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Robustness of a bisimulation-type faster-than relation

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Report 2009-08 Mai 2009

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1 Introduction

This thesis is based on Gerald Lüttgen's and Walter Vogler's paper 'Bisimulation on speed: worst case efficiency' [LV04] in which they introduce a novel bisimulation-based faster-than preorder. Their study is conducted for the process algebra TACS (Timed Asynchronous Communication System) that includes time as an aspect of system behaviour. TACS is obtained from Milner's process algebra CCS [Mil89], which is extended by a discrete time step, denoted with 'σ'. This clock prefix stands for a time unit to pass or a single clock tick that specifies an upper bound for delays and therefore determines an upper time bound on action occurrences. The process $\sigma.P$ can let time pass and then behave like P or skip the time step and at once behave like P, hence $\sigma.P$ can at most delay one time unit before behaving like P. As a result, asynchronous processes can be compared concerning their performance by taking their worst-case timing behaviour into account. Thereby only the worst case behaviour is regarded where the relative speeds of system components are indeterminate. Time is added to CCS solely to compare asynchronous systems and to evaluate the performance of processes, but not in order to influence the functional behaviour of CCS-processes. Particularly, actions are neither enabled nor disabled during the progress of time. Moreover, action transitions are regarded as instantaneous. In [LV04] a simple, concise faster-than preorder on processes, called naive faster-than preorder, is defined. Its justification as a good and elegant candidate for a faster–than preorder is formally underpinned, as it coincides with some alternative, more complicated definitions of preorders, which also formalize the idea of faster–than and possibly are intuitively more convincing. These preorders are called delayed faster-than preorder and indexed faster-than preorder. As it turns out the naive faster-than preorder is not a precongruence, hence [LV04] defines the coarsest precongruence contained in it and axiomatizes this precongruence for the class of finite and sequential processes. Further, [LV04] investigates a corresponding 'weak' preorder, which abstracts from internal, unobservable actions, and characterizes the coarsest precongruence contained in it.

In this thesis, we will introduce further alternatives for a process to let time pass without however influencing its functional behaviour. While in [LV04], a process may skip any desired number of time steps when performing an action transition, in our new setting a process may also skip any desired number of σ-prefixes when performing a time step; for example the process $P \equiv \sigma.\sigma.\sigma.a.\mathbf{0}$ may now skip one σ-prefix when performing a time step and behave like $\sigma.a.\mathbf{0}$ afterwards. Thereby the concept of time determinism as an important property in [LV04] is lost and we will justify this with sensible reasons.
The main aim in this thesis is to study the candidates for faster-than preorders that are established in [LV04] in our new setting and prove that they coincide with the original preorders. Thus, we formally underpin the robustness of the

technically simple naive faster-than relation of [LV04]. For this, the crucial technical device is the syntactic coherence that relates the resulting processes of the original and new time steps. Moreover, we will introduce a third variant of a naive faster-than preorder whose definition combines new and old clock transitions and that provides small relations to demonstrate a naive faster-than relationship. In our search for reducing the size of relations, we as well introduce a 'naive faster-than relation up to' which is inspired by Milner's 'up-to'-technique. Regrettably, it will turn out that the indexed faster-than preorder as a third candidate for a faster-than preorder is not robust against the transition extension and this can be explained by the absence of time determinism. Further, we will study the precongruence in our new setting and show that it coincides with the original precongruence. Finally, it is quite easy to carry over our proof techniques to the weak variants.

This thesis is organised as follows. The next section presents the syntax and semantics of the process algebra TACS, based on [LV04] and extended by the novel transitions. Moreover, we get familiar with the nature of the new transitions and point out some important properties. In Section 3, we compare the previous time steps to the newly established time steps and develop differences as well as the syntactic and semantic coherences. Subsequently, we demonstrate the robustness of the naive faster-than preorder against the transition extension in Section 4. Furthermore, we introduce a third variant of a naive faster-than preorder, which also coincides with the original naive faster-than preorder, and a 'naive faster-than relation up to' in Section 4. Both relations are a technique for reducing the size of relations needed to demonstrate a naive faster-than relationship. Section 5 proves the robustness of the delayed faster-than preorder, while Section 6 demonstrates the defect of the extended indexed faster-than preorder. In Section 7 and 8, we will establish the robustness of the precongruence and its corresponding weak variant. Finally, we draw a short conclusion in Section 9.

2 TACS

In this section we introduce the syntax and the semantics of the process algebra TACS as defined in [LV04].
As explained in the introduction, TACS extends CCS by a discrete time step, denoted with 'σ'. The progress of time manifests itself in a clock transition as a recurrent global synchronization event. Yet, we have to distinguish a clock transition from a σ-prefix, which locally provides a process with the possibility to wait.

Definition 1 *Let Λ be a countable set of action names or ports a, b, c.*
Let $\overline{\Lambda} =_{df} \{\overline{a} | a \in \Lambda\}$ be the set of complementary action names $\overline{a}, \overline{b}, \overline{c}$.
Let $\mathcal{A} =_{df} \Lambda \cup \overline{\Lambda} \cup \{\tau\}$ be the set of all actions α, β, γ.
We define $\overline{\overline{a}} =_{df} a$ for all $a \in \Lambda$.

With every action $a \in \Lambda$ we associate a complementary action $\overline{a} \in \overline{\Lambda}$. As in CCS [Mil89] a complementary action pair a and $\overline{a}$ is a means for synchronized actions, or local handshakes. If a shared transition of two processes occurs in such a way that an action a is synchronised with an action $\overline{a}$, this leads to the unobservable, internal action τ. Being unobservable, τ has no complementary action. As an example for a local handshake consider the transition
$a.\mathbf{0} \mid (b.\mathbf{0} \, + \, \overline{a}.c.\mathbf{0}) \xrightarrow{\tau} \mathbf{0} \mid c.\mathbf{0}$.

A TACS term is defined as follows:

Definition 2 *Let $\mathcal{V}$ be a countably infinite set of variables. A* term *is defined as*

(1) $\boldsymbol{0}$
(2) $x \in \mathcal{V}$
(3) $\alpha.P$, $\alpha \in \mathcal{A}$ *(prefix)*
(4) $\sigma.P$ *(clock prefix)*
(5) $P_1 + P_2$ *(choice)*
(6) $P_1 \,|\, P_2$ *(parallel composition)*
(7) $P \setminus L$ *(restriction), where*
$L \subseteq \Lambda \cup \overline{\Lambda}$ *is a finite restriction set.*
(8) $P[f]$ *(relabeling), where*
f *is a finite relabeling function satisfying* $f(a) = a' \Rightarrow f(\overline{a}) = \overline{a'}$; $f(\tau) = \tau$;
$|\{\alpha \,|\, f(\alpha) \neq \alpha\}| < \infty$
(9) $\mu x.P$ *(recursion)*

in which P, P_1 and P_2 are terms.
$\widehat{\mathcal{P}}$ is the set of all terms.

$\mathbf{0}$ denotes the inactive process nil.

The binding strength of a σ-prefix is the same as for action-prefixes, i.e. the binding of a σ-prefix is weaker than the binding of restriction and relabeling operators, but dominates the binding strength of parallel composition, which again is stronger than the binding of choice. The recursion operator $\mu x.$ binds as strong as a σ-prefix.

In TACS terms *free* and *bound* variables are defined in the usual fashion, where the recursion operator $\mu x.$ binds the variable x. The occurrence of a variable $x \in \mathcal{V}$ is bound in a TACS term $P \in \widehat{\mathcal{P}}$ if it falls within the scope of a recursion operator μx. *Closed* terms are TACS terms which only contain bound occurrences of variables. Conversely, *open* terms may include free variables. A variable $x \in \mathcal{V}$ is called *guarded* in a TACS term P, if each occurrence of the variable falls within the scope of an α-prefix for an $\alpha \in \mathcal{A}$. A TACS term P is guarded, if all occurring variables are guarded. As an example, the term $\mu x.(a.(c.\mathbf{0}+x) \mid \mu y.y)$ is not guarded, as y is not guarded. However, it is closed as all occurrences of x and y are bound. The term $\mu x.a.(c.y+x)$ is open, since y is not bound, but guarded as x and y are guarded.
We require for terms of the form $\mu x.P$ that x is guarded in P.

The set of *processes*, denoted with $\mathcal{P}$, is defined as the set of all closed and guarded TACS terms.

$P[Q/x]$ denotes the *simultaneous syntactic substitution* of Q for all *free* occurrences of x, which is done by Barendregt's convention with the usual care to avoid the capturing of free variables, i.e. we may assume that no free variable of Q is bound in P.

2.2 Semantics of TACS

The *operational semantics* of TACS is described by a labelled transition system, which is defined as $\langle \widehat{\mathcal{P}}, \mathcal{A} \cup \{\sigma\}, \longrightarrow \rangle$ where $\widehat{\mathcal{P}}$ is the set of states, $\mathcal{A} \cup \{\sigma\}$ the alphabet and $\longrightarrow \subseteq \widehat{\mathcal{P}} \times (\mathcal{A} \cup \{\sigma\}) \times \widehat{\mathcal{P}}$ the transition relation. In this thesis, the transition relation defined in [LV04] is denoted with $\longrightarrow_1$. Further, we call the newly established, extended transition relation $\longrightarrow_2$. The naming is justified as the type-1-transitions were introduced at an earlier point of time than the type-2-transitions and since the type-2-transitions extend the type-1-transitions. For both types, the operational semantics involves two kind of transitions, action transitions and clock transitions.

As the *structural operational rules* that describe the operational semantics refer to the urgent actions of processes, it is convenient to introduce the *urgent* set of a process before. As in [LV04], we define the urgent set $\mathcal{U}(P)$ of a term $P \in \widehat{\mathcal{P}}$, which includes the set of urgent actions of P, in Table 1. The urgent actions of a processes are those actions in which a process can initially engage and that are not in the scope of a σ-prefix. Note that an urgent τ also results from the occurrence of the matching urgent communication actions a and $\overline{a}$ according to the inductive definition for $\mathcal{U}(P \mid Q)$ in Table 1.

Table 1
Urgent action sets

$\mathcal{U}(\sigma.P)$	$=_{\mathrm{df}} \emptyset$	$\mathcal{U}(\mathbf{0}) = \mathcal{U}(x)$	$=_{\mathrm{df}} \emptyset$	$\mathcal{U}(P \setminus L)$	$=_{\mathrm{df}} \mathcal{U}(P) \setminus (L \cup \overline{L})$
$\mathcal{U}(\alpha.P)$	$=_{\mathrm{df}} \{\alpha\}$	$\mathcal{U}(P+Q)$	$=_{\mathrm{df}} \mathcal{U}(P) \cup \mathcal{U}(Q)$	$\mathcal{U}(P[f])$	$=_{\mathrm{df}} \{f(\alpha) \mid \alpha \in \mathcal{U}(P)\}$
$\mathcal{U}(\mu x.P)$	$=_{\mathrm{df}} \mathcal{U}(P)$	$\mathcal{U}(P\vert Q)$	$=_{\mathrm{df}} \mathcal{U}(P) \cup \mathcal{U}(Q) \cup \{\tau \mid \mathcal{U}(P) \cap \overline{\mathcal{U}(Q)} \neq \emptyset\}$		

Table 2
Operational semantics for TACS (action transitions)

$$\text{Act}\ \frac{-}{\alpha.P \xrightarrow{\alpha}_i P} \qquad \text{Pre}\ \frac{P \xrightarrow{\alpha}_i P'}{\sigma.P \xrightarrow{\alpha}_i P'} \qquad \text{Rec}\ \frac{P \xrightarrow{\alpha}_i P'}{\mu x.P \xrightarrow{\alpha}_i P'[\mu x.P/x]}$$

$$\text{Sum1}\ \frac{P \xrightarrow{\alpha}_i P'}{P+Q \xrightarrow{\alpha}_i P'} \qquad \text{Sum2}\ \frac{Q \xrightarrow{\alpha}_i Q'}{P+Q \xrightarrow{\alpha}_i Q'}$$

$$\text{Com1}\ \frac{P \xrightarrow{\alpha}_i P'}{P|Q \xrightarrow{\alpha}_i P'|Q} \qquad \text{Com2}\ \frac{Q \xrightarrow{\alpha}_i Q'}{P|Q \xrightarrow{\alpha}_i P|Q'} \qquad \text{Com3}\ \frac{P \xrightarrow{a}_i P' \quad Q \xrightarrow{\overline{a}}_i Q'}{P|Q \xrightarrow{\tau}_i P'|Q'}$$

$$\text{Rel}\ \frac{P \xrightarrow{\alpha}_i P'}{P[f] \xrightarrow{f(\alpha)}_i P'[f]} \qquad \text{Res}\ \frac{P \xrightarrow{\alpha}_i P'}{P \setminus L \xrightarrow{\alpha}_i P' \setminus L}\ \alpha \notin L \cup \overline{L}$$

Table 3
Operational semantics for TACS (clock transitions)

$$\text{tNil}\ \frac{-}{\mathbf{0} \xrightarrow{\sigma}_i \mathbf{0}} \qquad \text{tRec}\ \frac{P \xrightarrow{\sigma}_i P'}{\mu x.P \xrightarrow{\sigma}_i P'[\mu x.P/x]} \qquad \text{tRes}\ \frac{P \xrightarrow{\sigma}_i P'}{P \setminus L \xrightarrow{\sigma}_i P' \setminus L}$$

$$\text{tAct}\ \frac{-}{a.P \xrightarrow{\sigma}_i a.P} \qquad \text{tSum}\ \frac{P \xrightarrow{\sigma}_i P' \quad Q \xrightarrow{\sigma}_i Q'}{P+Q \xrightarrow{\sigma}_i P'+Q'} \qquad \text{tRel}\ \frac{P \xrightarrow{\sigma}_i P'}{P[f] \xrightarrow{\sigma}_i P'[f]}$$

$$\text{tPre}\ \frac{-}{\sigma.P \xrightarrow{\sigma}_i P} \qquad \text{tCom}\ \frac{P \xrightarrow{\sigma}_i P' \quad Q \xrightarrow{\sigma}_i Q'}{P|Q \xrightarrow{\sigma}_i P'|Q'}\ \tau \notin \mathcal{U}(P|Q)$$

$$\text{tnew}\ \frac{P \xrightarrow{\sigma}_2 P'}{\sigma.P \xrightarrow{\sigma}_2 P'}$$

Now we are able to define the structural operational rules, abbreviated by SOS-rules where SOS stands for Structured Operational Semantics. The SOS-rules for action transitions are displayed in Table 2, the ones for clock transitions in

Table 3 for $i \in \{1, 2\}$. Both tables are adopted from [LV04] with the exception of the new rule (tnew).
In the sequel we will use indexed SOS-rules in order to, for example, clearly distinguish the SOS-rule $(\text{tNil})_1$ for type-1-transitions from the SOS-rule $(\text{tNil})_2$ for type-2-transitions.

Furthermore, $\longrightarrow_i{}^+$ denotes the transitive closure of the transition relation $\longrightarrow_i$ for $i \in \{1, 2\}$ and $\longrightarrow_i{}^*$ stands for the reflexive and transitive closure.

The action transitions are exactly the same rules as for standard CCS, with the exception of the new rule (Pre). The rule (Pre) is intuitively sensible as $\sigma.P$ has the same functional behaviour as P since $\sigma.P$ can skip the time step and functionally behave like P.
Now we have a closer look at some of the clock transition SOS-rules. The inactive process may idle, i.e. perform a clock transition to itself. Analogously, the term $a.P$ for an $a \in \Lambda \cup \overline{\Lambda}$ may wait for some communication partner when performing a time step without changing its state. The process $\sigma.P$ can wait one time unit and afterwards behave like P. A process of the form $P + Q$ can only perform a time step if time progresses equally on both sides, whence the progress of time does not determine choice but delays it. Processes involving parallel composition of the form $P \,|\, Q$ can only wait if all components are able to let time pass and if the side condition $\tau \notin \mathcal{U}(P \,|\, Q)$ is fulfilled.

In [LV04] the *Maximal Progress Assumption* is employed. According to the Maximal Progress Assumption a system can only let time pass, if it cannot do any internal computation, but has to wait for a communication partner. The approach in [LV04] differs from this strong concept insofar that a process P can only perform a time step if it cannot engage in any *urgent* internal computation or communication with the environment. As an example, a is non-urgent in $\sigma.a.\mathbf{0}$. The process can let time pass and thus can delay any enabled communication on port a by one time unit, even a communication with a process with an urgent $\overline{a}$. However a is urgent in $a.\mathbf{0}$, i.e. the process $a.\mathbf{0}$ must at once engage in a communication with a process with an urgent action $\overline{a}$. If such a communication partner does not exist or such a possible communication partner performs an unshared action $\overline{a}$ or engages in an internal communication with a third port, the process is allowed to wait until some other communication partner is ready. In summary, the system can always let time pass if no urgent internal τ inhibits a time step, i.e. $P \xrightarrow{\sigma}$ if and only if $\tau \notin \mathcal{U}(P)$. This approach concretises the intuition of upper–time bounds and performance guarantee.

The newly established type-2-transitions extend the type-1-time steps by the additional SOS-Rule (tnew). Intuitively, this new rule enables the processes to skip σ-prefixes when performing a time step. As an example consider the process $P \equiv \sigma.\sigma.\sigma.a.b.\mathbf{0}$ that may now either skip one or two time steps when performing a time step by type-2-transitions of the form $\sigma.\sigma.\sigma.a.b.\mathbf{0} \xrightarrow{\sigma}_2 \sigma.a.b.\mathbf{0}$ or $\sigma.\sigma.\sigma.a.b.\mathbf{0} \xrightarrow{\sigma}_2 a.b.\mathbf{0}$. Obviously, there is not an analogous type-1-time step for every type-2 time step. In order to get an impression of the timing behaviour in our new setting, consider the following two examples displaying the labelled transition system for the start processes $\sigma.\sigma\sigma.a.\mathbf{0}$ and $\sigma\sigma.a.\mathbf{0} \mid \sigma.\sigma.a.\mathbf{0}$, restricted to clock transitions. The 'real' type-2-time steps, that are not as well type-1-time steps are illustrated as dashed arrows.

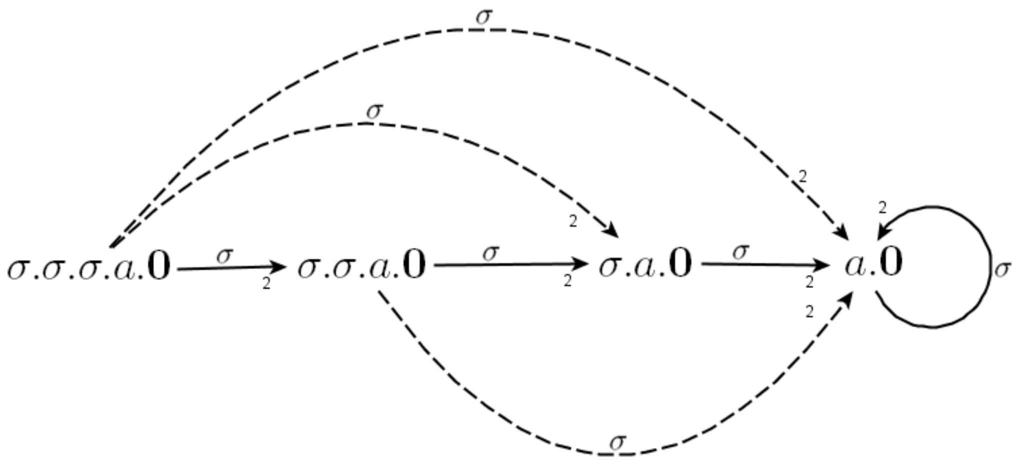


Fig. 1. new transitions

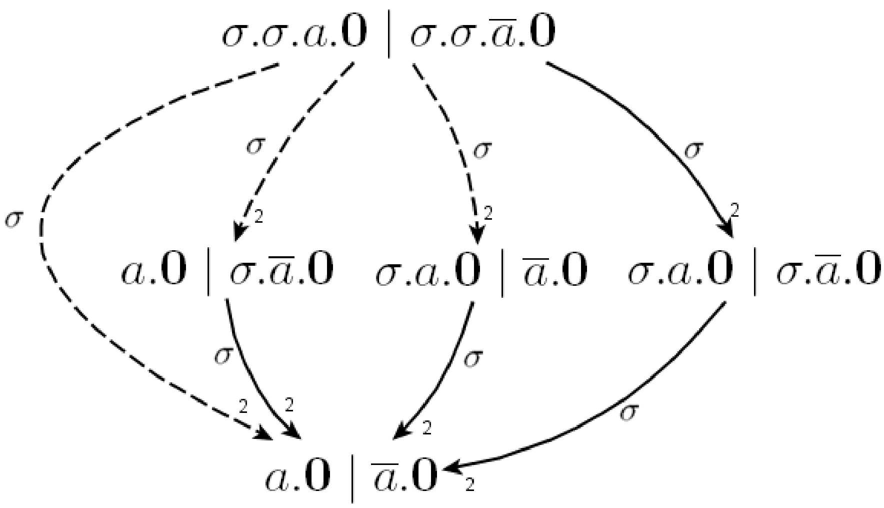


Fig. 2. new transitions for parallel composition

Observe, that processes involving parallel composition are often able to perform a variety of different type-2-time steps as pointed out in Figure 2.

Furthermore, both figures support the conjecture that the transition relation $\longrightarrow_2$ restricted to clock transitions is transitive, which we will state and prove in one of our next lemmas. Before proceeding, it is convenient to establish a lemma that highlights the interplay between our transition relation and substitution and that will be employed in some of the following proofs. Part (2) and Part (3) are given in [LV04] for type-1-transitions and are studied now in our new setting. Thereby, Part (3) is adopted in a simplified formulation. Part (1) is a technical lemma that concerns the preservation of guarded variables under substitution and which is needed for the proofs of Part (2) and Part (3).

Lemma 3 *Let $P, P', Q \in \widehat{\mathcal{P}}$ and $\gamma \in \mathcal{A} \cup \{\sigma\}$.*

(1) [LV04][Lemma 8(1)] *If x is guarded in P, then $\mathcal{U}(P[Q/x]) = \mathcal{U}(P)$.*
(2) $P \xrightarrow{\gamma}_2 P'$ implies $P[\mu y.Q/y] \xrightarrow{\gamma}_2 P'[\mu y.Q/y]$.
(3) y guarded in P and $P[\mu y.Q/y] \xrightarrow{\gamma}_2 P'$ implies
$$\exists P'' \in \widehat{\mathcal{P}}.\, P \xrightarrow{\gamma}_2 P'' \text{ and } P' \equiv P''[\mu y.Q/y].$$

Proof.

Part (1) is taken from [LV04], hence a proof is not necessary. Since Part (2) and Part (3) are not proven in [LV04], we will prove it here at least for $i = 2$. The proofs for case $i = 1$ are contained in them.
The proof of Part (2) is by induction on the inference length of $P \xrightarrow{\gamma}_2 P'$.

(1) Act If $\alpha.P_1 \xrightarrow{\alpha}_2 P_1$ by (Act), we can as well conclude $(\alpha.P_1)[\mu y.Q/y] \equiv \alpha.P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1[\mu y.Q/y]$ by (Act).
(2) Pre If $\sigma.P_1 \xrightarrow{\alpha}_2 P'$, due to $P_1 \xrightarrow{\alpha}_2 P'$ by Rule (Pre), then, by the induction hypothesis, $P_1 \xrightarrow{\alpha}_2 P'$ implies $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P'[\mu y.Q/y]$ By using (Pre) we can conclude $(\sigma.P_1)[\mu y.Q/y] \equiv \sigma.P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P'[\mu y.Q/y]$.
(3) Sum1, Sum2 If $P_1 + P_2 \xrightarrow{\alpha}_2 P_1'$, due to $P_1 \xrightarrow{\alpha}_2 P_1'$ by (Sum1), then by the induction hypothesis, we get $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1'[\mu y.Q/y]$ and $(P_1 + P_2)[\mu y.Q/y] \equiv P_1[\mu y.Q/y] + P_2[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1'[\mu y.Q/y]$ by (Sum1). The case for Rule (Sum2) is analogous.
(4) Com1, Com2 Theses cases are analogous to (Sum1) and (Sum2).
(5) Com3 Consider the case that $P_1|P_2 \xrightarrow{\tau}_2 P_1'|P_2'$, due to $P_1 \xrightarrow{a}_2 P_1'$ and $P_2 \xrightarrow{\overline{a}}_2 P_2'$ by (Com3). We may assume $P_1[\mu y.Q/y] \xrightarrow{a}_2 P_1'[\mu y.Q/y]$ and $P_2[\mu y.Q/y] \xrightarrow{\overline{a}}_2 P_2'[\mu y.Q/y]$ by induction.
$(P_1|P_2)[\mu y.Q/y] \equiv P_1[\mu y.Q/y]|P_2[\mu y.Q/y] \xrightarrow{\tau}_2 P_1'[\mu y.Q/x]|P_2'[\mu y.Q/x] \equiv (P_1'|P_2')[\mu y.Q/y]$ follows by application of Rule (Com3).

(6) Res Let $P_1 \setminus L \xrightarrow{\alpha}_2 P_1' \setminus L$, due to $P_1 \xrightarrow{\alpha}_2 P_1'$ by Rule (Res), where the side condition $\alpha \notin L \cup \overline{L}$ is fulfilled. Then, by the induction hypothesis, $P_1 \xrightarrow{\alpha}_2 P_1'$ implies $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1'[\mu y.Q/y]$.
By using (Res) we can conclude $(P_1 \setminus L)[\mu y.Q/y] \equiv P_1[\mu y.Q/y] \setminus L \xrightarrow{\alpha}_2 P_1'[\mu y.Q/y] \setminus L \equiv (P_1' \setminus L)[\mu y.Q/y]$, since $\alpha \notin L \cup \overline{L}$.

(7) Rel If $P_1[f] \xrightarrow{f(\alpha)}_2 P_1'[f]$, due to $P_1 \xrightarrow{\alpha}_2 P_1'$ by Rule (Rel), then, by the induction hypothesis, $P_1 \xrightarrow{\alpha}_2 P_1'$ implies $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1'[\mu y.Q/y]$.
By using (Rel) we can conclude $(P_1[f])[\mu y.Q/y] \equiv P_1[\mu y.Q/y][f] \xrightarrow{f(\alpha)}_2 P_1'[\mu y.Q/y][f] \equiv (P_1'[f])[\mu y.Q/y]$.

