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On Kepler's Geometric Approach to Consonances

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Kepler's 1618 book *The Harmonies of the World* (*Harmonices mundi*) [9] is famous because it is the work in which Kepler states his third law of planetary motion, namely that the cubes of the semimajor axis of the elliptic orbits of the planets are proportional to the squares of their periods. However, mention of this crucial discovery does not appear until the fifth book, and in fact, it remains rather mysterious how Kepler actually arrived at his third law.

Kepler's thinking is highly original and rather unique, and one characteristic feature of it is that he is looking for geometric and not arithmetic explanations. His geometric approach had already played a crucial role in his 1596 work *Cosmographic Mystery* (*Mysterium Cosmographicum*) [7]. According to the Alexandrian astronomer Ptolemy (second century CE), there were seven planets orbiting the Earth, namely the Moon, Mercury, Venus, the Sun, Mars, Jupiter, and Saturn. But in the sixteenth century, Nicolaus Copernicus interchanged the roles of Earth and Sun. The planets, Earth included, now orbited the Sun, and thus the Earth became a planet, while the Sun was a planet no longer. The Moon, too, lost its status as a planet, since it was now orbiting the Earth. Therefore, the number of planets dropped by one, from seven to six. How could one explain this number? In his *Narratio prima*, in which he explained Copernicus's theory to the world, Copernicus's student Georg Joachim de Porris, known as Rheticus, gave the following explanation:

$$6 = 1 + 2 + 3, \quad 6 = 1 \cdot 2 \cdot 3,$$

that is, 6 is a perfect number, since it equals the sum of its proper divisors. Kepler didn't like this arithmetic explanation. His geometric explanation in the *Cosmographic Mystery* was the following. If there are six planets, there are five intervals between them, and five is the number of the Platonic solids.

To get the relative sizes of the Platonic solids to correspond with observational data on the distances between the planets was challenging, especially after Kepler had discovered his first law, which appears in his 1609 *New Astronomy* (*Astronomia Nova*) [8], which states that each planet moves in an elliptical orbit about the Sun, which is at rest at a focus of the ellipse. In his *Harmonies of the World*, he was therefore looking for an even deeper explanation for the structure of the universe based on musical consonances. We refer to [4] for a thorough analysis of Kepler's geometric cosmology.

According to Kepler, there are seven consonant intervals, the octave $\frac{1}{2}$, the perfect fifth $\frac{2}{3}$, the perfect fourth $\frac{3}{4}$, the major third $\frac{4}{5}$, the minor third $\frac{5}{6}$, the major sixth $\frac{3}{5}$, and the minor sixth $\frac{5}{8}$. That thirds and sixths were considered consonant was a rather new trend in Renaissance music [10]. So Kepler distinguished himself from Ptolemy not just in that in his cosmology the Sun is at the center and not the Earth, but in that unlike Ptolemy—who was also a music theorist—he considered thirds and sixths to be consonant.

The third book of Kepler's *Harmonices Mundi* is devoted to a geometric explanation of why only the seven above-mentioned intervals are consonant. Kepler's idea was that there is a relationship between consonance and construction with straightedge and compass. Thus the fact that the 7-gon cannot be constructed by straightedge and compass fits with the fact that the interval $\frac{6}{7}$ does not sound consonant. On the other hand, the 15-gon can be constructed with straightedge and compass, and Kepler makes rather ad hoc arguments as to why the 15-gon should not be considered consonant. Another issue with Kepler's approach is that he was aware of only the constructions using straightedge and compass that can be found in Euclid's *Elements*. The world needed to wait almost two centuries until the 19-year-old Gauss discovered that the 17-gon can be constructed by straightedge and compass as well.

In this article, based on Kepler's ideas, we discuss how Kepler's seven consonant intervals can be characterized mathematically with the help of the numbers of edges of the n-gons that were constructed in Euclid's *Elements*. We give a mathematical definition of Euclidean consonant and then prove a theorem that the Euclidean consonants are precisely Kepler's seven consonant intervals.

The limited impact of Kepler's geometric approach to the development of music is discussed in Dickreiter's book [2]. Kepler had a particularly strong influence on Andreas Werckmeister [12], the inventor of well-tempered tuning, which inspired Johann Sebastian Bach to compose his famous *Well-Tempered Clavier*.

