psi_1, \ldots, \Psi_k)'$ replaced by $(\tilde{\Psi}_1, \ldots, \tilde{\Psi}_k)'$. By construction we have $\left(\tilde{\Psi}_a : a \in T_i^{(1)}\right) \sim P_{T_i^{(1)}}$, that is, $\left(\tilde{\Psi}_a : a \in T_i^{(1)}\right) \stackrel{\mathrm{d}}{=} \left(\Psi_a : a \in T_i^{(1)}\right)$, $i = 1, 2$. Hence the fact that $\Sigma_{\mathrm{dec}}^{(1)}$ and $\mathcal{C}_{\mathrm{dec}}^{(1)}$ are completely determined by the marginals $P_{T_1^{(1)}}$ and $P_{T_2^{(1)}}$ implies $\tilde{\Sigma}_{\mathrm{dec}}^{(1)} = \Sigma_{\mathrm{dec}}^{(1)}$ and $\tilde{\mathcal{C}}_{\mathrm{dec}}^{(1)} = \mathcal{C}_{\mathrm{dec}}^{(1)}$. In particular we have

$$\mathrm{Cum}_k(\tilde{\Psi}_1, \ldots, \tilde{\Psi}_k) = \tilde{\Sigma}_{\mathrm{indec}} + \tilde{\Sigma}^{(1)} + \Sigma_{\mathrm{dec}}^{(1)} - \tilde{\mathcal{C}}^{(1)} - \mathcal{C}_{\mathrm{dec}}^{(1)}. \tag{4.5}$$

Clearly, $\left(\tilde{\Psi}_a : a \in T_1^{(1)}\right)$ and $\left(\tilde{\Psi}_a : a \in T_2^{(1)}\right)$ are independent by construction. This implies the left-hand side in (4.5) to be equal to zero. Since the mixed moment

$$M(\tilde{\Psi}_1,\dots,\tilde{\Psi}_k) = \mathbb{E}\Big[\prod_{i=1}^{k}\tilde{\Psi}_i\Big] = \mathbb{E}\Big[\prod_{i\in T_1^{(1)}}\tilde{\Psi}_i\Big]\cdot\mathbb{E}\Big[\prod_{i\in T_2^{(1)}}\tilde{\Psi}_i\Big] = \tilde{\Sigma}_{\text{dec}}^{(1)}$$

is decomposable with respect to the decomposition $T^{(1)}$ we also have $\tilde{\Sigma}_{\text{indec}} = 0$ and $\tilde{\Sigma}^{(1)} = 0$.

Finally the independence of $\big(\tilde{\Psi}_a : a \in T_1^{(1)}\big)$ and $\big(\tilde{\Psi}_a : a \in T_2^{(1)}\big)$ yields $\mathrm{Cum}_k(\tilde{\Psi}_a : a \in K) = 0$ for all $K \subseteq \{1,\dots,k\}$ with $K\cap T_1^{(1)} \neq \emptyset$ and $K\cap T_2^{(1)} \neq \emptyset$. Since every summand in $\tilde{\mathcal{C}}^{(1)}$ contains a factor of this type we obtain $\tilde{\mathcal{C}}^{(1)} = 0$.

Altogether this proves (4.4) by equation (4.5). As a result we have

$$\mathrm{Cum}_k(\Psi_1,\dots,\Psi_k) = \Sigma_{\text{indec}} + \Sigma^{(1)} - \mathcal{C}^{(1)}.$$

Now we go through all possible decompositions of I in this manner. Since every term of Σ_{dec} is decomposable with respect to some decomposition and every term of $\mathcal{C}$ factorizes with respect to some decomposition, this yields

$$\Sigma_{\text{dec}} = \mathcal{C}$$

and hence

$$\mathrm{Cum}_k(\Psi_1,\dots,\Psi_k) = \Sigma_{\text{indec}}.$$

In summary we have

$$\mathrm{Cum}_k\big(\underbrace{\Psi^{(p_1)}(f_1),\dots,\Psi^{(p_1)}(f_1)}_{k_1},\dots,\underbrace{\Psi^{(p_j)}(f_j),\dots,\Psi^{(p_j)}(f_j)}_{k_j}\big) = \mu^*_{k_1,\dots,k_j}$$

for all $k_1,\dots,k_j \in \{0,\dots,k\}$ with $\sum_{i=1}^{j} k_i = k$. This completes the proof. ∎

5

Estimators for product densities and the pair correlation function

Kernel-type estimators for product densities were introduced by Krickeberg [44], page 245, in 1982. Since then, especially the second-order product density and its isotropic analogue, the pair correlation function, have been of particular interest for spatial data analysis. In the first section we define kernel functions, present some conditions that will be helpful in the following chapters for deriving convergence rates of the mean and the variance of the estimators, and give some examples of kernel functions satisfying these conditions. The second and the third section introduce estimators for product densities and the pair correlation function, respectively. The chapter concludes with an interpretation of the second-order product density, the pair correlation function, and their estimators.

For asymptotic considerations we assume that a single realization of a point process Ψ is given in an expanding convex observation window. For 1-stationary point processes this corresponds to an increase in the expected number of points.

5.1 Kernel functions

The estimators for product densities that will be introduced in the following section are based on *kernel functions*. After the definition of kernel functions we will state some conditions that can be used for deriving rates of convergence for the estimators. The section ends with some examples of kernel functions.

Definition 5.1.1 *Let $\ell \geq 2$. The function $k^{(\ell,d)} : (\mathbb{R}^d)^{\ell-1} \to \mathbb{R}$ is called a* kernel function *if it satisfies*

(i) $\displaystyle\int_{(\mathbb{R}^d)^{\ell-1}} k^{(\ell,d)}(x)\mathrm{d}x = 1,$

(ii) $k^{(\ell,d)}(x) = 0$ *for every* $x \in (\mathbb{R}^d)^{\ell-1}$ *with* $\|x\| > R$ *and some* $R \in (0,\infty)$*,*

(iii) $|k^{(\ell,d)}(x)| \leq M$ *for every* $x \in (\mathbb{R}^d)^{\ell-1}$ *and some* $M \in (0,\infty)$*, and*

(iv) $k^{(\ell,d)}(x) = k^{(\ell,d)}(-x)$ *for every* $x \in (\mathbb{R}^d)^{\ell-1}$. □

The second condition in the above definition says that the support of the kernel function satisfies $\operatorname{supp}\left(k^{(\ell,d)}\right) \subseteq b_{\ell-1}(o,R)$, implying the kernel function to have bounded support. (Here, $b_{\ell-1}(x,r)$ denotes the closed ball in $(\mathbb{R}^d)^{\ell-1}$ with radius $r > 0$ and midpoint $x \in (\mathbb{R}^d)^{\ell-1}$.) This condition is not essential for our further investigations but simplifies proofs in the following chapters. The condition could be replaced by integrability conditions to get the same results.

From now on we will suppress the superscript and write only $k = k^{(\ell,d)}$ for a kernel function wherever the appropriate superscript follows from the context.

The following condition on the kernel function is helpful for deriving rates of convergence of mean and variance of the estimated product densities and pair correlation function that will be defined in the next two sections. For the kernel function $k^{(2,d)}$ this condition has been defined in Heinrich and Liebscher [30]. An analogous condition is also needed for probability density estimators, see Nadaraya [47], page 19, and Horváth [37].

Condition 5.1.2 $\mathcal{K}((\ell-1)d, s)$
The kernel function $k = k^{(\ell,d)}$ satisfies Condition $\mathcal{K}((\ell-1)d, s)$ ($d \geq 1$, $s \geq 1$), see Heinrich and Liebscher [30], if for $i_1, \ldots, i_j \in \{1, \ldots, (\ell-1)d\}$, $j = 1, \ldots, s-1$ (with $s \geq 2$)

$$\int_{(\mathbb{R}^d)^{(\ell-1)}} x_{i_1} \cdot \ldots \cdot x_{i_j} k(x_1, \ldots, x_{(\ell-1)d}) \mathrm{d}(x_1, \ldots, x_{(\ell-1)d}) = 0$$

holds. □

Condition $\mathcal{K}((\ell-1)d, 1)$ is satisfied for every kernel $k^{(\ell,d)}$ as defined in 5.1.1 since there are no further requirements for $s = 1$. Obviously, if a kernel function satisfies Condition $\mathcal{K}((\ell-1)d, s)$ for some $s \geq 2$, it satisfies $\mathcal{K}((\ell-1)d, j)$ for all $j \leq s$.

The conus kernel,

$$k(x) = \frac{d(d+1)\Gamma(d/2)}{2\pi^{d/2}R^d}\left(1 - \frac{\|x\|}{R}\right)\mathbb{1}_{b(o,R)}(x),$$

and the Epanechnikov kernel,

$$k(x) = \frac{d(d+2)\Gamma(d/2)}{4\pi^{d/2}R^d}\left(1 - \frac{\|x\|^2}{R^2}\right)\mathbb{1}_{b(o,R)}(x),$$

satisfy Condition $\mathcal{K}(d,2)$, see Heinrich and Liebscher [30].

The following condition can be used for deriving rates of convergence of the mean and the variance of the empirical second-order product density.

Condition 5.1.3 $\mathcal{K}^2(d,2)$
The kernel function $k = k^{(2,d)}$ satisfies Condition $\mathcal{K}^2(d,2)$ if

$$\int_{\mathbb{R}^d} x_i k^2(x_1,\ldots,x_d)\mathrm{d}(x_1,\ldots,x_d) = 0$$

holds for all $i \in \{1,\ldots,d\}$. □

The *convolution of the kernel function* $k = k^{(\ell,d)}$ defined by

$$\tilde{k} : (\mathbb{R}^d)^{\ell-1} \to \mathbb{R}, \qquad t \mapsto \int_{(\mathbb{R}^d)^{\ell-1}} k(x)k(t-x)\mathrm{d}x$$

will occur in Chapter 7. Due to the symmetry of the kernel k we have

$$\tilde{k}(t) = \int_{(\mathbb{R}^d)^{\ell-1}} k(x)k(t-x)\mathrm{d}x = \int_{(\mathbb{R}^d)^{\ell-1}} k(x)k(x-t)\mathrm{d}x = \int_{(\mathbb{R}^d)^{\ell-1}} k(x+t)k(x)\mathrm{d}x = \tilde{k}(-t)$$

for all $t \in (\mathbb{R}^d)^{\ell-1}$, that is, the convolution $\tilde{k}$ is symmetric.

Table 5.1 provides some examples of univariate kernel functions $k^{(2,1)}$, along with their values of $\int_{\mathbb{R}} k^2(x)\mathrm{d}x$ and $\int_{\mathbb{R}} \tilde{k}^2(y)\mathrm{d}y$. The column labeled $\mathcal{K}(1,s)$ shows the maximum value of s for which $\mathcal{K}(1,s)$ is satisfied by the respective kernel function. Except for the cosine kernel all kernels and their value of $\int_{\mathbb{R}} k^2(x)\mathrm{d}x$ can be found in Nadaraya [47], page 176. Two misprints in Nadaraya [47], $\frac{2625}{2048\sqrt{\pi}}$ instead of $\frac{2265}{2048\sqrt{\pi}}$ and $(0.54 + 0.46\cos\pi x)\,\mathbb{1}_{[-1,1]}(x)$ instead of $\left(\frac{1}{2} + \frac{1}{2}\cos\pi x\right)\mathbb{1}_{[-1,1]}(x)$, are corrected.

For all even $s \in \mathbb{N}$ the function

$$k(x) = \sum_{\substack{i=0 \\ i \text{ even}}}^{s-2} a_i x^i \mathbb{1}_{[-1,1]}(x)$$

is a kernel function that satisfies Condition 5.1.2 $\mathcal{K}(1,s)$, where $(a_0, a_2, \ldots, a_{s-2})$ is the solution of the linear system of equations

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}_{\frac{s}{2}\times 1} = \begin{pmatrix} 1 & \frac{1}{3} & \cdots & \frac{1}{s-1} \\ \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{s+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{s-1} & \frac{1}{s+1} & \cdots & \frac{1}{2s-3} \end{pmatrix}_{\frac{s}{2}\times\frac{s}{2}} \begin{pmatrix} a_0 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{s-2} \end{pmatrix}_{\frac{s}{2}\times 1}.$$

A multivariate kernel function $k^{(\ell,d)}$ can be obtained by setting

$$k^{(\ell,d)}(x_1, \ldots, x_{(\ell-1)d}) = \prod_{i=1}^{(\ell-1)d} k_i(x_i)$$

("product kernels") where $k_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, (\ell-1)d$, are univariate kernel functions, see e.g. Table 5.1. If k_i satisfies Condition $\mathcal{K}(1, s_i)$, $i = 1, \ldots, (\ell-1)d$, then the product kernel $k^{(\ell,d)}$ satisfies Condition $\mathcal{K}((\ell-1)d, \min_{i=1,\ldots,d} s_i)$. Furthermore we have

$$\int_{(\mathbb{R}^d)^{\ell-1}} (k^{(\ell,d)}(x))^2 \mathrm{d}x = \prod_{i=1}^{(\ell-1)d} \int_{\mathbb{R}} k_i^2(x_i)\mathrm{d}x_i$$

and

$$\int_{(\mathbb{R}^d)^{\ell-1}} (\tilde{k}^{(\ell,d)}(y))^2 \mathrm{d}y = \prod_{i=1}^{(\ell-1)d} \int_{\mathbb{R}} \tilde{k}_i^2(y_i)\mathrm{d}y_i.$$

Further examples of multivariate kernels are the conus kernel and the Epanechnikov kernel.

Table 5.1: Kernel functions

$k(x)$	$\int_{\mathbb{R}} k^2(x)\mathrm{d}x$	$\int_{\mathbb{R}} \tilde{k}^2(y)\mathrm{d}y$	$\mathcal{K}(1,s)$
$\frac{1}{2}\mathbb{1}_{[-1,1]}(x)$ (rectangular kernel)	$\frac{1}{2}$	$\frac{1}{3}$	$s=2$
$(1-\lvert x\rvert)\mathbb{1}_{[-1,1]}(x)$ (triangular kernel)	$\frac{2}{3}$	$\frac{151}{315}$	$s=2$
$\left(\frac{1}{\sqrt{6}}-\frac{\lvert x\rvert}{6}\right)\mathbb{1}_{[-\sqrt{6},\sqrt{6}]}(x)$	$\frac{\sqrt{6}}{9}$	$\frac{151\sqrt{6}}{1890}$	$s=2$
$\left(\frac{3}{4\sqrt{5}}-\frac{3x^2}{20\sqrt{5}}\right)\mathbb{1}_{[-\sqrt{5},\sqrt{5}]}(x)$ (Epanechnikov kernel)	$\frac{3}{5\sqrt{5}}$	$\frac{167}{385\sqrt{5}}$	$s=2$
$\left(\frac{1}{2}+\frac{1}{2}\cos\pi x\right)\mathbb{1}_{[-1,1]}(x)$	$\frac{3}{4}$	$\frac{3(8\pi^2+35)}{64\pi^2}$	$s=2$
$\frac{\pi}{4}\cos\frac{\pi}{2}x\mathbb{1}_{[-1,1]}(x)$ (cosine kernel)	$\frac{\pi^2}{16}$	$\frac{\pi^4(8\pi^2+15)}{1536\pi^2}$	$s=2$
$\frac{9}{8}\left(1-\frac{5}{3}x^2\right)\mathbb{1}_{[-1,1]}(x)$	$\frac{9}{8}$	$\frac{543}{616}$	$s=4$
$\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$ (Gaussian kernel)	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{\sqrt{2\pi}}$	$s=2$
$\frac{1}{2}\exp(-\lvert x\rvert)$	$\frac{1}{4}$	$\frac{5}{32}$	$s=2$
$\frac{3}{2}\left(1-\frac{x^2}{3}\right)\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$	$\frac{27}{32\sqrt{\pi}}$	$\frac{7881}{8192\sqrt{2\pi}}$	$s=4$
$\frac{15}{8}\left(1-\frac{2}{3}x^2+\frac{1}{15}x^4\right)\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$	$\frac{2265}{2048\sqrt{\pi}}$	$\frac{711122385}{536870912\sqrt{2\pi}}$	$s=6$

Examples for univariate kernel functions k and associated values of $\int_{\mathbb{R}} k^2(x)\mathrm{d}x$ and $\int_{\mathbb{R}} \tilde{k}^2(y)\mathrm{d}y$. The rightmost column indicates the maximal value of s for which Condition 5.1.2 $\mathcal{K}(1,s)$ is satisfied. The last four functions do not satisfy our definition of kernel functions since their support is unbounded.

5.2 Estimators for product densities

The definition of a kernel-type estimator for the ℓth-order product density, $\ell \geq 2$, goes back to Krickeberg [44]. Let

$$\rho(W) \quad := \quad \sup\{r \geq 0 : b(x,r) \subset W, x \in \mathbb{R}^d\}$$

denote the *inscribed radius of the set* $W \subseteq \mathbb{R}^d$.

Definition 5.2.1 *Let* $(W_n)_{n\in\mathbb{N}}$ *be a sequence of convex sets in* $\mathbb{R}^d$ *satisfying* $\rho(W_n) \xrightarrow[n\to\infty]{} \infty$. *Let* $(b_n)_{n\in\mathbb{N}}$ *be a descending sequence of positive real numbers satisfying* $b_n \xrightarrow[n\to\infty]{} 0$ *and let* $(b_n^d)^{\ell-1}|W_n| \xrightarrow[n\to\infty]{} \infty$. *Let* $k = k^{(\ell,d)}$ *be a kernel function with support contained in* $[-R,R]^{d(\ell-1)}$.

Let the point process $\Psi \sim P$ *in* $\mathbb{R}^d$ *be 2-stationary and assume its* ℓ*th-order product density* $\varrho^{(\ell)}$ *to exist.*

Then we define

$$\hat{\varrho}_n^{(\ell)}(t_1,\ldots,t_{\ell-1}) := \frac{1}{(b_n^d)^{\ell-1}|W_n|} \sum_{x_1,\ldots,x_\ell\in\Psi}^{*} \mathbb{1}_{W_n}(x_1) k\left(\frac{x_2-x_1-t_1}{b_n},\ldots,\frac{x_\ell-x_1-t_{\ell-1}}{b_n}\right)$$

as an estimator for $\lambda\varrho^{(\ell)}(t)$ *for* $t = (t_1,\ldots,t_{\ell-1}) \in (\mathbb{R}^d)^{\ell-1}$. □

The above-mentioned sequence $(b_n)_{n\in\mathbb{N}}$ is called the sequence of *bandwidths*. Evaluating the estimator $\hat{\varrho}_n^{(\ell)}(t)$ requires observing a realization of Ψ in the window $W_n \oplus b(o, b_n R + \|t\|)$. (Here, $A \oplus B = \{x+y : x \in A, y \in B\}$ denotes Minkowski addition of two convex sets $A, B \subseteq \mathbb{R}^d$.) Given a realization of a point process in a window W_n, the estimator $\hat{\varrho}_n^{(\ell)}$ can still be used by applying minus sampling, see Stoyan et al. [57], page 133, which is a simple form of edge-correction. Another edge-corrected version of the above-mentioned estimator—analogous to edge-corrected versions of estimators for the kth-order factorial moment measure as considered in Hanisch [24]—, is given by the following definition.

Definition 5.2.2 *Let* $(W_n)_{n\in\mathbb{N}}$ *be a sequence of convex sets in* $\mathbb{R}^d$ *satisfying* $\rho(W_n) \xrightarrow[n\to\infty]{} \infty$. *Let* $(b_n)_{n\in\mathbb{N}}$ *be a descending sequence of positive real numbers satisfying* $b_n \xrightarrow[n\to\infty]{} 0$ *and let* $(b_n^d)^{\ell-1}|W_n| \xrightarrow[n\to\infty]{} \infty$. *Let* $k = k^{(\ell,d)}$ *be a kernel function with support lying in* $[-R,R]^{d(\ell-1)}$.

Let the point process $\Psi \sim P$ *in* $\mathbb{R}^d$ *be 2-stationary and assume its* ℓ*th-order product density* $\varrho^{(\ell)}$ *to exist.*

Then we define

$$\tilde{\varrho}_n^{(\ell)}(t_1,\dots,t_{\ell-1}) := \sum_{x_1,\dots,x_\ell\in\Psi}^{*} \frac{\mathbb{1}_{W_n}(x_1)\dots\mathbb{1}_{W_n}(x_\ell)k\left(\frac{x_2-x_1-t_1}{b_n},\dots,\frac{x_\ell-x_1-t_{\ell-1}}{b_n}\right)}{(b_n^d)^{\ell-1}\,|(W_n-x_1)\cap\dots\cap(W_n-x_\ell)|}$$

as an estimator for $\lambda\varrho^{(\ell)}(t)$ *for* $t=(t_1,\dots,t_{\ell-1})\in(\mathbb{R}^d)^{\ell-1}$. □

The estimator $\hat{\varrho}_n^{(\ell)}$ has already been studied by Jolivet [40]. We will study its asymptotic behavior and investigate statistics based on this estimator in Chapters 6 and 7.

5.3 Estimators for the pair correlation function

Now we define estimators for the pair correlation function. Let $\omega_d = \frac{\sqrt{\pi^d}}{\Gamma(1+d/2)}$ denote the volume of the unit ball in $\mathbb{R}^d$.

Definition 5.3.1 *Let* $(W_n)_{n\in\mathbb{N}}$ *be a sequence of convex sets in* $\mathbb{R}^d$ *satisfying* $\rho(W_n) \xrightarrow[n\to\infty]{} \infty$. *Let* $(b_n)_{n\in\mathbb{N}}$ *be a descending sequence of positive real numbers satisfying* $b_n \xrightarrow[n\to\infty]{} 0$ *and let* $b_n|W_n| \xrightarrow[n\to\infty]{} \infty$. *Let* $k=k^{(2,1)}$ *be a kernel function with support lying in* $[-R,R]$.

Let the point process $\Psi\sim P$ *in* $\mathbb{R}^d$ *be 2-stationary and assume its pair correlation function to exist.*

Then we define

$$\hat{g}_n(r) = \frac{1}{b_n\,|W_n|\,d\,\omega_d}\sum_{x_1,x_2\in\Psi}^{*}\frac{\mathbb{1}_{W_n}(x_1)}{\|x_2-x_1\|^{d-1}}\,k\Big(\frac{\|x_2-x_1\|-r}{b_n}\Big)$$

as an estimator for $\lambda^2 g(r)$ *for* $r\in[0,\infty)$. □

Definition 5.3.2 *Let* $(W_n)_{n\in\mathbb{N}}$ *be a sequence of convex sets in* $\mathbb{R}^d$ *satisfying* $\rho(W_n) \xrightarrow[n\to\infty]{} \infty$. *Let* $(b_n)_{n\in\mathbb{N}}$ *be a descending sequence of positive real numbers satisfying* $b_n \xrightarrow[n\to\infty]{} 0$ *and let* $b_n|W_n| \xrightarrow[n\to\infty]{} \infty$. *Let* $k=k^{(2,1)}$ *be a kernel function with support lying in* $[-R,R]$.

Let the point process $\Psi\sim P$ *in* $\mathbb{R}^d$ *be 2-stationary and assume its pair correlation function to exist.*

Then we define

$$\hat{g}_{n,2}(r) = \frac{1}{b_n\,|W_n|\,d\,\omega_d\,r^{d-1}} \sum_{x_1,x_2\in\Psi}^{*} \mathbb{1}_{W_n}(x_1)\,k\Big(\frac{\|x_2-x_1\|-r}{b_n}\Big)$$

as an estimator for $\lambda^2 g(r)$ for $r \in (0,\infty)$. □

As in the case of the estimated product density $\hat{\varrho}_n$ evaluation of the estimators $\hat{g}_n(r)$ and $\hat{g}_{n,2}(r)$ requires observing a realization of Ψ in $W_n \oplus b(o, b_n R + r)$.

Just like for the estimator for the second-order product density we can define edge-corrected versions of the estimators $\hat{g}_n$ and $\hat{g}_{n,2}$, see Fiksel [17].

Definition 5.3.3 *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of convex sets in $\mathbb{R}^d$ satisfying $\rho(W_n) \xrightarrow[n\to\infty]{} \infty$. Let $(b_n)_{n\in\mathbb{N}}$ be a descending sequence of positive real numbers satisfying $b_n \xrightarrow[n\to\infty]{} 0$ and let $b_n|W_n| \xrightarrow[n\to\infty]{} \infty$. Let $k = k^{(2,1)}$ be a kernel function with support lying in $[-R,R]$.*

Let the point process $\Psi \sim P$ in $\mathbb{R}^d$ be 2-stationary and assume its pair correlation function to exist. Then we define

$$\tilde{g}_n(r) = \sum_{x_1,x_2\in\Psi}^{*} \frac{\mathbb{1}_{W_n}(x_1)\mathbb{1}_{W_n}(x_2)k\left(\frac{\|x_2-x_1\|-r}{b_n}\right)}{b_n\,|(W_n-x_1)\cap(W_n-x_2)|\,d\,\omega_d\|x_2-x_1\|^{d-1}}$$

as an estimator for $\lambda^2 g(r)$ for $r \in [0,\infty)$. □

Definition 5.3.4 *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of convex sets in $\mathbb{R}^d$ with the inscribed radii satisfying $\rho(W_n) \xrightarrow[n\to\infty]{} \infty$. Let $(b_n)_{n\in\mathbb{N}}$ be a descending sequence of positive real numbers satisfying $b_n \xrightarrow[n\to\infty]{} 0$ and let $b_n|W_n| \xrightarrow[n\to\infty]{} \infty$. Let $k = k^{(2,1)}$ be a kernel function with support lying in $[-R,R]$.*

Let the point process $\Psi \sim P$ in $\mathbb{R}^d$ be 2-stationary and assume its pair correlation function to exist. Then we define

$$\tilde{g}_{n,2}(r) = \sum_{x_1,x_2\in\Psi}^{*} \frac{\mathbb{1}_{W_n}(x_1)\mathbb{1}_{W_n}(x_2)k\left(\frac{\|x_2-x_1\|-r}{b_n}\right)}{b_n\,|(W_n-x_1)\cap(W_n-x_2)|\,d\,\omega_d\,r^{d-1}}$$

as an estimator for $\lambda^2 g(r)$ for $r \in (0,\infty)$. □

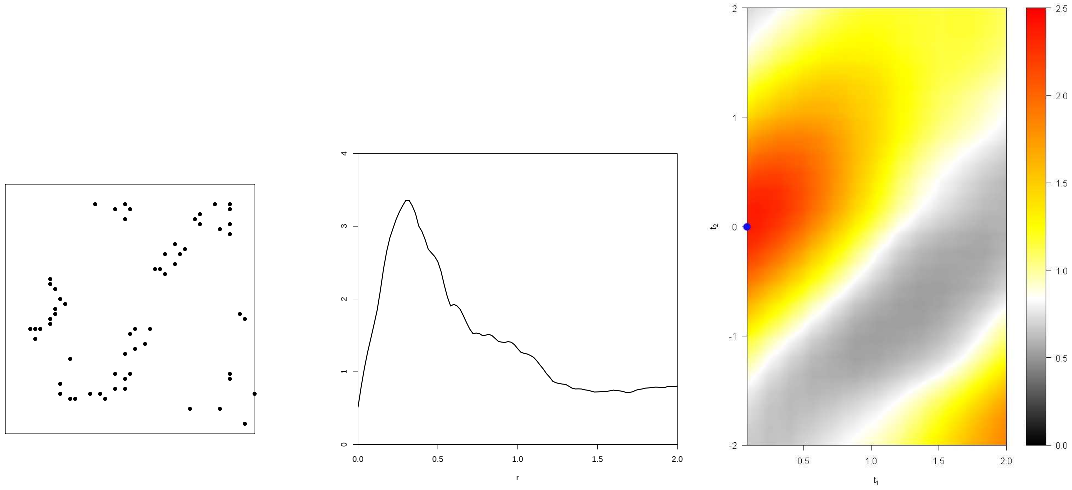
Stoyan and Stoyan [60] suggest further estimators for $g(r)$ using adapted distance-dependent intensity estimators and refined estimators of the squared intensity. Simulation studies for Poisson processes, Matérn's cluster processes, and Matérn's hard-core processes show that these estimators are superior to the above estimators in the sense of reduced bias and variance for a fixed bandwidth. The fact that these estimators involve a division by an estimator for the squared intensity makes the analytic investigation of these estimators' asymptotic behavior a difficult problem. Therefore we will restrict our investigations to the estimators from Definitions 5.3.1–5.3.4.