(8) Rec Consider the case that $\mu x.P_1 \xrightarrow{\alpha}_2 P_1'[\mu x.P_1/x]$, due to $P_1 \xrightarrow{\alpha}_2 P_1'$ by (tRec). Since x is neither free in $\mu x.P_1$ nor in $P_1'[\mu x.P_1/x]$ we can assume that $x \not\equiv y$. By Barendregt's convention we can assume that there is no free occurrence of x in Q. We may assume by induction that $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1'[\mu y.Q/y]$. By application of rule (Rec) we obtain $\mu x.(P_1[\mu y.Q/y]) \xrightarrow{\alpha}_2 P_1'[\mu y.Q/y][\mu x.(P_1[\mu y.Q/y])/x]$. Since $x \not\equiv y$ and there is no free occurrence of x in Q, we get
$P_1'[\mu y.Q/y][\mu x.P_1[\mu y.Q/y]/x] \equiv P_1'[\mu x.P_1/x][\mu y.Q/y]$.
Further, $\mu x.(P_1[\mu y.Q/y]) \equiv (\mu x.P_1)[\mu y.Q/y]$ obviously holds, since $x \not\equiv y$ and there is no free occurrence of x in Q.

(9) tNil If $\mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0}$ by Rule (tNil) we infer $\mathbf{0}[\mu y.Q/y] \equiv \mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0} \equiv \mathbf{0}[\mu y.Q/y]$ as well by Rule (tNil).

(10) tAct If $a.P_1 \xrightarrow{\sigma}_2 a.P_1$ by (tAct) we can as well conclude $(a.P_1)[\mu y.Q/y] \equiv a.P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 a.P_1[\mu y.Q/y] \equiv (a.P_1)[\mu y.Q/y]$ by (tAct).

(11) tPre If $\sigma.P_1 \xrightarrow{\sigma}_2 P_1$ by (tPre), then the time step $(\sigma.P_1)[\mu y.Q/y] \equiv \sigma.P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 P_1[\mu y.Q/y]$ can be as well inferred by (tPre).

(11a) (tnew) If $\sigma.P_1 \xrightarrow{\sigma}_2 P'$, due to $P_1 \xrightarrow{\sigma}_2 P'$ by (tnew), then we may assume $P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 P'[\mu y.Q/y]$ by induction and $(\sigma.P_1)[\mu y.Q/y] \equiv \sigma.P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 P'[\mu y.Q/y]$ by (tnew).

(12) tSum If $P_1 + P_2 \xrightarrow{\sigma}_2 P_1' + P_2'$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ and $P_2 \xrightarrow{\sigma}_2 P_2'$ by (tSum), then by the induction hypothesis, we get $P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 P_1'[\mu y.Q/y]$ as well as $P_2[\mu y.Q/y] \xrightarrow{\sigma}_2 P_2'[\mu y.Q/y]$ and hence conclude that $(P_1 + P_2)[\mu y.Q/y] \equiv P_1[\mu y.Q/y] + P_2[\mu y.Q/y] \xrightarrow{\sigma}_2 P_1'[\mu y.Q/y] + P_2'[\mu y.Q/y] \equiv (P_1' + P_2')[\mu y.Q/y]$ by (tSum).

(13) tCom If $P_1|P_2 \xrightarrow{\sigma}_2 P_1' \mid P_2'$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ and $P_2 \xrightarrow{\sigma}_2 P_2'$ by (tCom), then by the induction hypothesis, we get $P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 P_1'[\mu y.Q/y]$ as well as $P_2[\mu y.Q/y] \xrightarrow{\sigma}_2 P_2'[\mu y.Q/y]$ and hence conclude that
$(P_1|P_2)[\mu y.Q/y] \equiv P_1[\mu y.Q/y] \mid P_2[\mu y.Q/y] \xrightarrow{\sigma}_2 P_1'[\mu y.Q/y]|P_2'[\mu y.Q/y] \equiv (P_1'|P_2')[\mu y.Q/y]$ by (tCom). Additionally we must check the side condition, which is fulfilled as $\tau \notin \mathcal{U}(P_1|P_2)$ implies $\tau \notin \mathcal{U}(P_1[\mu y.Q/y]|P_2[\mu y.Q/y])$ by Part (1) of this lemma as y is guarded in P_1 and P_2.

(14) tRes Analogous to (Res) without regarding the side condition.

(15) tRel Analogous to (Rel).

(16) tRec Analogous to (Rec)

The proof of Part (3) is by induction on the structure of P.

(1) $\underline{P \equiv \mathbf{0}}$. The inactive process $\mathbf{0}[\mu y.Q/y] \equiv \mathbf{0}$ is not able to perform an action transition. Any clock transitions is of the form $\mathbf{0}[\mu y.Q/y] \xrightarrow{\sigma}_2 \mathbf{0}$. Then $P'' \equiv \mathbf{0}$ since $\mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0}$ and $\mathbf{0} \equiv \mathbf{0}[\mu y.Q/y]$.

(2) $\underline{P \equiv x}$. If $x \neq y$, then $x[\mu y.Q/y] \equiv x$. Otherwise, if $x = y$, then y is not guarded in $P \equiv x$.

(3) $\underline{P \equiv \alpha.P_1}$. Any α-transition is of the form $(\alpha.P_1)[\mu y.Q/y] \equiv \alpha.P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1[\mu y.Q/y]$ by (Act). Thus, $P'' \equiv P_1$ as $\alpha.P_1 \xrightarrow{\alpha}_2 P_1$ by Rule (Act) and $P_1[\mu y.Q/y] \equiv P_1[\mu y.Q/y]$ obviously holds.
If $(\alpha.P_1)[\mu y.Q/y]$ performs a time step $(\alpha.P_1)[\mu y.Q/y] \equiv \alpha.P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 \alpha.P_1[\mu y.Q/y] \equiv (\alpha.P_1)[\mu y.Q/y]$ by (tAct), then $\alpha \neq \tau$ and $P'' \equiv \alpha.P_1$ with $\alpha.P_1 \xrightarrow{\sigma}_2 \alpha.P_1$ by Rule (tAct) and $(\alpha.P_1)[\mu y.Q/y] \equiv (\alpha.P_1)[\mu y.Q/y]$.

(4) $\underline{P \equiv \sigma.P_1}$ Any action transition is of the form $\sigma.P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P'$, due to $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P'$ by (Pre). We can assume by induction that there exists a P'' with $P_1 \xrightarrow{\alpha}_2 P''$ and $P' \equiv P''[\mu y.Q/y]$. By (Pre) we infer $\sigma.P_1 \xrightarrow{\alpha}_2 P''$ from $P_1 \xrightarrow{\alpha}_2 P''$.
Any clock transition of $\sigma.P_1[\mu y.Q/y]$ can either be performed by Rule (tPre) or Rule (tnew).
Let $\sigma.P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 P_1[\mu y.Q/y]$ by (tPre).
Then, $P'' \equiv P_1$ with $\sigma.P_1 \xrightarrow{\sigma}_2 P_1$ by (tPre) and $P_1[\mu y.Q/y] \equiv P_1[\mu y.Q/y]$.
Consider the case that $\sigma.P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 P'$, due to $P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 P'$ by (tnew). We may assume that there exists some P'' in such a way that $P_1 \xrightarrow{\sigma}_2 P''$ as well as $P' \equiv P''[\mu y.Q/y]$ is fullfilled. Hence, $\sigma.P_1 \xrightarrow{\sigma}_2 P'$ by application of Rule (tnew).

(5) $\underline{P \equiv P_1 + P_2}$ Firstly, any α-transition of $(P_1 + P_2)[\mu y.Q/y]$ is of the form $(P_1 + P_2)[\mu y.Q/y] \equiv P_1[\mu y.Q/y] + P_2[\mu y.Q/y] \xrightarrow{\alpha}_2 P'$ by (Sum1) or by (Sum2). We consider only the case for (Sum1), since the case for (Sum2) is analogous. If $(P_1+P_2)[\mu y.Q/y] \xrightarrow{\alpha}_2 P'$, due to $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P'$ we may assume $P_1 \xrightarrow{\alpha}_2 P''$ as well as $P' \equiv P''[\mu y.Q/y]$ for some P''. Finally, we get $P_1 + P_2 \xrightarrow{\alpha}_2 P''$ by (Sum1).
Secondly, any enabled clock transition is of the form $(P_1+P_2)[\mu y.Q/y] \equiv P_1[\mu y.Q/y] + P_2[\mu y.Q/y] \xrightarrow{\sigma}_2 P_1' + P_2'$, due to $P_1[\mu y.Q/y] \xrightarrow{\sigma}_2 P_1'$ and $P_2[\mu y.Q/y] \xrightarrow{\sigma}_2 P_2'$ by (tSum). Hence, by the induction hypothesis we may assume that $P_1 \xrightarrow{\sigma}_2 P_1''$ and $P_1' \equiv P_1''[\mu y.Q/y]$ for some P_1'' as well as $P_2 \xrightarrow{\sigma}_2 P_2''$ and $P_2' \equiv P_2''[\mu y.Q/y]$ for some P_2''. In conclusion, we define $P'' \equiv P_1'' + P_2''$, get $P_1 + P_2 \xrightarrow{\sigma}_2 P_1'' + P_2''$ by (tSum) and may write $P_1' + P_2' \equiv P_1''[\mu y.Q/y] + P_2''[\mu y.Q/y] \equiv (P_1'' + P_2'')[\mu y.Q/y]$.

(6) $\underline{P \equiv P_1|P_2}$ The cases for (Com1) and (Com2) are analogous to (Sum1) and (Sum2). The case for (Com3) is similar and therefore omitted here. The treatment of a clock transition by (tCom) follows in analogy to (tSum) besides that we additionally must take into consideration the

side condition. For this, observe that $\tau \notin \mathcal{U}(P_1[\mu y.Q/y]|P_2[\mu y.Q/y])$ implies $\tau \notin \mathcal{U}(P_1|P_2)$, since y is guarded in $P_1|P_2$ by using Part (1) of this lemma.

(7) $\underline{P \equiv P_1 \setminus L}$ Any α-transition of $(P_1 \setminus L)[\mu y.Q/y]$ is of the form $(P_1 \setminus L)[\mu y.Q/y] \equiv P_1[\mu y.Q/y] \setminus L \xrightarrow{\alpha}_2 P_1' \setminus L$, due to $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1'$ by (Res) and $\alpha \notin L \cup \overline{L}$. Then, we get $P_1 \xrightarrow{\alpha}_2 P_1''$ and $P_1' \equiv P_1''[\mu y.Q/y]$ for some P_1'' by the induction hypothesis. We are ready since we can define $P'' \equiv P_1'' \setminus L$, obtain $P_1 \setminus L \xrightarrow{\alpha}_2 P_1'' \setminus L$ by (Res), as $\alpha \notin L \cup \overline{L}$, and can infer $P_1' \setminus L \equiv P_1''[\mu y.Q/y] \setminus L \equiv (P_1'' \setminus L)[\mu y.Q/y]$ from $P_1' \equiv P_1''[\mu y.Q/y]$. The case for (tRes) is analogous and even easier as we do not have to regard a side condition.

(8) $\underline{P \equiv P_1[f]}$ Any α-transition of $(P_1[f])[\mu y.Q/y]$ is of the form $(P_1[f])[\mu y.Q/y] \equiv P_1[\mu y.Q/y][f] \xrightarrow{f(\alpha)}_2 P_1'[f]$, due to $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1'$ by (Rel). Then, we get $P_1 \xrightarrow{f(\alpha)}_2 P_1''$ and $P_1' \equiv P_1''[\mu y.Q/y]$ for some P_1'' by the induction hypothesis. We are ready since we can define $P'' \equiv P_1''[f]$, obtain $P_1[f] \xrightarrow{\alpha}_2 P_1''[f]$ by (Rel) and can infer $P_1'[f] \equiv P_1''[\mu y.Q/y][f] \equiv (P_1''[f])[\mu y.Q/y]$ from $P_1' \equiv P_1''[\mu y.Q/y]$.
The case for (tRel) is analogous.

(9) $\underline{\mu x.P_1}$ We may assume that there is no free occurence of x in Q by Barendregt's convention and further may assume that $x \not\equiv y$, since x is not free in $\mu x.P_1$. Any α-transition of $(\mu x.P_1)[\mu y.Q/y]$ is of the form $(\mu x.P_1)[\mu y.Q/y] \equiv \mu x.(P_1[\mu y.Q/y]) \xrightarrow{\alpha}_2 P_1'[\mu x.P_1[\mu y.Q/y]/x]$, due to $P_1[\mu y.Q/y] \xrightarrow{\alpha}_2 P_1'$ by (Rec) and by using the above statements. By induction hypothesis, we may assume $P_1 \xrightarrow{\alpha}_2 P_1''$ and $P_1' \equiv P_1''[\mu y.Q/y]$ for some P_1''. We define $P'' \equiv P_1''[\mu x.P_1/x]$ and obtain $\mu x.P_1 \xrightarrow{\alpha}_2 P_1''[\mu x.P_1/x]$, due to $P_1 \xrightarrow{\alpha}_2 P_1''$ by (Rec). Thus, we are left with showing $P_1'[\mu x.P_1[\mu y.Q/y]/x] \equiv P_1''[\mu x.P_1/x][\mu y.Q/y]$. As $P_1' \equiv P_1''[\mu y.Q/y]$ holds by induction hypothesis, we may conclude $P_1'[\mu x.P_1[\mu y.Q/y]/x]$ $\equiv P_1''[\mu y.Q/y][\mu x.P_1[\mu y.Q/y]/x] \equiv P_1''[\mu x.P_1/x][\mu y.Q/y]$ using the above statements. The treatment of the case for Rule (tRes) is analogous. □

Now we are able to introduce and prove the lemma stating that the transition relation $\xrightarrow{\sigma}_2$ is transitive. In the proof of this lemma and in other following proofs, we will use a property of time steps, which is concerned with the preservation of guardness under a time step: $P \xrightarrow{\sigma}_i P'$ for some $i \in \{1, 2\}$ and x guarded in P implies that x is also guarded in P'. The correctness of this statement is obvious, as a process does not change its functional behaviour during the progress of time. Thus, any action prefix that guards a variable x in P, also guards x in P'. Therefore, we dispense with a proof by induction on the length of inference of $P \xrightarrow{\sigma}_i P'$ for $i \in \{1, 2\}$.

Proposition 4 *Let* $P, P', P'' \in \widehat{\mathcal{P}}$.
$P \xrightarrow{\sigma}_2 P' \xrightarrow{\sigma}_2 P''$ *implies* $P \xrightarrow{\sigma}_2 P''$.

Proof. This proposition can be proved by induction on the structure of P.

(1) $\underline{\mathbf{0}}$: Let $P \equiv \mathbf{0}$. Any clock-transition of P as well as of P' is by (tNil), hence $P' \equiv P'' \equiv \mathbf{0}$. We may conclude $P \equiv \mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0} \equiv P''$ by (tNil).
(2) $\underline{x}$: No clock transitions can be derived by SOS-rules.
(3) $\underline{\alpha.P_1}$: Let $P \equiv \alpha.P_1$.
If $\alpha \equiv \tau$, $\alpha.P_1$ cannot perform a time step. If $\alpha \equiv a$ where $a \in \Lambda \cup \overline{\Lambda}$, any type-2-clock transition of P and P' is by (tAct), hence $P' \equiv P'' \equiv \alpha.P_1$. We are ready, since we get $P \equiv a.P_1 \xrightarrow{\sigma}_2 a.P_1 \equiv P''$ by (tAct).
(4) $\underline{\sigma.P_1}$: Let $P \equiv \sigma.P_1$.
If $\sigma.P_1 \xrightarrow{\sigma}_2 P_1 \equiv P'$ by (tPre) and $P' \xrightarrow{\sigma}_2 P''$ by an arbitrary SOS-rule for clock transitions, we can infer $\sigma.P_1 \equiv \sigma.P' \xrightarrow{\sigma}_2 P''$, due to $P' \xrightarrow{\sigma}_2 P''$ by (tnew) .
Let now be $\sigma.P_1 \xrightarrow{\sigma}_2 P'$, due to $P_1 \xrightarrow{\sigma}_2 P'$ by (tnew) and $P' \xrightarrow{\sigma}_2 P''$ by an arbitrary SOS-rule for clock transitions. Then, by the induction hypothesis, $P_1 \xrightarrow{\sigma}_2 P' \xrightarrow{\sigma}_2 P''$ implies $P_1 \xrightarrow{\sigma}_2 P''$. Using (tnew) we may conclude $\sigma.P_1 \xrightarrow{\sigma}_2 P''$.
(5) $\underline{P_1 + P_2}$: Let $P \equiv P_1 + P_2$.
Any time step of P as well as of P' is of the form $P_1+P_2 \xrightarrow{\sigma}_2 P_1'+P_2' \xrightarrow{\sigma}_2 P_1''+P_2''$, due to $P_1 \xrightarrow{\sigma}_2 P_1' \xrightarrow{\sigma}_2 P_1''$ and $P_2 \xrightarrow{\sigma}_2 P_2' \xrightarrow{\sigma}_2 P_2''$ by (tSum). $P_1 \xrightarrow{\sigma}_2 P_1' \xrightarrow{\sigma}_2 P_1''$ leads to $P_1 \xrightarrow{\sigma}_2 P_1''$ by induction. Analogously, we get $P_2 \xrightarrow{\sigma}_2 P_2''$ and may conclude $P_1 + P_2 \xrightarrow{\sigma}_2 P_1'' + P_2''$ by application of Rule (tSum).
(6) $\underline{P_1|P_2}$: Analogous to (5), observing the side condition $\tau \notin \mathcal{U}(P_1|P_2)$ in the case of (tCom).
(7) $\underline{P_1 \setminus L}$: Any time step of P as well as of P' is of the form $P_1 \setminus L \xrightarrow{\sigma}_2 P_1' \setminus L \xrightarrow{\sigma}_2 P_1'' \setminus L$, due to $P_1 \xrightarrow{\sigma}_2 P_1' \xrightarrow{\sigma}_2 P_1''$ by (tRes). By induction hypothesis, $P_1 \xrightarrow{\sigma}_2 P_1' \xrightarrow{\sigma}_2 P_1''$ implies $P_1 \xrightarrow{\sigma}_2 P_1''$. By application of (tRes), we get $P_1 \setminus L \xrightarrow{\sigma}_2 P_1'' \setminus L$.
(8) $\underline{P_1[f]}$: The treatment of this case is analogous to (7).
(9) $\underline{\mu x.P_1}$: Let $P \equiv \mu x.P_1$.
Consider the case $\mu x.P_1 \xrightarrow{\sigma}_2 P_1'[\mu x.P_1/x]$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ by (tRec) and $P_1'[\mu x.P_1/x] \xrightarrow{\sigma}_2 P''$ by an arbitrary SOS-rule.
Since we require for terms of the form $\mu x.P_1$ that x is guarded in P_1, x is also guarded in P_1', due to $P_1 \xrightarrow{\sigma}_2 P_1'$, as explained above.
Using Lemma 3(3), we obtain $P_1' \xrightarrow{\sigma}_2 P_1''$ and $P'' \equiv P_1''[\mu x.P_1/x]$ for some $P_1'' \in \mathcal{P}$, since x is guarded in P_1'.
By induction, we infer $P_1 \xrightarrow{\sigma}_2 P_1''$ from $P_1 \xrightarrow{\sigma}_2 P_1' \xrightarrow{\sigma}_2 P_1''$. Finally, we may conclude $\mu x.P_1 \xrightarrow{\sigma}_2 P_1''[\mu x.P_1/x] \equiv P''$ by (tRec). □

The following proposition describes the interplay between clock transitions and actions transitions and is adopted from [LV04] for $i = 1$.

Proposition 5 [LV04][Lemma 1 for $i = 1$] *Let P, P', P'' be processes, with no* *occurrence of parallel composition in P, and let $\alpha \in \mathcal{A}$, $i \in \{1,2\}$.*

(1) $P \xrightarrow{\sigma}_i P' \xrightarrow{\alpha}_i P''$ implies $P \xrightarrow{\alpha}_i P''$.
(2) $P \xrightarrow{\sigma}_i P'$ and $P \xrightarrow{\alpha}_i P''$ implies $P' \xrightarrow{\alpha}_i P''$.

Part (1) points out that only upper time bounds are considered in TACS. A process P which is enabled to perform a time step can either let time pass or skip the time step and perform any action transition that P' can engage in.
Part (2) highlights the persistence property of processes, stating that while its functional behaviour is not influenced during the progress of time, the structure of a process can change.
Both parts are given in [LV04] for $i = 1$ (without proof) and are also valid when we include our newly established type-2-transitions.

Proof. Part (1) and Part (2) are not proven in [LV04], we will prove them here for $i = 2$ by induction on the structure of P. The proofs for $i = 1$ are containd in the proofs for $i = 2$.
Proof of Part (1):

(1) $\underline{P \equiv \mathbf{0}}$
The only possible time step of P is of the form $\mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0}$ by (tNil). However the inactive process $\mathbf{0}$ is not able to perform any action transition.

(2) $\underline{P \equiv x}$
No clock transitions can be derived from SOS rules.

(3) $\underline{P \equiv \alpha.P_1}$
If $\alpha \equiv \tau$, then P is not able to perform a time step. Any time step of P is of the form $a.P_1 \xrightarrow{\sigma}_2 a.P_1$ for $a \in \Lambda \cup \overline{\Lambda}$ by Rule (tAct). $a.P_1 \xrightarrow{a}_2 P_1$ by (Act) is the only action transition $a.P_1$ can perform. $a.P_1 \xrightarrow{a}_2 P_1$ is obvious by rule (Act).

(4) $\underline{P \equiv \sigma.P_1}$
$\sigma.P_1$ can perform a time step by Rule (tPre) or (tnew).
If $\sigma.P_1 \xrightarrow{\sigma}_2 P_1 \xrightarrow{\alpha}_2 P''$ by application of Rule (tPre) for the time step, then we conclude $\sigma.P_1 \xrightarrow{\alpha}_2 P''$ from $P_1 \xrightarrow{\alpha}_2 P''$ by Rule (Pre).
If $\sigma.P_1 \xrightarrow{\sigma}_2 P' \xrightarrow{\alpha}_2 P''$, due to $P_1 \xrightarrow{\sigma}_2 P'$ by using (tnew), then, by the induction hypothesis, $P_1 \xrightarrow{\sigma}_2 P' \xrightarrow{\alpha}_2 P''$ implies $P_1 \xrightarrow{\alpha}_2 P''$. Thus, we conclude that $\sigma.P_1 \xrightarrow{\alpha}_2 P''$ by application of Rule (Pre).

(5) $\underline{P \equiv P_1 + P_2}$
Any clock transition of $P_1 + P_2$ is of the form $P_1 + P_2 \xrightarrow{\sigma}_2 P_1' + P_2'$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ as well as $P_2 \xrightarrow{\sigma}_2 P_2'$ by (tSum). Further, $P_1' + P_2'$ is able to perform an action transition by (Sum1) or (Sum2). If $P_1' + P_2' \xrightarrow{\alpha}_2 P_1''$, due to $P_1' \xrightarrow{\alpha}_2 P_1''$ by (Sum1), then, by the induction hypothesis, $P_1 \xrightarrow{\sigma}_2 P_1' \xrightarrow{\alpha}_2 P_1''$ implies $P_1 \xrightarrow{\alpha}_2 P_1''$. Using (Sum1) we conclude $P_1 + P_2 \xrightarrow{\alpha}_2 P_1''$.
The case for (Sum2) is analogous to the previous case.

(6) $\underline{P \equiv P_1 \setminus L}$
Any clock transition of $P_1 \setminus L$ is of the form $P_1 \setminus L \xrightarrow{\sigma}_2 P_1' \setminus L$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ by (tRes). The only action transitions of $P_1' \setminus L$ are of the form $P_1' \setminus L \xrightarrow{\alpha}_2 P_1'' \setminus L$, due to $P_1' \xrightarrow{\alpha}_2 P_1''$ by (Res), provided that $\alpha \notin L \cup \overline{L}$. Then, according to the induction hypothesis, $P_1 \xrightarrow{\sigma}_2 P_1' \xrightarrow{\alpha}_2 P_1''$ implies $P_1 \xrightarrow{\alpha}_2 P_1''$. We conclude $P_1 \setminus L \xrightarrow{\alpha}_2 P_1'' \setminus L$ by (Res) since $\alpha \notin L \cup \overline{L}$.

(7) $\underline{P \equiv P_1[f]}$
This case is similar to the previous case and therefore omitted here.

(8) $\underline{P \equiv \mu x.P_1}$
$\mu x.P_1$ can only perform clock transitions of the form $\mu x.P_1 \xrightarrow{\sigma}_2 P_1'[\mu x.P_1/x]$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ by (tRec). Further, $P_1'[\mu x.P_1/x]$ may perform an action-transition by an arbitrary SOS rule for action transitions of the form $P_1'[\mu x.P_1/x] \xrightarrow{\alpha}_2 P_1''$. Note, that we require for terms of the form $\mu x.P_1$ that x is guarded in P_1. Due to to $P_1 \xrightarrow{\sigma}_2 P_1'$, x is as well guarded in P_1' using the usual argumentation. Hence, we obtain $P_1' \xrightarrow{\alpha}_2 P''$ and $P_1'' \equiv P''[\mu x.P_1/x]$ for some P'' by using Lemma 3(3). Consequently we may assume $P_1 \xrightarrow{\alpha}_2 P''$ by induction hypothesis and infer $\mu x.P_1 \xrightarrow{\alpha}_2 P''[\mu x.P_1/x] \equiv P_1''$ by (Rec). $\square$

Proof of Part (2):

(1) $\underline{P \equiv \mathbf{0}}$
Any possible clock transition of P is of the form $\mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0}$ by (tNil) However, the inactive process is not able to perform an action transition.

(2) $\underline{P \equiv x}$
Neither clock transitions nor action transitions can be derived from SOS rules.

(3) $\underline{P \equiv \alpha.P_1}$.
Any clock transition of P is of the form $\alpha.P_1 \xrightarrow{\sigma}_2 \alpha.P_1$ for $\alpha \equiv a$ by Rule (tAct) and any action transition is of the form $\alpha.P_1 \xrightarrow{\alpha}_2 P_1$ by (Act). $\alpha.P_1 \xrightarrow{\alpha}_2 P_1$ is obvious by rule (Act).

