Stabilisation by rough noise for an epitaxial growth model

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Symbols and Notations

Symbols

$a \lesssim b$	There exists a constant $C > 0$ such that $a \leqslant Cb$.
$a \gtrsim b$	There exists a constant $C > 0$ such that $a \geqslant Cb$.
$a \sim b$	If both $a \lesssim b$ and $a \gtrsim b$ hold.
$a_\varepsilon \ll b_\varepsilon$	$a_\varepsilon \in o(b_\varepsilon)$, i.e. $\frac{a_\varepsilon}{b_\varepsilon} \to 0$ as $\varepsilon \to 0$.
$x \wedge y$	$\min\{x, y\}$.
$x \vee y$	$\max\{x, y\}$.
$\lfloor z \rfloor$	Largest integer y with $y \leqslant z$.
$x \cdot y$	$\sum_{i=1}^{d} x_i y_i$ $x, y \in \mathbb{R}^d$, $d \in \mathbb{N}$.
$\|x\|$	$\sqrt{x \cdot x}$ for $x \in \mathbb{R}^d$, $d \in \mathbb{N}$.
	(Note that ∇ and f are clear indicators that the expression is an element of $\mathbb{R}^2$.)
$\mathrm{d}x$	$\mathrm{d}\lambda^d(x)$, for $x \in \mathbb{R}^d$ and the Lebesgue measure $\lambda^d(\cdot)$ on the d-dimensional Lebesgue space, $d \in \mathbb{N}$.
$\mathbf{0}$	$(0, \cdots, 0) \in \mathbb{R}^d$, $d \in \mathbb{N}$.
C	Positive constant that may vary from line to line but depend only on the parameters indicated in their indices.

Notations

$\mathbb{T}^2 := \mathbb{R}^2/L\mathbb{Z}^2$	Two-dimensional torus.
$\mathcal{Z} := \mathbb{Z}^2 \setminus \{(0,0)\}$	Index set for Fourier series.
$f : \mathbb{R}^2 \to \mathbb{R}^2$	$f(x) := \frac{x}{1+\lvert x\rvert^2}$
$\mathfrak{F} : \mathcal{H}^1 \to \mathcal{H}^{-1}$	$\mathfrak{F}(u) := \nabla \cdot f(\nabla u) = \nabla \cdot \frac{\nabla u}{1+\lvert\nabla u\rvert^2}$
A	$A := -\Delta^2 := -\Delta \circ \Delta :$ Bilaplace operator
A_δ	$A_\delta := -\delta\Delta^2 - \Delta :$ Linear operator in the context of linearisation
$L^p := L^p(\mathbb{T}^2)$	Standard Lebesgue space
$\mathcal{H}^\alpha := \mathcal{H}^\alpha(\mathbb{T}^2)$	Fractional Sobolev space in a moving frame: $\mathcal{H}^\alpha := \left\{ u \in L^2(\mathbb{T}^2) \mid (-\Delta)^{\frac{\alpha}{2}} u_k^2 < \infty,\ \int_{\mathbb{T}^2} u(x)\,\mathrm{d}x = 0 \right\}.$ (Unless referenced otherwise, we refer to function spaces from $\mathbb{T}^2$ to $\mathbb{R}$.)
$u \in \mathcal{H}^{\alpha-}$	For every $\gamma > 0$ we have $u \in \mathcal{H}^{\alpha-\gamma}$.
$\mathcal{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2.
$\mathcal{N}(\mathbf{0}, Q)$	Normal distribution with mean $\mathbf{0} \in \mathbb{R}^d$ and covariance matrix $Q \in \mathbb{R}^{d\times d}$, $d \in \mathbb{N}$.
$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P}\right)$	Filtered probability space.
$g(t) \leqslant t^{\gamma+}$	For each $\varepsilon > 0$ we have $g(t) \leqslant t^{\gamma+\varepsilon}$.
$g(t) \leqslant t^{\gamma-}$	For each $\varepsilon > 0$ we have $g(t) \leqslant t^{\gamma-\varepsilon}$.

1. Introduction

This thesis builds upon results from our joint works [14, 15] as well as two forthcoming papers [13, 16]. Parts of this introduction are adapted from [14], with additional extensions and revisions for the present thesis.

The study of epitaxial thin-film growth plays a central role in understanding the formation and stability of crystalline surfaces. A key physical mechanism is the *Schwoebel barrier* [72], which hinders atoms from crossing step edges, thereby influencing the morphology of the growing surface. This barrier induces an effective uphill current and is responsible for the formation of mounds and other growth patterns. A phenomenological model for such growth processes is given by the stochastic partial differential equation (SPDE)

$$\partial_t u = -\delta \Delta^2 u - \nabla \cdot \frac{\nabla u}{1 + |\nabla u|^2} + \sigma \, \xi, \tag{1.1}$$

where $u(t, x)$ represents the surface height over $x \in \mathbb{T}^d$ at time $t > 0$. In this model, $\delta > 0$ is a surface diffusion parameter, $\sigma > 0$ is the noise intensity, also denoted as diffusion coefficient, and $\xi = \partial_t W$ denotes a space-time white noise, formally given by the time derivative of a cylindrical Wiener process W. The stochastic forcing models thermal fluctuations of the incoming particles.
In physics literature the equation is usually studied on the full space $\mathbb{R}^d$ or, as in this work, under periodic boundary conditions on the torus $\mathbb{T}^d$ with spatial dimension $d \in \{1, 2\}$. The physically relevant case is $d = 2$. Since the model describes growth in a moving frame, we impose the mean-zero condition

$$\int_{\mathbb{T}^d} u(t, x) \, \mathrm{d}x = 0, \qquad \text{for all } t \geqslant 0$$

for the surface height function u. Equation (1.1) has its origins in the physics literature, where it was first introduced by Hunt, Sander, et. al. for $d = 1$ in [48] and subsequently for $d = 2$ in [49]. For general surveys on surface growth, we refer to [5, 55, 56, 57].
Molecular Beam Epitaxy (MBE) is a central technique for depositing atoms from vapor onto crystalline surfaces. Atoms diffuse across terraces and settle into energetically favourable sites, making molecular beam epitaxy crucial for the fabrication of quantum wires and other nanostructures.

A continuum model for the one-dimensional molecular beam epitaxy growth incorporating the Schwoebel barrier was proposed in [48]:

$$\partial_t u = -\partial_x^4 u - \partial_x \left(\frac{\partial_x u}{1 + (\partial_x u)^2} \right). \tag{1.2}$$

This equation replaces the KPZ model earlier studied for similar phenomena. However, (1.2) admits a free energy functional

$$E = \frac{1}{2} \int_{\mathbb{T}} |\Delta u|^2 - \ln(1 + |\nabla u|^2) \, \mathrm{d}x,$$

indicating the existence of a ground state toward which the surface evolves. Experimental observations in [48] revealed the formation of Super-Large Unstable Growths (SLUGs), that is, mounds with nearly constant slopes and lateral sizes between 1 and 10 micrometres. These structures coarsen over time, with surface roughness increasing mainly due to larger spacing between SLUGs, while their slopes remain nearly constant or grow only slowly under given conditions.

The two-dimensional generalisation of the molecular beam epitaxy model was studied in [49]. In two spatial dimensions, a planar surface is typically unstable and develops mounds with slopes exceeding a critical threshold. By contrast, vicinal surfaces with sufficiently large miscut angles can remain stable up to small statistical fluctuations. Surface roughness is quantified by the *root-mean-square width*

$$w(t) := \left(\fint_{\mathbb{T}^2} \big(u(x,t) - \bar{u}(t)\big)^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

which typically grows in time according to a power law $w(t) \sim t^{\beta}$, with positive exponent $\beta < 1$. At higher substrate temperatures, films display only weak roughening, and under optimal conditions, step-flow growth is observed. As emphasised in [74], step-flow descriptions are not suitable for the early stages of molecular beam epitaxy, where a directed surface current dominates. This regime arises due to nearly complete sticking of incoming atoms, negligible desorption, and the absence of overhangs or vacancies.

While the stochastic equation (1.1) has received little direct mathematical attention so far, its deterministic counterpart (obtained by setting $\sigma = 0$) and related modifications have been studied extensively in recent years. The deterministic model reads

$$\partial_t u = -\delta \Delta^2 u - \nabla \cdot \frac{\nabla u}{1 + |\nabla u|^2}.$$

Due to the global Lipschitz continuity of the nonlinearity f, the deterministic equation is well posed. Existence and uniqueness results in more general settings can be found in [1]. Well-posedness and spectral Galerkin methods have been analysed in [61, 62, 63], where the dynamics of phase separation were investigated both numerically and via asymptotic scaling. Error estimates for different numerical schemes have been addressed in numerous works. We mention, for instance, [26, 60, 70, 76], where also more general models of the type (1.1) were considered. The long time dynamics of the deterministic version of (1.1) have been studied in detail in [2, 32, 37]. The solution generates a dissipative dynamical system that possesses a compact global attractor governing the long-time behaviour of the dynamics. In fact, the equation can be interpreted as a gradient flow with respect to the energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{T}^d} |\Delta u|^2 - \ln \left(1 + |\nabla u|^2\right) \mathrm{d}x,$$

whose variational derivative is

$$\frac{\delta E}{\delta u} = \Delta^2 u + \nabla \cdot \frac{\nabla u}{1 + |\nabla u|^2}.$$

Such gradient-flow structures are classical in phase-separation models and go back to the seminal work of Cahn and Hilliard [24, 25]. See also [17] for a stochastic variant and related energy formulations. By the Poincaré inequality, E is bounded from below. It follows that, on bounded domains, the attractor consists solely of equilibria and heteroclinic orbits connecting them.
For small $\delta > 0$, the flat surface $u = 0$ ceases to minimise the energy, and instead an (up to translation) unique, nontrivial minimiser emerges. However, the detailed structure of the attractor and the global energy landscape remain far from being fully understood. This also raises interesting open questions for the corresponding stochastic model, in particular concerning the stochastic dynamics of surface patterns, which have not yet been systematically explored.

Over the past two decades, substantial progress has been made in establishing the well-posedness of irregular SPDEs forced by rough noise. The pioneering work of Da Prato and Debussche [28, 29] introduced a strategy based on splitting the solution into a rough stochastic convolution and a more regular remainder. Later developments include the theory of paracontrolled distributions by Gubinelli, Imkeller, and Perkowski [38] and the celebrated theory of regularity structures by Hairer [40, 41, 42]. These approaches address equations that are otherwise ill-posed because of interactions between noise and nonlinear terms.
Most results in this area, however, focus on models with polynomial nonlinearities, such as Burgers-type equations, the KPZ equation, or stochastic quantisation

models. Other examples include FitzHugh–Nagumo SPDEs [7], the Φ^4_3 model [43], and the dynamical sine-Gordon equation [45]. Equation (1.1), in contrast, features a nonlinear drift that does not fit directly into this polynomial framework. A natural point of comparison for the epitaxial growth model is the *Kardar–Parisi–Zhang (KPZ) equation*, a paradigmatic model for stochastic surface growth, given by

$$\partial_t h = \nu \Delta h + \frac{\lambda}{2} \big(\nabla h\big)^2 + \sigma \xi,$$

where ν is the diffusivity, λ the growth rate, σ the noise amplitude, and ξ denotes space-time white noise. The linear term $\nu\Delta h$ provides surface smoothing, the quadratic nonlinearity $\frac{\lambda}{2}(\nabla h)^2$ drives instability, and the noise introduces stochastic fluctuations.
In the one-dimensional case, the KPZ equation is well understood due to the groundbreaking work of Martin Hairer and has become a universal model for stochastic interface growth (see [40]). However, in higher dimensions ($d \geqslant 2$) the nonlinearity becomes too singular, and the model must be interpreted in a renormalised sense. One introduces diverging constants $C_\varepsilon \sim \nu^{-2}\, \mathbb{E}[\xi_\varepsilon^2(x)]$, yielding the renormalised equation

$$\partial_t h_\varepsilon = \nu \Delta h_\varepsilon + \frac{\lambda_\varepsilon}{2} \left[(\nabla h_\varepsilon)^2 - C_\varepsilon \right] + \sigma \xi_\varepsilon,$$

where $\lambda_\varepsilon = \sqrt{\varepsilon}$ and ξ_ε is a spatially regularised noise for some $\varepsilon > 0$ (cf. [38, 41, 42]). In one dimension, the Cole–Hopf transformation,

$$\theta_t = e^{-h_t},$$

transforms the KPZ equation into the stochastic heat equation (SHE)

$$\mathrm{d}\theta_t = \frac{1}{2}\Delta\theta_t \,\mathrm{d}t - \theta_t \,\mathrm{d}W_t,$$

which allows for rigorous analysis. In higher dimensions, no comparable transformations exist, and the singularity of the nonlinearity becomes the central challenge.

The main mathematical difficulty of (1.1) in the physically relevant case $d = 2$ arises from the lack of regularity of the solution. With additive space–time white noise, the solution is expected to be at most Hölder continuous with exponent strictly below one, so that the gradient ∇u cannot be interpreted as a function. Thus, direct approaches such as fixed-point arguments or Galerkin methods are not applicable.

Nevertheless, the regularity of the noise is only slightly too rough to define the nonlinearity in a classical way. This makes it possible to treat the problem by a straightforward Fourier analysis of a regularised version of the equation, following the idea of Da Prato and Debussche [28, 29], without invoking the advanced frameworks of regularity structures [41] or paracontrolled distributions [38]. In the context of surface growth, similar approaches were used in [18, 19], where the nonlinearity is quadratic and differs from (1.1), although the linearisation around the flat surface remains the same.

In contrast to the KPZ equation, the epitaxial thin–film model (1.1) involves a globally bounded and Lipschitz-continuous nonlinearity

$$\mathfrak{F}(u) \coloneqq \nabla \cdot f(\nabla u) \coloneqq \nabla \cdot \frac{\nabla u}{1 + |\nabla u|^2}, \tag{1.3}$$

which even vanishes as $|\nabla u| \to \infty$. Consequently, the equation remains analytically tractable without any renormalisation. A suitable regularisation of the space–time white noise ensures that the nonlinearity is well defined, and in the limit of vanishing regularisation the nonlinear term becomes self-regularising, naturally preventing divergences and constraining the evolution.

Stabilisation of stochastic systems by noise is a broad research area, both for finite-dimensional SDEs and infinite-dimensional SPDEs. Our focus here is on stabilisation caused by the spatial roughness of the driving noise. This phenomenon is closely related to the notion of *triviality of SPDEs*, introduced in the work of Hairer, Ryser, and Weber [44]. In this article, the authors studied the Allen–Cahn (or Φ^4-) equation in two spatial dimensions with periodic boundary conditions, driven by space-time white noise. Since the solution is only a distribution, the cubic nonlinearity cannot be defined without renormalisation. A common approach is to introduce regularised noise with covariance operator $\mathcal{Q}_\varepsilon$, e.g. via a Fourier cutoff, together with a noise strength $\sigma_\varepsilon > 0$, so that the approximating equation reads

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + f(u_\varepsilon) + \sigma_\varepsilon \mathcal{Q}_\varepsilon^{1/2}\, \partial_t W,$$

with $f(u) = u - u^3$. Here, the linear part u generates an instability, while the cubic term $-u^3$ has a stabilising effect.
Depending on the scaling of σ_ε, three regimes occur. For small noise strength, the noise vanishes in the limit. For intermediate scaling, the limit yields a deterministic Allen–Cahn equation with an additional stabilising linear term $-Cu$ for some constant $C > 0$. Most strikingly, if σ_ε is kept fixed, the interaction of the noise with the cubic term produces a diverging linear counterterm $-C_\varepsilon u_\varepsilon$ with $C_\varepsilon \to \infty$ as $\varepsilon \to 0$. As a consequence, u_ε converges to 0 in negative Sobolev or Besov norms. The solution becomes *trivial*, as the rough noise destroys the instability mechanism of the nonlinearity.

Similar results were obtained in other contexts. For example, Oh, Okamoto, and Robert [66] established triviality for the stochastic nonlinear wave equation. In [12], such stabilisation effects were analysed in a more general setting where both the linear operator and the nonlinearity vary with the regularisation parameter. Hairer and Weber [46] investigated how the balance between vanishing noise strength and decreasing regularity of the noise leads to different limiting behaviours.
The model (1.1) shows a related mechanism. Although the nonlinearity is structurally different from the cubic Allen–Cahn term, the roughness of the driving noise suppresses nonlinear effects in the limit of vanishing regularisation. Unlike in [44], no renormalisation is required, but the outcome is similar. The nonlinear interactions vanish, and the limiting equation is purely linear, preventing the growth of surface instabilities.

The Cahn–Hilliard equation is a prototypical fourth-order nonlinear PDE that originates from the seminal works of Cahn and Hilliard [24, 25]. It models phase separation in binary alloys by means of a free energy functional that combines bulk and interfacial contributions. In its stochastic extension, random fluctuations are added to capture microscopic uncertainties and thermal noise. This leads to the *stochastic Cahn–Hilliard equation*, a widely studied model in both analysis and computation [4, 27, 30, 71].
In the absence of noise, the equation describes the dynamics of a concentration field that evolves to form distinct phases over time. The deterministic Cahn–Hilliard equation has been extensively studied, both analytically and numerically, and provides the foundation for its stochastic generalisations.

The inclusion of noise reflects thermal fluctuations in real materials. This modification not only improves the physical realism of the model, but also introduces significant mathematical challenges due to the interaction between rough stochastic forcing and the fourth-order operator.

The numerical analysis of SPDEs of Cahn–Hilliard and Allen–Cahn type has been the subject of extensive research. Early contributions focused on stochastic Burgers equations [8, 10], while subsequent works investigated stochastic Allen–Cahn equations and established various convergence results [6, 20, 23, 36, 65]. More general numerical frameworks for SPDEs can be found in [50, 52, 53, 64].

These contributions demonstrate both the breadth of SPDE models studied in the literature and the importance of developing accurate and robust numerical methods. This thesis continues this line of research by applying numerical methods to approximate the regularised version of (1.1). In particular, we have studied a spectral Galerkin scheme in space and an Euler scheme in time in [15] and our forthcoming works [13, 16], which are discussed in more detail in Chapters 5 and 6.

The structure of the thesis is outlined below.

In Chapter 2, we provide an introduction to essential results from stochastic processes, establishing the framework required for the analysis of the stochastic convolution and the space-time white noise. In Chapter 3, we rigorously establish that, after a suitable regularisation of the space-time white noise, the nonlinearity in the epitaxial thin-film growth model vanishes in the limit. In particular, we prove convergence of the solution of the regularised equation to the solution of the linear limiting equation. This shows that even arbitrarily small noise is sufficient to suppress nonlinear effects. In Chapter 4, we define a decomposition approach for the linearisation of (1.1) and derive growth results up to a stopping time. In Chapter 5, we present the spectral Galerkin method for the Euler scheme and establish convergence rates for numerical discretisation in time and space, including an error analysis. To handle the singularities caused by rough noise, we project the equation onto a finite-dimensional subspace of the Fourier space. This spectral Galerkin scheme allows us to derive uniform estimates and to establish convergence to the mild solution of the limiting stochastic partial differential equation. Furthermore, we present numerical experiments that illustrate the theoretical findings. In particular, the simulations confirm the vanishing of nonlinear effects in the limit and demonstrate the stabilising role of noise in the surface growth model. In Chapter 6, we propose an alternative approach for the numerical approximation of a Cahn–Hilliard-type SPDE, where more advanced analytical techniques, such as the stochastic sewing lemma, are applicable. In Chapter 7, we provide a brief summary of the main results of this thesis. Finally, Appendix A contains essential definitions and theorems that are used throughout this thesis.

2. Fundamentals in Stochastics

Throughout this thesis, we assume the setting of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions of completeness and right-continuity. All stochastic processes are assumed to be adapted to this filtration.

2.1. Gaussian Random Variables

The results presented in this subsection and Section 2.2 are strongly inspired by and thus discussed in greater detail in [27].

Let $\mathcal{H}$ be a separable real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and orthonormal basis $(e_k)_{k\in\mathbb{N}}$.

Definition 2.1. *A real-valued random variable X is called Gaussian if there exist $a, \sigma \in \mathbb{R}$ such that its characteristic function is for every $\lambda \in \mathbb{R}$ given by*

$$\varphi_X(\lambda) = \mathbb{E}\left[e^{i\lambda X}\right] = e^{i\lambda a - \frac{1}{2}\sigma^2\lambda^2}.$$

Definition 2.2. *An $\mathcal{H}$-valued random variable is called Gaussian, if and only if for all $h \in \mathcal{H}$ the real-valued random variable $\langle X, h \rangle$ is Gaussian.*

Theorem 2.3 (Fernique's theorem)**.** *Let E be a separable Banach space, $\mathcal{B}(E)$ be its Borel σ-algebra and $\|.\|_E$ be its norm. Let μ be an arbitrary symmetric Gaussian measure on $(E, \mathcal{B}(E))$ and $\lambda > 0$, $r > 0$ such that*

$$\ln\left(\frac{1 - \mu\left(\overline{B_r(0)}\right)}{\mu\left(\overline{B_r(0)}\right)}\right) + 32\lambda r^2 \leqslant -1.$$

Then we have

$$\int_E e^{\lambda\|x\|_E^2}\,\mathrm{d}\mu(x) \leqslant e^{16\lambda r^2} + \frac{e^2}{e^2-1}.$$

As an immediate consequence of Theorem 2.3 we obtain the following corollary, which is the formulation relied upon this work.

