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Nucleation in the One-Dimensional Stochastic Cahn-Hilliard Model

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Abstract

Despite their misleading label, rare events in stochastic systems are central to many applied phenomena. In this paper, we concentrate on one such situation — phase separation through homogeneous nucleation in binary alloys as described by the stochastic partial differential equation model due to Cahn, Hilliard, and Cook. We show that in the limit of small noise intensity, nucleation can be explained by the stochastically driven exit from the domain of attraction of an asymptotically stable homogeneous equilibrium state for the associated deterministic model. Furthermore, we provide insight into the subsequent nucleation dynamics via the structure of the attractor of the model in the absence of noise.

AMS subject classifications: 35B40, 35B41, 35K55

Keywords: Cahn-Hilliard-Cook model, nucleation, stochastic partial differential equation, domain exit

1 Introduction

In the study of phase separation in metal alloys one can distinguish between two distinct mechanisms — spinodal decomposition and nucleation [22, 35]. In both cases, one is interested in how the components of an alloy separate immediately after the quenching of a homogeneous high-temperature mixture of the alloy components. Deep quenches lead to spinodal decomposition, which results in sudden phase separation throughout the material and often leads to complicated snake-like microstructures with a characteristic wave length [8, 10, 19, 32, 33, 37, 38, 44]. On the other hand, shallow quenches lead to the somewhat time-delayed formation of droplets throughout the material. In order to describe these two mechanisms, Cahn and Hilliard [12, 13, 14] proposed a fourth-order parabolic partial differential equation model in which the onset of either mechanism is determined by the stability properties of spatially constant equilibrium solutions, which model the homogeneous component mixture at high temperature for various concentration ratios of the involved components. If such an equilibrium is unstable, small initial imperfections in the homogeneous state are quickly amplified, leading to complicated microstructures throughout the material, i.e., to spinodal decomposition. In contrast, nucleation corresponds to the case that the homogeneous state is stable — and at first glance, this stability seems to preclude the occurrence of any type of phase separation.

The first mathematical explanation of nucleation in the Cahn-Hilliard model was obtained by Bates and Fife [5]. In their paper, they consider the one-dimensional Cahn-Hilliard model, which is given by the parabolic partial differential equation

$$\begin{aligned} \partial_t u + \frac{1}{\lambda^2} \cdot \partial_x^4 u \;&=\; \partial_x^2 f(m+u) \qquad \text{for} \qquad x \in [0,1] \;\text{ and }\; t \ge 0 \;, \\ \int_0^1 u(t,x)\,dx = 0 \qquad &\text{for} \qquad t \ge 0 \;, \\ \partial_x u = \partial_x^3 u = 0 \qquad &\text{for} \qquad x \in \{0,1\} \;\text{ and }\; t \ge 0 \;. \end{aligned} \tag{1}$$

In this model, the nonlinearity f is the derivative of a double-well potential, the standard example being $f(u) = u^3 - u$, and the large parameter λ is a measure for the inverse interaction length in the alloy. The constant m describes the total mass of the system, which is automatically conserved by the evolution equation. Notice that our form of the Cahn-Hilliard equation slightly deviates from the one of [5]. In order to simplify our presentation, the solution u of (1) measures the deviation from the mean mass m, and therefore the above-mentioned homogeneous initial state can be taken as the constant function $h^0 \equiv 0$ without loss of generality.

One can easily see that the stability of the vanishing initial state h^0 is determined by the value of the derivative $f'(m)$. If the derivative is negative this state is unstable, i.e., one is in the spinodal decomposition regime. On the other hand, for $f'(m) > 0$ the

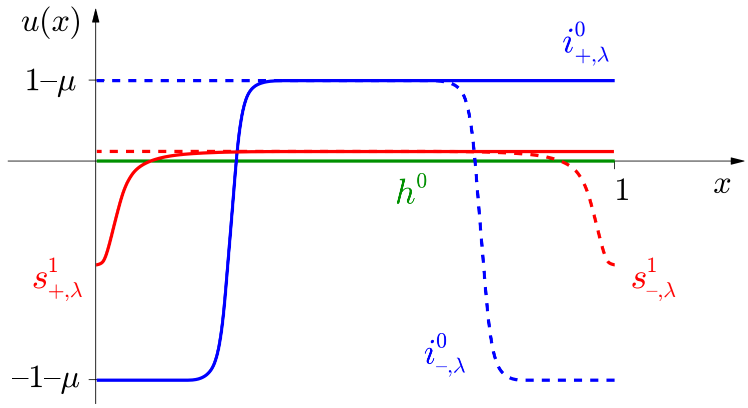

Figure 1: The set of equilibrium solutions of the Cahn-Hilliard model with $f'(m) > 0$ consists of the stable state $h^0 \equiv 0$, as well as spike solutions $s^k_{\pm,\lambda}$ for $k \in \mathbb{N}$, and transition layer solutions $i^k_{\pm,\lambda}$ for $k \in \mathbb{N}_0$, which have index k. The figure shows the solutions $s^1_{\pm,\lambda}$ and $i^0_{\pm,\lambda}$, the ones for larger k are obtained from these via even reflections and rescaling.

state h^0 is asymptotically stable, and in the following we will always assume this to be the case. This asymptotic stability implies that any solution which originates in a small neighborhood of the homogeneous state will be attracted exponentially towards it, thereby — as mentioned above — seemingly ruling out the possibility of phase separation.

To overcome this problem, Bates and Fife [5] proposed that nucleation is triggered by considering initial states which are sufficiently far away from the homogeneous state. In order to quantify the size of this perturbation, they prove the existence of two spike-like unstable equilibria solutions $s^1_{\pm,\lambda}$ called canonical nuclei, which are close to the homogeneous state h^0 for large values of λ, and which have a one-dimensional unstable manifold. It is then conjectured in [5] that one branch of each unstable manifold converges to the homogeneous stable state, while the other branch converges to a globally stable transition layer solution $i^0_{\pm,\lambda}$, which would correspond to the droplets appearing during nucleation. The general shape of these equilibrium solutions is shown in Figure 1.

While the results by Bates and Fife identify the threshold that has to be overcome to observe nucleation, their results cannot provide a dynamical description of the initial stage of nucleation starting from initial conditions which are close to the homogeneous state. At least heuristically, this can be achieved if one assumes the effects of external noise. In this case it is reasonable to assume that the cumulative effects of the noise might drive the solution out of the domain of attraction of the homogeneous state, thereby leading to the onset of nucleation. See for example the discussion in [9], as well as the applied literature on transition state theory [36].

As a deterministic model, the classical Cahn-Hilliard equation (1) ignores thermal fluctuations which are present in any material, and we therefore turn our attention to a stochastic extension of the model to address the deficiencies of [5]. Such an extension was proposed by Cook [15], see also [20, 28, 29, 30, 34]. In the physics literature, this model is also known as Model B in the classification of Hohenberg and Halperin [27]. We consider the Cahn-Hilliard-Cook model in the form

$$\begin{aligned} \partial_t u + \frac{1}{\lambda^2} \cdot \partial_x^4 u \; &= \; \partial_x^2 f(m+u) + \sigma \cdot \xi \qquad \text{for} \qquad x \in [0,1] \;\text{ and }\; t \ge 0 \;, \\ \int_0^1 u(t,x)\, dx = 0 & \qquad \text{for} \qquad t \ge 0 \;, \\ \partial_x u = \partial_x^3 u = 0 & \qquad \text{for} \qquad x \in \{0,1\} \;\text{ and }\; t \ge 0 \;. \end{aligned} \tag{2}$$

This stochastic partial differential equation differs from (1) by the additive noise term $\sigma \cdot \xi$, where ξ denotes the distributional derivative of a Hilbert space valued Wiener process, and $\sigma \ge 0$ denotes the intensity of the noise. Specifically, we focus on the noise process $\xi = \partial_x \eta$ which is the spatial derivative of space-time white noise with

$$\mathbb{E}\eta(t,x) = 0 \qquad \text{and} \qquad \mathbb{E}\eta(t,x)\eta(s,y) = \delta(t-s)\delta(x-y) \;.$$

This is one of the standard models in the physics literature for mass-conservative noise.

In this paper, we present rigorous mathematical results which explain nucleation in the stochastic Cahn-Hilliard-Cook model (2). This is accomplished by combining stochastic large deviations type results for the Cahn-Hilliard-Cook model with results on the attractor structure of the deterministic model (1) from [24]. In the latter work it is shown that the boundary spikes are in fact the minimizers of the energy associated with the Cahn-Hilliard model on the boundary of the domain of attraction of the homogeneous state. Then, using the theory of Freidlin and Wentzell [17, 23] we verify that for sufficiently small noise, the solution of (2) will in fact develop a boundary spike. While the detailed formulation of our main result will be given later in Theorem 4.2, it can be roughly summarized as follows.

Theorem 1.1 *Consider the stochastic Cahn-Hilliard-Cook model (2) and denote its solution originating at an initial condition $u_0 \approx h^0$ by $u_\sigma^{u_0}(t)$. Let*

$$\tau = \inf \left\{ t > 0 \;:\; u_\sigma^{u_0}(t) \notin \mathcal{D} \right\} \;,$$

where $\mathcal{D}$ denotes the domain of attraction of h^0 under the deterministic evolution (1). Under some conditions that will be made precise later, one then has for any small $\delta > 0$ the identity

$$\lim_{\sigma \to 0} \mathbb{P} \left(\| u_\sigma^{u_0}(\tau) - s^1_{+,\lambda} \| < \delta \quad \textit{or} \quad \| u_\sigma^{u_0}(\tau) - s^1_{-,\lambda} \| < \delta \right) = 1 \;,$$

i.e., for small noise intensity $\sigma > 0$, most solutions exit the basin of attraction of h^0 close to the spikes $s^1_{\pm,\lambda}$.

This result is the stochastic extension of the deterministic result of Bates and Fife [5]. Related results on the Allen-Cahn equation can be found in [11, 43]. Notice, however, that in the Allen-Cahn case the result is much easier, since it is known a priori that the constants ± 1 are the global minimizers of the energy, and the heteroclinic connection between these homogeneous states is well-understood.

The remainder of this paper is organized as follows. In Section 2 we recall the basic functional analytic framework for the deterministic Cahn-Hilliard equation (1), including its energy functional and gradient structure. Furthermore, we have to extend these results to allow for the phase space of continuous functions, rather than the usually considered Hilbert space setting. The deeper reason for this is the delicate interplay between the regularity of the stochastic convolution associated with our noise process, the fact that the Cahn-Hilliard system is a gradient system with respect to the H^{-1}-norm, and the goal of being able to use the standard energy functional as the quasi-potential in the Freidlin-Wentzell theory. Finally, Section 2 presents results from [24] concerning the attractor structure of the deterministic Cahn-Hilliard model. Section 3 is devoted to the stochastic Cahn-Hilliard-Cook model and addresses basic functional analytic questions such as existence of solutions in our various phase spaces, spatial regularity of the stochastic convolution, as well as a discussion of the law of the mild solution and the basic Freidlin-Wentzell estimates. Finally, Section 4 establishes the standard Cahn-Hilliard energy as the quasi-potential associated with the deterministic domain of attraction of the homogeneous state, and states and proves our main result on the boundary spike creation.

In order to make the presentation of this paper as accessible as possible, we restrict our attention exclusively to the case of binary alloys as described by (2). We would like to point out, however, that the stochastic parts of our arguments equally apply to the case of multi-component alloys as studied in [18] in the context of Cahn-Morral systems. In other words, as soon as one can describe the deterministic attractor of Cahn-Morral systems, or at least the boundary of the domain of attraction of the homogeneous state, then our main result applies to this situation as well. From a numerical point of view, this description of the attractor is the subject of [18].

2 The Deterministic Equation and its Attractor

In this section we present the basic functional analytic setting as well as important auxiliary results that are necessary for our study of the stochastic Cahn-Hilliard-Cook model, all in the deterministic setting. More precisely, Section 2.1 collects well-known definitions and results from the Hilbert space theory for (1), including the associated energy, and then introduces the phase space of continuous functions which will be crucial for our large deviation results. Section 2.2 establishes existence and uniqueness of solutions to a non-autonomous generalization of (1) in the space of continuous functions, and derives some basic properties of an extended energy functional in this space. Finally, Section 2.3 presents results from [24] on the attractor structure of (1) which will be needed for our main result.