(4) $\underline{P \equiv \sigma.P_1}$
$\sigma.P_1$ can perform a time step by Rule (tPre) or (tnew).
Firstly, we consider the case that $\sigma.P_1 \xrightarrow{\sigma}_2 P_1$ by application of Rule (tPre) for the time step. Since the α-transitions of P are of the form $\sigma.P_1 \xrightarrow{\alpha}_2 P''$, for $P_1 \xrightarrow{\alpha}_2 P''$ by the assumption of Rule (Pre), we are done.
Secondly, we consider the case that $\sigma.P_1 \xrightarrow{\sigma}_2 P'$, due to $P_1 \xrightarrow{\sigma}_2 P'$ by using (tnew). Analogously to the previous case, the α-transitions of P are of the form $\sigma.P_1 \xrightarrow{\alpha}_2 P''$ and we get $P_1 \xrightarrow{\alpha}_2 P''$ by Rule (Pre). Further, we conclude $P' \xrightarrow{\alpha}_2 P''$ from $P_1 \xrightarrow{\sigma}_2 P'$ and $P_1 \xrightarrow{\alpha}_2 P''$, by the induction hypothesis.

(5) $\underline{P \equiv P_1 + P_2}$
Any clock transition of $P_1 + P_2$ is of the form $P_1 + P_2 \xrightarrow{\sigma}_2 P_1' + P_2'$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ as well as $P_2 \xrightarrow{\sigma}_2 P_2'$ by (tSum). Moreover, the process

$P_1 + P_2$ is able to perform an action transition by (Sum1) or (Sum2). If $P_1 + P_2 \xrightarrow{\alpha}_2 P_1''$, due to $P_1 \xrightarrow{\alpha}_2 P_1''$ by (Sum1), then, by the induction hypothesis, $P_1 \xrightarrow{\sigma}_2 P_1'$ and $P_1 \xrightarrow{\alpha}_2 P_1''$ implies $P_1' \xrightarrow{\alpha}_2 P_1''$. Using (Sum1) we conclude $P_1' + P_2' \xrightarrow{\alpha}_2 P_1''$.
The case for (Sum2) is analogous to the previous case.

(6) $\underline{P \equiv P_1 \setminus L}$
$P_1 \setminus L$ can only perform a time step of the form $P_1 \setminus L \xrightarrow{\sigma}_2 P_1' \setminus L$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ by (tRes). Any action transition of $P_1 \setminus L$ is of the form $P_1 \setminus L \xrightarrow{\alpha}_2 P_1'' \setminus L$, due to $P_1 \xrightarrow{\alpha}_2 P_1''$ by (Res), provided that $\alpha \notin L \cup \overline{L}$. Then, according to the induction hypothesis, $P_1 \xrightarrow{\sigma}_2 P_1'$ and $P_1 \xrightarrow{\alpha}_2 P_1''$ implies $P_1' \xrightarrow{\alpha}_2 P_1''$. We conclude $P_1' \setminus L \xrightarrow{\alpha}_2 P_1'' \setminus L$ by (Res) since $\alpha \notin L \cup \overline{L}$.

(7) $\underline{P \equiv P_1[f]}$
This case is similar to the previous case and therefore omitted here.

(8) $\underline{P \equiv \mu x.P_1}$
Any clock transition of $\mu x.P_1$ is of the form $\mu x.P_1 \xrightarrow{\sigma}_2 P_1'[\mu x.P_1/x]$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ by (tRec). Further, any α-transition is of the form $\mu x.P_1 \xrightarrow{\alpha}_2 P_1''[\mu x.P_1/x]$, due to $P_1 \xrightarrow{\alpha}_2 P_1''$ by (Rec). We obtain $P_1' \xrightarrow{\alpha}_2 P_1''$ by induction hypothesis and are ready since we can infer
$P_1'[\mu x.P_1/x] \xrightarrow{\alpha}_2 P_1''[\mu x.P_1/x]$ by application of Lemma 3(2). $\square$

As we have not gained any knowledge about the coherence between the two different types of transitions so far, we had to prove Proposition 5 for $i = 2$ independently of Proposition 5 for $i = 1$. Later in this work, a more elegant proof exploiting this coherence could be given.

Note that both statements are not valid for processes involving parallel composition. $P \xrightarrow{\sigma}_i P' \xrightarrow{\alpha}_i P''$ implies $P \xrightarrow{\alpha}_i P'''$ but $P'' \equiv P'''$ is not always valid as components of P''' may have additional leading σ-prefixes.
Similarly, we can infer from $P \xrightarrow{\sigma}_i P'$ and $P \xrightarrow{\alpha}_i P''$ that $P' \xrightarrow{\alpha}_i P'''$, but we may have $P'' \not\equiv P'''$ due to potential additional leading σ-prefix of P'''.
As an example consider the following transitions:

$\sigma.\sigma.a.\mathbf{0} \mid \sigma.\sigma.b.\mathbf{0} \xrightarrow{\sigma}_2 a.\mathbf{0} \mid \sigma.b.\mathbf{0} \xrightarrow{b}_2 a.\mathbf{0} \mid \mathbf{0}$ but only
$\sigma.\sigma.a.\mathbf{0} \mid \sigma.\sigma.b.\mathbf{0} \xrightarrow{b}_2 \sigma\sigma.a.\mathbf{0} \mid \mathbf{0}$.

$\sigma.\sigma.a.\mathbf{0} \mid \sigma.\sigma.b.\mathbf{0} \xrightarrow{\sigma}_2 a.\mathbf{0} \mid \sigma.b.\mathbf{0}$ and $\sigma.\sigma.a.\mathbf{0} \mid \sigma.\sigma.b.\mathbf{0} \xrightarrow{b}_2 \sigma.\sigma.a.\mathbf{0} \mid \mathbf{0}$ but
$a.\mathbf{0} \mid \sigma.b.\mathbf{0} \not\xrightarrow{b}_2 \sigma\sigma.a.\mathbf{0} \mid \mathbf{0}$.

3 Coherence between $\longrightarrow_1$ and $\longrightarrow_2$

The objective of this section is to develop the syntactic and semantic coherence between type-1-transitions and our newly established type-2-transitions.

Type-1-transitions $\longrightarrow_1$ and type-2-transitions $\longrightarrow_2$ do not differ in their functional behaviour. Moreover, the analogous type-2-time step to every type-1 time step exists. These two characteristics are stated in a first simple lemma:

Lemma 6 *Let $P, P' \in \widehat{\mathcal{P}}$, $\alpha \in \mathcal{A}$.*

(1) $P \xrightarrow{\alpha}_1 P'$ if and only if $P \xrightarrow{\alpha}_2 P'$
(2) $P \xrightarrow{\sigma}_1 P'$ implies $P \xrightarrow{\sigma}_2 P'$.

Proof. The statement of Part (1) follows from the fact that the set of SOS-rules for action transitions of type-1 and type-2 are identical. In particular, this statement is true since the type-2-transitions only rely on action transitions.

Part (2) is valid since the set of SOS-rules for type-2-time steps extends the SOS-rules for type-1-time steps by an additional type-2-transition rule. Hence, every type-1-time step can be derived from the analogous SOS-rules for clock transitions of type-2. $\square$

3.1 The syntactic relation $\succ$

In [LV04] the syntactic relation $\succ$ on terms is introduced as a useful technical handle for proving coincidence results between the different candidates for a faster-than preorder.

Definition 7 [LV04, Definition 6]
The relation $\succ \subseteq \widehat{\mathcal{P}} \times \widehat{\mathcal{P}}$ is defined as the smallest relation satisfying the following properties, for all $P, P', Q, Q' \in \widehat{\mathcal{P}}$.

$$\begin{array}{rll} \textit{Always:} & (1)\ P \succ P & (2)\ P \succ \sigma.P \\ \textit{If } P' \succ P,\ Q' \succ Q\textit{:} & (3)\ P'|Q' \succ P|Q & (4)\ P' + Q' \succ P + Q \\ & (5)\ P' \setminus L \succ P \setminus L & (6)\ P'[f] \succ P[f] \\ \textit{If } P' \succ P,\ x \textit{ guarded in } P\textit{:} & (7)\ P'[\mu x.\, P/x] \succ \mu x.\, P & \end{array}$$

Note that the syntactic relation is defined for arbitrary open terms and is not restricted to processes. Later in this work, we will claim and prove that the syntactic relation, restricted to processes, satisfies the definition of a naive-faster-than-relation, which is itself restricted to processes.

Yet, we must require that the definition of the syntactic relation is not restricted to processes. Consider the following example of a derivation:

(1) $a.x \succ \sigma.a.x$ by Def. 7(2)
(2) $a.x[\mu x.\,\sigma.a.x/x] \equiv a.\mu x.\,\sigma.a.x \succ \mu x.\,\sigma.a.x$ by Def. 7(7)

We have open, guarded terms in the first derivation step, in order to obtain processes in the second derivation step. The syntactic relationship $a.x[\mu x.\,\sigma.a.x/x] \succ \mu x.\,\sigma.a.x$ could not be derived, if the definition of the syntactic relation was restricted to processes.

The syntactic relation $\succ$ is not transitive. As this property is however desired in the following, we need to define the transitive closure $\succ^+$.

Definition 8
$\succ^+ \subseteq \widehat{\mathcal{P}} \times \widehat{\mathcal{P}}$ is defined as the transitive closure of $\succ$, i.e. $P \succ^+ P'$ if and only if $\exists n \geq 1\ \exists P_0, \ldots, P_n \in \widehat{\mathcal{P}} : P \equiv P_0 \succ P_1 \succ \ldots \succ P_n \equiv P'$.

The objective in [LV04] was to construct the syntactic relation in such a way that the following coherence between a type-1-time step and the speed of TACS terms holds:

$$P \xrightarrow{\sigma}_1 P' \ \text{ implies } \ P' \succ P \ \text{ for all } P, P' \in \widehat{\mathcal{P}}$$

(cf. Proposition 12(1))

In addition, the coherence between type-2-time steps and speed will be established in this work, namely

$$P \xrightarrow{\sigma}_2 P' \ \text{ implies } \ P' \succ^+ P \ \text{ for all } P, P' \in \widehat{\mathcal{P}}$$

(cf. Proposition 12(2))

Before proceeding, we need to establish two technical lemmas that are in parts adopted from [LV04].

Lemma 9 (Preservation results) *Let $P, P' \in \widehat{\mathcal{P}}$ such that $P' \succ P$, and let $y \in \mathcal{V}$.*

(1) [LV04, Lemma 7(1)] *Then y is guarded in P if and only if y is guarded in P'.*
(2) *Then y is free in P if and only if y is free in P'*
(3) *Then $P \in \mathcal{P}$ if and only if $P' \in \mathcal{P}$*
(4) [LV04, Lemma 7(2)] *Then $P'[Q/y] \succ P[Q/y]$.*

Proof. The proofs of Part (1) and Part (4) can be found in Appendix A in [LV04]. Part (2) follows by induction on the inference length of $P' \succ P$. The only interesting case is Case (7) of Def. 7, the other cases are obvious:

(7) $\underline{P'[\mu x.P/x] \succ \mu x.P}$ due to $P' \succ P$, x guarded in P.
If $y = x$, x is neither free in $\mu x.P$ nor in $P'[\mu x.P/x]$. Any occurrence of x is bound in $\mu x.P$ and each free occurrence of x in P' is substituted by $\mu x.P$, hence, all occurrences of x are bound in $P'[\mu x.P/x]$.
Next, we consider the case $y \neq x$.
If there exists a free occurrence of y in $P'[\mu x.P/x]$, then there is also one in P or P'. If there is a free occurrence of y in P, then this occurrence is also present after falling into the scope of the recursion operator μx. If y occurs free in P', then, by the induction hypothesis, y occurs free in P and as well free in $\mu x.P$.
If there is a free occurrence of y in $\mu x.P$, then y is as well free in P, and by the induction hypothesis in P' and therefore in $P'[\mu x.P/x]$.

Proving Part (3), no variables occur free in P if P is a process. By using Part (2) of this lemma, no variables occur free in P' as well. Consequently, P' is as well a process.
If P' is a process, then there are no free occurrences of variables in P' and no variables occur free in P by Part (2), whence P is a process. $\square$

Lemma 10 [LV04, Lemma 8(2)]
Let $P, Q \in \widehat{\mathcal{P}}$.
If $Q \succ P$, then $\mathcal{U}(Q) \supseteq \mathcal{U}(P)$.

The following lemma establishes several important properties of $\succ^+$ similar to those of the syntactic relation in Def. 7.

Lemma 11 *Let* $P, P', Q, Q' \in \widehat{\mathcal{P}}$.

If $P' \succ^+ P$, $Q' \succ^+ Q$ *then:*	(1) $P'\|Q' \succ^+ P\|Q$
	(2) $P' + Q' \succ^+ P + Q$
If $P' \succ^+ P$ *then:*	(3) $P' \setminus L \succ^+ P \setminus L$
	(4) $P'[f] \succ^+ P[f]$
If $P' \succ^+ P$, x *guarded in* P *then:*	(5) $P'[\mu x.\, P/x] \succ^+ \mu x.\, P$

Proof.

(1) $P' \succ^+ P$ and $Q' \succ^+ Q$ implies
$\exists n \geq 1\ \exists P_0, ...P_n \in \widehat{\mathcal{P}} : P' \equiv P_0 \succ P_1 \cdots \succ P_n \equiv P$ and
$\exists m \geq 1\ \exists Q_0, ...Q_m \in \widehat{\mathcal{P}} : Q' \equiv Q_0 \succ Q_1 \cdots \succ Q_m \equiv Q$
by definition of the transitive closure of $\succ$.
If $n \leq m$, we have $P_{i-1} \succ P_i$ for $1 \leq i \leq n$ and $Q_{i-1} \succ Q_i$ for $1 \leq i \leq n$.
Then, $P_{i-1} \,|\, Q_{i-1} \succ P_i \,|\, Q_i$ for $1 \leq i \leq n$ follows by Def. 7(3).
Further, $P_n \succ P_n$ is valid by Part (1) of Def. 7.
Since $Q_{j-1} \succ Q_j$ for $n < j \leq m$, we get $P_n \,|\, Q_{j-1} \succ P_n \,|\, Q_j$ for $n < j \leq m$ by using Part (3) of Def. 7.
In summary, we obtain $P' \,|\, Q' \equiv P_0 \,|\, Q_0 \succ^+ P_n \,|\, Q_m \equiv P \,|\, Q$.
The case $n < m$ leads to an argumentation analogous to the previous case.

(2) The proof of this case is analogous to Part (1) and uses Def. 7(4).

(3) $P' \succ^+ P$ means that
$\exists n \geq 1\ \exists P_0, ...P_n \in \widehat{\mathcal{P}} : P' \equiv P_0 \succ P_1 \cdots \succ P_n \equiv P$.
Then, by repeated application of Def. 7(5), we get $P \setminus L \equiv P_0 \setminus L \succ P_1 \setminus L \succ \cdots \succ P_{n-1} \setminus L \succ P_n \setminus L \equiv P' \setminus L$.
Finally, we conclude $P \setminus L \succ^+ P' \setminus L$ by the definition of $\succ^+$.

(4) The proof follows in analogy to Part (3), using Def. 7(6).

(5) $P' \succ^+ P$ is defined as $\exists n \geq 1\ \exists P_0, ...P_n \in \widehat{\mathcal{P}} : P' \equiv P_0 \succ P_1 \cdots \succ P_n \equiv P$. If x is guarded in P, we can infer $P_{n-1}[\mu x.\, P_n/x] \succ \mu x.\, P_n$ from $P_{n-1} \succ P_n$ by using Def. 7(7).
Further, we obtain $P_{i-1}[\mu x.\, P_n/x] \succ P_i[\mu x.\, P_n/x]$, due to $P_{i-1} \succ P_i$ for $1 \leq i \leq n-1$ by Lemma 9(4). To sum up, we have $P'[\mu x.\, P/x] \equiv P_0[\mu x.\, P_n/x] \succ \cdots \succ P_{n-1}[\mu x.\, P_n/x] \succ \mu x.\, P_n \equiv \mu x.\, P$ and therefore $P'[\mu x.\, P/x] \succ^+ \mu x.\, P$. $\square$

Now we have established the machinery needed to prove the above coherence between time steps and speed:

Lemma 12 *Let $P, P' \in \widehat{\mathcal{P}}$.*

(1) [LV04, Prop. 9(1)] *$P \xrightarrow{\sigma}_1 P'$ implies $P' \succ P$, for all terms $P, P' \in \widehat{\mathcal{P}}$.*
(2) $P \xrightarrow{\sigma}_2 P'$ implies $P' \succ^+ P$, for all terms $P, P' \in \widehat{\mathcal{P}}$.

Proof. We give a more detailed proof of Part (1) here, it is an induction on the length of inference of $P \xrightarrow{\sigma}_1 P'$.

(1) <u>tNil:</u> $P \equiv \mathbf{0} \equiv P'$. $\mathbf{0} \succ \mathbf{0}$ trivially holds by Def. 7(1).
(2) <u>tAct:</u> $P \equiv a.P'' \equiv P'$. $a.P'' \succ a.P''$ trivially holds by Def. 7(1).
(3) <u>tPre:</u> $P \equiv \sigma.P'$. Then $P' \succ \sigma.P'$ by Def. 7(2).
(4) <u>tRec:</u> $P \equiv \mu x.\, P_1$ and $P' \equiv P_2[\mu x.\, P_1/x]$.
Let $\mu x.\, P_1 \xrightarrow{\sigma}_1 P_2[\mu x.\, P_1/x]$, due to $P_1 \xrightarrow{\sigma}_1 P_2$. We obtain $P_2 \succ P_1$ by the induction hypothesis. Since x is guarded in P_1, we infer $P_2[\mu x.\, P_1/x] \succ \mu x.\, P_1$ from $P_2 \succ P_1$ by using Def. 7(7).
(5) <u>tSum:</u> $P \equiv P_1 + Q_1$ and $P' \equiv P_2 + Q_2$. Since $P \xrightarrow{\sigma}_1 P'$, we have $P_1 \xrightarrow{\sigma}_1 P_2$ and $Q_1 \xrightarrow{\sigma}_1 Q_2$. According to the induction hypothesis, we obtain $P_2 \succ P_1$ and $Q_2 \succ Q_1$ and hence may conclude $P_2 + Q_2 \succ P_1 + Q_1$ by application of Def. 7(4).
(6) <u>tCom:</u> The treatment of this case follows in analogy to case (5) by using Def. 7(3)
(7) <u>tRes:</u> $P \equiv P_1 \setminus L$ and $P' \equiv P_2 \setminus L$. Then, $P_1 \xrightarrow{\sigma}_1 P_2$ implies $P_2 \succ P_1$ by induction. Using Def. 7(5), we obtain $P_2 \setminus L \succ P_1 \setminus L$.
(8) <u>tRel:</u> This case is analogous to case (7), using Def. 7(6).

Part (2) can be proved by induction on the length of inference of $P \xrightarrow{\sigma}_2 P'$.

(1) <u>tNil:</u> $P \equiv \mathbf{0} \equiv P'$. Since $\succ \subseteq \succ^+$, $\mathbf{0} \succ^+ \mathbf{0}$ holds by using Def. 7(1).
(2) <u>tAct:</u> $P \equiv a.P'' \equiv P'$. Since $\succ \subseteq \succ^+$, $a.P'' \succ^+ a.P''$ holds by using Def. 7(1).
(3) <u>tPre:</u> $P \equiv \sigma.P'$. Since $\succ \subseteq \succ^+$, $P' \succ^+ \sigma.P'$ holds by using Def. 7(2).
(3a) <u>tnew:</u> $P \equiv \sigma.P_1$. Let $\sigma.P_1 \xrightarrow{\sigma}_2 P'$, due to $P_1 \xrightarrow{\sigma}_2 P'$.
Then, $P_1 \xrightarrow{\sigma}_2 P'$ implies $P' \succ^+ P_1$ by the induction hypothesis. Finally, we get $P_1 \succ \sigma.P_1$ by Def. 7(2) and may conclude $P' \succ^+ P_1 \succ \sigma.P_1$ and therefore $P' \succ^+ \sigma.P_1$.
(4) <u>tRec:</u> $P \equiv \mu x.\, P_1$ and $P' \equiv P_2[\mu x.\, P_1/x]$.
Let $\mu x.\, P_1 \xrightarrow{\sigma}_2 P_2[\mu x.\, P_1/x]$, due to $P_1 \xrightarrow{\sigma}_2 P_2$. By the induction hypothesis, $P_1 \xrightarrow{\sigma}_2 P_2$ implies $P_2 \succ^+ P_1$. Since x is guarded in P_1, we can infer $P_2[\mu x.\, P_1/x] \succ^+ \mu x.\, P_1$ from $P_2 \succ^+ P_1$ by Lemma 11(5) .
(5) <u>tSum:</u> $P \equiv P_1 + Q_1$ and $P' \equiv P_2 + Q_2$.
Since $P \xrightarrow{\sigma}_2 P'$, we have $P_1 \xrightarrow{\sigma}_2 P_2$ and $Q_1 \xrightarrow{\sigma}_2 Q_2$. $P_2 \succ^+ P_1$ and $Q_2 \succ^+ Q_1$ follows by induction hypothesis and $P_2 + Q_2 \succ^+ P_1 + Q_1$ results by application of Lemma 11(2).
(6) <u>tCom:</u> The treatment of this case is analogous to case (5) and uses

Lemma 11(1).

(7) tRes: $P \equiv P_1 \setminus L$ and $P' \equiv P_2 \setminus L$.
We obtain $P_2 \succ^+ P_1$ by induction and $P_2 \setminus L \succ^+ P_1 \setminus L$ by Lemma 11(3).

(8) tRel: This case follows in analogy to case (7), using Lemma 11(4). $\square$

In the follwing two subsections, we will work out the differences as well as the syntactic and semantic coherences between our two transition types.

3.2 Syntactic coherence

The definition of the type-2-time steps extends those of the type-1-time steps by one additional SOS-rule (tnew). Thus, as described in Lemma 6(2), any type-1-time step obviously is as well a type-2-time step.
Now we investigate in what way there exists a corresponding type-1-time step to a type-2-time step.

In many cases a type-2-time step of the form $P \xrightarrow{\sigma}_2 P'$ can be matched by a corresponding sequence of type-1-time steps of the form $P \xrightarrow{\sigma}_1{}^+ P'$. As an example, $\sigma.P \xrightarrow{\sigma}_2 P$ is valid by (tPre). We get $\sigma.\sigma.\sigma.P \xrightarrow{\sigma}_2 P$ by using (tnew) twice. The type-1-time step sequence $\sigma.\sigma.\sigma.P \xrightarrow{\sigma}_1 \sigma.\sigma.P \xrightarrow{\sigma}_1 \sigma.P \xrightarrow{\sigma}_1 P$ matches this type-2-time step.

That this is not always the case, is exemplarily illustrated by the process $P =_{\mathrm{df}} \sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, \sigma.a.\mathbf{0}$, which may perform the type-2-time step $\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, \sigma.a.\mathbf{0} \xrightarrow{\sigma}_2 a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \,{}_{\mathrm{df}} = P'$. The only enabled type-1-time step is of the form $\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, \sigma.a.\mathbf{0} \xrightarrow{\sigma}_1 \sigma.\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \,{}_{\mathrm{df}} = P''$. Now $P'' \not\xrightarrow{\sigma}$ as $\tau \in \mathcal{U}(P'')$. Thus, we observe that there exists a corresponding type-1-time step of the form $P \xrightarrow{\sigma}_1 P''$ to every type-2-time step $P \xrightarrow{\sigma}_2 P'$, but we may have $P' \neq P''$ for the result processes. A comparison of the result processes $P' \equiv a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}$ and $P'' \equiv \sigma.\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}$ of the above example supports the conjecture that P' results from P'' by removing none, one or several leading σ-prefixes from each component.

This relationship seems to correspond to the transitive closure $\succ^+$ of the syntactic relation and, as a matter of fact, we are able to show:

Lemma 13 *Let $P, P' \in \widehat{\mathcal{P}}$.*
$P \xrightarrow{\sigma}_2 P'$ implies $\exists P'' \in \widehat{\mathcal{P}}.\ P \xrightarrow{\sigma}_1 P''$ and $P' \succ^+ P''$.

Proof. The proof is an induction on the length of inference of $P \xrightarrow{\sigma}_2 P'$.

(1) tNil: $P \equiv \mathbf{0} \equiv P'$.
Let $\mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0}$ by (tNil_2).
Then, $\mathbf{0} \xrightarrow{\sigma}_1 \mathbf{0}$ holds by (tNil_1). Since $\succ \subseteq \succ^+$, we get $\mathbf{0} \succ^+ \mathbf{0}$ by Def. 7(1).

(2) tAct: $P \equiv a.P'' \equiv P'$.
Let $a.P'' \xrightarrow{\sigma}_2 a.P''$ by (tAct_2).
Then, $a.P'' \xrightarrow{\sigma}_1 a.P''$ holds by (tAct_1). As $\succ \subseteq \succ^+$, we obtain $a.P'' \succ^+ a.P''$ by Def. 7(1).

(3) tPre: $P \equiv \sigma.P'$.
Let $\sigma.P' \xrightarrow{\sigma}_2 P'$ by (tPre_2).
Then, we can derive $\sigma.P' \xrightarrow{\sigma}_1 P'$ by (Pre_1) and as well get $P' \succ^+ P'$ by Def. 7(1) and $\succ \subseteq \succ^+$.

(3a) tnew: $P \equiv \sigma.P''$.
Let $\sigma.P'' \xrightarrow{\sigma}_2 P'$, due $P'' \xrightarrow{\sigma}_2 P'$ by (tnew).
Then, $\sigma.P'' \xrightarrow{\sigma}_1 P''$ holds by (Pre_1). We are ready since we can conclude $P' \succ^+ P''$ from $P'' \xrightarrow{\sigma}_2 P'$ by using Lemma 12(2).

(4) tRec: $P \equiv \mu x.\, P_1$ and $P' \equiv P_1'[\mu x.\, P_1/x]$.
Let $\mu x.\, P_1 \xrightarrow{\sigma}_2 P_1'[\mu x.\, P_1/x]$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ by (tRec_2).
By the induction hypothesis, there exists a P_2' such that $P_1 \xrightarrow{\sigma}_1 P_2'$ and $P_1' \succ^+ P_2'$.
Thus, we may infer $\mu x.\, P_1 \xrightarrow{\sigma}_1 P_2'[\mu x.\, P_1/x]$ by (tRec_1). $P_1' \succ^+ P_2'$ is defined as $\exists n \geq 1\ \exists P_0, ...P_n \in \widehat{\mathcal{P}} : P_1' \equiv P_0 \succ P_1 \cdots \succ P_n \equiv P_2'$.
Further, $P_1'[\mu x.\, P_1/x] \equiv P_0[\mu x.\, P_1/x] \succ P_1[\mu x.\, P_1/x] \cdots \succ P_n[\mu x.\, P_1/x] \equiv P_2'[\mu x.\, P_1/x]$ follows by application of Lemma 9(4) and we may hence conclude $P_1'[\mu x.\, P_1/x] \succ^+ P_2'[\mu x.\, P_1/x]$.