5.4 Interpretation

In Section 2.3 we have given an interpretation of the second-order product density and the pair correlation function. We will now continue this discussion and present an example of a real data set. The interpretation of the pair correlation function has been discussed in detail in Stoyan et al. [57], page 130. For the Poisson process the pair correlation function g is the constant 1. High values of g indicate frequent occurrences of inter-point distances; likewise, low values of g indicate inhibition at these inter-point distances. The second-order product density can be interpreted analogously, with directed distances $t \in \mathbb{R}^d$ instead of (undirected) distances. That is, high values of the second-order product density $\varrho^{(2)}$ indicate frequent occurrences of directed distances between points, whereas low values of $\varrho^{(2)}$ indicate inhibition at these directed distances.

Figure 5.1 shows the estimated pair correlation function and second-order product density for the data set **redwood** provided in the **R** package **spatstat**, see Baddeley and Turner [2]. This point field's anisotropy is not visible in the estimated pair correlation function but is evident in the estimated second-order product density.

Figure 5.1: Interpretation of estimated pair correlation function and second-order product density



Locations of 62 seedlings and saplings of California redwood trees (left) and the estimated pair correlation function of this point field up to the radius 2 (middle). In the level plot of the estimated second-order product density for $t = (t_1, t_2)' \in [0, 2] \times [-2, 2]$ (right) a blue dot marks the origin.

6

Central limit theorems for empirical product densities and the empirical pair correlation function

This chapter deals with the asymptotic behavior of estimators for product densities and the pair correlation function. It is divided into results for the empirical second-order product density and the empirical pair correlation function in the first section and analogous results for the estimators for product densities of higher order in the second section. The asymptotic behavior of these estimators has already been studied, see for instance Jolivet [40], Heinrich [26], and Heinrich and Liebscher [30]. Jolivet [40] studies the speed of convergence of empirical product densities of order two and higher, and presents a short proof of the normal convergence for the estimated product densities of order two and higher by determining the asymptotic order of their cumulants. However, the assumptions stated in Jolivet [40] are not sufficient for deriving this representation, which may be seen by deriving the asymptotic order of the variance. In the setting of Poisson cluster processes Heinrich [26] proves a central limit theorem for the empirical second-order product density. Heinrich and Liebscher [30] prove almost sure convergence of the estimators of the second-order product density and the pair correlation function in the setting of β-mixing point processes and also give rates of convergence.

All results in this chapter are stated only for one of the product density estimators and one of the pair correlation estimators defined in Sections 5.2 and 5.3, respectively. The results carry over to the other estimators from these sections. Proofs parallel those to be given in this chapter. In the following we will use the abbreviations $\varrho = \varrho^{(2)}$ and $\hat{\varrho}_n = \hat{\varrho}_n^{(2)}$.

6.1 Central limit theorems for the empirical second-order product density and the empirical pair correlation function

After presenting asymptotic representations of the mean and the variance of the estimators for the second-order product density (see Definitions 5.2.1 and 5.2.2 with $\ell = 2$) and the estimator for the pair correlation function (see Section 5.3) under mild mixing conditions we derive central limit theorems for these estimators in the setting of Brillinger-mixing point processes. Throughout this section we tacitly assume the second-order product density to exist.

6.1.1 Asymptotic representation of the mean and the variance of the estimators

In this section we derive asymptotic representations for the mean and the variance of the estimators for the second-order product density and the pair correlation function under mild mixing conditions. Analogous results for these estimators in the setting of β-mixing point processes can be found in Heinrich and Liebscher [30].

The following theorem gives an asymptotic representation for the mean of the estimator for the second-order product density. We will use the *Landau notation* $f(n) = O(h(n))$ as $n \to \infty$ for error terms $f(n)$ satisfying $\limsup_{n\to\infty} \left|\frac{f(n)}{h(n)}\right| < \infty$, and $f(n) = o(h(n))$ as $n \to \infty$ for error terms $f(n)$ satisfying $\lim_{n\to\infty} \left|\frac{f(n)}{h(n)}\right| = 0$.

Theorem 6.1.1 *Let $\Psi \sim P$ be a 2-stationary point process in $\mathbb{R}^d$ with intensity λ. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\hat{\varrho}_n(t) = \lambda\varrho(t)$$

in every point of continuity $t \in \mathbb{R}^d$ of ϱ. In addition, let the kernel function satisfy Condition 5.1.2 $\mathcal{K}(d,s)$ and let ϱ have bounded and continuous partial derivatives of order s in $b^o(t,\varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$.

Then we have

$$\mathbb{E}\hat{\varrho}_n(t) = \lambda\varrho(t) + O(b_n^s)$$

as $n \to \infty$.

Proof: Due to the 2-stationarity of Ψ and the existence of the second-order product density we have

$$\mathbb{E}\hat{\varrho}_n(t) = \int_{\mathbb{R}^d} k(y)\lambda\varrho(b_n y + t)\mathrm{d}y.$$

The continuity of ϱ in t and the boundedness conditions on the kernel function yield the first conjecture by Lebesgue's dominated convergence theorem.

By Taylor's expansion we get

$$\varrho(t + b_n z) = \varrho(t) + \sum_{i=1}^{s-1} \frac{1}{i!} \left(\frac{\partial}{\partial t_1} b_n z_1 + \ldots + \frac{\partial}{\partial t_d} b_n z_d \right)^i \varrho(t_1, \ldots, t_d) + R_s(z, t)$$

where $R_s(z,t) = \frac{1}{s!} \left(\frac{\partial}{\partial t_1} b_n z_1 + \ldots + \frac{\partial}{\partial t_d} b_n z_d \right)^s \varrho(t_1 + \theta b_n z_1, \ldots, t_d + \theta b_n z_d)$ for some $\theta \in (0,1)$. Together with condition $\mathcal{K}(d,s)$ and the boundedness of the partial derivatives of order s of ϱ in $b^o(t, \varepsilon)$ for some $\varepsilon > 0$ we obtain the claimed rate of convergence. ■

The following theorem states the assumptions under which the variance of the estimator for the second-order product density vanishes. Since asymptotic unbiasedness does not require any further assumptions (see the first part of the previous theorem), the assumptions in Theorem 6.1.2 are solely responsible for ensuring weak consistency of $\hat{\varrho}_n(.)$ for $\lambda\varrho(.)$.

Theorem 6.1.2 *Let $\Psi \sim P$ be a 4-stationary point process in $\mathbb{R}^d$ with intensity λ with finite total variations $\|\gamma_{\mathrm{red}}^{(k)}\| < \infty$, $k = 2,3,4$. Let one of the assumptions*

(i) the bandwidth satisfies $b_n^{2d}|W_n| \xrightarrow[n\to\infty]{} \infty$

(ii) the third-order and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist and satisfy $\sup_{u,v\in b(t,\varepsilon)\cup b(-t,\varepsilon)} |c^{(3)}(u,v)| < \infty$ and $\sup_{u,v\in b(t,\varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u,w,v+w)|\mathrm{d}w < \infty$ for some $\varepsilon > 0$

be satisfied.

Then we have $\lim_{n\to\infty} \mathrm{Var}(\hat{\varrho}_n(t)) = 0$ in every point of continuity $t \in \mathbb{R}^d$ of ϱ.

We will prove Theorem 6.1.2 together with the following theorem, which presents an asymptotic representation of the covariance $\mathrm{Cov}(\hat{\varrho}_n(s), \hat{\varrho}_n(t))$ of the estimated second-order product density in two points $s, t \in \mathbb{R}^d$. In Heinrich [26], Theorem 5, the limit $b_n^d |W_n| \mathrm{Cov}(\hat{\varrho}_n(s), \hat{\varrho}_n(t))$ has already been determined, based on assumptions on the densities $p^{(2)}$, $p^{(3)}$ and $p^{(4)}$ of the moment measures of order two, three and four, not on assumptions on cumulant densities as we have here. Note that Heinrich [26], Theorem 5 misstates the limiting variance of the second-order product density estimator in zero. An extra factor 2 has to be added for correction.

Theorem 6.1.3 *Let $\Psi \sim P$ be a 4-stationary point process in $\mathbb{R}^d$ with intensity λ, and let $\|\gamma_{\text{red}}^{(k)}\| < \infty$, $k = 2, 3, 4$. Let the third-order and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist and satisfy*

$$\sup_{u \in b(s,\varepsilon) \cup b(-s,\varepsilon), v \in b(t,\varepsilon) \cup b(-t,\varepsilon)} |c^{(3)}(u, v)| < \infty$$

and

$$\sup_{u \in b(s,\varepsilon), v \in b(t,\varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u, w, v + w)| \mathrm{d}w < \infty$$

for some $\varepsilon > 0$.

Then we have

$$b_n^d |W_n| \operatorname{Cov}\big(\hat{\varrho}_n(s), \hat{\varrho}_n(t)\big) = \begin{cases} \lambda \varrho(s) \int_{\mathbb{R}^d} k^2(x) \mathrm{d}x + o(1), & s = \pm t \neq o, \\ 2\lambda \varrho(o) \int_{\mathbb{R}^d} k^2(x) \mathrm{d}x + o(1), & s = t = o, \\ 0 + O(b_n^d), & s \neq \pm t, \end{cases}$$

in every point of continuity $s \in \mathbb{R}^d$ of ϱ as $n \to \infty$. This result can be refined in the following two ways.

(i) If, in addition, the second-order product density ϱ has bounded and continuous first-order partial derivatives in $b^o(t, \varepsilon)$ for some $\varepsilon > 0$, then we have

$$b_n^d |W_n| \operatorname{Cov}\big(\hat{\varrho}_n(s), \hat{\varrho}_n(t)\big) = \begin{cases} \lambda \varrho(s) \int_{\mathbb{R}^d} k^2(x) \mathrm{d}x + O(b_n), & s = \pm t \neq o, \\ 2\lambda \varrho(o) \int_{\mathbb{R}^d} k^2(x) \mathrm{d}x + O(b_n), & s = t = o, \\ 0 + O(b_n^d), & s \neq \pm t, \end{cases}$$

as $n \to \infty$.

(ii) If, in addition, the second-order product density ϱ has bounded and continuous second-order partial derivatives in $b^o(t,\varepsilon)$ for some $\varepsilon > 0$ and the kernel function satisfies Condition 5.1.3 $\mathcal{K}^2(d,2)$, then we have

$$b_n^d |W_n| \operatorname{Cov}\big(\hat{\varrho}_n(s), \hat{\varrho}_n(t)\big) = \begin{cases} \lambda\varrho(s) \int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x + O(b_n^2) + O(b_n^d), & s = \pm t \neq o, \\ 2\lambda\varrho(o) \int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x + O(b_n^2) + O(b_n^d), & s = t = o, \\ 0 + O(b_n^d), & s \neq \pm t, \end{cases}$$

as $n \to \infty$.

Proof of Theorems 6.1.2 and 6.1.3: Concerning the proof of Theorem 6.1.2, we only consider assumption (i). Under assumption (ii) Theorem 6.1.2 is a special case of Theorem 6.1.3. By equation (4.2) we have

$$\begin{aligned} &\operatorname{Cov}\big(\hat{\varrho}_n(s), \hat{\varrho}_n(t)\big) \\ &= \frac{1}{b_n^{2d}|W_n|^2} \\ &\quad \times \Bigg[\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) k\left(\frac{y-x-s}{b_n}\right) k\left(\frac{y-x-t}{b_n}\right) \alpha^{(2)}(\mathrm{d}(x,y)) \\ &\qquad + \int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y) k\left(\frac{y-x-s}{b_n}\right) k\left(\frac{x-y-t}{b_n}\right) \alpha^{(2)}(\mathrm{d}(x,y)) \\ &\qquad + \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) k\left(\frac{y-x-s}{b_n}\right) k\left(\frac{z-x-t}{b_n}\right) \alpha^{(3)}(\mathrm{d}(x,y,z)) \\ &\qquad + \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y) k\left(\frac{y-x-s}{b_n}\right) k\left(\frac{z-y-t}{b_n}\right) \alpha^{(3)}(\mathrm{d}(x,y,z)) \\ &\qquad + \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(z) k\left(\frac{y-x-s}{b_n}\right) k\left(\frac{x-z-t}{b_n}\right) \alpha^{(3)}(\mathrm{d}(x,y,z)) \\ &\qquad + \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(z) k\left(\frac{y-x-s}{b_n}\right) k\left(\frac{y-z-t}{b_n}\right) \alpha^{(3)}(\mathrm{d}(x,y,z)) \\ &\qquad + \int\limits_{(\mathbb{R}^d)^4} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(z) k\left(\frac{y-x-s}{b_n}\right) k\left(\frac{v-z-t}{b_n}\right) [\gamma^{(4)}(\mathrm{d}(x,y,z,v)) \\ &\qquad\quad + \gamma^{(1)}(\mathrm{d}x)\gamma^{(3)}(\mathrm{d}(y,z,v)) + \gamma^{(1)}(\mathrm{d}y)\gamma^{(3)}(\mathrm{d}(x,z,v)) + \gamma^{(1)}(\mathrm{d}z)\gamma^{(3)}(\mathrm{d}(x,y,v)) \end{aligned}$$

$$+\gamma^{(1)}(\mathrm{d}v)\gamma^{(3)}(\mathrm{d}(x,y,z)) + \gamma^{(2)}(\mathrm{d}(x,z))\gamma^{(2)}(\mathrm{d}(y,v)) + \gamma^{(2)}(\mathrm{d}(x,v))\gamma^{(2)}(\mathrm{d}(y,z))$$
$$+\gamma^{(2)}(\mathrm{d}(x,z))\gamma^{(1)}(\mathrm{d}y)\gamma^{(1)}(\mathrm{d}v) + \gamma^{(2)}(\mathrm{d}(x,v))\gamma^{(1)}(\mathrm{d}y)\gamma^{(1)}(\mathrm{d}z)$$
$$\left.+\gamma^{(2)}(\mathrm{d}(y,z))\gamma^{(1)}(\mathrm{d}x)\gamma^{(1)}(\mathrm{d}v) + \gamma^{(2)}(\mathrm{d}(y,v))\gamma^{(1)}(\mathrm{d}x)\gamma^{(1)}(\mathrm{d}z)]\right].$$

When multiplied by $b_n^d|W_n|$, only the first two integrals do not converge to zero for $s \neq t$. For the first integral we have

$$\begin{aligned}
&\frac{1}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)k\left(\frac{y-x-s}{b_n}\right)k\left(\frac{y-x-t}{b_n}\right)\alpha^{(2)}(\mathrm{d}(x,y))\\
&= \frac{\lambda}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)k\left(\frac{y-s}{b_n}\right)k\left(\frac{y-t}{b_n}\right)\alpha^{(2)}_{\mathrm{red}}(\mathrm{d}y)\mathrm{d}x\\
&= \lambda\int\limits_{\mathbb{R}^d} k(y)k\left(y+\frac{s-t}{b_n}\right)\varrho(b_ny+s)\mathrm{d}y\\
&= \begin{cases}\lambda\varrho(s)\int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x + o(1), & s=t,\\ 0, & s\neq t,\end{cases}
\end{aligned}$$

by continuity of ϱ in s and the bounded support of the kernel function k as $n\to\infty$. Note that in the case $s\neq t$ the error term equals zero for sufficiently large n since k has bounded support. Using Taylor's formula we find, under the additional assumption (i) in Theorem 6.1.3,

$$\lambda\int\limits_{\mathbb{R}^d} k(y)k\left(y+\frac{s-t}{b_n}\right)\varrho(b_ny+s)\mathrm{d}y = \begin{cases}\lambda\varrho(s)\int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x + O(b_n), & s=t,\\ 0, & s\neq t,\end{cases}$$

as $n\to\infty$. Likewise the additional assumption (ii) in Theorem 6.1.3 yield

$$\lambda\int\limits_{\mathbb{R}^d} k(y)k\left(y+\frac{s-t}{b_n}\right)\varrho(b_ny+s)\mathrm{d}y = \begin{cases}\lambda\varrho(s)\int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x + O(b_n^2), & s=t,\\ 0, & s\neq t,\end{cases}$$

as $n\to\infty$. Again, in the case $s\neq t$ the error terms equal zero for sufficiently large n because k has bounded support.

Multiplying the second integral by $b_n^d|W_n|$ we get

$$\begin{aligned}
&\frac{1}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y)k\left(\frac{y-x-s}{b_n}\right)k\left(\frac{x-y-t}{b_n}\right)\alpha^{(2)}(\mathrm{d}(x,y))\\
&= \frac{\lambda}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y+x)k\left(\frac{y-s}{b_n}\right)k\left(\frac{y+t}{b_n}\right)\alpha_{\mathrm{red}}^{(2)}(\mathrm{d}y)\mathrm{d}x\\
&= \lambda\int\limits_{\mathbb{R}^d} \frac{|W_n\cap(W_n-b_ny-s)|}{|W_n|}k(y)k\left(y+\frac{s+t}{b_n}\right)\varrho(b_ny+s)\mathrm{d}y\\
&= \begin{cases}\lambda\varrho(s)\int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x+o(1), & s=-t,\\ 0, & s\neq -t,\end{cases}
\end{aligned}$$

in every point of continuity $s\in\mathbb{R}^d$ as $n\to\infty$. As before, in the case $s\neq -t$ the error term equals zero for sufficiently large n since k has bounded support. The rates of convergence for this integral under the additional assumptions (i) and (ii) in Theorem 6.1.3 are the same as those for the first integral.

We will now show that all the other integrals vanish under the assumptions of Theorem 6.1.2. Moreover we will prove them to be of order $O(b_n^d)$ when multiplied with $b_n^d|W_n|$ given the assumptions of Theorem 6.1.3. For the first integral with respect to the third-order factorial moment measure $\alpha^{(3)}$ we have

$$\begin{aligned}
&\frac{1}{b_n^{2d}|W_n|^2}\int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x)k\left(\frac{y-x-s}{b_n}\right)k\left(\frac{z-x-t}{b_n}\right)\alpha^{(3)}(\mathrm{d}(x,y,z))\\
&= \frac{1}{b_n^{2d}|W_n|^2}\int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x)k\left(\frac{y-x-s}{b_n}\right)k\left(\frac{z-x-t}{b_n}\right)[\gamma^{(3)}(\mathrm{d}(x,y,z))\\
&\qquad +\lambda\mathrm{d}x\gamma^{(2)}(\mathrm{d}(y,z))+\lambda\mathrm{d}y\gamma^{(2)}(\mathrm{d}(x,z))+\lambda\mathrm{d}z\gamma^{(2)}(\mathrm{d}(x,y))+\lambda^3\mathrm{d}x\mathrm{d}y\mathrm{d}z].
\end{aligned}$$

In order to prove the integral with respect to $\gamma^{(3)}$ to converge to zero we use either $\|\gamma_{\mathrm{red}}^{(3)}\|<\infty$ and assumption (i) in Theorem 6.1.2 or the existence of the third-order cumulant density. Multiplying with $b_n^d|W_n|$ we find

$$\begin{aligned}
&\frac{1}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x)k\left(\frac{y-s}{b_n}\right)k\left(\frac{z-t}{b_n}\right)\gamma_{\mathrm{red}}^{(3)}(\mathrm{d}(y,z))\mathrm{d}x\\
&= \frac{1}{b_n^d}\lambda\int\limits_{(\mathbb{R}^d)^2} k\left(\frac{y-s}{b_n}\right)k\left(\frac{z-t}{b_n}\right)\gamma_{\mathrm{red}}^{(3)}(\mathrm{d}(y,z))
\end{aligned}$$

$$= b_n^d \lambda \int\limits_{(\mathbb{R}^d)^2} k(y)k(z)c^{(3)}(b_n y + s, b_n z + t)\mathrm{d}y\mathrm{d}z$$

which is of order $O(b_n^d)$ by our assumption on the third-order cumulant density $c^{(3)}$. Analogously, one may show the other integrals with respect to the third-order factorial moment measure $\alpha^{(3)}$ to vanish or to be of order $O(b_n^d)$.

The integrals with respect to the factorial cumulant measures can be treated analogously. In order to prove the integral with respect to $\gamma^{(4)}$ to converge to zero we use either $\|\gamma_{\mathrm{red}}^{(4)}\| < \infty$ and assumption (i) in Theorem 6.1.2 or the existence of the fourth-order cumulant density. Multiplying with $b_n^d|W_n|$ we get

$$\frac{1}{b_n^d|W_n|} \int\limits_{(\mathbb{R}^d)^4} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(z)k\left(\frac{y-x-s}{b_n}\right)k\left(\frac{v-z-t}{b_n}\right)\gamma^{(4)}(\mathrm{d}(x,y,z,v))$$

$$= \frac{\lambda}{b_n^d|W_n|} \int\limits_{(\mathbb{R}^d)^4} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(z+x)k\left(\frac{y-s}{b_n}\right)k\left(\frac{v-z-t}{b_n}\right)\gamma_{\mathrm{red}}^{(4)}(\mathrm{d}(y,z,v))\mathrm{d}x$$

$$= \frac{\lambda}{b_n^d} \int\limits_{(\mathbb{R}^d)^4} \frac{|W_n \cap (W_n - z)|}{|W_n|} k\left(\frac{y-s}{b_n}\right)k\left(\frac{v-z-t}{b_n}\right)\gamma_{\mathrm{red}}^{(4)}(\mathrm{d}(y,z,v))$$

$$= \frac{\lambda}{b_n^d} \int\limits_{(\mathbb{R}^d)^4} \frac{|W_n \cap (W_n - z)|}{|W_n|} k\left(\frac{y-s}{b_n}\right)k\left(\frac{v-z-t}{b_n}\right)c^{(4)}(y,z,v)\mathrm{d}y\mathrm{d}z\mathrm{d}v$$

$$= b_n^d \lambda \int\limits_{(\mathbb{R}^d)^4} \frac{|W_n \cap (W_n - z)|}{|W_n|} k(y)k(v)c^{(4)}(b_n y + s, z, b_n v + z + t)\mathrm{d}y\mathrm{d}z\mathrm{d}v$$

which is of asymptotic order $O(b_n^d)$ due to the assumption on the fourth-order cumulant density $c^{(4)}$. Similar arguments show that the other integrals converge to zero or are of asymptotic order $O(b_n^d)$, respectively. ∎

The above results for the estimated second-order product density can be shown analogously for the estimated pair correlation function. We will briefly sketch the first proof, and point out the main differences. We start with the asymptotic representation of the mean.

Theorem 6.1.4 *Let $\Psi \sim P$ be a 2-stationary point process in $\mathbb{R}^d$ with intensity λ. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\,\tilde{g}_n(r) = \lambda^2 g(r)$$

in every point of continuity $r \in (0,\infty)$ of g.

Let the kernel function satisfy Condition 5.1.2 $\mathcal{K}(1,s)$ and let the derivative of order s of g be bounded and continuous in $(r-\varepsilon, r+\varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$.

Then we have

$$\mathbb{E}\,\tilde{g}_n(r) = \lambda^2 g(r) + O(b_n^s)$$

as $n \to \infty$.

Proof: The mean of the estimated pair correlation function satisfies

$$\begin{aligned}
\mathbb{E}\,\tilde{g}_n(r) &= \mathbb{E}\sum_{x,y\in\Psi}^{*} \frac{\mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y)k\left(\frac{\|y-x\|-r}{b_n}\right)}{b_n\,|(W_n-x)\cap(W_n-y)|\,d\,\omega_d\|y-x\|^{d-1}} \\
&= \int_{(\mathbb{R}^d)^2} \frac{\mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y)k\left(\frac{\|y-x\|-r}{b_n}\right)}{b_n\,|(W_n-x)\cap(W_n-y)|\,d\,\omega_d\|y-x\|^{d-1}}\;\alpha^{(2)}(\mathrm{d}(x,y)) \\
&= \lambda^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y)k\left(\frac{\|y-x\|-r}{b_n}\right)}{b_n\,|(W_n-x)\cap(W_n-y)|\,d\,\omega_d\|y-x\|^{d-1}}\;g(\|y-x\|)\mathrm{d}x\mathrm{d}y \\
&= \lambda^2 \int_{\mathbb{R}^d} \frac{k\left(\frac{\|z\|-r}{b_n}\right)}{b_n\,d\,\omega_d\|z\|^{d-1}}\;g(\|z\|)\mathrm{d}z \\
&= \lambda^2 \int_{-r/b_n}^{\infty} k(s)g(r+b_n z)\mathrm{d}z \\
&\xrightarrow[n\to\infty]{} \lambda^2 g(r)
\end{aligned}$$

in every point of continuity $r \in (0,\infty)$ of g, see Fiksel [17]. Now we turn to the second assertion. First we use Taylor's formula to obtain

$$g(r+b_n z) = g(r) + \sum_{i=1}^{s-1}\frac{1}{i!}\left(\frac{\mathrm{d}}{\mathrm{d}r}b_n z\right)^i g(r) + R_s(z,r),$$

where $R_s(z,r) = b_n^s \frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}r^s}g(r+\theta b_n z)$ for some $\theta \in (0,1)$. Together with Condition 5.1.2 $\mathcal{K}(1,s)$ and the boundedness of the derivative of order s of g in $(r-\varepsilon, r+\varepsilon)$ for some $\varepsilon > 0$ this completes the proof. ■

The following two theorems state the assumptions under which the variance of the estimated pair correlation function vanishes and give asymptotic representations of that variance. Since these results are shown along the same lines as Theorem 6.1.2 and Theorem 6.1.3 the proofs are omitted.

Theorem 6.1.5 *Let $\Psi \sim P$ be a 4-stationary point process in $\mathbb{R}^d$ with intensity λ with finite total variations $\|\gamma_{\text{red}}^{(k)}\| < \infty$, $k = 2, 3, 4$. Let one of the assumptions*

(i) the bandwidth satisfies $b_n^2 |W_n| \xrightarrow[n\to\infty]{} \infty$

(ii) the third-order and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist and satisfy $\sup_{u,v \in b(\mathbf{r},\varepsilon)\cup b(-\mathbf{r},\varepsilon)} |c^{(3)}(u,v)| < \infty$ and $\sup_{u,v\in b(\mathbf{r},\varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u,w,v+w)|\mathrm{d}w < \infty$ for some $\varepsilon > 0$ and all $\mathbf{r} \in \mathbb{R}^d$ with $\|\mathbf{r}\| = r$

be satisfied.

Then we have $\lim_{n\to\infty} \operatorname{Var}\big(\hat{g}_n(r)\big) = 0$ in every point of continuity $r \in (0,\infty)$ of g. ■

Theorem 6.1.6 *Let $\Psi \sim P$ be a 4-stationary point process in $\mathbb{R}^d$ with intensity λ with finite total variations $\|\gamma_{\text{red}}^{(k)}\| < \infty$, $k = 2, 3, 4$. Let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist and satisfy*

$$\sup_{u\in b(\mathbf{s},\varepsilon)\cup b(-\mathbf{s},\varepsilon),\, v\in b(\mathbf{t},\varepsilon)\cup b(-\mathbf{t},\varepsilon)} |c^{(3)}(u,v)| < \infty$$

and

$$\sup_{u\in b(\mathbf{s},\varepsilon),\, v\in b(\mathbf{t},\varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u,w,v+w)|\mathrm{d}w < \infty$$

for some $\varepsilon > 0$ and all $\mathbf{s},\mathbf{t} \in \mathbb{R}^d$ with $\|\mathbf{s}\| = s$ and $\|\mathbf{t}\| = t$.

Then we have

$$b_n|W_n| \operatorname{Cov}\big(\hat{g}_n(s), \hat{g}_n(t)\big) = \begin{cases} 2\lambda^2 \dfrac{g(s)}{d\omega_d s^{d-1}} \displaystyle\int_{\mathbb{R}} k^2(x)\mathrm{d}x + o(1), & s = t, \\ 0 + O(b_n), & s \neq t, \end{cases}$$

in every point of continuity $s \in (0,\infty)$ of g as $n \to \infty$. This result can be refined in the following two ways.