2.1 Functional-Analytic Framework and Phase Spaces

Let $\dot{L}^2(0,1)$ denote the Hilbert space of all square integrable functions $u : (0,1) \to \mathbb{R}$ with zero average and standard inner product $(\cdot,\cdot)_0$. Then the functions

$$e_k(x) = \sqrt{2} \cdot \cos(k\pi x) \quad \text{for} \quad k \in \mathbb{N} \tag{3}$$

form a complete orthonormal set in $\dot{L}^2(0,1)$, and for every $u \in \dot{L}^2(0,1)$ its Fourier coefficients are given by $u_k = (u, e_k)_0$ for $k \in \mathbb{N}$. Consider the operator

$$A = \partial_x^4 \; : \; \dot{L}^2(0,1) \to \dot{L}^2(0,1)$$

subject to homogeneous Neumann boundary conditions. Then A is a positive selfadjoint operator with compact resolvent and spectrum

$$\sigma(A) = \left\{ \mu_k = k^4 \pi^4 \; : \; k \in \mathbb{N} \right\} ,$$

with corresponding normalized eigenfunctions e_k as defined above. The domain of A is given by $D(A) = \mathcal{H}^4$, where we define the scale of Hilbert spaces

$$\mathcal{H}^\alpha = \left\{ u = \sum_{k=1}^{\infty} u_k e_k \; : \; |u|_\alpha^2 = \sum_{k=1}^{\infty} \mu_k^{\alpha/2} u_k^2 < \infty \right\} , \quad \alpha \in \mathbb{R} .$$

Notice that for $\alpha \geq 0$ we have $\dot{L}^2(0,1) = \mathcal{H}^0 \subset \mathcal{H}^\alpha$, and that for $\alpha < 0$ the space $\mathcal{H}^\alpha$ is the topological dual of $\mathcal{H}^{-\alpha}$. Since A is positive and sectorial, one can define its fractional powers via

$$A^\beta u = \sum_{k=1}^{\infty} \mu_k^\beta u_k e_k \quad \text{for all} \quad u \in D(A^\beta) = \mathcal{H}^{4\beta} , \quad \text{and} \quad \beta \geq 0 .$$

The operator $A^\beta : \mathcal{H}^{\alpha+4\beta} \to \mathcal{H}^\alpha$ is an isometry for arbitrary $\alpha \in \mathbb{R}$ and $\beta \geq 0$, i.e., we have $(A^\beta u, A^\beta v)_\alpha = (u, v)_{\alpha+4\beta}$. Furthermore, the identity $(A^\beta u, A^\beta v)_\alpha = (A^{2\beta} u, v)_\alpha$ holds for all $u \in \mathcal{H}^{\alpha+8\beta}$ and $v \in \mathcal{H}^{\alpha+4\beta}$. For integer exponents $\alpha \in \mathbb{N}_0$ the spaces $\mathcal{H}^\alpha$ are closely related to the standard Sobolev spaces $H^\alpha(0,1) = W^{\alpha,2}(0,1)$. Indeed, $\mathcal{H}^\alpha$ is a closed subspace of $H^\alpha(0,1)$ and its norm is equivalent to the standard norm. This relation can be extended to arbitrary exponents and leads to Besov spaces, see [1, 2]. Furthermore, for real $\alpha_1 < \alpha_2$ one has the compact embedding $\mathcal{H}^{\alpha_2} \hookrightarrow \mathcal{H}^{\alpha_1}$. Moreover, we remark that on the space $\mathcal{H}^\alpha$ we have

$$A^{1/2} = -\partial_x^2 , \quad \text{with} \quad D(A^{1/2}) = \mathcal{H}^{\alpha+2} . \tag{4}$$

Finally, we introduce the projection operator $P_0 : H^\alpha(0,1) \to \mathcal{H}^\alpha$ which projects a function of the form $u = \sum_{k=0}^\infty u_k e_k$ to its mass free part $P_0 u = \sum_{k=1}^\infty u_k e_k$.

In the above Hilbert space setting one can easily rewrite the Cahn-Hilliard model (1) as an abstract evolution equation in $\mathcal{H}^\alpha$ for any $\alpha \in \mathbb{N}$. For this, define

$$A_\lambda u = \frac{1}{\lambda^2} \cdot Au \qquad \text{and} \qquad G(u) = -A^{1/2} P_0 f(m+u) ,$$

and consider the evolution equation

$$\partial_t u + A_\lambda u = G(u) , \quad \text{with initial condition} \quad u_0 \in \mathcal{H}^\alpha . \tag{5}$$

Assuming that f is either a polynomial or linearly bounded at infinity, one can then easily show that G is a continuously differentiable and Lipschitz continuous function from $\mathcal{H}^\alpha$ into $\mathcal{H}^{\alpha-2}$, and standard results imply that for any $\alpha \in \mathbb{N}$ the evolution equation (5) has a unique mild solution $u : [0,\infty) \to \mathcal{H}^\alpha$, i.e., a solution which satisfies the variation of constants formula. Furthermore, this solution satisfies $u(t) \in \mathcal{H}^{\alpha+2}$ for all $t > 0$ and is automatically a strong solution, i.e., it satisfies the differential equation in (5) for almost all t in $\mathcal{H}^{\alpha-2}$.

While the local unique existence is automatic, global existence follows from the fact that (1) is a gradient system. For any $u \in \mathcal{H}^1$ we define the energy functional

$$E_\lambda(u) = \int_0^1 \left(\frac{1}{2\lambda^2} \cdot |\partial_x u|^2 + F(m+u) \right) dx , \tag{6}$$

where F is an antiderivative of the nonlinearity f. For the nonlinearity $f(u) = u^3 - u$, the function F is the double well potential $F(u) = (u^2-1)^2/4$. It will at times be convenient to rewrite $E_\lambda(u)$ in the form

$$E_\lambda(u) = \frac{1}{2\lambda^2} \cdot |u|_1^2 + V(u) , \quad \text{where} \quad V(u) = \int_0^1 F(m+u)\, dx . \tag{7}$$

One can easily show that $E_\lambda : \mathcal{H}^1 \to \mathbb{R}$ is continuously differentiable and Lipschitz continuous with

$$DE_\lambda(u)h = \frac{1}{\lambda^2} \cdot (u,h)_1 + (f(m+u), h)_0 ,$$

as well as

$$\frac{1}{\lambda^2} \cdot (u,h)_1 = (A_\lambda u, h)_{-1} \quad \text{and} \quad DV(u)h = (f(m+u),h)_0 = (-G(u),h)_{-1} .$$

These identities finally furnish

$$-A_\lambda u + G(u) = -\nabla_{\mathcal{H}^{-1}} E_\lambda(u) , \tag{8}$$

i.e., the Cahn-Hilliard model (5) is the $\mathcal{H}^{-1}$-gradient system associated with E_λ. From this observation it can readily be shown that the mild solution u of (5) satisfies

$$E_\lambda(u(t)) = E_\lambda(u_0) - \int_0^t |\partial_t u(s)|_{-1}^2 \, ds \tag{9}$$

for all $\alpha \in \mathbb{N}$. In other words, the energy decreases along solutions of the Cahn-Hilliard model.

While the above-described Hilbert space setting is well-known, it does not suffice for our applications to the Cahn-Hilliard-Cook model. One of the main problems of the stochastic equation (2) is its lack of spatial regularity. Under the considered noise process ξ, we simply do not have a solution in the space $C([0,T], \mathcal{H}^1)$, for any $T > 0$. The deeper reason for this will be described in Section 3 and is due to the lack of regularity of the stochastic convolution. Thus, we have to introduce a second phase space, and our results will be obtained by combining the Hilbert space approach with this new phase space. Specifically, we consider the Banach space

$$\mathcal{C} = \left\{ u \in C[0,1] \; : \; \int_0^1 u(x) \, dx = 0 \right\} ,$$

equipped with the maximum norm $|\cdot|_\infty$, where $C[0,1]$ denotes the space of all continuous functions on $[0,1]$. This space will be our default phase space.

Remark 2.1 *In the following, solutions of evolution equations are generally understood as taking values in the phase space $\mathcal{C}$, unless explicitly stated otherwise. In addition, we generally use the topology of $\mathcal{C}$ to describe δ-neighborhoods of subsets $\mathcal{A} \subset \mathcal{C}$ and we write*

$$B_\delta(\mathcal{A}) = \{ u \in \mathcal{C} \; : \; |u - a|_\infty < \delta \; , \quad \text{for some} \quad a \in \mathcal{A} \} \; .$$

The boundary $\partial\mathcal{A}$, the closure $cl\,\mathcal{A}$, and the complement $\mathcal{A}^c$ are taken with respect to the space $\mathcal{C}$ and its topology.

In order to study evolution equations on the phase space $\mathcal{C}$, we need to state some basic properties of the Laplacian on this space. For this, let $B = -\partial_x^2$ on $\mathcal{C}$ subject to homogeneous Neumann boundary conditions and with domain

$$\begin{aligned} D(B) \;\; = \;\; & \Bigg\{ u \in \bigcap_{p \geq 1} W^{2,p}(0,1) \; : \\ & \qquad \int_0^1 u(x) \, dx = 0 \; , \;\; \partial_x^2 u \in \mathcal{C} \; , \;\; \partial_x u(0) = \partial_x u(1) = 0 \Bigg\} \; . \end{aligned} \tag{10}$$

Then the operator B is positive and sectorial in $\mathcal{C}$, see for example [31, Section 7.3.4]. Consequently, the extrapolated fractional power scale of order $m \in \mathbb{N}$ generated by $(\mathcal{C}, B)$ according to [4, Chapter 5] is well-defined. For our applications it suffices to consider the case $m = 1$ and the scale of spaces $[\mathcal{C}_\alpha,\ \alpha \in [-1, \infty)]$. In particular, one has $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_1 = D(B)$ with norm $|u|_{\mathcal{C}_1} = |Bu|_{\mathcal{C}_0} = |Bu|_\infty$. More generally, the fractional powers B^β, $\beta \in \mathbb{R}$, are well-defined. For $\beta > 0$ they are isometric isomorphisms from $\mathcal{C}_{\alpha+\beta}$ to $\mathcal{C}_\alpha$. Moreover, B considered as an operator on $\mathcal{C}_{-1}$ is positive and sectorial, see for example [4, Chapter 5, Lemma 1.3.7]. Finally, by slightly abusing notation we write $A_\lambda = \lambda^{-2} \cdot B^2$ for the linear part in (5), and will make clear in the following whether we consider this operator in the Hilbert space setting or on $\mathcal{C}$.

2.2 Deterministic Flow in the New Phase Space

In this section we will collect basic results for a generalized deterministic Cahn-Hilliard equation, which can then later be used for the stochastic Cahn-Hilliard-Cook equation as well, see Section 3.1. For the main results of this paper we are only interested in deterministic and stochastic solutions as long as they stay in a certain bounded neighborhood of the stable equilibrium $h^0 \equiv 0$ mentioned in the introduction. Therefore, in order to simplify our presentation for the stochastic model, we replace the standard nonlinearity $f(u)$ in (2) by a sufficiently smooth function $\hat{f}$ which satisfies

$$\hat{f}(m+u) = f(m+u) \quad \text{for} \quad |u| < R \,, \qquad \hat{f}(m+u) = 0 \quad \text{for} \quad |u| > 2R \,, \tag{11}$$

for sufficiently large $R > 0$. The specific choice of $R > 0$ will be presented in Section 2.3 below. Using this new nonlinearity, consider the nonautonomous evolution equation

$$\partial_t u + A_\lambda u = H_w(u,t) \,, \quad t \in [0,T] \,, \qquad u(0) = u_0 \,, \tag{12}$$

where the right-hand side H_w depends on a given $w \in C([0,T], \mathcal{C})$ and is defined by

$$H_w(u,t) = -BP_0 \hat{f}(m + u - w(t)) \,. \tag{13}$$

The nonlinearity $\hat{f}$ was defined in (11) and the projection P_0 was introduced after (4). We are interested in the existence of mild solutions of (12), i.e., solutions u satisfying

$$u(t) = e^{-A_\lambda t} u_0 + \int_0^t e^{-A_\lambda (t-s)} H_w(u(s), s)\, ds \,.$$

Their existence is established in the following result.

Theorem 2.2 *Consider the abstract evolution equation (12) in the phase space $\mathcal{C}$. Then for every $u_0 \in \mathcal{C}$ and $w \in C([0,T], \mathcal{C})$ this problem has a unique mild solution u in $\mathcal{C}$ which satisfies*

$$u \ \in \ C([0,T], \mathcal{C}) \ \cap \ C^{0,\theta_1}_{loc}([0,T), \mathcal{C}_{\alpha_1}) \ \cap \ C^{0,\theta_2}_{loc}((0,T), \mathcal{C}_{\alpha_2})$$

for all $\alpha_1, \alpha_2 \in \mathbb{R}$ with $-1 \le \alpha_1 < 0$ and $-1 \le \alpha_2 < 1$, where $\theta_1 > 0$, and $\theta_2 > 0$. Furthermore, the solution map $\mathcal{C} \times C([0,T], \mathcal{C}) \times [0,T] \to \mathcal{C}$ given by $(u_0, w, t) \mapsto u(t)$ is continuous.

Proof: This theorem is a direct consequence of [39, Lemma 47.1, Theorem 47.5, Theorem 47.7]. One can easily verify that $\sup_{t\in[0,T]} |H_w(u,t)-H_w(v,t)|_{\mathcal{C}_{-1}} \le C|u-v|_\infty$. Moreover, the nonlinearity is globally bounded, and thus there exists a constant $c > 0$ such that $|H_w(u,t)|_{\mathcal{C}_{-1}} \le c$ for all $(u,t) \in \mathcal{C} \times \mathbb{R}^+$. Note also that $|B \cdot|_{\mathcal{C}_{-1}} = |\cdot|_\infty$. $\diamond$

Remark 2.3 *For the autonomous special case $w(t) \equiv 0$ this theorem provides the semiflow $S_\lambda(t)$ on $\mathcal{C}$ for the truncated Cahn-Hilliard model. Using a standard bootstrap argument one can show that in this case the inclusions $u(t) = S_\lambda(t)u_0 \in \mathcal{C}_\alpha$ are valid for all $\alpha > -1$ and $t > 0$. Due to the embeddings $\mathcal{C}_1 \hookrightarrow \mathcal{H}^2 \hookrightarrow \mathcal{H}^1$ one can therefore see that the trajectories $\{u(t) \; : \; t > 0\}$ coincide with those constructed in the Hilbert space setting, as long as they stay in $B_R(0)$.*

Remark 2.4 *It will be shown in Section 3.1 that choosing w as the stochastic convolution associated with A_λ naturally leads to an existence result for solutions of (2) in $\mathcal{C}$, if the standard nonlinearity f is replaced by the truncated nonlinearity $\hat{f}$.*

We close this section with a few comments concerning the energy functional $E_\lambda(u)$ defined in Section 2.1. Since, in general, continuous functions are not contained in $\mathcal{H}^1$, we extend the definition of $E_\lambda(u)$ given in (6) via

$$E_\lambda(u) \;=\; \begin{cases} \dfrac{1}{2\lambda^2} \cdot |u|_1^2 + \displaystyle\int_0^1 F(m+u)\,dx \;, & \text{for } u \in \mathcal{H}^1 \;, \\ +\infty \;, & \text{otherwise} \;, \end{cases} \tag{14}$$

where F is again an antiderivative of f. Note that on $B_R(0)$, this energy functional coincides with the one obtained by using the truncated nonlinearity $\hat{f}$. The following lemma collects some basic properties of the extended energy functional.