Corollary 2.4. *Let X be a centered Gaussian random variable in the Hilbert-space $\mathcal{H}$. Then there exists an $\lambda > 0$ such that*

$$\mathbb{E}\left[e^{\lambda \|x\|_E^2}\right] < \infty.$$

By Theorem 2.3 the following corollary is a direct conclusion:

Corollary 2.5 (Moment bounds for Gaussian processes)**.** *For all $p > 0$ there exists a constant $C_p > 0$ only depending on p, such that for any centered Gaussian random variable X in $\mathcal{H}$ it holds that*

$$\mathbb{E}\,\|X\|_{\mathcal{H}}^{2p} \leqslant C_p \left(\mathbb{E}\,\|X\|_{\mathcal{H}}^2\right)^p.$$

In particular, moments of centered Gaussians are finite, i.e.

$$\mathbb{E}\,\|X\|_{\mathcal{H}}^p < \infty.$$

2.2. Cylindrical Wiener Process and Space-Time White Noise

In this subsection, we state the basic properties of *space-time white noise* ξ.
Let $Q \in L(\mathcal{H}, \mathcal{H})$ be a nonnegative and self-adjoint linear operator, i.e. the inequality $\langle Qx, x\rangle \geqslant 0$ holds for each $x \in \mathcal{H}$.

Definition 2.6 (Q-Wiener process)**.** *Assume that $Q \in L(\mathcal{H}, \mathcal{H})$ is a trace class operator, i.e. $\operatorname{tr}(Q) := \sum_{k=1}^{\infty} \langle Qe_k, e_k\rangle < \infty$ holds for every orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$. An $\mathcal{H}$-valued stochastic process $(W(t))_{t\geqslant 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called* (Standard) Q-Wiener process *if it fulfills the following properties:*

- *$W_0 = 0$ holds $\mathbb{P}$-almost surely.*
- *W has $\mathbb{P}$-almost surely continuous trajectories.*
- *W has independent increments, i.e. the random variables*

$$W(t_1) - W(0), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$$

  are independent for all $0 \leqslant t_1 < \cdots < t_n < \infty$, $n \in \mathbb{N}$.

- *The increments are Gaussian in $\mathcal{H}$*

$$(W(t) - W(s)) \sim \mathcal{N}(0, (t-s)Q)$$

  for all $0 \leqslant s \leqslant t < \infty$.

Proposition 2.7 (Representation of the Q-Wiener process)**.** *Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, consisting of eigenvectors of Q with corresponding eigenvalues $(\alpha_k^2)_{k\in\mathbb{N}}$. Then an $\mathcal{H}$-valued stochastic process $(W(t))_{t\in[0,T]}$ is a Q-Wiener process if and only if the series*

$$W(t) = \sum_{k\in\mathbb{N}} \alpha_k \beta_k(t) e_k,$$

converges in $L^2(\Omega, \mathcal{H})$ for each $t \geqslant 0$ and defines a process with continuous paths, where $(\beta_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of real-valued Wiener processes.

Definition 2.8 (Cylindrical Wiener process)**.** *Let $Q \in L(\mathcal{H}, \mathcal{H})$ be a bounded operator (not necessarily of trace class) with an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$, consisting of eigenfunctions of Q and a bounded sequence of eigenvalues $(\alpha_k^2)_{k\in\mathbb{N}}$. A family $W = (W(t))$ is called a* cylindrical Q-Wiener process *if for each $h \in \mathcal{H}$*

$$\langle W(t), h\rangle := \sum_{k\in\mathbb{N}} \alpha_k \beta_k(t) \langle e_k, h\rangle$$

where $(\beta_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of real-valued Wiener processes, defines a centered real-valued Gaussian process with covariance

$$\mathbb{E}\left[\langle W(t), h\rangle \langle W(s), g\rangle\right] = (t \wedge s)\langle Qh, g\rangle.$$

Remark 2.9. *The cylindrical Wiener process in Definition 2.8 does not necessarily converge in $\mathcal{H}$, since*

$$\begin{aligned}\mathbb{E}\left[\|W(t)\|_{\mathcal{H}}^2\right] &= \mathbb{E}\left[\sum_{k\in\mathbb{N}} \beta_k^2(t)\alpha_k^2\right] = \sum_{k\in\mathbb{N}} \mathbb{E}\left[\beta_k^2(t)\right]\alpha_k^2 \\ &= t\sum_{k\in\mathbb{N}} \alpha_k^2 = t\sum_{k\in\mathbb{N}} \langle Qe_k, e_k\rangle = t\,\mathrm{tr}(Q)\end{aligned}$$

is finite for each $t \geqslant 0$ if and only if Q is a trace class operator. By Fernique's theorem (cf. Theorem 2.3), this moment must be finite.

From now on, we use the orthonormal basis $(e_k)_{k\in\mathcal{Z}}$ of the Hilbert space $L^2(\mathbb{T}^2, \mathbb{R})$ as defined in Appendix A.1. We consider the solution of (1.1) in a moving frame (cf. Assumption A.3) to utilise the characterisation of fractional Sobolev spaces via fractional powers of $(-\Delta)$, as defined in Definition A.4. We use the index set $\mathcal{Z} := \mathbb{Z}^2 \setminus \{(0,0)\}$ since, in a moving frame, the zeroth Fourier mode of a function is zero. Space-time white noise can be formally seen as the time derivative of a Wiener process. However, since the sample paths of a Wiener process are only α-Hölder continuous for $\alpha < \frac{1}{2}$, its derivative does not exist in the classical sense.

Definition 2.10 (Space-time white noise)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A real-valued centered Gaussian random distribution ξ is called* (Gaussian) space-time white noise *on $[0,T] \times \mathbb{T}^2$ if, for all $\varphi, \psi \in L^2([0,T] \times \mathbb{T}^2)$, if the following holds:*

$$\mathbb{E}[\xi(\varphi)\xi(\psi)] = \langle \varphi, \psi \rangle_{L^2([0,T]\times\mathbb{T}^2)},$$

where $\xi(\varphi) \coloneqq \langle \xi, \varphi \rangle$ denotes the dual pairing between ξ and φ.
In particular, let

$$W(t,x) \coloneqq \sum_{k \in \mathcal{Z}} \beta_k(t) e_k(x),$$

$x \in \mathbb{T}^2$, $t \in [0,T]$, be a cylindrical Wiener process with the $L^2(\mathbb{T}^2)$-orthonormal basis $(e_k)_{k\in\mathcal{Z}}$, defined in Appendix A.1, and an i.i.d. sequence of real-valued Wiener processes $(\beta_k)_{k\in\mathcal{Z}}$. Then the distributional derivative in time $\xi \coloneqq \partial_t W$ is called space-time white noise*.*

Remark 2.11. *We call $\xi \coloneqq \partial_t W$* space-time white noise*, as we have*

$$\begin{aligned} \xi(\varphi) &= \int_0^T \langle \varphi(t,\cdot),\, \mathrm{d}W(t) \rangle_{L^2(\mathbb{T}^2)} \\ &= \sum_{k\in\mathcal{Z}} \int_0^T \langle \varphi(t,\cdot), e_k \rangle_{L^2(\mathbb{T}^2)} \,\mathrm{d}\beta_k(t). \end{aligned}$$

and therefore using the independence of the sequence $(\beta_k)_{k\in\mathcal{Z}}$, Itô's isometry, and Parseval's identity, we derive

$$\begin{aligned} \mathbb{E}\left[\xi(\varphi)\xi(\psi)\right] &= \sum_{k\in\mathcal{Z}} \int_0^T \langle \varphi(t,\cdot), e_k \rangle_{L^2(\mathbb{T}^2)} \langle \psi(t,\cdot), e_k \rangle_{L^2(\mathbb{T}^2)} \,\mathrm{d}t \\ &= \int_0^T \langle \varphi(t,\cdot), \psi(t,\cdot) \rangle_{L^2(\mathbb{T}^2)} \,\mathrm{d}t \\ &= \langle \varphi, \psi \rangle_{L^2([0,T]\times\mathbb{T}^2)}. \end{aligned}$$

2.3. Stochastic Convolution

For the linear operator $A \coloneqq -\Delta^2$ and its eigenvalues $-\mu_k^2$, as mentioned in (A.3), we define the stochastic convolution, for $x \in \mathbb{T}^2$ and $t \in [0,T]$, as

$$Z(t,x) \coloneqq \int_0^t e^{(t-s)A} \,\mathrm{d}W(s,x) = \sum_{k\in\mathcal{Z}} \int_0^t e^{-2(t-s)\mu_k^2} d\beta_k(s) e_k(x) \tag{2.1}$$

which is the mild solution of the linearised SPDE

$$\begin{cases} \partial_t u &= -\Delta^2 u + \partial_t W, \\ u_0 &= 0. \end{cases}$$

Since space-time white noise is modeled by a cylindrical Wiener process, it is highly irregular. As a result, the stochastic convolution has limited regularity.

Lemma 2.12. *The stochastic convolution Z has the regularity $C^0\big([0,T],\mathcal{H}^\alpha(\mathbb{T}^2)\big)$ if and only if $\alpha < 1$.*

Proof. As the stochastic convolution Z is a Gaussian random variable taking values in a separable Hilbert space, the finiteness of its second moment and the fact that it is almost surely an element of the Hilbert space $\mathcal{H}^\alpha(\mathbb{T}^2)$ are equivalent, which is a consequence of Fernique's theorem (see [27]). For $\alpha < 1$, the independence of the Wiener processes and the Itô isometry yields

$$\begin{aligned} \mathbb{E}\left[\|Z(t)\|^2_{\mathcal{H}^\alpha}\right] &= \mathbb{E}\left[\sum_{k\in\mathcal{Z}} \mu_k^\alpha \left(\int_0^t e^{-(t-s)\mu_k^2}\,\mathrm{d}\beta_k(s)\right)^2\right] \\ &= \sum_{k\in\mathcal{Z}} \mu_k^\alpha\, \frac{1-e^{-2t\mu_k^2}}{2\mu_k^2} < \infty \end{aligned} \tag{2.2}$$

for each $t \in [0,T]$. Furthermore, for $\alpha = 1$, we obtain the diverging series

$$\sum_{k\in\mathcal{Z}} \frac{1-e^{-2t\mu_k^2}}{2\mu_k} \geqslant \left(1-e^{-2t\mu_{(1,0)}^2}\right)\sum_{k\in\mathcal{Z}}\frac{1}{2\mu_k} = \infty$$

for each $t > 0$. This shows that $Z(t) \in \mathcal{H}^\alpha(\mathbb{T}^2)$ holds almost surely for each $t \in [0,T]$ if and only if $\alpha < 1$. □

In the context of studying (1.1), we require at least the regularity $Z(t) \in \mathcal{H}^1(\mathbb{T}^2)$ for each $t \in [0,T]$ so that the derivative ∇Z — and consequently ∇u as well as the nonlinearity $f(\nabla u)$ — is well-defined not only in the distributional sense, but also in a stronger function space. Therefore, we introduce a regularised stochastic convolution.

Definition 2.13 (Regularisation)**.** *Let $\alpha := \big(\alpha_k^{(\varepsilon)}\big)_{k\in\mathcal{Z}} \subset \mathcal{O}\big((|k|^{-\delta(\varepsilon)})_{k\in\mathcal{Z}}\big)$ be a uniformly bounded sequence for $\varepsilon > 0$, such that for all k, $\alpha_k^{(\varepsilon)} \to 1$ as $\varepsilon \to 0$, and $\delta(\varepsilon) > 0$ depending on ε. For technical reasons, given below, we also assume that the sequence $\big(\alpha_k^{(\varepsilon)}\big)_{k\in\mathcal{Z}}$ is radially symmetric, meaning it depends only on $|k|$. For $t \in [0,T]$ and $x \in \mathbb{T}^2$ we then define*

$$Z_\varepsilon(t,x) := \int_0^t e^{-(t-s)A}\,\mathrm{d}W_\varepsilon(s,x) := \sum_{k\in\mathcal{Z}} \alpha_k^{(\varepsilon)} \int_0^t e^{-(t-s)\mu_k^2}\, d\beta_k(s)\, e_k(x). \tag{2.3}$$

Remark 2.14. *We require the regularising sequence $\left(\alpha_k^{(\varepsilon)}\right)_{k\in\mathcal{Z}}$ to be radially symmetric in order to ensure that the covariance operator*

$$\begin{aligned}\operatorname{Cov}\left(\nabla Z_\varepsilon(t,x)\right) &= \mathbb{E}\left[\sum_{k\in\mathcal{Z}}\left(\left(\alpha_k^{(\varepsilon)}\right)\int_0^t e^{-(t-s)\mu_k^2}\,\mathrm{d}\beta_k(s)\right)^2 \nabla e_k(x)\nabla e_k(x)^T\right]\\ &= \sum_{k\in\mathcal{Z}}\left(\alpha_k^{(\varepsilon)}\right)^2 \mathbb{E}\left[\left(\int_0^t e^{-(t-s)\mu_k^2}\,\mathrm{d}\beta_k(s)\right)^2\right]\nabla e_k(x)\nabla e_k(x)^T\\ &= \sum_{k\in\mathcal{Z}}\frac{\left(\alpha_k^{(\varepsilon)}\right)^2}{2\mu_k^2}\left[1-e^{-2t\mu_k^2}\right]\nabla e_k(x)\nabla e_k(x)^T\end{aligned}$$

is independent of $x\in\mathbb{T}^2$ which we will proof in Section 3.3.

Applying the same calculations as in the proof of Lemma 2.12, we obtain the following

Corollary 2.15. *For each $\varepsilon>0$ and $t\in[0,T]$, we have*

$$\begin{aligned}\mathbb{E}\left[\|Z_\varepsilon(t)\|_{L^2}^2\right] &< \infty,\\ \mathbb{E}\left[\|\nabla Z_\varepsilon(t)\|_{L^2(\mathbb{T}^2,\mathbb{R}^2)}^2\right] &< \infty.\end{aligned}$$

Proof. For each $\varepsilon>0$ and $t\in[0,T]$ we immediately obtain by Itô's isometry

$$\mathbb{E}\left[\|Z_\varepsilon(t)\|_{L^2}^2\right] = \sum_{k\in\mathcal{Z}}\left[1-e^{-2t\mu_k^2}\right]\frac{\left(\alpha_k^{(\varepsilon)}\right)^2}{2\mu_k^2} \leqslant \frac{1}{2}\sum_{k\in\mathcal{Z}}\frac{\left(\alpha_k^{(\varepsilon)}\right)^2}{\mu_k^2} < \infty.$$

Furthermore, as in Lemma 2.12, by applying Itô's isometry, we also obtain

$$\begin{aligned}\mathbb{E}\left[\|\nabla Z_\varepsilon(t)\|_{L^2(\mathbb{T}^2,\mathbb{R}^2)}^2\right] &= \mathbb{E}\left[\|Z_\varepsilon(t)\|_{\mathcal{H}^1}^2\right]\\ &= \mathbb{E}\left[\sum_{k\in\mathcal{Z}}\mu_k\left(\alpha_k^{(\varepsilon)}\right)^2\left(\int_0^t e^{-(t-s)\mu_k^2}\,\mathrm{d}\beta_k(s)\right)^2\right]\\ &= \sum_{k\in\mathcal{Z}}\left(\alpha_k^{(\varepsilon)}\right)^2\left[1-e^{-2t\mu_k^2}\right]\frac{1}{2\mu_k}\\ &< \infty.\end{aligned}$$

This concludes the proof. ∎

Thereby, for all $\varepsilon > 0$, the stochastic convolution Z_ε possesses a well-defined spatial gradient. This property is essential for the application of Banach's fixed point theorem, as it ensures that the gradient of the stochastic convolution is well-defined. Recall that the Fourier series of $Z(t,x)$, defined in (2.1), and $Z_\varepsilon(t,x)$, defined in (2.3), are pointwise well-defined real-valued random variables for each $t \in [0,T]$ and $x \in \mathbb{T}^2$.

Lemma 2.16. *For every $x \in \mathbb{T}^2$ and $t \in [0,T]$ the series defining $Z(t,x)$ and $Z_\varepsilon(t,x)$ converge in $L^2(\Omega,\mathbb{R})$ and is a well-defined real-valued Gaussian random variable.*

Proof. We prove this statement only for Z, as the proof for Z_ε, with $\varepsilon > 0$, is completely analogous, as the sequence $\big(\alpha_k^{(\varepsilon)}\big)_{k\in\mathcal{Z}}$ is uniformly bounded. To show this, we truncate to a finite sum

$$P_n Z(t,x) \coloneqq \sum_{k\in B_{\mathcal{Z}}(0,n)} \int_0^t e^{-(t-\tau)\mu_k^2}\,\mathrm{d}\beta_k(\tau) e_k(x)$$

by using the restriction on the ball $B_{\mathcal{Z}}(0,n) \coloneqq \{y \in \mathcal{Z} : |y| \leqslant n\} \subset \mathcal{Z}$ with radius $n \in \mathbb{N}$. Now $P_n Z(t,x)$ is a well-defined real-valued Gaussian random variable. By Itô's isometry and since $\big(\beta_k\big)_{k\in\mathcal{Z}}$ is a sequence of independent processes, we obtain

$$\begin{aligned}
\mathbb{E}\left[|P_n Z(t,x)|^2\right] &= \mathbb{E}\left[\left|\sum_{k\in B_{\mathcal{Z}}(0,n)} \int_0^t e^{-(t-\tau)\mu_k^2}\,\mathrm{d}\beta_k(\tau) e_k(x)\right|^2\right] \\
&= \sum_{k\in B_{\mathcal{Z}}(0,n)} e_k^2(x) \frac{1}{2\mu_k^2}\left(1 - e^{-2t\mu_k^2}\right) \\
&\leqslant C \sum_{k\in B_{\mathcal{Z}}(0,n)} \frac{1}{\mu_k^2} < \infty
\end{aligned}$$

for each $t \in [0,T]$, $x \in \mathbb{T}^2$, $n \in \mathbb{N}$ and a constant $C > 0$. In a similar way we can bound for $n > m$ the difference

$$\begin{aligned}
\mathbb{E}\left[|P_n Z(t,x) - P_m Z(t,x)|^2\right] &\leqslant C \sum_{k\in B_{\mathcal{Z}}(0,n)\setminus B_{\mathcal{Z}}(0,m)} \frac{1}{\mu_k^2} \\
&\leqslant C \sum_{k\in\mathcal{Z},\, m<|k|\leqslant n} \frac{1}{\mu_k^2}
\end{aligned}$$

of two elements of this sequence and show that $(P_n Z(t,x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega,\mathbb{R})$ and therefore convergent. □

Our next goal is to provide bounds for Z, Z_ε, ∇Z_ε, and $Z - Z_\varepsilon$. Since these quantities are structurally similar, we first establish a general result that encompasses all of these cases.