(i) If, in addition, the first-order derivative of the pair correlation function g is bounded and continuous in $(s-\varepsilon, s+\varepsilon)$ for some $\varepsilon > 0$, then we have

$$b_n|W_n| \operatorname{Cov}\big(\hat{g}_n(s), \hat{g}_n(t)\big) = \begin{cases} 2\lambda^2 \dfrac{g(s)}{d\omega_d s^{d-1}} \displaystyle\int_{\mathbb{R}} k^2(x)\mathrm{d}x + O(b_n), & s = t, \\ 0 + O(b_n), & s \neq t, \end{cases}$$

as $n \to \infty$.

(ii) If, in addition, the second-order derivative of the pair correlation function g is bounded and continuous in $(s-\varepsilon, s+\varepsilon)$ for some $\varepsilon > 0$ and the kernel function satisfies Condition 5.1.3 $\mathcal{K}^2(1,2)$, then we have

$$b_n|W_n| \operatorname{Cov}\big(\hat{g}_n(s), \hat{g}_n(t)\big) = \begin{cases} 2\lambda^2 \dfrac{g(s)}{d\omega_d s^{d-1}} \displaystyle\int_{\mathbb{R}} k^2(x)\mathrm{d}x + O(b_n), & s = t, \\ 0 + O(b_n) & s \neq t, \end{cases}$$

as $n \to \infty$. ■

6.1.2 Central limit theorems

For Poisson cluster processes Heinrich [26] proves a central limit theorem for the sequence

$$\Delta_n(t) := (b_n^d|W_n|)^{1/2}\big(\hat{\varrho}_n(t) - \mathbb{E}\hat{\varrho}_n(t)\big)$$

by using methods for m-dependent point fields. However, these methods cannot be applied in the setting of Brillinger-mixing point processes. In the latter case we will prove a central limit theorem by showing the k-th cumulants of the above-mentioned sequence to converge to zero for $k \geq 3$. We will also investigate the corresponding sequence for the estimators of the pair correlation function.

Jolivet [40] determines the order of the kth cumulant of the ℓth-order product density $\hat{\varrho}_n^{(\ell)}$, $\ell \geq 2$, by using methods by Leonov and Shiryaev [45] and Jolivet [39]. However, Jolivet [40] only investigates the terms of highest order and does not take into account that some assumptions on the cumulant densities have to be made in order to prove that the lower-order terms converge to zero.

In the following $\xrightarrow[n\to\infty]{\mathrm{d}}$ denotes weak convergence, while χ_q^2 and $\mathrm{N}(\mu, \Sigma_q)$ denote the χ^2-distribution with q degrees of freedom and the q-variate normal distribution with mean vector $\mu \in \mathbb{R}^q$ and positive-semidefinite $(q \times q)$-covariance matrix Σ_q, respectively.

Theorem 6.1.7 *Let $\Psi \sim P$ be a Brillinger-mixing point process in $\mathbb{R}^d$ with intensity λ and let all cumulant densities of order 2 and higher exist. Let the q-tuple $(u_1, \dots, u_q) \in (\mathbb{R}^d)^q$ be chosen such that $u_i \neq \pm u_j$ for $i \neq j$, and let u_i be a point of continuity of ϱ for every $i = 1, \dots, q$.*

(i) Let the cumulant densities $c^{(\ell)}$, $\ell \geq 3$, satisfy

$$\sup_{u \in b(u_i, \varepsilon)} \int_{(\mathbb{R}^d)^{\ell-2}} |c^{(\ell)}(u, x_2, \dots, x_{\ell-1})| \mathrm{d}(x_2, \dots, x_{\ell-1}) < \infty$$

for some $\varepsilon > 0$ and $i = 1, \dots, q$, and

(ii) let the third-order and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ satisfy

$$\sup_{u \in b(u_i, \varepsilon) \cup b(-u_i, \varepsilon), v \in b(u_j, \varepsilon) \cup b(-u_j, \varepsilon)} |c^{(3)}(u, v)| < \infty$$

and

$$\sup_{u \in b(u_i, \varepsilon), v \in b(u_j, \varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u, w, v + w)| \mathrm{d}w < \infty$$

for some $\varepsilon > 0$ and $i, j = 1, \dots, q$.

Then we have $(\Delta_n(u_i))_{i=1}^q \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}(0, \Sigma_q)$, where the covariance matrix $\Sigma_q = (\sigma_{ij})_{i,j=1}^q$ is given by

$$\sigma_{ii} = \begin{cases} \lambda \varrho(u_i) \int_{\mathbb{R}^d} k^2(x) \mathrm{d}x, & u_i \neq 0, \\ 2\lambda \varrho(u_i) \int_{\mathbb{R}^d} k^2(x) \mathrm{d}x, & u_i = 0, \end{cases}$$

for $i = 1, \dots, q$, and $\sigma_{ij} = 0$ for $i \neq j$. Furthermore we have

$$\sum_{i=1}^q \frac{(\Delta_n(u_i))^2}{\sigma_{ii}} \xrightarrow[n\to\infty]{\mathrm{d}} \chi_q^2.$$

Proof: The asymptotic covariance has already been determined in Theorem 6.1.3. In order to show asymptotic normality of $(\Delta_n(u_i))_{i=1}^q$ we use the method of Cramér-Wold and consider the linear combination $a_1\Delta_n(u_1)+\ldots+a_q\Delta_n(u_q)$ for an arbitrary q-tuple $(a_1,\ldots,a_q)' \in \mathbb{R}^q$. Asymptotic normality of this linear combination will be established by showing that its cumulants of order k converge to zero for all $k \geq 3$.

Applying Lemma 4.2.1 and using the notation given there the kth cumulant of $a_1\Delta_n(u_1)+\ldots+a_q\Delta_n(u_q)$, $k \geq 2$, satisfies

$$\begin{aligned}
&\Gamma_k\left(a_1\Delta_n(u_1)+\ldots+a_q\Delta_n(u_q)\right)\\
&= \Gamma_k\left(\sum_{i=1}^q (b_n^d|W_n|)^{-1/2} a_i \Psi^{(2)}(f_i)\right)\\
&= (b_n^d|W_n|)^{-k/2} \sum_{\substack{k_1+\ldots+k_q=k\\ k_1,\ldots,k_q\geq 0}} \frac{k!}{k_1!\cdot\ldots\cdot k_q!}\, a_1^{k_1}\cdot\ldots\cdot a_q^{k_q}\, \mu^*_{k_1,\ldots,k_q}
\end{aligned}$$

with $f_i : (\mathbb{R}^d)^2 \to \mathbb{R}$, $(x_1,x_2) \mapsto \mathbb{1}_{W_n}(x_1) k\left(\frac{x_2-x_1-u_i}{b_n}\right)$, $i=1,\ldots,q$.

Now we have to determine the order of $\mu^*_{k_1,\ldots,k_q}$ for $k_1,\ldots,k_q \geq 0$ with $k_1+\ldots+k_q = k$. Since $\mu^*_{k_1,\ldots,k_q}$ consists only of indecomposable integrals it can be seen by disintegration and substitution that the highest order of integrals with respect to at least two factorial cumulant measures is $O(b_n^d|W_n|)$ due to $\|\gamma_{\mathrm{red}}^{(k)}\| < \infty$ for $k \geq 2$ and the boundedness assumptions on the kernel function. Together with the factor $(b_n^d|W_n|)^{-k/2}$ this yields the asymptotic order $O((b_n^d|W_n|)^{1-k/2})$ of these terms. For integrals taken with respect to only one factorial cumulant measure $\gamma^{(\ell)}$, disintegration and the finiteness of the total variations yield the asymptotic order $O(|W_n|)$. Since this result is insufficient for our purposes we use the additional assumption that the cumulant densities exist. Due to this assumption we may substitute $z = \frac{x_j-x_\ell-u_i}{b_n}$ in one instance of the kernel function k. In the case $\ell \geq 3$ assumption (i) on the cumulant densities yields the asymptotic order $O(b_n^d|W_n|)$ of this integral by Lebesgue's dominated convergence theorem. For an integral taken with respect to $\gamma^{(2)}$ the continuity of the second-order product density in u_i, $i=1,\ldots,q$, (which implies the continuity of $c^{(2)}$ in these points due to $\varrho(.) = c^{(2)}(.)+\lambda$) and Lebesgue's dominated convergence theorem again yield the asymptotic order $O(b_n^d|W_n|)$.

Altogether we have

$$\Gamma_k\left(a_1\Delta_n(u_1)+\ldots+a_q\Delta_n(u_q)\right) = O((b_n^d|W_n|)^{1-k/2})$$

as $n\to\infty$. Hence the cumulants of order three and higher converge to zero as $n\to\infty$ which implies the claimed normal convergence. The weak convergence of

$$\sum_{i=1}^{q} \frac{(\Delta_n(u_i))^2}{\sigma_{ii}}$$

to a χ^2-distributed random variable with q degrees of freedom follows immediately by the Continuous Mapping Theorem (see Pollard [49]). ■

The previous theorem assumed the existence of and boundedness conditions on all cumulant densities. In the following theorem these assumptions are only made on the cumulant densities up to order ℓ, supplemented with the condition $(b_n^d)^\ell |W_n|^{\ell-2} \xrightarrow[n\to\infty]{} \infty$ on the bandwidth, for some $\ell \geq 3$.

Theorem 6.1.8 *Let $\Psi \sim P$ be a Brillinger-mixing point process in $\mathbb{R}^d$ with intensity λ. Let the q-tuple $(u_1, \ldots, u_q) \in (\mathbb{R}^d)^q$ be chosen such that $u_i \neq \pm u_j$ for $i \neq j$, and let u_i be a point of continuity of ϱ for every $i = 1, \ldots, q$.*

(i) Let there be an $\ell \geq 3$ with $(b_n^d)^\ell |W_n|^{\ell-2} \xrightarrow[n\to\infty]{} \infty$ and let the cumulant densities $c^{(k)}$, $k = 3, \ldots, 2(\ell-1)$, exist and satisfy

$$\sup_{u \in b(u_i,\varepsilon)} \int\limits_{(\mathbb{R}^d)^{k-2}} |c^{(k)}(u, x_2, \ldots, x_{k-1})| \mathrm{d}(x_2, \ldots, x_{k-1}) < \infty$$

for some $\varepsilon > 0$, $i = 1, \ldots, q$, and $k = 3, \ldots, 2(\ell-1)$, and

(ii) let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ satisfy

$$\sup_{u \in b(u_i,\varepsilon) \cup b(-u_i,\varepsilon), v \in b(u_j,\varepsilon) \cup b(-u_j,\varepsilon)} |c^{(3)}(u, v)| < \infty$$

and

$$\sup_{u \in b(u_i,\varepsilon), v \in b(u_j,\varepsilon)} \int\limits_{\mathbb{R}^d} |c^{(4)}(u, w, v+w)| \mathrm{d}w < \infty$$

for some $\varepsilon > 0$ and $i, j = 1, \ldots, q$.

Then we have $(\Delta_n(u_i))_{i=1}^q \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}(0, \Sigma_q)$, where the covariance matrix $\Sigma_q = (\sigma_{ij})_{i,j=1}^q$ is given by

$$\sigma_{ii} = \begin{cases} \lambda \varrho(u_i) \int\limits_{\mathbb{R}^d} k^2(x) \mathrm{d}x, & u_i \neq 0, \\ 2\lambda \varrho(u_i) \int\limits_{\mathbb{R}^d} k^2(x) \mathrm{d}x, & u_i = 0, \end{cases}$$

for $i = 1, \ldots, q$, and $\sigma_{ij} = 0$ for $i \neq j$. Furthermore we have

$$\sum_{i=1}^{q} \frac{(\Delta_n(u_i))^2}{\sigma_{ii}} \xrightarrow[n\to\infty]{\mathrm{d}} \chi_q^2.$$

Proof: The result can be proved analogously to the previous theorem. The only difference to Theorem 6.1.7 is that we only use assumptions on the cumulant densities up to order $2(\ell - 1)$. This leads to the asymptotic order $O((b_n^d|W_n|)^{1-k/2})$ of the kth cumulant for $k = 3, \ldots, \ell - 1$. For the cumulants of order $k \geq \ell$ we obtain the asymptotic order $O((b_n^d)^{-k/2}|W_n|^{1-k/2})$ by disintegration and $\|\gamma_{\mathrm{red}}^{(j)}\| < \infty$ for all $j \leq 2$. Here the assumptions $(b_n^d)^\ell |W_n|^{\ell-2} \xrightarrow[n\to\infty]{} \infty$ and $b_n^d|W_n| \xrightarrow[n\to\infty]{} \infty$ imply the cumulants of order three and higher to converge to zero. This completes the proof. ■

Corollary 6.1.9 *Let $\Psi \sim P$ be a Brillinger-mixing point process in $\mathbb{R}^d$ with intensity λ and let all cumulant densities of order 2 and higher exist. Let the q-tuple $(u_1, \ldots, u_q) \in (\mathbb{R}^d)^q$ be chosen such that $u_i \neq \pm u_j$ for $i \neq j$, and let ϱ have bounded and continuous partial derivatives of order s in $b^o(u_i, \varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$ and every $i = 1, \ldots, q$. Let the kernel function satisfy Condition 5.1.2 $\mathcal{K}(d, s)$. Further, let the bandwidth satisfy $b_n^{2s+d}|W_n| \xrightarrow[n\to\infty]{} 0$.*

(i) Let the cumulant densities $c^{(\ell)}$, $\ell \geq 3$, satisfy

$$\sup_{u \in b(u_i,\varepsilon)} \int_{(\mathbb{R}^d)^{\ell-2}} |c^{(\ell)}(u, x_2, \ldots, x_{\ell-1})| \mathrm{d}(x_2, \ldots, x_{\ell-1}) < \infty$$

for some $\varepsilon > 0$ and $i = 1, \ldots, q$, and

(ii) let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ satisfy

$$\sup_{u \in b(u_i,\varepsilon) \cup b(-u_i,\varepsilon), v \in b(u_j,\varepsilon) \cup b(-u_j,\varepsilon)} |c^{(3)}(u, v)| < \infty$$

and

$$\sup_{u \in b(u_i,\varepsilon), v \in b(u_j,\varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u, w, v + w)| \mathrm{d}w < \infty$$

for some $\varepsilon > 0$ and $i, j = 1, \ldots, q$.

Then we have

$$\left(\tilde{\Delta}_n(u_i)\right)_{i=1}^q := \left((b_n^d|W_n|)^{1/2}\left(\hat{\varrho}_n(u_i) - \lambda\varrho(u_i)\right)\right)_{i=1}^q \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}(0, \Sigma_q),$$

where the covariance matrix $\Sigma_q = (\sigma_{ij})_{i,j=1}^q$ *is given by*

$$\sigma_{ii} = \begin{cases} \lambda\varrho(u_i) \int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x, & u_i \neq 0, \\ 2\lambda\varrho(u_i) \int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x, & u_i = 0, \end{cases}$$

for $i = 1, \ldots, q$, *and* $\sigma_{ij} = 0$ *for* $i \neq j$. *Furthermore we have*

$$\sum_{i=1}^{q} \frac{\left(\tilde{\Delta}_n(u_i)\right)^2}{\sigma_{ii}} \xrightarrow[n\to\infty]{\mathrm{d}} \chi_q^2.$$

Proof: The result is an immediate consequence Theorem 6.1.7, the second part of Theorem 6.1.1, and $b_n^{2s+d}|W_n| \xrightarrow[n\to\infty]{} 0$. ■

In the following corollary we apply a variance-stabilizing transformation to the sequence $\tilde{\Delta}_n$.

Corollary 6.1.10 *Let* $\Psi \sim P$ *be a Brillinger-mixing point process in* $\mathbb{R}^d$ *with intensity* λ. *Let the assumptions of Corollary 6.1.9 be satisfied. Then we have*

$$\left(\tilde{\Delta}_n^*(u_i)\right)_{i=1}^q := \left((b_n^d|W_n|)^{1/2}\left(\sqrt{\hat{\varrho}_n(u_i)} - \sqrt{\lambda\varrho(u_i)}\right)\right)_{i=1}^q \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}(0, \Sigma_q),$$

where the covariance matrix $\Sigma_q = (\sigma_{ij})_{i,j=1}^q$ *is given by*

$$\sigma_{ii} = \begin{cases} \int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x, & u_i \neq 0, \\ 2\int\limits_{\mathbb{R}^d} k^2(x)\mathrm{d}x, & u_i = 0, \end{cases}$$

for $i = 1, \ldots, q$, *and* $\sigma_{ij} = 0$ *for* $i \neq j$. *Furthermore we have*

$$\sum_{i=1}^{q} \frac{\left(\tilde{\Delta}_n^*(u_i)\right)^2}{\sigma_{ii}} \xrightarrow[n\to\infty]{\mathrm{d}} \chi_q^2.$$

Proof: The claim is established based on the weak consistency of the estimated product density, on Corollary 6.1.9, and on variance-stabilization by a square-root transformation (see Witting and Müller-Funk [66], page 107). ■

The above central limit theorems can be proved analogously for the sequence

$$\Delta_n^{(g)}(r) := (b_n|W_n|)^{1/2}\big(\hat{g}_n(r) - \mathbb{E}\hat{g}_n(r)\big),$$

and for the corresponding sequences based on the other estimators for the pair correlation function suggested in Section 5.3.

Theorem 6.1.11 *Let $\Psi \sim P$ be a Brillinger-mixing point process in $\mathbb{R}^d$ with intensity λ and let all cumulant densities of order 2 and higher exist. Let the q-tuple $(u_1, \ldots, u_q) \in (0,\infty)^q$ be chosen such that $u_i \neq u_j$ for $i \neq j$, and let u_i be a point of continuity of g for every $i = 1, \ldots, q$.*

(i) Let the cumulant densities $c^{(\ell)}$, $\ell \geq 4$, satisfy

$$\sup_{u \in b(\mathbf{u_i},\varepsilon)} \int\limits_{(\mathbb{R}^d)^{\ell-2}} |c^{(\ell)}(u, x_2, \ldots, x_{\ell-1})| \mathrm{d}(x_2, \ldots, x_{\ell-1}) < \infty$$

for some $\varepsilon > 0$ and all $\mathbf{u_i} \in \mathbb{R}^d$ with $\|\mathbf{u_i}\| = u_i$, $i = 1, \ldots, q$, and

(ii) let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ satisfy

$$\sup_{u \in b(\mathbf{u_i},\varepsilon) \cup b(-\mathbf{u_i},\varepsilon),\, v \in b(\mathbf{u_j},\varepsilon) \cup b(-\mathbf{u_j},\varepsilon)} |c^{(3)}(u, v)| < \infty$$

and

$$\sup_{u \in b(\mathbf{u_i},\varepsilon),\, v \in b(\mathbf{u_j},\varepsilon)} \int\limits_{\mathbb{R}^d} |c^{(4)}(u, w, v + w)| \mathrm{d}w < \infty$$

for some $\varepsilon > 0$ and all $\mathbf{u_i}, \mathbf{u_j} \in \mathbb{R}^d$ with $\|\mathbf{u_i}\| = u_i$ and $\|\mathbf{u_j}\| = u_j$, $i, j = 1, \ldots, q$.

Then we have $\left(\Delta_n^{(g)}(u_i)\right)_{i=1}^q \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}(0, \Sigma_q)$, where the covariance matrix $\Sigma_q = (\sigma_{ij})_{i,j=1}^q$ is given by

$$\sigma_{ii} = 2\lambda^2 \frac{g(u_i)}{d\omega_d u_i^{d-1}} \int\limits_{\mathbb{R}} k^2(x) \mathrm{d}x$$

for $i = 1, \ldots, q$, and $\sigma_{ij} = 0$ for $i \neq j$. Furthermore we have

$$\sum_{i=1}^q \frac{\left(\Delta_n^{(g)}(u_i)\right)^2}{\sigma_{ii}} \xrightarrow[n\to\infty]{\mathrm{d}} \chi_q^2.$$

■

Theorem 6.1.12 *Let $\Psi \sim P$ be a Brillinger-mixing point process in $\mathbb{R}^d$ with intensity λ. Let the q-tuple $(u_1, \dots, u_q) \in (0,\infty)^q$ be chosen such that $u_i \neq u_j$ for $i \neq j$, and let u_i be a point of continuity of g for every $i = 1, \dots, q$.*

(i) Let there be an $\ell \geq 3$ with $b_n^\ell |W_n|^{\ell-2} \xrightarrow[n\to\infty]{} \infty$ and let the cumulant densities $c^{(k)}$, $k = 3, \dots, 2(\ell-1)$, exist and satisfy

$$\sup_{u \in b(\mathbf{u_i},\varepsilon)} \int\limits_{(\mathbb{R}^d)^{k-2}} |c^{(k)}(u, x_2, \dots, x_{k-1})| \mathrm{d}(x_2, \dots, x_{k-1}) < \infty$$

for some $\varepsilon > 0$ and all $\mathbf{u_i} \in \mathbb{R}^d$ with $\|\mathbf{u_i}\| = u_i$, $i = 1, \dots, q$, and $k = 3, \dots, 2(\ell-1)$, and

(ii) let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ satisfy

$$\sup_{u \in b(\mathbf{u_i},\varepsilon)\cup b(-\mathbf{u_i},\varepsilon), v \in b(\mathbf{u_j},\varepsilon)\cup b(-\mathbf{u_j},\varepsilon)} |c^{(3)}(u,v)| < \infty$$

and

$$\sup_{u \in b(\mathbf{u_i},\varepsilon), v \in b(\mathbf{u_j},\varepsilon)} \int\limits_{\mathbb{R}^d} |c^{(4)}(u, w, v+w)| \mathrm{d}w < \infty$$

for some $\varepsilon > 0$ and all $\mathbf{u_i}, \mathbf{u_j} \in \mathbb{R}^d$ with $\|\mathbf{u_i}\| = u_i$ and $\|\mathbf{u_j}\| = u_j$, $i, j = 1, \dots, q$.

Then we have $\left(\Delta_n^{(g)}(u_i)\right)_{i=1}^q \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}(0, \Sigma_q)$, where the covariance matrix $\Sigma_q = (\sigma_{ij})_{i,j=1}^q$ is given by

$$\sigma_{ii} = 2\lambda^2 \frac{g(u_i)}{d\omega_d u_i^{d-1}} \int\limits_{\mathbb{R}} k^2(x) \mathrm{d}x$$

for $i = 1, \dots, q$, and $\sigma_{ij} = 0$ for $i \neq j$. Furthermore we have

$$\sum_{i=1}^q \frac{\left(\Delta_n^{(g)}(u_i)\right)^2}{\sigma_{ii}} \xrightarrow[n\to\infty]{\mathrm{d}} \chi_q^2.$$

■

Remark 6.1.13 Results analogous to Corollaries 6.1.9 and 6.1.10 also hold for the sequences

$$\left((b_n|W_n|)^{1/2}\left(\hat{g}(u_i) - \lambda^2 g(u_i)\right)\right)_{i=1}^q$$

and

$$\left((b_n|W_n|)^{1/2}\left(\sqrt{\hat{g}(u_i)} - \sqrt{\lambda^2 g(u_i)}\right)\right)_{i=1}^q.$$

Again this can be shown by the asymptotic representation of the mean, the weak consistency of the estimated pair correlation function, and by variance-stabilization by a square-root transformation. □

6.2 Central limit theorems for empirical product densities of higher order

This section generalizes the results on the second-order product density in the previous section to product densities $\varrho^{(\ell)}$ of order $\ell \geq 2$. Throughout the whole section we assume the product density $\varrho^{(\ell)}$ to exist. Brillinger [5] proved normal convergence of product densities for point processes in $\mathbb{R}^1$. As already mentioned in Section 6.1.2, the normal convergence has already been stated in Jolivet [40] without mentioning the necessary conditions on the cumulant densities.

6.2.1 Asymptotic representation of the mean and the variance of the estimators

First we derive asymptotic representations for the mean and the variance of the estimators for product densities of order two and higher in the setting of Brillinger-mixing point processes.

Theorem 6.2.1 *Let $\Psi \sim P$ be an ℓ-stationary point process in $\mathbb{R}^d$ with intensity λ. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\hat{\varrho}_n^{(\ell)}(t) = \lambda\varrho^{(\ell)}(t)$$

in every point of continuity $t \in (\mathbb{R}^d)^{\ell-1}$ of $\varrho^{(\ell)}$.

Let the kernel function satisfy Condition 5.1.2 $\mathcal{K}((\ell-1)d, s)$ and let $\varrho^{(\ell)}$ have bounded and continuous partial derivatives of order s in $b^o_{\ell-1}(t,\varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$, where $b^o_{\ell-1}(x,r)$ denotes the open ball in $(\mathbb{R}^d)^{\ell-1}$ with radius $r > 0$ and midpoint $x \in (\mathbb{R}^d)^{\ell-1}$. Then we have

$$\mathbb{E}\hat{\varrho}_n^{(\ell)}(t) = \lambda\varrho^{(\ell)}(t) + O(b_n^s)$$

as $n \to \infty$.

Proof: Since Ψ is ℓ-stationary we have

$$\begin{aligned}
&\mathbb{E}\hat{\varrho}_n^{(\ell)}(t) \\
&= \frac{1}{(b_n^d)^{\ell-1}|W_n|} \int\limits_{(\mathbb{R}^d)^\ell} \mathbb{1}_{W_n}(x_1) k\left(\frac{x_2 - x_1 - t_1}{b_n}, \dots, \frac{x_\ell - x_1 - t_{\ell-1}}{b_n}\right) \alpha^{(\ell)}(\mathrm{d}(x_1, \dots, x_\ell)) \\
&= \lambda \int\limits_{(\mathbb{R}^d)^{\ell-1}} k(z) \varrho^{(\ell)}(b_n z + t) \mathrm{d}z
\end{aligned}$$

for $t = (t_1, \dots, t_{\ell-1})' = (t_{11}, t_{12}, \dots, t_{(\ell-1)d})' \in (\mathbb{R}^d)^{\ell-1}$, see Jolivet [40]. The boundedness conditions on the kernel imply the asymptotic unbiasedness in every point of continuity of $\varrho^{(\ell)}$ by Lebesgue's dominated convergence theorem.

If $\varrho^{(\ell)}$ has continuous partial derivatives of order s in $b^o(t, \varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$, then we get

$$\begin{aligned}
&\varrho^{(\ell)}(t + b_n z) = \\
&\qquad \varrho^{(\ell)}(t) + \sum_{i=1}^{s-1} \frac{1}{i!} \left(\frac{\partial}{\partial t_{11}} b_n z_{11} + \ldots + \frac{\partial}{\partial t_{(\ell-1)d}} b_n z_{(\ell-1)d}\right)^i \varrho(t_{11}, \dots, t_{(\ell-1)d}) + R_s(z, t)
\end{aligned}$$

by Taylor's expansion, where

$$R_s(z, t) = \frac{1}{s!} \left(\frac{\partial}{\partial t_{11}} b_n z_1 + \ldots + \frac{\partial}{\partial t_{(\ell-1)d}} b_n z_{(\ell-1)d}\right)^s \varrho(t_{11} + \theta b_n z_{11}, \dots, t_{(\ell-1)d} + \theta b_n z_{(\ell-1)d})$$

for some $\theta \in (0, 1)$. Together with Condition 5.1.2 $\mathcal{K}((\ell-1)d, s)$ the boundedness of the partial derivatives of order s imply the second assertion. ■

Theorem 6.2.2 *Let $\Psi \sim P$ be a 2ℓ-stationary point process in $\mathbb{R}^d$ with intensity λ with finite total variations $\|\gamma_{\mathrm{red}}^{(j)}\| < \infty$, $j = 2, \dots, 2\ell$. Let one of the assumptions*

(i) the bandwidth satisfies $b_n^{2(\ell-1)d}|W_n| \to \infty$

(ii) the cumulant densities up to order 2ℓ exist and satisfy

$$\sup_{x_1, \dots, x_{j-1} \in \bigcup_{i=1}^{\ell-1} b(t_i, \varepsilon) \cup b(-t_i, \varepsilon)} |c^{(j)}(x_1, \dots, x_{j-1})| < \infty$$

for $j = \ell+1, \ldots, 2\ell-1$, *and*

$$\sup_{\substack{x_1, x_\ell \in b(t_1,\varepsilon), \ldots, \\ x_{\ell-1}, x_{2\ell-2} \in b(t_{\ell-1},\varepsilon)}} \int_{\mathbb{R}^d} |c^{(2\ell)}(x_1, \ldots, x_{\ell-1}, x+x_\ell, x+x_{\ell+1}, \ldots, x+x_{2\ell-2}, x)| \mathrm{d}x < \infty$$

for some $\varepsilon > 0$

be satisfied.