Euclidean and Gaussian Consonances

On the set of positive rational numbers

$$\mathbb{Q}_+ = \left\{ q \in \mathbb{Q} : q > 0 \right\}$$

we consider the equivalence relation $\sim$ defined for $q_1, q_2 \in \mathbb{Q}_+$ by the requirement

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$$q_1 \sim q_2 \Longleftrightarrow q_1 = 2^n q_2, \quad n \in \mathbb{Z}.$$

The connection to music is the following. The set $\mathbb{Q}_+$ models rational intervals, and two intervals are equivalent if they coincide after octave reduction. The quotient

$$\mathfrak{S} = \mathbb{Q}_+/\sim$$

is then an abelian group whose operation is induced by multiplication on $\mathbb{Q}_+$. The neutral element in this group,

$$\mathfrak{o} = [1] = \left[\frac{1}{2}\right],$$

then corresponds in the musical interpretation to the octave.

For $q \in \mathbb{Q}_+$ we denote by $[q] \in \mathfrak{S}$ the equivalence class of q. We refer to the group $\mathfrak{S}$ as the group of musical intervals, and to its elements as musical intervals. Note that for a musical interval $\sigma \in \mathfrak{S}$, there exist unique positive integers $n_\sigma, m_\sigma \in \mathbb{N}$ characterized by the following properties:

$$\sigma = \left[\frac{m_\sigma}{n_\sigma}\right], \quad \frac{1}{2} \leq \frac{m_\sigma}{n_\sigma} < 1, \quad \gcd(n_\sigma, m_\sigma) = 1, \qquad (1)$$

where gcd stands for greatest common divisor. In other words, the third equality means that n_σ and m_σ are relatively prime.

We obtain the two maps

$$\mathcal{N} : \mathfrak{S} \to \mathbb{N}, \quad \sigma \mapsto n_\sigma,$$
$$\mathcal{M} : \mathfrak{S} \to \mathbb{N}, \quad \sigma \mapsto m_\sigma.$$

We further introduce the map

$$K : \mathfrak{S} \to \mathfrak{S}, \quad \sigma \mapsto \left[\frac{n_\sigma - m_\sigma}{m_\sigma}\right],$$

and we refer to it as the Kepler map.

We first check that the Kepler map is well defined. To see this, we need to show that

$$\frac{n_\sigma - m_\sigma}{m_\sigma} \in \mathbb{Q}_+.$$

The number is obviously rational. To convince ourselves that it is positive, we infer from the second equation in (1) that

$$1 < \frac{n_\sigma}{m_\sigma} \leq 2,$$

which implies that

$$0 < \frac{n_\sigma - m_\sigma}{m_\sigma} \leq 1. \qquad (2)$$

This implies positivity. Actually, for positivity we need only the first inequality in (2).

The following lemma tells us that the map $\mathcal{N}$ is monotonically decreasing under the Kepler map.

Lemma 1. *For every $\sigma \in \mathfrak{S}$, we have $\mathcal{N} \circ K(\sigma) \leq \mathcal{N}(\sigma)$, and equality holds if and only if $\sigma = \mathfrak{o}$.*

Proof. We first consider the case $\sigma \neq \mathfrak{o}$, in which we have

$$\frac{1}{2} < \frac{m_\sigma}{n_\sigma},$$

implying that $n_\sigma < 2m_\sigma$, and therefore $n_\sigma - m_\sigma < m_\sigma$, so that

$$\frac{n_\sigma - m_\sigma}{m_\sigma} < 1.$$

Hence there exists a unique nonnegative integer $\ell \in \mathbb{N}_0$ such that

$$\frac{1}{2} \leq \frac{2^\ell (n_\sigma - m_\sigma)}{m_\sigma} < 1. \qquad (3)$$

We may decompose m_σ uniquely into the product of an odd number and a power of 2, i.e.,

$$m_\sigma = 2^k \mu_\sigma,$$

where $k \in \mathbb{N}_0$ and μ_σ is odd. Define

$$\rho := \max\{0, k - \ell\} \in \mathbb{N}_0,$$
$$\tau := \max\{0, \ell - k\} \in \mathbb{N}_0.$$