Lemma 2.5 *The functional E_λ defined in (14) and considered as a mapping $E : \mathcal{C} \to \bar{\mathbb{R}}$ is lower semicontinuous, i.e., for any sequence $(u_n) \in \mathcal{C}$ converging to $u \in \mathcal{C}$ one has*

$$E_\lambda(u) \le \liminf_{n\to\infty} E_\lambda(u_n) \;.$$

In particular, the functional E_λ attains its minimum on closed subsets of $\mathcal{C}$. Moreover, for $w(t) \equiv 0$ and $u_0 \in \mathcal{C}$ the mapping $t \mapsto E_\lambda(S_\lambda(t)u_0)$ is continuous in $t > 0$, where $S_\lambda(t)$ was introduced in Remark 2.3. If in addition one has $u_0 \in B_R(0)$, then the function $E_\lambda(S_\lambda(\cdot)u_0)$ is monotone decreasing as long as $S_\lambda(t)u_0$ stays in $B_R(0)$.

Proof: Lower semicontinuity follows from [23, Chapter 3, Lemma 2.1]. Due to the compact embedding $\mathcal{H}^1 \hookrightarrow \mathcal{C}$, combined with the estimate

$$\frac{1}{2\lambda^2} \cdot |u|_1^2 - c_1 \le E_\lambda(u) \;, \tag{15}$$

it is clear that E_λ attains its minimum on closed sets. The remaining statement follows from the fact that for $t > 0$ the solution $S_\lambda(t)u_0$ coincides with the solution constructed in the Hilbert space setting, at least as long as it stays in $B_R(0)$, where the modified nonlinearity has no effect. Recall also that in the Hilbert space setting, the functional E_λ is a Lyapunov function for the semiflow $S_\lambda(t)$ due to (9). $\diamond$

Remark 2.6 *Since the energy E_λ is a Lyapunov functional for the flow generated by (1) in $\mathcal{H}^1$, one can readily see that for $u_0 \notin \mathcal{H}^1 \cap B_R(0)$ we have $E(S_\lambda(t)u_0) \to +\infty$ as $t \to 0^+$.*

2.3 Some Results on the Deterministic Attractor Structure

In this section we recall results on the attractor structure of the deterministic Cahn-Hilliard model (1) for the case $f'(m) > 0$, which is assumed throughout this paper. It is well-known that the semiflow generated by (1) in $\mathcal{H}^1$ has a compact global attractor $\mathcal{A}_\lambda$, see for example [42]. As we mentioned in Section 2.1, the deterministic Cahn-Hilliard model is a gradient system with respect to the energy functional E_λ, and thus the attractor $\mathcal{A}_\lambda$ consists of equilibrium solutions and heteroclinic connections between them.

The set of equilibrium solutions for the deterministic Cahn-Hilliard model has been determined completely by Grinfeld and Novick-Cohen in [25]. They showed that in the nucleation regime, for fixed $\lambda > 0$ there exists an integer N_λ, such that all equilibria of the Cahn-Hilliard model (1) are given by the spike solutions $s^k_{\pm,\lambda}$, for $k = 1, \ldots, N_\lambda$, the transition layer solutions $i^k_{\pm,\lambda}$, for $k = 0, \ldots, N_\lambda - 1$, and the homogeneous state $h^0 \equiv 0$, see also Figure 1. The integer N_λ converges to infinity as $\lambda \to \infty$. In the following, we use the abbreviation

$$\mathcal{E}_\lambda = \left\{ h^0 \ , \ \ s^k_{\pm,\lambda} \ , \ \ i^{k-1}_{\pm,\lambda} \ \ : \ \ k = 1, \ldots, N_\lambda \right\}$$

for the set of equilibria of (1). A sketch of the complete equilibrium bifurcation diagram is shown in Figure 2. The specific shape of the solutions $s^1_{\pm,\lambda}$ and $i^0_{\pm,\lambda}$ has already been shown in Figure 1. Notice that $s^1_{+,\lambda}$ and $s^1_{-,\lambda}$ are related via $s^1_{+,\lambda}(x) = s^1_{-,\lambda}(1-x)$, and similarly for $i^0_{\pm,\lambda}$. The solutions $s^k_{\pm,\lambda}$ and $i^k_{\pm,\lambda}$ for larger k are obtained from these via even reflections, extensions, and rescaling.

Unfortunately, while the stationary states are completely known, a complete description of the attractor structure does not exist. Partial results can be found in [26]. For the purpose of this paper, we are mainly interested in a deeper understanding of the domain of attraction of the homogeneous state h^0, particularly, its boundary. Such a description has been recently obtained by one of the authors in [24]. Before presenting the precise result, we need to introduce the following assumption.

Assumption 2.7 *Consider the energy-based bifurcation diagram for the Cahn-Hilliard model (1) shown in Figure 2. Suppose that there exists an interval $\Lambda \subset \mathbb{R}^+$ of parameter*

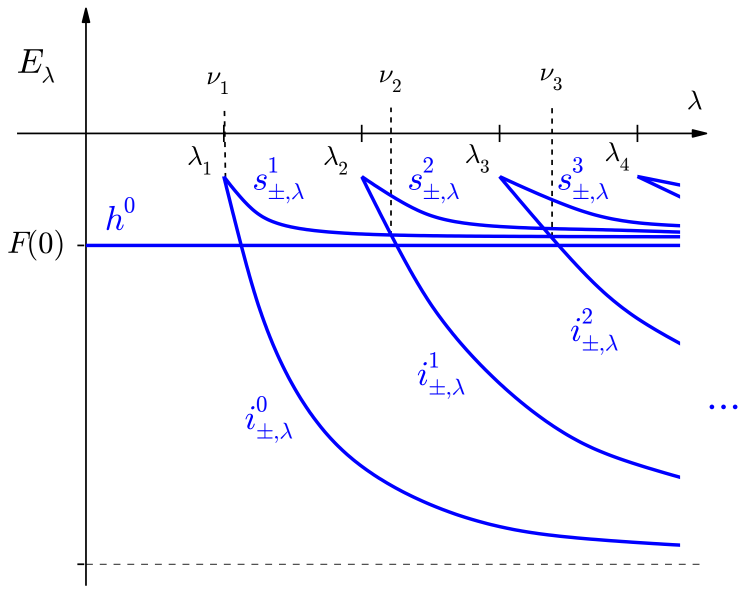

Figure 2: Sketch of the equilibrium bifurcation diagram for the deterministic Cahn-Hilliard model on the unit interval. In the diagram, the energy E_λ is plotted against λ.

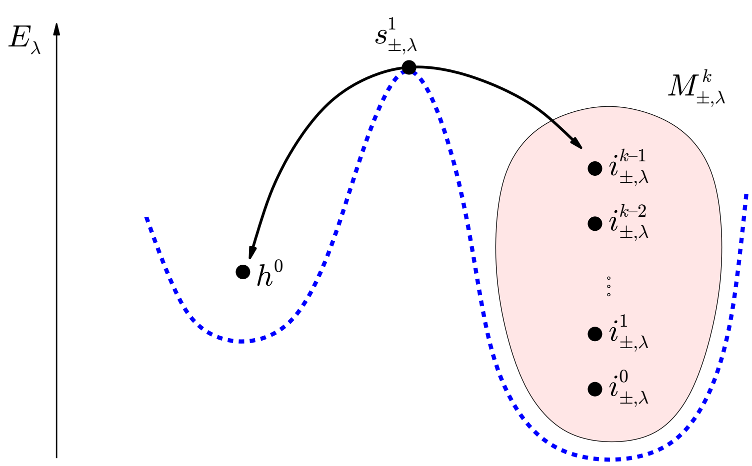

Figure 3: Sketch of the attractor structure result in Theorem 2.8. The boundary spikes $s^1_{\pm,\lambda}$ are saddle solutions on the boundary of attraction of the homogeneous state h^0, while the equilibria in $M^k_{\pm,\lambda}$ are contained in a different attracting set.

values λ such that on Λ, the $s^1_{\pm,\lambda}$-branch intersects the $i^{m-1}_{\pm,\lambda}$-branch for $m \geq 1$ in a unique point, i.e., there exist unique $\nu_m \in \Lambda$ such that

$$E_{\nu_m}\left(s^1_{\pm,\nu_m}\right) = E_{\nu_m}\left(i^{m-1}_{\pm,\nu_m}\right) \ .$$

While we conjecture that this assumption holds with $\Lambda = \mathbb{R}^+$, no proof exists at the present time. However, Assumption 2.7 has been verified numerically for $\Lambda = (0, 100)$.

Theorem 2.8 *Suppose that Assumption 2.7 holds, let $\lambda \in (\nu_k, \nu_{k+1}] \cap \Lambda$ be fixed, and let $M^k_{\pm,\lambda}$ denote the Morse set containing the equilibrium solutions $i^0_{\pm,\lambda}, \ldots, i^{k-1}_{\pm,\lambda}$. Then one branch of the unstable manifold of $s^1_{\pm,\lambda}$ converges to the homogeneous state h^0, while the other branch converges to the Morse set $M^k_{\pm,\lambda}$. Moreover, there are no connecting orbits between $M^k_{\pm,\lambda}$ and h^0.*

This theorem, whose proof can be found in [24], establishes all the attractor information that is necessary for our application. Its statement is illustrated in Figure 3.

As mentioned before, understanding the boundary $\partial\mathcal{D}$ of the open domain of attraction $\mathcal{D} \subset \mathcal{H}^1$ of the homogeneous state h^0 lies at the heart of our stochastic nucleation result. Yet, the above result is for the original Cahn-Hilliard model (1) with nonlinearity $f(u) = u^3 - u$, and for our stochastic application it is more convenient to work with the truncated system (12) with $w \equiv 0$. Moreover, while the classical model is usually studied in the Hilbert space setting, we have to consider the phase space $\mathcal{C}$. The remainder of this section describes the framework that will be used to bring together these different points of view.

Due to the compactness of the attractor $\mathcal{A}_\lambda$ in $\mathcal{H}^1$ and Sobolev's embedding theorem, there exists a constant $\rho > 0$ such that $\mathcal{A}_\lambda \subset B_\rho(0) \subset \mathcal{C}$. Recall from Remark 2.1 that balls are implicitly understood to be defined with respect to the maximum norm in $\mathcal{C}$. Furthermore, choose the constant R in the definition of $\hat{f}$ in (11) in such a way that $R > 2\rho$. Now consider the truncated semiflow S_λ on $\mathcal{C}$ which was introduced in Remark 2.3. Let $\mathcal{B}$ denote the open basin of attraction of the homogeneous state h^0 with respect to this semiflow S_λ. Since $\mathcal{B}$ will significantly differ from the $\mathcal{D}$ introduced above, we define a set $\mathcal{D}(\rho) \subset \mathcal{C}$ via

$$\mathcal{D}(\rho) = \{u \in \mathcal{B} \ : \ S_\lambda(t)u \in B_\rho(0) \quad \text{for all} \quad t \geq 0\} \ , \tag{16}$$

see also Figure 4.

It is clear that $\mathcal{D}(\rho)$ is a bounded open set in $\mathcal{C}$, and that each point $u \in \mathcal{D}(\rho)$ is attracted by h^0 in both semiflows. According to our choice of $\rho > 0$, the attractor $\mathcal{A}_\lambda$ of the non-truncated semiflow is contained in $B_\rho(0)$. In other words, the set $\mathcal{E}_\lambda$ of equilibria together with their connecting orbits are contained in $B_\rho(0)$. Due to Lemma 2.5 and (15), and after possibly increasing ρ further, one can assume that all points on $\partial B_\rho(0)$ have larger energy E_λ than any of the equilibria in $\mathcal{E}_\lambda$, i.e., one has

$$\bar{E}_\lambda \ = \ \min_{u \in \partial B_\rho(0)} \{E_\lambda(u)\} \ > \ \max_{e \in \mathcal{E}_\lambda} \{E_\lambda(e)\} \ . \tag{17}$$

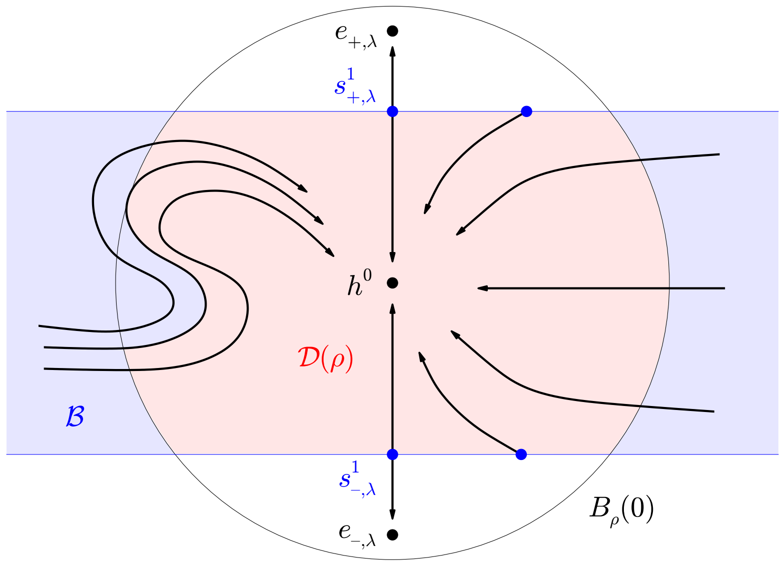

Figure 4: The special attracting set $\mathcal{D}(\rho)$ as a subset of the domain of attraction $\mathcal{B}$.