Then we have

$$\lim_{n\to\infty} \operatorname{Var}\big(\hat{\varrho}_n^{(\ell)}(t)\big) = 0$$

in every point of continuity $t = (t_1, \ldots, t_{\ell-1}) \in (\mathbb{R}^d)^{\ell-1}$ *of* $\varrho^{(\ell)}$. ■

As in the case $\ell = 2$ (see Theorem 6.1.2) we prove the above theorem together with the following one.

Theorem 6.2.3 *Let* $\Psi \sim P$ *be a* 2ℓ*-stationary point process in* $\mathbb{R}^d$ *with intensity* λ *with finite total variations* $\|\gamma_{\mathrm{red}}^{(j)}\| < \infty$, $j = 2, \ldots, 2\ell$. *Let the cumulant densities up to order* 2ℓ *exist and satisfy*

$$\sup_{x_1,\ldots,x_{j-1} \in \bigcup_{i=1}^{\ell-1} b(t_i,\varepsilon)\cup b(-t_i,\varepsilon)\cup b(s_i,\varepsilon)\cup b(-s_i,\varepsilon)} |c^{(j)}(x_1, \ldots, x_{j-1})| < \infty$$

for $j = \ell+1, \ldots, 2\ell-1$, *and*

$$\sup_{\substack{x_1 \in b(t_1,\varepsilon), \ldots, x_{\ell-1} \in b(t_{\ell-1},\varepsilon), \\ x_\ell \in b(s_1,\varepsilon), \ldots, x_{2\ell-2} \in b(s_{\ell-1},\varepsilon)}} \int_{\mathbb{R}^d} |c^{(2\ell)}(x_1, \ldots, x_{\ell-1}, x+x_\ell, x+x_{\ell+1}, \ldots, x+x_{2\ell-2}, x)| \mathrm{d}x < \infty$$

for some $\varepsilon > 0$.

Then we have

$$\lim_{n\to\infty} b_n^{(\ell-1)d} |W_n| \operatorname{Cov}\big(\hat{\varrho}_n^{(\ell)}(s), \hat{\varrho}_n^{(\ell)}(t)\big) = \begin{cases} \lambda \varrho^{(\ell)}(s) \displaystyle\int_{(\mathbb{R}^d)^{\ell-1}} k^2(x)\mathrm{d}x, & s = t, \\ 0, & s \neq t, \end{cases}$$

in every point of continuity $s = (s_1, \ldots, s_{\ell-1}) \in (\mathbb{R}^d)^{\ell-1}$ *of* $\varrho^{(\ell)}$ *satisfying* $s_i \neq o$ *and* $s_i \neq \pm s_j$ *for all* $i, j \in \{1, \ldots, \ell-1\}$, $i \neq j$.

Proof of Theorems 6.2.2 and 6.2.3: For Theorem 6.2.2 we only have to consider assumption (i) since Theorem 6.2.3 generalizes Theorem 6.2.2 under assumption (ii). As in equation (4.2) the covariance $\operatorname{Cov}\left(\hat{\varrho}_n^{(\ell)}(s), \hat{\varrho}_n^{(\ell)}(t)\right)$ for $s = (s_1, \dots, s_{\ell-1})' \in (\mathbb{R}^d)^{\ell-1}$ and $t = (t_1, \dots, t_{\ell-1})' \in (\mathbb{R}^d)^{\ell-1}$ is a sum of indecomposable integrals. One of the terms of highest order of the covariance is

$$\begin{aligned}
&\frac{1}{(b_n^d)^{2\ell-2}|W_n|^2} \int_{(\mathbb{R}^d)^\ell} \mathbb{1}_{W_n}(x_1) k\left(\frac{x_2 - x_1 - s_1}{b_n}, \dots, \frac{x_\ell - x_1 - s_{\ell-1}}{b_n}\right) \\
&\qquad\qquad \times k\left(\frac{x_2 - x_1 - t_1}{b_n}, \dots, \frac{x_\ell - x_1 - t_{\ell-1}}{b_n}\right) \alpha^{(\ell)}(\mathrm{d}(x_1, \dots, x_\ell)) \\
&= \frac{1}{(b_n^d)^{2\ell-2}|W_n|} \int_{(\mathbb{R}^d)^{\ell-1}} k\left(\frac{x_2 - s_1}{b_n}, \dots, \frac{x_\ell - s_{\ell-1}}{b_n}\right) k\left(\frac{x_2 - t_1}{b_n}, \dots, \frac{x_\ell - t_{\ell-1}}{b_n}\right) \\
&\qquad\qquad \times \varrho^{(\ell)}(x_2, \dots, x_\ell)\, \mathrm{d}(x_2, \dots, x_\ell) \\
&= \frac{1}{(b_n^d)^{\ell-1}|W_n|} \int_{(\mathbb{R}^d)^{\ell-1}} k(x_2, \dots, x_\ell) k\left(x_2 + \frac{s_1 - t_1}{b_n}, \dots, x_\ell + \frac{s_{\ell-1} - t_{\ell-1}}{b_n}\right) \\
&\qquad\qquad \times \varrho^{(\ell)}(b_n x_2 + s_1, \dots, b_n x_\ell + s_{\ell-1})\, \mathrm{d}(x_2, \dots, x_\ell).
\end{aligned}$$

Since the ℓth-order product density is continuous in s, Lebesgue's dominated convergence theorem and assumption (i) of Theorem 6.2.2 imply this term to converge to zero. For Theorem 6.2.3 we multiply with $b_n^{(\ell-1)d}|W_n|$. Then the continuity of $\varrho^{(\ell)}$ and Lebesgue's dominated convergence theorem entail

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^{\ell-1}} k(x_2, \dots, x_\ell) k\left(x_2 + \frac{s_1 - t_1}{b_n}, \dots, x_\ell + \frac{s_{\ell-1} - t_{\ell-1}}{b_n}\right) \\
&\qquad\qquad \times \varrho^{(\ell)}(b_n x_2 + s_1, \dots, b_n x_\ell + s_{\ell-1})\, \mathrm{d}(x_2, \dots, x_\ell) \\
&\xrightarrow[n\to\infty]{} \begin{cases} \lambda \varrho^{(\ell)}(s) \displaystyle\int_{(\mathbb{R}^d)^{\ell-1}} k^2(x)\mathrm{d}x, & s = t \\ 0, & s \neq t. \end{cases}
\end{aligned}$$

All other integrals converge to zero. ■

6.2.2 Central limit theorems

In this section we will show normal convergence of the sequence

$$\Delta_n(t) := (b_n^{(\ell-1)d}|W_n|)^{1/2}\big(\hat{\varrho}_n^{(\ell)}(t) - \mathbb{E}\hat{\varrho}_n^{(\ell)}(t)\big)$$

for $\ell \geq 2$. The following theorem is proved along the same lines as Theorem 6.1.7.

Theorem 6.2.4 *Let $\Psi \sim P$ be a Brillinger-mixing point process in $\mathbb{R}^d$ with intensity λ and let all cumulant densities of order 2 and higher exist. Let the q-tuple $(u_1, \ldots, u_q) \in (\mathbb{R}^{(\ell-1)d})^q$ with $u_i = (u_{i1}, \ldots, u_{i(\ell-1)})'$ be chosen such that $u_i \neq o$ and $u_i \neq \pm u_j$ for all $i, j \in \{1, \ldots, \ell-1\}$, $i \neq j$, and let u_i be a point of continuity of $\varrho^{(\ell)}$ for every $i = 1, \ldots, q$.*

(i) Let the cumulant densities $c^{(j)}$, $j \geq \ell + 1$, satisfy

$$\sup_{\substack{y_1 \in b(u_{i1},\varepsilon),\ldots,\\ y_{\ell-1} \in b(u_{i(\ell-1)},\varepsilon)}} \int_{(\mathbb{R}^d)^{j-\ell}} |c^{(j)}(y_1, \ldots, y_{\ell-1}, x_\ell, \ldots, x_{j-1})| \mathrm{d}(x_\ell, \ldots, x_{j-1}) < \infty$$

for $i = 1, \ldots, q$, and

(ii) let the cumulant densities up to order 2ℓ exist and satisfy

$$\sup_{x_1,\ldots,x_{k-1} \in \bigcup_{i=1}^q \bigcup_{j=1}^{\ell-1} b(u_{ij},\varepsilon) \cup b(-u_{ij},\varepsilon)} |c^{(k)}(x_1, \ldots, x_{k-1})| < \infty$$

for $k = \ell+1, \ldots, 2\ell-1$, and

$$\sup_{\substack{x_1 \in b(u_{i1},\varepsilon),\ldots,x_{\ell-1} \in b(u_{i(\ell-1)},\varepsilon),\\ x_\ell \in b(u_{j1},\varepsilon),\ldots,x_{2\ell-2} \in b(u_{j(\ell-1)},\varepsilon)}} \int_{\mathbb{R}^d} |c^{(2\ell)}(x_1, \ldots, x_{\ell-1}, x + x_\ell, x + x_{\ell+1}, \ldots, x + x_{2\ell-2}, x)| \mathrm{d}x < \infty$$

for some $\varepsilon > 0$ and $i, j = 1, \ldots, q$.

Then we have $(\Delta_n(u_i))_{i=1}^q \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}(0, \Sigma_q)$, where the covariance matrix $\Sigma_q = (\sigma_{ij})_{i,j=1}^q$ is given by

$$\sigma_{ii} = \lambda \varrho^{(\ell)}(u_i) \int_{(\mathbb{R}^d)^{\ell-1}} k^2(x_1, \ldots, x_{\ell-1}) \mathrm{d}(x_1, \ldots, x_{\ell-1})$$

for $i = 1, \ldots, q$, and $\sigma_{ij} = 0$ for $i \neq j$. Furthermore we have

$$\sum_{i=1}^q \frac{(\Delta_n(u_i))^2}{\sigma_{ii}} \xrightarrow[n\to\infty]{\mathrm{d}} \chi_q^2.$$

Proof: The asymptotic covariance has been derived in Theorem 6.2.3; assumption (ii) provides the conditions on the cumulant densities needed in this derivation. To prove normal convergence of the random vector $(\Delta_n(u_i))_{i=1}^q$ we proceed as in Theorem 6.1.7. Using the method of Cramér-Wold we prove asymptotic normality of the linear combination $a_1\Delta_n(u_1) + \ldots + a_q\Delta_n(u_q)$, $(a_1, \ldots, a_q)' \in \mathbb{R}^q$, by showing its cumulants of order k to converge to zero for all $k \geq 3$.

Lemma 4.2.1 (with the notation given there) entails that the kth cumulant of $a_1\Delta_n(u_1)+\ldots+a_q\Delta_n(u_q)$, $k\geq 2$, satisfies

$$\begin{aligned}\Gamma_k\left(a_1\Delta_n(u_1)+\ldots+a_q\Delta_n(u_q)\right) &= \Gamma_k\left(\sum_{i=1}^{q}(b_n^{(\ell-1)d}|W_n|)^{-1/2}a_i\Psi^{(\ell)}(f_i)\right)\\ &= (b_n^{(\ell-1)d}|W_n|)^{-k/2}\sum_{\substack{k_1+\ldots+k_q=k\\ k_1,\ldots,k_q\geq 0}}\frac{k!}{k_1!\cdot\ldots\cdot k_q!}\,a_1^{k_1}\cdot\ldots\cdot a_q^{k_q}\,\mu^*_{k_1,\ldots,k_q}\end{aligned}$$

with $f_i:(\mathbb{R}^d)^\ell\to\mathbb{R}$, $(x_1,\ldots,x_\ell)\mapsto \mathbb{1}_{W_n}(x_1)k\left(\frac{x_2-x_1-u_i}{b_n},\ldots,\frac{x_\ell-x_1-u_i}{b_n}\right)$, $i=1,\ldots,q$.

Now we have to determine the order of $\mu^*_{k_1,\ldots,k_q}$ for $k_1,\ldots,k_q\geq 0$ with $k_1+\ldots+k_q=k$. Since $\mu^*_{k_1,\ldots,k_q}$ consists only of indecomposable integrals it can be seen by disintegration and substitution that the highest order of integrals with respect to at least two factorial cumulant measures is $O(b_n^{(\ell-1)d}|W_n|)$, due to $\left\|\gamma^{(k)}_{\mathrm{red}}\right\|<\infty$ for $k\geq 2$ and the boundedness assumptions on the kernel function. Together with the factor $(b_n^{(\ell-1)d}|W_n|)^{-k/2}$ this yields the asymptotic order $O((b_n^{(\ell-1)d}|W_n|)^{1-k/2})$ of these terms. For integrals with respect to only one factorial cumulant measure $\gamma^{(j)}$, disintegration and the finiteness of the total variations yield the asymptotic order $O(|W_n|)$. As in the proof of Theorem 6.1.7 we may obtain a stricter asymptotic order by assuming the existence of the cumulant densities and substituting $z=\frac{x_m-x_j-u_i}{b_n}$ in one instance of the kernel function k. Assumption (i) on the cumulant densities leads to the asymptotic order $O(b_n^{(\ell-1)d}|W_n|)$ of this integral if $j\geq \ell+1$ by Lebesgue's dominated convergence theorem. For an integral with respect to $\gamma^{(\ell)}$ the continuity of the ℓth-order product density in u_i, $i=1,\ldots,q$, (which implies the continuity of $c^{(\ell)}$ in these points due to $\varrho(.)=c^{(2)}(.)+\lambda$) and Lebesgue's dominated convergence theorem again yield the asymptotic order $O(b_n^{(\ell-1)d}|W_n|)$.

Altogether we obtain

$$\Gamma_k\left(a_1\Delta_n(u_1)+\ldots+a_q\Delta_n(u_q)\right)=O((b_n^{(\ell-1)d}|W_n|)^{1-k/2})$$

as $n\to\infty$, so the cumulants of order three and higher converge to zero as $n\to\infty$. This shows normal convergence of the random vector $(\Delta_n(u_i))_{i=1}^q$. The weak convergence of

$$\sum_{i=1}^{q}\frac{(\Delta_n(u_i))^2}{\sigma_{ii}}$$

is an immediate consequence using the Continuous Mapping Theorem. ■

Remark 6.2.5 The normal convergence can also be shown for the sequences

$$\left(\tilde{\Delta}_n(u_i)\right)_{i=1}^q := \left((b_n^d|W_n|)^{1/2}\left(\hat{\varrho}_n^{(\ell)}(u_i) - \lambda\varrho^{(\ell)}(u_i)\right)\right)_{i=1}^q$$

and

$$\left(\tilde{\Delta}_n^*(u_i)\right)_{i=1}^q = \left((b_n^d|W_n|)^{1/2}\left(\sqrt{\hat{\varrho}_n^{(\ell)}(u_i)} - \sqrt{\lambda\varrho^{(\ell)}(u_i)}\right)\right)_{i=1}^q$$

using the same arguments as in Corollaries 6.1.9 and 6.1.10. □

7

Central limit theorems for the integrated squared error of the empirical second-order product density and the empirical pair correlation function

The integrated squared error (henceforth abbreviated as ISE) is often used in simulation studies to measure the performance of probability density estimators, see Hall [22]. In this chapter we will study the ISE of estimators for the second-order product density and for the pair correlation function. More specifically we derive central limit theorems for the ISE of the estimators introduced in Sections 5.2 and 5.3 in the setting of Brillinger-mixing point processes. These central limit theorems may be used for constructing goodness-of-fit tests.

7.1 Central limit theorems for the integrated squared error of the second-order product density estimator

After some preliminary considerations concerning the cumulants of the ISE of the empirical second-order product density we will derive asymptotic representations for mean and variance of this estimator's ISE under mild mixing conditions. The chapter concludes with central limit theorems for the ISE of the estimated second-order product density in the setting of Poisson processes, Poisson cluster processes and Brillinger-mixing processes.

All results in this chapter are presented for the ISE based on the estimator $\hat{\varrho}_n = \hat{\varrho}_n^{(2)}$ from Definition 5.2.1 and the estimator $\hat{g}_n$ from Definition 5.3.1 but also hold for the variants of these estimators introduced in Section 5.2 and 5.3.

7.1.1 The integrated squared error and some preliminary considerations

In Theorem 6.1.7 we have shown that under certain mixing conditions the sequence $(\Delta_n(u_i))_{i=1}^q$ with $\Delta_n(\,.\,) = (b_n^d|W_n|)^{1/2}\big(\hat{\varrho}_n(\,.\,) - \mathbb{E}\hat{\varrho}_n(\,.\,)\big)$ and pairwise distinct $u_1, \ldots u_q$ converges weakly to a Gaussian vector with independent components. This entails that the potential limiting process of $\Delta_n(.)$ is a Gaussian white noise. Since this process is not measurable, see Kallianpur [41], page 10, this result is not useful for finding the limiting distribution of the estimated second-order product density's ISE defined by

$$I_n(K) := \int\limits_K (\hat{\varrho}_n(t) - \lambda\varrho(t))^2 \mathrm{d}t,$$

where $K \in \mathfrak{B}(\mathbb{R}^d)$, $|K| > 0$, is a bounded set. For this reason we will investigate the integrated squared error $I_n(K)$ directly. For Brillinger-mixing point processes the normal convergence of the centered and suitably scaled ISE is proved by showing all cumulants of order three and higher to converge to zero. Lemma 7.1.1 will give a representation for the cumulants of the ISE of the estimated second-order product density. This representation can also be used for deriving an asymptotic representation for the variance of the ISE.

The cumulants of the ISE can be represented by a sum of indecomposable (see page 36) and *irreducible* integrals. First we give a definition of an irreducible integral. This definition is closely related to the special form of the functions

$$f_1 : (\mathbb{R}^d)^4 \to \mathbb{R},$$
$$(x_1, x_2, x_3, x_4) \mapsto \mathbb{1}_{W_n}(x_1)\mathbb{1}_{W_n}(x_3)\mathbb{1}_{\{x_1 \neq x_2, x_3 \neq x_4\}} \int\limits_K k\left(\frac{x_2 - x_1 - t}{b_n}\right) k\left(\frac{x_4 - x_3 - t}{b_n}\right) \mathrm{d}t,$$

and

$$f_2 : (\mathbb{R}^d)^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto \mathbb{1}_{W_n}(x_1)\mathbb{1}_{\{x_1 \neq x_2\}} \int\limits_K k\left(\frac{x_2 - x_1 - t}{b_n}\right) \lambda\varrho(t)\mathrm{d}t.$$

An integral $I_{\mathcal{P},\mathcal{Q}}(\underbrace{f_1,\ldots,f_1}_{k-j},\underbrace{f_2,\ldots,f_2}_{j})$, $j=0,\ldots,k-1$, with $\mathcal{P}=\{P_1,\ldots,P_q\}$ and $\mathcal{Q}=\{Q_1,\ldots,Q_r\}$ (see page 35) is *reducible* if there are indices $a,b\in\{1,\ldots,q\}$, $c,d\in\{1,\ldots,r\}$, and an odd number $i\in\{1,\ldots,4k-4j\}$ such that $P_a=\{i\}$, $P_b=\{i+1\}$, and $Q_c=\{a,b\}$ or $Q_c=\{a\}$ and $Q_d=\{b\}$. In other words, a reducible integral contains one of the terms

$$\int\limits_{(\mathbb{R}^d)^2} f_1(x_i,x_{i+1},x,y)\gamma^{(2)}(\mathrm{d}(x_i,x_{i+1})),$$

$$\int\limits_{(\mathbb{R}^d)^2} f_1(x,y,x_i,x_{i+1})\gamma^{(2)}(\mathrm{d}(x_i,x_{i+1})),$$

$$\int\limits_{(\mathbb{R}^d)^2} f_1(x_i,x_{i+1},x,y)\gamma^{(1)}(\mathrm{d}x_i)\gamma^{(1)}(\mathrm{d}x_{i+1}),$$

$$\int\limits_{(\mathbb{R}^d)^2} f_1(x,y,x_i,x_{i+1})\gamma^{(1)}(\mathrm{d}x_i)\gamma^{(1)}(\mathrm{d}x_{i+1}),$$

with $x,y\notin\{x_i,x_{i+1}\}$, and the remaining functions contain neither x_i nor x_{i+1}. We will call this term the *reducible part* of the integral. An integral can have more than one reducible part. An integral that is not reducible is called *irreducible*. For instance, the integral

$$\begin{aligned}&I_{\{\{1\},\{2\},\{3,5\},\{4,6\}\},\{\{1,2\},\{3,4\}\}}(f_1,f_2)\\&=\int\limits_{(\mathbb{R}^d)^2}\int\limits_{(\mathbb{R}^d)^2} f_1(z_1,z_2,z_3,z_4)f_2(z_3,z_4)\gamma^{(2)}(\mathrm{d}(z_1,z_2))\gamma^{(2)}(\mathrm{d}(z_3,z_4))\end{aligned}$$

is reducible with reducible part

$$\int\limits_{(\mathbb{R}^d)^2} f_1(z_1,z_2,z_3,z_4)\gamma^{(2)}(\mathrm{d}(z_1,z_2)),$$

whereas the integral

$$\begin{aligned}&I_{\{\{1,5\},\{2\},\{3,6\},\{4\}\},\{\{1,2\},\{3,4\}\}}(f_1,f_2)\\&=\int\limits_{(\mathbb{R}^d)^2}\int\limits_{(\mathbb{R}^d)^2} f_1(z_1,z_2,z_3,z_4)f_2(z_1,z_3)\gamma^{(2)}(\mathrm{d}(z_1,z_2))\gamma^{(2)}(\mathrm{d}(z_3,z_4))\end{aligned}$$

is irreducible.

Recall the sum of indecomposable integrals $\mu^*_{k-j,j}$, $j = 0, \ldots, k$, see (4.3). We denote the sum of irreducible integrals in $\mu^*_{k-j,j}$ by $\mu^{**}_{k-j,j}$, $j = 0, \ldots, k$. We will write $\mu^{**a}_{k-(j+r),j+r}$, $a = 1, \ldots, r$, for the term obtained from $\mu^{**}_{k-(j+r),j+r}$ by replacing a instances of f_2 with $\tilde{f}_2$, where the function $\tilde{f}_2$ is given by

$$\tilde{f}_2(x,y) := \mathbb{1}_{W_n}(x) \int\limits_K \left[k\left(\frac{y-x-t}{b_n}\right) \lambda \left(\int\limits_{\mathbb{R}^d} R_s(z,t) k(z) \mathrm{d}z \right) \right] \mathrm{d}t$$

with $R_s(z,t) = \frac{1}{s!}\left(\frac{\partial}{\partial t_1} z_1 + \ldots + \frac{\partial}{\partial t_d} z_d\right)^s \varrho(t_1 + \theta b_n z_1, \ldots, t_d + \theta b_n z_d)$ and $\theta = \theta(t) \in (0,1)$. As a result, $\mu^{**a}_{k-(j+r),j+r}$ contains only $j + r - a$ instances of f_2.

Now we can state the lemma giving a representation of the kth cumulant of the ISE in terms of indecomposable and irreducible integrals for $k \geq 2$.

Lemma 7.1.1 *Let $k \geq 2$, and let $\Psi \sim P$ be a $4k$-stationary point process in $\mathbb{R}^d$ with intensity λ and $\|\gamma^{(j)}_{\mathrm{red}}\| < \infty$ for $j = 2, \ldots, 4k$. Let the second-order product density ϱ have bounded and continuous partial derivatives of order s in $K \oplus b^o(o, \varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$. Furthermore let $b_n^{2s+d}|W_n| \to 0$, and let the kernel function satisfy Condition 5.1.2 $\mathcal{K}(d,s)$.*

Then the kth cumulant of the ISE $I_n(K)$ satisfies

$$\Gamma_k(I_n(K)) \quad = \quad \sum_{j=0}^{k} \binom{k}{j} 2^j (b_n^s)^j (b_n^d |W_n|)^{j-2k} \mu^{**j}_{k-j,j}.$$

Proof: In the first part of the proof we apply Lemma 4.2.1 in order to express the kth cumulant by a sum of indecomposable integrals. Due to the smoothness conditions on the second-order product density this representation can be further simplified. This is shown in the second part of the proof.

Due to the semi-invariance of the cumulants of order 2 and higher the kth cumulant of $\int\limits_K \left(\hat{\varrho}_n^2(t) - 2\lambda\varrho(t)\hat{\varrho}_n(t)\right) \mathrm{d}t$ is identical to the kth cumulant of $I_n(K) - \mathbb{E}I_n(K)$ for $k \geq 2$. Therefore we investigate the kth cumulant of $\int\limits_K \left(\hat{\varrho}_n^2(t) - 2\lambda\varrho(t)\hat{\varrho}_n(t)\right) \mathrm{d}t$.

I Representation of the kth cumulant by indecomposable integrals

First we rewrite $\int\limits_K \left(\hat{\varrho}_n^2(t) - 2\lambda\varrho(t)\hat{\varrho}_n(t)\right) \mathrm{d}t$. We have

$$
\begin{aligned}
&\int\limits_K \left(\hat{\varrho}_n^2(t) - 2\lambda\varrho(t)\hat{\varrho}_n(t)\right) \mathrm{d}t \\
&\quad = \sum_{\substack{x_1,x_2,x_3,x_4\in\Psi \\ x_1\neq x_2, x_3\neq x_4}} (b_n^d|W_n|)^{-2}\mathbb{1}_{W_n}(x_1)\mathbb{1}_{W_n}(x_3)\int\limits_K k\left(\frac{x_2-x_1-t}{b_n}\right)k\left(\frac{x_4-x_3-t}{b_n}\right)\mathrm{d}t \\
&\quad\quad - \sum_{x_1,x_2\in\Psi}^{*} 2(b_n^d|W_n|)^{-1}\mathbb{1}_{W_n}(x_1)\int\limits_K k\left(\frac{x_2-x_1-t}{b_n}\right)\lambda\varrho(t)\mathrm{d}t \\
&\quad = \sum_{x_1,x_2,x_3,x_4\in\Psi} C_1 f_1(x_1,x_2,x_3,x_4) + \sum_{x_1,x_2\in\Psi} C_2 f_2(x_1,x_2),
\end{aligned}
$$

with functions

$$
\begin{aligned}
&f_1 : (\mathbb{R}^d)^4 \to \mathbb{R}, \\
&\quad (x_1,x_2,x_3,x_4) \mapsto \mathbb{1}_{W_n}(x_1)\mathbb{1}_{W_n}(x_3)\mathbb{1}_{\{x_1\neq x_2, x_3\neq x_4\}}\int\limits_K k\left(\frac{x_2-x_1-t}{b_n}\right)k\left(\frac{x_4-x_3-t}{b_n}\right)\mathrm{d}t,
\end{aligned}
$$

and

$$
f_2 : (\mathbb{R}^d)^2 \to \mathbb{R}, \qquad (x_1,x_2) \mapsto \mathbb{1}_{W_n}(x_1)\mathbb{1}_{\{x_1\neq x_2\}}\int\limits_K k\left(\frac{x_2-x_1-t}{b_n}\right)\lambda\varrho(t)\mathrm{d}t,
$$

and constants $C_1 := (b_n^d|W_n|)^{-2}$ and $C_2 := -2(b_n^d|W_n|)^{-1}$. Since we have $|K| < \infty$ and since k is bounded with bounded support the moments $\mathbb{E}\left[|\sum_{x_1,x_2,x_3,x_4\in\Psi} f_1(x_1,x_2,x_3,x_4)|^k\right]$ and $\mathbb{E}\left[|\sum_{x_1,x_2\in\Psi} f_2(x_1,x_2)|^k\right]$ are finite. Hence we can apply Lemma 4.2.1. Therefore the kth cumulant $\Gamma_k(I_n(K))$ of the ISE $I_n(K)$ satisfies

$$
\begin{aligned}
\Gamma_k(I_n(K)) &= \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} \frac{k!}{k_1!k_2!}(-1)^{k_2}2^{k_2}(b_n^d|W_n|)^{-2k_1-k_2}\mu^*_{k_1,k_2} \\
&= \sum_{j=0}^{k}\binom{k}{j}(-1)^j 2^j (b_n^d|W_n|)^{j-2k}\mu^*_{k-j,j}. \qquad (7.1)
\end{aligned}
$$

II Representation of the cumulants by indecomposable and irreducible integrals

The special form of the functions f_1 and f_2 allows a further simplification of the representation (7.1) for the kth cumulant. This simplification is based on the approximate identity

$$\int_{(\mathbb{R}^d)^2} f_1(x_1,x_2,x_3,x_4)\alpha^{(2)}(\mathrm{d}(x_1,x_2)) = \int_{(\mathbb{R}^d)^2} f_1(x_3,x_4,x_1,x_2)\alpha^{(2)}(\mathrm{d}(x_1,x_2)) \approx b_n^d|W_n|f_2(x_3,x_4)$$

for $x_3,x_4 \notin \{x_1,x_2\}$, which implies the reducible integrals of $\mu^*_{k,0}$ (except for the error terms) and integrals in $\mu^*_{k-\ell,\ell}$, $\ell = 1,\dots,k$, to cancel.