Lemma 2.17. *Let $\left(X_k\right)_{k\in\mathcal{Z}}$ be a sequence of independent real-valued processes such that for each $k \in \mathcal{Z}$ and $t \in [0,T]$ the random variable with distribution $X_k(t)$ is a centered Gaussian with variance $\sigma_k^2(t)$ and*

$$\sup_{t\in[0,T]} \sup_{k\in\mathcal{Z}} \mu_k^2 \sigma_k^2(t) < \infty.$$

Assume that, for every $\delta \in \left(0, \frac{1}{2}\right)$, there exist constants $K_T, C_{T,\delta} > 0$ such that

$$\mathbb{E}\left[|X_k(t) - X_k(s)|^2\right] \leqslant C_{T,\delta}\left[|t-s|^\delta \mu_k^{2\delta-2}\right] \wedge K_T \mu_k^{-2} \tag{2.4}$$

holds for all $k \in \mathcal{Z}$ and $t, s \in [0,T]$. Let $(\alpha_k)_{k\in\mathcal{Z}}$ be a real-valued sequence such that $\sum_{k\in\mathcal{Z}} \alpha_k^2 \mu_k^{2\delta-2} < \infty$. Define the stochastic process for $t \in [0,T]$, $x \in \mathbb{T}^2$

$$X(t,x) \coloneqq \sum_{k\in\mathcal{Z}} \alpha_k X_k(t) e_k(x).$$

Let $0 < \delta < \frac{1}{2}$. Then, for each $p \geqslant 1$ we have $X \in L^p\left(\Omega, C^0\left([0,T]\times\mathbb{T}^2\right)\right)$ and there exists a constant $C = C_{\delta,p,L,T} > 0$, depending only on δ, p, L, and T, such that we obtain the bound

$$\mathbb{E}\left[\|X\|^p_{C^0([0,T]\times\mathbb{T}^2)}\right]^{\frac{1}{p}} \leqslant C\left[\sum_{k\in\mathcal{Z}} \alpha_k^2 \mu_k^{2\delta-2}\right]^{\frac{1}{2}} < \infty.$$

Proof. First, recall that for each $\eta > 0$ there is a constant $C_\eta > 0$ such that

$$|e_k(x) - e_k(y)| \leqslant C_\eta \mu_k^{\frac{\eta}{2}} |x-y|^\eta \tag{2.5}$$

holds for each $x, y \in \mathbb{T}^2$. This can be shown using standard calculus techniques, starting from the basic estimate

$$|e_k(x) - e_k(y)| \leqslant C\left(1 \wedge \mu_k^{\frac{1}{2}}|x-y|\right).$$

For $p > 20$ and $0 < \alpha < \frac{\delta}{2} < \frac{1}{4}$ such that $p\left(\frac{\delta}{2} - \alpha\right) > 2$ and $p\alpha > 3$ hold, we establish from the stochastic independence of the basis $\left(X_k\right)_{k\in\mathcal{Z}}$ the following expression:

$$\begin{aligned}
&\mathbb{E}\left[|X(t,x) - X(s,y)|^2\right]^{\frac{p}{2}} \\
&\leqslant\ 2^{\frac{p}{2}}\left[\mathbb{E}\left[|X(t,x) - X(s,x)|^2\right] + \mathbb{E}\left[|X(s,x) - X(s,y)|^2\right]\right]^{\frac{p}{2}} \\
&= 2^{\frac{p}{2}}\left[\sum_{k\in\mathcal{Z}}\alpha_k^2\mathbb{E}\left[|X_k(t) - X_k(s)|^2\right]e_k^2(x) + \sum_{k\in\mathcal{Z}}\alpha_k^2\,\mathbb{E}\left[X_k^2(s)\right]\left[e_k(x) - e_k(y)\right]^2\right]^{\frac{p}{2}} \\
&\leqslant C_{\delta,p,L,T}\left[\sum_{k\in\mathcal{Z}}\alpha_k^2\mu_k^{2\delta-2}|t-s|^\delta + \sum_{k\in\mathcal{Z}}\alpha_k^2\sigma_k^2(s)\mu_k^{\frac{\delta}{2}}|x-y|^\delta\right]^{\frac{p}{2}} \\
&\leqslant C_{\delta,p,L,T}\left[|t-s|^\delta\sum_{k\in\mathcal{Z}}\alpha_k^2\mu_k^{2\delta-2} + |x-y|^\delta\sum_{k\in\mathcal{Z}}\alpha_k^2\mu_k^{\frac{\delta}{2}-2}\right]^{\frac{p}{2}} \\
&\leqslant C_{\delta,p,L,T}\left[|t-s| + |x-y|\right]^{\delta\frac{p}{2}}\left[\sum_{k\in\mathcal{Z}}\alpha_k^2\mu_k^{2\delta-2}\right]^{\frac{p}{2}}.
\end{aligned}$$

In order to be able to calculate the fractional Sobolev-norm of X we investigate a last auxiliary calculation of the L^p–norm

$$\begin{aligned}
\int_{[0,T]\times\mathbb{T}^2}\mathbb{E}\left[|X(t,x)|^p\right]\mathrm{d}t\,\mathrm{d}x &\leqslant C_p\int_{[0,T]\times\mathbb{T}^2}\left[\sum_{k\in\mathcal{Z}}\alpha_k^2\,\mathbb{E}\left[X_k^2(t)\right]e_k^2(x)\right]^{\frac{p}{2}}\mathrm{d}t\,\mathrm{d}x \\
&\leqslant C_{p,L}T\left[\sum_{k\in\mathcal{Z}}\alpha_k^2\mu_k^{-2}\right]^{\frac{p}{2}} < \infty,
\end{aligned}$$

where we used the Fubini–Tonelli theorem combined with the following general moment bound for Gaussian processes from Corollary 2.5. We are therefore able to examine the norm of X.

The first part of the inequality below directly follows by Morrey's inequality (Theorem A.23) as $\alpha p > 3$ holds:

$$\begin{aligned}
&\mathbb{E}\left[\|X\|^p_{C^0([0,T]\times\mathbb{T}^2)}\right] \leqslant C_p \mathbb{E}\left[\|X\|^p_{W^{\alpha,p}([0,T]\times\mathbb{T}^2)}\right] \\
&= C_p \mathbb{E}\left[\int_{[0,T]\times\mathbb{T}^2}\int_{[0,T]\times\mathbb{T}^2} \frac{|X(t,x)-X(s,y)|^p}{(|t-s|+|x-y|)^{3+\alpha p}} \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\right] \\
&\quad + C_p \mathbb{E}\left[\int_{[0,T]\times\mathbb{T}^2} |X(t,x)|^p \,\mathrm{d}t\,\mathrm{d}x\right] \\
&= C_p \int_{[0,T]\times\mathbb{T}^2}\int_{[0,T]\times\mathbb{T}^2} \frac{\mathbb{E}\left[|X(t,x)-X(s,y)|^2\right]^{\frac{p}{2}}}{(|t-s|+|x-y|)^{3+\alpha p}} \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y \\
&\quad + C_p \int_{[0,T]\times\mathbb{T}^2} \mathbb{E}\left[|X(t,x)|^2\right]^{\frac{p}{2}} \,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant C \int_{[0,T]\times\mathbb{T}^2}\int_{[0,T]\times\mathbb{T}^2} (|t-s|+|x-y|)^{p\left(\frac{\delta}{2}-\alpha\right)-3} \left[\sum_{k\in\mathcal{Z}} \alpha_k^2 \mu_k^{2\delta-2}\right]^{\frac{p}{2}} \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y \\
&\quad + C_{\delta,p,L,T} T^{\frac{\delta p}{2}+1} \left[\sum_{k\in\mathcal{Z}} \alpha_k^2 \mu_k^{-2}\right]^{\frac{p}{2}} \\
&\leqslant C_{\alpha,\delta,p,L,T} \left[\sum_{k\in\mathcal{Z}} \alpha_k^2 \mu_k^{2\delta-2}\right]^{\frac{p}{2}} \\
&< \infty,
\end{aligned}$$

since $p\left(\frac{\delta}{2}-\alpha\right)-3 > -1$. Strictly speaking, we only showed the bound

$$\mathbb{E}\left[\|X\|^p_{C^0([0,T]\times\mathbb{T}^2)}\right] < \infty,$$

but we have not shown yet that $X \in L^p\left(\Omega, C^0\left([0,T]\times\mathbb{T}^2\right)\right)$. Nevertheless, we can redo the same argument for finite subsets $A \subset \mathcal{Z}$ and obtain

$$\mathbb{E}\left[\left\|\sum_{k\in A} \alpha_k X_k e_k\right\|^p_{C^0([0,T]\times\mathbb{T}^2)}\right] \leqslant C\left(\sum_{k\in A} \alpha_k^2 \mu_k^{2\delta-2}\right)^{\frac{p}{2}}.$$

Now we define the process

$$X_N := \sum_{k\in B_{\mathcal{Z}}(0,N)} X_k e_k$$

whereby $B_{\mathcal{Z}}(0,N)$ is the centered ball in $\mathcal{Z}$ with radius $N\in\mathbb{N}$. Thus, it is an element of $L^p\left(\Omega, C^0\left([0,T]\times\mathbb{T}^2\right)\right)$ and we obtain by the previous argument and by choosing $A = B_{\mathcal{Z}}(0,N) \setminus B_{\mathcal{Z}}(0,M)$ that $\left(X_N\right)_{N\in\mathbb{N}}$ is a Cauchy-sequence in $L^p\left(\Omega, C^0\left([0,T]\times\mathbb{T}^2\right)\right)$, which converges to X. By applying Hölder's inequality this concludes the argument for each $p \geqslant 1$. □

Remark 2.18. *In the context of Lemma 2.17, it is not necessary for the sequence $\left(\alpha_k\right)_{k\in\mathcal{Z}}$ to be a null sequence. However, within the regularisation by noise approach, it is always assumed to be a sequence that converges to 0.*

Lemma 2.19. *Let $X(t) = \sum_{k\in\mathcal{Z}} \alpha_k X_k(t) e_k$ as in Lemma 2.17. For each $t \geqslant 0$ and $p \geqslant 1$ we have*

$$\mathbb{E}\left[\|X(t)\|^p_{L^p(\mathbb{T}^2)}\right]^{\frac{1}{p}} \leqslant C_p L^2 \mathbb{E}\left[\|X(t)\|^2_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{2}}.$$

Proof. By applying the moment bound for Gaussian processes, Minkowski inequality, and using the independence of the sequence $\left(X_k\right)_{k\in\mathcal{Z}}$, we conclude that

$$\begin{aligned}
\mathbb{E}\left[\|X(t)\|^p_{L^p}\right]^{\frac{1}{p}} &= \mathbb{E}\left[\int_{\mathbb{T}^2} |X(t,x)|^p \,\mathrm{d}x\right]^{\frac{1}{p}} \\
&\leqslant \int_{\mathbb{T}^2} \mathbb{E}\left[|X(t,x)|^p\right]^{\frac{1}{p}} \,\mathrm{d}x \\
&\leqslant C_p \int_{\mathbb{T}^2} \mathbb{E}\left[|X(t,x)|^2\right]^{\frac{1}{2}} \,\mathrm{d}x \\
&\leqslant C_p \int_{\mathbb{T}^2} \left[\sum_{k\in\mathcal{Z}} \alpha_k^2 \mathbb{E}\left[X_k^2\right] e_k(x)^2\right]^{\frac{1}{2}} \,\mathrm{d}x \\
&\leqslant C_p L^2 \mathbb{E}\left[\sum_{k\in\mathcal{Z}} \alpha_k^2 X_k^2\right]^{\frac{1}{2}} \,\mathrm{d}x \\
&= C_p L^2 \mathbb{E}\left[\|X(t)\|^2_{L^2}\right]^{\frac{1}{2}}.
\end{aligned}$$

This verifies the lemma. □

Definition 2.20. *We define the random variable Z_ε as the regularisation of the stochastic convolution Z (cf. (2.1)), as follows:*

$$Z_\varepsilon(t,x) := \int_0^t e^{(t-s)A} \,\mathrm{d}W_\varepsilon(s,x) = \sum_{k\in\mathcal{Z}} \alpha_k^{(\varepsilon)} Z_k(t) e_k(x), \qquad \varepsilon > 0,$$

where for each $k \in \mathcal{Z}$ we have

$$Z_k(t) := \int_0^t e^{-(t-s)\mu_k^2} \,\mathrm{d}\beta_k(s).$$

Theorem 2.21. *Let $\varepsilon > 0$ and $0 < \delta < \frac{1}{2}$. Then, for each $p \geqslant 1$ the stochastic processes Z and Z_ε belong to the space $L^p\left(\Omega, C^0\left([0,T] \times \mathbb{T}^2\right)\right)$ and there is a constant $C > 0$, depending on p, L, and T, such that*

$$\mathbb{E}\left[\|Z_\varepsilon\|^p_{C^0([0,T]\times\mathbb{T}^2)}\right]^{\frac{1}{p}} \leqslant C\left[\sum_{k\in\mathcal{Z}} \alpha_k^2 \mu_k^{2\delta-2}\right]^{\frac{1}{2}} < \infty.$$

Proof. As $\left(Z_k\right)_{k\in\mathcal{Z}}$ are independent Gaussian random variables with $\mathbb{E}\left[Z_k(t)\right] = 0$ and $\mathbb{E}\left[Z_k(t)^2\right] \leqslant \frac{1}{2}\mu_k^{-2}$ for each $k \in \mathcal{Z}$, we only need to verify (2.4) in order to apply Lemma 2.17, using the independence of the Itô integrals for disjoint intervals,

$$\begin{aligned}
&\mathbb{E}\left[|Z_k(t) - Z_k(s)|^2\right] \\
&= \mathbb{E}\left[\left|\int_s^t e^{-(t-\tau)\mu_k^2} d\beta_k(\tau) - \left(1 - e^{-(t-s)\mu_k^2}\right)\int_0^s e^{-(s-\tau)\mu_k^2} d\beta_k(\tau)\right|^2\right] \\
&= \mathbb{E}\left[\left|\int_s^t e^{-(t-\tau)\mu_k^2} d\beta_k(\tau)\right|^2 + \left(1 - e^{-(t-s)\mu_k^2}\right)^2\left|\int_0^s e^{-(s-\tau)\mu_k^2} d\beta_k(\tau)\right|^2\right] \\
&= \frac{1}{\mu_k^2}\left[\left(1 - e^{-2(t-s)\mu_k^2}\right) + \left(1 - e^{-(t-s)\mu_k^2}\right)^2\left(1 - e^{-2s\mu_k^2}\right)\right] \\
&\leqslant \frac{C}{\mu_k^2}\left[\left(1 - e^{-2(t-s)\mu_k^2}\right)\left(1 + \left(1 - e^{-2s\mu_k^2}\right)\right)\right] \\
&\leqslant \frac{C}{\mu_k^2}\left[\left(1 \wedge 2|t-s|\mu_k^2\right)\left(1 + \left(1 \wedge 2s\mu_k^2\right)\right)\right] \\
&\leqslant \frac{C_\delta}{\mu_k^2}\left(1 \wedge \mu_k^{2\delta}|t-s|^\delta\right),
\end{aligned}$$

whereby we derived the last inequality, as in (2.5), for every δ in the interval $\left(0, \frac{1}{2}\right)$, such that $2\delta - 2 < -1$. Finally, by applying Lemma 2.17, we obtain the result. □

Moreover, in a similar way, by studying the partial derivatives $D_i Z_\varepsilon$ we obtain the following theorem.

Theorem 2.22. *Let $0 < \delta < \frac{1}{2}$. Then, for each $p \geqslant 1$ we have*

$$\nabla Z_\varepsilon \in L^p\left(\Omega, C^0\left([0,T] \times \mathbb{T}^2\right)\right).$$

In particular, there is a constant $C > 0$ depending on p, δ, α, T, and L such that

$$\mathbb{E}\left[\|\nabla Z_\varepsilon\|^p_{C^0([0,T]\times\mathbb{T}^2,\mathbb{R}^2)}\right]^{\frac{1}{p}} \leqslant C\left(\sum_{k\in\mathcal{Z}}\left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^{2\delta-1}\right)^{\frac{1}{2}}.$$

Proof. By combining Lemma 2.17, Lemma 2.19 and the fact that the linear operator $\nabla : \mathcal{H}^1(\mathbb{T}^2) \to L^2(\mathbb{T}^2, \mathbb{R}^2)$ is an isometry, we obtain the result of the theorem. $\square$

Finally, we study the difference between

$$X_\varepsilon := Z - Z_\varepsilon = \sum_{k\in\mathcal{Z}} \left(1 - \alpha_k^{(\varepsilon)}\right) Z_k(t) e_k(x).$$

As in Lemma 2.17, we obtain the following bound for each $p \geqslant 1$ and $0 < \delta < \frac{1}{2}$:

$$\mathbb{E}\left[\|Z - Z_\varepsilon\|^p_{C^0([0,T]\times\mathbb{T}^2)}\right]^{\frac{1}{p}} \leqslant C_{\delta,p,L,T} \left(\sum_{k\in\mathcal{Z}} \left(1 - \alpha_k^{(\varepsilon)}\right)^2 \mu_k^{2\delta-2}\right)^{\frac{1}{2}}. \tag{2.6}$$

Theorem 2.23. *Let $T > 0$. For each $p \geqslant 1$ the regularised stochastic convolution Z_ε converges to Z in $L^p(\Omega, C^0([0,T] \times \mathbb{T}^2))$ as $\varepsilon \to 0$.*

Proof. This result follows directly by applying dominated convergence with the bound (2.6), given that the regularising sequence $\left(\alpha_k^{(\varepsilon)}\right)_{k\in\mathcal{Z}}$ is uniformly bounded. $\square$

To complete this chapter, we introduce two embedding results for the stochastic convolution.

Lemma 2.24. *For each $p \geqslant 1$ and $t \geqslant 0$ we have*

$$\mathbb{E}\left[\|Z_\varepsilon(t)\|^p_{L^p}\right]^{\frac{1}{p}} \leqslant C_p L^2 \mathbb{E}\left[\|Z_\varepsilon(t)\|^2_{L^2}\right]^{\frac{1}{2}}.$$

Proof. Applying Itô's isometry and the independence of $(\beta_k)_{k\in\mathcal{Z}}$ confirms that Lemma 2.19 applies, which completes the proof of the lemma. $\square$

Theorem 2.25. *For each $\eta > 0$ and $p \geqslant 1$ there exists a constant $C = C_{p,\eta} > 0$ such that*

$$\mathbb{E}\left[\|Z_\varepsilon(t)\|^p_{C^0}\right]^{\frac{1}{p}} \leqslant C\mathbb{E}\left[\|Z_\varepsilon(t)\|^2_{\mathcal{H}^\eta}\right]^{\frac{1}{2}}$$

holds for every $t \geqslant 0$.

Proof. First, recall that for each $\eta > 0$ there is a constant $C_\eta > 0$ such that

$$|e_k(x) - e_k(y)| \leqslant C_\eta \mu_k^{\frac{\eta}{2}} |x - y|^\eta$$

holds for each $x, y \in \mathbb{T}^2$.

Let $\eta > 0$ and $p > 2$ such that $\eta p > 2$. By applying Itô's isometry, we obtain

$$\begin{aligned}
&\mathbb{E}\left[|Z_\varepsilon(t,x) - Z_\varepsilon(t,y)|^2\right] \\
&= \mathbb{E}\left[\left|\int_0^t e^{(t-s)A}\,\mathrm{d}W_\varepsilon(s,x) - \int_0^t e^{(t-s)A}\,\mathrm{d}W_\varepsilon(s,y)\right|^2\right] \\
&= \mathbb{E}\left[\left|\sum_{k\in\mathcal{Z}} \alpha_k^{(\varepsilon)}\left[e_k(x) - e_k(y)\right]\int_0^t e^{-(t-s)\mu_k^2}\,\mathrm{d}\beta_k(s)\right|^2\right] \\
&= \sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \left[e_k(x) - e_k(y)\right]^2 \int_0^t e^{-2(t-s)\mu_k^2}\,\mathrm{d}s \\
&\leqslant C_\eta |x-y|^{2\eta} \sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{\mu_k^\eta}{2\mu_k^2}\left[1 - e^{-2t\mu_k^2}\right]
\end{aligned}$$

for each $x, y \in \mathbb{T}^2$, $t \geqslant 0$. Therefore, by *Morrey's inequality* (Theorem A.23) for $\alpha p > 2$ with $\eta > \alpha > 0$ and $p > 2$, and a bound for p-th Gaussian moments (see Corollary 2.5), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\|Z_\varepsilon(t)\|_{C^0}^p\right] \leqslant C_p \mathbb{E}\left[\|Z_\varepsilon(t)\|_{W^{\alpha,p}}^p\right] \\
&\leqslant C_p \mathbb{E}\left[\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \frac{|Z_\varepsilon(t,x) - Z_\varepsilon(t,y)|^p}{|x-y|^{2+\alpha p}}\,\mathrm{d}x\,\mathrm{d}y\right] + \mathbb{E}\left[\|Z_\varepsilon(t)\|_{L^p}^p\right] \\
&\leqslant C_p \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \frac{\mathbb{E}\left[|Z_\varepsilon(t,x) - Z_\varepsilon(t,y)|^2\right]^{\frac{p}{2}}}{|x-y|^{2+\alpha p}}\,\mathrm{d}x\,\mathrm{d}y + \mathbb{E}\left[\|Z_\varepsilon(t)\|_{L^p}^p\right] \\
&\leqslant C_{p,\eta} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \frac{\left[|x-y|^\eta \sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{\mu_k^\eta}{2\mu_k^2}\left[1 - e^{-2t\mu_k^2}\right]\right]^{\frac{p}{2}}}{|x-y|^{2+\alpha p}}\,\mathrm{d}x\,\mathrm{d}y + \mathbb{E}\left[\|Z_\varepsilon(t)\|_{L^p}^p\right] \\
&= C_{p,\eta} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |x-y|^{p(\eta-\alpha)-2}\,\mathrm{d}x\,\mathrm{d}y \left[\sum_{k\in\mathcal{Z}} \mu_k^\eta \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{1}{2\mu_k^2}\left[1 - e^{-2t\mu_k^2}\right]\right]^{\frac{p}{2}} \\
&\quad + \mathbb{E}\left[\|Z_\varepsilon(t)\|_{L^p}^p\right] \\
&\leqslant C_{p,\eta} \mathbb{E}\left[\|Z_\varepsilon(t)\|_{\mathcal{H}^\eta}^2\right]^{\frac{p}{2}},
\end{aligned}$$

whereby we used $p(\eta - \alpha) - 2 > -2$. In particular, we have

$$\mathbb{E}\left[\|Z_\varepsilon(t)\|_{C^0}^p\right]^{\frac{1}{p}} \leqslant C_{p,\eta} \mathbb{E}\left[\|Z_\varepsilon(t)\|_{\mathcal{H}^\eta}^2\right]^{\frac{1}{2}}$$

By applying Hölder's inequality, this concludes the argument for each $p \geqslant 1$. $\square$

3. Solution of the Regularised Stochastic Partial Differential Equation

This chapter is based on collaborative work with Dirk Blömker, as published in [14]. The main result of this chapter is that, for any regularisation as defined in Definition 2.13, the nonlinearity vanishes from the equation in the limit as ε approaches zero.