Then the following holds.

Lemma 2.9 *Suppose that Assumption 2.7 holds, let $\lambda \in \Lambda$, and consider the set $\mathcal{D}(\rho)$ defined in (16). Then the first spikes $s^1_{\pm,\lambda}$ are the unique minimizers of the problem*

$$E_\lambda(u) \to \min \qquad \textit{for} \qquad u \in \partial\mathcal{D}(\rho) ,$$

where E_λ denotes the extended energy introduced in (14).

Proof: Since $s^1_{\pm,\lambda} \in \partial\mathcal{D}(\rho)$, it is clear that the minimum can be at most $E_\lambda(s^1_{\pm,\lambda})$. Assume that there exists an element $u \in \partial\mathcal{D}(\rho) \setminus \{s^1_{\pm,\lambda}\}$ with $E_\lambda(u) < E_\lambda(s^1_{\pm,\lambda})$. Then one has $u \in \mathcal{H}^1$ and the solution $S_\lambda(t)u$ stays in $\partial\mathcal{B}$ for all $t \geq 0$. Since S_λ is a gradient semiflow, in combination with the fact that the Cahn-Hilliard model (1) has only finitely many equilibrium solutions for fixed λ, see again [25], one obtains that for some $e \in \mathcal{E}^0_\lambda$ with $E_\lambda(e) \leq E_\lambda(s^1_{\pm,\lambda})$, where $\mathcal{E}^0_\lambda = \mathcal{E}_\lambda \cap \partial\mathcal{B}$, we have $S_\lambda(t)u \to e$. Due to the positive invariance of the boundary $\partial\mathcal{B}$ this finally furnishes $e \in \partial\mathcal{D}(\rho)$, which contradicts Theorem 2.8. $\diamond$

Essential information on the set $\mathcal{D}(\rho)$ is shown in Figure 4. Note that the orbits starting from the first spikes $s^1_{\pm,\lambda}$ which are not attracted by the homogeneous state h^0 necessarily are attracted by some equilibria $e_\pm$, respectively. These equilibrium solutions satisfy both $e_\pm \notin \operatorname{cl}\mathcal{B}$ and $E_\lambda(e_\pm) < E_\lambda(s^1_{\pm,\lambda})$. We close this section with an auxiliary result which will be needed later.

Lemma 2.10 *Suppose that Assumption 2.7 holds and let $\lambda \in \Lambda$. Fix an arbitrary $\delta > 0$ and let $B_\delta(\mathcal{E}_\lambda^0 \cup \{h^0\})$ denote the open δ-neighborhood of the boundary equilibria and h^0. Then the first hitting time of the deterministic trajectories $\{S_\lambda(t)u_0 \,:\, t \geq 0\}$ with this neighborhood is bounded uniformly in $u_0 \in$ cl $\mathcal{D}(\rho)$, i.e., there exists $0 < T_0 < +\infty$ such that*

$$\sup_{u_0 \in cl\ \mathcal{D}(\rho)} \inf \left\{ t \geq 0 \ :\ S_\lambda(t)u_0 \in B_\delta\left(\mathcal{E}_\lambda^0 \cup \{h^0\}\right)\right\} \leq T_0 \ .$$

Proof: Suppose that T_0 does not exist. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in cl $\mathcal{D}(\rho)$ such that $S_\lambda(t)u_n \notin B_\delta(\mathcal{E}_\lambda^0 \cup \{h^0\})$ for all $0 \leq t \leq n$. Due to the regularization of the truncated semigroup S_λ and the compact embedding $\mathcal{C}_1 \hookrightarrow \mathcal{C}$, there exists a point $u \in$ cl $\mathcal{D}(\rho)$ with $u_n \to u$ in $\mathcal{C}$ as $n \to \infty$, possibly after passing to a subsequence. Then $S_\lambda(t)u \notin B_\delta(\mathcal{E}_\lambda^0 \cup \{h^0\})$ for all $t \geq 0$. On the other hand, cl $\mathcal{D}(\rho)$ is positively invariant, which contradicts the fact that in a gradient system every point is attracted by an equilibrium. ◇

3 The Stochastic Equation and Large Deviations

In this section we present the precise mathematical framework for the stochastic Cahn-Hilliard equation (2). In Section 3.1 we present the probabilistic framework for our noise process ξ, introduce the associated stochastic convolution, determine its regularity properties, and then use this to establish the existence of a mild solution of the stochastic Cahn-Hilliard-Cook equation in the phase space $\mathcal{C}$. The law of the mild solution is the subject of Section 3.2, where we derive the Freidlin-Wentzell large deviation estimates.

3.1 The Stochastic Convolution and Mild Solutions

It was mentioned earlier that in studying the stochastic Cahn-Hilliard-Cook model, one has to be careful about the various topologies that are involved. While the deterministic equation is usually studied in the spaces $\mathcal{H}^\alpha$ with $\alpha \in \mathbb{N}$, it is a gradient system with respect to E_λ only in the $\mathcal{H}^{-1}$-topology. Furthermore, for our stochastic application we need to consider the phase space $\mathcal{C}$. In this section, we have to introduce yet another topology, namely that of $\mathcal{H}^{-2}$, which is necessary for obtaining the appropriate large deviations estimates. Furthermore, this topology will be important for relating the quasi-potential in Section 4.1 to the energy E_λ.

We specify the noise process in (2) in terms of an $\mathcal{H}^{-2}$-valued Wiener process W. For this, let e_k, for $k \in \mathbb{N}$, be defined as in (3), and let $\mu_k = k^4\pi^4$. Then the e_k form an orthonormal basis in $\mathcal{H}^0$, and they are exactly the eigenfunctions of the operators A_λ and $\mathcal{B}$ defined in Section 2.1, with

$$Be_k = \sqrt{\mu_k} \cdot e_k \quad \text{and} \quad A_\lambda e_k = \frac{\mu_k}{\lambda^2} \cdot e_k \ , \quad \text{where} \quad \mu_k = k^4\pi^4 \ .$$

We define the stochastic process W as the derivative of space-time white noise, i.e.,

$$W(t) = \sum_{k=1}^{\infty} \mu_k^{1/4} \cdot \beta_k(t) \cdot e_k \,, \tag{18}$$

where $\{\beta_k : k \in \mathbb{N}\}$ denotes a family of independent real Brownian motions over a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus, the process W is an $\mathcal{H}^{-2}$-valued Q-Wiener process, where the covariance operator $Q = A^{-1/2}$ is of trace-class, see also (4). For details on Q-Wiener processes we refer the reader to [17, Chapter 4]. The stochastic convolution process associated with W and A_λ is defined by

$$W_{A_\lambda}(t) = \int_0^t e^{-A_\lambda(t-s)}\, dW(s) \,, \tag{19}$$

where $e^{-A_\lambda t}$ denotes the analytic semigroup generated by $-A_\lambda$ on $\mathcal{H}^{-2}$. One can easily see that the stochastic convolution is the mild solution of the linear stochastic equation in $\mathcal{H}^{-2}$ which is given by

$$dv = -A_\lambda v dt + dW(t) \,, \qquad v(0) = 0 \,. \tag{20}$$

Furthermore, using the system of eigenfunctions one can readily see that

$$W_{A_\lambda}(t,x) = \sum_{k=1}^{\infty} \mu_k^{1/4} \cdot \int_0^t e^{-\mu_k \cdot (t-s)/\lambda^2}\, d\beta_k(s) \cdot e_k(x) \,.$$

The following theorem, which is essentially taken from [16, Proposition 1.1] and relies on the celebrated Kolmogorov test, addresses the regularity of the stochastic convolution. Similar results can be found in [6, Section 2.2.2] and [7, Lemma 5.1].

Theorem 3.1 *Let W be the process defined in (18), and let A_λ be defined as in Section 2.1. Then the stochastic convolution W_{A_λ} has a version which is α-Hölder continuous with respect to (t,x), for $t \geq 0$ and $x \in [0,1]$, for any $\alpha \in [0,1/8)$.*

Using the above regularity of the stochastic convolution, one can now address the existence of solutions for the stochastic Cahn-Hilliard-Cook model. In contrast to the linear equation (20), the nonlinear Cahn-Hilliard-Cook equation (2) is not well-posed as an abstract evolution equation in $\mathcal{H}^{-2}$, and we therefore have to use the new phase space $\mathcal{C}$. The abstract stochastic evolution equation is given by

$$du = (-A_\lambda u + H_0(u))\, dt + \sigma dW(t) \,, \qquad u(0) = u_0 \in \mathcal{C} \,, \tag{21}$$

where the truncated nonlinearity H_0 was defined in (13). A $\mathcal{C}$-valued process $u = u_\sigma^{u_0}$ is called mild solution of (21), if for arbitrary $t \geq 0$ we have

$$u_\sigma^{u_0}(t) = e^{-A_\lambda t} u_0 + \int_0^t e^{-A_\lambda(t-s)} H_0\left(u_\sigma^{u_0}(s)\right)\, ds + \sigma W_{A_\lambda}(t) \,, \qquad \mathbb{P}-\text{a.s.} \,. \tag{22}$$

It can readily be shown that problem (21) is well-posed in this Banach space setting. In fact, we have the following result.

Theorem 3.2 *The stochastic Cahn-Hilliard-Cook model, using the formulation (21) and (22), and with initial condition* $u_0 \in \mathcal{C}$*, has a unique mild solution* $u = u_\sigma^{u_0}$ *in* $C([0,\infty),\mathcal{C})$*. Moreover, there exists a constant* $M > 0$ *such that for all* $u_0, u_1 \in \mathcal{C}$ *and all* $t \geq 0$ *we have*

$$|u_\sigma^{u_0}(t) - u_\sigma^{u_1}(t)|_\infty \leq e^{Mt} \cdot |u_0 - u_1|_\infty \ . \tag{23}$$

The proof of this result is straightforward using pathwise fixed-point arguments and the regularity of the stochastic convolution W_{A_λ} established in Theorem 3.1. One just has to employ Theorem 2.2 with $w = \sigma W_{A_\lambda}$. Recall that the nonlinearity H_0 is globally Lipschitz, and that for the analytic semigroup one has the estimates

$$\left\| e^{-A_\lambda t} \right\|_{\mathcal{L}(\mathcal{C})} \leq C \quad \text{as well as} \quad \left\| A_\lambda^{1/2} e^{-A_\lambda t} \right\|_{\mathcal{L}(\mathcal{C})} \leq C \cdot \left(1 + t^{-1/2}\right) \ .$$

3.2 The Law of the Mild Solution

In this section we study the law of the mild solution. Particularly, we derive Freidlin-Wentzell type large deviation estimates in the phase space of continuous functions. Our results are aimed at providing the appropriate framework for studying the domain exit problem in the next section. Similar results, but in a different setting, have been obtained in [21, 40].

As in Section 3.1, we consider the Cahn-Hilliard-Cook equation in the abstract form (21), and denote its unique mild solution by $u_\sigma^{u_0}(t)$. According to Theorem 3.2, this mild solution induces a law $\mathscr{L}(u_\sigma^{u_0}(\cdot))$ on the Banach space

$$\mathscr{C}_T = C([0,T],\mathcal{C}) \quad \text{with norm} \quad |u|_{\mathscr{C}_T} = \max_{t\in[0,T]} |u(t)|_\infty \ . \tag{24}$$

In order to derive the large deviation estimates of Freidlin and Wentzell for the law $\mathscr{L}(u_\sigma^{u_0}(\cdot))$, we follow the lines of [17] and proceed in three steps. First of all, the estimates are established for the linear problem on the Hilbert space

$$\mathscr{H}_T = L^2\left((0,T),\mathcal{H}^{-2}\right) \quad \text{with norm} \quad |u|_{\mathscr{H}_T} = \left(\int_0^T |u(s)|_{-2}^2 \, ds\right)^{1/2} \ . \tag{25}$$

Secondly, and using the fact that the linear solutions already live in the smaller space $\mathscr{C}_T$, these results can be readily lifted to the new space $\mathscr{C}_T$. Finally, the estimates are lifted to the full nonlinear situation via a transformation argument.