More precisely we start by combining two reducible integrals in $\mu^*_{k-j,j}$, $j = 0,\dots,k-1$. These integrals differ only by their reducible parts, in two possible ways. Either the two integrals' reducible parts are

$$\int_{(\mathbb{R}^d)^2} f_1(x_i,x_{i+1},x,y)\gamma^{(2)}(\mathrm{d}(x_i,x_{i+1})) \quad \text{and} \quad \int_{(\mathbb{R}^d)^2} f_1(x_i,x_{i+1},x,y)\gamma^{(1)}(\mathrm{d}x_i)\gamma^{(1)}(\mathrm{d}x_{i+1})$$

or they are

$$\int_{(\mathbb{R}^d)^2} f_1(x,y,x_i,x_{i+1})\gamma^{(2)}(\mathrm{d}(x_i,x_{i+1})) \quad \text{and} \quad \int_{(\mathbb{R}^d)^2} f_1(x,y,x_i,x_{i+1})\gamma^{(1)}(\mathrm{d}x_i)\gamma^{(1)}(\mathrm{d}x_{i+1}).$$

The sum of these two reducible integrals in $\mu^*_{k-j,j}$ is hence an integral which emerges from either of the two aforementioned integrals by replacing the respective reducible parts by

$$\int_{(\mathbb{R}^d)^2} f_1(x_i,x_{i+1},x,y)\alpha^{(2)}(\mathrm{d}(x_i,x_{i+1})) \tag{7.2}$$

or

$$\int_{(\mathbb{R}^d)^2} f_1(x,y,x_i,x_{i+1})\alpha^{(2)}(\mathrm{d}(x_i,x_{i+1})), \tag{7.3}$$

depending on the above distinction. If the integral has more than one reducible part, then we iterate the above procedure, eventually obtaining an irreducible integral. In the following, we will only consider irreducible integrals and integrals which arise from the above-mentioned combination and summation of reducible integrals. The latter integrals are also called reducible parts. Now we simplify one of the reducible parts (7.2) and (7.3) of a reducible integral by disintegration and Taylor's expansion, that is,

$$\begin{aligned}
&\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x_i)\mathbb{1}_{W_n}(x) \int\limits_K k\left(\frac{x_{i+1}-x_i-t}{b_n}\right) k\left(\frac{y-x-t}{b_n}\right) \mathrm{d}t\alpha^{(2)}(\mathrm{d}(x_i,x_{i+1}))\\
&= \lambda \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \mathbb{1}_{W_n}(x_i)\mathbb{1}_{W_n}(x) \int\limits_K k\left(\frac{x_{i+1}-t}{b_n}\right) k\left(\frac{y-x-t}{b_n}\right) \mathrm{d}t\varrho(x_{i+1})\mathrm{d}x_{i+1}\mathrm{d}x_i\\
&= b_n^d|W_n|\mathbb{1}_{W_n}(x) \int\limits_K k\left(\frac{y-x-t}{b_n}\right) \lambda \left(\int\limits_{\mathbb{R}^d} k(x_{i+1})\varrho(b_n x_{i+1}+t)\mathrm{d}x_{i+1}\right) \mathrm{d}t\\
&= b_n^d|W_n|f_2(x,y) + b_n^s b_n^d|W_n|\tilde{f}_2(x,y), \qquad (7.4)
\end{aligned}$$

where

$$\tilde{f}_2(x,y) = \mathbb{1}_{W_n}(x)\int\limits_K \left[k\left(\frac{y-x-t}{b_n}\right)\lambda\left(\int\limits_{\mathbb{R}^d} R_s(z,t)k(z)\mathrm{d}z\right)\right]\mathrm{d}t$$

with $R_s(z,t) = \frac{1}{s!}\left(\frac{\partial}{\partial t_1}z_1 + \ldots + \frac{\partial}{\partial t_d}z_d\right)^s \varrho(t_1+\theta b_n z_1,\ldots,t_d+\theta b_n z_d)$ and $\theta = \theta(t) \in (0,1)$. Here we have used Condition 5.1.2 $\mathcal{K}(d,s)$ so that only $\varrho(t)$ and the error term remain from Taylor's expansion. In the following we will refer to the above simplification by disintegration and Taylor's expansion as *reduction* of the integral.

An integral in $\mu^*_{k-j,j}$ is called r-*reducible* if it can be reduced exactly r times (that is, if reduction as defined above can be applied exactly r times), with $r \in \{0,\ldots,k-j\}$. Reducing an r-reducible integral r times yields a sum of two parts. The first part is an integral in $\mu^{**}_{k-(j+r),j+r}$ multiplied by $(b_n^d|W_n|)^r$ while the second part is a sum of integrals containing the error terms from all Taylor expansions performed in the reductions. Note that within this iterative scheme reductions can also be applied to error terms obtained from earlier reductions. We illustrate this procedure by an example involving three reductions of a 3-reducible integral in $\mu^*_{3,0}$:

$$\int\limits_{(\mathbb{R}^d)^8} f_1(x_1,x_2,x_3,x_4)f_1(x_1,x_2,x_5,x_6)f_1(x_1,x_2,x_7,x_8)$$
$$\gamma^{(2)}(\mathrm{d}(x_1,x_2))\alpha^{(2)}(\mathrm{d}(x_3,x_4))\alpha^{(2)}(\mathrm{d}(x_5,x_6))\alpha^{(2)}(\mathrm{d}(x_7,x_8))$$

$$= b_n^d|W_n|\Bigg(\int\limits_{(\mathbb{R}^d)^6} f_2(x_1,x_2)f_1(x_1,x_2,x_5,x_6)f_1(x_1,x_2,x_7,x_8)$$
$$\gamma^{(2)}(\mathrm{d}(x_1,x_2))\alpha^{(2)}(\mathrm{d}(x_5,x_6))\alpha^{(2)}(\mathrm{d}(x_7,x_8))$$
$$+ b_n^s \int\limits_{(\mathbb{R}^d)^6} \tilde{f}_2(x_1,x_2)f_1(x_1,x_2,x_5,x_6)f_1(x_1,x_2,x_7,x_8)$$
$$\gamma^{(2)}(\mathrm{d}(x_1,x_2))\alpha^{(2)}(\mathrm{d}(x_5,x_6))\alpha^{(2)}(\mathrm{d}(x_7,x_8))\Bigg)$$
$$= b_n^{2d}|W_n|^2\Bigg(\int\limits_{(\mathbb{R}^d)^4} (f_2(x_1,x_2))^2 f_1(x_1,x_2,x_7,x_8)\gamma^{(2)}(\mathrm{d}(x_1,x_2))\alpha^{(2)}(\mathrm{d}(x_7,x_8))$$
$$+ 2b_n^s \int\limits_{(\mathbb{R}^d)^4} f_2(x_1,x_2)\tilde{f}_2(x_1,x_2)f_1(x_1,x_2,x_7,x_8)\gamma^{(2)}(\mathrm{d}(x_1,x_2))\alpha^{(2)}(\mathrm{d}(x_7,x_8))$$
$$+ b_n^{2s} \int\limits_{(\mathbb{R}^d)^4} (\tilde{f}_2(x_1,x_2))^2 f_1(x_1,x_2,x_7,x_8)\gamma^{(2)}(\mathrm{d}(x_1,x_2))\alpha^{(2)}(\mathrm{d}(x_7,x_8))\Bigg)$$
$$= b_n^{3d}|W_n|^3\Bigg(\int\limits_{(\mathbb{R}^d)^2} (f_2(x_1,x_2))^3\gamma^{(2)}(\mathrm{d}(x_1,x_2))$$
$$+ 3b_n^s \int\limits_{(\mathbb{R}^d)^4} (f_2(x_1,x_2))^2\tilde{f}_2(x_1,x_2)\gamma^{(2)}(\mathrm{d}(x_1,x_2))$$
$$+ 3b_n^{2s} \int\limits_{(\mathbb{R}^d)^4} f_2(x_1,x_2)(\tilde{f}_2(x_1,x_2))^2\gamma^{(2)}(\mathrm{d}(x_1,x_2))$$
$$+ b_n^{3s} \int\limits_{(\mathbb{R}^d)^4} (\tilde{f}_2(x_1,x_2))^3\gamma^{(2)}(\mathrm{d}(x_1,x_2))\Bigg).$$

In the remaining terms a instances of the function f_2 are replaced by $\tilde{f}_2$, $a = 1,\dots,r$. For each integral in $\mu^{**a}_{k-(j+r),j+r}$ the number of r-reducible integrals in $\mu^*_{k-j,j}$ leading to this integral is $2^r\binom{k-j}{r}$. Hence we obtain the representation

$$\mu^*_{k-j,j} = \sum_{r=0}^{k-j} 2^r \binom{k-j}{r} (b_n^d|W_n|)^r \sum_{a=0}^{r} (b_n^s)^a \binom{r}{a} \mu^{**a}_{k-(j+r),j+r} \tag{7.5}$$

for $j = 0,\dots,k$. The main terms are $\mu^{**}_{k-(j+r),j+r} \equiv \mu^{**0}_{k-(j+r),j+r}$, $r = 0,\dots,k-j$, and the remaining terms are $\mu^{**a}_{k-(j+r),j+r}$, $a = 1, \ldots, r$. Equations (7.1) and (7.5) imply

$$\begin{aligned}
\Gamma_k(I_n(K)) &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} 2^i (b_n^d |W_n|)^{i-2k} \mu^{*}_{k-i,i} \\
&= \sum_{i=0}^{k} \sum_{r=0}^{k-i} \sum_{a=0}^{r} (-1)^i \frac{k!}{i!a!(r-a)!(k-(i+r))!} 2^{i+r} (b_n^d |W_n|)^{i+r-2k} (b_n^s)^a \mu^{**a}_{k-(i+r),i+r} \\
&= \sum_{j=0}^{k} \sum_{i=0}^{j} \sum_{a=0}^{j-i} (-1)^i \frac{k!}{i!a!(j-i-a)!(k-j)!} 2^j (b_n^d |W_n|)^{j-2k} (b_n^s)^a \mu^{**a}_{k-j,j} \\
&= \sum_{j=0}^{k} \binom{k}{j} 2^j (b_n^s)^j (b_n^d |W_n|)^{j-2k} \mu^{**j}_{k-j,j}. \qquad (7.6)
\end{aligned}$$

For the last line we consider the summands indexed by $j \in \{0, \ldots, k\}$. This yields

$$\begin{aligned}
&\sum_{i=0}^{j} \sum_{a=0}^{j-i} (-1)^i \frac{k!}{i!a!(j-i-a)!(k-j)!} 2^j (b_n^d |W_n|)^{j-2k} (b_n^s)^a \mu^{**a}_{k-j,j} \\
&= \frac{k!}{(k-j)!} 2^j (b_n^d |W_n|)^{j-2k} \sum_{a=0}^{j} \frac{1}{a!} (b_n^s)^a \mu^{**a}_{k-j,j} \sum_{i=0}^{j-a} (-1)^i \frac{1}{i!(j-a-i)!}.
\end{aligned}$$

Due to $\sum_{i=0}^{j-a} (-1)^i \frac{1}{i!(j-a-i)!} = 0$ for $a = 0, \ldots, j-1$ and $\sum_{i=0}^{0} (-1)^i \frac{1}{i!(j-i)!} = 1$ the identity (7.6) follows and the proof is complete. ■

For the stationary Poisson process the kth cumulant of the ISE $I_n(K)$ takes a simpler form, as is seen in the following corollary.

Corollary 7.1.2 *Let $k \geq 2$, and let $\Psi \sim P$ be a stationary Poisson process in $\mathbb{R}^d$ with intensity λ. Then the kth cumulant of the ISE $I_n(K)$ satisfies*

$$\Gamma_k(I_n(K)) \quad = \quad (b_n^d |W_n|)^{-2k} \mu^{**}_{k,0}.$$

Proof: For the stationary Poisson process the kth-order product density is the constant λ^{k-1}. Hence equation (7.4) simplifies to

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x_i) \mathbb{1}_{W_n}(x) \int_K k\left(\frac{x_{i+1} - x_i - t}{b_n}\right) k\left(\frac{y - x - t}{b_n}\right) \mathrm{d}t \alpha^{(2)}(\mathrm{d}(x_i, x_{i+1})) \\
&= \lambda^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{W_n}(x_i) \mathbb{1}_{W_n}(x) \int_K k\left(\frac{x_{i+1} - t}{b_n}\right) k\left(\frac{y - x - t}{b_n}\right) \mathrm{d}t \mathrm{d}x_{i+1} \mathrm{d}x_i
\end{aligned}$$

$$= b_n^d |W_n| \mathbb{1}_{W_n}(x) \lambda^2 \int\limits_K k\left(\frac{y-x-t}{b_n}\right) \mathrm{d}t$$
$$= b_n^d |W_n| f_2(x,y),$$

where we have used the same notation as in Lemma 7.1.1. As a result, Condition 5.1.2 $\mathcal{K}(d,s)$ is not needed for deriving the representation (7.5), which reduces to

$$\mu^*_{k-j,j} = \sum_{r=0}^{k-j} 2^r \binom{k-j}{r} (b_n^d|W_n|)^r \mu^{**}_{k-(j+r),j+r}.$$

Thus equation (7.6) simplifies to

$$\begin{aligned}
\Gamma_k(I_n(K)) &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} 2^i (b_n^d|W_n|)^{i-2k} \mu^*_{k-i,i} \\
&= \sum_{i=0}^{k} \sum_{r=0}^{k-i} (-1)^i \frac{k!}{i!r!(k-(i+r))!} 2^{i+r} (b_n^d|W_n|)^{i+r-2k} \mu^{**}_{k-(i+r),i+r} \\
&= \sum_{j=0}^{k} \sum_{i=0}^{j} (-1)^i \frac{k!}{i!(j-i)!(k-j)!} 2^j (b_n^d|W_n|)^{j-2k} \mu^{**}_{k-j,j} \\
&= (b_n^d|W_n|)^{-2k} \mu^{**}_{k,0},
\end{aligned}$$

where the last equation holds by the same argument as in (7.6). ■

7.1.2 Asymptotic representation of the mean and the variance

In this section we derive asymptotic representations of the mean and the variance of the ISE $I_n(K)$ for the Poisson process and under some mild mixing conditions. The following lemma gives an asymptotic representation for the mean of the ISE $I_n(K)$ for the Poisson process.

Lemma 7.1.3 *Let $\Psi \sim \Pi_\lambda$ be a stationary Poisson process in $\mathbb{R}^d$ with intensity λ. Then we have*

$$b_n^d |W_n| \mathbb{E} \int\limits_K (\hat{\varrho}_n(t) - \lambda^2)^2 \mathrm{d}t = \lambda^2 |K| \int\limits_{\mathbb{R}^d} k^2(y) \mathrm{d}y + O(b_n^d)$$

as $n \to \infty$.

Proof: For the Poisson process the estimator for the second-order product density is unbiased for λ^2, that is,

$$\begin{aligned}
\mathbb{E}\,\hat{\varrho}_n(t) &= \frac{1}{b_n^d|W_n|}\mathbb{E}\sum_{x,y\in\Psi}^{*} \mathbb{1}_{W_n}(x)k\left(\frac{y-x-t}{b_n}\right) \\
&= \frac{1}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)k\left(\frac{y-x-t}{b_n}\right)\alpha^{(2)}(\mathrm{d}(x,y)) \\
&= \frac{\lambda}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)k\left(\frac{y-x-t}{b_n}\right)\alpha_{\mathrm{red}}^{(2)}(\mathrm{d}(y-x))\mathrm{d}x \\
&= \frac{\lambda^2}{b_n^d|W_n|}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \mathbb{1}_{W_n}(x)k\left(\frac{y-t}{b_n}\right)\mathrm{d}x\mathrm{d}y \\
&= \lambda^2.
\end{aligned}$$

Applying Fubini's theorem the mean of the ISE is

$$\mathbb{E}\int\limits_K(\hat{\varrho}_n(t)-\lambda^2)^2\mathrm{d}t = \int\limits_K \mathrm{Var}(\hat{\varrho}_n(t))\mathrm{d}t.$$

For the Poisson process we have $\gamma^{(k)} \equiv 0$ for $k \geq 2$. This implies $\alpha^{(k)}(\mathrm{d}(x_1,\ldots,x_k)) = \lambda^k\mathrm{d}(x_1,\ldots,x_k)$. Hence the representation (4.2) simplifies to

$$\begin{aligned}
&\mathrm{Cov}\left(\sum_{x,y\in\Psi}^{*} h_1(x,y), \sum_{z,v\in\Psi}^{*} h_2(z,v)\right) \\
&= \int\limits_{(\mathbb{R}^d)^2} h_1(x,y)[h_2(x,y)+h_2(y,x)]\lambda^2\mathrm{d}(x,y) \\
&\quad + \int\limits_{(\mathbb{R}^d)^3} h_1(x,y)[h_2(x,z)+h_2(y,z)+h_2(z,x)+h_2(z,y)]\lambda^3\mathrm{d}(x,y,z).
\end{aligned}$$

Thus we have

$$\mathrm{Var}\big(\hat{\varrho}_n(t)\big) = \frac{1}{b_n^{2d}|W_n|^2}\mathrm{Cov}\left(\sum_{x,y\in\Psi}^{*} \mathbb{1}_{W_n}(x)k\left(\frac{y-x-t}{b_n}\right), \sum_{x,y\in\Psi}^{*} \mathbb{1}_{W_n}(x)k\left(\frac{y-x-t}{b_n}\right)\right)$$

$$\begin{aligned}
&= \frac{1}{b_n^{2d}|W_n|^2}\Bigg(\lambda^2 \int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) k^2\left(\frac{y-x-t}{b_n}\right) \mathrm{d}(x,y) \\
&\qquad + \lambda^2 \int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y) k\left(\frac{y-x-t}{b_n}\right) k\left(\frac{x-y-t}{b_n}\right) \mathrm{d}(x,y) \\
&\qquad + \lambda^3 \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) k\left(\frac{y-x-t}{b_n}\right) k\left(\frac{z-x-t}{b_n}\right) \mathrm{d}(x,y,z) \\
&\qquad + \lambda^3 \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y) k\left(\frac{y-x-t}{b_n}\right) k\left(\frac{z-y-t}{b_n}\right) \mathrm{d}(x,y,z) \\
&\qquad + \lambda^3 \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(z) k\left(\frac{y-x-t}{b_n}\right) k\left(\frac{x-z-t}{b_n}\right) \mathrm{d}(x,y,z) \\
&\qquad + \lambda^3 \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(z) k\left(\frac{y-x-t}{b_n}\right) k\left(\frac{y-z-t}{b_n}\right) \mathrm{d}(x,y,z)\Bigg) \\
&= \frac{1}{b_n^d|W_n|}\lambda^2 \int\limits_{\mathbb{R}^d} k^2(y)\mathrm{d}y \\
&\quad + \frac{1}{b_n^d|W_n|^2}\lambda^2 \int\limits_{\mathbb{R}^d} |W_n \cap (W_n - b_n y - t)| k(y) k\left(y - \frac{2t}{b_n}\right) \mathrm{d}y \\
&\quad + \frac{\lambda^3}{|W_n|} \\
&\quad + \frac{2}{|W_n|^2}\lambda^3 \int\limits_{\mathbb{R}^d} |W_n \cap (W_n - b_n y - t)| k(y)\mathrm{d}y \\
&\quad + \frac{1}{|W_n|^2}\lambda^3 \int\limits_{(\mathbb{R}^d)^2} |W_n \cap (W_n - b_n x + b_n y)| k(x) k(y) \mathrm{d}(x,y).
\end{aligned}$$

This entails

$$\begin{aligned}
& b_n^d|W_n| \int\limits_K \mathrm{Var}(\hat{\varrho}_n(t))\mathrm{d}t \\
&\quad = \lambda^2|K| \int\limits_{\mathbb{R}^d} k^2(y)\mathrm{d}y \\
&\qquad + b_n^d \int\limits_K \lambda^2 \int\limits_{\mathbb{R}^d} \mathbb{1}_K(b_n t) \frac{|W_n \cap (W_n - b_n y - b_n t)|}{|W_n|} k(y) k(y-2t) \mathrm{d}y\mathrm{d}t \\
&\qquad + b_n^d \lambda^3 |K|
\end{aligned}$$

$$+ b_n^d 2\lambda^3 \int\limits_K \int\limits_{\mathbb{R}^d} \frac{|W_n \cap (W_n - b_n y - t)|}{|W_n|} k(y) \mathrm{d}y \mathrm{d}t$$
$$+ b_n^d \lambda^3 |K| \int\limits_{(\mathbb{R}^d)^2} \frac{|W_n \cap (W_n - b_n x + b_n y)|}{|W_n|} k(x) k(y) \mathrm{d}(x,y).$$

Due to $|K| < \infty$ and since the kernel function is bounded with bounded support this completes the proof. ■

The following two lemmata give an asymptotic representation of the mean of the ISE $I_n(K)$. In the first lemma we assume the second-order product density to be Lipschitz-continuous whereas in the second lemma we assume the second-order product density to have bounded and continuous partial derivatives of order s. The latter condition implies a slightly faster convergence for $s \geq 2$.

Lemma 7.1.4 *Let $\Psi \sim P$ be a 4-stationary point process in $\mathbb{R}^d$ with intensity λ and $\|\gamma_{\mathrm{red}}^{(k)}\| < \infty$ for $k = 2, 3, 4$. Let the second-order product density ϱ be Lipschitz-continuous in $K \oplus b^o(o, \varepsilon)$ for some $\varepsilon > 0$. Let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist.*

Then we have

$$b_n^d |W_n| \mathbb{E} \int\limits_K \big(\hat{\varrho}_n(t) - \lambda \varrho(t)\big)^2 \mathrm{d}t = \lambda \int\limits_K \varrho(t) \mathrm{d}t \int\limits_{\mathbb{R}^d} k^2(x) \mathrm{d}x + O(b_n) + O(b_n^{d+2}|W_n|)$$

as $n \to \infty$.

Proof: By Fubini's theorem we have

$$\begin{aligned}
\mathbb{E} \int\limits_K (\hat{\varrho}_n(t) - \lambda \varrho(t))^2 \mathrm{d}t &= \int\limits_K \mathbb{E}\big(\hat{\varrho}_n(t) - \mathbb{E}\hat{\varrho}_n(t) + \mathbb{E}\hat{\varrho}_n(t) - \lambda\varrho(t)\big)^2 \mathrm{d}t \\
&= \int\limits_K \mathrm{Var}\big(\hat{\varrho}_n(t)\big) \mathrm{d}t + \int\limits_K \big(\mathbb{E}\hat{\varrho}_n(t) - \lambda \varrho(t)\big)^2 \mathrm{d}t.
\end{aligned}$$

For the second summand we find

$$\begin{aligned}
b_n^d |W_n| \int\limits_K \big(\mathbb{E}\hat{\varrho}_n(t) - \lambda\varrho(t)\big)^2 \mathrm{d}t &= b_n^d |W_n| \lambda^2 \int\limits_K \left(\int\limits_{\mathbb{R}^d} \big(\varrho(t + b_n z) - \varrho(t)\big) k(z) \mathrm{d}z \right)^2 \mathrm{d}t \\
&= O(b_n^{d+2}|W_n|)
\end{aligned}$$

as $n \to \infty$, due to the Lipschitz-continuity of ϱ in $K \oplus b^o(o, \varepsilon)$ for some $\varepsilon > 0$ and the boundedness of the kernel function. Now we will prove the asymptotic representation

$$b_n^d |W_n| \int\limits_K \operatorname{Var}\big(\hat{\varrho}_n(t)\big) \mathrm{d}t = \lambda \int\limits_K \varrho(t) \mathrm{d}t \int\limits_{\mathbb{R}^d} k^2(x) \mathrm{d}x + O(b_n).$$

Using the representation (4.2) we obtain

$$\begin{aligned}
\operatorname{Var}\big(\hat{\varrho}_n(t)\big) &= \frac{1}{b_n^{2d} |W_n|^2} \\
&\times \Bigg[\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) k^2 \left(\frac{y-x-t}{b_n} \right) \alpha^{(2)}(\mathrm{d}(x,y)) \\
&\quad + \int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(y) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{x-y-t}{b_n} \right) \alpha^{(2)}(\mathrm{d}(x,y)) \\
&\quad + \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{z-x-t}{b_n} \right) \alpha^{(3)}(\mathrm{d}(x,y,z)) \\
&\quad + \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(y) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{z-y-t}{b_n} \right) \alpha^{(3)}(\mathrm{d}(x,y,z)) \\
&\quad + \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{x-z-t}{b_n} \right) \alpha^{(3)}(\mathrm{d}(x,y,z)) \\
&\quad + \int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{y-z-t}{b_n} \right) \alpha^{(3)}(\mathrm{d}(x,y,z)) \\
&\quad + \int\limits_{(\mathbb{R}^d)^4} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{v-z-t}{b_n} \right) [\gamma^{(4)}(\mathrm{d}(x,y,z,v)) \\
&\qquad + \gamma^{(1)}(\mathrm{d}x)\gamma^{(3)}(\mathrm{d}(y,z,v)) + \gamma^{(1)}(\mathrm{d}y)\gamma^{(3)}(\mathrm{d}(x,z,v)) \\
&\qquad + \gamma^{(1)}(\mathrm{d}z)\gamma^{(3)}(\mathrm{d}(x,y,v)) + \gamma^{(1)}(\mathrm{d}v)\gamma^{(3)}(\mathrm{d}(x,y,z)) \\
&\qquad + \gamma^{(2)}(\mathrm{d}(x,z))\gamma^{(2)}(\mathrm{d}(y,v)) + \gamma^{(2)}(\mathrm{d}(x,v))\gamma^{(2)}(\mathrm{d}(y,z)) \\
&\qquad + \gamma^{(2)}(\mathrm{d}(x,z))\gamma^{(1)}(\mathrm{d}y)\gamma^{(1)}(\mathrm{d}v) + \gamma^{(2)}(\mathrm{d}(x,v))\gamma^{(1)}(\mathrm{d}y)\gamma^{(1)}(\mathrm{d}z) \\
&\qquad + \gamma^{(2)}(\mathrm{d}(y,z))\gamma^{(1)}(\mathrm{d}x)\gamma^{(1)}(\mathrm{d}v) + \gamma^{(2)}(\mathrm{d}(y,v))\gamma^{(1)}(\mathrm{d}x)\gamma^{(1)}(\mathrm{d}z)] \Bigg].
\end{aligned}$$

First we consider the two integrals with respect to the second-order factorial moment measure $\alpha^{(2)}$. For the first one we obtain

$$\begin{aligned}
&\frac{1}{b_n^d|W_n|}\int\limits_K\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)k^2\left(\frac{y-x-t}{b_n}\right)\alpha^{(2)}(\mathrm{d}(x,y))\mathrm{d}t \\
&= \frac{\lambda}{b_n^d|W_n|}\int\limits_K\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)k^2\left(\frac{y-t}{b_n}\right)\alpha^{(2)}_{\mathrm{red}}(\mathrm{d}y)\mathrm{d}x\mathrm{d}t \\
&= \lambda\int\limits_K\int\limits_{\mathbb{R}^d} k^2(y)\varrho(b_ny+t)\mathrm{d}y\mathrm{d}t \\
&= \lambda\int\limits_K \varrho(t)\mathrm{d}t\int\limits_{\mathbb{R}^d} k^2(y)\mathrm{d}y + O(b_n)
\end{aligned} \tag{7.7}$$

as $n\to\infty$, due to the Lipschitz-continuity of ϱ in $K\oplus b^o(o,\varepsilon)$ for some $\varepsilon>0$ and the conditions on the boundedness of the kernel function. For the second integral with respect to the second-order factorial moment measure $\alpha^{(2)}$ we have

$$\begin{aligned}
&\frac{1}{b_n^d|W_n|}\int\limits_K\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y)k\left(\frac{y-x-t}{b_n}\right)k\left(\frac{x-y-t}{b_n}\right)\alpha^{(2)}(\mathrm{d}(x,y))\mathrm{d}t \\
&= \frac{\lambda}{b_n^d|W_n|}\int\limits_K\int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y+x)k\left(\frac{y-t}{b_n}\right)k\left(\frac{y+t}{b_n}\right)\alpha^{(2)}_{\mathrm{red}}(\mathrm{d}y)\mathrm{d}x\mathrm{d}t \\
&= \lambda\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{|W_n\cap(W_n-b_ny-t)|}{|W_n|}k(y)k\left(y-\frac{2t}{b_n}\right)\varrho(b_ny+t)\mathbb{1}_K(t)\mathrm{d}y\mathrm{d}t \\
&= b_n^d\lambda\int\limits_K\int\limits_{\mathbb{R}^d}\frac{|W_n\cap(W_n-b_ny-b_nt)|}{|W_n|}k(y)k(y-2t)\varrho(b_ny+b_nt)\mathbb{1}_K(b_nt)\mathrm{d}y\mathrm{d}t \\
&= O(b_n^d)
\end{aligned}$$

as $n\to\infty$. Here we have used the Lipschitz-continuity of ϱ in $K\oplus b^o(o,\varepsilon)$ for some $\varepsilon>0$ as well as the symmetry of the kernel function.