We claim that

$$n_{K(\sigma)} = 2^\rho \mu_\sigma, \quad m_{K(\sigma)} = 2^\tau (n_\sigma - m_\sigma). \qquad (4)$$

To prove (4), we need, according to (1), to check the following:

$$K(\sigma) = \left[\frac{2^\tau (n_\sigma - m_\sigma)}{2^\rho \mu_\sigma}\right], \qquad (5a)$$

$$\frac{1}{2} \leq \frac{2^\tau (n_\sigma - m_\sigma)}{2^\rho \mu_\sigma} < 1, \qquad (5b)$$

$$\gcd\left(2^\rho \mu_\sigma, 2^\tau (n_\sigma - m_\sigma)\right) = 1. \qquad (5c)$$

By substituting for m_σ in (3) and canceling powers of 2, we obtain

$$\frac{2^\ell (n_\sigma - m_\sigma)}{m_\sigma} = \frac{2^\ell (n_\sigma - m_\sigma)}{2^k \mu_\sigma} = \frac{2^\tau (n_\sigma - m_\sigma)}{2^\rho \mu_\sigma}.$$

Hence assertion (5b) follows from (3). Moreover, (5a) follows from

$$K(\sigma) = \left[\frac{n_\sigma - m_\sigma}{m_\sigma}\right] = \left[\frac{2^\ell (n_\sigma - m_\sigma)}{m_\sigma}\right] = \left[\frac{2^\tau (n_\sigma - m_\sigma)}{2^\rho \mu_\sigma}\right].$$

It remains to check (5c). First note that since n_σ and m_σ are relatively prime, the same is true for m_σ and $n_\sigma - m_\sigma$, so that we have

$$\gcd(m_\sigma, n_\sigma - m_\sigma) = 1. \qquad (6)$$

Since μ_σ is a divisor of m_σ, it follows that

$$\gcd(\mu_\sigma, n_\sigma - m_\sigma) = 1. \qquad (7)$$

We now consider the case $\rho = 0$. Since μ_σ is odd, assertion (5c) follows in this case directly from (7). It remains to check the case $\rho \neq 0$, which implies that $\tau = 0$, $k > 0$. In particular, since $k > 0$, we conclude that $m_\sigma = 2^k \mu_\sigma$ is even. From (6), we conclude that $n_\sigma - m_\sigma$ is odd. Hence again assertion (5c) follows from (7). Therefore, the three assertions (5a), (5b), (5c) are established, and (4) follows.

From the first assertion in (4), we conclude that

$$\begin{aligned} \mathcal{N} \circ K(\sigma) &= n_{K(\sigma)} = 2^\rho \mu_\sigma \leq 2^k \mu_\sigma \\ &= m_\sigma < n_\sigma = \mathcal{N}(\sigma), \end{aligned} \tag{8}$$

where the last inequality follows from the second property in (1). This proves the lemma in the case $\sigma \neq \mathfrak{o}$.

It remains to discuss the case $\sigma = \mathfrak{o}$. We have $n_\mathfrak{o} = 2$, $m_\mathfrak{o} = 1$. It follows that

$$K(\mathfrak{o}) = \left[\frac{n_\mathfrak{o} - m_\mathfrak{o}}{m_\mathfrak{o}} \right] = [1] = \mathfrak{o}, \tag{9}$$

so that $\mathfrak{o}$ is a fixed point of K. In particular, $\mathcal{N} \circ K(\mathfrak{o}) = \mathcal{N}(\mathfrak{o})$. This finishes the proof of the lemma. □

Corollary 2. *Given $\sigma \in \mathfrak{S}$, there exists $\ell \in \mathbb{N}$ such that $K^\ell(\sigma) = \mathfrak{o}$.*

Proof. As we have seen in (9), the octave is a fixed point of the Kepler map, so we can assume without loss of generality that $\sigma \neq \mathfrak{o}$. By Lemma 1, there exists $\ell \in \mathbb{N}$ such that $\mathcal{N} \circ K^\ell(\sigma) \leq 2$. By the second assertion in (1), it follows that

$$\mathcal{N} \circ K^\ell(\sigma) = 2, \quad \mathcal{M} \circ K^\ell(\sigma) = 1,$$

so that $K^\ell(\sigma) = \mathfrak{o}$. This proves the corollary. □

In view of Corollary 2, we can make the following definition.