We begin by considering the linearized Cahn-Hilliard-Cook model in a Hilbert space setting. More precisely, consider the phase space $\mathcal{H}^{-2}$ and the linear stochastic equation

$$dv = -A_\lambda v dt + \sigma dW(t) \ , \qquad v(0) = v_0 \ , \tag{26}$$

where W was defined in (18). If we let W_{A_λ} denote the stochastic convolution introduced in (19), which solves the linear evolution equation (20), then for any $v_0 \in \mathcal{H}^{-2}$ the mild solution $v_\sigma^{v_0}$ of (26) is given by

$$v_\sigma^{v_0}(t) = e^{-A_\lambda t} v_0 + \sigma W_{A_\lambda}(t) \ , \tag{27}$$

and the law of this solution is a symmetric Gaussian measure on the Hilbert space $\mathscr{H}_T$ defined in (25). In fact, it follows from [17, Section 5.1.2] that

$$\mathscr{M}_\sigma^{v_0} = \mathscr{L}(v_\sigma^{v_0}(\cdot)) = \mathscr{N}(e^{-A_\lambda t} v_0 \ , \ \ \sigma^2 \mathscr{K}) \ , \tag{28}$$

where $\mathscr{K}$ denotes the covariance operator of the stochastic convolution W_{A_λ}, which is explicitly given by

$$(\mathscr{K}\varphi)(t) = \int_0^T g(t,s)\varphi(s)\, ds \ , \qquad \text{with} \qquad g(t,s) = \int_0^{t\wedge s} A^{-1/2} e^{-A_\lambda (t+s-2r)}\, dr \ . \tag{29}$$

For more details we refer the reader to [17, Section 5.1.2]. The family of measures $\mathscr{M}_\sigma^{v_0}$ satisfies the large deviation principle. We begin by considering the case $v_0 = 0$.

Lemma 3.3 *The family* $\{\mathscr{M}_\sigma^0\}_{\sigma>0}$ *defined in (28) satisfies the large deviation principle with rate function* $I_T^0 \ : \ \mathscr{H}_T \to [0,\infty]$ *defined by*

$$I_T^0(h) = \begin{cases} \dfrac{1}{2} \cdot \left|\mathscr{K}^{-1/2} h\right|_{\mathscr{H}_T}^2 \ , & \text{for} \quad h \in Im(\mathscr{K}^{1/2}) \ , \\ \infty \ , & \text{otherwise} \ . \end{cases} \tag{30}$$

Furthermore, let $K_T^0(r) = \{h \in \mathscr{H}_T \ : \ I_T^0(h) \leq r\}$ *and let* $B_\delta(\cdot)$ *denote the open* δ*-neighborhood with respect to the norm* $|\cdot|_{\mathscr{H}_T}$*. Then for all* $r_0 > 0$*,* $\delta > 0$*, and* $\gamma > 0$ *there exists a* $\sigma_0 > 0$ *such that for arbitrary* $\sigma \in (0,\sigma_0)$ *and* $r \in (0,r_0)$ *one has*

$$\mathscr{M}_\sigma^0 \left(B_\delta\left(K_T^0(r)\right)\right) \geq 1 - e^{-(r-\gamma)/\sigma^2} \ .$$

In addition, for all $r_0 > 0$*,* $\delta > 0$*, and* $\gamma > 0$ *there exists a* $\sigma_0 > 0$ *such that for arbitrary* $\sigma \in (0,\sigma_0)$ *and all* $h \in K_T^0(r_0)$ *one has*

$$\mathscr{M}_\sigma^0 \left(B_\delta(h)\right) \geq e^{-\left(I_T^0(h)+\gamma\right)/\sigma^2} \ .$$

In other words, the estimates of Freidlin and Wentzell hold.

For the proof of this lemma, as well as for the precise definition of the large deviation principle, we refer the reader to [17, Proposition 12.8, Chapter 12].

In view of our further applications of Lemma 3.3, it is convenient to establish a control-theoretic representation of the rate function $I_T^0(h)$ defined in (30). For this, consider the control system associated with equation (26), which is given by

$$\partial_t \mathbf{v} = -A_\lambda \mathbf{v} + Q^{1/2} c , \qquad \mathbf{v}(0) = v_0 , \tag{31}$$

where the control function c satisfies $c \in \mathscr{H}_T$. Recall from Section 3.1 that for our specific noise process we have $Q = A^{-1/2}$, see also (4). One can readily verify that for initial conditions $v_0 \in \mathcal{H}^{-2}$ the mild solution $\mathbf{v}_c^{v_0} \in C([0,T], \mathcal{H}^{-2})$ of (31) exists and is given by

$$\mathbf{v}_c^{v_0}(t) = e^{-A_\lambda t} v_0 + \int_0^t e^{-A_\lambda (t-s)} A^{-1/4} c(s)\, ds . \tag{32}$$

Specifically for $v_0 = 0$ we denote the corresponding solution operator $\mathscr{S}$ by

$$\mathscr{S} \;:\; \mathscr{H}_T \to \mathscr{H}_T , \qquad \mathscr{S} c = \mathbf{v}_c^0 .$$

One can show that the covariance operator $\mathscr{K}$ defined in (29) satisfies $\mathscr{K} = \mathscr{S}\mathscr{S}^*$, where $\mathscr{S}^*$ denotes the adjoint of $\mathscr{S}$. Furthermore, the operator $\mathscr{S}$ is one-to-one, and one has

$$Im(\mathscr{S}) \;=\; Im(\mathscr{K}^{1/2}) \;=\; W^{1,2}\left((0,T), \mathcal{H}^{-1}\right) \cap L^2\left((0,T), \mathcal{H}^3\right) \cap \{\mathbf{v}(0) = 0\} \;,$$

as well as

$$\left|\mathscr{K}^{-1/2} h\right|_{\mathscr{H}_T} \;=\; |c|_{\mathscr{H}_T} \;, \quad \text{where} \quad \mathscr{S} c = h \quad \text{and} \quad h \in Im(\mathscr{S}) \;,$$

see [17, Appendix B]. This immediately furnishes the following lemma.

Lemma 3.4 *Consider the rate function I_T^0 defined in (30), as well as the control problem (31) with $v_0 = 0$ and corresponding mild solution $\mathbf{v}_c^0$. Then we have*

$$I_T^0(\mathbf{v}) = \begin{cases} \dfrac{1}{2} \cdot |c|_{\mathscr{H}_T}^2 , & \text{for} \quad \mathbf{v} = \mathbf{v}_c^0 , \\ +\infty , & \text{for} \quad \mathbf{v} \notin Im(\mathscr{S}) . \end{cases}$$

We are now in a position to formulate the large deviation principle for general $v_0 \in \mathcal{H}^{-2}$. For this, define

$$I_T^{v_0}(\mathbf{v}) = I_T^0(\mathbf{v} - e^{-A_\lambda t} v_0) \;, \;\; \mathbf{v} \in \mathscr{H}_T \quad \text{and} \quad K_T^{v_0}(r) = \{\mathbf{v} \in \mathscr{H}_T \;:\; I_T^{v_0}(\mathbf{v}) \le r\} \;.$$

Due to the identity $\mathbf{v}_c^0(t) = \mathbf{v}_c^{v_0}(t) - e^{-A_\lambda t} v_0$, the function $\mathbf{v}$ is contained in $K_T^{v_0}(r)$ if and only if $\mathbf{v} - e^{-A_\lambda t} v_0 \in K_T^0(r)$. Thus, Lemmas 3.3 and 3.4 furnish the following large deviation result in the space $\mathscr{H}_T$.

Theorem 3.5 *The family $\{\mathscr{M}_\sigma^{v_0}\}_{\sigma>0}$ defined in (28) satisfies the large deviation principle with rate function $I_T^{v_0} : \mathscr{H}_T \to [0,\infty]$. Furthermore, let $B_\delta(\cdot)$ denote the open δ-neighborhood with respect to the norm $|\cdot|_{\mathscr{H}_T}$. Then for all $r_0 > 0$, $\delta > 0$, and $\gamma > 0$ there exists a $\sigma_0 > 0$ such that for all $\sigma \in (0, \sigma_0)$ and $r \in (0, r_0)$ we have*

$$\mathscr{M}_\sigma^{v_0}\left(B_\delta\left(K_T^{v_0}(r)\right)\right) \geq 1 - e^{-(r-\gamma)/\sigma^2} ,$$

and for all $r_0 > 0$, $\delta > 0$, and $\gamma > 0$ there exists a $\sigma_0 > 0$ such that for all $\sigma \in (0, \sigma_0)$ and $\mathbf{v} \in K_T^{v_0}(r_0)$ one has

$$\mathscr{M}_\sigma^{v_0}\left(B_\delta(\mathbf{v})\right) \geq e^{-\left(I_T^{v_0}(\mathbf{v})+\gamma\right)/\sigma^2} .$$

We now turn our attention to the second step mentioned above, i.e., we establish the Freidlin-Wentzell estimates in the space of continuous functions. One can readily see that for initial conditions $v_0 \in \mathcal{C}$ the solutions $v_\sigma^{v_0}$ of (26) and the solutions $\mathbf{v}_c^{v_0}$ of (31) are already contained in the smaller space $\mathscr{C}_T \subset \mathscr{H}_T$. Due to the embedding $\mathcal{C} \hookrightarrow \mathcal{H}^{-2}$ it follows easily that the space $\mathscr{C}_T$ is continuously embedded in $\mathscr{H}_T$. Furthermore, a nontrivial result in [3, Theorem 10.28] or [41, Theorem 4.5.4] then implies that $\mathscr{C}_T$ is in fact a Borel set in $\mathscr{H}_T$. Thus, one can combine Theorem 3.5 with [17, Theorem 12.14] to establish the following result.

Theorem 3.6 *Let $v_0 \in \mathcal{C}$ and let $\mathbf{v}_c^{v_0}$ denote the mild solution (32) of the associated control problem (31). As before, define*

$$K_T^{v_0}(r) = \{\mathbf{v} \in \mathscr{H}_T \;:\; I_T^{v_0}(\mathbf{v}) \leq r\} = \left\{\mathbf{v}_c^{v_0} \in \mathscr{C}_T \;:\; |c|^2_{\mathscr{H}_T} \leq 2r\right\} .$$

Then the mild solution $v_\sigma^{v_0}$ of (26), as defined in (27), satisfies the following Freidlin-Wentzell estimates. For all $r_0 > 0$, $\delta > 0$, and $\gamma > 0$ there exists a $\sigma_0 > 0$ such that for all $v_0 \in \mathcal{C}$, $\sigma \in (0, \sigma_0)$, and $r \in (0, r_0)$ one has

$$\mathbb{P}\left(\mathrm{dist}_{\mathscr{C}_T}\left(v_\sigma^{v_0},\; K_T^{v_0}(r)\right) < \delta\right) \geq 1 - e^{-(r-\gamma)/\sigma^2} .$$

In addition, for all $r_0 > 0$, $\delta > 0$, and $\gamma > 0$ there exists a $\sigma_0 > 0$ such that for arbitrary $v_0 \in \mathcal{C}$, $\sigma \in (0, \sigma_0)$, and all $\mathbf{v} \in \mathscr{C}_T$ satisfying $I_T^{v_0}(\mathbf{v}) \leq r_0$ one has

$$\mathbb{P}\left(|v_\sigma^{v_0} - \mathbf{v}|_{\mathscr{C}_T} < \delta\right) \geq e^{-\left(I_T^{v_0}(\mathbf{v})+\gamma\right)/\sigma^2} .$$

In the final step we now consider solutions $u_\sigma^{u_0}$ of the nonlinear problem (21) with initial conditions $u_0 \in \mathcal{C}$. The associated control system is given by

$$\partial_t \mathbf{u} = -A_\lambda \mathbf{u} + H_0(\mathbf{u}) + Q^{1/2} c , \qquad \mathbf{u}(0) = u_0 , \tag{33}$$

with control function $c \in \mathscr{H}_T$ and mild solution $\mathbf{u}_c^{u_0} \in \mathscr{C}_T$ satisfying

$$\mathbf{u}_c^{u_0}(t) = \int_0^t e^{-A_\lambda(t-s)} H_0(\mathbf{u}_c^{u_0}(s))\, ds + \mathbf{v}_c^{u_0}(t) , \tag{34}$$

where $\mathbf{v}_c^{u_0}(t)$ denotes the solution of (31) — which due to $u_0 \in \mathcal{C}$ is contained in $\mathscr{C}_T$. Analogous to Theorem 3.2, one can show that for any $u_0 \in \mathcal{C}$ and $c \in \mathscr{H}_T$ the integral equation (34) has a unique solution, and that in addition one has

$$|u_\sigma^{u_0} - \mathbf{u}_c^{u_0}|_{\mathscr{C}_T} \leq L \cdot |v_\sigma^{u_0} - \mathbf{v}_c^{u_0}|_{\mathscr{C}_T} \ . \tag{35}$$

Together with Theorem 3.6 and the definition

$$I_T^{u_0}(\mathbf{u}) = \begin{cases} \dfrac{1}{2} \cdot |c|^2_{\mathscr{H}_T} \ , & \text{if there exists a} \quad c \in \mathscr{H}_T \quad \text{with} \quad \mathbf{u} = \mathbf{u}_c^{u_0} \ , \\ +\infty \ , & \text{otherwise} \ , \end{cases} \tag{36}$$

we finally obtain the main result of this section.