Now we consider the integrals with respect to the third-order factorial moment measure $\alpha^{(3)}$. For the first of these integrals we have

$$\frac{1}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^4}\mathbb{1}_K(t)\mathbb{1}_{W_n}(x)k\left(\frac{y-x-t}{b_n}\right)k\left(\frac{z-x-t}{b_n}\right)\mathrm{d}t\alpha^{(3)}(\mathrm{d}(x,y,z))$$

$$
\begin{aligned}
&= \frac{1}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^4} \mathbb{1}_K(t)\mathbb{1}_{W_n}(x)k\left(\frac{y-x-t}{b_n}\right)k\left(\frac{z-x-t}{b_n}\right)\mathrm{d}t[\gamma^{(3)}(\mathrm{d}(x,y,z)) + \\
&\qquad \lambda\mathrm{d}x\gamma^{(2)}(\mathrm{d}(y,z)) + \lambda\mathrm{d}y\gamma^{(2)}(\mathrm{d}(x,z)) + \lambda\mathrm{d}z\gamma^{(2)}(\mathrm{d}(x,y)) + \lambda^3\mathrm{d}x\mathrm{d}y\mathrm{d}z] \\
&= b_n^d\lambda\int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_K(t)k(y)k(z)c^{(3)}(b_ny+t, b_nz+t)\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
&\quad + b_n^d\lambda^2\int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_K(t)k(y)k(y+z)c^{(2)}(b_nz)\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
&\quad + 2b_n^d\lambda^2\int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_K(t)k(y)k(z)c^{(2)}(b_nz+t)\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
&\quad + b_n^d\lambda^3\int\limits_{(\mathbb{R}^d)^3} \mathbb{1}_K(t)k(y)k(z)\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
&= O(b_n^d)
\end{aligned}
$$

as $n \to \infty$. For deriving this asymptotic order we have used Lebesgue's dominated convergence theorem which is applicable due to $|K| < \infty$, the boundedness assumptions on the kernel function, the Lipschitz-continuity of $c^{(2)}$, and $\|\gamma_{\text{red}}^{(3)}\| < \infty$ (see Remark 2.2.5). By analogous arguments we can show the asymptotic order of the other integrals with respect to the third-order factorial moment measure $\alpha^{(3)}$ to be $O(b_n^d)$, too.

Let us now consider the integrals with respect to the factorial cumulant measures. Due to the finiteness of the total variations of order two and three the asymptotic order of the integrals with respect to $\gamma^{(2)}$ and $\gamma^{(3)}$ is $O(b_n^d)$. The integral with respect to $\gamma^{(4)}$ is

$$
\begin{aligned}
&\frac{1}{b_n^d|W_n|}\int\limits_{(\mathbb{R}^d)^5} \mathbb{1}_K(t)\mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(z)k\left(\frac{y-x-t}{b_n}\right)k\left(\frac{v-z-t}{b_n}\right)\gamma^{(4)}(\mathrm{d}(x,y,z,v))\mathrm{d}t \\
&= b_n^d\lambda\int\limits_{(\mathbb{R}^d)^4} \frac{|W_n\cap(W_n-z)|}{|W_n|}\mathbb{1}_K(t)k(y)k(v)c^{(4)}(b_ny+t, z, b_nv+z+t)\mathrm{d}y\mathrm{d}z\mathrm{d}v\mathrm{d}t.
\end{aligned}
$$

Considering Remark 2.2.5 we find that this integral is of asymptotic order $O(b_n^d)$ due to $\|\gamma_{\text{red}}^{(4)}\| < \infty$, using Lebesgue's dominated convergence theorem. This completes the proof. ■

Lemma 7.1.5 *Let $\Psi \sim P$ be a 4-stationary point process in $\mathbb{R}^d$ with intensity λ and $\|\gamma_{\text{red}}^{(k)}\| < \infty$ for $k = 2, 3, 4$. Let the second-order product density ϱ have bounded and continuous partial derivatives of order s in $K \oplus b^o(o, \varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$. Let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist, and let the kernel function satisfy Condition 5.1.2 $\mathcal{K}(d, s)$.*

Then we have

$$b_n^d |W_n| \mathbb{E} \int_K (\hat{\varrho}_n(t) - \lambda \varrho(t))^2 \mathrm{d}t = \lambda \int_K \varrho(t) \mathrm{d}t \int_{\mathbb{R}^d} k^2(x) \mathrm{d}x + O(b_n) + O(b_n^{d+2s} |W_n|)$$

as $n \to \infty$. If $s = 2$ and, in addition, Condition 5.1.3 $\mathcal{K}^2(d, 2)$ is satisfied, then we have

$$b_n^d |W_n| \mathbb{E} \int_K (\hat{\varrho}_n(t) - \lambda \varrho(t))^2 \mathrm{d}t = \lambda \int_K \varrho(t) \mathrm{d}t \int_{\mathbb{R}^d} k^2(x) \mathrm{d}x + O(b_n^2) + O(b_n^d) + O(b_n^{d+4} |W_n|)$$

as $n \to \infty$.

Proof: By Fubini's theorem we have

$$\begin{aligned} \mathbb{E} \int_K (\hat{\varrho}_n(t) - \lambda \varrho(t))^2 \mathrm{d}t &= \int_K \mathbb{E} \left(\hat{\varrho}_n(t) - \mathbb{E}\hat{\varrho}_n(t) + \mathbb{E}\hat{\varrho}_n(t) - \lambda \varrho(t) \right)^2 \mathrm{d}t \\ &= \int_K \operatorname{Var}(\hat{\varrho}_n(t)) \mathrm{d}t + \int_K (\mathbb{E}\hat{\varrho}_n(t) - \lambda \varrho(t))^2 \mathrm{d}t. \end{aligned}$$

For the second summand we get

$$\int_K (\mathbb{E}\hat{\varrho}_n(t) - \lambda \varrho(t))^2 \mathrm{d}t = \lambda^2 \int_K \left(\int_{\mathbb{R}^d} \big(\varrho(t + b_n z) - \varrho(t) \big) k(z) \mathrm{d}z \right)^2 \mathrm{d}t.$$

Using Taylor's expansion

$$\varrho(t + b_n z) = \varrho(t) + \sum_{i=1}^{s-1} \frac{1}{i!} \left(\frac{\partial}{\partial t_1} b_n z_1 + \ldots + \frac{\partial}{\partial t_d} b_n z_d \right)^i \varrho(t_1, \ldots, t_d) + b_n^s R_s(z, t)$$

with $R_s(z, t) = \frac{1}{s!} \left(\frac{\partial}{\partial t_1} z_1 + \ldots + \frac{\partial}{\partial t_d} z_d \right)^s \varrho(t_1 + \theta b_n z_1, \ldots, t_d + \theta b_n z_d)$ and $\theta = \theta(t) \in (0, 1)$ we have

$$\Big| \int_{\mathbb{R}^d} \big(\varrho(t + b_n z) - \varrho(t) \big) k(z) \mathrm{d}z \Big| = b_n^s \Big| \int_{\mathbb{R}^d} R_s(z, t) k(z) \mathrm{d}z \Big| \leq b_n^s C_0$$

for some $C_0 < \infty$ due to Condition 5.1.2 $\mathcal{K}(d,s)$ and the boundedness of the partial derivatives of order s. This implies

$$\int_K (\mathbb{E}\hat{\varrho}_n(t) - \lambda\varrho(t))^2 \mathrm{d}t = O(b_n^{2s})$$

as $n \to \infty$. Now we will show

$$b_n^d |W_n| \int_K \mathrm{Var}(\hat{\varrho}_n(t))\mathrm{d}t = \lambda \int_K \varrho(t)\mathrm{d}t \int_{\mathbb{R}^d} k^2(x)\mathrm{d}x + O(b_n)$$

as $n \to \infty$; for the case $s = 2$ and with the additional assumption of Condition 5.1.3 $\mathcal{K}^2(d,2)$ we will also show

$$b_n^d |W_n| \int_K \mathrm{Var}(\hat{\varrho}_n(t))\mathrm{d}t = \lambda \int_K \varrho(t)\mathrm{d}t \int_{\mathbb{R}^d} k^2(x)\mathrm{d}x + O(b_n^2) + O(b_n^d).$$

The derivation of the asymptotic order of $b_n^d |W_n| \int_K \mathrm{Var}(\hat{\varrho}_n(t))\mathrm{d}t$ in the proof of Lemma 7.1.4 was solely based on the Lipschitz-continuity in (7.7). Hence we only have to consider the term

$$\lambda \int_K \int_{\mathbb{R}^d} k^2(y)\varrho(b_n y + t)\mathrm{d}y\mathrm{d}t.$$

By Taylor's expansion and the boundedness of the partial derivatives of order s this integral satisfies

$$\lambda \int_K \int_{\mathbb{R}^d} k^2(y)\varrho(b_n y + t)\mathrm{d}y\mathrm{d}t = \lambda \int_K \varrho(t)\mathrm{d}t \int_{\mathbb{R}^d} k^2(y)\mathrm{d}y + O(b_n)$$

as $n \to \infty$. In the case $s = 2$ and under the additional assumption of Condition 5.1.3 $\mathcal{K}^2(d,2)$ we have

$$\lambda \int_K \int_{\mathbb{R}^d} k^2(y)\varrho(b_n y + t)\mathrm{d}y\mathrm{d}t = \lambda \int_K \varrho(t)\mathrm{d}t \int_{\mathbb{R}^d} k^2(y)\mathrm{d}y + O(b_n^2) + O(b_n^d)$$

as $n \to \infty$. This completes the proof. ■

In the following lemma we consider the reflection $\check{K} := \{x \in \mathbb{R}^d : -x \in K\}$ of the set K as well as the convolution of the kernel function k,

$$\tilde{k} : \mathbb{R}^d \to \mathbb{R}, \qquad t \mapsto \int_{\mathbb{R}^d} k(x)k(t-x)\mathrm{d}x,$$

see page 43. The lemma presents an asymptotic representation for the variance of the ISE $I_n(K)$ for the Poisson process.

Lemma 7.1.6 *Let $\Psi \sim P$ be a stationary Poisson process in $\mathbb{R}^d$ with intensity λ. Then we have*

$$\operatorname{Var}\Big(b_n^{d/2}|W_n| \int\limits_K (\hat{\varrho}_n(t) - \lambda^2)^2 \mathrm{d}t\Big) \xrightarrow[n\to\infty]{} 2\lambda^4 \left(|K| + |K \cap \check{K}|\right) \int\limits_{\mathbb{R}^d} \tilde{k}^2(t)\mathrm{d}t.$$

Proof: We use the representation of the cumulant $\Gamma_2(I_n(K)) = (b_n^d|W_n|)^{-4}\mu_{2,0}^{**}$ of the ISE $I_n(K)$ derived in Corollary 7.1.2. The terms of highest order in $\Gamma_2(I_n(K))$, scaled with $b_n^d|W_n|^2$, are

$$\begin{aligned}
&\frac{\lambda^4}{b_n^{3d}|W_n|^2} \int\limits_{(\mathbb{R}^d)^4} f_1(x_1,x_2,x_3,x_4)[f_1(x_1,x_2,x_3,x_4) + f_1(x_3,x_4,x_1,x_2) \\
&\qquad\qquad + f_1(x_2,x_1,x_4,x_3) + f_1(x_4,x_3,x_2,x_1)]\mathrm{d}(x_1,x_2,x_3,x_4) \\
&= 2\frac{\lambda^4}{b_n^d} \int\limits_{(\mathbb{R}^d)^4} \mathbb{1}_K(t_1)\mathbb{1}_K(t_2)k(x)k\left(x + \frac{t_1 - t_2}{b_n}\right)k(y)k\left(y + \frac{t_1 - t_2}{b_n}\right)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}x\mathrm{d}y \\
&\quad + 2\frac{\lambda^4}{b_n^d} \int\limits_{(\mathbb{R}^d)^4} \frac{|W_n \cap (W_n - b_n x - t_1)||W_n \cap (W_n - b_n y - t_2)|}{|W_n|^2} \\
&\qquad\qquad \times \mathbb{1}_K(t_1)\mathbb{1}_K(t_2)k(x)k\left(x + \frac{t_1 + t_2}{b_n}\right)k(y)k\left(y + \frac{t_1 + t_2}{b_n}\right)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}x\mathrm{d}y \\
&= 2\lambda^4 \int\limits_{(\mathbb{R}^d)^4} \mathbb{1}_K(b_n t_1 + t_2)\mathbb{1}_K(t_2)k(x)k(x+t_1)k(y)k(y+t_1)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}x\mathrm{d}y \\
&\quad + 2\lambda^4 \int\limits_{(\mathbb{R}^d)^4} \frac{|W_n \cap (W_n - b_n x - b_n t_1 + t_2)||W_n \cap (W_n - b_n y - t_2)|}{|W_n|^2} \\
&\qquad\qquad \times \mathbb{1}_K(b_n t_1 - t_2)\mathbb{1}_K(t_2)k(x)k(x+t_1)k(y)k(y+t_1)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

For the first integral we have

$$\begin{aligned}
&2\lambda^4 \int\limits_{(\mathbb{R}^d)^4} \mathbb{1}_K(b_n t_1 + t_2)\mathbb{1}_K(t_2)k(x)k(x+t_1)k(y)k(y+t_1)\mathrm{d}t_1\mathrm{d}t_1\mathrm{d}x\mathrm{d}y \\
&\xrightarrow[n\to\infty]{} 2\lambda^4|K| \int\limits_{(\mathbb{R}^d)^3} k(x)k(x+t)k(y)k(y+t)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= 2\lambda^4|K| \int\limits_{\mathbb{R}^d} \tilde{k}^2(t)\mathrm{d}t,
\end{aligned}$$

and for the second integral we find

$$2\lambda^4 \int\limits_{(\mathbb{R}^d)^4} \frac{|W_n \cap (W_n - b_n x - b_n t_1 + t_2)||W_n \cap (W_n - b_n y - t_2)|}{|W_n|^2}$$

$$\mathbb{1}_K(b_n t_1 - t_2)\mathbb{1}_K(t_2)k(x)k(x+t_1)k(y)k(y+t_1)\mathrm{d}t_1\mathrm{d}t_1\mathrm{d}x\mathrm{d}y$$

$$\xrightarrow[n\to\infty]{} \quad 2\lambda^4|K \cap \check{K}| \int\limits_{(\mathbb{R}^d)^3} k(x)k(x+t)k(y)k(y+t)\mathrm{d}x\mathrm{d}y\mathrm{d}t$$

$$= \quad 2\lambda^4|K \cap \check{K}| \int\limits_{\mathbb{R}^d} \tilde{k}^2(t)\mathrm{d}t,$$

where $\check{K} = \{-x \in \mathbb{R}^d : x \in K\}$ is the reflection of K, as introduced above the lemma. The remaining part of the scaled variance $b_n^d|W_n|^2\ \Gamma_2(I_n(K))$ converges to zero. ■

Now we derive an asymptotic representation for the variance of the ISE under some mild mixing conditions.

Lemma 7.1.7 *Let $\Psi \sim P$ be an 8-stationary point process in $\mathbb{R}^d$ with intensity λ and $\|\gamma_{\mathrm{red}}^{(k)}\| < \infty$ for $k = 2, \ldots, 8$. Let the second-order product density ϱ have bounded and continuous partial derivatives of order s in $K \oplus b^o(o,\varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$. Let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist. Furthermore let $b_n^{2s+d}|W_n| \to 0$, and let the kernel function satisfy Condition 5.1.2 $\mathcal{K}(d,s)$.*

Then we have

$$\mathrm{Var}\Big(b_n^{d/2}|W_n| \int\limits_K (\hat{\varrho}_n(t) - \lambda\varrho(t))^2\mathrm{d}t\Big)$$

$$\xrightarrow[n\to\infty]{} \quad 2\lambda^2 \int\limits_{(\mathbb{R}^d)^3} k(x)k(x+t)k(y)k(y+t)\mathrm{d}x\mathrm{d}y\mathrm{d}t \left(\int\limits_K \varrho^2(t)\mathrm{d}t + \int\limits_{K\cap\check{K}} \varrho^2(t)\mathrm{d}t \right)$$

$$= \quad 2\lambda^2 \int\limits_{\mathbb{R}^d} \tilde{k}^2(t)\mathrm{d}t \left(\int\limits_K \varrho^2(t)\mathrm{d}t + \int\limits_{K\cap\check{K}} \varrho^2(t)\mathrm{d}t \right).$$

Proof: We use the representation of the second cumulant of the ISE

$$\Gamma_2(I_n(K)) = \sum_{j=0}^{2} \binom{2}{j} 2^j (b_n^s)^j (b_n^d|W_n|)^{j-4} \mu_{2-j,j}^{**j} \tag{7.8}$$

derived in Lemma 7.1.1. Now we will determine the asymptotic order of $\mu^{**j}_{2-j,j}$, $j = 0, 1, 2$. The highest-order terms in $\mu^{**0}_{2,0}$ are

$$\int\limits_{(\mathbb{R}^d)^4} f_1(x_1,x_2,x_3,x_4)\big[f_1(x_1,x_2,x_3,x_4) + f_1(x_3,x_4,x_1,x_2) + f_1(x_2,x_1,x_4,x_3) + f_1(x_4,x_3,x_2,x_1)\big] \Big[\gamma^{(2)}(\mathrm{d}(x_1,x_2))\gamma^{(2)}(\mathrm{d}(x_3,x_4)) + \gamma^{(2)}(\mathrm{d}(x_1,x_2))\gamma^{(2)}(\mathrm{d}(x_3,x_4)) + \gamma^{(1)}(\mathrm{d}x_1)\gamma^{(1)}(\mathrm{d}x_2)\gamma^{(2)}(\mathrm{d}(x_3,x_4)) + \gamma^{(1)}(\mathrm{d}x_3)\gamma^{(1)}(\mathrm{d}x_4)\gamma^{(2)}(\mathrm{d}(x_3,x_4)) + \gamma^{(1)}(\mathrm{d}x_1)\gamma^{(1)}(\mathrm{d}x_2)\gamma^{(1)}(\mathrm{d}x_3)\gamma^{(1)}(\mathrm{d}x_4)\Big].$$

Combining the factorial cumulant measures to factorial moment measures, see (2.1), and multiplying with the scaling factor $\frac{1}{b_n^{3d}|W_n|^2}$ we obtain

$$\begin{aligned}
&\frac{1}{b_n^{3d}|W_n|^2}\int\limits_{(\mathbb{R}^d)^4} f_1(x_1,x_2,x_3,x_4)[f_1(x_1,x_2,x_3,x_4) + f_1(x_3,x_4,x_1,x_2) \\
&\qquad + f_1(x_2,x_1,x_4,x_3) + f_1(x_4,x_3,x_2,x_1)]\alpha^{(2)}(\mathrm{d}(x_1,x_2))\alpha^{(2)}(\mathrm{d}(x_3,x_4)) \\
= \;& 2\lambda^2 \int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_K(b_nt_1+t_2)\mathbb{1}_K(t_2)\Bigg(\int\limits_{\mathbb{R}^d} k(x)k(x+t_1)\varrho(b_nx+b_nt_1+t_2)\mathrm{d}x\Bigg)^2 \mathrm{d}t_1\mathrm{d}t_2 \\
&+ 2\lambda^2 \int\limits_{(\mathbb{R}^d)^2} \mathbb{1}_K(b_nt_1-t_2)\mathbb{1}_K(t_2) \\
&\times \Bigg(\int\limits_{\mathbb{R}^d} \frac{|W_n\cap(W_n-b_nx+b_nt_1-t_2)|}{|W_n|}k(x)k(x+t_1)\varrho(b_nx+b_nt_1-t_2)\mathrm{d}x\Bigg)^2 \mathrm{d}t_1\mathrm{d}t_2 \\
\xrightarrow[n\to\infty]{} \;& 2\lambda^2\int\limits_K \varrho^2(t)\mathrm{d}t\int\limits_{\mathbb{R}^d}\tilde{k}^2(x)\mathrm{d}x + 2\lambda^2\int\limits_{\check{K}\cap K}\varrho^2(-t)\mathrm{d}t\int\limits_{\mathbb{R}^d}\tilde{k}^2(x)\mathrm{d}x \\
= \;& 2\lambda^2\int\limits_{\mathbb{R}^d}\tilde{k}^2(x)\mathrm{d}x\Bigg(\int\limits_K\varrho^2(t)\mathrm{d}t + \int\limits_{K\cap\check{K}}\varrho^2(t)\mathrm{d}t\Bigg).
\end{aligned}$$

The remaining part of $\mu^{**0}_{2,0}$, scaled with $(b_n^{3d}|W_n|^2)^{-1}$, is of order $O\big(b_n^d + (b_n^d|W_n|)^{-1}\big)$ as $n\to\infty$. For integrals in $\mu^{**0}_{2,0}$ containing an integration with respect to $\gamma^{(5)}$, $\gamma^{(6)}$, $\gamma^{(7)}$ and $\gamma^{(8)}$, this is due to the finiteness of these measures' total variation. For the other integrals one uses the existence of the cumulant densities up to order four and the finiteness of the total variations $\|\gamma^{(k)}_{\mathrm{red}}\|$, $k = 2, 3, 4$. For example, if we do not assume the existence of the fourth-order cumulant density $c^{(4)}$, then the integral

$$\frac{1}{b_n^{3d}|W_n|^2} \int\limits_{(\mathbb{R}^d)^8} f_1(x_1,x_2,x_3,x_4) f_1(x_5,x_6,x_7,x_8) \gamma^{(4)}(\mathrm{d}(x_1,x_2,x_5,x_6)) \gamma^{(4)}(\mathrm{d}(x_3,x_4,x_7,x_8))$$
$$= \frac{1}{b_n^{3d}} \int\limits_{(\mathbb{R}^d)^8} \frac{|W_n \cap (W_n - x_5)||W_n \cap (W_n - x_7)|}{|W_n|^2} \mathbb{1}_K(t_1) \mathbb{1}_K(t_2) \times k\left(\frac{x_2 - t_1}{b_n}\right) k\left(\frac{x_4 - t_1}{b_n}\right) k\left(\frac{x_6 - x_5 - t_2}{b_n}\right) k\left(\frac{x_8 - x_7 - t_2}{b_n}\right) \mathrm{d}t_1 \mathrm{d}t_2 \gamma_{\mathrm{red}}^{(4)}(\mathrm{d}(x_2,x_5,x_6)) \gamma_{\mathrm{red}}^{(4)}(\mathrm{d}(x_4,x_7,x_8))$$

occurring in $(b_n^{3d}|W_n|^2)^{-1}\mu_{2,0}^{**0}$ can only be shown to be of asymptotic order $O(b_n^{-d})$. Showing that the above integral converges to zero hence requires the assumption that the fourth-order cumulant density $c^{(4)}$ exists. By substitution the above integral turns into

$$b_n^d \int\limits_{(\mathbb{R}^d)^8} \frac{|W_n \cap (W_n - x_5)||W_n \cap (W_n - x_7)|}{|W_n|^2} \mathbb{1}_K(t_1) \mathbb{1}_K(t_2) k(x_2) k(x_4) k(x_6) k(x_8) \times c^{(4)}(b_n x_2 + t_1, x_5, b_n x_6 + x_5 + t_2) c^{(4)}(b_n x_4 + t_1, x_7, b_n x_8 + x_7 + t_2) \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}x_2 \mathrm{d}x_4 \mathrm{d}x_5 \mathrm{d}x_6 \mathrm{d}x_7 \mathrm{d}x_8.$$

This term is of order $O(b_n^d)$ as $n \to \infty$, due to $\|\gamma_{\mathrm{red}}^{(4)}\| < \infty$ (see Remark 2.2.5) and since the kernel function is bounded with bounded support. Likewise, the third-order cumulant density $c^{(3)}$ is needed for showing that the integral

$$\frac{1}{b_n^{3d}|W_n|^2} \int\limits_{(\mathbb{R}^d)^6} f_1(x_1,x_2,x_3,x_4) f_1(x_1,x_5,x_3,x_6) \gamma^{(3)}(\mathrm{d}(x_1,x_2,x_5)) \gamma^{(3)}(\mathrm{d}(x_3,x_4,x_6))$$

is of order $O(b_n^d)$ as $n \to \infty$.

Both terms $\mu_{1,1}^{**1}$ and $\mu_{0,2}^{**2}$ are of order $O(b_n^{2d}|W_n|)$ which can be shown using the finiteness of the total variations $\|\gamma_{\mathrm{red}}^{(k)}\|$, $k = 2, \ldots, 6$. Together with the representation (7.8) this leads to the asymptotic representation

$$b_n^d |W_n|^2 \, \Gamma_2(I_n(K)) = 2\lambda^2 \int\limits_{\mathbb{R}^d} \tilde{k}^2(t) \mathrm{d}t \left(\int\limits_K \varrho^2(t) \mathrm{d}t + \int\limits_{K \cap \check{K}} \varrho^2(t) \mathrm{d}t \right) + O(b_n^d) + O(b_n^s) + O(b_n^{2s+d}|W_n|).$$

Now the assumption $b_n^{2s+d}|W_n| \to 0$ implies the claim. ∎

7.1.3 Central limit theorems

In this section we derive a central limit theorem for the integrated squared error of the estimated second-order product density in the setting of Brillinger-mixing point processes. The result will be proved by showing the cumulants of order $k \geq 3$ of the suitably scaled integrated squared error to converge to zero. Since the normalizing factor for the deviation $\big(\hat{\varrho}_n(t) - \lambda\varrho(t)\big)$ is $(b_n^d|W_n|)^{1/2}$ (see Section 6.1) one might expect $b_n^d|W_n|$ as the normalizing factor for the integrated squared error $I_n(K) = \int\limits_K (\hat{\varrho}_n(t) - \lambda\varrho(t))^2 \mathrm{d}t$. However, the asymptotic variance of $I_n(K)$ shows that the suitable normalizing factor is $b_n^{d/2}|W_n|$. Note also that there is an analogous normalizing factor for the integrated squared error of estimated probability densities, see Horváth [37].