(5) tSum: $P \equiv P_1 + Q_1$ and $P' \equiv P_1' + Q_1'$.
Let $P_1 + Q_1 \xrightarrow{\sigma}_2 P_1' + Q_1'$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ and $Q_1 \xrightarrow{\sigma}_2 Q_1'$ by (tSum_2).
By the induction hypothesis, there exists some P_2', such that $P_1 \xrightarrow{\sigma}_1 P_2'$ and $P_1' \succ^+ P_2'$, as well as some Q_2', satisfying $Q_1 \xrightarrow{\sigma}_1 Q_2'$ and $Q_1' \succ^+ Q_2'$.
Consequently, $P_1 + Q_1 \xrightarrow{\sigma}_1 P_2' + Q_2'$ can be derived by (tSum_1) and $P_1' + Q_1' \succ^+ P_2' + Q_2'$ can be inferred by Lemma 11(2).

(6) tCom: $P \equiv P_1 | Q_1$ and $P' \equiv P_1' | Q_1'$.
Let $P_1 | Q_1 \xrightarrow{\sigma}_2 P_1' | Q_1'$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ and $Q_1 \xrightarrow{\sigma}_2 Q_1'$ by (tCom_2).
Thereby the side condition ensures $\tau \notin \mathcal{U}(P_1 | Q_1)$.
By the induction hypothesis, $P_1 \xrightarrow{\sigma}_1 P_2'$ and $P_1' \succ^+ P_2'$ for some P_2', as well as $Q_1 \xrightarrow{\sigma}_1 Q_2'$ and $Q_1' \succ^+ Q_2'$ for some Q_2'.
As $\tau \notin \mathcal{U}(P_1 | Q_1)$, we may derive $P_1 | Q_1 \xrightarrow{\sigma}_1 P_2' | Q_2'$ using (tCom_1).
Moreover, application of Lemma 11(1) leads to $P_1' | Q_1' \succ^+ P_2' | Q_2'$.

(7) tRes: $P \equiv P_1 \setminus L$ and $P' \equiv P_1' \setminus L$.
Let $P_1 \setminus L \xrightarrow{\sigma}_2 P_1' \setminus L$, due to $P_1 \xrightarrow{\sigma}_2 P_1'$ by (tRes_2).
By the induction hypothesis, there exists some P_2' with $P_1 \xrightarrow{\sigma}_1 P_2'$ and $P_1' \succ^+ P_2'$. Then, $P_1 \setminus L \xrightarrow{\sigma}_1 P_2' \setminus L$ can be derived by (tRes_1).
In addition, we can infer $P_1' \setminus L \succ^+ P_2' \setminus L$ by Part (3) of Lemma 11.

(8) tRel: This case is analogous to the previous case (7) and uses Lemma 11(4). □

Every type-1-time step sequence of the form $P \xrightarrow{\sigma}_1{}^+ P'$ can be mimicked with one single matching type-2-time step $P \xrightarrow{\sigma}_2 P'$. This statement can easily be proved, involving Proposition 4: since every type-1-time step is also a type-2-time step by using Lemma 6(2), $P \xrightarrow{\sigma}_1{}^+ P'$ implies $P \xrightarrow{\sigma}_2{}^+ P'$. Further, we may conclude $P \xrightarrow{\sigma}_2 P'$ by application of Proposition 4.
Conversely, there are type-2-time steps with no matching sequence of type-1-time steps. This phenomenon relies exclusively on the occurrence of parallel composition or choice in processes. Crucial are 'real' type-2-time steps as for example the time step $\sigma.\sigma.\sigma.a.\mathbf{0}|\sigma.\overline{a}.\mathbf{0}|\sigma.a.\mathbf{0} \xrightarrow{\sigma}_2 a.\mathbf{0}|\overline{a}.\mathbf{0}|a.\mathbf{0}$, where a different number of time steps are skipped in at least two components. The process in the example lets time pass, thereby leaving out two time steps in the first component and no time step in the second component. Due to these observations, we claim that type-2-time steps of processes that neither involve parallel composition nor choice can be matched by a sequence of type-1-time steps:

Proposition 14 *Let P, P' be processes with no occurrence of parallel composition and choice in P; then $P \xrightarrow{\sigma}_2 P'$ implies $P \xrightarrow{\sigma}_1{}^+ P'$.*

Proof. The proof is an induction on the structure of P.

(1) $\underline{\mathbf{0}}$: Let $P \equiv \mathbf{0}$.
The only time step that can be derived is that by Rule (tNil$_2$) of the form $\mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0}$. Trivially, we obtain $\mathbf{0} \xrightarrow{\sigma}_1 \mathbf{0}$ by using (tNil$_1$).

(2) $\underline{x}$: There is no applicable SOS-rule for time steps.

(3) $\underline{\alpha.P_1}$: Let $P \equiv \alpha.P_1$.
If $\alpha \equiv \tau$, then $\alpha.P_1$ cannot perform a time step. If $\alpha \equiv a$, the only derivable time step is of the form $a.P_1 \xrightarrow{\sigma}_2 a.P_1$ by (tAct$_2$). Trivially, $a.P_1 \xrightarrow{\sigma}_1 a.P_1$ holds by Rule (tAct$_1$).

(4) $\underline{\sigma.P_1}$: Let $P \equiv \sigma.P_1$.
(tPre$_2$) and (tnew) are the only applicable rules that allow a time step of $\sigma.P_1$.
If the time step is of the form $\sigma.P_1 \xrightarrow{\sigma}_2 P_1$ by (tPre$_2$), we may as well derive $\sigma.P_1 \xrightarrow{\sigma}_1 P_1$, using (tPre$_1$).
Otherwise, any clock transition is of the form $\sigma.P_1 \xrightarrow{\sigma}_2 P'$ for some P', due to the clock transition $P_1 \xrightarrow{\sigma}_2 P'$ and application of (tnew). Then, $P_1 \xrightarrow{\sigma}_2 P'$ implies $P_1 \xrightarrow{\sigma}_1{}^+ P'$ by the induction hypothesis. Due to Rule (tPre$_1$), $\sigma.P_1 \xrightarrow{\sigma}_1 P_1$ obviously holds. Hence, we have $\sigma.P_1 \xrightarrow{\sigma}_1 P_1 \xrightarrow{\sigma}_1{}^+ P'$ and now may conclude $\sigma.P_1 \xrightarrow{\sigma}_1{}^+ P'$.

(5) $\underline{P_1 \setminus L}$: Let $P \equiv P_1 \setminus L$.
Any clock transition is of the form $P_1 \setminus L \xrightarrow{\sigma}_2 P_n \setminus L$, due to $P_1 \xrightarrow{\sigma}_2 P_n$ by (tRes$_2$). By the induction hypothesis, $P_1 \xrightarrow{\sigma}_1{}^+ P_n$, i.e. $P_1 \xrightarrow{\sigma}_1 \ldots \xrightarrow{\sigma}_1 P_n$ for some $P_2, \ldots, P_n \in \widehat{\mathcal{P}}$ and some $n \geq 2$. We get $P_1 \setminus L \xrightarrow{\sigma}_1 P_2 \setminus L \xrightarrow{\sigma}_1 \ldots \xrightarrow{\sigma}_1 P_n \setminus L$ by repeated application of (tRes$_1$) and altogether

$P_1 \setminus L \xrightarrow{\sigma}_1{}^+ P_n \setminus L$.

(6) $\underline{P_1[f]}$: This case follows in analogy to case (5).

(7) $\underline{\mu x.P_1}$: Let $P \equiv \mu x.P_1$.

Any time step is of the form $\mu x.P_1 \xrightarrow{\sigma}_2 P_n[\mu x.P_1/x]$, due to $P_1 \xrightarrow{\sigma}_2 P_n$ by (tRec$_2$). Then, $P_1 \xrightarrow{\sigma}_1{}^+ P_n$ follows by the induction hypothesis and means $P_1 \xrightarrow{\sigma}_1 \ldots \xrightarrow{\sigma}_1 P_n$ for some $P_2, \ldots, P_n \in \widehat{\mathcal{P}}$ and some $n \geq 2$.

We may infer $\mu x.P_1 \xrightarrow{\sigma}_1 P_2[\mu x.P_1/x]$ from $P_1 \xrightarrow{\sigma}_1 P_2$ by application of Rule (tRec$_1$).

By Lemma 3(2) we get $P_2[\mu x.P_1/x] \xrightarrow{\sigma}_1 P_3[\mu x.P_1/x]$, due to $P_2 \xrightarrow{\sigma}_1 P_3$. This finishes the proof, since we obtain $\mu x.P_1 \xrightarrow{\sigma}_1 P_2[\mu x.P_1/x] \xrightarrow{\sigma}_1 \ldots \xrightarrow{\sigma}_1 P_n[\mu x.P_1/x]$ by repeated application of Lemma 3(2) and hence $\mu x.P_1 \xrightarrow{\sigma}_1{}^+ P_n[\mu x.P_1/x]$. □

If we try to involve terms of the form $P_1 \mid P_1'$ in the structural induction, any time step is of the form $P_1 \mid P_1' \xrightarrow{\sigma}_2 P_n \mid P_m'$, due to $P_1 \xrightarrow{\sigma}_2 P_n$ and $P_1' \xrightarrow{\sigma}_2 P_m'$ by Rule (tCom). Thereby the side condition ensures $\tau \notin \mathcal{U}(P_1 \mid P_1')$. Then, by the induction hypothesis, we may assume that there exist the corresponding type-1-time step sequences $P_1 \xrightarrow{\sigma}_1 \ldots \xrightarrow{\sigma}_1 P_n$ and $P_1' \xrightarrow{\sigma}_1 \ldots \xrightarrow{\sigma}_1 P_m'$. The successive application of (tCom$_2$) over the complete sequence of time steps is only possible, if $n = m$. Otherwise the proof attempt fails. This fact again confirms the above claim that only those type-2-time steps that skip different numbers of time steps in at least two components cannot be simulated by a type-1-time step sequence.

3.3 Semantic coherence

When waiting, a process does not change its functional behaviour, nevertheless the structure of the process may change during the progress of time. As an example consider the process $P =_{\mathrm{df}} \sigma.\sigma.\sigma.\tau.\mathbf{0}$ that may perform a time step of the form $\sigma.\sigma.\sigma.\tau.\mathbf{0} \xrightarrow{\sigma}_2 \tau.\mathbf{0}$ and afterwards behaves like $\tau.\mathbf{0}$.

Obviously, the waiting behaviour of processes has changed in our new type-2-transition system, since we introduced further alternatives for a process to let time pass. As an example consider again the process P.

P can delay the execution of τ at most for three time steps in both transition systems. Taken as a whole, it can wait between 0 and 3 time steps in both transition systems. However, $\longrightarrow_2$ involves more forms for a process to wait these 0 to 3 time steps than $\longrightarrow_1$. If we consider a process P that can only perform type-1-time steps, then P may at each point of time either take the next time step or skip all remaining σ-prefixes and at once perform τ.

In the new setting, alternatively to these time steps, P can make a better progress during the first time step by performing a transition of the form $\sigma.\sigma.\sigma.\tau.\mathbf{0} \xrightarrow{\sigma}_2 \sigma.\tau.\mathbf{0}$. In practice, this could be the case if it is known right from the start that the process will not use the complete delay spectrum of three time steps, but a delay time of one or two time steps is sufficient. The way in which a process can wait could be regarded as more flexible and foresighted.

In [LV04] time-determinism is pointed out as an important property of the operational semantics of processes. If a process P can perform a time step $P \xrightarrow{\sigma}_1 P'$, the result process P' is uniquely defined. Thus, processes react deterministically to clock ticks. From a purely syntactic point of view, this is true as there exists only one possible derivation for every type-1-time step. In contrast to this, there exist different derivable time steps for some processes in the new transition system and therefore a set of possible result processes. The deterministic concept of time is abolished here, which is however justified as argued above. [LV04] requires that the progress of time should not resolve choice, but delay it. By application of Rule (tSum$_1$), a process of the form $P =_{\mathrm{df}} P_1 + P_2$ can perform a time step if and only if both P_1 and P_2 can. The functional behaviour of the subprocesses P_1 and P_2 is not influenced by the progress of time. It is quite obvious that this principle is as well retained in our new transition system $\xrightarrow{\sigma}_2$, since the SOS-rule (tSum$_2$) is the only derivation rule which is in charge of waiting behaviour for processes of the form $P =_{\mathrm{df}} P_1 + P_2$.

4 The naive faster-than preorders

In [LV04] the naive faster-than preorder is introduced as an elegant and concise candidate for a faster-than preorder. It is (bi)-simulation based as the faster and the slower process are linked by a relation that is a simulation for time steps and a strong bisimulation for actions. The definition of the 1-naive faster-than preorder is adopted from [LV04] and is extended to a second variant by including our new type-2-transitions:

Definition 15 (i-naive faster-than preorder) [LV04, Def. 4 for $i = 1$] *For $i \in \{1, 2\}$, a relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is an* i-naive faster-than relation *if the following conditions hold for all $\langle P, Q\rangle \in \mathcal{R}$ and $\alpha \in \mathcal{A}$.*

(1) $P \xrightarrow{\alpha}_i P'$ implies $\exists Q'.\, Q \xrightarrow{\alpha}_i Q'$ and $\langle P', Q'\rangle \in \mathcal{R}$.
(2) $Q \xrightarrow{\alpha}_i Q'$ implies $\exists P'.\, P \xrightarrow{\alpha}_i P'$ and $\langle P', Q'\rangle \in \mathcal{R}$.
(3) $P \xrightarrow{\sigma}_i P'$ implies $\exists Q'.\, Q \xrightarrow{\sigma}_i Q'$ and $\langle P', Q'\rangle \in \mathcal{R}$.

We write $P \sqsupseteq\!\!\!\sim_{i\text{-}nv} Q$ if $\langle P, Q\rangle \in \mathcal{R}$ for some i-naive faster-than relation $\mathcal{R}$, which means that

$$\sqsupseteq\!\!\!\sim_{i\text{-}nv} = \bigcup\{\mathcal{R} : \mathcal{R} \textit{ is an i-naive faster-than relation}\}.$$

We call $\sqsupseteq\!\!\!\sim_{i\text{-}nv}$ i-naive faster-than preorder.

Part (1) and (2) require that the faster and the slower process are functionally equivalent in the sense of a strong bisimulation. The i-naive faster-than relation refines the strong bisimulation by additionally requiring that any enabled time step of the faster process can be simulated by a time step of the slower process. Conversely, clock transitions of the slower process Q are not required to be matched by the faster process. Extra time steps Q can possibly perform need not to be considered, particularly as the functional behaviour of Q before the time step is the same as after the time step.
Faster in the sense of a naive faster-than preorder means that the faster process P is at least as fast as the slower process Q. If we refer to the 'faster process' in the sequel, it does not have to be strictly faster.

In the first part of this section we will demonstrate that the 2-naive faster-than preorder coincides with the 1-naive faster-than preorder. In the second part we will compare the *relations* of the two naive faster-than preorders and develop a third variant of a naive faster-than preorder that as well coincides with the previous naive faster-than preorders. Its definition combines new and old transitions and provides small relations in order to demonstrate a naive faster-than relationship.

We develop the idea of a 'naive faster-than relation up to $\sqsupseteq\!\!\!\!\sim_{i\text{-}nv}$' as another technique for reducing the size of relations needed to demonstrate a naive faster-than relationship in the last part of this section.
We are left with establishing that $\sqsupseteq\!\!\!\!\sim_{i\text{-}nv}$ is the largest i-naive faster-than relation and that it is moreover a preorder. The proof follows from the fact that the union and composition of i-naive faster-than relations are still i-naive faster-than relations and also the identity relation on processes is an i-naive faster-than relation. This is established in the next proposition, based on [Mil89]:

Proposition 16 *Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_j$ be i-naive faster-than relations. Then*

(1) $\emptyset$
(2) $id_{\mathcal{P}}$
(3) $\mathcal{R}_1\mathcal{R}_2$.
(4) $\bigcup_{j\in J}\mathcal{R}_j$.

are as well i-naive faster-than relations.

Proof. Part (1) and Part (2) trivially hold.
To prove Part (3), consider some arbitrary pair of processes (P, Q) in $\mathcal{R}_1\mathcal{R}_2$, hence $\exists R \in \mathcal{P} : (P, R) \in \mathcal{R}_1 \wedge (R, Q) \in \mathcal{R}_2$ by the definition of composition. If $P \xrightarrow{\alpha}_i P'$ for some P', then $R \xrightarrow{\alpha}_i R'$ for some R' such that $(P', R') \in \mathcal{R}_1$ by Definition 15(1). Due to $R \xrightarrow{\alpha}_i R'$, there exists some Q' such that $Q \xrightarrow{\alpha}_i Q'$ and $(R', P') \in \mathcal{R}_2$. Consequently, we have $(P', Q') \in \mathcal{R}_1\mathcal{R}_2$.

The treatment of the cases $Q \xrightarrow{\alpha}_i Q'$ and $P \xrightarrow{\sigma}_i P'$ is analogous to the previous case and uses Definition 15(2) and 15(3), respectively.
Again Part (4) is obvious. $\Box$

By Proposition 16(4), $\gtrsim_{i\text{-}nv}$ is an i-naive faster-than relation. Including every i-naive faster-than relation, $\gtrsim_{i\text{-}nv}$ is the largest i-naive faster-than relation. Further, $\gtrsim_{i\text{-}nv}$ is reflexive, as it includes $id_{\mathcal{P}}$.

Since $\gtrsim_{i\text{-}nv}$ is an i-naive faster-than relation, $\gtrsim_{i\text{-}nv} \circ \gtrsim_{i\text{-}nv}$ is still an i-naive faster-than relation, employing Proposition 16(3). As $\gtrsim_{i\text{-}nv} \circ \gtrsim_{i\text{-}nv}$ is contained in the largest i-naive faster-than relation $\gtrsim_{i\text{-}nv}$, we obtain $\gtrsim_{i\text{-}nv} \circ \gtrsim_{i\text{-}nv} \subseteq \gtrsim_{i\text{-}nv}$. Hence, $\gtrsim_{i\text{-}nv}$ is transitive and we have proved that $\gtrsim_{i\text{-}nv}$ is indeed a preorder.

The i-naive faster-than preorder is defined on processes and not on arbitrary open terms. Otherwise, the relation $\{(a.x, a.y), (x, y)\}$ would satisfy the defining clauses of an i-naive faster-than relation. This would not be sensible, as we do not know anything about the behaviour of the process variables x and y. As it is however sensible to involve pairs of terms such as $\sigma.x$ and x or $\sigma.(x+y)$ and $\sigma.y+\sigma.x$ in our behavioural $\gtrsim_{i\text{-}nv}$-relations, the naive faster-than preorders can be extended to open terms by means of closed substitution as usual [Mil89]. A closed substitution maps variables in $\mathcal{V}$ to closed terms, here processes. We include terms P and Q with free variables in $\vec{x} = (x_1, \ldots, x_n)$ in our naive faster-than relations such that $(P, Q) \in \gtrsim_{i\text{-}nv}$, provided that $P[\vec{R}/\vec{x}] \gtrsim_{i\text{-}nv} Q[\vec{R}/\vec{x}]$ for all tuples $\vec{R} = (R_1, \ldots, R_n)$.

4.1 The preorders

In the sequel, we will prove that the faster-than preorders $\gtrsim_{1\text{-}nv}$ and $\gtrsim_{2\text{-}nv}$ coincide. For this, it is convenient to first establish an auxiliary result, stating that the syntactic relation satisfies the definition of a 1-naive faster-than relation as well as the one of a 2-naive faster-than relation. Furthermore we are able to establish that the transitive closure of the syntactic relation as well fulfils the definition of an i-naive faster-than relation for $i \in \{1, 2\}$. Before establishing these desired properties of $\succ$, we have to introduce the abstract closure operator.

Definition 17 *Let A be a set and let $\mathcal{P}(A)$ be the set of all subsets of A. A closure operator h is a function $h : \mathcal{P}(A) \to \mathcal{P}(A)$ which satisfies the following conditions for all $A_1, A_2 \subseteq A$:*

(1) <u>extensive</u>: $A_1 \subseteq h(A_1)$
(2) <u>monotone</u>: $A_1 \subseteq A_2$ implies $h(A_1) \subseteq h(A_2)$
(3) <u>idempotent</u>: $h(h(A_1)) = h(A_1)$

The extensitive property states that the closure of a set must at least include the set itself. The monotone property points out that, if a set includes another set, then this inclusion is also valid for their closures. Repeated application of the closure operator does not add any further elements according to the idempotence property.

Proposition 18

(1) [LV04, Prop. 9(2) for $i = 1$] *The relation $\succ$ satisfies the defining clauses of an i-naive faster-than relation for $i \in \{1, 2\}$, also on open terms; hence, $\succ$ restricted to processes is an i-naive faster-than relation for $i \in \{1, 2\}$ and $\succ_{|\mathcal{P}\times\mathcal{P}} =_{df} \succ \cap (\mathcal{P} \times \mathcal{P}) \subseteq \sqsupseteq\!\!\!\!\sim_{i\text{-}nv}$.*

(2) The relation $\succ^+$ satisfies the defining clauses of an i-naive faster-than relation for $i \in \{1, 2\}$, also on open terms; hence, $\succ^+$ restricted to processes is an i-naive faster-than relation for $i \in \{1, 2\}$ and $\succ^+_{|\mathcal{P}\times\mathcal{P}} =_{df} \succ^+ \cap (\mathcal{P} \times \mathcal{P}) \subseteq \sqsupseteq\!\!\!\!\sim_{i\text{-}nv}$.

Proof.

The proof of Part (1) for $i = 1$ is, shortened to the interesting cases, given in [LV04]. To prove Part (1) for $i = 2$ we show that for $P' \succ P$ the definition of $\sqsupseteq\!\!\!\!\sim_{2\text{-}nv}$ is satisfied, by induction on the inference length of $P' \succ P$.

(1) $\underline{P \succ P}$ $P \xrightarrow{\alpha}_2 P'$ if and only if $P \xrightarrow{\alpha}_2 P'$ and $P' \succ P'$ holds by Def. 7(1).
$P \xrightarrow{\sigma}_2 P'$ trivially implies $P \xrightarrow{\sigma}_2 P'$ and $P' \succ P'$ holds by Def. 7(1).

(2) $\underline{P \succ \sigma.P}$ $P \xrightarrow{\alpha}_2 P'$ if and only if $\sigma.P \xrightarrow{\alpha}_2 P'$ for some P' by (Pre) and $P' \succ P'$ by Def. 7(1).
$P \xrightarrow{\sigma}_2 P'$ for some P' implies $\sigma.P \xrightarrow{\sigma}_2 P'$ by (tnew) and $P' \succ P'$ by Def. 7(1).

(3) $\underline{P'|Q' \succ P|Q}$ due to $P' \succ P$ and $Q' \succ Q$.
If $P'|Q' \xrightarrow{\alpha}_2 P'_1|Q'$ due to $P' \xrightarrow{\alpha}_2 P'_1$ by (Com1), then we may conclude $P \xrightarrow{\alpha}_2 P_1$ for some P_1 such that $P'_1 \succ P_1$ from $P' \succ P$ by the induction hypothesis. Further, we get $P|Q \xrightarrow{\alpha}_2 P'_1|Q$ by application of Rule (Com1) and $P'_1|Q' \succ P_1|Q$ results from $P'_1 \succ P_1$ and $Q' \succ Q$ by Definition 7(3). The treatment of the case $P'|Q' \xrightarrow{\alpha}_2 P'|Q'_1$ by (Com2) follows in analogy to the case for (Com1).
If $P'|Q' \xrightarrow{\tau}_2 P'_1|Q'_1$, due to $P' \xrightarrow{a}_2 P'_1$ and $Q' \xrightarrow{\overline{a}}_2 Q'_1$ by (Com3), then $P \xrightarrow{a}_2 P_1$ and $P'_1 \succ P_1$ for some P_1 as well as $Q \xrightarrow{\overline{a}}_2 Q_1$ and $Q'_1 \succ Q_1$ for some Q_1 by the induction hypothesis. Using (Com3), we may infer $P|Q \xrightarrow{\tau}_2 P_1|Q_1$. Further, we obtain $P'_1|Q'_1 \succ P_1|Q_1$, due to $P'_1 \succ P_1$ and $Q'_1 \succ Q_1$ by Definition 7(3) .
The cases if $P|Q \xrightarrow{\alpha}_2 P_1|Q_1$ by (Com1), (Com2) and (Com3) are analogous to the above cases.
If $P'|Q' \xrightarrow{\sigma}_2 P'_1|Q'_1$, due to $P' \xrightarrow{\sigma}_2 P'_1$ and $Q' \xrightarrow{\sigma}_2 Q'_1$ by (tCom),

then $\tau \notin \mathcal{U}(P'|Q')$. Thus, by the induction hypothesis, we may assume $P \xrightarrow{\sigma}_2 P_1$ for some P_1 such that $P'_1 \succ P_1$ as well as $Q \xrightarrow{\sigma}_2 Q_1$ for some Q_1 such that $Q'_1 \succ Q_1$. We may conclude that $\tau \notin \mathcal{U}(P|Q)$ by using Lemma 10. Therefore, we may infer $P|Q \xrightarrow{\sigma}_2 P_1|Q_1$ by (tCom) and $P'_1|Q'_1 \succ P_1|Q_1$ from $P'_1 \succ P_1$ and $Q'_1 \succ Q_1$ by Definition 7(3).

(4) $\underline{P' + Q' \succ P + Q}$
This case is treated in analogy to Rule (Com1), (Com2) and (tCom) in the previous case.

(5) $\underline{P' \setminus L \succ P \setminus L}$ Let $P' \setminus L \xrightarrow{\alpha}_2 P'_1 \setminus L$ due to $P' \xrightarrow{\alpha}_2 P'_1$ by (Res), whereby the side condition $\alpha \notin L \cup \overline{L}$ holds. Due to $P' \succ P$ and $P' \xrightarrow{\alpha}_2 P'_1$, we may assume that there exists some P_1 such that $P \xrightarrow{\alpha}_2 P_1$ and $P'_1 \succ P_1$ is satisfied, by induction. With regard to the side condition $\alpha \notin L \cup \overline{L}$, we may infer $P \setminus L \xrightarrow{\alpha}_2 P_1 \setminus L$ by (Res). Moreover, we obtain $P'_1 \setminus L \succ P_1 \setminus L$ by Def. 7(5).
The cases if $P \setminus L \xrightarrow{\alpha}_2 P_1 \setminus L$ and $P' \setminus L \xrightarrow{\sigma}_2 P'_1 \setminus L$ are analogous.