Theorem 3.7 *Let $u_0 \in \mathcal{C}$, let $\mathbf{u}_c^{u_0}$ denote the mild solution (34) of the associated control problem (33), and let*

$$K_T^{u_0}(r) = \{\mathbf{u} \in \mathscr{H}_T \ : \ I_T^{u_0}(\mathbf{u}) \leq r\} = \left\{\mathbf{u}_c^{u_0} \in \mathscr{C}_T \ : \ |c|^2_{\mathscr{H}_T} \leq 2r\right\} \ .$$

Then the mild solution $u_\sigma^{u_0}$ of (21) as defined in (22) satisfies the Freidlin-Wentzell estimates in the following form. For all $r_0 > 0$, $\delta > 0$, and $\gamma > 0$ there exists a $\sigma_0 > 0$ such that for all $u_0 \in \mathcal{C}$, $\sigma \in (0, \sigma_0)$, and $r \in (0, r_0)$ one has

$$\mathbb{P}\left(\mathrm{dist}_{\mathscr{C}_T}\left(u_\sigma^{u_0}, \ K_T^{u_0}(r)\right) < \delta\right) \geq 1 - e^{-(r-\gamma)/\sigma^2} \ .$$

In addition, for all $r_0 > 0$, $\delta > 0$, and $\gamma > 0$ there exists a $\sigma_0 > 0$ such that for arbitrary $u_0 \in \mathcal{C}$, $\sigma \in (0, \sigma_0)$, and for all $\mathbf{u} \in \mathscr{C}_T$ satisfying $I_T^{u_0}(\mathbf{u}) \leq r_0$ one has

$$\mathbb{P}\left(|u_\sigma^{u_0} - \mathbf{u}|_{\mathscr{C}_T} < \delta\right) \geq e^{-\left(I_T^{u_0}(\mathbf{u}) + \gamma\right)/\sigma^2} \ .$$

4 Stochastic Domain Exit

In this final section of the paper we return to the nucleation phenomenon as described by the stochastic Cahn-Hilliard-Cook model. It will be shown that trajectories of this stochastic equation which originate at the homogeneous state h^0 can overcome the attracting influence of the underlying deterministic system, in which h^0 is an attracting equilibrium. In fact, a trajectory of the stochastic equation leaves any bounded neighborhood $\mathcal{U}$ of h^0 almost surely, see for instance [17, Proposition 12.17]. Thus, one can ask whether there is an exit set $\mathcal{E}$ which occupies only a small portion of the boundary $\partial\mathcal{U}$ where the stochastic trajectories leave $\mathcal{U}$ with high probability. In this section, we determine this exit set for the canonical candidate for $\mathcal{U}$, i.e., the open basin of attraction of the stable equilibrium h^0 in the non-truncated semiflow. This is accomplished in two steps. In Section 4.1 we establish a link between the Cahn-Hilliard energy defined in (6) and the quasi-potential for (21). After that, Section 4.2 addresses the domain exit problem.

4.1 The Energy as Quasi-Potential

In this first section it will be shown that the peculiar functional-analytic setup used in this paper allows us to relate the classical Cahn-Hilliard energy as described by (6) or (7) to the quasi-potential for the stochastic Cahn-Hilliard-Cook model (21) which describes the likelihood of certain stochastic orbits in the small-noise limit. As we mentioned several times before, this connection is not obvious due to the fact that the Cahn-Hilliard model is the $\mathcal{H}^{-1}$-gradient of the energy. To begin with, let $u_0, u_1 \in \mathcal{C}$ be arbitrary. Then the quasi-potential $I(u_0, u_1)$ for the problem (21) is defined by

$$I(u_0, u_1) = \inf \left\{ \frac{1}{2} \cdot \int_0^T |c(s)|_{-2}^2 \, ds \ : \ \mathbf{u}_c^{u_0}(T) = u_1 \ , \ \text{ for some } \ T > 0 \right\} \ ,$$

where $\mathbf{u}_c^{u_0}$ denotes the solution of the associated nonlinear control problem (33). Note that in this definition the time T is not fixed. Using (36) one can see that

$$I(u_0, u_1) = \inf \left\{ I_T^{u_0}(\mathbf{u}) \ : \ \mathbf{u}(T) = u_1 \ , \ \text{ for some } \ T > 0 \right\} \ .$$

Interpreting the value $\int_0^T |c(s)|_{-2}^2 \, ds/2$ as the energy dissipated by the control c, one can say that $I(u_0, u_1)$ is the minimal energy required by the nonlinear control system (33) to transfer the point u_0 to the point u_1.

For our application to nucleation, we need to understand the quasi-potential centered at the homogeneous state h^0, i.e., we need to be able to compute $I(h^0, u_1)$, for any point $u_1 \in \mathcal{D}(\rho)$ with $E_\lambda(u_1) < \bar{E}_\lambda$, see also (16) and (17). This is accomplished in the following lemma.

Lemma 4.1 *For $u_1 \in \mathcal{D}(\rho)$ with $E_\lambda(u_1) \le \bar{E}_\lambda$ the quasi-potential can be determined from the Ginzburg-Landau free energy (6) via the identity*

$$I\left(h^0, u_1\right) \ = \ 2 \cdot \left(E_\lambda\left(u_1\right) - E_\lambda\left(h^0\right)\right) \ .$$

Proof: We begin by establishing the inequality $I(h^0, u_1) \le 2(E_\lambda(u_1) - E_\lambda(h^0))$. For this, let $c(t)$ be an arbitrary control function and define $\tilde{c}(t) = A^{-1/4}c(t)$. Using the setup of Section 2.3, as well as Remark 2.1, and recalling that the energy functional (6) satisfies (8), one obtains for all $u \in B_{2\rho}(0) \cap \mathcal{H}^1$ the identity

$$H_0(u) = G(u) = -\nabla_{\mathcal{H}^{-1}} V(u) \ ,$$

where V was defined in (7). Due to the regularity of the initial condition h^0 the solution $\mathbf{u}_c^{h^0}$ is in fact a strong solution in $C([0,T], \mathcal{H}^1)$, and on the set $B_{2\rho}(0)$ the control system (33) is equivalent to the $\mathcal{H}^{-1}$-gradient system

$$\partial_t \mathbf{u} = -A_\lambda \mathbf{u} - \nabla_{\mathcal{H}^{-1}} V(\mathbf{u}) + \tilde{c} \ , \qquad \mathbf{u}(0) = h^0 \ . \tag{37}$$

Now let $\mathbf{u} = \mathbf{u}_c^{h^0}$ denote the solution of (37) and let $T > 0$ be such that $\mathbf{u}$ stays in $B_{2\rho}(0)$ on $[0,T]$. Then one can rewrite the action $I_T^{h^0}(\mathbf{u})$ without reference to the control as

$$\begin{aligned} I_T^{h^0}(\mathbf{u}) \;=\; I_T^{h^0}\left(\mathbf{u}_c^{h^0}\right) \;&=\; \frac{1}{2}\cdot\int_0^T |c(s)|_{-2}^2\,ds \;=\; \frac{1}{2}\cdot\int_0^T |\tilde{c}(s)|_{-1}^2\,ds \\ &=\; \frac{1}{2}\cdot\int_0^T |\dot{\mathbf{u}}(s) + A_\lambda \mathbf{u}(s) + \nabla_{\mathcal{H}^{-1}} V(\mathbf{u}(s))|_{-1}^2\,ds \\ &=\; \frac{1}{2}\cdot\int_0^T |\dot{\mathbf{u}}(s) - A_\lambda \mathbf{u}(s) - \nabla_{\mathcal{H}^{-1}} V(\mathbf{u}(s))|_{-1}^2\,ds \\ &\qquad +2\cdot\left(E_\lambda(\mathbf{u}(T)) - E_\lambda\left(h^0\right)\right)\ . \end{aligned}$$

We now construct a path $\mathbf{u} = \mathbf{u}_c^{h^0}$ from h^0 to u_1 which stays in the set $B_\rho(0)$, and for which the integral term in the last line above can be made arbitrarily small. More precisely, we construct the path $\mathbf{u} : [0,T] \to \mathcal{C}$ by concatenating paths $\mathbf{u}_1 : [0,T_1] \to \mathcal{C}$ and $\mathbf{u}_2 : [0,T_2] \to \mathcal{C}$, where $T = T_1 + T_2$ and $\mathbf{u}_1(T_1) = \mathbf{u}_2(0)$. The second path is obtained from the deterministic flow S_λ defined in Remark 2.3 via $\mathbf{u}_2(t) = S_\lambda(T_2 - t)u_1$, i.e., one considers the time-reversed orbit $\{S_\lambda(t)u_1 \;:\; 0 \le t \le T_2\}$. The time T_2 is chosen sufficiently large, see below. Now define $\mathbf{u}_1(t) = t \cdot S_\lambda(T_2)u_1/T_1$, i.e., the first path is the linear interpolation between the homogeneous state h^0 and the point $\mathbf{u}_2(0)$. For $T_2 \to \infty$ we have $|S_\lambda(T_2)u_1|_3 \to 0$, and the action of the first path, which is given by $I_{T_1}^{h^0}(\mathbf{u}_1)$, converges to 0. In the action of the second path the integral term in the last line above vanishes, and one obtains $I_{T_2}^{h^0}(\mathbf{u}_2) = 2\cdot(E_\lambda(u_1) - E_\lambda(u_2))$. Since this term converges to $2\cdot(E_\lambda(u_1) - E_\lambda(h^0))$ as $T_2 \to \infty$ one finally obtains

$$I_T^{h^0}(\mathbf{u}) = I_{T_1}^{h^0}(\mathbf{u}_1) + I_{T_2}^{h^0}(\mathbf{u}_2) \;\to\; 2\cdot\left(E_\lambda(u_1) - E_\lambda\left(h^0\right)\right) \qquad \text{for} \qquad T_2 \to \infty\ .$$

It remains to prove $I(h^0, u_1) \ge 2\cdot(E_\lambda(u_1) - E_\lambda(h^0))$. Note that a path $\mathbf{u} : [0,T] \to \mathcal{C}$ from h^0 to u_1 either stays in $B_\rho(0)$ on all of $[0,T]$, or there exists a time $T_0 \in (0,T)$ with $\mathbf{u}(T_0) \in \partial B_\rho(0)$ and $\mathbf{u}(t) \in B_\rho(0)$ on $[0,T_0)$. In the first case, the action can be estimated similar to the previous paragraph by $I_T^{h^0}(\mathbf{u}) \ge 2\cdot(E_\lambda(u_1) - E_\lambda(h^0))$. In the second case, notice that $I_T^{h^0}(\mathbf{u}) \ge I_{T_0}^{h^0}(\mathbf{u}) \ge 2\cdot(E_\lambda(\mathbf{u}(T_0)) - E_\lambda(h^0))$. Due to (17) and $E_\lambda(u_1) \le \bar{E}_\lambda$ we then obtain $I_T^{h^0}(\mathbf{u}) \ge 2\cdot(\bar{E}_\lambda - E_\lambda(h^0)) \ge 2\cdot(E_\lambda(u_1) - E_\lambda(h^0))$. This completes the proof of the lemma. $\diamond$

4.2 Exit from the Deterministic Attracting Set

In this final section we address the domain exit problem which lies at the heart of our approach to nucleation. We begin by a slight change of setting. For fixed $\sigma > 0$ the solutions $\{u_\sigma^{u_0}(t)\}_{t\ge 0,\, u_0\in\mathcal{C}}$ of the Cahn-Hilliard-Cook model form a strong Markov family, see for example [17, Chapter 9], and it turns out to be more convenient to work within this framework. Thus, we consider in the following the strong Markov process $(u_\sigma(t), \mathbb{P}_{u_0})$ which is defined by

$$\mathbb{P}_{u_0}(u_\sigma(t) \in \Gamma) = \mathbb{P}(u_\sigma^{u_0}(t) \in \Gamma) ,$$

see [23, Chapter 1, Paragraph 2] for more details. For arbitrary $T > 0$, we can then reformulate the large deviations estimates of Theorem 3.7. For all $r_0 > 0$, $\delta > 0$, and all $\gamma > 0$ there exists a $\sigma_0 > 0$ such that for all $u_0 \in \mathcal{C}$, $\sigma \in (0, \sigma_0)$, and $r \in (0, r_0)$ the estimate

$$\mathbb{P}_{u_0}\left(\operatorname{dist}_{\mathscr{C}_T}(u_\sigma,\ K_T^{u_0}(r)) < \delta\right) \geq 1 - e^{-(r-\gamma)/\sigma^2} \tag{38}$$

holds. In addition, for all $r_0 > 0$, $\delta > 0$, and $\gamma > 0$ there exists a $\sigma_0 > 0$ such that for all $u_0 \in \mathcal{C}$, $\sigma \in (0, \sigma_0)$, and for arbitrary $\mathbf{u} \in \mathscr{C}_T$ satisfying $I_T^{u_0}(\mathbf{u}) \leq r_0$ one has

$$\mathbb{P}_{u_0}\left(|u_\sigma - \mathbf{u}|_{\mathscr{C}_T} < \delta\right) \geq e^{-\left(I_T^{u_0}(\mathbf{u})+\gamma\right)/\sigma^2} . \tag{39}$$

In Section 2.3 we constructed a special subset $\mathcal{D}(\rho)$ of the deterministic basin of attraction $\mathcal{B}$ of the homogeneous state h^0, see (16). In this section, we are interested in the points where the stochastic trajectories $\{u_\sigma^{u_0}(t) \ :\ t \geq 0\}$ originating at $u_0 \in \mathcal{D}(\rho)$ leave the set $\mathcal{D}(\rho)$. To keep our presentation and the resulting proofs as simple as possible, it is convenient to consider a slightly smaller set than $\mathcal{D}(\rho)$. For this, let $\kappa > 0$ be a small constant, let $\rho > 0$ denote the generally large constant introduced in Section 2.3, and define

$$\mathcal{D}(\kappa, \rho) = \mathcal{D}(\rho) \setminus B_\kappa\left(s^1_{\pm,\lambda}\right) . \tag{40}$$

This definition is illustrated in Figure 5. The exit points from the set $\mathcal{D}(\kappa, \rho)$ can be defined in terms of the stopping time

$$\tau = \tau\left(\sigma,\ \mathcal{D}(\kappa, \rho)^c\right) , \tag{41}$$

where in general we define for Borel sets $\mathcal{G}$ and the given Markov process $(u_\sigma(t), \mathbb{P}_{u_0})$ the corresponding hitting time as $\tau(\sigma, \mathcal{G}) = \inf\{t \geq 0 \ :\ u_\sigma(t) \in \mathcal{G}\}$. Using this notation, the exit points are then given by $u_\sigma(\tau)$. In Lemma 2.9 we observed that the first spikes $s^1_{\pm,\lambda}$ are the unique minimizers of the energy E_λ on the boundary $\partial\mathcal{D}(\rho)$. Recalling that in $\mathcal{D}(\rho)$ the energy $E_\lambda(u)$ essentially coincides with the quasi-potential $I(h^0, u)$, as shown in Lemma 4.1, we can then formulate the following main result of this paper.