Recall the notation $\xrightarrow[n\to\infty]{\mathrm{d}}$ for weak convergence and $\mathrm{N}(\mu, \sigma^2)$ for the univariate normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \geq 0$.

Theorem 7.1.8 *Let $\Psi \sim P$ be a Brillinger-mixing point process in $\mathbb{R}^d$ with intensity λ. Let the second-order product density ϱ have bounded and continuous partial derivatives of order s in $K \oplus b^o(o,\varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$. Further let all cumulant densities $c^{(k)}$, $k \geq 2$, exist. Let the kernel function k satisfy Condition 5.1.2 $\mathcal{K}(d,s)$, and let the bandwidth satisfy $b_n^{2s+d}|W_n| \xrightarrow[n\to\infty]{} 0$.*

Then we have

$$b_n^{d/2}|W_n|(I_n(K) - \mathbb{E}I_n(K)) \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}\left(0, 2\lambda^2 \left(\int\limits_K \varrho^2(t)\mathrm{d}t + \int\limits_{K\cap\check{K}} \varrho^2(t)\mathrm{d}t\right) \int\limits_{\mathbb{R}^d} \tilde{k}^2(t)\mathrm{d}t\right).$$

Proof: The asymptotic variance of $b_n^{d/2} n^d (I_n(K) - \mathbb{E}I_n(K))$ has already been determined in Lemma 7.1.7. We will prove asymptotic normality by showing that the kth cumulant of $b_n^{d/2}|W_n|\big(I_n(K) - \mathbb{E}I_n(K)\big)$ converges to zero for all $k \geq 3$.

In Lemma 7.1.1 we derived a representation of the kth cumulant of $I_n(K)$ by indecomposable and irreducible integrals. Now we will show that the kth cumulant of $b_n^{d/2}|W_n|\big(I_n(K) - \mathbb{E}I_n(K)\big)$ is of order $O\big((b_n^d)^{k/2-1} + b_n^{2s+\frac{k}{2}d}|W_n|\big)$ as $n \to \infty$ for $k \geq 2$. This implies the cumulants of order three and higher to converge to zero.

We will use the representation (7.6) derived in part II of the proof of Lemma 7.1.1 and determine the asymptotic order of the terms $\mu_{k-j,j}^{**j}$ for $j = 0, \dots, k$. It is essential that the integrals in $\mu_{k-j,j}^{**j}$ are neither decomposable nor reducible.

Consider an integral $I_{\mathcal{P},\mathcal{Q}}(.)$ in $\mu^{**j}_{k-j,j}$, $j = 0,\ldots,k$, see (4.3). Let V be the set of integration variables occurring in the integral and define the set of *argument pairs*

$$\begin{aligned}\mathcal{V} \;:=\; \big\{\{v,w\} \subseteq V :\ & \text{the integrand of } I_{\mathcal{P},\mathcal{Q}}(.) \text{ contains} \\ & \text{a term } f_1(v,w,.,.), \text{ a term } f_1(.,.,v,w), \text{ or a term } f_2(v,w)\big\}.\end{aligned}$$

Now we define a *linkage* relation on $\mathcal{V}$. Two argument pairs $\{v,w\},\{x,y\} \in \mathcal{V}$ are said to be *linked* (notation: $\{v,w\} \smile \{x,y\}$) if at least one of the following conditions is satisfied:

(i) The argument pairs $\{v,w\},\{x,y\}$ have a common element, that is, $\{v,w\} \cap \{x,y\} \neq \emptyset$.

(ii) The integral $I_{\mathcal{P},\mathcal{Q}}(.)$ involves an integration $\gamma^{(i)}(\mathrm{d}(v_1,\ldots,v_i))$ for some $i \geq 2$ and some $v_1,\ldots,v_i \in V$ such that $\{v,w\} \cap \{v_1,\ldots,v_i\} \neq \emptyset$ and $\{x,y\} \cap \{v_1,\ldots,v_i\} \neq \emptyset$.

(iii) The integral $I_{\mathcal{P},\mathcal{Q}}(.)$ involves an integration $\gamma^{(1)}(\mathrm{d}v_0)\,\gamma^{(i)}(\mathrm{d}(v_1,\ldots,v_i))$ for some $i \geq 1$ and $v_0,\ldots,v_i \in V$ such that $\{v,w\} \cap \{v_0,\ldots,v_i\} \neq \emptyset$ and $\{x,y\} \cap \{v_0,\ldots,v_i\} \neq \emptyset$.

Note that the relation $\smile$ is reflexive and symmetric.

The maximal order of each integration of linked argument pairs with ℓ arguments is $O\big((b_n^d)^{\lceil \frac{\ell}{2} \rceil}|W_n|\big)$. After reduction of the factorial cumulant measures we make use of the existence of the cumulant densities. There are at least $\lceil \frac{\ell}{2} \rceil$ kernel functions k. By substitution of the arguments of the kernel functions k we get a factor b_n^d for each function. Furthermore there is exactly one variable occurring only in the indicator functions $\mathbb{1}_{W_n}$ (this is due to the integral's indecomposability and irreducibility). Integration over this variable yields the factor $|W_n|$. Because of the boundedness of the total variations the integrals over the cumulant densities are also bounded. Therefore we obtain the order $O((b_n^d)^{\lceil \frac{\ell}{2} \rceil}|W_n|)$ for each integration over ℓ linked argument pairs. Note that without the existence of the cumulant densities one can only derive the order $O(|W_n|)$. For determining the order of the whole integral we also have to take into account that some of the arguments t of the functions $\mathbb{1}_K(t)$ can be substituted, where each substitution produces a factor b_n^d. Thus the highest-order terms are those in which as many argument pairs as possible are not linked.

We will now use the concept of a *cyclic linkage*. Consider a product

$$\prod_{i=1}^{m} f_1(p_i,q_i)$$

occurring in the integrand of $I_{\mathcal{P},\mathcal{Q}}(.)$ and involving the argument pairs $p_1, q_1, \dots, p_m, q_m \in \mathcal{V}$. (Here $f_1(p,q)$ with argument pairs $p = \{u,v\}$, $q = \{x,y\}$ is understood as $f_1(u,v,x,y)$.) This product is said to be *cyclically linked* if there are an enumeration $r_1, \dots, r_{2m}$ of $\{p_1, q_1, \dots, p_m, q_m\}$ and a permutation π of $\{1, \dots, m\}$ such that $\{r_{2i-1}, r_{2i}\} = \{p_{\pi(i)}, q_{\pi(i)}\}$ for all $i = 1, \dots, m$ and such that

$$r_{2i} \smile r_{2i+1 \bmod 2m} \quad \text{for all } i \in \{1, \dots, m\}$$

is an exhaustive list of the links between the argument pairs $p_1, q_1, \dots, p_m, q_m$.

We will now investigate the highest-order terms in $\mu^{**j}_{k-j,j}$ for $j = 0, \dots, k$.

Let $j = 0$. Then the integrands of all highest-order integrals in $\mu^{**0}_{k,0}$ are cyclically linked. As an example consider the integral

$$\int\limits_{(\mathbb{R}^d)^{2k}} \prod_{\substack{a=1\\ a \text{ odd}}}^{2k-3} f_1(x_a, x_{a+1}, x_{a+2}, x_{a+3}) f_1(x_{2k-1}, x_{2k}, x_1, x_2) \prod_{\substack{a=1\\ a \text{ odd}}}^{2k-1} \gamma^{(2)}(\mathrm{d}(x_a, x_{a+1})).$$

By disintegration and substitution we get

$$\begin{aligned}
&\lambda^k |W_n|^k \int\limits_{(\mathbb{R}^d)^k} \prod_{\substack{a=1\\ a \text{ odd}}}^{2k-3} \int\limits_K k\left(\frac{x_{a+1} - t_a}{b_n}\right) k\left(\frac{x_{a+3} - t_a}{b_n}\right) \mathrm{d}t_a \\
&\qquad \times \int\limits_K k\left(\frac{x_{2k} - t_{2k-1}}{b_n}\right) k\left(\frac{x_2 - t_{2k-1}}{b_n}\right) \mathrm{d}t_{2k-1} \prod_{\substack{a=1\\ a \text{ odd}}}^{2k-1} \gamma^{(2)}_{\mathrm{red}}(\mathrm{d}x_{a+1}) \\
= \;& (b_n^d)^k |W_n|^k \lambda^k \int\limits_{(\mathbb{R}^d)^{2k}} \prod_{\substack{a=1\\ a \text{ odd}}}^{2k-3} \mathbb{1}_K(t_a) k(x_{a+1}) k\left(x_{a+3} + \frac{t_{a+2} - t_a}{b_n}\right) \mathbb{1}_K(t_{2k-1}) k(x_{2k}) \\
&\qquad \times k\left(x_2 + \frac{t_1 - t_{2k-1}}{b_n}\right) \prod_{\substack{a=1\\ a \text{ odd}}}^{2k-1} c^{(2)}(b_n x_{a+1} + t_a) \\
&\qquad\qquad \mathrm{d}x_2 \mathrm{d}x_4 \dots \mathrm{d}x_{2k} \mathrm{d}t_1 \mathrm{d}t_3 \dots \mathrm{d}t_{2k-1}.
\end{aligned}$$

By substituting $\tilde{t}_{a+2} = \frac{t_{a+2}-t_a}{b_n}$, $a = 3, 5, \ldots, 2k-1$, we see that this is equal to

$$(b_n^d)^{2k-1}|W_n|^k\lambda^k \int\limits_{(\mathbb{R}^d)^{2k}} \prod_{\substack{a=1\\a \text{ odd}}}^{2k-3} \mathbb{1}_K(b_n t_{a+2}+t_1)k(x_{a+1})k(x_{a+3}+t_{a+2}-t_a)\mathbb{1}_K(t_1)k(x_{2k})$$
$$\times k(x_2-t_{2k-1}) \prod_{\substack{a=3\\a \text{ odd}}}^{2k-1} c^{(2)}(b_n x_{a+1}+b_n t_a+t_1)c^{(2)}(b_n x_2+t_1)$$
$$\mathrm{d}x_2\mathrm{d}x_4\ldots\mathrm{d}x_{2k}\mathrm{d}t_1\mathrm{d}t_3\ldots\mathrm{d}t_{2k-1}.$$

The second-order cumulant density $c^{(2)}$ is continuous since the second-order product density is continuous in $K \oplus b^o(o,\varepsilon)$ for some $\varepsilon > 0$. Hence the above-mentioned integral is of order $O\big((b_n^d)^{2k-1}|W_n|^k\big)$. Analogous arguments apply to the other terms in $\mu_{k,0}^{**}$.

Now let $j = 1$. Then each integrand of a highest-order term in $\mu_{k-1,1}^{**1}$ is a product of two parts: First, a cyclically linked product of $k-1$ instances of f_1, and second, one instance of the function $\tilde{f}_2$ whose argument pair is linked to at least one argument pair from the first part. One of these highest-order integrals is

$$\int\limits_{(\mathbb{R}^d)^{2k-2}} \prod_{\substack{a=1\\a \text{ odd}}}^{2k-5} f_1(x_a,x_{a+1},x_{a+2},x_{a+3})f_1(x_{2k-3},x_{2k-2},x_1,x_2)\tilde{f}_2(x_1,x_2) \prod_{\substack{a=1\\a \text{ odd}}}^{2k-3} \alpha^{(2)}(\mathrm{d}(x_a,x_{a+1})).$$

By applying disintegration and substitution as above and taking advantage of the boundedness of the partial derivatives of order s of ϱ in $K \oplus b^o(o,\varepsilon)$ for some $\varepsilon > 0$, one finds the above-mentioned integral to be order $O\big((b_n^d)^{2k-2}|W_n|^{k-1}\big)$ as $n \to \infty$. Analogous arguments apply to the remaining integrals.

Next let $j = 2$. Then each integrand of a highest-order term in $\mu_{k-2,2}^{**2}$ is a product of two parts: First, a cyclically linked product of $k-2$ instances of f_1, and second, a product of two instances of the function $\tilde{f}_2$ whose argument pairs are both linked to argument pairs from the first part. For example, the integral

$$\int\limits_{(\mathbb{R}^d)^{2k-4}} \prod_{\substack{a=1\\a \text{ odd}}}^{2k-7} f_1(x_a,x_{a+1},x_{a+2},x_{a+3})\tilde{f}_2(x_{2k-5},x_{2k-4})\tilde{f}_2(x_1,x_2) \prod_{\substack{a=1\\a \text{ odd}}}^{2k-5} \alpha^{(2)}(\mathrm{d}(x_a,x_{a+1}))$$

is of asymptotic order $O((b_n^d)^{2k-2}|W_n|^{k-1})$ and hence one of the highest-order terms for the case $j = 2$.

For $j = 3, \ldots, k-1$ one obtains the asymptotic order $O((b_n^d)^{2k-j}|W_n|^{k-j+1})$ by analogous considerations.

Finally, in the case $j = k$ all integrands of the integrals in $\mu_{0,k}^{**k}$ are products of k instances of the function $\tilde{f}_2$. Since these integrals are indecomposable the argument pairs occurring in the integrand can be enumerated as $p_1, \ldots, p_k$ such that $p_i \smile p_{i+1}$ for $i = 1, \ldots, k-1$. Hence the term $\mu_{0,k}^{**k}$ is of order $O((b_n^d)^k |W_n|)$.

Altogether we have

$$\mu_{k-j,j}^{**j} = O((b_n^d)^{2k-j-1} |W_n|^{k-j}) \qquad \text{for } j = 0, 1,$$

and

$$\mu_{k-j,j}^{**j} = O((b_n^d)^{2k-j} |W_n|^{k-j+1}) \qquad \text{for } j = 2, \ldots, k.$$

Together with (7.6) the kth cumulant hence satisfies

$$\Gamma_k(I_n(K)) = O(b_n^{-d} |W_n|^{-k}) + 2kO(b_n^{s-d} |W_n|^{-k}) + \sum_{j=2}^{k} \binom{k}{j} 2^j O(b_n^{sj} |W_n|^{1-k}).$$

As a result the kth cumulant of $b_n^{d/2} |W_n| \big(I_n(K) - \mathbb{E} I_n(K)\big)$ is of order $O\big((b_n^d)^{k/2-1} + b_n^{2s+\frac{k}{2}d} |W_n|\big)$ for $k \geq 2$. Due to the assumption $b_n^{2s+d} |W_n| \to 0$ the kth cumulant of $b_n^{d/2} |W_n| (I_n(K) - \mathbb{E} I_n(K))$ converges to zero for every $k \geq 3$. This proves normal convergence. ■

The following theorems state versions of the above central limit theorem for Poisson processes and Neyman-Scott processes.

Theorem 7.1.9 *Let $\Psi \sim \Pi_\lambda$ be a stationary Poisson process in $\mathbb{R}^d$ with intensity λ. Then we have*

$$b_n^{d/2} |W_n| \big(I_n(K) - \mathbb{E} I_n(K)\big) \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}\left(0, 2\lambda^4 \left(|K| + |K \cap \check{K}|\right) \int_{\mathbb{R}^d} \tilde{k}^2(t)\mathrm{d}t\right).$$

Proof: Using Corollary 7.1.2 it is easily seen that for the Poisson process the proof of Theorem 7.1.8 requires neither the assumptions of Condition 5.1.2 $\mathcal{K}(d, s)$ nor the assumption $b_n^{2s+d} |W_n| \xrightarrow[n\to\infty]{} 0$ on the bandwidth. The remaining parts of the proof are analogous to that of Theorem 7.1.8. ■

Theorem 7.1.10 *Let $\Psi \sim P$ be a Neyman-Scott process in $\mathbb{R}^d$ with intensity λ_{p} of the underlying Poisson process, random number of points $M \sim G$ in the typical cluster, and let f be the Lebesgue density of the distribution of the difference of two independent random points of the typical cluster (see page 13). The second-order product density ϱ is then given by (2.6). Let*

$\lambda = \lambda_{\mathrm{p}}\,\mathbb{E}M$ denote the intensity of Ψ. Let $\mathbb{E}M^k < \infty$ for all $k \geq 1$. Let f have bounded and continuous partial derivatives of order s in $K \oplus b^o(o,\varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$. Let the kernel function k satisfy Condition 5.1.2 $\mathcal{K}(d,s)$, and let $b_n^{2s+d}|W_n| \xrightarrow[n\to\infty]{} 0$.

Then we have

$$b_n^{d/2}|W_n|\big(I_n(K) - \mathbb{E}I_n(K)\big) \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}\left(0, 2\lambda^2\left(\int_K \varrho^2(t)\mathrm{d}t + \int_{K\cap\check{K}} \varrho^2(t)\mathrm{d}t\right)\int_{\mathbb{R}^d}\tilde{k}^2(t)\mathrm{d}t\right).$$

Proof: The assumption $\mathbb{E}M^k < \infty$ for all $k \geq 1$ implies Ψ to be Brillinger-mixing, see page 24. By equation (3.6) it can be seen that the existence of the cumulant densities of order two and higher is entailed by the existence of the density f. The assumptions on f imply boundedness and continuity of the partial derivatives of order s of ϱ in $K \oplus b^o(o,\varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$. Thus all assumptions of Theorem 7.1.8 are satisfied, and the claim follows. ■

An alternative approach for proving central limit theorems for the ISE in the setting of Poisson cluster processes might be the following. Taking advantage of the m-dependence of Poisson cluster processes with bounded cluster radius one may apply a truncation method to Poisson cluster processes with unbounded cluster radius (see Heinrich [25], Heinrich [26], and Heinrich and Werner [35]). Then the techniques for degenerate U-statistics used in Hall [22] and Fan and Li [16] might be put to use. This approach might also be useful for the above central limit theorems in the setting of absolutely regular point processes.

7.2 Central limit theorems for the integrated squared error of the pair correlation function estimator

In this chapter we derive asymptotic representations for the mean and the variance of the integrated squared error of the pair correlation function estimator under mild mixing conditions. We will then present central limit theorems for the integrated squared error of the pair correlation function estimator for Brillinger-mixing point processes and Poisson cluster processes. Since all results can be proved analogously to those for the second-order product density estimator in the previous section we will focus on some proofs for the above-mentioned asymptotic representations for the mean and the variance. These results are derived in the first section. In the second section we present the central limit theorems.

All results given in this chapter are valid for the estimators $\hat{g}_n$, $\hat{g}_{n,2}$, $\tilde{g}_n$ and $\tilde{g}_{n,2}$, see Section 5.3, despite the fact that we consider only one of these estimators in our proofs.

7.2.1 Asymptotic representation of the mean and the variance

In this section we derive asymptotic representations of the mean and the variance of the ISE of the estimated pair correlation function defined as

$$I_n(K) := \int\limits_K (\hat{g}_n(r) - \lambda^2 g(r))^2 \mathrm{d}r,$$

where $K \in \mathfrak{B}((0,\infty))$, $|K| > 0$, is a bounded set. Compared with the results for the second-order product density, the symmetry of the Euclidean norm $\|.\|$ causes an extra factor 2 for the mean and an extra factor 4 for the variance of the ISE $I_n(K)$.

The following lemma presents the desired asymptotic representation for the mean of the ISE $I_n(K)$ in the setting of Poisson processes.

Lemma 7.2.1 *Let $\Psi \sim \Pi_\lambda$ be a stationary Poisson process in $\mathbb{R}^d$ with intensity λ. Then we have*

$$b_n|W_n|\mathbb{E}\int\limits_K \left(\tilde{g}_n(r) - \lambda^2\right)^2 \mathrm{d}r = 2\lambda^2 \int\limits_K \frac{1}{d\omega_d r^{d-1}}\mathrm{d}r \int\limits_{\mathbb{R}} k^2(x)\mathrm{d}x + O(b_n)$$

as $n \to \infty$.

Proof: For the Poisson process the pair correlation function g is the constant λ^2. Hence the estimator for the pair correlation function satisfies

$$\mathbb{E}\,\tilde{g}_n(r) \quad = \quad \lambda^2 \int\limits_{-r/b_n}^{\infty} k(s)g(r+b_n s)\mathrm{d}s = \lambda^2 \int\limits_{-r/b_n}^{\infty} k(s)\mathrm{d}s,$$

see the proof of Theorem 6.1.4. Since the kernel function k has bounded support and integrates to one this entails $\mathbb{E}\,\tilde{g}_n(r) = \lambda^2$ for $r \in (0,\infty)$ and sufficiently large n. By Fubini's theorem the mean of the ISE is

$$\mathbb{E}\int\limits_K (\tilde{g}_n(r) - \lambda^2)^2 \mathrm{d}r \quad = \quad \int\limits_K \mathrm{Var}(\tilde{g}_n(r))\mathrm{d}r$$

for sufficiently large n. For the Poisson process we have $\gamma^{(k)} \equiv 0$ for $k \geq 2$, which implies $\alpha^{(k)}(\mathrm{d}(x_1,\ldots,x_k)) = \lambda^k \mathrm{d}(x_1,\ldots,x_k)$. Hence the variance satisfies

$$\mathrm{Var}\left(\sum_{x,y\in\Psi}^{*} h_1(x,y)\right) \quad = \quad \int\limits_{(\mathbb{R}^d)^2} h_1(x,y)[h_1(x,y) + h_1(y,x)]\lambda^2 \mathrm{d}(x,y)$$

$$+ \int_{(\mathbb{R}^d)^3} h_1(x,y)[h_1(x,z) + h_1(y,z) + h_1(z,x) + h_1(z,y)]\lambda^3 \mathrm{d}(x,y,z),$$

see equation (4.2) with

$$h_1(x,y) = h_2(x,y) = \frac{\mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y) k\left(\frac{\|y-x\|-r}{b_n}\right)}{|(W_n - x)\cap(W_n - y)|\,\|y-x\|^{d-1}}.$$

Due to $h_1(x,y) = h_1(y,x)$ this reduces to

$$\mathrm{Var}\left(\sum_{x,y\in\Psi}^{*} h_1(x,y)\right) = 2\int_{(\mathbb{R}^d)^2} h_1^2(x,y)\lambda^2 \mathrm{d}(x,y) + 4\int_{(\mathbb{R}^d)^3} h_1(x,y)h_1(x,z)\lambda^3 \mathrm{d}(x,y,z).$$

This yields

$$\begin{aligned}
&\mathrm{Var}(\tilde{g}_n(r))\\
&= \frac{1}{b_n^2 d^2 \omega_d^2}\mathrm{Cov}\left(\sum_{x,y\in\Psi}^{*} \frac{\mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y) k\left(\frac{\|y-x\|-r}{b_n}\right)}{|(W_n - x)\cap(W_n - y)|\,\|y-x\|^{d-1}},\right.\\
&\qquad\qquad \left.\sum_{x,y\in\Psi}^{*} \frac{\mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y) k\left(\frac{\|y-x\|-r}{b_n}\right)}{|(W_n - x)\cap(W_n - y)|\,\|y-x\|^{d-1}}\right)\\
&= \frac{2\lambda^2}{b_n^2 d^2 \omega_d^2}\int_{(\mathbb{R}^d)^2} \frac{\mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y) k^2\left(\frac{\|y-x\|-r}{b_n}\right)}{|(W_n - x)\cap(W_n - y)|^2\,\|y-x\|^{2(d-1)}}\,\mathrm{d}(x,y)\\
&\quad + \frac{4\lambda^3}{b_n^2 d^2 \omega_d^2}\int_{(\mathbb{R}^d)^3} \frac{\mathbb{1}_{W_n}(x)\mathbb{1}_{W_n}(y)\mathbb{1}_{W_n}(z) k\left(\frac{\|y-x\|-r}{b_n}\right) k\left(\frac{\|z-x\|-r}{b_n}\right)}{|(W_n - x)\cap(W_n - y)|\,|(W_n - x)\cap(W_n - z)|}\\
&\qquad\qquad \times \frac{1}{\|y-x\|^{d-1}\,\|z-x\|^{d-1}}\,\mathrm{d}(x,y,z)\\
&= \frac{2\lambda^2}{b_n^2 d^2 \omega_d^2}\int_{\mathbb{R}^d} \frac{k^2\left(\frac{\|z\|-r}{b_n}\right)}{|W_n \cap (W_n - z)|\,\|z\|^{2(d-1)}}\,\mathrm{d}z\\
&\quad + \frac{4\lambda^3}{b_n^2 d^2 \omega_d^2}\int_{(\mathbb{R}^d)^2} \frac{|W_n\cap(W_n - v)\cap(W_n - w)| k\left(\frac{\|v\|-r}{b_n}\right) k\left(\frac{\|w\|-r}{b_n}\right)}{|W_n\cap(W_n - v)|\,|W_n\cap(W_n - w)|\,\|v\|^{d-1}\,\|w\|^{d-1}}\,\mathrm{d}(v,w)
\end{aligned}$$

$$= \frac{2\lambda^2}{b_n^2 d^2 \omega_d^2} \int_0^\infty \int_{S^{d-1}} \frac{k^2\left(\frac{s-r}{b_n}\right)}{|W_n \cap (W_n - sx)|\, s^{d-1}} \mathcal{H}_{d-1}(\mathrm{d}x)\mathrm{d}s$$

$$+ \frac{4\lambda^3}{b_n^2 d^2 \omega_d^2} \int_0^\infty \int_0^\infty \int_{S^{d-1}} \int_{S^{d-1}} \frac{|W_n \cap (W_n - sy) \cap (W_n - tz)| k\left(\frac{s-r}{b_n}\right) k\left(\frac{t-r}{b_n}\right)}{|W_n \cap (W_n - sy)|\,|W_n \cap (W_n - tz)|}$$

$$\mathcal{H}_{d-1}(\mathrm{d}y)\mathcal{H}_{d-1}(\mathrm{d}z)\mathrm{d}s\,\mathrm{d}t$$

$$= \frac{2\lambda^2}{b_n d^2 \omega_d^2} \int_{-r/b_n}^\infty \int_{S^{d-1}} \frac{k^2(s)}{|W_n \cap (W_n - (b_n s + r)x)|\,(b_n s + r)^{d-1}} \mathcal{H}_{d-1}(\mathrm{d}x)\mathrm{d}s$$

$$+ \frac{4\lambda^3}{d^2 \omega_d^2} \int_{-r/b_n}^\infty \int_{-r/b_n}^\infty \int_{S^{d-1}} \int_{S^{d-1}} \frac{|W_n \cap (W_n - (b_n s + r)y) \cap (W_n - (b_n t + r)z)| k(s)k(t)}{|W_n \cap (W_n - (b_n s + r)y)|\,|W_n \cap (W_n - (b_n t + r)z)|}$$

$$\mathcal{H}_{d-1}(\mathrm{d}y)\mathcal{H}_{d-1}(\mathrm{d}z)\mathrm{d}s\,\mathrm{d}t.$$

Here $\mathcal{H}_{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure, while S^{d-1} is the $(d-1)$-dimensional unit sphere. Due to $|K| < \infty$, the boundedness of the kernel function, the boundedness of its support, and $\mathcal{H}_{d-1}(S^{d-1}) = d\omega_d$, Lebesgue's dominated convergence theorem yields

$$b_n |W_n| \int_K \mathrm{Var}(\tilde{g}_n(r))\mathrm{d}r = 2\lambda^2 \int_K \frac{1}{d\omega_d r^{d-1}} \mathrm{d}r \int_{\mathbb{R}} k^2(x)\mathrm{d}x + O(b_n)$$

as $n \to \infty$. This completes the proof. ■

The following two lemmata present an asymptotic representation of the mean of the ISE $I_n(K)$. In the first lemma we assume the pair correlation function to be Lipschitz-continuous, whereas the second lemma is based on the assumption that the derivative of order s of the pair correlation function is bounded and continuous for some $s \geq 1$. Compared to the first assumption the latter assumption will be seen to imply a faster convergence when $s \geq 2$.