Definition 3. We define the height of a musical interval $\sigma \in \mathfrak{S}$ to be

$$\mathfrak{h}(\sigma) := \min \left\{ \ell \in \mathbb{N}_0 \,:\, K^\ell(\sigma) = \mathfrak{o} \right\}.$$

Note that the octave $\mathfrak{o}$ can be characterized as the unique musical interval of height 0.

Given a musical interval $\sigma \in \mathfrak{S}$, we define its first Kepler sequence as

$$\mathfrak{K}_1(\sigma) := \left(\sigma, K(\sigma), \dots, K^{\mathfrak{h}(\sigma)}(\sigma) \right),$$

and its second Kepler sequence as

$$\mathfrak{K}_2(\sigma) := \mathcal{N}(\mathfrak{K}_1(\sigma)) := \left(\mathcal{N}(\sigma), \mathcal{N} \circ K(\sigma), \dots, \mathcal{N} \circ K^{\mathfrak{h}(\sigma)}(\sigma) \right).$$

We consider some examples. The first Kepler sequence for the minor sixth $\left[\frac{5}{8}\right]$ is

$$\mathfrak{K}_1\left(\left[\frac{5}{8}\right] \right) = \left(\left[\frac{5}{8}\right], \left[\frac{3}{5}\right], \left[\frac{2}{3}\right], \left[\frac{1}{2}\right] \right), \tag{10}$$

i.e., (minor sixth, major sixth, perfect fifth, octave).

The second Kepler sequence for the minor sixth is then

$$\mathfrak{K}_2\left(\left[\frac{5}{8}\right] \right) = (8, 5, 3, 2). \tag{11}$$

For the minor third $\left[\frac{5}{6}\right]$, we get as the first Kepler sequence

$$\mathfrak{K}_1\left(\left[\frac{5}{6}\right] \right) = \left(\left[\frac{5}{6}\right], \left[\frac{4}{5}\right], \left[\frac{1}{2}\right] \right), \tag{12}$$

i.e., (minor third, major third, octave), and as the second Kepler sequence, we have

$$\mathfrak{K}_2\left(\left[\frac{5}{6}\right] \right) = (6, 5, 2). \tag{13}$$

For the perfect fourth $\left[\frac{3}{4}\right]$, the first Kepler sequence is

$$\mathfrak{K}_1\left(\left[\frac{3}{4}\right] \right) = \left(\left[\frac{3}{4}\right], \left[\frac{2}{3}\right], \left[\frac{1}{2}\right] \right), \tag{14}$$

i.e., (perfect fourth, perfect fifth, octave), and the second Kepler sequence is

$$\mathfrak{K}_2\left(\left[\frac{3}{4}\right] \right) = (4, 3, 2). \tag{15}$$

Inspired by Kepler, our next goal is to define a musical interval as consonant if the members of its second Kepler sequence are integers that give rise to constructible polygons. For this purpose, recall that a prime number p is called a Fermat prime if it is of the form $p = 2^{2^n} + 1$. There are five known Fermat primes,

$$F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537, \tag{16}$$

and there are some probabilistic considerations suggesting that these might be the only ones [1]. The Gauss–Wantzel theorem tells us that a regular n-gon is constructible by ruler and compass if and only if n is the product of a power of 2 and any number of distinct Fermat primes [6, 11]. Gauss proved that these n-gons can be constructed, while Wantzel showed that no other n-gon is constructible. We make the following definition.

Definition 4. A positive integer $n > 1$ is called a Gauss–Wantzel number if

$$n = 2^k p_1 \cdots p_\ell,$$

where $k, \ell \in \mathbb{N}_0$ and $p_1, \dots, p_\ell$ are distinct Fermat primes.

We introduce

$$\mathfrak{G} := \left\{ n \in \mathbb{N} \,:\, n \text{ Gauss-Wantzel} \right\},$$

the subset of Gauss–Wantzel numbers. With this notion, the Gauss–Wantzel theorem just tells us that an n-gon is constructible with straightedge and compass if and only if $n \in \mathfrak{G}$.