(6) $\underline{P'[f] \succ P[f]}$ analogous to (5)

(7) $\underline{P'[\mu x.\, P/x] \succ \mu x.\, P}$ due to $P' \succ P$, x guarded in P.
As x is guarded in P, it is also guarded in P' by using Lemma 9(1).If $P'[\mu x.\, P/x] \xrightarrow{\alpha}_2 P'_1$, then $P' \xrightarrow{\alpha}_2 P'_2$ for some P'_2 such that $P'_1 \equiv P'_2[\mu x.\, P/x]$ by application of Lemma 3(3).
Thus, $P \xrightarrow{\alpha}_2 P_1$ and $P'_2 \succ P_1$, due to $P' \succ P$ by the induction hypothesis. Further, we may infer $\mu x.\, P \xrightarrow{\alpha}_2 P_1[\mu x.\, P/x]$ from $P \xrightarrow{\alpha}_2 P_1$ by Rule (Rec). We are ready since we may conclude $P'_1 \equiv P'_2[\mu x.\, P/x] \succ P_1[\mu x.\, P/x]$, due to $P'_2 \succ P_1$ by application of Lemma 9(4).If $\mu x.\, P \xrightarrow{\alpha}_2 P_1[\mu x.\, P/x]$, due to $P \xrightarrow{\alpha}_2 P_1$ by Rule (Rec), then we may assume $P' \xrightarrow{\alpha}_2 P'_1$ for some P'_1 such that $P'_1 \succ P_1$, due to $P' \succ P$ and $P \xrightarrow{\alpha}_2 P_1$ by the induction hypothesis.
Moreover, we may infer $P'[\mu x.\, P/x] \xrightarrow{\alpha}_2 P'_1[\mu x.\, P/x]$ from $P' \xrightarrow{\alpha}_2 P'_1$ by using Lemma 3(2). By application of Lemma 9(4), we obtain $P'_1[\mu x.\, P/x] \succ P_1[\mu x.\, P/x]$, since $P'_1 \succ P_1$ holds.If $P'[\mu x.\, P/x] \xrightarrow{\sigma}_2 P'_1$, then we know as shown above that x is guarded in P'. Thus, we may conclude $P' \xrightarrow{\sigma}_2 P'_2$ for some P'_2 satisfying $P'_1 \equiv P'_2[\mu x.\, P/x]$ by Lemma 3(3).
Then, we obtain $P \xrightarrow{\sigma}_2 P_1$ and $P'_2 \succ P_1$, due to $P' \succ P$ by the induction hypothesis. Further, we infer $\mu x.\, P \xrightarrow{\sigma}_2 P_1[\mu x.\, P/x]$ from $P \xrightarrow{\sigma}_2 P_1$ by using Rule (tRec).
We are done since $P'_2 \succ P_1$ implies $P'_1 \equiv P'_2[\mu x.\, P/x] \succ P_1[\mu x.\, P/x]$ by using Lemma 9(4).

We are left with ensuring that each transition of a process again leads to a process, i.e. $P \in \mathcal{P}$ and $P \xrightarrow{\gamma}_i P'$ for $\gamma \in \mathcal{A} \cup \{\sigma\}$ and $i \in \{1, 2\}$ implies $P' \in \mathcal{P}$. Clearly, this is true for $\gamma \equiv \sigma$. Since $P \xrightarrow{\sigma}_1 P'$ implies $P' \succ P$ and $P \xrightarrow{\sigma}_2 P'$ implies $P' \succ^+ P$ by Lemma 12, P' is as well a process by application of Lemma 9(3). As the statement obviously also holds for $\gamma \equiv \alpha$, we dispense with conducting a proof on the length of inference of $P \xrightarrow{\alpha}_i P'$ here.

Proving Part (2), we want to show that $\succ^+_{|\mathcal{P}\times\mathcal{P}}$ satisfies the definition of a 1-naive faster-than relation. We already know that $\succ_{|\mathcal{P}\times\mathcal{P}}$ is a 1-naive faster-than relation by Proposition (18)(1) for $i = 1$. Since $\succ^+_{|\mathcal{P}\times\mathcal{P}}$ is defined as $\succ^0_{|\mathcal{P}\times\mathcal{P}} \cup \succ^1_{|\mathcal{P}\times\mathcal{P}} \cup \cdots \cup \succ^n_{|\mathcal{P}\times\mathcal{P}}$, we may conclude that $\succ^+_{|\mathcal{P}\times\mathcal{P}}$ is as well a 1-naive faster-than relation by repeated application of Proposition 16(3) and (4).

The particularly important second statement of Part (2) can as well be proved as follows:
$\succ_{|\mathcal{P}\times\mathcal{P}} = \succ \cap (\mathcal{P}\times\mathcal{P}) \subseteq \gtrsim_{\text{1-nv}}$ holds by Proposition 18(1) for $i = 1$.
Further, the operator $(.)^+$ is known to be a closure operator on relations. We get $(\succ \cap (\mathcal{P}\times\mathcal{P}))^+ \subseteq \gtrsim^+_{\text{1-nv}}$ by using the monotonicity property of the closure operator (Definition 17(2)).
Since the 1-naive faster-than preorder is transitive, the transitive closure does not add any further elements, i.e. $\gtrsim_{\text{1-nv}} = \gtrsim^+_{\text{1-nv}}$ holds. In the following, we will show that $\succ^+ \cap (\mathcal{P}\times\mathcal{P}) \subseteq (\succ \cap (\mathcal{P}\times\mathcal{P}))^+$ holds.
Therefore we consider some arbitrary element $(P, P') \in \succ^+ \cap (\mathcal{P}\times\mathcal{P})$ for which $\exists n \geq 1\ \exists P_0, ...P_n \in \widehat{\mathcal{P}} : P \equiv P_0 \succ P_1 \cdots \succ P_n \equiv P'$ and $P, P' \in \mathcal{P}$ holds. Since $P_0 \in \mathcal{P}$, we know that as well $P_1 \in \mathcal{P}$ and successively $P_{n-1} \in \mathcal{P}$ by using Lemma 9(3). Hence, $(P, P') \in (\succ \cap (\mathcal{P}\times\mathcal{P}))^+$ and in summary $\succ^+ \cap (\mathcal{P}\times\mathcal{P}) \subseteq (\succ \cap (\mathcal{P}\times\mathcal{P}))^+ \subseteq \gtrsim^+_{\text{1-nv}} = \gtrsim_{\text{1-nv}}$.

The proof of Part (2) for $i = 2$ is treated analogously, using Proposition 18(1) for $i = 2$. □

Theorem 19 (Coincidence I) *The preorders $\gtrsim_{\text{1-nv}}$ and $\gtrsim_{\text{2-nv}}$ coincide.*

Proof.

To prove $\gtrsim_{\text{1-nv}} \subseteq \gtrsim_{\text{2-nv}}$, we show that $\gtrsim_{\text{1-nv}}$ satisfies the definition of a 2-naive faster-than relation. Hence, consider some arbitrary processes P and Q such that $P \gtrsim_{\text{1-nv}} Q$.

(1) If $P \xrightarrow{\alpha}_2 P'$ for some process P' and some action $\alpha \in \mathcal{A}$, then as well $P \xrightarrow{\alpha}_1 P'$ by using Lemma 6(1). By the definition of $\gtrsim_{\text{1-nv}}$, we obtain $Q \xrightarrow{\alpha}_1 Q'$ for some Q' such that $P' \gtrsim_{\text{1-nv}} Q'$. Finally, we get $Q \xrightarrow{\alpha}_2 Q'$ by application of Lemma 6(1).
(2) The case $Q \xrightarrow{\alpha}_2 Q'$ for some process Q' and some action $\alpha \in \mathcal{A}$ is analogous to the previous case.
(3) If $P \xrightarrow{\sigma}_2 P'$ for some process P', then $P \xrightarrow{\sigma}_1 P''$ for some process P'' satisfying $P' \succ^+ P''$ by Lemma 13. According to the definition of $\gtrsim_{\text{1-nv}}$, there exists some Q'' with $Q \xrightarrow{\sigma}_1 Q''$ and $P'' \gtrsim_{\text{1-nv}} Q''$. By Lemma 6(2),

we obtain $Q \xrightarrow{\sigma}_2 Q''$. $P' \succ^+ P''$ is defined as $P' \equiv P_0 \succ P_1 \succ \ldots \succ P_n \equiv P''$ for some $n \geq 1$ and some $P_0, \ldots, P_n \in \widehat{\mathcal{P}}$. Further, we conclude $P' \equiv P_0 \sqsupseteq\!\!\sim_{\text{1-nv}} P_1 \sqsupseteq\!\!\sim_{\text{1-nv}} \cdots \sqsupseteq\!\!\sim_{\text{1-nv}} P_n \equiv P''$ by Proposition 18(1) for $i = 1$. We are done since $P' \sqsupseteq\!\!\sim_{\text{1-nv}} P''$ and $P'' \sqsupseteq\!\!\sim_{\text{1-nv}} Q''$ implies $P' \sqsupseteq\!\!\sim_{\text{1-nv}} Q''$ by the transitivity of $\sqsupseteq\!\!\sim_{\text{1-nv}}$.

To prove the inverse inclusion $\sqsupseteq\!\!\sim_{\text{2-nv}} \subseteq \sqsupseteq\!\!\sim_{\text{1-nv}}$, we demonstrate that $\sqsupseteq\!\!\sim_{\text{2-nv}}$ satisfies the defining clauses of 1-naive faster-than relation. Hence, consider P and Q such that $P \sqsupseteq\!\!\sim_{\text{2-nv}} Q$.

(1,2) The cases $P \xrightarrow{\alpha}_1 P'$ for some P' and $Q \xrightarrow{\alpha}_1 Q'$ for some Q' can be treated in analogy to the corresponding cases above.

(3) If $P \xrightarrow{\sigma}_1 P'$ for some process P', then $P \xrightarrow{\sigma}_2 P'$ is as well a valid time step by Lemma 6(2). Further, $Q \xrightarrow{\sigma}_2 Q'$ for some Q' such that $P' \sqsupseteq\!\!\sim_{\text{2-nv}} Q'$ by the definition of $\sqsupseteq\!\!\sim_{\text{2-nv}}$. By Lemma 13, there exists some process Q'' such that $Q \xrightarrow{\sigma}_1 Q''$ and $Q' \succ^+ Q''$, which means that $Q' \equiv Q_0 \succ Q_1 \succ \ldots \succ Q_n \equiv Q''$ for some $n \geq 1$ and some $Q_0, ..., Q_n \in \widehat{\mathcal{P}}$. We get $Q' \equiv Q_0 \sqsupseteq\!\!\sim_{\text{1-nv}} Q_1 \sqsupseteq\!\!\sim_{\text{1-nv}} \cdots \sqsupseteq\!\!\sim_{\text{1-nv}} Q_n \equiv Q''$ by application of Proposition 18(1) for $i = 1$. As we have proved $\sqsupseteq\!\!\sim_{\text{1-nv}} \subseteq \sqsupseteq\!\!\sim_{\text{2-nv}}$ above, we may conclude $Q' \sqsupseteq\!\!\sim_{\text{2-nv}} Q''$ by the transitivity of $\sqsupseteq\!\!\sim_{\text{2-nv}}$. Finally, together with $P' \sqsupseteq\!\!\sim_{\text{2-nv}} Q'$, we have $P' \sqsupseteq\!\!\sim_{\text{2-nv}} Q''$. $\square$

As an alternative, one could prove the last part applying Proposition 18(2) for $i = 1$ instead of using $\sqsupseteq\!\!\sim_{\text{1-nv}} \subseteq \sqsupseteq\!\!\sim_{\text{2-nv}}$ and Proposition 18(1) for $i = 1$. Yet, we favour the approach using Proposition 18(1) for $i = 1$ since this result can simply be taken from [LV04].
Since $\sqsupseteq\!\!\sim_{\text{1-nv}}$ and $\sqsupseteq\!\!\sim_{\text{2-nv}}$ coincide, we will now write $\sqsupseteq\!\!\sim_{\text{nv}}$.

4.2 The naive faster-than relations

In the previous subsection we proved as a main result that the 1-naive faster-than preorder coincides with our newly established 2-naive faster-than preorder. Leaving aside the preorders, we take a closer look at the two different types of naive faster-than *relations* now. First, we demonstrate that it is neither true that each 2-naive faster-than-relation is also a 1-naive faster-than relation, nor that each 1-naive faster-than relation satisfies the defining clauses of a 2-naive faster-than relation. Consider the relation R_1 which serves as an example of a relation which is a 2-naive faster-than relation, but not a 1-naive faster-than relation:

$R_1 =_{\text{df}} \{(\sigma.\sigma.a.\mathbf{0}, \sigma.\sigma.\sigma.a.\mathbf{0}), (\sigma.a.\mathbf{0}, \sigma.a.\mathbf{0}), (a.\mathbf{0}, a.\mathbf{0}), (\mathbf{0}, \mathbf{0})\}$.

R_1 does not satisfy the definition of a 1-naive faster-than relation since

$\sigma.\sigma.a.\mathbf{0} \xrightarrow{\sigma}_1 \sigma.a.\mathbf{0}$ and $\sigma.\sigma.\sigma.a.\mathbf{0} \xrightarrow{\sigma}_1 \sigma.\sigma.a.\mathbf{0}$, but $(\sigma.a.\mathbf{0}, \sigma.\sigma.a.\mathbf{0}) \notin R_1$. We observe that R_1 is not a 1-naive faster-than relation since a type-1-time step occurs in the first component and is only matched by a 'real' type-2-time step in the second component.

Altering the relation R_1 in such a way that it meets the requirements of a 1-naive faster-than relation, the following relation R_1' results as the smallest 1-naive faster-than relation with $(\sigma.\sigma.a.\mathbf{0}, \sigma.\sigma.\sigma.a.\mathbf{0}) \in R_1'$:

$R_1' =_{\text{df}} \{(\sigma.\sigma.a.\mathbf{0}, \sigma.\sigma.\sigma.a.\mathbf{0}), (\sigma.a.\mathbf{0}, \sigma.\sigma.a.\mathbf{0}), (a.\mathbf{0}, \sigma.a.\mathbf{0}), (a.\mathbf{0}, a.\mathbf{0}), (\mathbf{0}, \mathbf{0})\}$.

Since $|R_1'| > |R_1|$, providing the smaller relation R_1 as proof for $(\sigma.\sigma.a.\mathbf{0}, \sigma.\sigma.\sigma.a.\mathbf{0}) \in \mathrel{\underset{\sim}{\sqsupset}}_{nv}$, is profitable in this case.

As an example of a 1-naive faster-than relation which is not a 2-naive faster-than relation, consider the following relation R_2 where $L =_{\text{df}} \{a, \overline{a}\}$.

$R_2 =_{\text{df}} \{((\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} | \, \mathbf{0}) \setminus L),$
$((\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L),$
$((a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L)\}$

The point of this example is that R_2 does not satisfy the properties of a 2-naive faster-than relation since $\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L \xrightarrow{\sigma}_2 a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L$, but no Q' exists such that $(a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L, Q') \in R_2$ holds. At this point, observe that the process $\sigma.a.\mathbf{0} \,|\overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L$ is not allowed to perform a time step as it must engage in communication, due to the fact that a is urgent in $a.\mathbf{0}$ and $\overline{a}$ is urgent in $\overline{a}.\mathbf{0}$.

Crucial for this example is that the faster process is able to perform a type-2-time step $P \xrightarrow{\sigma}_2 P'$ but cannot mimic it with a corresponding sequence of type-1-time steps of the form $P \xrightarrow{\sigma}_1{}^+ P'$. Observe that this phenomenon relies exclusively on the occurrence of parallel composition or choice in processes (cf. Proposition 14).

Constructing a 2-naive faster-than relation for $(\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L, \sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L)$, we observe that any process on the left hand side of the pairs in relation R_2 as well can be reached in the type-2 setting as the analogous type-2-time step for any type-1-time step exists. Due to the fact that the left-hand-sides of all components differ from each other and that we must additionally cater for the above time step $\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L \xrightarrow{\sigma}_2 a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L$, any 2-naive faster-than relation is definitely larger than R_2.

R_2' is a candidate for a corresponding smallest 2-naive faster-than relation with $(\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L, \sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0} \setminus L) \in R_2'$:

$R_2' =_{\text{df}} \{((\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L)$
$((\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0}| \, \mathbf{0}) \setminus L),$
$((\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L),$
$((a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L)\}$

Now we define a new preorder, which we refer to as *combined-naive faster-than preorder* or *c-naive faster-than preorder* for short, as its definition combines type-1-time steps with type-2-time steps. We take $\sqsupset\!\!\!\sim_{c\text{-}nv}$ to denote the newly established preorder. In the following we will state and prove that the new preorder coincides with the two variants of naive faster-than preorders from above. Thus we provide another instrument to prove that $P \sqsupset\!\!\!\sim_{nv} Q$ by means of the new c-naive faster-than relations. Our goal is to increase the number of relations available for demonstrating a naive faster-than relationship and to reduce the size of these relations. The new c-naive faster-than preorder is defined as follows:

Definition 20 (c-naive faster-than preorder)
A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is a c-naive faster-than relation *if the following conditions hold for all $\langle P, Q\rangle \in \mathcal{R}$, $\alpha \in \mathcal{A}$.*

(1) $P \xrightarrow{\alpha}_1 P'$ *implies* $\exists Q'.\, Q \xrightarrow{\alpha}_1 Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.
(2) $Q \xrightarrow{\alpha}_1 Q'$ *implies* $\exists P'.\, P \xrightarrow{\alpha}_1 P'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.
(3) $P \xrightarrow{\sigma}_1 P'$ *implies* $\exists Q'.\, Q \xrightarrow{\sigma}_2 Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$

We write $P \sqsupset\!\!\!\sim_{c\text{-}nv} Q$ if $\langle P, Q\rangle \in \mathcal{R}$ for some c-naive faster-than relation $\mathcal{R}$ and call $\sqsupset\!\!\!\sim_{c\text{-}nv}$ c-naive faster-than preorder*.*

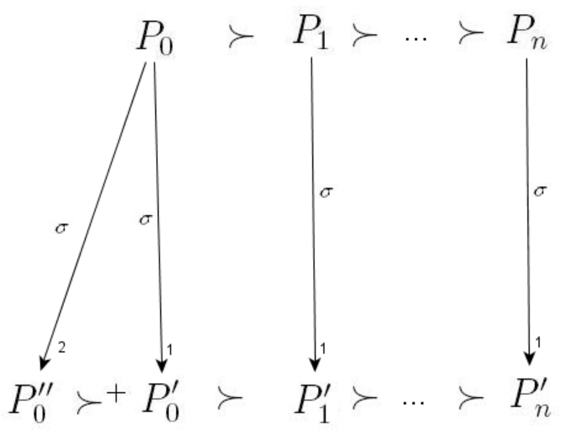
As usual, we can show that $\sqsupset\!\!\!\sim_{c\text{-}nv}$ is the largest c-naive faster-than relation. However, note that it is not clear that $\sqsupset\!\!\!\sim_{c\text{-}nv}$ is transitive. This will follow when we establish our next theorem stating that the c-naive faster-than preorder $\sqsupset\!\!\!\sim_{c\text{-}nv}$ coincides with our 1-naive-faster-preorder $\sqsupset\!\!\!\sim_{1\text{-}nv}$. The proof of this coincidence result is based on a technical lemma, highlighting two important properties of the transitive closure of the syntactic relation, which will become even more important in the next section. The first part is intuitively convincing because, if the faster process skips time steps by performing a 'real' type-2-time step, it can only become even faster than the slower process Q.

Lemma 21 *Let $P, P' \in \widehat{\mathcal{P}}$ such that $P \succ^+ P'$.*

(1) Then $P \xrightarrow{\sigma}_2 P_1$ *implies* $\exists P_1'.P' \xrightarrow{\sigma}_1 P_1'$ *and* $P_1 \succ^+ P_1'$.
(2) Then $P \xrightarrow{\alpha}_2 P_1$ *implies* $\exists P_1'.P' \xrightarrow{\alpha}_1 P_1'$ *and* $P_1 \succ^+ P_1'$.

Proof.

Proof of Part (1):
If $P \succ^+ P'$ for some process P', then $P \equiv P_0 \succ P_1 \cdots \succ P_n \equiv P'$ for some $n \geq 1$ and some processes $P_0, ...P_n \in \widehat{\mathcal{P}}$ by the definition of $\succ^+$. If $P_0 \xrightarrow{\sigma}_2 P_0''$, then $P_0 \xrightarrow{\sigma}_1 P_0'$ for some process P_0' such that $P_0'' \succ^+ P_0'$ by using Lemma 13. Further, $\succ$ satisfies the defining clauses of a 1-naive faster-than relation by Proposition 18(1) for $i = 1$. Hence, using Definition 15(3) for $i = 1$, we conclude from $P_0 \xrightarrow{\sigma}_1 P_0'$ that $P_1 \xrightarrow{\sigma}_1 P_1'$ for some P_1' with $P_0' \succ P_1'$. Successively for $2 \leq i \leq n$, we infer $P_i \xrightarrow{\sigma}_1 P_i'$ and $P_{i-1}' \succ P_i'$ for some P_i' by repeated application of Definition 15(3). In summary, $P_0' \succ \cdots \succ P_n'$ and $P_0'' \succ^+ P_0'$ implies $P_0'' \succ^+ P_n'$, cf. figure below



The proof of Part (2) is analogous to the one of Part (1). Considering action transitions in this part, P_0' coincides with P_0'' by Lemma 6 (1), cf. figure below. $\square$

$$\begin{tikzcd}
P_0 \arrow[r, phantom, "\succ"] \arrow[d, "\alpha", "2{,}1"'] & P_1 \arrow[r, phantom, "\succ"] \arrow[d, "\alpha", "1"'] & \cdots \arrow[r, phantom, "\succ"] & P_n \arrow[d, "\alpha", "1"'] \\
P'_0 \arrow[r, phantom, "\succ"] & P'_1 \arrow[r, phantom, "\succ"] & \cdots \arrow[r, phantom, "\succ"] & P'_n
\end{tikzcd}$$

As an alternative, one could prove both parts applying Proposition 18(1) for $i = 2$ or Proposition 18(2) instead of Proposition 18(1) for $i = 1$. Yet, we favour the approach using Proposition 18(1) for $i = 1$, since this result can simply be taken from [LV04].

We are now able to state and prove the main result of this section:

Theorem 22 (Coincidence II) *The preorders* $\underset{\sim}{\sqsupset}_{\textit{1-nv}}$ *and* $\underset{\sim}{\sqsupset}_{\textit{c-nv}}$ *coincide.*

Proof. In order to prove the inclusion $\underset{\sim}{\sqsupset}_{\textit{1-nv}} \subseteq \underset{\sim}{\sqsupset}_{\textit{c-nv}}$, consider some arbitrary processes P and Q such that $P \underset{\sim}{\sqsupset}_{\textit{1-nv}} Q$ and check that this pair satisfies clauses (1) to (3) in Def. 20.

(1,2) The case $P \xrightarrow{\alpha}_1 P'$ for some process P' and some action α, as well as the case $Q \xrightarrow{\alpha}_1 Q'$ for some process Q' and some action α, is obvious.

(3) If $P \xrightarrow{\sigma}_1 P'$, we obtain $Q \xrightarrow{\sigma}_1 Q'$ for some Q' satisfying $P' \underset{\sim}{\sqsupset}_{\textit{1-nv}} Q'$ by definition of $\underset{\sim}{\sqsupset}_{\textit{1-nv}}$. Using Lemma 6(2), $Q \xrightarrow{\sigma}_1 Q'$ implies $Q \xrightarrow{\sigma}_2 Q'$.

For the reverse inclusion $\underset{\sim}{\sqsupset}_{\textit{c-nv}} \subseteq \underset{\sim}{\sqsupset}_{\textit{1-nv}}$, define the relation $\mathcal{R}$ by $(P, Q) \in \mathcal{R}$ if and only if $\exists R \in \mathcal{P}.\ P \underset{\sim}{\sqsupset}_{\textit{c-nv}} R \succ^+ Q$ for $P, Q \in \mathcal{P}$. In the following we check that this relation $\mathcal{R}$ satisfies the definition of a 1-naive faster-than relation; consider $P \underset{\sim}{\sqsupset}_{\textit{c-nv}} R \succ^+ Q$.

(1) If $P \xrightarrow{\alpha}_1 P'$ for some P', the definition of $\underset{\sim}{\sqsupset}_{\textit{c-nv}}$ shows $R \xrightarrow{\alpha}_1 R'$ for some process R' with $P' \underset{\sim}{\sqsupset}_{\textit{c-nv}} R'$. Due to R, Q being processes, the syntactic relation is restricted to processes here. Since $\succ^+_{|\mathcal{P}\times\mathcal{P}}$ is a 1-naive faster-than relation by Proposition 18(2) for $i = 1$, we conclude $Q \xrightarrow{\alpha}_1 Q'$ for

some Q' such that $R' \succ^+ Q'$.

(2) The case $Q \xrightarrow{\alpha}_1 Q'$ for some Q' is analogous to Part (1).

(3) If $P \xrightarrow{\sigma}_1 P'$ for some P', then $R \xrightarrow{\sigma}_2 R'$ for some process R' with $P' \gtrsim_{c\text{-}nv} R'$ by definition of $\gtrsim_{c\text{-}nv}$. Due to $R \xrightarrow{\sigma}_2 R'$, Lemma 21(1) implies $Q \xrightarrow{\sigma}_1 Q'$ for some Q' satisfying $R' \succ^+ Q'$.

This finishes the proof, since $\succ^+$ is reflexive and hence $\gtrsim_{c\text{-}nv} \subseteq \gtrsim_{c\text{-}nv} \circ \succ^+ = \mathcal{R} \subseteq \gtrsim_{1\text{-}nv}$ is valid. In summary, we obtain $\gtrsim_{c\text{-}nv} \subseteq \gtrsim_{1\text{-}nv}$. $\square$

One can state that both any 1-naive faster-than relation and any 2-naive-faster-relation is a c-naive faster-than relation. Consequently, the set of c-naive faster-than relations includes the union of 1-naive faster-than relations and 2-naive faster-than relations.

Proposition 23

(1) Every 1-naive faster-than relation $\mathcal{R}$ is a c-naive faster-than relation.

(2) Every 2-naive faster-than-relation $\mathcal{R}$ is a c-naive faster-than relation.

Proof. To prove Part (1) we consider some arbitrary processes P and Q such that $(P, Q) \in \mathcal{R}$. $\mathcal{R}$ satisfies the definition of a 1-naive faster-than relation.

(1) If $P \xrightarrow{\alpha}_1 P'$ for some process P' and some $\alpha \in \mathcal{A}$, then $Q \xrightarrow{\alpha}_1 Q'$ for some process Q' by Definition of $\mathcal{R}$. We are done since we may conclude $Q \xrightarrow{\alpha}_2 Q'$ by using Lemma 6(1).