Definition 3.1. *Let $A := -\Delta^2$ and $\mathfrak{F}(u_\varepsilon) := \nabla \cdot f(\nabla u_\varepsilon) := \nabla \cdot \frac{\nabla u_\varepsilon}{1+|\nabla u_\varepsilon|^2}$ and Z_ε be the stochastic convolution, as defined in Definition 2.13. Let $u_0 \in \mathcal{H}^1(\mathbb{T}^2)$. For $T > 0$, we call a process $(u_\varepsilon(t))_{t\in[0,T]}$ with continuous paths in $\mathcal{H}^1(\mathbb{T}^2)$ such that for all $t \in [0,T]$ and $x \in \mathbb{T}^2$*

$$u_\varepsilon(t,x) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\mathfrak{F}(u_\varepsilon(s,x))\,\mathrm{d}s + Z_\varepsilon(t,x),$$

a mild solution *of the regularised SPDE*

$$\partial_t u_\varepsilon(t,x) = -\Delta^2 u_\varepsilon(t,x) + \nabla \cdot \frac{\nabla u_\varepsilon(t,x)}{1+|\nabla u_\varepsilon(t,x)|^2} + \partial_t W_\varepsilon(t,x), \quad u(0) = u_0. \tag{3.1}$$

Theorem 3.2. *For $T > 0$ let u_ε be the mild solution of (3.1). Then $u_\varepsilon \to u$ for $\varepsilon \to 0$ in $L^p(\Omega, C^0([0,T]\times\mathbb{T}^2))$ for each $p \geqslant 1$, where u solves*

$$\partial_t u = -\Delta^2 u + \partial_t W, \quad u(0) = u_0. \tag{3.2}$$

Remark 3.3. *A hill formation is expected for (3.1), due to the linear instability of the equation, given by the approximation*

$$-\nabla \cdot f(\nabla u) \approx -\Delta u$$

for ∇u very small. This approximation yields the linearised equation

$$\partial_t u = -\left(\Delta^2 - \Delta\right)u + \partial_t W.$$

For further analysis of the linear instability effects, see [9] for a discussion in the context of the Cahn–Hilliard equation. In contrast, when the nonlinearity vanishes, no hills are observed in the solution of (3.2), where solutions are expected to be merely of order σ. In particular, this yields to (3.2). However, we will analyse this linearisation in more detail in Chapter 4.

3.1. Existence and Uniqueness of the Mild Solution

Based on *Banach's fixed-point theorem*, it is straightforward to prove the following:

Theorem 3.4. *For $u_0 \in \mathcal{H}^1$ the stochastic partial differential equation (3.1) given by Definition 2.13 has a unique mild solution u_ε in the sense of Definition 3.1 in the space $\mathcal{H}^1$.*

Remark 3.5. *It is possible to consider initial conditions with $u_0 \notin \mathcal{H}^1$ by employing weighted spaces in time, e.g.*

$$X_{\alpha,\delta}(0,T) = \left\{ u \in C^0\left((0,T], \mathcal{H}^\alpha\right) : \sup_{t\in(0,T]} t^\alpha \|u(t)\|_{\mathcal{H}^\alpha} < \infty \right\}.$$

However, we refrain from giving further details here. For an application of such initial conditions, see [11].

We now prove Theorem 3.4 based on Banach's fixed-point theorem. First, we define the mapping

$$\mathfrak{G}\left(u_\varepsilon\right)(t) := e^{tA}u_0 + \int_0^t e^{(t-s)A}\mathfrak{F}\left(u_\varepsilon(s)\right)\,\mathrm{d}s + Z_\varepsilon(t).$$

In order to apply Banach's fixed-point theorem, it is necessary to first establish the following result.

Lemma 3.6. *The mapping $f \circ \nabla : \mathcal{H}^1\left(\mathbb{T}^2\right) \to L^2\left(\mathbb{T}^2, \mathbb{R}^2\right)$, $h \mapsto \frac{\nabla h}{1+|\nabla h|^2}$ (cf. (1.3)), is bounded and globally Lipschitz continuous with Lipschitz constant 1.*

Proof. The boundedness follows immediately from the boundedness of the function $f : \mathbb{R}^2 \to \mathbb{R}^2$. To obtain the global Lipschitz continuity of f we use the continuous differentiability and therefore, by *multidimensional mean value theorem* we establish for each $x, y \in \mathbb{R}^2$ that

$$|f(x) - f(y)| \leqslant \sup_{t\in[0,1]} \left\{\|Df\left(x + t(y-x)\right)\|\right\} |x - y|,$$

whereby

$$Df(z) := \frac{1}{(1+|z|^2)^2} \begin{pmatrix} 1-z_1^2 & -2z_1z_2 \\ -2z_2z_1 & 1-z_2^2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

denotes the Jacobian of the function f for $z \in \mathbb{R}^2$. It holds that $\|Df\|_\infty \leqslant 1$, which implies the global Lipschitz continuity of f. By applying Poincaré's inequality and the characterisation of the Sobolev spaces $\mathcal{H}^\alpha = \mathcal{H}^\alpha(\mathbb{T}^2)$ via the operator $(-\Delta)^{\frac{\alpha}{2}}$ (see Appendix A.2), the linear operator $\nabla : \mathcal{H}^1(\mathbb{T}^2,\mathbb{R}) \to L^2(\mathbb{T}^2,\mathbb{R}^2)$ is an isometry. Consequently, the mapping

$$f \circ \nabla : \mathcal{H}^1(\mathbb{T}^2,\mathbb{R}) \to L^2(\mathbb{T}^2,\mathbb{R}^2), \quad h \mapsto f(\nabla h)$$

is globally Lipschitz continuous. For each $u, v \in \mathcal{H}^1$ we obtain the bound

$$\left\| \frac{\nabla u}{1+|\nabla u|^2} - \frac{\nabla v}{1+|\nabla v|^2} \right\|_{L^2} \leqslant \|u-v\|_{\mathcal{H}^1}.$$

This verifies the lemma. □

Additionally, in preparation for applying Banach's fixed-point theorem, we recall several key properties of the semigroup $\left(e^{tA}\right)_{t\geqslant 0}$:

- **Smoothing estimate:** For $\alpha < \beta$ and $t \in (0,T]$, the operator norm satisfies

$$\left\|e^{tA}\right\|_{L\left(\mathcal{H}^\alpha,\mathcal{H}^\beta\right)} \leqslant \left(\frac{\beta-\alpha}{4e}\right)^{\frac{\beta-\alpha}{4}} t^{\frac{\alpha-\beta}{4}}. \tag{3.3}$$

  This inequality is integrable at $t=0$ whenever $\alpha < \beta < \alpha+4$ (cf. (A.8)).

- **Isometry of the divergence:** The divergence operator is an isometry,

$$\nabla\cdot : \mathcal{H}^1(\mathbb{T}^2,\mathbb{R}^2) \to L^2(\mathbb{T}^2,\mathbb{R}).$$

- **Continuity of the semigroup:** Since A is self-adjoint, $\left(e^{tA}\right)_{t\geqslant 0}$ forms an analytic semigroup on $L^p(\mathbb{T}^2)$ for each $p \in [1,\infty)$, and thus we have the continuity

$$t \mapsto e^{tA}u_0 \in C^0([0,T],\mathcal{H}^1) \tag{3.4}$$

  for all $u_0 \in \mathcal{H}^1$ (see [68, Section 2.6.]).

Lemma 3.7. *Given $w \in C^0\left([0,T],\mathcal{H}^1\right)$, then*

$$t \mapsto \int_0^t e^{(t-s)A}\mathfrak{F}\left(w(s)\right)\,\mathrm{d}s$$

has the regularity $C^0\left([0,T],\mathcal{H}^{3-}\right)$.

Proof. Because the nonlinearity is Lipschitz continuous, $\mathfrak{F}:\mathcal{H}^1 \to \mathcal{H}^{-1}$ has Lipschitz constant 1. Therefore, $\mathfrak{F}(w)$ belongs to $C^0([0,T],\mathcal{H}^{-1})$. By choosing $\alpha = -1$ and $\beta < 3$ and applying (3.3), we obtain the regularity

$$t \mapsto \int_0^t e^{(t-s)A}\mathfrak{F}\left(w(s)\right)\,\mathrm{d}s \;\in C^0\left([0,T],\mathcal{H}^{3-}\right).$$

The proof is now complete. □

Finally, by (3.4) and Lemma 3.7 we conclude that the operator $\mathfrak{G}$ is a self-mapping on $C^0\left([0,T],\mathcal{H}^1\right)$ in the case of $Z_\varepsilon \in C^0\left([0,T],\mathcal{H}^1\right)$, which we already established in Theorem 2.22.

Theorem 3.8. *The n-th iteration $\mathfrak{G}^n := \overbrace{\mathfrak{G}\circ\cdots\circ\mathfrak{G}}^{n\ \text{times}}$ of the operator $\mathfrak{G}$ is a contraction on $C^0\left([0,T],\mathcal{H}^1\right)$ and thus there is a (pathwise) unique fixed-point.*

Proof. We consider the difference between two solutions $u, v \in C^0([0,T],\mathcal{H}^1)$ with the same initial condition u_0. Thus, by using (3.3) and the Lipschitz continuity of $\mathfrak{F}$, we obtain

$$\begin{aligned}\|\mathfrak{G}(u)(t)-\mathfrak{G}(v)(t)\|_{\mathcal{H}^1} &= \left\|\int_0^t e^{(t-s)A}\mathfrak{F}\left(u(s)\right)\,\mathrm{d}s - \int_0^t e^{(t-s)A}\mathfrak{F}\left(v(s)\right)\,\mathrm{d}s\right\|_{\mathcal{H}^1}\\ &\leqslant \sqrt{\frac{1}{2e}}\int_0^t \frac{1}{\sqrt{t-s}}\|u(s)-v(s)\|_{\mathcal{H}^1}\,\mathrm{d}s.\end{aligned}$$

By analysing the supremum-norm in time, we obtain

$$\begin{aligned}\sup_{t\in[0,T]}\left\{\|\mathfrak{G}(u)(t)-\mathfrak{G}(v)(t)\|_{\mathcal{H}^1}\right\} &\leqslant \sup_{t\in[0,T]}\left\{\sqrt{\frac{1}{2e}}\int_0^t \frac{1}{\sqrt{t-s}}\|u(s)-v(s)\|_{\mathcal{H}^1}\,\mathrm{d}s\right\}\\ &= \sqrt{\frac{1}{2e}}\sup_{t\in[0,T]}\left\{\|u(t)-v(t)\|_{\mathcal{H}^1}\right\}\int_0^T \frac{1}{\sqrt{s}}\,\mathrm{d}s\\ &= \sqrt{\frac{2T}{e}}\sup_{t\in[0,T]}\left\{\|u(t)-v(t)\|_{\mathcal{H}^1}\right\}.\end{aligned}$$

Since the constant $\sqrt{2T/e}$ does not yet ensure that $\mathfrak{G}$ satisfies the contraction property, we examine the n–th iteration of this term, where $n \in \mathbb{N}$ is sufficiently large.

For this examination, we first need to compute the integral via substitution

$$\begin{aligned}\int_0^t \sqrt{\frac{z^k}{t-z}}\,\mathrm{d}z &= t^{\frac{k+1}{2}} \int_0^1 \sqrt{\frac{x^k}{1-x}}\,\mathrm{d}x \\ &= t^{\frac{k+1}{2}} \int_0^1 x^{\frac{k}{2}}(1-x)^{-\frac{1}{2}}\,\mathrm{d}x \\ &= t^{\frac{k+1}{2}} B\left(\frac{k+2}{2},\frac{1}{2}\right),\end{aligned}$$

for $k \in \mathbb{N}_0$, where

$$B\left(x,y\right) \coloneqq \frac{\Gamma\left(y\right)\Gamma\left(z\right)}{\Gamma\left(y+z\right)},$$

for $y, z > 0$ is the *Beta function* and Γ the *Gamma function*. Furthermore, the product of these functions creates a telescoping product

$$\prod_{k=1}^{n-1} B\left(\frac{k+2}{2},\frac{1}{2}\right) = \prod_{k=1}^{n-1} \frac{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+3}{2}\right)} = \left(\Gamma\left(\frac{1}{2}\right)\right)^n \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.$$

By combining these calculations, we can show that the n-th iteration satisfies the following contraction property

$$\begin{aligned}&\sup_{t\in[0,T]} \|\mathfrak{G}^n(u)(t) - \mathfrak{G}^n(v)(t)\|_{\mathcal{H}^1} \\ &\leqslant \left(\frac{1}{2e}\right)^{\frac{1}{2}} \int_0^t \frac{1}{\sqrt{t-s}} \left\|\mathfrak{G}^{n-1}(u)(s) - \mathfrak{G}^{n-1}(v)(s)\right\|_{\mathcal{H}^1} \mathrm{d}s \\ &\leqslant \left(\frac{1}{2e}\right)^{n/2} \sup_{s\in[0,T]} \|u(s)-v(s)\|_{\mathcal{H}^1} \int_0^t \frac{1}{\sqrt{t-t_1}} \cdots \int_0^{t_{n-1}} \frac{1}{\sqrt{t_{n-1}-t_n}}\,\mathrm{d}t_1 \ldots \mathrm{d}t_n \\ &\leqslant 2\Gamma\left(\frac{3}{2}\right)\left(\frac{T\Gamma\left(\frac{1}{2}\right)^2}{2e}\right)^{n/2} \frac{1}{\Gamma\left(\frac{n+2}{2}\right)} \sup_{s\in[0,T]} \|u(s)-v(s)\|_{\mathcal{H}^1}.\end{aligned}$$

This expression converges to 0 as $n \to \infty$. Thus, $\mathfrak{G}^n$ is a contraction for sufficiently large $n \in \mathbb{N}$. We now apply a corollary of Banach's fixed-point theorem [75] for fixed-point iterations. Since the stochastic process Z_ε is also an element of $C^0([0,T],\mathcal{H}^1)$, we obtain the existence and uniqueness of a fixed-point $\mathfrak{G}(u) = u$ in $C^0([0,T],\mathcal{H}^1)$. □

By applying Banach's fixed-point theorem, we establish the existence and uniqueness of the mild solution $u_\varepsilon \in C^0([0,T],\mathcal{H}^1)$, which for each $t \in [0,T]$ is given by

$$u_\varepsilon(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\mathfrak{F}\left(u_\varepsilon(s)\right)\,\mathrm{d}s + Z_\varepsilon(t).$$

3.2. Uniform Boundedness

In the following, we establish a significant deterministic uniform bound on ∇v_ε, which is crucial for proving the vanishing of the nonlinearity. For this purpose, consider the standard transformation

$$v_\varepsilon := u_\varepsilon - Z_\varepsilon - e^{tA}u_0,$$

where v_ε solves the stochastic partial differential equation

$$\partial_t v_\varepsilon = -\Delta^2 v_\varepsilon + \nabla \cdot f\left(\nabla v_\varepsilon + \nabla Z_\varepsilon + \nabla e^{tA}u_0\right), \quad v_\varepsilon(0) = 0. \tag{3.5}$$

Theorem 3.9. *For all $T > 0$ there is a constant $C = C_{L,T} > 0$ such that*

$$|\nabla v_\varepsilon(\omega, t, x)| \leqslant C$$

holds for each $\omega \in \Omega$, $t \in [0,T]$, $x \in \mathbb{T}^2$ and $\varepsilon > 0$.

Proof. First of all, note that $\|f\|_{L^\infty(\mathbb{R}^2,\mathbb{R}^2)}$ is bounded by 1. Let $\delta \in (0,1)$. For each $t \in [0,T]$ and $\varepsilon > 0$, the function v_ε suffices

$$\begin{aligned}
&\|\nabla v_\varepsilon(t)\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)} \\
&\leqslant C_\delta \|v_\varepsilon(t)\|_{\mathcal{H}^{2+\delta}} \\
&= C_\delta \left\| \int_0^t e^{(t-s)A} \mathfrak{F}\left(v_\varepsilon(s) + Z_\varepsilon(s) + e^{sA}u_0\right) \mathrm{d}s \right\|_{\mathcal{H}^{2+\delta}} \\
&\leqslant C_\delta \int_0^t \left\| e^{(t-s)A}\nabla\cdot \right\|_{L\left(L^2(\mathbb{T}^2,\mathbb{R}^2),\mathcal{H}^{2+\delta}\right)} \left\| f\left(\nabla v_\varepsilon(s) + \nabla Z_\varepsilon(s) + \nabla e^{sA}u_0\right)\right\|_{L^2} \mathrm{d}s \\
&\leqslant C_\delta \int_0^t (t-s)^{-(3+\delta)/4} L \,\mathrm{d}s \\
&\leqslant C_\delta L T^{(1-\delta)/4}.
\end{aligned}$$

This establishes the desired result. □

3.3. Vanishing of the Nonlinearity

In this subsection, we prove the convergence $f(\nabla u_\varepsilon(t,x)) \to 0$ for $\varepsilon \to 0$ in $L^p(\Omega, \mathbb{R}^2)$, uniformly in $x \in \mathbb{T}^2$ for all $t \in [0,T]$. First, we present a short auxiliary statement, which relies on the periodicity of the domain and the type of regularisation.

Lemma 3.10. *For each $t \in [0,T]$ the covariance operator of the process $Z_\varepsilon(t,x)$ does not depend on $x \in \mathbb{T}^2$.*

Proof. Without loss of generality, assume $k_1, k_2 > 0$. By applying the *Pythagorean identity* to (A.1), we obtain for each $x = (x_1, x_2) \in \mathbb{T}^2$

$$\begin{aligned}
&e^2_{(k_1,k_2)}(x) + e^2_{(-k_1,k_2)}(x) + e^2_{(k_1,-k_2)}(x) + e^2_{(-k_1,-k_2)}(x) \\
&= \frac{2}{L}\left[\sin^2\left(\frac{2\pi l}{L}x_1\right)\sin^2\left(\frac{2\pi l}{L}x_2\right) + \sin^2\left(\frac{2\pi l}{L}x_1\right)\cos^2\left(\frac{2\pi l}{L}x_2\right)\right] \\
&\quad + \frac{2}{L}\left[\cos^2\left(\frac{2\pi l}{L}x_1\right)\sin^2\left(\frac{2\pi l}{L}x_2\right) + \cos^2\left(\frac{2\pi l}{L}x_1\right)\cos^2\left(\frac{2\pi l}{L}x_2\right)\right] \\
&= \frac{2}{L}
\end{aligned}$$

which does not depend on $x \in \mathbb{T}^2$. Since $\omega_0(x_i) = L^{-\frac{1}{2}}$ is a constant function, the other cases with $k_1 = 0$ or $k_2 = 0$ follow analogously. By the independence of sequence of Wiener processes $(\beta_k)_{k\in\mathcal{Z}}$, we can apply the Itô isometry. Furthermore, due to the radial symmetry of the sequences $(\mu_k)_{k\in\mathcal{Z}}$ and $\left(\alpha_k^{(\varepsilon)}\right)_{k\in\mathcal{Z}}$, we conclude for each $\varepsilon > 0$ that

$$\begin{aligned}
\mathbb{E}\left[|Z_\varepsilon(t,x)|^2\right] &= \mathbb{E}\left[\left(\sum_{k\in\mathcal{Z}} \alpha_k^{(\varepsilon)} e_k(x) \int_0^t e^{-(t-s)\mu_k^2}\,\mathrm{d}\beta_k(s)\right)^2\right] \\
&= \sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 e_k^2(x) \int_0^t e^{-2(t-s)\mu_k^2}\,\mathrm{d}s \\
&= 4 \sum_{\substack{k\in\mathcal{Z}\\ k_1,k_2>0}} \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{2}{L} \int_0^t e^{-2(t-s)\mu_k^2}\,\mathrm{d}s \\
&\quad + \sum_{i=1}^2 \sum_{\substack{k\in\mathcal{Z}\\ k_i=0}} \frac{1}{\sqrt{L}} \omega_{k_i}(x_i) \left(\alpha_k^{(\varepsilon)}\right)^2 \int_0^t e^{-2(t-s)\mu_k^2}\,\mathrm{d}s \\
&= \frac{16}{L} \sum_{\substack{k\in\mathcal{Z}\\ k_1,k_2>0}} \left(\alpha_k^{(\varepsilon)}\right)^2 \int_0^t e^{-2(t-s)\mu_k^2}\,\mathrm{d}s \\
&\quad + \frac{2}{L} \sum_{l\in\mathbb{N}} \left(\alpha_{(l,0)}^{(\varepsilon)}\right)^2 \int_0^t e^{-2(t-s)\mu_{(l,0)}^2}\,\mathrm{d}s \\
&\quad \times \left[\cos^2\left(\frac{2\pi l}{L}x_1\right) + \sin^2\left(\frac{2\pi l}{L}x_1\right) + \cos^2\left(\frac{2\pi l}{L}x_2\right) + \cos^2\left(\frac{2\pi l}{L}x_2\right)\right] \\
&= \frac{16}{L} \sum_{\substack{k\in\mathcal{Z}\\ k_1,k_2>0}} \left(\alpha_k^{(\varepsilon)}\right)^2 \int_0^t e^{-2(t-s)\mu_k^2}\,\mathrm{d}s + \frac{2}{L} \sum_{l\in\mathbb{N}} \left(\alpha_{(l,0)}^{(\varepsilon)}\right)^2 \int_0^t e^{-2(t-s)\mu_{(l,0)}^2}\,\mathrm{d}s,
\end{aligned}$$

which does not depend on $x \in \mathbb{T}^2$. This establishes the desired result. □