In Euclid's *Elements* [3], one can find the construction of n-gons using straightedge and compass in the special case in which the only allowed Fermat primes are 3 and 5. We make the following definition.

Definition 5. A Gauss–Wantzel number is called Euclidean if

$$n = 2^k 3^\ell 5^m,$$

where $k \in \mathbb{N}_0$ and $\ell, m \in \{0, 1\}$.

The first eleven Euclidean Gauss–Wantzel numbers are

$$2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20.$$

The smallest Gauss–Wantzel number that is not Euclidean is 17. It was a big surprise when the 19-year-old Gauss announced in 1796 in the *Intelligenzblatt der allgemeinen Literaturzeitung* a construction of the 17-gon by straightedge and compass [5]. Five years later, he published this construction in his *Disquisitiones Arithmeticae* [6].

We abbreviate

$$\mathfrak{E} := \{n \in \mathfrak{G} : n \text{ Euclidean}\},$$

the set of Euclidean Gauss–Wantzel numbers. We therefore have a nested sequence of subsets

$$\mathfrak{E} \subset \mathfrak{G} \subset \mathbb{N}.$$

We are now in position to define two versions of consonant musical intervals.

Definition 6. A musical interval $\sigma \in \mathfrak{S}$ is said to be a Euclidean consonant if

$$\mathfrak{K}(\sigma) \in \mathfrak{E}^{\mathfrak{h}(\sigma)+1},$$

i.e., all members of its second Kepler sequence are Euclidean Gauss–Wantzel numbers.

Definition 7. A musical interval $\sigma \in \mathfrak{S}$ is said to be a Gaussian consonant if

$$\mathfrak{K}(\sigma) \in \mathfrak{G}^{\mathfrak{h}(\sigma)+1},$$

i.e., all members of its second Kepler sequence are Gauss–Wantzel numbers.

Since $\mathfrak{E} \subset \mathfrak{G}$, every Euclidean consonant is also a Gaussian consonant. Further note that if σ is a Euclidean, respectively Gaussian, consonant, then every musical interval in the first Kepler sequence $\mathfrak{K}_1(\sigma)$ is also a Euclidean, respectively Gaussian, consonant.

We have now the following theorem, which goes back to the ideas of Kepler.

Theorem 8. *There are precisely seven Euclidean consonants, namely the minor third, major third, perfect fourth, perfect fifth, minor sixth, major sixth, and octave, i.e.,*

$$\left\{\left[\frac{5}{6}\right], \left[\frac{4}{5}\right], \left[\frac{3}{4}\right], \left[\frac{2}{3}\right], \left[\frac{5}{8}\right], \left[\frac{3}{5}\right], \left[\frac{1}{2}\right]\right\} \subset \mathfrak{S}.$$

Proof. That these seven musical intervals are Euclidean consonants follows from (10)–(15). It remains to show that there are no others. Suppose that

$$\sigma = \left[\frac{m_\sigma}{n_\sigma}\right]$$

is a Euclidean consonant. We necessarily have $n_\sigma \in \mathfrak{E}$. From (8) in the proof of Lemma 1, we conclude that there exists $\ell \in \mathbb{N}_0$ such that $2^\ell n_{K(\sigma)} = m_\sigma$. Since σ is a Euclidean consonant, it follows that $n_{K(\sigma)} \in \mathfrak{E}$. Since $\mathfrak{E}$ is invariant under multiplication by 2, we conclude that $2^\ell n_{K(\sigma)} \in \mathfrak{E}$, and therefore $m_\sigma \in \mathfrak{E}$.