(2) If $Q \xrightarrow{\alpha}_2 Q'$ for some process P' and some $\alpha \in \mathcal{A}$, then $Q \xrightarrow{\alpha}_1 Q'$ by using Lemma 6(1). Hence, we obtain $P \xrightarrow{\alpha}_1 P'$ for some P' such that $P' \,\mathcal{R}\, Q'$ is satisfied by the definition of $\mathcal{R}$.

(3) If $P \xrightarrow{\sigma}_1 P'$, then $Q \xrightarrow{\sigma}_1 Q'$ for some process Q' with $(P', Q') \in \mathcal{R}$ by definition of $\mathcal{R}$. Further, Lemma 6(2) implies $Q \xrightarrow{\sigma}_2 Q'$.

To prove Part (2), consider P and Q such that $(P, Q) \in \mathcal{R}$. $\mathcal{R}$ satisfies the definition of a 2-naive faster-than relation.

(1) If $P \xrightarrow{\alpha}_1 P'$ for some process P' and some $\alpha \in \mathcal{A}$, then $P \xrightarrow{\alpha}_2$ by application of Lemma 6(1). Thus, we may conclude $Q \xrightarrow{\alpha}_2 Q'$ such that $P' \,\mathcal{R}\, Q'$ by the definition of $\mathcal{R}$.

(2) If $Q \xrightarrow{\alpha}_2 Q'$ for some process Q', then we obtain $P \xrightarrow{\alpha}_2 P'$ for some P' such that $P' \,\mathcal{R}\, Q'$ is satisfied by the definition of $\mathcal{R}$. Finally, we may conclude $P \xrightarrow{\alpha}_1 P'$ by using Lemma 6(1).

(3) If $P \xrightarrow{\sigma}_1 P'$ for some process P', then $P \xrightarrow{\sigma}_2 P'$ by using Lemma 6(2). Hence, we obtain $Q \xrightarrow{\sigma}_2 Q'$ for some Q' such that $(P', Q') \in \mathcal{R}$ by the definition of $\mathcal{R}$. $\square$

Conversely, c-naive faster-than relations exist that are neither 1-naive faster-than relations, nor 2-naive faster-than relations. This fact shows that the set of relations, available for proving a naive faster-than relationship, was effectively increased. To give an example, consider the relation R_3 which neither satisfies the conditions of a 1-naive faster-than relation, nor the conditions of a 2-naive faster-than relation:

$R_3 =_{\text{df}} \{((\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} | \, \mathbf{0}) \setminus L),$
$((\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L),$
$((a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L)\}$

R_3 is not a 1-naive faster-than preorder as
$(\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L \xrightarrow{\sigma}_1 (\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L$ and
$(\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L \xrightarrow{\sigma}_1 (\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L$
but $((\sigma.a.\mathbf{0} \,|\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L) \notin R_3$.

Further, R_3 does not satisfy the definition of a 2-naive faster-than relation, since $(\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L \xrightarrow{\sigma}_2 (a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L$
but $((a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, Q') \in R''_3$ is not valid for any process Q'.

Finally, we investigate to what extent the newly-established relations are small ones. Once again, we consider the c-naive faster-than relation R_3 introduced above for which it will turn out that it is smaller than both the smallest corresponding 1-naive faster-than relation and every smallest 2-naive faster-than preorder.

R'_3 is the smallest 1-naive faster-than relation for
$((\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L)$,
while R''_3 is a smallest 2-naive faster-than relation for it:

$R'_3 =_{\text{df}} \{((\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} | \, \mathbf{0}) \setminus L),$
$((\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L),$
$((a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L),$
$((a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L)\}$

$R''_3 =_{\text{df}} \{((\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L)$
$((\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} | \, \mathbf{0}) \setminus L),$

$((\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L),$
$((a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L)\}$

Thus, there are c-naive faster-than relations that are smaller than the corresponding smallest naive faster-than relations of type-1 and type-2. The arising question if there exist 1-naive faster-than relations that are smaller than each corresponding c-naive-faster-relation can easily be answered in the negative. Such a 1-naive faster-than relation would be as well a c-naive faster-than relation by Lemma 23(1). The analogous argumentation involving Lemma 23(2) leads to the result that each smallest 2-naive faster-than relation is at least as large as each smallest corresponding c-naive faster-than relation.

If we constructed a preorder inversely to Definition 20 in such a way that a type-2-transition in the first component is simulated by a type-1-transition in the second component, this preorder would as well coincidence with our previous naive faster-than preorders. Each of these new relations would be a 1-naive faster-than relation as well as a 2-naive faster-than relation, hence the set of these new relations is contained in the intersection of the type-1-relations and type-2-relations. As a consequence, there are fewer of these relations and they are rather large ones, due to the type-2-transition which is regarded in the first component.

We come back to the c-naive faster-than preorder when we investigate the coarsest precongruence for type-2-transitions.

4.3 Naive faster-than relation up to

In the sequel, we define a new relation, called 'i-naive faster-than relation up to $\sqsupseteq_{i\text{-}nv}$', which is inspired by Milner's notion of 'bisimulation up to'. 'Naive faster-than relation up to $\sqsupseteq_{i\text{-}nv}$' is a technique for reducing the size of the relation needed to demonstrate a naive-faster-than-relationship.

Definition 24 *A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is an* i-naive faster-than relation up to $\sqsupseteq_{i\text{-}nv}$ *if the following conditions hold for all $\langle P, Q \rangle \in \mathcal{R}$, $\alpha \in \mathcal{A}$ and $i \in \{1, 2\}$.*

(1) $P \xrightarrow{\alpha}_i P'$ implies $\exists Q'. \, Q \xrightarrow{\alpha}_i Q'$ and $P' \sqsupseteq_{i\text{-}nv} \mathcal{R} \sqsupseteq_{i\text{-}nv} Q'$.
(2) $Q \xrightarrow{\alpha}_i Q'$ implies $\exists P'. \, P \xrightarrow{\alpha}_i P'$ and $P' \sqsupseteq_{i\text{-}nv} \mathcal{R} \sqsupseteq_{i\text{-}nv} Q'$.
(3) $P \xrightarrow{\sigma}_i P'$ implies $\exists Q'. \, Q \xrightarrow{\sigma}_i Q'$ and $P' \sqsupseteq_{i\text{-}nv} \mathcal{R} \sqsupseteq_{i\text{-}nv} Q'$.

Definition 25 *A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is a* c-naive faster-than relation up to $\sqsupseteq_{c\text{-}nv}$ *if the following conditions hold for all $\langle P, Q \rangle \in \mathcal{R}$ and $\alpha \in \mathcal{A}$.*

(1) $P \xrightarrow{\alpha}_1 P'$ *implies* $\exists Q'.\, Q \xrightarrow{\alpha}_1 Q'$ *and* $P' \underset{\sim}{\sqsupset}_{c\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{c\text{-}nv} Q'$.
(2) $Q \xrightarrow{\alpha}_1 Q'$ *implies* $\exists P'.\, P \xrightarrow{\alpha}_1 P'$ *and* $P' \underset{\sim}{\sqsupset}_{c\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{c\text{-}nv} Q'$.
(3) $P \xrightarrow{\sigma}_1 P'$ *implies* $\exists Q'.\, Q \xrightarrow{\sigma}_2 Q'$ *and* $P' \underset{\sim}{\sqsupset}_{c\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{c\text{-}nv} Q'$.

Lemma 26 *Let $\mathcal{R}$ be an i-naive faster-than relation up to $\underset{\sim}{\sqsupset}_{i\text{-}nv}$ for $i \in \{1, 2, c\}$. Then $\underset{\sim}{\sqsupset}_{i\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{i\text{-}nv}$ is an i-naive faster-than relation.*

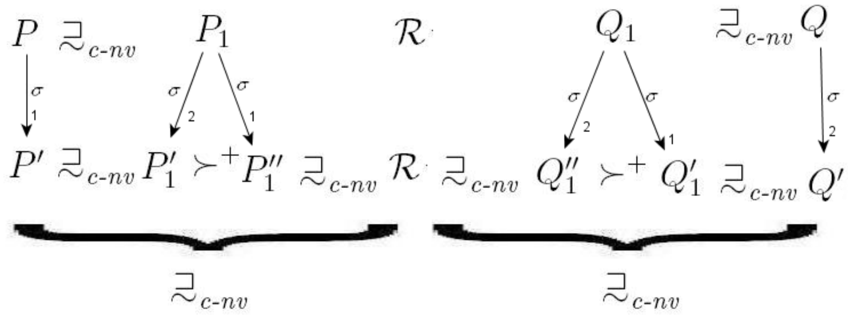
Proof. We combine the proofs for $i = 1$ and $i = 2$:
Consider $P \underset{\sim}{\sqsupset}_{i\text{-}nv} P_1 \; \mathcal{R} \; Q_1 \underset{\sim}{\sqsupset}_{i\text{-}nv} Q$.

(1) If $P \xrightarrow{\alpha}_i P'$ for some process P', then $P_1 \xrightarrow{\alpha}_i P'_1$ for some process P'_1 such that $P' \underset{\sim}{\sqsupset}_{i\text{-}nv} P'_1$ by the definition of $\underset{\sim}{\sqsupset}_{i\text{-}nv}$. Since $P_1 \xrightarrow{\alpha}_i P'_1$, we obtain $Q_1 \xrightarrow{\alpha}_i Q'_1$ for some process Q'_1 such that $P'_1 \underset{\sim}{\sqsupset}_{i\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{i\text{-}nv} Q'_1$ by Definition 24. Further, $Q_1 \xrightarrow{\alpha}_i Q'_1$ implies that there exists some Q' such that $Q \xrightarrow{\alpha}_i Q'$ and $Q'_1 \underset{\sim}{\sqsupset}_{i\text{-}nv} Q'$ by the definition of $\underset{\sim}{\sqsupset}_{i\text{-}nv}$. We conclude $P' \underset{\sim}{\sqsupset}_{i\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{i\text{-}nv} Q'$ from $P' \underset{\sim}{\sqsupset}_{i\text{-}nv} P'_1 \underset{\sim}{\sqsupset}_{i\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{i\text{-}nv} Q'_1 \underset{\sim}{\sqsupset}_{i\text{-}nv} Q'$ and the transitivity of $\underset{\sim}{\sqsupset}_{i\text{-}nv}$, cf. figure below.
(2) The case $Q \xrightarrow{\alpha}_i Q'$ for some Q' is analogous to case (1).
(3) The case $P \xrightarrow{\sigma}_i P'$ for some P' is as well analogous to the previous cases.

$$\begin{array}{ccccccccc}
P & \underset{\sim}{\sqsupset}_{i\text{-}nv} & P_1 & & \mathcal{R} & & Q_1 & \underset{\sim}{\sqsupset}_{i\text{-}nv} & Q \\
\Big\downarrow{\scriptstyle \alpha \atop i} & & \Big\downarrow{\scriptstyle \alpha \atop i} & & & & \Big\downarrow{\scriptstyle \alpha \atop i} & & \Big\downarrow{\scriptstyle \alpha \atop i} \\
P' & \underset{\sim}{\sqsupset}_{i\text{-}nv} & P'_1 & \underset{\sim}{\sqsupset}_{i\text{-}nv} & \mathcal{R} & \underset{\sim}{\sqsupset}_{i\text{-}nv} & Q'_1 & \underset{\sim}{\sqsupset}_{i\text{-}nv} & Q'
\end{array}$$
$$\underbrace{\underset{\sim}{\sqsupset}_{i\text{-}nv} P'_1 \underset{\sim}{\sqsupset}_{i\text{-}nv}}_{\underset{\sim}{\sqsupset}_{i\text{-}nv}} \qquad \underbrace{\underset{\sim}{\sqsupset}_{i\text{-}nv} Q'_1 \underset{\sim}{\sqsupset}_{i\text{-}nv}}_{\underset{\sim}{\sqsupset}_{i\text{-}nv}}$$

Proof for $i = c$:
Consider $P \underset{\sim}{\sqsupset}_{c\text{-}nv} P_1 \; \mathcal{R} \; Q_1 \underset{\sim}{\sqsupset}_{c\text{-}nv} Q$.

(1,2) The cases $P \xrightarrow{\alpha}_1 P'$ for some P' and $Q \xrightarrow{\alpha}_1 Q'$ for some Q' are treated in analogy to the corresponding cases in the proof above.
(3) If $P \xrightarrow{\sigma}_1 P'$ for some process P', then $P_1 \xrightarrow{\sigma}_2 P'_1$ for some process P'_1 such that $P' \underset{\sim}{\sqsupset}_{c\text{-}nv} P'_1$ by the definition of $\underset{\sim}{\sqsupset}_{c\text{-}nv}$. Further, $P_1 \xrightarrow{\sigma}_2 P'_1$ implies $P_1 \xrightarrow{\sigma}_1 P''_1$ such that $P'_1 \succ^+ P''_1$ by using Lemma 13. Since $P_1 \xrightarrow{\sigma}_1 P''_1$, we obtain $Q_1 \xrightarrow{\sigma}_2 Q''_1$ for some process Q''_1 such that $P''_1 \underset{\sim}{\sqsupset}_{c\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{c\text{-}nv} Q''_1$ by Definition 25. Again, $Q_1 \xrightarrow{\sigma}_2 Q''_1$ leads to $Q_1 \xrightarrow{\sigma}_1 Q'_1$ for some Q'_1 satisfying $Q''_1 \succ^+ Q'_1$. Finally, since $Q_1 \xrightarrow{\sigma}_1 Q'_1$, we infer $Q \xrightarrow{\sigma}_2 Q'$ for some Q' with $Q'_1 \underset{\sim}{\sqsupset}_{c\text{-}nv} Q'$. Since $\succ^+ \subseteq \underset{\sim}{\sqsupset}_{1\text{-}nv}$ by Proposition 18(2) for $i = 1$ and $\underset{\sim}{\sqsupset}_{1\text{-}nv} = \underset{\sim}{\sqsupset}_{c\text{-}nv}$, we know that $\succ^+ \subseteq \underset{\sim}{\sqsupset}_{c\text{-}nv}$. Hence, we conclude $P' \underset{\sim}{\sqsupset}_{c\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{c\text{-}nv} Q'$ from $P' \underset{\sim}{\sqsupset}_{c\text{-}nv} P'_1 \succ^+ P''_1 \underset{\sim}{\sqsupset}_{c\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{c\text{-}nv} Q''_1 \succ^+ Q'_1 \underset{\sim}{\sqsupset}_{c\text{-}nv} Q'$ by the transitivity of $\underset{\sim}{\sqsupset}_{c\text{-}nv}$, cf. figure below.



$\square$

Proposition 27 *Let $\mathcal{R}$ be an i-naive faster-than relation up to $\underset{\sim}{\sqsupset}_{i\text{-}nv}$ for $i \in \{1, 2, c\}$. Then $\mathcal{R} \subseteq \underset{\sim}{\sqsupset}_{i\text{-}nv}$.*

Proof. Let $\mathcal{R}$ be an i-naive faster-than relation up to $\underset{\sim}{\sqsupset}_{i\text{-}nv}$. According to the definition of the i-naive faster-than preorder we have $id_{\mathcal{P}} \subseteq \underset{\sim}{\sqsupset}_{i\text{-}nv}$. We conclude $\mathcal{R} = id_{\mathcal{P}} \circ \mathcal{R} \circ id_{\mathcal{P}} \subseteq \underset{\sim}{\sqsupset}_{i\text{-}nv} \circ \mathcal{R} \circ \underset{\sim}{\sqsupset}_{i\text{-}nv}$. Using Lemma 26 we know that $\underset{\sim}{\sqsupset}_{i\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{i\text{-}nv}$ is an i-naive faster-than relation and hence is contained in the largest i-naive faster-than relation $\underset{\sim}{\sqsupset}_{i\text{-}nv}$, i.e. $\underset{\sim}{\sqsupset}_{i\text{-}nv} \mathcal{R} \underset{\sim}{\sqsupset}_{i\text{-}nv} \subseteq \underset{\sim}{\sqsupset}_{i\text{-}nv}$. $\square$

Consequently it is sufficient to give an i-naive faster-than relation up to $\underset{\sim}{\sqsupset}_{i\text{-}nv}$ in order to prove a naive faster-than relationship.
Consider the following relation R_1 as an example for a c-naive faster-than relation up to $\underset{\sim}{\sqsupset}_{c\text{-}nv}$ in order to get sure that such a relation is really profitable.

$R_1 =_{\text{df}} \{((\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.\sigma.a.\mathbf{0} \,|\, \sigma.\sigma.\overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\sigma.\sigma.a.\mathbf{0} \,|\, \overline{a}.\mathbf{0} \,|\, a.\mathbf{0}) \setminus L),$
$((\sigma.\sigma.a.\mathbf{0} \,|\, \mathbf{0} \,|\, \mathbf{0}) \setminus L, (a.\mathbf{0} \,|\, \mathbf{0} | \,\mathbf{0}) \setminus L),$
$((\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L, (\mathbf{0} \,|\, \mathbf{0} \,|\, a.\mathbf{0}) \setminus L)\}$

Compare the relation R_1 to the relation R_3 in Subsection 4.2 and observe that we could save two further elements. Yet, this example is not valid for the precongruence that will be established later.

5 The delayed-faster-than-preorders

The next alternative candidate for a faster-than relation on processes in the course of design choices in [LV04] is the delayed faster-than preorder. Its nature is less strict, since it allows the slower process to perform any number of extra time steps when simulating an action transition or a time step of the faster process. We define the i-delayed faster-than preorder as follows, whereby the definition for $i = 1$ is adopted from [LV04]. We will study the delayed faster-than preorder in our new setting.

Definition 28 (i-delayed faster-than preorder) [LV04, Def. 5 for $i = 1$] *A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is an* i-delayed faster-than relation *if the following conditions hold for all $\langle P, Q\rangle \in \mathcal{R}$, $\alpha \in \mathcal{A}$ and $i \in \{1, 2\}$.*

(1) $P \xrightarrow{\alpha}_i P'$ *implies* $\exists Q'.\, Q \xrightarrow{\sigma}_i{}^* \xrightarrow{\alpha}_i \xrightarrow{\sigma}_i{}^* Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.
(2) $Q \xrightarrow{\alpha}_i Q'$ *implies* $\exists P'.\, P \xrightarrow{\alpha}_i P'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.
(3) $P \xrightarrow{\sigma}_i P'$ *implies* $\exists Q'.\, Q \xrightarrow{\sigma}_i{}^+ Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.

We write $P \gtrsim_{\textit{i-dly}} Q$ if $\langle P, Q\rangle \in \mathcal{R}$ for some i-delayed faster-than relation $\mathcal{R}$ and call $\gtrsim_{\textit{i-dly}}$ i-delayed faster-than preorder*.*

As usual, we can show that $\gtrsim_{\textit{i-dly}}$ is the largest delayed faster-than relation. Moreover, we can show that $\gtrsim_{\textit{i-dly}}$ is a preorder, where the proof for the transitivity is based on the following proposition.

Proposition 29 *Let $\mathcal{R}_1, \mathcal{R}_2$ be i-delayed faster-than relations for $i \in \{1, 2\}$. Then $\mathcal{R}_1 \circ \mathcal{R}_2$ is as well an i-delayed faster-than relation.*

Proof. We only consider the interesting Part (1) of Definition 28.
Consider some arbitrary pair of processes (P, Q) in $\mathcal{R}_1\mathcal{R}_2$, hence $\exists R \in \mathcal{P}$: $(P, R) \in \mathcal{R}_1 \wedge (R, Q) \in \mathcal{R}_2$ by the definition of composition.
If $P \xrightarrow{\alpha}_i P'$ for some P', then $R \xrightarrow{\sigma}_i{}^* R'' \xrightarrow{\alpha}_i R''' \xrightarrow{\sigma}_i{}^* R'$ for some processes R', R'' and R''' such that $(P', R') \in \mathcal{R}_1$ by Definition 28(1). $R \xrightarrow{\sigma}_i{}^* R''$ is defined as $R \equiv R_0 \xrightarrow{\sigma}_i \ldots \xrightarrow{\sigma}_i R_n \equiv R''$ for some processes $R_0, \ldots, R_n$ and some $n \geq 0$. Since $(R, Q) \in \mathcal{R}_2$, we obtain $Q \equiv Q_0 \xrightarrow{\sigma}_i{}^+ \ldots \xrightarrow{\sigma}_i{}^+ Q_n \equiv Q''$ for some processes $Q_0, \ldots, Q_n$ such that $\{(R_0, Q_0) \ldots (R_n, Q_n)\} \in \mathcal{R}_2$. Thus, $Q \xrightarrow{\sigma}_i{}^* Q''$ and $(R'', Q'') \in \mathcal{R}_2$. Further, we may assume $Q'' \xrightarrow{\sigma}_i{}^* \xrightarrow{\alpha}_i \xrightarrow{\sigma}_i{}^*$ Q''' such that $(R''', Q''') \in \mathcal{R}_2$ due to $(R'', Q'') \in \mathcal{R}_2$ and $R'' \xrightarrow{\alpha}_i R'''$. As $(R''', Q''') \in \mathcal{R}_2$ and $R''' \xrightarrow{\sigma}_i{}^* R'$ we know that $Q''' \xrightarrow{\sigma}_i{}^* Q'$ and $(R', Q') \in \mathcal{R}_2$ as above. This finishes the proof since $Q \xrightarrow{\sigma}_i{}^* \xrightarrow{\sigma}_i{}^* \xrightarrow{\alpha}_i \xrightarrow{\sigma}_i{}^* \xrightarrow{\sigma}_i{}^* Q'$ implies $Q \xrightarrow{\sigma}_i{}^* \xrightarrow{\alpha}_i \xrightarrow{\sigma}_i{}^* Q'$ and (P', Q') in $\mathcal{R}_1\mathcal{R}_2$. $\square$

In the following we want to show that the 2-delayed faster-than preorder $\gtrsim_{\textit{2-dly}}$ coincides with the 2-naive faster-than preorder $\gtrsim_{\textit{2-nv}}$ and the 1-delayed faster-than preorder $\gtrsim_{\textit{1-dly}}$. Before proceeding, we need to introduce one of the main results of [LV04], stating that the 1-naive faster-than preorder coincides with the 1-delayed faster-than preorder:

Theorem 30 (Coincidence III) [LV04][Theorem 10]
The preorders $\gtrsim_{\textit{1-nv}}$ *and* $\gtrsim_{\textit{1-dly}}$ *coincide.*

No we are able to state and prove that the two variants of delayed faster-than preorders coincide:

Theorem 31 (Coincidence IV) *The preorders* $\gtrsim_{\textit{1-dly}}$ *and* $\gtrsim_{\textit{2-dly}}$ *coincide.*

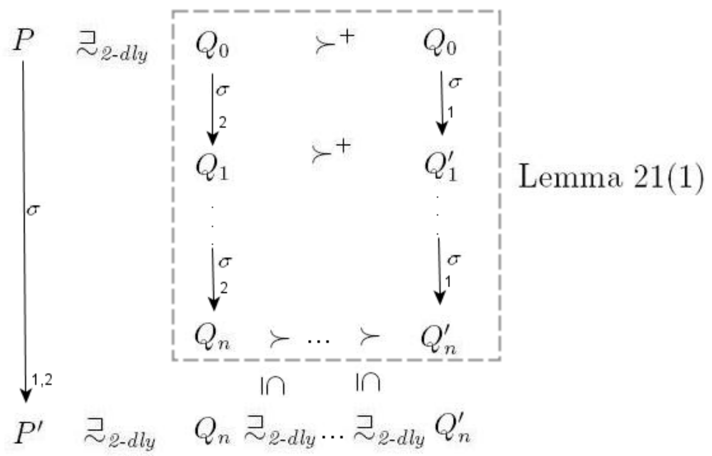
Proof. First, we prove that the 1-delayed faster-than preorder is as well a 2-delayed faster-than relation. Hence, consider some arbitrary processes P and Q such that $P \gtrsim_{\textit{1-dly}} Q$.

(1) If $P \xrightarrow{\alpha}_2 P'$ for some process P' and some action α, then $P \xrightarrow{\alpha}_1 P'$ is as well a valid transition by application of Lemma 6(1). Further, $Q \xrightarrow{\sigma}_1{}^* \xrightarrow{\alpha}_1 \xrightarrow{\sigma}_1{}^* Q'$ for some Q' such that $P' \gtrsim_{\textit{1-dly}} Q'$ by definition of $\gtrsim_{\textit{1-dly}}$. By repeated application of Lemma 6(2) and Lemma 6(1), we conclude $Q \xrightarrow{\sigma}_2{}^* \xrightarrow{\alpha}_2 \xrightarrow{\sigma}_2{}^* Q'$.
(2) If $Q \xrightarrow{\alpha}_2 Q'$ for some process Q' and some action α, we know $Q \xrightarrow{\alpha}_1 Q'$ by Lemma 6(1). Then, $P \xrightarrow{\alpha}_1 P'$ for some P' such that $P' \gtrsim_{\textit{1-dly}} Q'$ by definition of $\gtrsim_{\textit{1-dly}}$. We are done since we obtain $P \xrightarrow{\alpha}_2 P'$ again by Lemma 6(1).
(3) If $P \xrightarrow{\sigma}_2 P'$ for some P', we have $P \xrightarrow{\sigma}_1 P''$ for some P'' such that $P' \succ^+ P''$ by Lemma 13. Thus, $Q \xrightarrow{\sigma}_1{}^+ Q''$ for some Q'' with $P'' \gtrsim_{\textit{1-dly}} Q''$ by the definition of $\gtrsim_{\textit{1-dly}}$. We can infer $Q \xrightarrow{\sigma}_2{}^+ Q''$ by repeated application of Lemma 6(2).
Further, $P' \succ^+ P''$ is defined as $P' \equiv P_0 \succ P_1 \succ \ldots \succ P_n \equiv P''$ for some $n \geq 1$ and some $P_0, \ldots, P_n \in \widehat{\mathcal{P}}$.
We may conclude $P' \equiv P_0 \gtrsim_{\textit{1-dly}} P_1 \gtrsim_{\textit{1-dly}} \cdots \gtrsim_{\textit{1-dly}} P_n \equiv P''$, since $\succ \subseteq \gtrsim_{\textit{1-nv}}$ holds by Proposition 18(1) for $i = 1$ and $\gtrsim_{\textit{1-nv}} \subseteq \gtrsim_{\textit{1-dly}}$ by Theorem 30.
We are done since $P' \gtrsim_{\textit{1-dly}} Q''$ follows by the transitivity of $\gtrsim_{\textit{1-dly}}$ and $P' \gtrsim_{\textit{1-dly}} P''$.