Theorem 3.11. *For each $p \geqslant 1$ and $t \in (0,T]$ we have*

$$\sup_{x\in\mathbb{T}^2} \mathbb{E}\left[|f\left(\nabla u_\varepsilon(t,x)\right)|^p\right] \to 0 \qquad \text{if } \varepsilon \to 0.$$

Proof. Fix an arbitrary $t \in (0,T]$. Due to Theorem 3.9 we fix a constant $M > 0$ such that $M \geqslant \|\nabla v_\varepsilon\|_{L^\infty([0,T]\times\mathbb{T}^2)}$. We also choose $K_\varepsilon \geqslant M$ by

$$K_\varepsilon := \left[\sum_{k_1,-k_2\in\mathbb{N}} \left(\frac{k_1}{|k|^2}\right)^2 \left(\alpha_k^{(\varepsilon)}\right)^2\right]^\eta \vee M,$$

where η is a constant in the interval $(0,1)$ and thus we derive $K_\varepsilon \to \infty$, for $\varepsilon \to 0$. First, let us note that we can directly estimate

$$|f(\nabla u_\varepsilon(t,x))| \leqslant \frac{2}{1+|\nabla u_\varepsilon(t,x)|}.$$

As $\nabla u_\varepsilon = \nabla v_\varepsilon + \nabla Z_\varepsilon$, we obtain for $|\nabla Z_\varepsilon| > K_\varepsilon$ the inequality

$$\frac{2}{1+|\nabla u_\varepsilon(t,x)|} \leqslant \frac{2}{1+|\nabla Z_\varepsilon(t,x)| - |\nabla v_\varepsilon(t,x)|} \leqslant \frac{2}{1+K_\varepsilon - M.}$$

Let $B_1(0)$ be the ball with radius 1 in $\mathbb{R}^2$. By the boundedness of f, we obtain

$$\begin{aligned}
\mathbb{E}\left[|f\left(\nabla u_\varepsilon(t,x)\right)|^p\right] &\leqslant \mathbb{E}\left[|f\left(\nabla u_\varepsilon(t,x)\right)|^p \mid |\nabla Z_\varepsilon(t,x)| > K_\varepsilon\right] \mathbb{P}\left(|\nabla Z_\varepsilon(t,x)| > K_\varepsilon\right) \\
&\quad + \mathbb{E}\left[|f\left(\nabla u_\varepsilon(t,x)\right)|^p \mid |\nabla Z_\varepsilon(t,x)| \leqslant K_\varepsilon\right] \mathbb{P}\left(|\nabla Z_\varepsilon(t,x)| \leqslant K_\varepsilon\right) \\
&\leqslant \left(\frac{2}{1+K_\varepsilon - M}\right)^p + C\mathbb{P}\left(|\nabla Z_\varepsilon(t,x)| \leqslant K_\varepsilon\right) \\
&= \left(\frac{2}{1+K_\varepsilon - M}\right)^p + C\mathbb{P}\left(\frac{\nabla Z_\varepsilon(t,x)}{K_\varepsilon} \in \overline{B_1(0)}\right) \\
&\leqslant \left(\frac{2}{1+K_\varepsilon - M}\right)^p + C \max_{y\in\overline{B_1(0)}} \{\varphi_\varepsilon(t,x,y)\} \operatorname{vol}\left(B_1(0)\right),
\end{aligned}$$

where $C \in (0,1)$ is a constant and

$$\varphi_\varepsilon(t,x,y) := \frac{1}{2\pi\sqrt{\det(\Sigma_\varepsilon(t,x))}} e^{-\frac{1}{2}y^T\Sigma_\varepsilon^{-1}(t,x)y},$$

is the density function of $\frac{\nabla Z_\varepsilon(t,x)}{K_\varepsilon} \sim \mathcal{N}\left(\mathbf{0}, \Sigma_\varepsilon(t,x)\right)$.

To this end, we denote by Σ_ε the covariance matrix as follows:

$$\Sigma_\varepsilon(t,x) \coloneqq \operatorname{Cov}\left(\frac{\nabla Z_\varepsilon(t,x)}{K_\varepsilon}\right) = \frac{1}{K_\varepsilon^2}\sum_{k\in\mathcal{Z}}\left[1-e^{-2t\mu_k^2}\right]\frac{\left(\alpha_k^{(\varepsilon)}\right)^2}{2\mu_k^2}\nabla e_k(x)\nabla e_k(x)^T$$
$$= \frac{1}{K_\varepsilon^2}\sum_{k\in\mathcal{Z}}\left[1-e^{-2t\mu_k^2}\right]\frac{\left(\alpha_k^{(\varepsilon)}\right)^2}{2\mu_k^2}\begin{pmatrix} D_1e_k(x)D_1e_k(x) & D_1e_k(x)D_2e_k(x)\\ D_2e_k(x)D_1e_k(x) & D_2e_k(x)D_2e_k(x)\end{pmatrix}.$$

By Lemma 3.10, the covariance of Z_ε, and thus the covariance of ∇Z_ε, does not depend on the variable $x\in\mathbb{T}^2$. We investigate the determinant of the covariance matrix at $(0,0) =: \mathbf{0}$, which simplifies the argument. Furthermore, by the definition of $(e_k)_{k\in\mathcal{Z}}$, we obtain

$$D_1e_k(\mathbf{0})\,D_2e_k(\mathbf{0}) = \frac{2}{L}\left(\frac{2\pi}{L}\right)^2 k_1k_2\sin(0)\cos(0)\sin(0)\cos(0) = 0$$

for each $k=(k_1,k_2)\in\mathcal{Z}$ and thus we derive

$$\begin{aligned}
&\det\left(\Sigma_\varepsilon(t,\mathbf{0})\right) \qquad (3.6)\\
&= \frac{1}{4K_\varepsilon^4}\sum_{k,l\in\mathcal{Z}}\left(\frac{\alpha_k^{(\varepsilon)}\alpha_l^{(\varepsilon)}}{\mu_k\mu_l}\right)^2\left[1-e^{-2t\mu_k^2}\right]\left[1-e^{-2t\mu_l^2}\right]\\
&\qquad \left[\left|D_1e_k(\mathbf{0})\right|^2\left|D_2e_l(\mathbf{0})\right|^2 - D_1e_k(\mathbf{0})\,D_1e_l(\mathbf{0})\,D_2e_k(\mathbf{0})\,D_2e_l(\mathbf{0})\right]\\
&= \frac{1}{4K_\varepsilon^4}\sum_{k,l\in\mathcal{Z}}\left(\frac{\alpha_k^{(\varepsilon)}\alpha_l^{(\varepsilon)}}{\mu_k\mu_l}\right)^2\left|D_1e_k(\mathbf{0})\right|^2\left|D_2e_l(\mathbf{0})\right|^2\left[1-e^{-2t\mu_k^2}\right]\left[1-e^{-2t\mu_l^2}\right]\\
&= \frac{1}{4K_\varepsilon^4}\left[\sum_{k\in\mathcal{Z}}\left(\frac{\alpha_k^{(\varepsilon)}}{\mu_k}\right)^2\left|D_1e_k(\mathbf{0})\right|^2\left[1-e^{-2t\mu_k^2}\right]\right]^2\\
&= \frac{1}{4K_\varepsilon^4}\left[\sum_{\substack{k_1\in\mathbb{N}\\ -k_2\in\mathbb{N}_0}}\left(\frac{\alpha_k^{(\varepsilon)}}{\mu_k}\right)^2\frac{16\pi^2}{L^3}k_1^2\omega_{k_2}(0)^2\left[1-e^{-2t\mu_k^2}\right]\right]^2\\
&= \frac{M}{K_\varepsilon^4}\frac{4}{\pi^4}\left[\sum_{\substack{k_1\in\mathbb{N}\\ -k_2\in\mathbb{N}_0}}\frac{k_1^2}{|k|^4}\left(\alpha_k^{(\varepsilon)}\right)^2\left[1-e^{-2t\mu_k^2}\right]\right]^2,
\end{aligned}$$

for $M\in\{1,2\}$, as $\omega_{k_2}(0)\in\left\{L^{-\frac{1}{2}}\,,\;2L^{-\frac{1}{2}}\right\}$ and $1-e^{-2t(2\pi/L)^4}\leqslant 1-e^{-2t\mu_k^2}\leqslant 1$ hold true for each $t\geqslant 0$ and $k=(k_1,k_2)\in\mathcal{Z}$. For $t=0$, this determinant is equal to 0.

For fixed $t > 0$, the convergence of this series is equivalent to the convergence of the following series due to the non-negativity of its summands

$$\frac{1}{K_\varepsilon^2} \sum_{k_1, -k_2 \in \mathbb{N}} \left(\frac{k_1}{|k|^2} \right)^2 \left(\alpha_k^{(\varepsilon)} \right)^2$$

which converges if and only if $\varepsilon > 0$. This divergence follows directly from the *integral comparison criterion* by analysing the convergence or divergence of

$$\begin{aligned} \int_1^\infty \int_1^\infty \frac{x_1^2}{\left(|x_1| + |x_2|\right)^4} \, \mathrm{d}x_1 \, \mathrm{d}x_2 &= \int_1^\infty \frac{3 + x_2(3 + x_2)}{3(1 + x_2)^3} \, \mathrm{d}x_2 \\ &= \int_1^\infty \frac{3 + 3x_2 + x_2^2}{3(1 + x_2)^3} \, \mathrm{d}x_2 = \infty. \end{aligned}$$

Since this series diverges, we conclude that

$$\begin{aligned} \mathbb{E}\left[|f\left(\nabla u_\varepsilon(t, x)\right)|^p \right] &\leqslant \left(\frac{2}{1 + K_\varepsilon - M} \right)^p + C \operatorname{vol}\left(B_1(0)\right) \max_{y \in B_1(0)} \{\varphi_\varepsilon(t, x, y)\} \\ &\leqslant \left(\frac{2}{1 + K_\varepsilon - M} \right)^p + \frac{C}{\sqrt{\det\left(\Sigma_\varepsilon(t, x)\right)}} \xrightarrow{\varepsilon \to 0} 0 \end{aligned}$$

also vanishes in the limit for $t \in (0, T]$ and $x \in \mathbb{T}^2$. □

Therefore, by pointwise convergence and the uniform boundedness of f, we directly conclude the following corollary.

Corollary 3.12. *For each $p \geqslant 1$ holds*

$$\mathbb{E}\left[\|f\left(\nabla u_\varepsilon\right)\|_{L^p([0,T] \times \mathbb{T}^2, \mathbb{R}^2)}^p \right] \to 0 \quad \text{if } \varepsilon \to 0.$$

Proof. By pointwise convergence from Theorem 3.11

$$\mathbb{E}\left[|f\left(\nabla u_\varepsilon(t, x)\right)|^p \right] \xrightarrow{\varepsilon \to 0} 0$$

for each $x \in \mathbb{T}^2$ and $t \in (0, T]$ and the uniform boundedness

$$\|f\left(\nabla u_\varepsilon\right)\|_{L^\infty([0,T] \times \mathbb{T}^2)} \leqslant 1,$$

we directly derive the following result by applying Theorem A.24:

$$
\begin{aligned}
\mathbb{E}\left[\|f(\nabla u_\varepsilon)\|^p_{L^p([0,T]\times\mathbb{T}^2,\mathbb{R}^2)}\right]^{\frac{1}{p}} &= \mathbb{E}\left[\int_{[0,L]^2}\int_0^T |f(\nabla u_\varepsilon(t,x))|^p \,\mathrm{d}t\,\mathrm{d}x\right]^{\frac{1}{p}} \\
&\leqslant C\int_{[0,L]^2}\int_0^T \mathbb{E}\left[|f(\nabla u_\varepsilon(t,x))|^p\right]^{\frac{1}{p}} \,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant C\int_0^T \sup_{x\in\mathbb{T}^2}\mathbb{E}\left[|f(\nabla u_\varepsilon(t,x))|^p\right]^{\frac{1}{p}} \,\mathrm{d}t
\end{aligned}
$$

where constants $C = C_{L,T} > 0$. This last expression tends to zero if ε goes to zero. This finishes the proof. □

3.4. Proof of the Main Result

Finally, we are able to prove the convergence of the mild solutions from Theorem 3.2 using Corollary 3.12. Let us first consider $v_\varepsilon = u_\varepsilon - Z_\varepsilon - e^{tA}u_0$, as the mild solution of (3.5). In order to verify the main result of this chapter, we have to prove the following convergence for each $p \geqslant 1$:

$$
v_\varepsilon \xrightarrow{\varepsilon\to 0} 0 \quad \text{in} \quad L^p\left(\Omega, C^0([0,T]\times\mathbb{T}^2)\right).
$$

Proof of Theorem 3.2. For each $p > 2$ there exists a $\delta \in (0,1)$ such that $p > \frac{4}{2-\delta}$ or equivalently $\frac{(2+\delta)p}{4(p-1)} \in (0,1)$ is fulfilled. Now, by applying Morrey's inequality (Theorem A.23) and Hölder's inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\|v_\varepsilon\|^p_{C^0([0,T]\times\mathbb{T}^2)}\right] \leqslant C_{T,L,\delta}\mathbb{E}\left[\sup_{t\in[0,T]}\|v_\varepsilon(t)\|^p_{\mathcal{H}^{1+\delta}}\right] \\
&\leqslant C_{T,L,\delta}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t e^{(t-s)A}\nabla\cdot f(\nabla u_\varepsilon(s))\,\mathrm{d}s\right\|^p_{\mathcal{H}^{1+\delta}}\right] \\
&\leqslant C_{T,L,\delta}\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t (t-s)^{\frac{-(2+\delta)}{4}}\|f(\nabla u_\varepsilon(s))\|_{L^2}\,\mathrm{d}s\right)^p\right] \\
&\leqslant C_{T,L,\delta}\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t (t-s)^{\frac{-(2+\delta)p}{4(p-1)}}\,\mathrm{d}s\right)^{(p-1)}\int_0^t\|f(\nabla u_\varepsilon(s))\|^p_{L^2}\,\mathrm{d}s\right] \\
&\leqslant C_{T,L,\delta}\mathbb{E}\left[\int_0^T\|f(\nabla u_\varepsilon(s))\|^p_{L^2}\,\mathrm{d}s\right] \\
&\leqslant C_{p,T,L,\delta}\mathbb{E}\left[\|f(\nabla u_\varepsilon)\|^p_{L^p([0,T]\times\mathbb{T}^2)}\right] \xrightarrow[\varepsilon\to 0]{} 0
\end{aligned}
$$

which follows immediately from Corollary 3.12. Finally, by applying Hölder's inequality, this establishes the desired result for each $p \geqslant 1$. □

This convergence gives us the reduction to the linearised form of the SPDE

$$\partial_t u \approx -\Delta^2 u + \sigma \partial_t W$$

which will be the subject of our investigation in the next chapter.

4. Growth Results of the Linearised Mild Solution

In this chapter, we analyse regularity results and the growth behaviour of the mild stochastic solution of the partial differential equation

$$\partial_t u_\varepsilon(t,x) = -\delta\Delta^2 u_\varepsilon(t,x) - \nabla \cdot \frac{\nabla u_\varepsilon(t,x)}{1+|\nabla u_\varepsilon(t,x)|^2} + \sigma\partial_t W_\varepsilon(t,x), \tag{4.1}$$

introduced in (1.1) and in a regularised form in (3.1). In the following analysis, we multiply the operator $-\Delta^2$ by a small factor $\delta > 0$. This adjustment affects the number and magnitude of positive eigenvalues of the linear operator $-\delta\Delta^2 - \Delta$, which appears after linearisation since $\nabla \cdot f(\nabla u) \approx \Delta u$ and represents the linear part of (4.1). As a result, we can make precise statements about the stability of the equation, since positive eigenvalues lead to linear instability (cf. Definition 4.7) which results in a positive growth rate. To analyse this instability, we will start with the comparison of the mild solution

$$u_\varepsilon(t) := e^{-t\delta\Delta^2}u_0 + \int_0^t e^{-(t-s)\delta\Delta^2}\mathfrak{F}\left(u_\varepsilon(s)\right)\,\mathrm{d}s + \sigma\int_0^t e^{-(t-s)\delta\Delta^2}\,\mathrm{d}W_\varepsilon(s) \tag{4.2}$$

of (4.1) with the mild solution of the partial differential equation without additive space-time white noise term.

4.1. Linearisation and Decomposition

Before presenting the main results, we outline a decomposition approach to the linearised form of (4.1), which we use to analyse the growth rate of the mild solution u_ε, defined in Definition 3.1. We first examine the growth rate in $\mathcal{H}^1$ using a Grönwall approach (cf. Lemma A.19) to compare the mild solution u_ε with the deterministic mild solution u_{det}, which excludes the noise term. To establish the main growth result in C^1, we follow a decomposition approach detailed in Section 4.2. Nevertheless, we use a different approach for the growth rate of the C^1-norm up to a stopping time, applying the factorisation method (see [27], Section 4.3.2).

4.1.1. Grönwall Argument

Definition 4.1 (With high probability)**.** *We say that an event $E_\sigma \in \mathcal{F}$ holds* with high probability *if and only if*

$$1 - \mathbb{P}(E_\sigma) \ll 1,$$

i.e. the probability of the complement tends to zero with respect to the underlying parameter $1 - \mathbb{P}(E_\sigma) \to 0$ if σ tends to 0.

To compare the two mild solutions u_{det} and u_ε, we first need the following result to apply the *Henry–Grönwall lemma* (cf. Lemma A.19).