We first discuss the case that n_σ is odd, i.e., $n_\sigma \in \{3, 5, 15\}$. By the second assertion in (1), we have

$$\frac{n_\sigma}{2} < m_\sigma < n_\sigma. \tag{17}$$

Hence for $n_\sigma = 3$, the only possibility is the perfect fifth $\left[\frac{2}{3}\right]$, and for $n_\sigma = 5$, the only possibilities are the major sixth $\left[\frac{3}{5}\right]$ and the major third $\left[\frac{4}{5}\right]$. In the odd case, it therefore suffices to discuss the case $n_\sigma = 15$. Since n_σ and m_σ are relatively prime by the third assertion in (1), we do not need to worry about the cases $m_\sigma \in \{9, 10, 12\}$. So the only case left to discuss in the odd case is

$$\sigma = \left[\frac{8}{15}\right].$$

We compute

$$K\left(\left[\frac{8}{15}\right]\right) = \left[\frac{7}{8}\right], \quad K^2\left(\left[\frac{8}{15}\right]\right) = \left[\frac{4}{7}\right],$$

so that

$$\mathcal{N} \circ K^2\left(\left[\frac{8}{15}\right]\right) = 7 \notin \mathfrak{E}. \tag{18}$$

We conclude that $\left[\frac{8}{15}\right]$ is not a Euclidean consonant. In particular, there are no further Euclidean consonants in the odd case.

It remains to discuss the case in which n_σ is even, i.e., $n_\sigma = 2^k 3^\ell 5^m$ for $k \in \mathbb{N}$ and $m, k \in \{0, 1\}$. Since n_σ and m_σ are relatively prime, it follows that m_σ has to be odd, i.e., $m_\sigma \in \{3, 5, 15\}$. In view of (17), the only case that appears for $m_\sigma = 3$ is the perfect fourth $\left[\frac{3}{4}\right]$. For $m_\sigma = 5$, there are only the minor third $\left[\frac{5}{6}\right]$ and the minor sixth $\left[\frac{5}{8}\right]$. If $m_\sigma = 15$, then since n_σ and m_σ are relatively prime, it follows that $n_\sigma = 2^k$ is a power of 2. In view of (17), the only case left to discuss is $n_\sigma = 16$, i.e., the musical interval $\left[\frac{15}{16}\right]$. We compute

$$K\left(\left[\frac{15}{16}\right]\right) = \left[\frac{8}{15}\right].$$

We have already seen in (18) that $\left[\frac{8}{15}\right]$ is not a Euclidean consonant. Therefore, $\left[\frac{15}{16}\right]$ is not a Euclidean consonant either. This finishes the proof of the theorem. □

Apart from the Euclidean consonants, there are additional Gaussian consonants. We have the following lemma.

Lemma 9. *If F is a Fermat prime, then* $\left[\frac{F-1}{F}\right]$ *is a Gaussian consonant.*

Proof. A Fermat prime is a prime number of the form $F = 2^{2^n} + 1$ for $n \in \mathbb{N}_0$. Therefore, we have

$$\frac{F-1}{F} = \frac{2^{2^n}}{2^{2^n}+1},$$

so that we get

$$K\left(\left[\frac{F-1}{F}\right]\right) = \left[\frac{1}{2^{2^n}}\right] = \left[\frac{1}{2}\right].$$

We therefore obtain as the first Kepler sequence

$$\mathfrak{K}_1\left(\left[\frac{F-1}{F}\right]\right) = \left(\left[\frac{F-1}{F}\right], \left[\frac{1}{2}\right]\right),$$

and as the second Kepler sequence,

$$\mathfrak{K}_2\left(\left[\frac{F-1}{F}\right]\right) = (F, 2),$$

all of whose members are Gauss–Wantzel numbers. This proves the lemma. □

In view of the known Fermat primes (16), we obtain as Gaussian consonants, apart from the perfect fifth $\left[\frac{2}{3}\right]$ and the major third $\left[\frac{4}{5}\right]$,

$$\left[\frac{F_2-1}{F_2}\right] = \left[\frac{16}{17}\right], \quad \left[\frac{F_3-1}{F_3}\right] = \left[\frac{256}{257}\right], \quad \left[\frac{F_4-1}{F_4}\right] = \left[\frac{65536}{65537}\right].$$

These three Gaussian consonants are not Euclidean consonants. Another example of a Gaussian consonant that is not a Euclidean consonant is

$$\sigma = \left[\frac{12}{17}\right].$$

Its image under the Kepler map

$$K(\sigma) = \left[\frac{5}{6}\right]$$

is the minor third, which is a Euclidean consonant. Therefore, since $\mathcal{N}(\sigma) = 17 \in \mathfrak{G}$, it follows that σ is a Gaussian consonant.

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