Theorem 4.2 *Suppose that Assumption 2.7 holds, let $\lambda \in \Lambda$, let $\mathcal{D}(\kappa, \rho)$ be defined as in (40), and consider the exit time τ defined in (41). Then for any $\delta > \kappa$ we have*

$$\mathbb{P}_{u_0}\left(u_\sigma(\tau) \notin B_\delta\left(s^1_{\pm,\lambda}\right)\right) \ \rightarrow\ 0 \quad \text{as} \quad \sigma \rightarrow 0 ,$$

uniformly in the initial condition $u_0 \in B_c(h^0)$, where $c > 0$ is sufficiently small to ensure that $B_c(h^0) \subset \mathcal{D}(\kappa, \rho)$, and that the deterministic orbits $\{S_\lambda(t)u_0 \ :\ t \geq 0\}$ remain bounded away from the boundary $\partial\mathcal{D}(\kappa, \rho)$ for all $u_0 \in B_c(h^0)$.

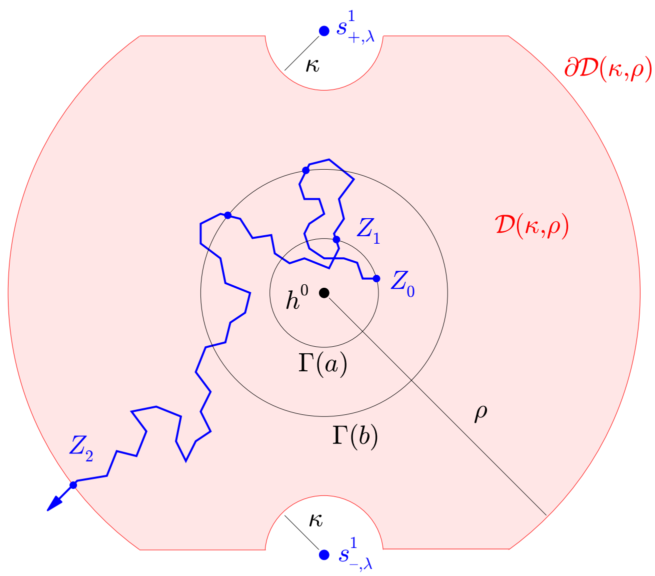

Figure 5: Construction of the Markov chain for the exit from $\mathcal{D}(\kappa, \rho)$.

For the proof of this result we follow the lines of [11], where a similar result was derived for the stochastic Allen-Cahn model subject to Dirichlet boundary conditions. The basic idea of this proof is illustrated in Figure 5 and dates back to Freidlin and Wentzell [23] in the finite-dimensional setting.

The remainder of this section is devoted to the proof of Theorem 4.2. We begin by briefly outlining this proof and introducing some notation. For each $r > 0$ define the sphere

$$\Gamma(r) \;=\; \partial B_r(h^0) \;=\; \{u \in \mathcal{C} \;:\; |u|_\infty = r\} \;,$$

and for positive numbers a and b with $a < b/2$, which will be chosen sufficiently small later on, consider the increasing sequence of Markov times given recursively by

$$\zeta_0 = 0\;, \qquad \eta_0 = \inf\{t > \zeta_0 \;:\; u_\sigma(t) \in \Gamma(b)\}\;, \tag{42}$$

and

$$\zeta_n = \inf\{t > \eta_{n-1} \;:\; u_\sigma(t) \in \Gamma(a) \cup \partial\mathcal{D}(\kappa,\rho)\}\;, \quad \eta_n = \inf\{t > \zeta_n \;:\; u_\sigma(t) \in \Gamma(b)\}\;,$$

for $n \in \mathbb{N}$. Using these Markov times we define both the Markov chain

$$Z_n = u_\sigma(\zeta_n)\;, \qquad \text{and the random variable} \qquad N = \inf\{n \geq 0 \;:\; Z_n \in \partial\mathcal{D}(\kappa,\rho)\}\;.$$

We are only interested in the process until it hits the set $\partial\mathcal{D}(\kappa, \rho)$ at time τ, hence we do not need to take care of the possibility of Z_n not being well-defined. Since the family $(u_\sigma(t), \mathbb{P}_{u_0})$ is a strong Markov process, one can argue as in [23, Chapter 4, Paragraph 2] to show that for $u_0 \in \Gamma(a)$ we have

$$\mathbb{P}_{u_0}\left(u_\sigma(\tau) \notin B_\delta(s^1_{\pm,\lambda})\right) \leq \sup_{u_0 \in \Gamma(a)} \frac{\mathbb{P}_{u_0}\left(Z_1 \in \partial\mathcal{D}(\kappa,\rho) \setminus B_\delta\left(s^1_{\pm,\lambda}\right)\right)}{\mathbb{P}_{u_0}\left(Z_1 \in \partial\mathcal{D}(\kappa,\rho)\right)} . \tag{43}$$

To complete the proof in the case $u_0 \in \Gamma(a)$, it suffices to establish a lower bound for the denominator and an upper bound for the numerator such that the quotient converges to zero as $\sigma \to 0$.

To begin with, we address the lower bound for the denominator. It is clear that the deterministic connecting orbit between $s^1_{+,\lambda}$ and h^0 hits the boundary $\partial B_{\kappa/2}(s^1_{+,\lambda})$ in a point u_3 with energy smaller than $E_\lambda(s^1_{+,\lambda})$. Define

$$\Delta E_\lambda = E_\lambda\left(s^1_{\pm,\lambda}\right) - E_\lambda\left(h^0\right) \quad \text{and} \quad \tilde{\Delta} E_\lambda = E_\lambda\left(u_3\right) - E_\lambda\left(h^0\right) < \Delta E_\lambda .$$

The following lemma provides the lower bound for the denominator in (43).

Lemma 4.3 *In the situation of Theorem 4.2, let $b > 0$ be such that $B_b(h^0) \subset \mathcal{D}(\kappa,\rho)$. Furthermore, let $a \in (0, b/2)$ be such that if u_0 belongs to $\Gamma(a)$, then the deterministic orbit $\{S_\lambda(t)u_0 \ : \ t \geq 0\}$ does not intersect $\Gamma(b/2)$. Then there exists a $\sigma_0 > 0$ such that for any $u_0 \in \Gamma(a)$ and $\sigma < \sigma_0$ one has*

$$\mathbb{P}_{u_0}\left(Z_1 \in \partial\mathcal{D}(\kappa,\rho)\right) \geq e^{-\left(2\tilde{\Delta}E_\lambda + kb\right)/\sigma^2} ,$$

where $k > 0$ is a constant that does not depend on b.

Proof: The idea of the proof is to construct a path $\mathbf{u} : [0,T] \to \mathcal{D}(\rho)$ from the point $u_0 \in \Gamma(a)$ to the point $u_3 \in \partial B_{\kappa/2}(s^1_{+,\lambda})$, which hits the set $\Gamma(b)$ in a single point $u_2 \in \Gamma(b)$. Then we can choose $\delta < \min\{\kappa/4,\ b/2\}$ and any sample path u_σ which satisfies $\mathrm{dist}_{\mathscr{C}_T}(u_\sigma, \mathbf{u}) < \delta$ enforces $Z_1 \in \partial\mathcal{D}(\kappa,\rho)$. The action of the path $\mathbf{u}$ can be estimated by $I^{u_0}_T(\mathbf{u}) \leq 2\tilde{\Delta}E_\lambda + kb/2$, and the lemma then follows from (39) by choosing δ as above and $\gamma = kb/2$.

We construct $\mathbf{u}$ as the union of three paths $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, see also Figure 6. Let $\mathbf{u}_1$ denote the deterministic solution $S_\lambda(t)u_0$, for $t \in [0, T_1] = [0, 1]$, between u_0 and $u_1 = S_\lambda(1)u_0$, and let $\mathbf{u}_2$ denote the linear interpolation between u_1 and u_2 in time $T_2 = |u_1 - u_2|_\infty$, where $u_2 = S_\lambda(T_3)u_3 \in \Gamma(b)$ is the first hitting point with the sphere $\Gamma(b)$ of the deterministic orbit starting at u_3. Finally the last path is exactly this deterministic orbit with reversed time, i.e., $\mathbf{u}_3(t) = S_\lambda(T_3 - t)u_3$. The estimate of the action $I^{u_0}_T(\mathbf{u})$ follows using similar arguments as in the proof of Lemma 4.1. ◇

In the next step one has to derive an estimate for the numerator in (43). Due to the strong Markov property, one obtains for arbitrary $u_0 \in \Gamma(a)$ the identity

$$\mathbb{P}_{u_0}\left(Z_1 \in \partial\mathcal{D}(\kappa,\rho) \setminus B_\delta\left(s^1_{\pm,\lambda}\right)\right) \ = \ \mathbb{E}_{u_0}\left(\mathbb{P}_{u_\sigma(\eta_0)}\left(u_\sigma(\tilde{\tau}) \in \partial\mathcal{D}(\kappa,\rho) \setminus B_\delta\left(s^1_{\pm,\lambda}\right)\right)\right) ,$$

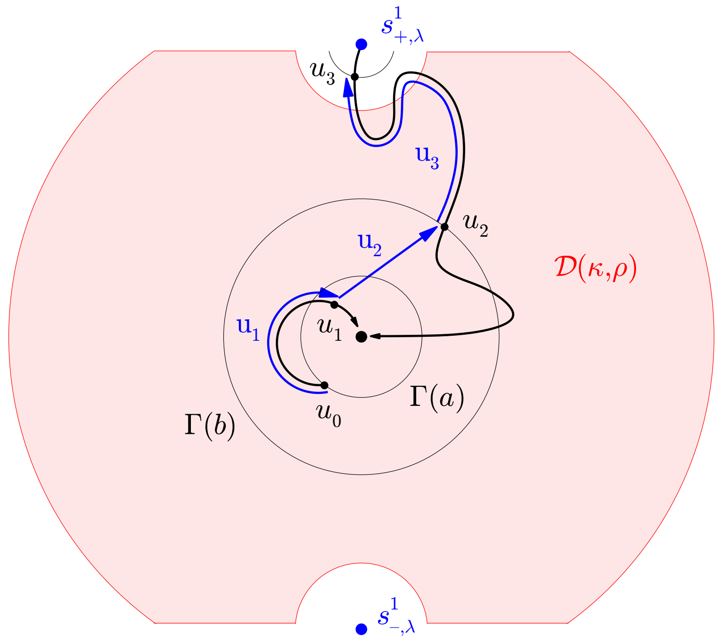

Figure 6: The path $\mathbf{u} = \mathbf{u}_1 \cup \mathbf{u}_2 \cup \mathbf{u}_3$ for the lower bound on the exit probability

where η_0 was introduced in (42) and we define

$$\tilde{\tau} = \tau\left(\sigma,\ \Gamma(a) \cup \partial\mathcal{D}(\kappa,\rho)\right) .$$

Hence it suffices to bound the probability

$$\mathbb{P}_{u_0}\left(u_\sigma(\tilde{\tau}) \in \partial\mathcal{D}(\kappa,\rho) \setminus B_\delta\left(s^1_{\pm,\lambda}\right)\right) , \tag{44}$$

uniformly in $u_0 \in \Gamma(b)$. To this end, consider Figure 7 which introduces the set

$$B_d = B_d\left(\partial\mathcal{D}(\kappa,\rho) \setminus B_\delta\left(s^1_{\pm,\lambda}\right)\right) ,$$

where the constant $d > 0$ is chosen sufficiently small to ensure that

$$\mathrm{cl}\, B_d \ \cap\ \mathrm{cl}\ \left\{B_\kappa\left(s^1_{\pm,\lambda}\right) \cup B_b\left(h^0\right) \cup B_{2d}\left(h^0\right)\right\} \ =\ \emptyset\, , \tag{45}$$

as well as

$$\inf_{u \in B_{2d}(\partial\mathcal{D}(\kappa,\rho)\setminus B_\delta(s^1_{\pm,\lambda}))} E_\lambda(u) \ >\ E_\lambda\left(s^1_{\pm,\lambda}\right) . \tag{46}$$

Moreover, define the hitting time of the set B_d via

$$\tau_d = \tau\left(\sigma,\ B_d\right) .$$

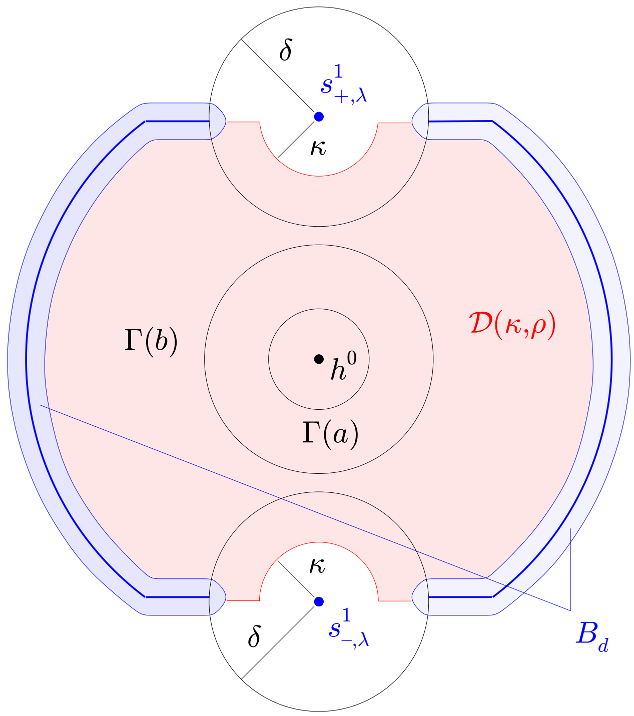

Figure 7: The set B_d

For fixed time $T > 0$ the event in (44) can now be decomposed into two disjoint sets. More precisely, one has

$$\underbrace{\left\{u_\sigma(\tilde{\tau}) \in \partial\mathcal{D}(\kappa,\rho) \setminus B_\delta\left(s^1_{\pm,\lambda}\right)\right\}}_{A} = \underbrace{(A \cap \{\tau_d > T\})}_{A_0} \cup \underbrace{(A \cap \{\tau_d \leq T\})}_{A_1} . \tag{47}$$

In order to derive estimates for the probabilities of the events A_0 and A_1 we need the following two lemmas.