Corollary 4.2. *Let Z_ε be the stochastic convolution corresponding to the linear operator $-\delta\Delta^2$ (see (4.2)). Then there is a constant $C = C_{L,\delta,\varepsilon} > 0$ such that*

$$\|Z_\varepsilon(t)\|_{\mathcal{H}^1} \leqslant C$$

holds for each $t \geqslant 0$ with high probability*.*

Proof. By applying the *Markov inequality* and *Itô's isometry*, we obtain

$$\begin{aligned}
\mathbb{P}\left(\|Z_\varepsilon(t)\|_{\mathcal{H}^1} \geqslant C\right) &\leqslant C^{-2}\mathbb{E}\|Z_\varepsilon(t)\|^2_{\mathcal{H}^1} \\
&= C^{-2}\sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k \int_0^t e^{-2(t-s)\delta\mu_k^2}\,\mathrm{d}s \\
&\leqslant C^{-2}\sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{1}{2\delta\mu_k} \\
&\leqslant C_{L,\delta,\varepsilon}C^{-2},
\end{aligned}$$

where $C_{L,\delta,\varepsilon} > 0$ is a constant that depends only on L and ε. By choosing C sufficiently large, the statement follows. □

We now compare the mild solution u_{det} of the deterministic partial differential equation

$$\partial_t u_{\mathrm{det}} = -\delta\Delta^2 u_{\mathrm{det}} - \nabla\cdot f\left(\nabla u_{\mathrm{det}}\right) \tag{4.3}$$

with the mild solution u_ε of the stochastic partial differential equation

$$\partial_t u_\varepsilon = -\delta\Delta^2 u_\varepsilon - \nabla\cdot f\left(\nabla u_\varepsilon\right) + \sigma\partial_t W_\varepsilon.$$

Thus, we require the operator norm of $\left(e^{tA}\right)_{t\geqslant 0}$, where we set $A := -\delta\Delta^2$. Hence, we obtain from (A.8) for $\beta > \alpha$ the following bound for the operator norm:

$$\left\|e^{tA}\right\|_{L\left(\mathcal{H}^\alpha,\mathcal{H}^\beta\right)} \leqslant \left(\frac{\beta-\alpha}{2e\delta}\right)^{\frac{\beta-\alpha}{4}} t^{\frac{\alpha-\beta}{4}}.$$

Theorem 4.3. *Let u_ε be the mild solution of* (4.1) *and u_{det} be the deterministic mild solution of* (4.3)*. Let $C_{\varepsilon,\delta} > 0$ be constants that only depend on ε and δ. Then we obtain*

$$\|u_{\text{det}}(t) - u_\varepsilon(t)\|_{\mathcal{H}^1} \leqslant \sigma C_\varepsilon e^{t\frac{\pi}{2e\delta}} \left[1 + t^{\frac{3}{2}} \left(\frac{\pi}{2e\delta}\right)^{\frac{3}{2}}\right]$$

with high probability *for each $t \geqslant 0$.*

Proof. For each $t \geqslant 0$ we obtain

$$\begin{aligned}
&\|u_{\text{det}}(t) - u_\varepsilon(t)\|_{\mathcal{H}^1} \\
&\leqslant \sigma\|Z_\varepsilon(t)\|_{\mathcal{H}^1} + \int_0^t \left\|e^{(t-s)A}\nabla\cdot\right\|_{L(L^2,\mathcal{H}^1)} \left\|f\left(u_{\text{det}}(s)\right) - f\left(u_\varepsilon(s)\right)\right\|_{L^2} \,\mathrm{d}s \\
&\leqslant \sigma\|Z_\varepsilon(t)\|_{\mathcal{H}^1} + \int_0^t \left\|e^{(t-s)A}\right\|_{L(\mathcal{H}^{-1},\mathcal{H}^1)} \|u_{\text{det}}(s) - u_\varepsilon(s)\|_{\mathcal{H}^1} \,\mathrm{d}s \\
&\leqslant \sigma\|Z_\varepsilon(t)\|_{\mathcal{H}^1} + \frac{1}{\sqrt{2e\delta}} \int_0^t (t-s)^{-\frac{1}{2}} \|u_{\text{det}}(s) - u_\varepsilon(s)\|_{\mathcal{H}^1} \,\mathrm{d}s.
\end{aligned}$$

Furthermore, to apply the *Henry–Grönwall lemma* (cf. Lemma A.19), we first need

$$\theta := \frac{\Gamma(\frac{1}{2})^2}{2e\delta} = \frac{\pi}{2e\delta}$$

and for $z > 0$ the inequality

$$\begin{aligned}
E'_{\frac{1}{2}}(z) &:= \frac{\partial}{\partial z} \sum_{n=0}^{\infty} \frac{z^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \\
&= \sum_{n=1}^{\infty} \frac{\frac{n}{2} z^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2}+1)} \\
&= \sum_{k=1}^{\infty} \frac{k z^{k-1}}{\Gamma(k+1)} + \sum_{l=0}^{\infty} \frac{\frac{2l+1}{2} z^{\frac{2l-1}{2}}}{\Gamma(l+\frac{3}{2})} \\
&= \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} + \sum_{l=0}^{\infty} \frac{z^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})} \\
&\leqslant e^z + \frac{1}{\sqrt{\pi z}} + \sqrt{z} \sum_{l=1}^{\infty} \frac{z^{l-1}}{\Gamma(l)} \\
&= e^z\left(1+\sqrt{z}\right) + \frac{1}{\sqrt{\pi z}}.
\end{aligned}$$

Now, by applying the *Henry–Grönwall lemma*, we obtain

$$
\begin{aligned}
&\|u_{\det}(t) - u_\varepsilon(t)\|_{\mathcal{H}^1} \\
&\leqslant \sigma\|Z_\varepsilon(t)\|_{\mathcal{H}^1} + \theta \int_0^t E'_{\frac{1}{2}}\left(\theta(t-s)\right) \sigma \left\|Z_\varepsilon(s)\right\|_{\mathcal{H}^1} \,\mathrm{d}s \\
&\leqslant \sigma\|Z_\varepsilon(t)\|_{\mathcal{H}^1} + \frac{\sigma\pi}{2e\delta} \int_0^t e^{\frac{\pi}{2e\delta}(t-s)} \left(1 + \sqrt{\frac{\pi}{2e\delta}(t-s)}\right) \|Z_\varepsilon(s)\|_{\mathcal{H}^1} \,\mathrm{d}s \\
&\qquad + \frac{\sigma\pi}{2e\delta} \int_0^t \sqrt{\frac{2e\delta}{\pi s}} \,\|Z_\varepsilon(s)\|_{\mathcal{H}^1} \,\mathrm{d}s \\
&\leqslant \sigma C_\varepsilon \left[1 + e^{t\frac{\pi}{2e\delta}} + \frac{2}{3}\, e^{t\frac{\pi}{2e\delta}} \left(\frac{\pi t}{2e\delta}\right)^{\frac{3}{2}} + \sqrt{\frac{\pi t}{2e\delta}}\right] \\
&\leqslant \sigma C_\varepsilon \left[1 + e^{t\frac{\pi}{2e\delta}} \left(1 + \left(\frac{\pi t}{2e\delta}\right)^{\frac{3}{2}}\right)\right] \\
&\leqslant \sigma C_\varepsilon e^{t\frac{\pi}{2e\delta}} \left[1 + t^{\frac{3}{2}} \left(\frac{\pi}{2e\delta}\right)^{\frac{3}{2}}\right]
\end{aligned}
$$

for each $t \geqslant 0$ *with high probability*. This shows the result of Theorem 4.3. □

4.1.2. Decomposition of the Eigenspaces

In this chapter, we consider the linearisation of the equation

$$\partial_t u_\varepsilon = -\delta\Delta^2 u_\varepsilon - \nabla \cdot \frac{\nabla u_\varepsilon}{1 + |\nabla u_\varepsilon|^2} + \sigma\partial_t W_\varepsilon.$$

Since $\nabla \cdot f(\nabla u) \approx \Delta u$ holds for sufficiently small ∇u, it is natural to linearise the equation by incorporating the nonlinear term into the linear operator. This yields the following stochastic partial differential equation:

$$\partial_t u = -\left(\delta\Delta^2 + \Delta\right) u_\varepsilon + \nabla \cdot \frac{\nabla u_\varepsilon |\nabla u_\varepsilon|^2}{1 + |\nabla u_\varepsilon|^2} + \partial_t W_\varepsilon. \tag{4.4}$$

We analyse the transformed linear operator $A_\delta := -\delta\Delta^2 - \Delta$. Within this framework, the remaining nonlinearity is of higher order, since

$$\nabla \cdot \frac{\nabla u_\varepsilon}{1 + |\nabla u_\varepsilon|^2} = \Delta u_\varepsilon + \frac{\nabla u_\varepsilon |\nabla u_\varepsilon|^2}{1 + |\nabla u_\varepsilon|^2}$$

holds. Since the new linear operator A_δ is a real linear combination of the Laplace and the Bilaplace operator, it is self-adjoint and thus generates a analytic semigroup (see [68, Section 2.6.]). Furthermore, we directly conclude $A_\delta e_k = \lambda_k^{(\delta)} e_k$ for each $k \in \mathcal{Z}$, where the corresponding eigenvalues are

$$\lambda_k := \lambda_k^{(\delta)} := -\delta\mu_k^2 + \mu_k.$$

Remark 4.4. *For simplicity, we do not denote the eigenvalues* $(\lambda_k)_{k\in\mathcal{Z}}$ *with an index* δ*, even though they certainly depend on the parameter* δ*. The same applies to explicit eigenvalues, such as the largest and smallest positive eigenvalues, as well as to upper bounds of the sequence of eigenvalues.*

We use this approach to precisely identify linear instability (cf. Definition 4.7) and to establish an upper bound on the growth rate. The following remark offers further insight and gives a motivation for Definition 4.7:

Remark 4.5. *Elementary analysis shows that*

$$(\lambda_k)_{k\in\mathcal{Z}} \subset \left(-\infty, \frac{1}{4\delta}\right],$$

since the eigenvalues $\lambda_k = -\delta\mu_k^2 + \mu_k$ *for* $k \in \mathcal{Z}$ *lie on the graph of the function*

$$g_\delta : (0,\infty) \to \left(-\infty, \frac{1}{4\delta}\right], \quad x \mapsto -\delta x^4 + x^2,$$

which is positive if and only if $x \in \left(0, \frac{1}{\sqrt{\delta}}\right)$ *holds. The graph of* g_δ *is illustrated in the following figure:*

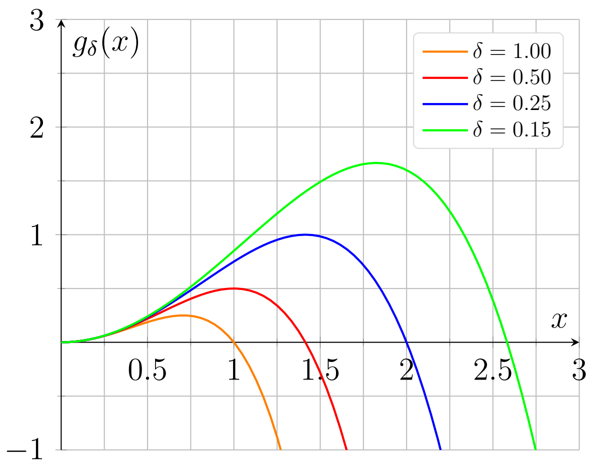

Figure 4.1.: Graph of Eigenvalues of A_δ According to δ.

In order for $A_\delta = -\delta\Delta^2 + \Delta$ *to have at least one positive eigenvalue, it is necessary that at least one eigenvalue* μ_k *of the negative Laplace operator* $-\Delta$ *is smaller than* $\frac{1}{\delta}$*, which is the zero of the function* g_δ *(cf. Figure 4.1). Hence, we need the parameter* δ *to be smaller than* $\frac{L^2}{4\pi^2|k|^2}$ *or equivalently*

$$\frac{2\pi|k|}{L} < \frac{1}{\sqrt{\delta}}$$

to hold for at least one k, and thus, by rotational symmetry of μ_k, for at least four $k \in \mathcal{Z}$. This leads to linear instability for $L \gg 1$ or, equivalently, $\delta \ll 1$, since there is at least one positive eigenvalue λ_k.

Remark 4.6. *For illustrative purpose, we approximate the unstable region in the Fourier space using $|k| := |k_1| + |k_2|$ for each $k \in \mathcal{Z}$. It then follows that the dimension of the subspace*

$$\hat{X}_p^+ := \left\{ \sum_{k \in \mathcal{Z}} u_k e_k \in L^p\left(\mathbb{T}^2\right) : u_k = 0 \text{ for } k \in \mathcal{Z} \text{ with } -\delta \left(\frac{2\pi|k|}{L}\right)^2 + 1 \leqslant 0 \right\} \tag{4.5}$$

depends on the parameters L and δ. More precisely, as δ decreases or L increases, the dimension increases. This is because the dimension corresponds to the number of positive eigenvalues of A_δ. In addition, for $\delta = 1$, the dimension of $\hat{X}_p^+$ equals the number of points up to a certain color in Figure 4.2. Specifically, the green, blue, and red points represent all indices k corresponding to the eigenvalues μ_k of Δ for $|k| = 1$, $|k| = 2$, and $|k| = 3$.

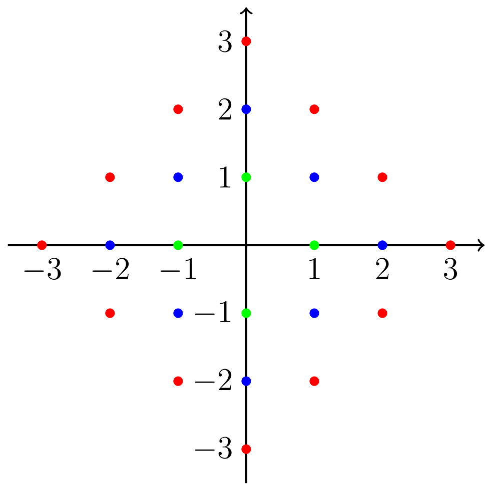

Figure 4.2.: Illustration of the indices $k \in \mathcal{Z}$.

Thus, for $n_* \in \mathbb{N}$, with $\frac{L}{2\pi\sqrt{\delta}} \in (n_*, n_* + 1]$, we obtain

$$\hat{n}_\delta := \dim\left(\hat{X}_p^+\right) = \sum_{l=1}^{n_*} 4l = 2n_*\left(n_* + 1\right), \tag{4.6}$$

since for each $l \in \mathbb{N} \setminus \{0\}$ there are $l + 1$ possible vectors $x \in \mathbb{N}^2$ with $|x| = l$, and therefore $4l$ possible vectors $y \in \mathcal{Z}$, by multiplying the number of vectors in $\mathbb{N}^2$ by 4 and subtracting the respective four corner points of the squares.

Definition 4.7 (Linear instability)**.** *We say that the mild solution u_ε (4.2) exhibits a* linear instability *if at least one eigenvalue λ_k of A_δ is positive, i.e., there exists at least one $k \in \mathcal{Z}$ such that*

$$\frac{2\pi|k|}{L} < \frac{1}{\sqrt{\delta}}.$$

Under this assumption, we define as $\lambda_{\max} := \lambda_{\max}^{(\delta)} \in (0, \frac{1}{4\delta}]$ the largest positive eigenvalue of A_δ which depends on the parameter $\delta > 0$.

In the following, we assume that the mild solution u_ε (cf. (4.2)) is *linearly unstable* such that the positive subspace $\hat{X}_p^+$ defined in (4.5) is non-empty.

4.1.3. Semigroup in the Linearisation Environment

In this subsection, we will analyse the family of operators $\left(e^{tA_\delta}\right)_{t\geqslant 0}$.

Lemma 4.8. *For each $p \in [1, \infty)$ the family of operators $\left(e^{tA_\delta}\right)_{t\geq 0}$ forms an analytic semigroup on $L^p(\mathbb{T}^2)$.*

Proof. Since A_δ is a real linear combination of $-\Delta^2$ and $-\Delta$, it is self-adjoint. Consequently, $\left(e^{tA_\delta}\right)_{t\geqslant 0}$ is an analytic semigroup on $L^p(\mathbb{T}^2)$ for each $p \in [1, \infty)$ (see [68, Section 2.6.]). □

Moreover, the family $\left(e^{tA_\delta}\right)_{t\geqslant 0}$ extends to an analytic semigroup on $W^{\alpha,p}(\mathbb{T}^2)$ for every $\alpha > 0$ and $p \in (1, \infty)$. This follows from interpolation estimates. In particular, by the *Brezis-Mironescu inequality* (cf. Theorem A.22), which is a corollary of *Gagliardo-Nirenberg interpolation inequality* (cf. Theorem A.21), we have

$$\|e^{tA_\delta}u\|_{W^{1+\alpha,p}} \leqslant C\|e^{tA_\delta}u\|_{W^{2,p}}^{\frac{1+\alpha}{2}}\|e^{tA_\delta}u\|_{L^p}^{\frac{1-\alpha}{2}}$$

for a constant $C > 0$. By applying Theorem A.21, we conclude that $\left(e^{tA_\delta}\right)_{t\geqslant 0}$ is an analytic semigroup on $W^{\alpha,p}(\mathbb{T}^2)$ for each $\alpha > 0$ and $p \in (1, \infty)$. Furthermore, an upper bound for the operator norm of the semigroup $\left(e^{tA_\delta}\right)_{t\geqslant 0}$ is established to enable analysis within a comparable framework as in Chapter 3.

Definition 4.9. *Let $\delta > 0$, $K_\delta \in (0, \delta)$, and $\theta \in (0, 1)$. We define*

$$\lambda_{\max}^{(\theta)} := \lambda_{\max} + \theta\left(\frac{1}{4K_\delta} - \lambda_{\max}\right).$$

From the bound $\lambda_{\max} \leqslant \frac{1}{4\delta} < \frac{1}{4K_\delta}$ we obtain $\lambda_{\max}^{(\theta)} > \lambda_{\max}$. To control the upper bound on the growth rate of $u_\varepsilon(t)$ in $\mathcal{H}^1$ and C^1, for $t \geqslant 0$, it is crucial to keep $\lambda_{\max}^{(\theta)}$ as small as possible. More precisely, we choose $K_\delta \in (\delta - \xi, \delta)$ for sufficiently small $\xi > 0$, and $\theta \in (0, 1)$ sufficiently small to ensure that $\lambda_{\max}^{(\theta)} - \lambda_{\max} > 0$ remains small.

Lemma 4.10. *For $\beta > \alpha$, $K_\delta \in (0,\delta)$, $\theta \in (0,1)$, and $\lambda_{\max}^{(\theta)}$ as in Definition 4.9, we have*

$$\left\| e^{t(-\delta\Delta^2-\Delta)} \right\|_{L\left(\mathcal{H}^\alpha,\mathcal{H}^\beta\right)} \leqslant e^{t\lambda_{\max}^{(\theta)}} \left(\frac{\beta-\alpha}{4e\theta\left(\delta-K_\delta\right)} \right)^{\frac{\beta-\alpha}{4}} t^{\frac{\alpha-\beta}{4}}$$

for each $t \geqslant 0$.

Proof. To show the upper bound, we need an upper bound for the sequence of the eigenvalues $\left(\lambda_k\right)_{k\in\mathcal{Z}}$. More precisely, for $K_\delta \in (0,\delta)$, we have

$$\lambda_k = -\delta\mu_k^2 + \mu_k \leqslant \left(-\delta + K_\delta\right)\mu_k^2 + \frac{1}{4K_\delta}$$

which is equivalent to the inequality

$$1 \leqslant K_\delta \mu_k + \frac{1}{4K_\delta \mu_k}.$$

The latter holds for each $k \in \mathcal{Z}$, as the auxiliary function

$$h : (0,\infty) \to \mathbb{R}, \quad h(z) := Cz + \frac{1}{4Cz}$$

has the global minimum of 1 independent of the choice of the constant $C > 0$. Moreover, this upper bound can be refined by employing a convex combination of the eigenvalues with $\theta \in (0,1)$. In particular, we obtain

$$\begin{aligned}
\lambda_k &= -\delta\mu_k^2 + \mu_k \\
&= (1-\theta)\left(-\delta\mu_k^2 + \mu_k\right) + \theta\left(-\delta\mu_k^2 + \mu_k\right) \\
&\leqslant (1-\theta)\lambda_{\max} + \theta\left[\left(-\delta + K_\delta\right)\mu_k^2 + \frac{1}{4K_\delta}\right] \\
&= \theta\left(-\delta + K_\delta\right)\mu_k^2 + \lambda_{\max} + \theta\left[\frac{1}{4K_\delta} - \lambda_{\max}\right] \\
&= \theta\left(-\delta + K_\delta\right)\mu_k^2 + \lambda_{\max}^{(\theta)}.
\end{aligned}$$

Finally, by applying this inequality to the operator norm of the semigroup, we derive

$$\begin{aligned}
\left\| e^{t(-\delta\Delta^2-\Delta)} \right\|_{L\left(\mathcal{H}^\alpha,\mathcal{H}^\beta\right)} &\leqslant e^{t\lambda_{\max}^{(\theta)}} \left\| e^{t\theta(-\delta+K_\delta)\Delta^2} \right\|_{L\left(\mathcal{H}^\alpha,\mathcal{H}^\beta\right)} \\
&\leqslant e^{t\lambda_{\max}^{(\theta)}} \left(\frac{\beta-\alpha}{4e\theta\left(\delta-K_\delta\right)} \right)^{\frac{\beta-\alpha}{4}} t^{\frac{\alpha-\beta}{4}}
\end{aligned}$$

for each $t \geqslant 0$. This finishes the proof. □

4.1.4. Growth of the Deterministic Solution

As established above, the difference $\|u_\varepsilon(t) - u_{\text{det}}(t)\|_{\mathcal{H}^1}$ grows at the rate

$$\mathcal{O}\left(\sigma\left(C + Ct^{\frac{1}{2}}\right)e^{t\frac{\pi}{2e\delta}}\right) \qquad \text{for each } t \geqslant 0.$$

The following subsection examines a growth condition for u_{det} in $\mathcal{H}^1(\mathbb{T}^2)$ with respect to the time variable t. To proceed, we define the stopping time

$$\hat{\tau} \coloneqq \inf\left\{s > 0 : \|u_\varepsilon(s)\|_{W^{1,6}} \geqslant r_\sigma\right\},$$

where r_σ denotes a radius that depends solely on σ and satisfies the condition

$$0 < r_\sigma^3 \ll \sigma \ll r_\sigma \ll 1.$$

Lemma 4.11. *Let $u_0 = 0$, $\alpha \in (-1, 3)$ and let $C = C_{\alpha,\delta,L,\theta} > 0$ be a constant that only depends on α, L, θ, δ, and K_δ. Then we obtain*

$$\|u_{\text{det}}(t)\|_{\mathcal{H}^\alpha} \leqslant Ce^{t\lambda_{\max}^{(\theta)}}t^{\frac{3-\alpha}{4}}r_\sigma^3,$$

for each $t \in (0, \hat{\tau}]$, where $\lambda_{\max}^{(\theta)}$ is the upper bound of the eigenvalues defined in Definition 4.9.