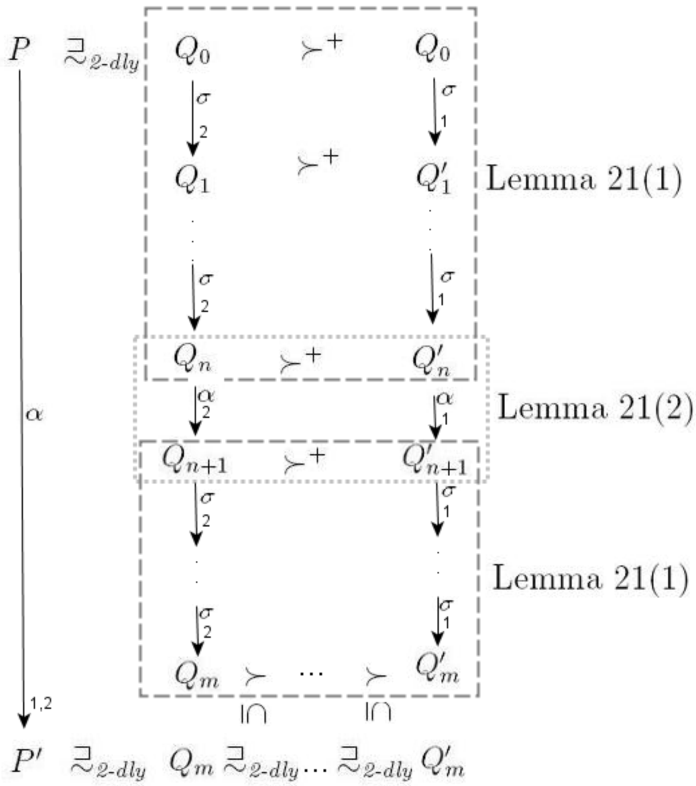
For the reverse inclusion, we show that the 2-delayed faster-than preorder is as well a 1-delayed faster-than relation and therefore consider $(P, Q) \in \gtrsim_{\textit{2-dly}}$. For a smooth presentation, we start with the simulation of a time step.

(3) If $P \xrightarrow{\sigma}_1 P'$, then $P \xrightarrow{\sigma}_2 P'$ is as well a valid time step by using Lemma 6(2). Then, we have $Q \xrightarrow{\sigma}_2{}^+ Q'$, i.e. $Q \equiv Q_0 \xrightarrow{\sigma}_2 \ldots \xrightarrow{\sigma}_2 Q_n \equiv Q'$ for some Q' and some $n \geq 1$ such that $P' \gtrsim_{\textit{2-dly}} Q'$ by definition of $\gtrsim_{\textit{2-dly}}$. Moreover, we obtain $Q_0 \succ^+ Q'_0 =_{\text{df}} Q_0$ by Definition 7(1). If now $Q_i \succ^+ Q'_i$ holds true, we may successively infer $Q'_i \xrightarrow{\sigma}_1 Q'_{i+1}$ for some Q'_{i+1} such that $Q_{i+1} \succ^+ Q'_{i+1}$ from $Q_i \xrightarrow{\sigma}_2 Q_{i+1}$ for $0 \leq i < n$ by application of

Lemma 21(1). Thus, we can conclude $Q \equiv Q'_0 \xrightarrow{\sigma}_1{}^+ Q'_n$ and $Q_n \succ^+ Q'_n$, which is defined as $Q_n \equiv Q''_0 \succ Q''_1 \succ \ldots \succ Q''_n \equiv Q'_n$ for some $n \geq 1$ and some $Q''_0, \ldots, Q''_n \in \widehat{\mathcal{P}}$. Further, $\succ \subseteq \precsim_{\textit{1-dly}}$ follows by Proposition 18(1) for $i = 1$ and by Theorem 30. As we showed $\precsim_{\textit{1-dly}} \subseteq \precsim_{\textit{2-dly}}$, we get $\succ \subseteq \precsim_{\textit{2-dly}}$. Hence, we obtain $Q_n \precsim_{\textit{2-dly}} Q'_n$ by the transitivity of $\precsim_{\textit{2-dly}}$ and may conclude $P' \precsim_{\textit{2-dly}} Q'_n$ from $P' \precsim_{\textit{2-dly}} Q'$ and $Q' \equiv Q_n \precsim_{\textit{2-dly}} Q'_n$, cf. figure below.



(1) If $P \xrightarrow{\alpha}_1 P'$, then as well $P \xrightarrow{\alpha}_2 P'$ by Lemma 6(1).
Then, $Q \equiv Q_0 \xrightarrow{\sigma}_2 \ldots \xrightarrow{\sigma}_2 Q_n \xrightarrow{\alpha}_2 Q_{n+1} \xrightarrow{\sigma}_2 \ldots \xrightarrow{\sigma}_2 Q_m \equiv Q'$ for some Q' and some n and m with $0 \leq n < m$ such that $P' \precsim_{\textit{2-dly}} Q'$ by definition of $\precsim_{\textit{2-dly}}$. Analogously to the argumentation in the previous case, we get $Q \xrightarrow{\sigma}_1{}^* Q'_n$ and $Q_n \succ^+ Q'_n$ for some Q'_n by application of Lemma 21(1). Further, we conclude $Q'_n \xrightarrow{\alpha}_1 Q'_{n+1}$ for some Q'_{n+1} such that $Q_{n+1} \succ^+ Q'_{n+1}$ by using Lemma 21(2). Then, $Q'_{n+1} \xrightarrow{\sigma}_1{}^* Q'_m$ for some Q'_m such that $Q_m \succ^+ Q'_m$ by using Lemma 21(1). Analogously to the previous case, $Q_m \precsim_{\textit{2-dly}} Q'_m$ results from $\succ \subseteq \precsim_{\textit{2-dly}}$. Altogether, we have $Q \xrightarrow{\sigma}_1{}^* \xrightarrow{\alpha}_1 \xrightarrow{\sigma}_1{}^* Q'_m$. Moreover, $P' \precsim_{\textit{2-dly}} Q'_m$ follows from $P' \precsim_{\textit{2-dly}} Q' \equiv Q_m$ and $Q_m \precsim_{\textit{2-dly}} Q'_m$.



(2) The case $Q \xrightarrow{\alpha}_1 Q'$ for some process Q' and some action $\alpha \in \mathcal{A}$ is analogous to the previous case. $\square$

Indirectly, we showed that the 2-naive faster-than preorder and the 2-delayed faster-than preorder coincide:

Corollary 32 (Coincidence V) *The preorders* $\underset{\sim}{\sqsupset}_{\textit{2-nv}}$ *and* $\underset{\sim}{\sqsupset}_{\textit{2-dly}}$ *coincide.*

Proof.

$\underset{\sim}{\sqsupset}_{\textit{2-nv}} \overset{\text{Coinc.I}}{=} \underset{\sim}{\sqsupset}_{\textit{1-nv}} \overset{\text{Coinc.II}}{=} \underset{\sim}{\sqsupset}_{\textit{1-dly}} \overset{\text{Coinc.III}}{=} \underset{\sim}{\sqsupset}_{\textit{2-dly}} \quad \square$

Clearly, any 1-naive faster-than relation is as well a 1-delayed one, as well as any 2-naive faster-than relation is a 2-delayed one. Comparing delayed faster-than relations of type-1 and type-2, we can establish that it is neither true that any 1-delayed faster-than relation is a 2-delayed faster-than relation, nor the reverse statement holds, in analogy to the naive faster-than relations. Similar to the previous section, we could define a third combined preorder that possesses properties equal to those of the c-naive faster-than preorder.

The fact that we have shown that the two variants of delayed faster-than preorders coincide with the two variants of naive faster-than preorders underpins the preference of the simple, concise naive faster-than preorder over the more complicated delayed faster-than preorder. Yet, delayed faster-than relations of both types can be useful in practice in order to demonstrate a faster-than relationship since there are delayed faster-than relations which are not naive faster-than relations and which are smaller than every corresponding naive faster-than relation.

6 Indexed faster-than preorder

The second variant of a faster-than preorder in [LV04] is the indexed faster-than preorder, formalizing the idea of an account for time steps for the faster process. If a time step of the slower process is not simulated immediately by the faster process, then this time step is credited and might be withdrawn if the process performs this time step later on. Obviously, the account balance may never be negative.

The xed faster-than preorder is defined as follows:

Definition 33 (Family of indexed faster-than preorders) [LV04, Def. 11 for $i = 1$]
For $i \in \{1, 2\}$, a family $(\mathcal{R}_j)_{j\in\mathbb{N}}$, of relations over $\mathcal{P}$, indexed by natural numbers (including 0), is a family of i-indexed faster-than relations *if, for all $j \in \mathbb{N}$, $\langle P, Q\rangle \in \mathcal{R}_{i,j}$, and $\alpha \in \mathcal{A}$:*

(1) $P \xrightarrow{\alpha}_i P'$ *implies* $\exists Q'.\, Q \xrightarrow{\alpha}_i Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}_j$.
(2) $Q \xrightarrow{\alpha}_i Q'$ *implies* $\exists P'.\, P \xrightarrow{\alpha}_i P'$ *and* $\langle P', Q'\rangle \in \mathcal{R}_j$.
(3) $P \xrightarrow{\sigma}_i P'$ *implies (a)* $\exists Q'.\, Q \xrightarrow{\sigma}_i Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}_j$, *or*
(b) $j > 0$ *and* $\langle P', Q\rangle \in \mathcal{R}_{j-1}$.
(4) $Q \xrightarrow{\sigma}_i Q'$ *implies (a)* $\exists P'.\, P \xrightarrow{\sigma}_i P'$ *and* $\langle P', Q'\rangle \in \mathcal{R}_j$, *or*
(b) $\langle P, Q'\rangle \in \mathcal{R}_{j+1}$.

We write $P \sqsupseteq\!\!\!\!\sim_{i,j} Q$ if $\langle P, Q\rangle \in \mathcal{R}_j$ for some family of i-indexed faster-than relations $(\mathcal{R}_j)_{j\in\mathbb{N}}$ and call $\sqsupseteq\!\!\!\!\sim_{i,j}$ i,j-indexed faster-than preorder.

Note, that it is not clear that the relations of the family of largest indexed-faster than relations are transitive. [LV04] proves that the 1-naive faster-than preorder coincides with $\sqsupseteq\!\!\!\!\sim_{1,0}$ and hence demonstrate that $\sqsupseteq\!\!\!\!\sim_{1,0}$ is transitive. We would have expected to be able to prove the same coincidence results in the new setting. Unfortunately, this has turned out to be wrong, due to the absence of time determinism. According to their definitions, $\sqsupseteq\!\!\!\!\sim_{2,0} \subseteq \sqsupseteq\!\!\!\!\sim_{2\text{-}nv}$ obviously holds. However, the reverse inclusion is not valid as one can inspect by studying the following counterexample:

Let $P =_{\mathrm{df}} \tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0}$ and $Q =_{\mathrm{df}} \sigma.\tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0}$.
Clearly, P is faster than Q in the sense of a naive faster-than relationship. In the sequel, we will try to build a family of relations such that $(P, Q) \in \gtrsim_{2,0}$ holds.

Hence, to define the family $(\mathcal{R}_j)_{j \in \mathbb{N}}$, we put (P, Q) into the relation $\mathcal{R}_0$ and first consider a 'real' type-2 time step of the process Q. If $\sigma.\tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0} \xrightarrow{\sigma}_2 \tau.\mathbf{0} \,|\, \tau.\mathbf{0}$, then P is not able to match this behaviour, as a time step is preempted by an urgent τ. Therefore we are forced to credit this time step and hence obtain:

$R_0 =_{\mathrm{df}} \{(\tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0},\ \sigma.\tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0}), \ldots\}$
$R_1 =_{\mathrm{df}} \{(\tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0},\ \tau.\mathbf{0} \,|\, \tau.\mathbf{0}), \ldots\}$

Further, we must take into consideration the action transition $\tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0} \xrightarrow{\tau}_2 \mathbf{0} |\, \sigma.\sigma.\tau.\mathbf{0}$ of P, which 'Q' can either mimic with the action transition $\tau.\mathbf{0} \,|\, \tau.\mathbf{0} \xrightarrow{\tau}_2 \mathbf{0} |\, \tau.\mathbf{0}$ or the transition $\tau.\mathbf{0} \,|\, \tau.\mathbf{0} \xrightarrow{\tau}_2 \tau.\mathbf{0} |\, \mathbf{0}$. Since the resulting processes have the same functional and waiting behaviour, we may consider any of them.
Thus, we so far know:

$R_0 =_{\mathrm{df}} \{(\tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0},\ \sigma.\tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0}), \ldots\}$
$R_1 =_{\mathrm{df}} \{(\tau.\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0},\ \tau.\mathbf{0} \,|\, \tau.\mathbf{0}), (\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0},\ \mathbf{0} \,|\, \tau.\mathbf{0}), \ldots\}$

Finally, $\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0}$ is enabled to perform a time step of the form $\mathbf{0} \,|\, \sigma.\sigma.\tau.\mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0} \,|\, \sigma.\tau.\mathbf{0}$. As this time step cannot be simulated by Q, we have to withdraw the credited time step and put $(\mathbf{0} \,|\, \sigma.\tau.\mathbf{0},\ \mathbf{0} \,|\, \tau.\mathbf{0})$ in the relation $\mathcal{R}_0$. This leads to a contradiction due to $\mathbf{0} \,|\, \sigma.\tau.\mathbf{0} \xrightarrow{\sigma}_2 \mathbf{0} \,|\, \tau.\mathbf{0}$ since $\mathbf{0} \,|\, \tau.\mathbf{0}$ is strictly faster than $\mathbf{0} \,|\, \sigma.\tau.\mathbf{0}$. Altogether, it is not possible to construct a family of relations satisfying $P \, \mathcal{R}_0 \, Q$.

Summarizing, the problem lies in the fact that the slower process Q performs a 'real' type-2 time step and skips a σ-prefix, but only one time step is credited for the faster process P so that the slower process Q is ahead in time. We leave the repair of this defect by altering the definition of the indexed faster-than preorder for future work.

7 Strong faster-than precongruence

According to [LV04] a shortcoming of the 1-naive faster-than preorder is that it is not a precongruence since it is not compositional. Clearly, $\gtrsim_{2\text{-}nv}$ and $\gtrsim_{c\text{-}nv}$

are as well not precongruences due to the fact that $\gtrsim_{\textit{1-nv}}$, $\gtrsim_{\textit{2-nv}}$ and $\gtrsim_{\textit{c-nv}}$ coincidence. As an example consider the processes $P =_{\mathrm{df}} \sigma.a.0$ and $Q =_{\mathrm{df}} a.0$ for which $P \gtrsim_{\textit{i-nv}} Q$ holds: the time step $\sigma.a.0 \xrightarrow{\sigma}_i a.0$ of P is matched by the time step $a.0 \xrightarrow{\sigma}_i a.0$,i.e. Q idles. Yet, if we compose both processes in parallel with a third process $R =_{\mathrm{df}} \overline{a}.\mathbf{0}$, then we observe that $P \,|\, R \gtrsim_{\textit{i-nv}} Q \,|\, R$ does not hold as the clock transition $P \,|\, R \xrightarrow{\sigma}_i a.|\overline{a}.\mathbf{0}$ cannot be matched with a clock transition of $Q \,|\, R$ as a time step is preempted due to an urgent τ. Anyway, it is intuitively suggestive to exclude such pairs of the form $\langle \sigma.P, P\rangle$ from our naive faster-than preorder. In [LV04] another preorder is defined that takes into account the urgent sets of processes in such a way that a time-step of the faster process P always implies that there are no urgent actions of the strictly slower process Q that are not urgent in P. Since this preorder turns out to be the largest precongruence contained in $\gtrsim_{\textit{1-nv}}$, it is called strong faster-than precongruence. We adopt the definition of the strong 1-faster-than precongruence from [LV04] and extend it for type-2-transitions and in order to get small relations we as well establish the strong c-faster-than precongruence:

Definition 34 (Strong i-faster-than precongruence) [LV04, Def. 18 for $i = 1$]
A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is an strong i-faster-than relation *for $i \in \{1, 2\}$ if the following condition hold for all $\langle P, Q\rangle \in \mathcal{R}$ and $\alpha \in \mathcal{A}$.*

(1) $P \xrightarrow{\alpha}_i P'$ *implies* $\exists Q'.\, Q \xrightarrow{\alpha}_i Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.
(2) $Q \xrightarrow{\alpha}_i Q'$ *implies* $\exists P'.\, P \xrightarrow{\alpha}_i P'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.
(3) $P \xrightarrow{\sigma}_i P'$ *implies* $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ *and* $\exists Q'.\, Q \xrightarrow{\sigma}_i Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.

We write $P \gtrsim_i Q$ if $\langle P, Q\rangle \in \mathcal{R}$ for some strong i-faster-than relation $\mathcal{R}$ and call $\gtrsim_i$ strong i-faster-than precongruence.

Definition 35 (Strong c-faster-than precongruence)
A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is a strong c-faster-than relation *if the following condition hold for all $\langle P, Q\rangle \in \mathcal{R}$ and $\alpha \in \mathcal{A}$.*

(1) $P \xrightarrow{\alpha}_1 P'$ *implies* $\exists Q'.\, Q \xrightarrow{\alpha}_1 Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.
(2) $Q \xrightarrow{\alpha}_1 Q'$ *implies* $\exists P'.\, P \xrightarrow{\alpha}_1 P'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.
(3) $P \xrightarrow{\sigma}_1 P'$ *implies* $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ *and* $\exists Q'.\, Q \xrightarrow{\sigma}_2 Q'$ *and* $\langle P', Q'\rangle \in \mathcal{R}$.

We write $P \gtrsim_c Q$ if $\langle P, Q\rangle \in \mathcal{R}$ for some strong c-faster-than relation $\mathcal{R}$ and call $\gtrsim_c$ strong c-faster-than precongruence.

Clearly, $\gtrsim_i$ is contained in $\gtrsim_{\textit{i-nv}}$ for $i \in \{1, 2, c\}$. As usual, it is easy to prove that $\gtrsim_i$ is the largest strong i-faster-than relation and that it is a preorder for $i \in \{1, 2\}$. However, note that it is not clear that $\gtrsim_c$ is transitive. This will follow when we establish our next theorems stating that the strong c-faster-than precongruence coincides with the two other strong faster-than precongruences.

To prove these theorems, we present the following statement:

Proposition 36
The relation $\succ$ satisfies the defining clauses of a strong 1-faster-than relation, also on open terms; hence, $\succ$ restricted to processes is a strong 1-faster-than relation and $\succ_{|\mathcal{P}\times\mathcal{P}} =_{df} \succ \cap (\mathcal{P}\times\mathcal{P}) \subseteq \sqsupseteq\!\!\!\!\sim_{\mathit{1}}$.

This statement is given in [LV04] and can be shown in analogy to the proof of Proposition 18(1) for $i = 1$, additionally using Lemma 10.
Now we can show the coincidence results similarly to the proof for the coincidence results I and II.

Theorem 37 (Coincidence VI) *The preorders $\sqsupseteq\!\!\!\!\sim_{\mathit{1}}$ and $\sqsupseteq\!\!\!\!\sim_{\mathit{2}}$ coincide.*

Proof. First, to see the inclusion $\sqsupseteq\!\!\!\!\sim_{\mathit{1}} \subseteq \sqsupseteq\!\!\!\!\sim_{\mathit{2}}$, we prove that $\sqsupseteq\!\!\!\!\sim_{\mathit{1}}$ is a strong 2-faster-than relation. Consider some arbitrary processes P and Q such that $P \sqsupseteq\!\!\!\!\sim_{\mathit{1}} Q$.

(1,2) The cases $P \xrightarrow{\alpha}_2 P'$ for some process P' and $Q \xrightarrow{\alpha}_2 Q'$ for some process Q' are treated in analogy to the corresponding cases in the coincidence result for the naive faster-than preorders of type-1 and type-2 in Theorem 19 and are therefore omitted here.

(3) If $P \xrightarrow{\sigma}_2 P''$ for some process P'', then $P \xrightarrow{\sigma}_1 P'$ for some P' such that $P'' \succ^+ P'$ by application of Lemma 13. Further, $P \xrightarrow{\sigma}_1 P'$ implies $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ and $Q \xrightarrow{\sigma}_1 Q'$ for some Q' satisfying $P' \sqsupseteq\!\!\!\!\sim_{\mathit{1}} Q'$ by definition of $\sqsupseteq\!\!\!\!\sim_{\mathit{1}}$. We may conclude $Q \xrightarrow{\sigma}_2 Q'$ by using Lemma 6(2). $P'' \succ^+ P'$ is defined as $P'' \equiv P_0 \succ \ldots \succ P_n \equiv P'$ for some processes $P_0, \ldots, P_n$ and some $n \geq 1$. Since $\succ$ satisfies the definition of a strong 1-faster-than relation by using Proposition 36, we conclude $P'' \equiv P_0 \sqsupseteq\!\!\!\!\sim_{\mathit{1}} \ldots \sqsupseteq\!\!\!\!\sim_{\mathit{1}} P_n \equiv P'$ and hence $P'' \sqsupseteq\!\!\!\!\sim_{\mathit{1}} P'$ by the transitivity of $\sqsupseteq\!\!\!\!\sim_{\mathit{1}}$. In summary, $P'' \sqsupseteq\!\!\!\!\sim_{\mathit{1}} Q'$ follows from $P'' \sqsupseteq\!\!\!\!\sim_{\mathit{1}} P'$ and $P' \sqsupseteq\!\!\!\!\sim_{\mathit{1}} Q'$.

To prove the inverse inclusion $\sqsupseteq\!\!\!\!\sim_{\mathit{2}} \subseteq \sqsupseteq\!\!\!\!\sim_{\mathit{1}}$, we analogously consider some arbitrary processes P and Q such that $P \sqsupseteq\!\!\!\!\sim_{\mathit{2}} Q$.

(3) As above, we only consider the case $P \xrightarrow{\sigma}_1 P'$ for some process P'.
As usual, we get $P \xrightarrow{\sigma}_2 P'$ by using Lemma 6(2). Then, $P \xrightarrow{\sigma}_2 P'$ implies $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ and $Q \xrightarrow{\sigma}_2 Q'$ for some Q' satisfying $P' \sqsupseteq\!\!\!\!\sim_{\mathit{2}} Q'$ by the definition of $\sqsupseteq\!\!\!\!\sim_{\mathit{2}}$.
Further, $Q \xrightarrow{\sigma}_2 Q'$ implies that $Q \xrightarrow{\sigma}_1 Q''$ for some Q'' with $Q' \succ^+ Q''$. Hence, $Q' \sqsupseteq\!\!\!\!\sim_{\mathit{2}} Q''$ follows from $\succ \subseteq \sqsupseteq\!\!\!\!\sim_{\mathit{1}} \subseteq \sqsupseteq\!\!\!\!\sim_{\mathit{2}}$ and the transitivity of $\sqsupseteq\!\!\!\!\sim_{\mathit{2}}$. We are done since $P' \sqsupseteq\!\!\!\!\sim_{\mathit{2}} Q'$ and $Q' \sqsupseteq\!\!\!\!\sim_{\mathit{2}} Q''$ leads to $P' \sqsupseteq\!\!\!\!\sim_{\mathit{2}} Q''$. $\Box$

Theorem 38 (Coincidence VII) *The preorders $\sqsupseteq\!\!\!\!\raisebox{-1ex}{$\sim$}_1$ and $\sqsupseteq\!\!\!\!\raisebox{-1ex}{$\sim$}_c$ coincide.*

Proof. First, we prove the inclusion $\gtrsim_1 \subseteq \gtrsim_c$ as above and hence consider some arbitrary processes P and Q such that $P \gtrsim_1 Q$.

(1,2) The cases $P \xrightarrow{\alpha}_1 P'$ for some process P' and $Q \xrightarrow{\alpha}_1 Q'$ for some process Q' are treated in analogy to the corresponding cases in the coincidence result for the naive faster-than preorders of type-1 and type-c in Theorem 22 and are therefore omitted here.

(3) If $P \xrightarrow{\sigma}_1 P'$ for some process P', then $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ and $Q \xrightarrow{\sigma}_1 Q'$ for some Q' satisfying $P' \gtrsim_1 Q'$ by definition of $\gtrsim_1$. We are done since $Q \xrightarrow{\sigma}_1 Q'$ implies $Q \xrightarrow{\sigma}_2 Q'$ by application of Lemma 6(2).

For the reverse inclusion $\gtrsim_c \subseteq \gtrsim_1$, define the relation $\mathcal{R}$ by $(P, Q) \in \mathcal{R}$ if and only if $\exists R \in \mathcal{P}.\ P \gtrsim_c R \succ^+ Q$ for $P, Q \in \mathcal{P}$. In the following we check that this relation $\mathcal{R}$ satisfies the definition of a strong 1-faster-than relation; consider $P \gtrsim_c R \succ^+ Q$.

(1) If $P \xrightarrow{\alpha}_1 P'$ for some P', the definition of $\gtrsim_c$ shows $R \xrightarrow{\alpha}_1 R'$ for some process R' with $P' \gtrsim_c R'$. Due to R, Q being processes, the syntactic relation is restricted to processes here. $R \succ^+ Q$ is defined as $R \succ \ldots \succ Q$. Since $\succ$ satisfies the definition of a strong 1-faster-than relation by using Proposition 36, we successively may conclude that $Q \xrightarrow{\alpha}_1 Q'$ for some Q' such that $R' \succ^+ Q'$.

(2) The case $Q \xrightarrow{\alpha}_1 Q'$ for some Q' is analogous to Part (1).

(3) If $P \xrightarrow{\sigma}_1 P'$ for some P', then $\mathcal{U}(R) \subseteq \mathcal{U}(P)$ and $R \xrightarrow{\sigma}_2 R'$ for some process R' with $P' \gtrsim_c R'$ by definition of $\gtrsim_c$.
Due to $R \xrightarrow{\sigma}_2 R'$, we may infer $Q \xrightarrow{\sigma}_1 Q'$ for some Q' satisfying $R' \succ^+ Q'$ by using Lemma 21(1).
Moreover, we know $\mathcal{U}(Q) \subseteq \mathcal{U}(R)$ due to $R \succ^+ Q$ by successive application of Lemma 10 and therefore $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ results from $\mathcal{U}(Q) \subseteq \mathcal{U}(R)$ and $\mathcal{U}(R) \subseteq \mathcal{U}(P)$.