Lemma 7.2.2 *Let $\Psi \sim P$ be a 4-stationary point process in $\mathbb{R}^d$ with intensity λ and $\|\gamma_{\mathrm{red}}^{(k)}\| < \infty$ for $k = 2,3,4$. Let the pair correlation function g be Lipschitz-continuous in $K \oplus (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist.*

Then we have

$$b_n |W_n| \mathbb{E} \int_K \left(\tilde{g}_n(r) - \lambda^2 g(r)\right)^2 \mathrm{d}r = 2\lambda^2 \int_K \frac{g(r)}{d\omega_d r^{d-1}} \mathrm{d}r \int_{\mathbb{R}} k^2(x)\mathrm{d}x + O(b_n) + O(b_n^3 |W_n|)$$

as $n \to \infty$.

Proof: By Fubini's theorem we have

$$\mathbb{E}\int_K \left(\tilde{g}_n(r)-\lambda^2 g(r)\right)^2 \mathrm{d}r = \int_K \mathbb{E}\left(\tilde{g}_n(r)-\mathbb{E}\tilde{g}_n(r)+\mathbb{E}\tilde{g}_n(r)-\lambda^2 g(r)\right)^2 \mathrm{d}r$$
$$= \int_K \mathrm{Var}(\tilde{g}_n(r))\mathrm{d}r + \int_K (\mathbb{E}\tilde{g}_n(r)-\lambda^2 g(r))^2 \mathrm{d}r.$$

For the second summand we find

$$b_n|W_n|\int_K \left(\mathbb{E}\tilde{g}_n(r)-\lambda^2 g(r)\right)^2 \mathrm{d}r = b_n|W_n|\lambda^2\int_K \left(\int_{-r/b_n}^{\infty} k(s)g(r+b_n s)\mathrm{d}s\right)^2 \mathrm{d}r$$
$$= O(b_n^3|W_n|)$$

as $n\to\infty$ due to the Lipschitz-continuity of g in $K\oplus(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$ and the boundedness of the kernel function. The asymptotic representation

$$b_n|W_n|\int_K \mathrm{Var}(\tilde{g}_n(r))\mathrm{d}r = 2\lambda^2\int_K \frac{g(r)}{d\omega_d r^{d-1}}\mathrm{d}r\int_{\mathbb{R}} k^2(x)\mathrm{d}x + O(b_n)$$

can be shown as in Lemma 7.1.4, where the differences regarding the pair correlation function can be handled as in Lemma 7.2.1. ■

The following result can be proved analogously to Lemma 7.1.5.

Lemma 7.2.3 *Let $\Psi\sim P$ be a 4-stationary point process in $\mathbb{R}^d$ with intensity λ and $\left\|\gamma_{\mathrm{red}}^{(k)}\right\|<\infty$ for $k=2,3,4$. Let the derivative of order s of the pair correlation function g be bounded and continuous in $K\oplus(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$ and some $s\geq 1$. Let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist, and let the kernel function satisfy Condition 5.1.2 $\mathcal{K}(1,s)$.*

Then we have

$$b_n|W_n|\mathbb{E}\int_K (\tilde{g}_n(r)-\lambda^2 g(r))^2\mathrm{d}r = 2\lambda^2\int_K \frac{g(r)}{d\omega_d r^{d-1}}\mathrm{d}r\int_{\mathbb{R}} k^2(x)\mathrm{d}x + O(b_n) + O(b_n^{1+2s}|W_n|)$$

as $n\to\infty$. ■

Recall the convolution

$$\tilde{k} : \mathbb{R} \to \mathbb{R}, \qquad t \mapsto \int_{\mathbb{R}} k(x)k(t-x)\mathrm{d}x$$

of the kernel function k, see page 43. The following lemma gives an asymptotic representation for the variance of the ISE $I_n(K)$ in the setting of Poisson processes.

Lemma 7.2.4 *Let $\Psi \sim \Pi_\lambda$ be a stationary Poisson process in $\mathbb{R}^d$ with intensity λ. Then we have*

$$\mathrm{Var}\left(b_n^{1/2} |W_n| \int_K \left(\hat{g}_n(r) - \lambda^2\right)^2 \mathrm{d}r \right) \xrightarrow[n\to\infty]{} 8\lambda^4 \int_K \left(\frac{1}{d\omega_d r^{d-1}}\right)^2 \mathrm{d}r \int_{\mathbb{R}} \tilde{k}^2(x)\mathrm{d}x.$$

Proof: In Corollary 7.1.2 we found the representation of the second cumulant $\Gamma_2(I_n(K)) = (b_n^d|W_n|)^{-4}\mu_{2,0}^{**}$, where $I_n(K)$ denoted the ISE of the estimated second-order product density. Now $I_n(K)$ denotes the ISE of the estimated pair correlation function, and the integrals in $\mu_{2,0}^{**}$ involve the functions

$$f_1 : (\mathbb{R}^d)^4 \to \mathbb{R}, \qquad (x_1,x_2,x_3,x_4) \mapsto \frac{\mathbb{1}_{W_n}(x_1)\mathbb{1}_{W_n}(x_3)\mathbb{1}_{\{x_1\neq x_2, x_3\neq x_4\}}}{d^2\,\omega_d^2\,\|x_2-x_1\|^{d-1}\|x_4-x_3\|^{d-1}} \times \int_K k\left(\frac{\|x_2-x_1\|-r}{b_n}\right) k\left(\frac{\|x_4-x_3\|-r}{b_n}\right)\mathrm{d}r,$$

and

$$f_2 : (\mathbb{R}^d)^2 \to \mathbb{R}, \qquad (x_1,x_2) \mapsto \frac{\mathbb{1}_{W_n}(x_1)\mathbb{1}_{\{x_1\neq x_2\}}}{d\,\omega_d\,\|x_2-x_1\|^{d-1}} \int_K k\left(\frac{\|x_2-x_1\|-r}{b_n}\right)\lambda^2 g(r)\mathrm{d}r,$$

instead of the functions f_1, f_2 from Lemma 7.1.1. Furthermore we have the constants $C_1 := (b_n|W_n|)^{-2}$ and $C_2 := -2(b_n|W_n|)^{-1}$ instead of C_1, C_2 from Lemma 7.1.1. Given these adaptations arguments analogous to those in Lemma 7.1.1 and Corollary 7.1.2 yield the representation $\Gamma_2(I_n(K)) = (b_n|W_n|)^{-4}\mu_{2,0}^{**}$.

The definitions of decomposability and irreducibility of integrals are adapted in a straightforward manner. The terms of highest order in $\Gamma_2(I_n(K))$, scaled with $b_n|W_n|^2$, are

$$\begin{aligned}\frac{\lambda^4}{b_n^3|W_n|^2} \int_{(\mathbb{R}^d)^4} & f_1(x_1,x_2,x_3,x_4)[f_1(x_1,x_2,x_3,x_4) + f_1(x_1,x_2,x_4,x_3) \\ & + f_1(x_2,x_1,x_3,x_4) + f_1(x_2,x_1,x_4,x_3) + f_1(x_3,x_4,x_1,x_2) \\ & + f_1(x_3,x_4,x_2,x_1) + f_1(x_4,x_3,x_1,x_2) + f_1(x_4,x_3,x_2,x_1)]\mathrm{d}(x_1,x_2,x_3,x_4).\end{aligned}$$

Due to symmetry this equals

$$\frac{8\lambda^4}{b_n^3|W_n|^2}\int\limits_{(\mathbb{R}^d)^4} f_1^2(x_1,x_2,x_3,x_4)\mathrm{d}(x_1,x_2,x_3,x_4)$$

$$= \frac{8\lambda^4}{b_n^3}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{\mathbb{1}_K(r_1)\mathbb{1}_K(r_2)}{d^4\omega_d^4\|x\|^{2(d-1)}\|y\|^{2(d-1)}}$$

$$\times k\left(\frac{\|x\|-r_1}{b_n}\right)k\left(\frac{\|y\|-r_1}{b_n}\right)k\left(\frac{\|x\|-r_2}{b_n}\right)k\left(\frac{\|y\|-r_2}{b_n}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r_1\,\mathrm{d}r_2$$

$$= \frac{8\lambda^4}{b_n}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\frac{\mathbb{1}_{(-r_1/b_n,\infty)}(s)\mathbb{1}_{(-r_1/b_n,\infty)}(t)\mathbb{1}_K(r_1)\mathbb{1}_K(r_2)}{d^2\omega_d^2(b_ns+r_1)^{d-1}(b_nt+r_1)^{d-1}}$$

$$\times k(s)k(t)k\left(s+\frac{r_1-r_2}{b_n}\right)k\left(t+\frac{r_1-r_2}{b_n}\right)\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}r_1\,\mathrm{d}r_2$$

$$= 8\lambda^4\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\frac{\mathbb{1}_{(-x-r_2/b_n,\infty)}(s)\mathbb{1}_{(-x-r_2/b_n,\infty)}(t)\mathbb{1}_K(b_nx+r_2)\mathbb{1}_K(r_2)}{d^2\omega_d^2(b_ns+b_nx+r_2)^{d-1}(b_nt+b_nx+r_2)^{d-1}}$$

$$\times k(s)k(t)k(s+x)k(t+x)\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}r_2$$

which converges to

$$8\lambda^4\int\limits_K\left(\frac{1}{d\omega_d r^{d-1}}\right)^2\mathrm{d}r\int\limits_{\mathbb{R}}\tilde{k}^2(x)\mathrm{d}x.$$

■

The asymptotic representation for the variance of the ISE under some mild mixing conditions is given without proof.

Lemma 7.2.5 *Let $\Psi\sim P$ be an 8-stationary point process in $\mathbb{R}^d$ with $\left\|\gamma_{\mathrm{red}}^{(k)}\right\|<\infty$ for $k=2,\ldots,8$ and intensity λ. Let the derivative of order s of the pair correlation function g be bounded and continuous in $K\oplus(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$ and some $s\geq 1$. Let the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist. Furthermore let $b_n^{2s+1}|W_n|\to 0$, and let the kernel function satisfy Condition 5.1.2 $\mathcal{K}(1,s)$.*

Then we have

$$\mathrm{Var}\left(b_n^{1/2}|W_n|\int\limits_K\left(\hat{g}_n(r)-\lambda^2 g(r)\right)^2\mathrm{d}r\right)\xrightarrow[n\to\infty]{}8\lambda^4\int\limits_K\left(\frac{g(r)}{d\omega_d r^{d-1}}\right)^2\mathrm{d}r\int\limits_{\mathbb{R}}\tilde{k}^2(x)\mathrm{d}x.$$

■

7.2.2 Central limit theorems

In this section we present central limit theorems for the ISE of the estimated pair correlation function for the Poisson process, for Poisson cluster processes and for Brillinger-mixing point processes. The proofs of the following results are analogous to those given in Section 7.1.3.

Theorem 7.2.6 *Let $\Psi \sim P$ be a Brillinger-mixing point process in $\mathbb{R}^d$ with intensity λ. Let the derivative of order s of the pair correlation function g be bounded and continuous in $K \oplus (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$. Furthermore let all cumulant densities $c^{(k)}$, $k \geq 2$, exist. Let the kernel function k satisfy Condition 5.1.2 $\mathcal{K}(1, s)$, and let $b_n^{2s+1}|W_n| \to 0$.*

Then we have

$$b_n^{1/2}|W_n|\big(I_n(K) - \mathbb{E}I_n(K)\big) \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}\left(0, 8\lambda^4 \int\limits_K \left(\frac{g(r)}{d\omega_d r^{d-1}}\right)^2 \mathrm{d}r \int\limits_{\mathbb{R}} \tilde{k}^2(x)\mathrm{d}x\right).$$

■

Theorem 7.2.7 *Let $\Psi \sim \Pi_\lambda$ be a stationary Poisson process in $\mathbb{R}^d$ with intensity λ. Then we have*

$$b_n^{1/2}|W_n|\big(I_n(K) - \mathbb{E}I_n(K)\big) \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}\left(0, 8\lambda^4 \int\limits_K \left(\frac{1}{d\omega_d r^{d-1}}\right)^2 \mathrm{d}r \int\limits_{\mathbb{R}} \tilde{k}^2(x)\mathrm{d}x\right).$$

■

Theorem 7.2.8 *Let $\Psi \sim P$ be a Neyman-Scott process in $\mathbb{R}^d$ with intensity λ_{p} of the underlying Poisson process, random number of points $M \sim G$ in the typical cluster, and let f be the Lebesgue density of the distribution of the distance between two independent random points of the typical cluster (see page 13). The pair correlation function g is then given by (2.7). Let $\lambda = \lambda_{\mathrm{p}}\, \mathbb{E}M$ denote the intensity of Ψ. Let $\mathbb{E}M^k < \infty$ for all $k \geq 1$. Let the derivative of order s of f be bounded and continuous in $K \oplus (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and some $s \geq 1$. Let the kernel function k satisfy Condition 5.1.2 $\mathcal{K}(1, s)$, and let the bandwidth satisfy $b_n^{2s+1}|W_n| \xrightarrow[n\to\infty]{} 0$.*

Then we have

$$b_n^{1/2}|W_n|\big(I_n(K) - \mathbb{E}I_n(K)\big) \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}\left(0, 8\lambda^4 \int\limits_K \left(\frac{g(r)}{d\omega_d r^{d-1}}\right)^2 \mathrm{d}r \int\limits_{\mathbb{R}} \tilde{k}^2(x)\mathrm{d}x\right).$$

■

8

Asymptotic goodness-of-fit tests

An important task for the analysis of point processes is model identification. Given a realization of a point process $\Psi \sim P$, one is interested in whether a hypothetical distribution P_0 of a point process is a "good fit" for the unknown "real" distribution P (see Diggle [13], for example). In this chapter we will explain how to use the limit theorems in Chapters 6 and 7 for constructing goodness-of-fit tests for point processes in order to get a decision rule for the test problem $H_0 : P = P_0$ versus $H_1 : P \neq P_0$.

Let $\Psi \sim P$ be a point process in $\mathbb{R}^d$ and let $(T_n)_{n\in\mathbb{N}}$ with $T_n = T_n(\Psi)$ be a sequence of random variables with

$$T_n \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{N}(0, \sigma^2) \qquad \text{for some } \sigma^2 > 0$$

under $P = P_0$. Let Φ^{-1}_{μ,σ^2} denote the quantile function of the normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \geq 0$. Then

$$\varphi(T_n) = \begin{cases} 1 \\ 0 \end{cases} \quad \text{for } |T_n| \begin{cases} > \\ \leq \end{cases} \quad \Phi^{-1}_{0,\sigma^2}(1 - \alpha/2)$$

is an asymptotic significance test at the level $\alpha \in (0, 1)$ for the test problem $H_0 : P = P_0$ versus $H_1 : P \neq P_0$. Likewise, the weak convergence

$$T_n \xrightarrow[n\to\infty]{\mathrm{d}} \chi^2_q \qquad \text{for some } q \in \mathbb{N}$$

under $P = P_0$ gives rise to the test

$$\varphi(T_n) = \begin{cases} 1 \\ 0 \end{cases} \text{for } T_n \begin{cases} \notin [F^{-1}_{\chi^2_q}(\alpha/2),\ F^{-1}_{\chi^2_q}(1-\alpha/2)], \\ \in [F^{-1}_{\chi^2_q}(\alpha/2),\ F^{-1}_{\chi^2_q}(1-\alpha/2)], \end{cases}$$

where $F^{-1}_{\chi^2_q}$ denotes the quantile function of the χ^2-distribution with q degrees of freedom. The test φ is an asymptotic significance test at the level $\alpha \in (0,1)$ for testing $H_0 : P = P_0$ versus $H_1 : P \neq P_0$.

In both cases the test φ is an asymptotic significance test at the level α in the sense that we have $\lim_{n\to\infty} \mathbb{E}_0\, \varphi(T_n) = \alpha$, where $\mathbb{E}_0$ denotes the mean under the null hypothesis. Note that for fixed n (especially for small n) we may have $\mathbb{E}_0\, \varphi(T_n) > \alpha$, which implies that φ is not a significance test at the level α. We may also have $\mathbb{E}_0\, \varphi(T_n) \ll \alpha$ for small n; then φ is still a significance test at the level α but will be hardly useful since it will reject the null hypothesis too rarely.

We will sketch how the above procedure is implemented for Theorem 7.2.7. This leads to a test for complete spatial randomness, that is, $H_0 : P = \Pi_\lambda$ versus $H_1 : P \neq \Pi_\lambda$.

Corollary 8.1 *Let $\Psi \sim P$ be a point process in $\mathbb{R}^d$ and let $K \in \mathfrak{B}((0,\infty))$ be a bounded Borel set with $|K| > 0$. Then*

$$\varphi(T_n) = \begin{cases} 1 \\ 0 \end{cases} \textit{for } |T_n| \begin{cases} > \\ \leq \end{cases} \Phi^{-1}_{0,\sigma^2}(1-\alpha/2)$$

with

$$T_n = b_n^{1/2}|W_n| \left(\int_K (\hat{g}_n(r) - \lambda^2)^2 \mathrm{d}r - \frac{1}{b_n|W_n|}\, 2\lambda^2 \int_K \frac{1}{d\omega_d r^{d-1}} \mathrm{d}r \int_{\mathbb{R}} k^2(x)\mathrm{d}x \right)$$

and

$$\sigma^2 = 8\lambda^4 \int_K \left(\frac{1}{d\omega_d r^{d-1}} \right)^2 \mathrm{d}r \int_{\mathbb{R}} \tilde{k}^2(x)\mathrm{d}x$$

is an asymptotic significance test at the level $\alpha \in (0,1)$ for testing $H_0 : P = \Pi_\lambda$ versus $H_1 : P \neq \Pi_\lambda$, where $\lambda > 0$.

Proof: This is an immediate consequence of the asymptotic representation for the mean (see Lemma 7.2.1) and the central limit theorem for the integrated squared error of the estimated pair correlation function (see Theorem 7.2.7). ■

9

Summary and outlook

In this work we have established normal convergence of empirical product densities of order two and higher (see Chapter 6) and the integrated squared error of the empirical second-order product density and of the empirical pair correlation function (see Chapter 7) for Brillinger-mixing point processes. The examples given in Section 3.4 show that the setting of Brillinger-mixing point processes covers many popular point process models under rather mild additional assumptions.

The estimated product densities of order two and higher are asymptotically normal, and their asymptotic variances depend only on the respective theoretical product density, the intensity, and the kernel function. Although the normal convergence itself is not a novel result (see Jolivet [40]), we have corrected the assumptions needed for this normal convergence, namely, Brillinger-mixing, continuity of the respective product density, and boundedness conditions on the cumulant densities. We have also presented asymptotic representations for means and variances of the estimated product densities. Furthermore we have derived the normal convergence of the integrated squared error $I_n(K)$ of the estimated second-order product density and of the estimated pair correlation function. The assumptions needed for the asymptotic normality of $I_n(K)$ of the estimated second-order product density are Brillinger-mixing, continuity of the second-order product density ϱ in a neighborhood of the set K, boundedness and continuity of the partial derivatives of order s of ϱ for some $s \geq 1$ in a neighborhood of the set K, existence of all cumulant densities, and conditions on the kernel and on the bandwidth (namely Condition 5.1.2 $\mathcal{K}(d, s)$ and $b_n^{2s+d}|W_n| \xrightarrow[n\to\infty]{} 0$). Analogous assumptions are needed for deriving the asymptotic normality of $I_n(K)$ of the estimated pair correlation function.

An analogue to the normal convergence of the integrated squared error of the estimated second-order product density is given by the asymptotic normality of the integrated squared error of probability density estimators, see Hall [22], Nadaraya [47], and Horváth [37]. The sufficient conditions stated in Nadaraya [47] and Horváth [37] are similar to the ones needed here.

An extension of the results on the asymptotic behavior of estimated product densities in Chapter 6 might be obtained by replacing the assumption of Brillinger-mixing by that of α- or β-mixing. The methods used in Hall [22] for proving a central limit theorem for the integrated squared error of nonparametric probability density estimators might be useful for deriving analogous results in the setting of β-mixing point processes.

In the present work we have studied summary statistics for stationary point processes only. An interesting question is whether the asymptotic approach used in the present work can be applied to summary statistics for inhomogeneous point processes, for instance, to the inhomogeneous K-function introduced in Baddeley et al. [1].

An important question concerning the applicability of the asymptotic goodness-of-fit tests presented in Chapter 8 is how large the observation window has to be for a satisfactory approximation in the central limit theorem. An answer may be found through simulation studies. The approximation will depend on several factors such as the distribution of the underlying point process—in particular the intensity and the second product density—, the choice of the bandwidth and the kernel function, and the choice of the points of evaluation $(u_1, \ldots, u_q)'$ for the test statistic $(\Delta_n(u_i))_{i=1}^q$ or the choice of the set K for the test statistic $I_n(K)$. Given a hypothetical distribution P_0 and the associated test problem $H_0 : P = P_0$ versus $H_1 : P \neq P_0$ it is obvious how to investigate the type-I error (that is, the probability of rejecting the null hypothesis when it is actually true) by simulation studies. The type-II error (that is, the probability of not rejecting the null hypothesis when the alternative hypothesis is actually true) is difficult to handle since the true distribution P can differ from P_0 in many different ways. Hence the type-II error can only be studied for some special cases. For example, if $P = \Pi_\lambda$ and $P_0 = \Pi_{\lambda_0}$ with $\lambda \neq \lambda_0$, an investigation of the type-II error for different combinations of λ and λ_0 is a sensitivity analysis of the test procedure with respect to the intensity of the underlying Poisson process. Another example of such a sensitivity analysis is given in Grabarnik and Chiu [18] who consider the null hypothesis of a Poisson process and the alternative hypothesis of a mixture of a conditional Strauss point process and Matérn's cluster process. Furthermore our tests should be compared with alternative tests for the null hypothesis of complete spatial randomness as well as with alternative test procedures (such as simulation tests, for example) involving null hypotheses other than the stationary Poisson process.

A

Properties of the sequence of observation windows

In Chapters 6 and 7 we implicitly used some properties of the sequence of observation windows $(W_n)_{n\in\mathbb{N}}$ to derive asymptotic results for the estimated product densities and the estimated pair correlation function. More precisely, these properties were needed for deriving the asymptotic order of some integrals occurring in the proofs in Chapters 6 and 7 by Lebesgue's dominated convergence theorem. These properties—which have already been studied in David [10]—are presented in the following.

Let ∂W denote the boundary of a set $W \subseteq \mathbb{R}^d$. For convex sets $A, B \subseteq \mathbb{R}^d$ let $A \ominus B = \{x \in \mathbb{R}^d : B + x \in A\}$ denote Minkowski subtraction and $A \oplus B = \{x + y : x \in A, y \in B\}$ Minkowski addition.

Lemma A.1 *Let $W \subseteq \mathbb{R}^d$ be a convex set and let $r \geq 0$. Then we have*

$$|W \setminus (W \ominus b(o,r))| \quad \leq \quad r\mathcal{H}_{d-1}(\partial W).$$

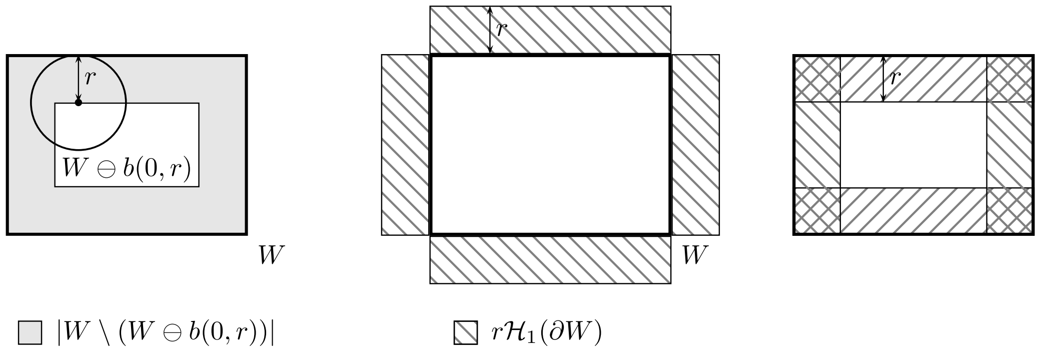
Proof: Since W is convex the function $V(x) = |W \setminus (W \ominus b(o,x))|$ is differentiable for $x \in [0, \rho(W))$ with $V'(x) = \mathcal{H}_{d-1}(\partial(W \ominus b(0,x)))$, see Hadwiger [19], page 207. Due to the monotonicity of $\mathcal{H}_{d-1}$ we have

$$|W \setminus (W \ominus b(o,r))| \;=\; \int\limits_0^r \mathcal{H}_{d-1}(\partial(W \ominus b(o,x)))dx \;\leq r\;\mathcal{H}_{d-1}(\partial W)$$

for $r \in [0, \rho(W))$. Because of the convexity of W we find $|W \setminus (W \ominus b(0,r))| = 0 \leq r\mathcal{H}_{d-1}(\partial W)$ for $r \geq \rho(W)$. ■

Lemma A.1 is illustrated in Figure A.1.

Figure A.1: Example to Lemma A.1



The inequality $|W \setminus (W \ominus b(o,r))| \leq r\mathcal{H}_{d-1}(\partial W)$ from Lemma A.1 for the special case of a rectangle W. On the left-hand side we see W with the set $W \setminus (W \ominus b(o,r))$ shaded in gray. In the middle $r\mathcal{H}_1(\partial W)$ corresponds to the union of the hatched regions. These hatched regions are flipped over along the edges of W, resulting in the figure on the right-hand side. Hence we see that we have $|W \setminus (W \ominus b(o,r))| = r\mathcal{H}_{d-1}(\partial W) + 4r^2$.

For a convex set W, the relation between $|W|$ and $\mathcal{H}_{d-1}(\partial W)$ is

$$\frac{\rho(W)}{d} \leq \frac{|W|}{\mathcal{H}_{d-1}(\partial W)} \leq \rho(W), \tag{A.1}$$

see Wills [65], Lemma 1 and 2.

The following lemma and remark justify the applicability of Lebesgue's dominated convergence theorem in the proofs of the asymptotic results in Chapters 6 and 7.

Lemma A.2 *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of convex sets in $\mathbb{R}^d$ with $\rho(W_n) \xrightarrow[n\to\infty]{} \infty$ and let $y, z \in \mathbb{R}^d$. Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers with $b_n \xrightarrow[n\to\infty]{} 0$. Then we have*

$$\frac{|W_n \cap (W_n - b_n y - z)|}{|W_n|} \xrightarrow[n\to\infty]{} 1.$$

Proof: Since W_n is convex we have $|W_n \cap (W_n - b_n y - z)| \geq |W_n \ominus b(o, \|b_n y + z\|)|$. Together with Lemma A.1 and the triangle inequality we find

$$\begin{aligned} 0 &\leq 1 - \frac{|W_n \cap (W_n - b_n y - z)|}{|W_n|} \\ &\leq \frac{|W_n \setminus (W_n \ominus b(o, \|b_n y + z\|))|}{|W_n|} \\ &\leq \|b_n y + z\| \frac{\mathcal{H}_{d-1}(\partial W_n)}{|W_n|} \\ &\leq b_n \|y\| \frac{\mathcal{H}_{d-1}(\partial W_n)}{|W_n|} + \|z\| \frac{\mathcal{H}_{d-1}(\partial W_n)}{|W_n|}. \end{aligned}$$

Due to formula (A.1) and the assumptions $\rho(W_n) \xrightarrow[n\to\infty]{} \infty$ and $b_n \xrightarrow[n\to\infty]{} \infty$ the right-hand side converges to zero. This proves the claim. ■

Remark A.1 Under the assumptions of Lemma A.2 the sequence $\frac{|W_n|}{|W_n \cap (W_n - b_n y - z)|}$ converges to 1 for arbitrary $y, z \in \mathbb{R}^d$ and is hence bounded. □

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Lebenslauf

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