Lemma 4.4 *In the situation of Theorem 4.2, let $\mathcal{F}$ denote a closed and bounded subset of $\mathcal{C}$ such that the deterministic orbits starting in $\mathcal{F}$ leave the neighborhood $B_\mu(\mathcal{F})$, for some $\mu > 0$, in time less than or equal to $T_0 > 0$. Then there exist constants $K > 0$ and $\sigma_0 > 0$ such that*

$$\mathbb{P}_{u_0}\left(\tau(\sigma,\ \mathcal{F}^c) > T_0\right) \leq e^{-K/\sigma^2} , \tag{48}$$

for all $u_0 \in \mathcal{F}$ and $0 < \sigma < \sigma_0$. Moreover, for arbitrary $M > 0$ there exist $T > 0$ and $\sigma_0 > 0$ such that

$$\mathbb{P}_{u_0}\left(\tau(\sigma,\ \mathcal{F}^c) > T\right) \leq e^{-M/\sigma^2} , \tag{49}$$

for all $u_0 \in \mathcal{F}$ and $0 < \sigma < \sigma_0$.

Proof: Denote the set of paths $\mathbf{u}$ which originate in $\mathcal{F}$ and stay in $B_{\mu/2}(\mathcal{F})$ during the time interval $[0,T_0]$ by $U(T_0) = \{\mathbf{u} : [0,T_0] \to B_{\mu/2}(\mathcal{F}) \ : \ \mathbf{u}(0) \in \mathcal{F}\}$. We begin by establishing the existence of a constant K such that

$$I_{T_0}^{\mathbf{u}(0)}(\mathbf{u}) \geq 3K > 0 \ , \qquad \text{for all} \qquad \mathbf{u} \in U(T_0) \ . \tag{50}$$

Assume that this were false. Then there exist paths $\mathbf{u}_n$ with $u_n = \mathbf{u}_n(0) \in \mathcal{F}$ and associated control functions c_n via (33), which remain in $B_{\mu/2}(\mathcal{F})$ and such that $I_{T_0}^{u_n}(\mathbf{u}_n) \to 0$ as $n \to \infty$. Together with (34) and (35) this furnishes

$$\sup_{t\in[0,T_0]} |S_\lambda(t)u_n - \mathbf{u}_n(t)|_\infty \ \leq \ L \cdot \sup_{t\in[0,T_0]} \left|\mathbf{v}_{c_n}^0(t)\right|_\infty \ .$$

Due to $2 \cdot I_{T_0}^{u_n}(\mathbf{u}_n) = \int_0^{T_0} |c_n(s)|_{-2}^2 \, ds \to 0$ one further has $\sup_{t\in[0,T_0]} |\mathbf{v}_{c_n}^0(t)|_\infty \to 0$, see for example [39, Theorem 42.12]. On the other hand, the deterministic orbits leave $B_\mu(\mathcal{F})$ during $[0,T_0]$, i.e., one has $\sup_{t\in[0,T_0]} |S_\lambda(t)u_n - \mathbf{u}_n(t)|_\infty \geq \mu/2$, which is a contradiction.

In order to complete the proof of the lemma, notice that due to (50) a path in $K_{T_0}^{u_0}(2K)$ leaves the set $B_{\mu/2}(\mathcal{F})$ during the time interval $[0,T_0]$, and we have the estimate

$$\mathbb{P}_{u_0}\left(\tau\left(\sigma, \ \mathcal{F}^c\right) > T_0\right) \ \leq \ \mathbb{P}_{u_0}\left(\mathrm{dist}_{\mathscr{C}_T}\left(u_\sigma, \ K_{T_0}^{u_0}(2K)\right) \geq \frac{\mu}{2}\right) \ .$$

Now (48) follows from (38) with $\delta = \mu/2$ and $\gamma = K$. Furthermore, for (49) we use (48) and the strong Markov property to obtain

$$\mathbb{P}_{u_0}\left(\tau\left(\sigma, \ \mathcal{F}^c\right) > NT_0\right) \ \leq \ \left(e^{-K/\sigma^2}\right)^N \ ,$$

and by choosing $N > M/K$ and $T = NT_0$ the desired estimate follows. $\diamond$

Lemma 4.5 *If, in the situation of Theorem 4.2, one has $\mathcal{F} \subset B_{2\rho}(0)$, and if for a certain point $u_1 \in B_{2\rho}(0)$ one has*

$$\inf_{u\in B_{2\mu}(\mathcal{F})} E_\lambda(u) \ > \ K \ > \ E_\lambda(u_1)$$

for some $\mu > 0$ such that dist$(u_1,\mathcal{F}) > 3\mu$, then for sufficiently large $R > 2\rho$ and arbitrary $T > 0$ there exists a $\sigma_0 > 0$ such that

$$\mathbb{P}_{u_1}\left(\tau\left(\sigma, \ B_{4\mu/3}(\mathcal{F})\right) \leq T\right) \ \leq \ e^{-(2K-2E_\lambda(u_1))/\sigma^2} \ , \tag{51}$$

for all $\sigma < \sigma_0$. Moreover, for sufficiently small $\nu > 0$ one has

$$\mathbb{P}_{u_0}\left(\tau\left(\sigma, \ B_\mu(\mathcal{F})\right) \leq T\right) \ \leq \ e^{-(2K-2E_\lambda(u_1))/\sigma^2} \ , \tag{52}$$

for all $u_0 \in B_\nu(u_1)$ and $0 < \sigma < \sigma_0$.

Proof: To begin with, choose $R > 2\rho$ sufficiently large such that $\inf_{u \in \Gamma(R)} E_\lambda(u) > K$, and that $B_{2\mu}(\mathcal{F}) \subset B_R$. Then as in the proof of Lemma 4.1 one can estimate the action of a path $\mathbf{u}$ starting in u_1 and hitting the neighborhood $B_{5\mu/3}(\mathcal{F})$ before time T by

$$I_T^{u_1}(\mathbf{u}) \ > \ 2K - 2E_\lambda(u_1) + \gamma \ ,$$

for some $\gamma > 0$. Now (51) follows from (38) with $\delta = \mu/3$ and $r = 2K - 2E_\lambda(u_1) + \gamma$. From (23) we deduce that for $\nu < e^{-MT}\mu/3$ one has

$$\sup_{t \in [0,T]} |u_\sigma^{u_0}(t) - u_\sigma^{u_1}(t)|_\infty \ < \ \frac{\mu}{3} \ ,$$

and consequently (52) follows from (51). $\diamond$

Proof of Theorem 4.2: Now we have all the necessary tools for the proof of our main result. To begin with, choose $b > 0$ small enough such that

$$2\tilde{\Delta} E_\lambda + kb \ \leq \ 2\Delta E_\lambda - \xi \ ,$$

for some $\xi > 0$. Then Lemma 4.3 implies

$$\mathbb{P}_{u_0}\left(Z_1 \in \partial\mathcal{D}(\kappa,\rho)\right) \ \geq \ e^{-(2\Delta E_\lambda - \xi)/\sigma^2} \ .$$

For the second step we provide the missing estimates for the probabilities of the events A_0 and A_1 defined in (47). First of all, consider A_0 and define

$$\mathcal{F} = \mathcal{D}(\kappa,\rho) \setminus \left(B_d \cup B_a\left(h^0\right)\right) \ .$$

Without loss of generality one can assume that $\delta > 0$ is sufficiently small such that

$$\operatorname{dist}\left(s^1_{\pm,\lambda},\ \mathcal{E}^0_\lambda \cup \left\{h^0\right\}\right) \ > \ \delta \ , \qquad \text{with} \qquad \mathcal{E}^0_\lambda = \mathcal{E}_\lambda \cap \partial\mathcal{B}$$

as in the proof of Lemma 2.9. Then $B_\mu(\mathcal{F}) \cap (\mathcal{E}^0_\lambda \cup \{h^0\}) = \emptyset$ for $0 < \mu < \min\{a, d, \kappa\}$. Under these conditions, Lemma 2.10 furnishes the existence of a time $T_0 > 0$ such that all deterministic orbits starting in $\mathcal{F}$ leave the set $B_\mu(\mathcal{F})$ in time less than or equal to T_0, and we can apply Lemma 4.4. This implies that for arbitrary $M > 0$ there exist constants $T > 0$ and $\sigma_0 > 0$ such that

$$\mathbb{P}_{u_0}\left(\tau\left(\sigma,\ \mathcal{F}^c\right) > T\right) \ \leq \ e^{-M/\sigma^2} \ ,$$

for all $u_0 \in \mathcal{F}$ and $0 < \sigma < \sigma_0$. In particular, the choice $M = 2\Delta E_\lambda$ leads to

$$\mathbb{P}_{u_0}(A_0) \ \leq \ \mathbb{P}_{u_0}\left(\tau\left(\sigma,\ \mathcal{F}^c\right) > T\right) \ \leq \ e^{-2\Delta E_\lambda/\sigma^2} \ ,$$

for all $u_0 \in \Gamma(b)$ and $0 < \sigma < \sigma_0$.

It remains to determine an estimate for the probability of A_1. Due to (45) and (46) one can apply Lemma 4.5 with $\mathcal{F} = \partial\mathcal{D}(\kappa, \rho) \setminus B_\delta(s^1_{\pm,\lambda})$, $u_1 = h^0$, $K = E_\lambda(s^1_{\pm,\lambda})$, as well as $\mu = d$. Therefore, there exists a $\sigma_0 > 0$ such that for sufficiently small $\nu > 0$ one has

$$\mathbb{P}_{u_0}(A_1) \;\leq\; \mathbb{P}_{u_0}(\tau_d \leq T) \;\leq\; e^{-2\Delta E_\lambda/\sigma^2} \;,$$

for all $u_0 \in B_\nu(h^0)$ and $0 < \sigma < \sigma_0$. If necessary, one can reduce $b > 0$ such that this estimate holds for all $u_0 \in \Gamma(b)$.

After these preparations, and recalling that one only has to bound the probability in (44), the estimates for $\mathbb{P}_{u_0}(A_0)$ and $\mathbb{P}_{u_0}(A_1)$ finally yield

$$\mathbb{P}_{u_0}\left(Z_1 \in \partial\mathcal{D}(\kappa, \rho) \setminus B_\delta\left(s^1_{\pm,\lambda}\right)\right) \;\leq\; 2e^{-2\Delta E_\lambda/\sigma^2} \;,$$

uniformly in $u_0 \in \Gamma(a)$ for all sufficiently small $\sigma > 0$, and (43) can be estimated by

$$\mathbb{P}_{u_0}\left(u_\sigma(\tau) \notin B_\delta\left(s^1_{\pm,\lambda}\right)\right) \;\leq\; 2e^{-\xi/\sigma^2} \;.$$

Since ξ is positive this probability converges to zero as $\sigma \to 0$, which completes the proof for the case $u_0 \in \Gamma(a)$. For general $u_0 \in B_c(h_0)$ note that

$$\begin{aligned} \mathbb{P}_{u_0}\left(u_\sigma(\tau) \notin B_\delta\left(s^1_{\pm,\lambda}\right)\right) \;\leq\;& \mathbb{P}_{u_0}\left(u_\sigma(\tilde\tau) \in \partial\mathcal{D}(\kappa, \rho)\right) \\ &+ \mathbb{P}_{u_0}\left(u_\sigma(\tilde\tau) \in \Gamma(a) \;\text{ and }\; u_\sigma(\tau) \notin B_\delta\left(s^1_{\pm,\lambda}\right)\right) \;. \end{aligned}$$

Using Lemmas 4.4 and 4.5 as above, it is then easy to see that the first probability converges to zero for $\sigma \to 0$. As for the second probability, one can employ the strong Markov property to obtain

$$\begin{aligned} &\mathbb{P}_{u_0}\left(u_\sigma(\tilde\tau) \in \Gamma(a) \;\text{ and }\; u_\sigma(\tau) \notin B_\delta\left(s^1_{\pm,\lambda}\right)\right) \\ &\quad = \mathbb{E}_{u_0}\left(u_\sigma(\tilde\tau) \in \Gamma(a)\,; \quad \mathbb{P}_{u_\sigma(\tilde\tau)}\left(u_\sigma(\tau) \notin B_\delta\left(s^1_{\pm,\lambda}\right)\right)\right) \;, \end{aligned}$$

and the probability inside the expectation can be estimated as above. This completes the proof of the theorem. ◇

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