Proof. For $t \in (0, \hat{\tau}]$ we have

$$\begin{aligned}
\|u_{\text{det}}(t)\|_{\mathcal{H}^\alpha} &\leqslant \int_0^t \left\| e^{(t-s)(-\delta\Delta^2 - \Delta)}\nabla\cdot\frac{\nabla u(s)\ |\nabla u(s)|^2}{1 + |\nabla u(s)|^2}\right\|_{\mathcal{H}^\alpha} \mathrm{d}s \\
&\leqslant \int_0^t \left\| e^{(t-s)(-\delta\Delta^2 - \Delta)}\right\|_{L(\mathcal{H}^{-1},\mathcal{H}^\alpha)} \left\|\frac{\nabla u(s)\ |\nabla u(s)|^2}{1 + |\nabla u(s)|^2}\right\|_{L^2} \mathrm{d}s \\
&\leqslant \int_0^t e^{(t-s)\lambda_{\max}^{(\theta)}} \left\| e^{(t-s)\theta(-\delta+K_\delta)\Delta^2}\right\|_{L(\mathcal{H}^{-1},\mathcal{H}^\alpha)} \left\|\nabla u(s)\ |\nabla u(s)|^2\right\|_{L^2} \mathrm{d}s \\
&\leqslant \int_0^t \left(\frac{\alpha+1}{4e\theta\,(\delta - K_\delta)}\right)^{\frac{\alpha+1}{4}} e^{(t-s)\lambda_{\max}^{(\theta)}}(t-s)^{\frac{-\alpha-1}{4}} \|u(s)\|_{W^{1,6}}^3 \,\mathrm{d}s \\
&= \frac{4}{3-\alpha}\left(\frac{\alpha+1}{4e\theta\,(\delta - K_\delta)}\right)^{\frac{\alpha+1}{4}} e^{t\lambda_{\max}^{(\theta)}}t^{\frac{3-\alpha}{4}} \sup_{s\in[0,t]}\left\{\|u(s)\|_{W^{1,6}}^3\right\} \\
&\leqslant C_{\alpha,\delta,L,\theta}e^{t\lambda_{\max}^{(\theta)}}t^{\frac{3-\alpha}{4}}r_\sigma^3.
\end{aligned}$$

This proves the claim. □

Assumption 4.12. *Let r_0 and r_σ denote radii that satisfy the following conditions:*

$$0 < r_\sigma^3 < r_0 < \sigma < r_\sigma \ll 1.$$

Theorem 4.13. *Let $\|u_0\|_{\mathcal{H}^1} < r_0$ and let $\delta > 0$ be sufficiently small such that we have linear instability (cf. Definition 4.7) and $\pi/2e < \lambda_{\max}^{(\theta)}$ holds.*
Then we obtain for $t \in (0, \hat{\tau}]$ the exponential growth rate

$$\|u_\varepsilon(t)\|_{\mathcal{H}^1} \lesssim \sigma e^{t\lambda_{\max}^{(\theta)}} \left(1 + \sqrt{t}\right)$$

with high probability, *where $\lambda_{\max}^{(\theta)}$ is the upper bound of the eigenvalues defined in Definition 4.9.*

Proof. By applying the triangle inequality, Theorem 4.3, and Lemma 4.11, there exist constants $C = C_{\alpha,\delta,L,\theta} > 0$ such that

$$\begin{aligned}
\|u_\varepsilon(t)\|_{\mathcal{H}^1} &\leqslant \left\|e^{tA}u_0\right\|_{\mathcal{H}^1} + \|u_{\det}(t)\|_{\mathcal{H}^1} + \|u_{\det}(t) - u_\varepsilon(t)\|_{\mathcal{H}^1} \\
&\leqslant e^{t\lambda_{\max}^{(\theta)}} \|u_0\|_{\mathcal{H}^1} + Ce^{t\lambda_{\max}^{(\theta)}} r_\sigma^3 \sqrt{t} + Ce^{t\frac{\pi}{2e}} \sigma \left(1 + \sqrt{t}\right) \\
&\leqslant e^{t\lambda_{\max}^{(\theta)}} C \left[r_0 + r_\sigma^3 \sqrt{t} + \sigma \left(1 + \sqrt{t}\right)\right] \\
&\leqslant Ce^{t\lambda_{\max}^{(\theta)}} \left[r_0 + \sigma\right] \left(1 + \sqrt{t}\right) \\
&\leqslant Ce^{t\lambda_{\max}^{(\theta)}} \sigma \left(1 + \sqrt{t}\right)
\end{aligned}$$

with high probability for each $t \in (0, \hat{\tau}]$. □

4.2. Exponential Growth up to a Stopping Time

This subsection examines more general arguments in $C^1(\mathbb{T}^2)$. Let r_0 and r_σ be radii as defined in Assumption 4.12. Let

$$\tau := \inf\left\{s > 0 : \|u_\varepsilon(s)\|_{C^1} \geqslant r_\sigma\right\} \tag{4.7}$$

be a stopping time on the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P}\right)$. For each $\theta \in (0, 1)$ and $\delta > 0$, the linear operator $A_\delta - \lambda_{\max}^{(\theta)} \operatorname{Id}$ is an infinitesimal generator of an analytic semigroup, whereby Id denotes the identity operator on $L^1(\mathbb{T}^2)$. Thus, there is a positive constant $C > 0$ such that

$$\|e^{tA_\delta}u_0\|_{C^1} \leqslant Ce^{t\lambda_{\max}^{(\theta)}} \|u_0\|_{C^1} \tag{4.8}$$

for each $t \geqslant 0$ (cf. [68, p.70]). This section focuses on the mild solution of the deterministic partial differential equation

$$\partial_t u = -\delta\Delta^2 u - \Delta u + \nabla \cdot \frac{\nabla u \, |\nabla u|}{1 + |\nabla u|^2}$$

with initial value $u_0 = 0$.

Lemma 4.14. *For each $t > 0$, $p \geqslant 1$ and $\alpha, \beta \in \mathbb{R}$, with $\alpha < \beta$, there exists a constant $C = C_{p,\alpha,\beta,\theta,\delta} > 0$ such that*

$$\left\| e^{t(-\delta\Delta^2 - \Delta)} \right\|_{L(W^{\alpha,p}, W^{\beta,p})} \leqslant C e^{t\lambda_{\max}^{(\theta)}} t^{-\frac{\beta-\alpha}{4}}.$$

Proof. Even though [38, Lemma A.7] is only formulated for semigroups generated by $-(-\Delta)^\sigma$ for $\sigma \in (0,1]$ on $\mathbb{T}^d$, the proof is applicable in its entirety to the case of $\sigma > 1$, as the integrability condition $(1 + |\cdot|^{d+1})\varphi \in L^1\left(\mathbb{T}^d\right)$ for the function $\varphi(\xi) = e^{-|\xi|^{2\sigma}}$, as required in [38, Lemma A.5] is even easier to verify. Hence, by using the inequality $\lambda_k \leqslant \theta\left(-\delta + K_\delta\right) \mu_k^2 + \lambda_{\max}^{(\theta)}$, with $\theta\left(-\delta + K_\delta\right) < 0$, as in the proof of Lemma 4.10 and by applying [38, Lemma A.7], we derive

$$\begin{aligned} \left\| e^{t(-\delta\Delta^2 - \Delta)} \right\|_{L(W^{\alpha,p}, W^{\beta,p})} &\leqslant e^{t\lambda_{\max}^{(\theta)}} \left\| e^{t\theta(-\delta + K_\delta)\Delta^2} \right\|_{L(W^{\alpha,p}, W^{\beta,p})} \\ &\leqslant C e^{t\lambda_{\max}^{(\theta)}} t^{-\frac{\beta-\alpha}{4}} \end{aligned}$$

for each $t > 0$. □

Corollary 4.15. *Let τ be the stopping time from (4.7) and let $\lambda_{\max}^{(\theta)}$, as defined in Definition 4.9. For $\alpha \in (0,1)$ we have the upper bound*

$$\|u_{\det}(t)\|_{C^1} \leqslant C e^{t\lambda_{\max}^{(\theta)}} t^{\frac{2-\alpha}{4}} r_\sigma^3$$

for each $t \in [0,\tau]$ and a constant $C = C_{\delta,\alpha,\theta,L} > 0$.

Proof. By applying *Morrey's inequality* (cf. Theorem A.23), for $\alpha > 0$ and $p > 2$ such that $\alpha p > d = 2$, and Lemma 4.14 we obtain for $t \in (0,\tau]$ the following inequality

$$\begin{aligned} \|u_{\det}(t)\|_{C^1} &\leqslant C \|u_{\det}(t)\|_{W^{1+\alpha,p}} \\ &\leqslant C \int_0^t \left\| e^{(t-s)(-\delta\Delta^2 - \Delta)} \nabla \cdot \frac{\nabla u(s)\, |\nabla u(s)|^2}{1 + |\nabla u(s)|^2} \right\|_{W^{1+\alpha,p}} \mathrm{d}s \\ &\leqslant C \int_0^t \left\| e^{(t-s)(-\delta\Delta^2 - \Delta)} \right\|_{L(W^{-1,p}, W^{1+\alpha,p})} \left\| \frac{\nabla u(s)\, |\nabla u(s)|^2}{1 + |\nabla u(s)|^2} \right\|_{L^p} \mathrm{d}s \\ &\leqslant e^{t\lambda_{\max}^{(\theta)}} \int_0^t \left\| e^{(t-s)\theta(-\delta + K_\delta)\Delta^2} \right\|_{L(W^{-1,p}, W^{1+\alpha,p})} \|u(s)\|_{W^{1,3p}}^3 \mathrm{d}s \\ &\leqslant C e^{t\lambda_{\max}^{(\theta)}} \int_0^t (t-s)^{\frac{-2-\alpha}{4}} \|u(s)\|_{W^{1,3p}}^3 \mathrm{d}s \\ &\leqslant C e^{t\lambda_{\max}^{(\theta)}} t^{\frac{2-\alpha}{4}} \sup_{s \in [0,t]} \|u(s)\|_{C^1}^3 \\ &\leqslant C e^{t\lambda_{\max}^{(\theta)}} t^{\frac{2-\alpha}{4}} r_\sigma^3. \end{aligned}$$

This proves the claim. □

4.3. Stochastic Convolution in the Context of Linearisation

In this section, we focus on the stochastic convolution $W_{A_\delta}^{(\varepsilon)}$ regarding the linear operator $A_\delta = -\delta\Delta^2 - \Delta$, as in the case of Definition 2.20, which we will introduce in Definition 4.16.

4.3.1. Upper Bounds and Auxiliary Results

Definition 4.16. *We define the stochastic convolution regarding the linear operator A_δ for each $t \geqslant 0$ and $x \in \mathbb{T}^2$, via*

$$W_{A_\delta}^{(\varepsilon)}(t,x) := \int_0^t e^{(t-s)A_\delta}\,\mathrm{d}W_\varepsilon(s,x) = \sum_{k\in\mathcal{Z}} \alpha_k^{(\varepsilon)} e_k(x) \int_0^t e^{(t-s)\lambda_k}\,\mathrm{d}\beta_k(s).$$

Theorem 4.17. *Let $m \in \{0,1\}$. Let $\varepsilon > 0$ and $\eta > 0$ such that there exists some $\psi > 0$ with*

$$\left(\left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^\eta\right)_{k\in\mathcal{Z}} \in \mathcal{O}\left(|k|^{-\psi}\right).$$

Then for each $p \geqslant 1$ there exists a constant $C = C_{p,\eta,L,\delta} > 0$ such that

$$\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^m}^p\right]^{\frac{1}{p}} \leqslant C\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{\mathcal{H}^{m+\eta}}^2\right]^{\frac{1}{2}}$$

holds for every $t \geqslant 0$.

Proof. Because the eigenvalues of the linear operator A_δ are distributed in a way that leads to a positive growth rate over time, the proof of Theorem 2.25 cannot be applied directly. However, the argument follows a similar approach.
For $\varepsilon > 0$ let $p > 2$ and $\eta > 0$ such that $\eta p > 2$, and that there exists some $\psi > 0$ such that

$$\left(\left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^\eta\right)_{k\in\mathcal{Z}} \in \mathcal{O}\left(|k|^{-\psi}\right).$$

Recall that there is a constant $K_\eta > 0$ such that

$$|e_k(x) - e_k(y)| \leqslant K_\eta \mu_k^{\frac{\eta}{2}} |x-y|^\eta$$

holds for each $x, y \in \mathbb{T}^2$, as can be shown by straightforward calculus using the definition of $(e_k)_{k\in\mathcal{Z}}$ (see (A.1)).

Applying Itô's isometry, we obtain the bound

$$
\begin{aligned}
&\mathbb{E}\left[\left|W_{A_\delta}^{(\varepsilon)}(t,x) - W_{A_\delta}^{(\varepsilon)}(t,y)\right|^2\right] \\
&= \mathbb{E}\left[\left|\int_0^t e^{(t-s)A_\delta}\,\mathrm{d}W_\varepsilon(s,x) - \int_0^t e^{(t-s)A_\delta}\,\mathrm{d}W_\varepsilon(s,y)\right|^2\right] \\
&= \mathbb{E}\left[\left|\sum_{k\in\mathcal{Z}} \alpha_k^{(\varepsilon)}\,[e_k(x) - e_k(y)] \int_0^t e^{(t-s)\lambda_k}\,\mathrm{d}\beta_k(s)\right|^2\right] \\
&\leqslant \sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 [e_k(x) - e_k(y)]^2 \int_0^t e^{2(t-s)\lambda_k}\,\mathrm{d}s \\
&= \sum_{k\in\mathcal{Z}^*} \left(\alpha_k^{(\varepsilon)}\right)^2 [e_k(x) - e_k(y)]^2 \,\frac{1}{2\lambda_k}\left[e^{2t\lambda_k} - 1\right] \\
&\qquad + \sum_{l\in\mathcal{Z}\setminus\mathcal{Z}^*} \left(\alpha_l^{(\varepsilon)}\right)^2 [e_l(x) - e_l(y)]^2\, t \\
&\leqslant K_\eta |x-y|^{2\eta}\left[\sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{\mu_k^\eta}{2\lambda_k}\left[e^{2t\lambda_k} - 1\right] + t\sum_{l\in\mathcal{Z}\setminus\mathcal{Z}^*} \mu_l^\eta\right] \\
&\leqslant K_\eta |x-y|^{2\eta}\left[\sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{\mu_k^\eta}{2\lambda_k}\left[e^{2t\lambda_k} - 1\right] + t\right]
\end{aligned}
$$

for each $x, y \in \mathbb{T}^2$, $t \geqslant 0$, whereby $\mathcal{Z}^* := \{k \in \mathcal{Z} : \lambda_k \neq 0\}$. Furthermore, Lemma 2.19 is also applicable to $W_{A_\delta}^{(\varepsilon)}$, which allows for control of its L^p-norm.

Therefore, by Morrey's inequality (cf. Theorem A.23) for $\eta > \alpha$ and $p > 2$ such that $\alpha p > 2$, and the bound for p -th Gaussian moments (cf. Corollary 2.5) we obtain

$$
\begin{aligned}
\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^0}^p\right] &\leqslant C_p \mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{W^{\alpha,p}}^p\right] \\
&\leqslant C_p \mathbb{E}\left[\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \frac{\left|W_{A_\delta}^{(\varepsilon)}(t,x) - W_{A_\delta}^{(\varepsilon)}(t,y)\right|^p}{|x-y|^{2+\alpha p}} \,\mathrm{d}x\,\mathrm{d}y\right] + C_p \mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{L^p}^p\right] \\
&\leqslant C_p \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \frac{\mathbb{E}\left[\left|W_{A_\delta}^{(\varepsilon)}(t,x) - W_{A_\delta}^{(\varepsilon)}(t,y)\right|^2\right]^{\frac{p}{2}}}{|x-y|^{2+\alpha p}} \,\mathrm{d}x\,\mathrm{d}y + C_p \mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{L^p}^p\right] \\
&\leqslant C_{p,\eta} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \frac{\left[|x-y|^{2\eta}\left(t + \sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{\mu_k^\eta}{2\lambda_k}\left[e^{2t\lambda_k} - 1\right]\right)\right]^{\frac{p}{2}}}{|x-y|^{2+\alpha p}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C_p \mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{L^p}^p\right] \\
&= C_{p,\eta} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |x-y|^{p(\eta-\alpha)-2} \,\mathrm{d}x\,\mathrm{d}y \left[t + \sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{\mu_k^\eta}{2\lambda_k}\left[e^{2t\lambda_k} - 1\right]\right]^{\frac{p}{2}} \\
&\quad + C_{p,L} \mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{L^2}^2\right]^{\frac{p}{2}} \\
&\leqslant C_{p,\eta,L,\delta} \mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{\mathcal{H}^\eta}^2\right]^{\frac{p}{2}}.
\end{aligned}
$$

By applying Hölder's inequality this concludes the proof for each $p \geqslant 1$. □

Remark 4.18. *Note that the set $\mathcal{Z} \setminus \mathcal{Z}^*$, used in the proof of Theorem 4.17, has, by rotational symmetry of λ_k according to the index k, either four or no elements.*

Corollary 4.19. *Let $\varepsilon > 0$ and $\eta > 0$ such that there exists some $\psi > 0$ with*

$$\left(\left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^\eta\right)_{k\in\mathcal{Z}} \in \mathcal{O}\left(|k|^{-\psi}\right).$$

Then for each $p \geqslant 1$ there exists a constant $C = C_{p,\eta,L,\delta} > 0$ such that the following upper bound holds for every $t \geqslant 0$:

$$\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^1}^p\right]^{\frac{1}{p}} \leqslant C\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{\mathcal{H}^{1+\eta}}^2\right]^{\frac{1}{2}}.$$

Proof. For $\varepsilon > 0$ let $p > 2$ and $\eta > 0$ with $\eta p > 2$, and that there exists some $\psi > 0$ such that

$$\left(\left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^\eta\right)_{k\in\mathcal{Z}} \in \mathcal{O}\left(|k|^{-\psi}\right).$$

By applying the proof of Theorem 4.17 for the $\mathbb{R}^2$-valued process $\nabla W_{A_\delta}^{(\varepsilon)}$ instead of $W_{A_\delta}^{(\varepsilon)}$, we obtain the bound

$$\mathbb{E}\left[\left|\nabla W_{A_\delta}^{(\varepsilon)}(t,x) - \nabla W_{A_\delta}^{(\varepsilon)}(t,y)\right|^2\right] \leqslant K_\eta |x-y|^{2\eta}\left[\sum_{k\in\mathcal{Z}}\left(\alpha_k^{(\varepsilon)}\right)^2 \frac{\mu_k^{1+\eta}}{2\lambda_k}\left[e^{2t\lambda_k}-1\right]+t\right]$$

for a constant $K_\eta > 0$ and for each $x, y \in \mathbb{T}^2$ and $t \geqslant 0$. For $\eta > \alpha > 0$ and $p > 2$ such that $\alpha p > 2$ we derive

$$\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^1}^p\right]^{\frac{1}{p}} \leqslant C_p \mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{W^{1+\alpha,p}}^p\right]^{\frac{1}{p}} \leqslant C_{p,\eta,L,\delta}\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{\mathcal{H}^{1+\eta}}^2\right]^{\frac{1}{2}}$$

for $C = C_{p,\eta,L,\delta} > 0$. By applying Hölder's inequality this shows the assertion for each $p \geqslant 1$. □

Lemma 4.20. *Let $\varepsilon > 0$ and $\eta > 0$ such that there exists some $\psi > 0$ with*

$$\left(\left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^\eta\right)_{k\in\mathcal{Z}} \in \mathcal{O}\left(|k|^{-\psi}\right).$$

Then for each $p \geqslant 1$ there exists a constant $C = C_{p,\delta,\varepsilon} > 0$ such that we have

$$\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^1}^p\right]^{\frac{1}{p}} \leqslant C_{p,\delta,\varepsilon} e^{t\lambda_{\max}}$$

for each $t \geqslant 0$.

Proof. For $\varepsilon > 0$ let $p > 2$ and $\eta > 0$ such that $\eta p > 2$, and that there exists some $\psi > 0$ such that

$$\left(\left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^\eta\right)_{k\in\mathcal{Z}} \in \mathcal{O}\left(|k|^{-\psi}\right).$$

By applying Itô's isometry, the bound for Gaussian moments (see Corollary 2.5), and Theorem 4.17, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^1}^p\right]^{\frac{1}{p}} &\leqslant C_p \mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{\mathcal{H}^{1+\eta}}^2\right]^{\frac{1}{2}} \\
&= C_p\left[\sum_{k\in\mathcal{Z}}\left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^{1+\eta}\int_0^t e^{2s\lambda_k}\,\mathrm{d}s\right]^{\frac{1}{2}} \\
&\leqslant C_{p,\delta,\varepsilon}\left[t+\sum_{k\in\mathcal{Z}^*}\left(\alpha_k^{(\varepsilon)}\right)^2 \frac{\mu_k^{1+\eta}}{\lambda_k}\left(e^{2t\lambda_k}-1\right)\right]^{\frac{1}{2}} \\
&\leqslant C_{p,\delta,\varepsilon}e^{t\lambda_{\max}}\left[1+\sum_{k\in\mathcal{Z}^*}\left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^{-1+\eta}\right]^{\frac{1}{2}} \\
&\leqslant C_{p,\delta,\varepsilon}e^{t\lambda_{\max}},
\end{aligned}
$$

whereby as above $\mathcal{Z}^* = \{k \in \mathcal{Z} : \lambda_k \neq 0\}$. By applying Hölder's inequality this finishes the proof for each $p \geqslant 1$. □

Lemma 4.21. *Let* $0 < \sigma \ll 1$. *Then for each* $t \geqslant 0$ *the estimate*

$$\sigma\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^1} \leqslant e^{t\lambda_{\max}}$$

holds true with high probability.