This finishes the proof, since $\succ^+$ is reflexive and hence $\gtrsim_c \subseteq \gtrsim_c \circ \succ^+ = \mathcal{R} \subseteq \gtrsim_1$ is valid. In summary, we obtain $\gtrsim_c \subseteq \gtrsim_1$. $\square$

[LV04] shows that the strong 1-faster-than preorder is a precongruence and that it is the largest precongruence (for all operators as well as for recursion). contained in the 1-naive-faster-than preorder.

Theorem 39 (Full abstraction) [LV04][Theorem 19 for $i = 1$]
The preorder $\gtrsim_i$ is the largest precongruence *contained in $\gtrsim_{i\text{-}nv}$ for $i \in \{1, 2, c\}$.*

Proof. The proof for $i = 1$ is given in [LV04].
Clearly, this statement as well holds for $\gtrsim_{2\text{-}nv}$ and $\gtrsim_{c\text{-}nv}$, since the coincidence of $\gtrsim_{i\text{-}nv}$, $\gtrsim_{2\text{-}nv}$ and $\gtrsim_{c\text{-}nv}$ immediately implies the coincidence of the largest precongruences in it. $\square$

Note, that the example for a small c-naive faster-than relation, that is given in Section 4.2, is also valid for the c-naive faster-than precongruence.

8 Weak variants

This section presents some variants of weak faster-than preorders, which abstract from internal, unobservable actions. Their specification is necessary since the strong faster-than precongruence is too discriminating to verify systems in practice. To demonstrate that the coincidence results carry over to the weak preorders, we show that the various types of weak preorders coincide for $i = 1$ and $i = 2$. It is worth pointing out that these coincidence proofs are exclusively based on technical devices that are already employed in previous coincidence proofs. First, it is convenient to introduce some notations that are used in the following definitions. For any action α we define $\hat{\alpha} =_{\text{df}} \epsilon$, if $\alpha = \tau$, and $\hat{\alpha} =_{\text{df}} \alpha$, otherwise. Further, we let $\stackrel{\epsilon}{\Longrightarrow} =_{\text{df}} \stackrel{\tau}{\longrightarrow}^*$ and write $P \stackrel{\alpha}{\Longrightarrow} Q$ if there exist R and S such that $P \stackrel{\epsilon}{\Longrightarrow} R \stackrel{\alpha}{\longrightarrow} S \stackrel{\epsilon}{\Longrightarrow} Q$.

In analogy to [LV04], we start off with the definition of an *i-naive weak faster-than preorder* where the faster and the slower process are linked by a relation which is a *weak bisimulation* for action transitions. One also allows the slower process to perform additional unobservable actions when simulating a time step of the faster process.

Definition 40 (i-naive weak faster-than preorder) [LV04, Def. 27 for $i = 1$]
A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is an i-naive weak faster-than relation *if the following conditions hold for all $\langle P, Q\rangle \in \mathcal{R}$, $\alpha \in \mathcal{A}$ and $i \in \{1, 2\}$.*

(1) $P \stackrel{\alpha}{\longrightarrow}_i P'$ implies $\exists Q'.\, Q \stackrel{\hat{\alpha}}{\Longrightarrow}_i Q'$ and $\langle P', Q'\rangle \in \mathcal{R}$.
(2) $Q \stackrel{\alpha}{\longrightarrow}_i Q'$ implies $\exists P'.\, P \stackrel{\hat{\alpha}}{\Longrightarrow}_i P'$ and $\langle P', Q'\rangle \in \mathcal{R}$.
(3) $P \stackrel{\sigma}{\longrightarrow}_i P'$ implies $\exists Q', Q'', Q'''.\, Q \stackrel{\epsilon}{\Longrightarrow}_i Q'' \stackrel{\sigma}{\longrightarrow}_i Q''' \stackrel{\epsilon}{\Longrightarrow}_i Q'$ and $\langle P', Q'\rangle \in \mathcal{R}$.

We write $P \gtrsim_{i\text{-}nv} Q$ if $\langle P, Q\rangle \in \mathcal{R}$ for some i-naive weak faster-than relation $\mathcal{R}$ and call $\gtrsim_{i\text{-}nv}$ i-naive weak faster-than preorder*.*

As usual, we can show that $\underset{\approx}{\sqsupset}_{\textit{i-nv}}$ is the largest i-naive weak faster-than relation and that it is a preorder.
Clearly, any and also the largest i-naive faster-than relation satisfies the definition of $\underset{\approx}{\sqsupset}_{\textit{i-nv}}$. Hence $\underset{\sim}{\sqsupset}_{\textit{i-nv}} \subseteq \underset{\approx}{\sqsupset}_{\textit{i-nv}}$, and the syntactic relation $\succ$ satisfies the definition of a weak 1-naive faster-than relation by using Proposition 18(1) for $i = 1$.
In the sequel we want to state and prove that the naive weak faster-than preorders of type-1 and type-2 coincide.

Theorem 41 (Coincidence VIII) *The preorders* $\underset{\approx}{\sqsupset}_{\textit{1-nv}}$ *and* $\underset{\approx}{\sqsupset}_{\textit{2-nv}}$ *coincide.*

Proof. First, we show the inclusion $\underset{\approx}{\sqsupset}_{\textit{1-nv}} \subseteq \underset{\approx}{\sqsupset}_{\textit{2-nv}}$ and hence consider some arbitrary processes P and Q such that $P \underset{\approx}{\sqsupset}_{\textit{1-nv}} Q$.

(1) If $P \xrightarrow{\alpha}_2 P'$ for some P', then $P \xrightarrow{\alpha}_1 P'$ by using Lemma 6(1). Further, $Q \xRightarrow{\hat{\alpha}}_1 Q'$ for some Q' such that $\langle P', Q' \rangle \in \underset{\approx}{\sqsupset}_{\textit{1-nv}}$ by definition of $\underset{\approx}{\sqsupset}_{\textit{1-nv}}$, which means that $Q \xrightarrow{\tau}_1{}^{*} \xrightarrow{\alpha}_1 \xrightarrow{\tau}_1{}^{*} Q'$ if $\alpha \equiv a$ for some $a \in \Lambda \cup \overline{\Lambda}$ or $Q \xrightarrow{\tau}_1{}^{*} Q'$ if $\alpha \equiv \tau$. In the first case we get $Q \xrightarrow{\tau}_2{}^{*} \xrightarrow{\alpha}_2 \xrightarrow{\tau}_2{}^{*} Q'$ by repeated application of Lemma 6(1). In the second case we obtain $Q \xrightarrow{\tau}_2{}^{*} Q'$ also by repeated application of Lemma 6(1).
(2) This case is treated in analogy to the previous case.
(3) If $P \xrightarrow{\sigma}_2 P''$ for some P'', then $P \xrightarrow{\sigma}_1 P'$ for some P' such that $P'' \succ^{+} P'$ by Lemma 13. Further, $Q \xRightarrow{\epsilon}_1 Q'' \xrightarrow{\sigma}_1 Q''' \xRightarrow{\epsilon}_1 Q'$, i.e. $Q \xrightarrow{\tau}_1{}^{*} Q'' \xrightarrow{\sigma}_1 Q''' \xrightarrow{\tau}_1{}^{*} Q'$ for some Q' such that $\langle P', Q' \rangle \in \underset{\approx}{\sqsupset}_{\textit{1-nv}}$ by the definition of $\underset{\approx}{\sqsupset}_{\textit{1-nv}}$. Then $Q \xrightarrow{\tau}_2{}^{*} Q'' \xrightarrow{\sigma}_2 Q''' \xrightarrow{\tau}_2{}^{*} Q'$ by repeated application of Lemma 6(1) and by Lemma 6(2). We are done since $P'' \succ^{+} P'$ implies $P'' \underset{\approx}{\sqsupset}_{\textit{1-nv}} P'$ and $P'' \underset{\approx}{\sqsupset}_{\textit{1-nv}} Q'$ follows from $P'' \underset{\approx}{\sqsupset}_{\textit{1-nv}} P'$ and $P' \underset{\approx}{\sqsupset}_{\textit{1-nv}} Q'$ by the transitivity of $\underset{\approx}{\sqsupset}_{\textit{1-nv}}$.

To show the inverse inclusion $\underset{\approx}{\sqsupset}_{\textit{2-nv}} \subseteq \underset{\approx}{\sqsupset}_{\textit{1-nv}}$, we consider some arbitrary processes P and Q such that $P \underset{\approx}{\sqsupset}_{\textit{2-nv}} Q$.

(1,2) The cases $P \xrightarrow{\alpha}_1 P'$ for some process P' and $Q \xrightarrow{\alpha}_1 Q'$ for some Q' are analogous to the corresponding cases in the proof for the inverse inclusion.
(3) If $P \xrightarrow{\sigma}_1 P'$ for some P', then $P \xrightarrow{\sigma}_2 P'$ by using Lemma 6(2). Then, $Q \xrightarrow{\tau}_2{}^{*} Q'' \xrightarrow{\sigma}_2 Q''' \xrightarrow{\tau}_2{}^{*} Q'$ for some Q' such that $\langle P', Q' \rangle \in \underset{\approx}{\sqsupset}_{\textit{2-nv}}$ by definition of $\underset{\approx}{\sqsupset}_{\textit{2-nv}}$. Further, $Q \xrightarrow{\tau}_2{}^{*} Q''$ implies $Q \xrightarrow{\tau}_1{}^{*} Q''$ by Lemma 6(1). $Q'' \xrightarrow{\sigma}_2 Q'''$ implies $Q'' \xrightarrow{\sigma}_1 Q_1'''$ such that $Q''' \succ^{+} Q_1'''$ by using Lemma 13. Moreover, $Q''' \xrightarrow{\tau}_2{}^{*} Q'$ implies $Q''' \xrightarrow{\tau}_1{}^{*} Q'$ by using Lemma 6(1).
Since $\succ$ satisfies the definition of a 1-naive faster-than relation by Proposition 18(1) for $i = 1$, $Q''' \succ^{+} Q_1'''$ and $Q''' \xrightarrow{\tau}_1{}^{*} Q'$ implies $Q_1''' \xrightarrow{\tau}_1{}^{*} Q_1'$ such that $Q' \succ^{+} Q_1'$ holds.

$Q' \succ^+ Q'_1$ implies $Q' \underset{\approx}{\sqsupseteq}_{\mathit{2\text{-}nv}} Q'_1$, since $\succ^+ \subseteq \underset{\approx}{\sqsupseteq}_{\mathit{1\text{-}nv}} \subseteq \underset{\approx}{\sqsupseteq}_{\mathit{2\text{-}nv}}$. We are done since $P' \underset{\approx}{\sqsupseteq}_{\mathit{2\text{-}nv}} Q'$ and $Q' \underset{\approx}{\sqsupseteq}_{\mathit{2\text{-}nv}} Q'_1$ implies $P' \underset{\approx}{\sqsupseteq}_{\mathit{2\text{-}nv}} Q'_1$ by the transitivity of $\underset{\approx}{\sqsupseteq}_{\mathit{2\text{-}nv}}$. $\square$

Obviously, the i-naive weak faster-than preorder is not a precongruence. Hence, [LV04] aims for characterizing the coarsest precongruence contained in it. Therefore, they introduce a preorder as a first candidate for a weak precongruence, called *weak faster-than preorder*. It turns out that this preorder is only the largest precongruence for all operators except summation operator, which is contained in $\underset{\approx}{\sqsupseteq}_{\mathit{i\text{-}nv}}$. As it is however used in the definition of the real precongruence, we have to introduce it and compare the corresponding variants of type-1 and type-2.

Definition 42 (Weak i-faster-than preorder) [LV04, Def. 28 for $i = 1$] *A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is a* weak i-faster-than relation *if the following conditions hold for all $\langle P, Q\rangle \in \mathcal{R}$, $\alpha \in \mathcal{A}$ and $i \in \{1, 2\}$:*

(1) $P \xrightarrow{\alpha}_i P'$ implies $\exists Q'.\, Q \overset{\hat{\alpha}}{\Longrightarrow}_i Q'$ and $\langle P', Q'\rangle \in \mathcal{R}$.
(2) $Q \xrightarrow{\alpha}_i Q'$ implies $\exists P'.\, P \overset{\hat{\alpha}}{\Longrightarrow}_i P'$ and $\langle P', Q'\rangle \in \mathcal{R}$.
(3) $P \xrightarrow{\sigma}_i P'$ implies $\exists Q', Q'', Q'''.\, Q \overset{\epsilon}{\Longrightarrow}_i Q'' \xrightarrow{\sigma}_i Q''' \overset{\epsilon}{\Longrightarrow}_i Q'$, $\mathcal{U}(Q'') \subseteq \mathcal{U}(P)$, and $\langle P', Q'\rangle \in \mathcal{R}$.

We write $P \underset{\approx}{\sqsupseteq}_i Q$ if $\langle P, Q\rangle \in \mathcal{R}$ for some weak i-faster-than relation $\mathcal{R}$ and call $\underset{\approx}{\sqsupseteq}_i$ weak i-faster-than preorder*.*

As usual, we can show that $\underset{\approx}{\sqsupseteq}_i$ is the largest weak i-faster-than relation and that it is a preorder. According to their definitions, the strong i-faster-than precongruence is contained in the weak i-faster-than preorder, hence we may conclude $\succ \subseteq \underset{\approx}{\sqsupseteq}_i$ from $\succ \subseteq \underset{\sim}{\sqsupseteq}_i \subseteq \underset{\approx}{\sqsupseteq}_i$. Now we are able to show that the weak faster-than preorders of type-1 and type-2 coincide:

Theorem 43 (Coincidence IX) *The preorders $\underset{\approx}{\sqsupseteq}_1$ and $\underset{\approx}{\sqsupseteq}_2$ coincide.*

Proof. First, we prove that $\underset{\approx}{\sqsupseteq}_1$ satisfies the definition of a weak 2-faster-than relation; hence consider some arbitrary processes P and Q satisfying $P \underset{\approx}{\sqsupseteq}_1 Q$.

(1,2) The treatment of the cases $P \xrightarrow{\alpha}_2 P'$ for some P' and $Q \xrightarrow{\alpha}_2 Q'$ for some Q' follows in analogy to the corresponding cases in the proof for the coincidence result VIII.

(3) If $P \xrightarrow{\sigma}_2 P''$ for some P'', then $P \xrightarrow{\sigma}_1 P'$ for some P' such that $P'' \succ^+ P'$ by using Lemma 13. Due to $P \xrightarrow{\sigma}_1 P'$ for some P', we may infer $Q \overset{\epsilon}{\Longrightarrow}_1 Q'' \xrightarrow{\sigma}_1 Q''' \overset{\epsilon}{\Longrightarrow}_1 Q'$ and $\mathcal{U}(Q'') \subseteq \mathcal{U}(P)$ for some Q' such that $P' \underset{\approx}{\sqsupseteq}_1 Q'$ by the definition of $\underset{\approx}{\sqsupseteq}_1$, i.e. $Q \xrightarrow{\tau}_1{}^* Q'' \xrightarrow{\sigma}_1 Q''' \xrightarrow{\tau}_1{}^* Q'$. Then,

$Q \xrightarrow{\tau}_2{}^* Q'' \xrightarrow{\sigma}_2 Q''' \xrightarrow{\tau}_2{}^* Q'$. follows by repeated application of Lemma 6(1) and by using Lemma 6(2). Since $\succ \subseteq \underset{\approx}{\sqsupseteq}_{1}$, we may conclude $P'' \underset{\approx}{\sqsupseteq}_{1} P'$ from $P'' \succ^+ P'$ and finally $P'' \underset{\approx}{\sqsupseteq}_{1} Q'$ results from $P'' \underset{\approx}{\sqsupseteq}_{1} P'$ and $P' \underset{\approx}{\sqsupseteq}_{1} Q'$.

Now, we show the inclusion $\underset{\approx}{\sqsupseteq}_{2} \subseteq \underset{\approx}{\sqsupseteq}_{1}$ and hence consider some arbitrary processes P and Q such that $P \underset{\approx}{\sqsupseteq}_{2} Q$.

(1,2) The cases $P \xrightarrow{\alpha}_1 P'$ for some P' and $Q \xrightarrow{\alpha}_1 Q'$ for some Q' are again in analogy to the proof for the coincidence result VIII.

(3) If $P \xrightarrow{\sigma}_1 P'$, then $P \xrightarrow{\sigma}_2 P'$ by Lemma 6(2). By the definition of $\underset{\approx}{\sqsupseteq}_{2}$, we obtain $\mathcal{U}(Q'') \subseteq \mathcal{U}(P)$ and $Q \xrightarrow{\tau}_2{}^* Q'' \xrightarrow{\sigma}_2 Q''' \xrightarrow{\tau}_2{}^* Q'$ for some Q' with $P' \underset{\approx}{\sqsupseteq}_{2} Q'$. $Q \xrightarrow{\tau}_2{}^* Q''$ implies $Q \xrightarrow{\tau}_1{}^* Q''$ by using Lemma 6(1). $Q'' \xrightarrow{\sigma}_2 Q'''$ implies $Q'' \xrightarrow{\sigma}_1 Q'''_1$ such that $Q''' \succ^+ Q'''_1$ by using Lemma 13. Moreover, $Q''' \xrightarrow{\tau}_2{}^* Q'$ implies $Q''' \xrightarrow{\tau}_1{}^* Q'$ by using Lemma 6(1). Since $\succ$ satisfies the definition of a 1-naive faster-than relation by Proposition 18(1) for $i = 1$, $Q''' \succ^+ Q'''_1$ and $Q''' \xrightarrow{\tau}_1{}^* Q'$ implies $Q'''_1 \xrightarrow{\tau}_1{}^* Q'_1$ such that $Q' \succ^+ Q'_1$ holds.
$Q' \succ^+ Q'_1$ implies $Q' \underset{\approx}{\sqsupseteq}_{2} Q'_1$, since $\succ^+ \subseteq \underset{\approx}{\sqsupseteq}_{1\text{-}nv} \subseteq \underset{\approx}{\sqsupseteq}_{2\text{-}nv}$. We are done since $P' \underset{\approx}{\sqsupseteq}_{2} Q'$ and $Q' \underset{\approx}{\sqsupseteq}_{2} Q'_1$ implies $P' \underset{\approx}{\sqsupseteq}_{2} Q'_1$ by the transitivity of $\underset{\approx}{\sqsupseteq}_{2}$. □

Now we define the *weak i-faster-than precongruence* which repairs the defect of the non-compositionality of the summation operator. In order to simplify its definition, we will combine $\underset{\approx}{\sqsupseteq}_{1}$ and $\underset{\approx}{\sqsupseteq}_{2}$ in $\underset{\approx}{\sqsupseteq}$, which is justified since $\underset{\approx}{\sqsupseteq}_{1}$ and $\underset{\approx}{\sqsupseteq}_{2}$ coincide.

Definition 44 (Weak i-faster-than precongruence) [LV04, Def. 30 for $i = 1$]
A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is a weak i-faster-than precongruence relation *if the following conditions hold for all $\langle P, Q\rangle \in \mathcal{R}$, $\alpha \in \mathcal{A}$ and $i \in \{1, 2\}$.*

(1) $P \xrightarrow{\alpha}_i P'$ implies $\exists Q'.\, Q \overset{\alpha}{\Longrightarrow}_i Q'$ and $P' \underset{\approx}{\sqsupseteq} Q'$.
(2) $Q \xrightarrow{\alpha}_i Q'$ implies $\exists P'.\, P \overset{\alpha}{\Longrightarrow}_i P'$ and $P' \underset{\approx}{\sqsupseteq} Q'$.
(3) $P \xrightarrow{\sigma}_i P'$ implies $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ and $\exists Q'.\, Q \xrightarrow{\sigma}_i Q'$ and $\langle P', Q'\rangle \in \mathcal{R}$.

We write $P \underset{\simeq}{\sqsupseteq}_i Q$ if $\langle P, Q\rangle \in \mathcal{R}$ for some weak i-faster-than precongruence relation $\mathcal{R}$ and call $\underset{\simeq}{\sqsupseteq}_i$ weak i-faster-than precongruence.

As usual, we can show that $\underset{\simeq}{\sqsupseteq}_i$ is the largest weak i-faster-than relation and is a preorder. The strong i-faster-than precongruence $\underset{\sim}{\sqsupseteq}_i$ is included in the weak i-faster-than precongruence $\underset{\simeq}{\sqsupseteq}_i$. Thus, we again know that $\succ \subseteq \underset{\simeq}{\sqsupseteq}_i$, due to $\succ \subseteq \underset{\sim}{\sqsupseteq}_i \subseteq \underset{\simeq}{\sqsupseteq}_i$. Using this device, we are able to prove that the weak faster-than precongruences of type-1 and type-2 coincide.

Theorem 45 (Coincidence X) *The precongruences $\underset{\simeq}{\sqsupseteq}_{1}$ and $\underset{\simeq}{\sqsupseteq}_{2}$ coincide.*

Proof. First, we show the inclusion $\underset{\sim}{\sqsupseteq}_1 \subseteq \underset{\sim}{\sqsupseteq}_2$ and hence consider some arbitrary processes P and Q such that $P \underset{\sim}{\sqsupseteq}_1 Q$.

(1) If $P \xrightarrow{\alpha}_2 P'$, then $P \xrightarrow{\alpha}_1 P'$ by using Lemma 6(1). Hence $Q \overset{\alpha}{\Longrightarrow}_1 Q'$ for some Q' such that $P' \underset{\approx}{\sqsupset} Q'$ by definition of $\underset{\sim}{\sqsupseteq}_1$. This means $Q \xrightarrow{\tau}_1{}^* \xrightarrow{\alpha}_1 \xrightarrow{\tau}_1{}^* Q'$ for $\alpha \equiv a$ and $Q \xrightarrow{\tau}_1{}^+ Q'$ for $\alpha \equiv \tau$. In the first case we get $Q \xrightarrow{\tau}_2{}^* \xrightarrow{\alpha}_2 \xrightarrow{\tau}_2{}^* Q'$ by repeated application of Lemma 6(1). Analogously, we get $Q \xrightarrow{\tau}_2{}^+ Q'$ in the second case.
(2) The case $Q \xrightarrow{\alpha}_2 Q'$ for some Q' is analogous to Case (1).
(3) If $P \xrightarrow{\sigma}_2 P''$ for some P'', then $P \xrightarrow{\sigma}_1 P'$ for some P' such that $P'' \succ^+ P'$. Further, we obtain $Q \xrightarrow{\sigma}_1 Q'$ for some Q' such that $P' \underset{\sim}{\sqsupseteq}_1 Q'$ and $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ by definition of $\underset{\sim}{\sqsupseteq}_1$. $Q \xrightarrow{\sigma}_1 Q'$ implies $Q \xrightarrow{\sigma}_2 Q'$ by using Lemma 6(2). Moreover $P'' \succ^+ P'$ implies $P'' \underset{\sim}{\sqsupseteq}_1 P'$ by successive application of $\succ \subseteq \underset{\sim}{\sqsupseteq}_1$ and the transitivity of $\underset{\sim}{\sqsupseteq}_1$. Hence, we may conclude $P'' \underset{\sim}{\sqsupseteq}_1 Q'$ from $P'' \underset{\sim}{\sqsupseteq}_1 P'$ and $P' \underset{\sim}{\sqsupseteq}_1 Q'$.

To prove the inverse inclusion $\underset{\sim}{\sqsupseteq}_2 \subseteq \underset{\sim}{\sqsupseteq}_1$, we consider some arbitrary processes P and Q such that $P' \underset{\sim}{\sqsupseteq}_2 Q'$.

(1,2) The cases $P \xrightarrow{\alpha}_1 P'$ for some P' and $Q \xrightarrow{\alpha}_2 Q'$ for some Q' are treated in analogy to the proof for the inverse inclusion.
(3) If $P \xrightarrow{\sigma}_1 P'$, then $P \xrightarrow{\sigma}_2 P'$ by using Lemma 6(1). Further, $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ and $Q \xrightarrow{\sigma}_2 Q'$ for some Q' such that $P' \underset{\sim}{\sqsupseteq}_2 Q'$ by the definition of $\underset{\sim}{\sqsupseteq}_2$. We may conclude $Q \xrightarrow{\sigma}_1 Q''$ for some Q'' such that $Q' \succ^+ Q''$ by using Lemma 13. Since $\succ \subseteq \underset{\sim}{\sqsupseteq}_1 \subseteq \underset{\sim}{\sqsupseteq}_2$, $Q' \succ^+ Q''$ leads to $Q' \underset{\sim}{\sqsupseteq}_2 Q''$. Finally, $P' \underset{\sim}{\sqsupseteq}_2 Q''$ results from $P' \underset{\sim}{\sqsupseteq}_2 Q'$ and $Q' \underset{\sim}{\sqsupseteq}_2 Q''$. $\square$

Theorem 46 (Full abstraction) [LV04][Theorem 32 for $i = 1$] *The relation $\underset{\sim}{\sqsupseteq}_i$ is the* largest *precongruence contained in $\underset{\approx}{\sqsupset}_i$ for $i \in \{1, 2\}$.*

Proof. The proof for $i = 1$ is given in [LV04].
Clearly, this statement as well holds for $i = 2$. Since $\underset{\sim}{\sqsupseteq}_1$ and $\underset{\sim}{\sqsupseteq}_2$ as well as $\underset{\approx}{\sqsupset}_1$ and $\underset{\approx}{\sqsupset}_2$ coincide, this immediately implies that $\underset{\approx}{\sqsupset}_2$ is the largest precongruence contained in $\underset{\sim}{\sqsupseteq}_2$. $\square$

We leave the definition of a corresponding weak preorder and precongruence of type-c for future work.

9 Conclusion and future work

In this thesis we extended the clock transitions of processes in the process algebra TACS by new time steps and studied the candidates for faster-than preorders that are established in [LV04] in the new setting. With the exception of the indexed faster-than preorder, we were able to prove that all the new preorders coincide with the old preorders. Summarizing, we one again formally underpinned that the concise and simple naive faster-than relation of [LV04] is a sensible candidate for a faster–than preorder. Even the precongruence and the corresponding weak variant have turned out to be robust against the transition extension. Throughout, all coincidence results were proved with the same proof technique. Apart from that, in search of obtaining small relations to demonstrate a faster-than relationship, we have proven that the combined-naive faster-preorder as well as the 'up to'-technique indeed provide small relations.

Future work should proceed along two different directions. First, we should intend to repair the defect of the largest indexed-faster than relation in our new setting by altering its definition. This approach is relevant since the indexed faster than preorder is quite a convincing candidate for a faster-than relation on processes. Another open issue for future work is to consider combined definitions of the weak variants. Moreover, it remains an open question, to what extent the newly established c-precongruence relations are small ones. Nevertheless, it seems to be worth investigating a combined variant of the delayed faster-than preorder with regard to infinite relations. There are reasons to believe that we are able to carry over the proof techniques gained so far to these investigations.

References

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