Proof. By applying Markov's inequality and Lemma 4.20, we derive

$$
\begin{aligned}
\mathbb{P}\left(\sigma\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^1} \geqslant e^{t\lambda_{\max}}\right) &\leqslant \left(\sigma e^{-t\lambda_{\max}^{(\delta)}}\right)^p \mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^1}^p\right] \\
&\leqslant C_p\sigma^p e^{-pt\lambda_{\max}}\,\mathbb{E}\left[\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^1}^2\right]^{\frac{p}{2}} \\
&\leqslant C_p\sigma^p
\end{aligned}
$$

for each $p > 0$ and $t \geqslant 0$. Hence, taking the limit $\sigma \to 0$, the claim follows. □

4.3.2. Factorisation Method

Since the integrand of the solution to (4.1) depends on the time variable t, the stochastic integral in this case does not constitute a martingale or submartingale. Therefore, we cannot make use of estimates from martingale theory, such as the *Doob's inequality*, in order to obtain uniform bounds in time. An alternative approach is provided by the *factorisation method* of Da Prato and Zabczyk [27]. To this end, we first examine the expression

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t e^{(t-s)A_\delta}\,\mathrm{d}W_\varepsilon(s)\right\|_{C^1}^p\right]^{\frac{1}{p}}, \tag{4.9}$$

for $p \geqslant 1$. To establish an upper bound, we first verify the independence of two summands $\big(Z_k(t)\big)_{t\geqslant 0}$ of the stochastic process

$$Z_\varepsilon(t,x) = \sum_{k\in\mathcal{Z}} \alpha_k^{(\varepsilon)} Z_k(t) e_k(x) = \sum_{k\in\mathcal{Z}} \alpha_k^{(\varepsilon)} \int_0^t e^{-(t-s)\delta\mu_k^2}\,\mathrm{d}\beta_k(s) e_k(x),$$

for which we use the representation

$$\begin{aligned} Z_k(t,x) &= \Big\langle \int_0^t e^{A_\delta(t-s)}\,\mathrm{d}W_\varepsilon(s)\ ,\ e_k(x)\Big\rangle \\ &= \Big\langle \alpha_k^{(\varepsilon)} e_k(x) \int_0^t e^{-(t-\tau)A_\delta}\,\mathrm{d}\beta_k(\tau)\ ,\ e_k(x)\Big\rangle. \end{aligned}$$

Since Z_k are Gaussian random variables, we verify their uncorrelatedness to establish independence. For arbitrary indices $k,l\in\mathcal{Z}$, $k\neq l$, we have

$$\begin{aligned} \mathbb{E}\left[Z_k(t,x)Z_l(t,x)\right] &= \mathbb{E}\left[\langle Z(t),e_k(x)\rangle\langle Z(t),e_l(x)\rangle\right] \\ &= \mathbb{E}\left[\langle Z(t),e_k(x)\rangle\langle Z(t),e_l(x)\rangle\right] \\ &= \mathbb{E}\left[\alpha_k^{(\varepsilon)}\int_0^t e^{-(t-\tau)\delta\mu_k^2}\,\mathrm{d}\beta_k(\tau)\alpha_l^{(\varepsilon)}\int_0^t e^{-(t-\tau)\delta\mu_l^2}\,\mathrm{d}\beta_l(\tau)\right] = 0, \end{aligned}$$

where the last equality holds because $\big(\beta_k\big)_{k\in\mathcal{Z}}$ are independent Wiener processes. The following identity serves as an essential component in applying the factorisation method to analyse the expression (4.9).

Corollary 4.22. *For $\gamma\in(0,1)$ let*

$$\begin{aligned} y(s) &\coloneqq \int_0^s (s-\tau)^{-\gamma}\, e^{(s-\tau)A_\delta}\,\mathrm{d}W_\varepsilon(\tau), \\ K_\gamma &\coloneqq \left(\int_0^1 (1-s)^{-\gamma}\, s^{\gamma-1}\,\mathrm{d}s\right)^{-1}. \end{aligned}$$

Then we obtain the following identity for each $t\geqslant 0$:

$$\int_0^t e^{(t-s)A_\delta}\,\mathrm{d}W_\varepsilon(s) = K_\gamma \int_0^t (t-s)^{\gamma-1} e^{(t-s)A_\delta} y(s)\,\mathrm{d}s.$$

Proof. Let $\gamma \in (0,1)$ be fixed. By applying *Fubini's theorem for stochastic integrals* (cf. Theorem A.25), we have

$$\begin{aligned}
\int_0^t (t-s)^{\gamma-1} e^{(t-s)A_\delta} y(s)\,\mathrm{d}s &= \int_0^t (t-s)^{\gamma-1} e^{(t-s)A_\delta} \int_0^s (s-\tau)^{-\gamma} e^{(s-\tau)A_\delta}\,\mathrm{d}W_\varepsilon(\tau)\,\mathrm{d}s \\
&= \int_0^t \int_\tau^t (t-s)^{\gamma-1} e^{(t-s)A_\delta} (s-\tau)^{-\gamma} e^{(s-\tau)A_\delta}\,\mathrm{d}s\;\,\mathrm{d}W_\varepsilon(\tau) \\
&= \int_0^t e^{(t-\tau)A_\delta} \int_\tau^t (t-s)^{\gamma-1}(s-\tau)^{-\gamma}\,\mathrm{d}s\,\mathrm{d}W_\varepsilon(\tau) \\
&= \int_0^t e^{(t-\tau)A_\delta} \int_0^{t-\tau} (t-\tau-s)^{\gamma-1} s^{\gamma}\,\mathrm{d}s\,\mathrm{d}W_\varepsilon(\tau) \\
&= \int_0^t e^{(t-\tau)A_\delta}\,\mathrm{d}W_\varepsilon(\tau) \int_0^1 (1-z)^{\gamma-1} z^{\gamma}\,\mathrm{d}z
\end{aligned}$$

for each $t \geqslant 0$. Multiplying this expression by K_γ yields the statement. □

Remark 4.23. *Let* $\Gamma(\cdot)$ *be the* Gamma function *and* $B(\cdot,\cdot)$ *be the* Euler Beta function*. Then we obtain*

$$K_\gamma^{-1} = B(\gamma, 1-\gamma)^{-1} = \frac{\Gamma(1)}{\Gamma(\gamma)\,\Gamma(1-\gamma)} = \left(\int_0^1 t^{\gamma-1}(1-t)^{-\gamma}\,\mathrm{d}t\right)^{-1} = \frac{\sin(\pi\gamma)}{\pi}.$$

Now we turn to the factorisation method.

Theorem 4.24. *Let* $\lambda_{\max}^{(\theta)}$ *be the upper bound of the eigenvalues, as defined in Definition 4.9. Moreover, for* $\varepsilon > 0$ *let* $\eta \in (0,1)$ *and* $\gamma \in (0,1)$ *such that*

$$\left(\mu_k^{\frac{\eta}{2}+2\gamma} \alpha_k^{(\varepsilon)}\right)_{k\in\mathcal{Z}} \in \mathcal{O}\left(|k|^{-\psi}\right)$$

holds for some $\psi > 0$*. Then, for each* $p \geqslant 1$ *there is a constant* $C = C_{p,\delta,\varepsilon,\eta,\gamma,\theta} > 0$ *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \int_0^t e^{(t-s)A_\delta}\,\mathrm{d}W_\varepsilon(s) \right\|_{C^1}^p \right]^{\frac{1}{p}} \leqslant C e^{T\lambda_{\max}^{(\theta)}}.$$

Proof. By using Corollary 4.22, Hölder's inequality with $(\gamma-1)\frac{p}{p-1} > -1$ or equivalently for each $p > \frac{1}{\gamma}$, and Minkowski's inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t e^{(t-s)A_\delta}\,\mathrm{d}W_\varepsilon(s)\right\|_{C^1}^p\right]^{\frac{1}{p}} \\
&= \mathbb{E}\left[\sup_{t\in[0,T]}\left\|K_\gamma\int_0^t (t-s)^{\gamma-1}e^{(t-s)A_\delta}y(s)\,\mathrm{d}s\right\|_{C^1}^p\right]^{\frac{1}{p}} \\
&\leqslant K_\gamma\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t (t-s)^{\gamma-1}e^{(t-s)\lambda_{\max}^{\left(\frac{\theta}{2}\right)}}\|y(s)\|_{C^1}\,\mathrm{d}s\right)^p\right]^{\frac{1}{p}} \\
&\leqslant K_\gamma e^{T\lambda_{\max}^{\left(\frac{\theta}{2}\right)}}\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t (t-s)^{(\gamma-1)\frac{p}{p-1}}\,\mathrm{d}s\right)^{p-1}\int_0^t e^{-sp\lambda_{\max}^{\left(\frac{\theta}{2}\right)}}\|y(s)\|_{C^1}^p\,\mathrm{d}s\right]^{\frac{1}{p}} \\
&\leqslant C_{p,\gamma}e^{T\lambda_{\max}^{\left(\frac{\theta}{2}\right)}}T^{\gamma-\frac{1}{p}}\int_0^T e^{-s\lambda_{\max}^{\left(\frac{\theta}{2}\right)}}\mathbb{E}\left[\|y(s)\|_{C^1}^p\right]^{\frac{1}{p}}\,\mathrm{d}s \\
&\leqslant C_{p,\gamma,\theta}e^{T\lambda_{\max}^{(\theta)}}\int_0^T e^{-s\lambda_{\max}^{\left(\frac{\theta}{2}\right)}}\mathbb{E}\left[\|y(s)\|_{C^1}^p\right]^{\frac{1}{p}}\,\mathrm{d}s.
\end{aligned}
$$

Here, $K_T > 0$ denotes a constant depending on T. The interchange of the integral, expectation, and supremum is justified by Tonelli's theorem and the dominated convergence theorem.

Similar as in Section 4.3.1, we derive the following bound by Theorem 4.17, Morrey's inequality (cf. Theorem A.23) and Itô's isometry:

$$
\begin{aligned}
&e^{-t\lambda_{\max}^{\left(\frac{\theta}{2}\right)}} \mathbb{E}\left[\|y(t)\|_{C^1}^p\right]^{\frac{1}{p}} \\
&\leqslant C_p e^{-t\lambda_{\max}^{\left(\frac{\theta}{2}\right)}} \mathbb{E}\left[\|y(t)\|_{\mathcal{H}^{1+\eta}}^2\right]^{\frac{1}{2}} \\
&= C_p e^{-t\lambda_{\max}^{\left(\frac{\theta}{2}\right)}} \left(\sum_{k\in\mathcal{Z}} \mu_k^{1+\eta} \left(\alpha_k^{(\varepsilon)}\right)^2 \int_0^t s^{-2\gamma} e^{2s\lambda_k}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leqslant C_p e^{-t\lambda_{\max}^{\left(\frac{\theta}{2}\right)}} \left(\sum_{k\in\mathcal{Z}} \mu_k^{1+\eta} \left(\alpha_k^{(\varepsilon)}\right)^2 e^{2t\lambda_{\max}^{\left(\frac{\theta}{2}\right)}} \int_0^t s^{-2\gamma} e^{-2s\left(\lambda_{\max}^{\left(\frac{\theta}{2}\right)}-\lambda_k\right)}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&= C_p \left(\sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \frac{\mu_k^{1+\eta}}{\left(\lambda_{\max}^{\left(\frac{\theta}{2}\right)}-\lambda_k\right)^{1-2\gamma}} \int_0^{t\left(\lambda_{\max}^{\left(\frac{\theta}{2}\right)}-\lambda_k\right)} s^{-2\gamma} e^{-2s}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leqslant C_{p,\delta,\gamma,\theta} \left(\sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^{-1+\eta+4\gamma} \int_0^\infty s^{-2\gamma} e^{-2s}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leqslant C_{p,\delta,\gamma,\theta} \left(\sum_{k\in\mathcal{Z}} \left(\alpha_k^{(\varepsilon)}\right)^2 \mu_k^{-1+\eta+4\gamma}\right)^{\frac{1}{2}} \\
&\leqslant C_{p,\delta,\varepsilon,\eta,\gamma,\theta}
\end{aligned}
$$

for $\eta,\gamma \in (0,1)$ such that $\left(\mu_k^{\frac{\eta}{2}+2\gamma}\alpha_k^{(\varepsilon)}\right)_{k\in\mathcal{Z}} \in \mathcal{O}\left(|k|^{-\psi}\right)$ is fulfilled, for some $\psi > 0$. Additionally, note that we have used

$$
\frac{\mu_k^{1+\eta}}{\left(\lambda_{\max}^{(\theta)}-\lambda_k\right)^{1-2\gamma}} \leqslant C \frac{\mu_k^{1+\eta}}{\mu_k^{2-4\gamma}}
$$

in the inequality above. To summarise, we obtain the estimate

$$
\mathbb{E}\left[\sup_{t\in[0,T]} \left\|\int_0^t e^{(t-s)A_\delta}\,\mathrm{d}W_\varepsilon(s)\right\|_{C^1}^p\right]^{\frac{1}{p}} \leqslant C_{p,\delta,\varepsilon,\eta,\gamma,T} e^{T\lambda_{\max}^{(\theta)}}.
$$

By applying Hölder's inequality, we obtain the claim for each $p \geqslant 1$. □

4.4. Upper Bound for the Mild Solution

The next theorem combines the results of Section 4.2 and Section 4.3.2 to establish the upper bound for the p-th moment of the C^1–norm of the mild solution u_ε to be smaller than the radius r_σ. We first examine this to show an upper bound for $\|u_\varepsilon\|_{C^1}$ *with high probability* (cf. Theorem 4.26).

Theorem 4.25. *Let $\|u_0\|_{C^1} < r_0$, where Assumption 4.12 holds. Then we have*

$$\mathbb{E}\left[\sup_{t\in[0,\tau\wedge T_\sigma]} \|u_\varepsilon(t)\|_{C^1}^p\right]^{\frac{1}{p}} \ll r_\sigma$$

for $T_\sigma \ll \frac{1}{\lambda_{\max}^{(\theta)}} \ln\left(\frac{r_\sigma}{\sigma}\right)$.

Proof. By applying (4.8), Corollary 4.15 and Theorem 4.24, we conclude that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,\tau\wedge T_\sigma]} \|u_\varepsilon(t)\|_{C^1}^p\right]^{\frac{1}{p}} \\
&\leqslant \mathbb{E}\left[\sup_{t\in[0,\tau\wedge T_\sigma]} \left\|e^{tA_\delta}u_0\right\|_{C^1}^p\right]^{\frac{1}{p}} + \mathbb{E}\left[\sup_{t\in[0,\tau\wedge T_\sigma]} \left\|\int_0^t e^{(t-s)A_\delta}\mathfrak{F}\left(u(s)\right)\,\mathrm{d}s\right\|_{C^1}^p\right]^{\frac{1}{p}} \\
&\quad + \mathbb{E}\left[\sup_{t\in[0,\tau\wedge T_\sigma]} \sigma\left\|W_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^1}^p\right]^{\frac{1}{p}} \\
&\leqslant C_{p,\alpha,\delta,\varepsilon} e^{T_\sigma\lambda_{\max}^{(\theta)}}\left[\|u_0\|_{C^0} + T_\sigma^{\frac{2-\alpha}{4}} r_\sigma^3 + \sigma\right] \\
&\leqslant C_{p,\alpha,\delta,\varepsilon} e^{T_\sigma\lambda_{\max}^{(\theta)}}\left[r_0 + T_\sigma^{\frac{2-\alpha}{4}} r_\sigma^3 + \sigma\right] \\
&\leqslant C_{p,\alpha,\delta,\varepsilon}\sigma\ e^{T_\sigma\lambda_{\max}^{(\theta)}}\left[1 + T^{\frac{2-\alpha}{4}}\right],
\end{aligned}$$

for constants $C_{p,\alpha,\delta,\varepsilon} > 0$. Now, by choosing $T_\sigma \ll \frac{3}{4\lambda_{\max}^{(\theta)}} \ln\left(\frac{r_\sigma}{\sigma}\right)$, which is equivalent to $T_\sigma \ll \frac{1}{\lambda_{\max}^{(\theta)}} \ln\left(\frac{r_\sigma}{\sigma}\right)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,\tau\wedge T_\sigma]} \|u_\varepsilon(t)\|_{C^1}^p\right]^{\frac{1}{p}} &\leqslant C_{p,\alpha,\delta,\varepsilon}\sigma e^{T_\sigma\lambda_{\max}^{(\theta)}}\left[1 + T^{\frac{2-\alpha}{4}}\right] \\
&\ll \frac{1}{C_{p,\rho,\gamma,\delta,\varepsilon}}\sigma\left(\frac{r_\sigma}{\sigma}\right)^{\frac{3}{4}}\left[1 + \left(\frac{1-\rho}{\lambda_{\max}^{(\theta)}}\ln\left(r_\sigma\right)\right)^{\frac{2-\alpha}{4}}\right] \\
&= \frac{1}{C_{p,\rho,\gamma,\delta,\varepsilon}}\sigma^{\frac{1}{4}} r_\sigma^{\frac{3}{4}}\left[1 + \left(\frac{1-\rho}{\lambda_{\max}^{(\theta)}}\ln\left(r_\sigma\right)\right)^{\frac{2-\alpha}{4}}\right] < r_\sigma
\end{aligned}$$

as for each $a > 0$ we have $r_\sigma^a \ln(r_\sigma) \to 0$ for $r_\sigma \to 0$. We have thus proved the result. □

Theorem 4.26. *Let $\|u_0\|_{C^1} < r_0$, where Assumption 4.12 holds. Then we have*

$$\sup_{t\in[0,T_\sigma]} \|u_\varepsilon(t)\|_{C^1} \leqslant r_\sigma$$

with high probability *for $T_\sigma \ll \frac{1}{\lambda_{\max}^{(\theta)}} \ln\left(\frac{r_\sigma}{\sigma}\right)$. In particular, we have $\tau = T_\sigma$* with high probability.

Proof. Using Theorem 4.25 and Markov's inequality, we derive

$$\begin{aligned}\mathbb{P}\left(\tau < T_\sigma\right) &= \mathbb{P}\left(\sup_{t\in[0,\tau\wedge T_\sigma]} \|u_\varepsilon(t)\|_{C^1} \geqslant r_\sigma\right)\\ &\leqslant \mathbb{E}\left[\sup_{t\in[0,\tau\wedge T_\sigma]} \|u_\varepsilon(t)\|_{C^1}^p\right] r_\sigma^{-p}\\ &\ll 1.\end{aligned}$$

Hence, $\tau = T_\sigma$ holds *with high probability* and we conclude that

$$\sup_{t\in[0,T_\sigma]} \|u_\varepsilon(t)\|_{C^1} \leqslant r_\sigma$$

holds *with high probability.* □

4.5. Lower Bound for the Mild Solution

4.5.1. Auxiliary Inequalities

This subsection examines a lower bound for the time-dependent growth of the solution of

$$\partial_t u_\varepsilon = -\left(\delta\Delta^2 + \Delta\right) u_\varepsilon + \nabla \cdot \frac{\nabla u_\varepsilon |\nabla u_\varepsilon|^2}{1 + |\nabla u_\varepsilon|^2} + \partial_t W_\varepsilon.$$

Notation 4.27. *For simplicity, we denote by $\Delta + \rho$ the linear operator*

$$\Delta + \rho \mathrm{Id} : W^{\alpha+2,p}(\mathbb{T}^2) \to W^{\alpha,p}(\mathbb{T}^2), \quad y \mapsto \Delta y + \rho y$$

for each $p \geqslant 1$ and $\alpha > 0$.

The following auxiliary lemma provides key properties of the operator A_δ on Sobolev spaces over $\mathbb{T}^2$.

Lemma 4.28. *For each $\rho > 0$, $p \geqslant 2$ there is a constant $C = C_{L,p,\rho} > 0$ such that*

$$\|-(\Delta + \rho)y\|_{L^p} \leqslant C \,\||\Delta + \rho|\, y\|_{L^p}$$

holds for each $y \in W^{2,p}(\mathbb{T}^2)$.

Proof. We begin by decomposing the space $W^{2,p}(\mathbb{T}^2)$ into the subspaces $\hat{X}_p^+$ and $\hat{X}_p^-$, given by

$$\hat{X}_p^+ := \left\{ \sum_{k \in \mathcal{Z}} y_k e_k \in W^{2,p}\left(\mathbb{T}^2\right) : y_k = 0 \text{ for each } k \in \mathcal{Z} \text{ with } -\mu_k + \rho \leqslant 0 \right\}$$

and the complementary subspace such that $\hat{X}_p^- \oplus \hat{X}_p^+ = L^p$ holds. For $y \in W^{2,p}$ and an L^2-decomposition $y = y^+ + y^-$ with $y^+ \in \hat{X}_2^+$ and the $y^- \in \hat{X}_2^-$ it even holds that $y^+ \in W^{2,p}$, as $\hat{X}_2^+$ is finite-dimensional. Thus, we also obtain that $y^- \in W^{2,p}$. On the subspace $\hat{X}_p^-$ the operators $|\Delta + \rho|$ and $-(\Delta + \rho)$ are equal, because of the equality of their eigenvalues

$$|-\mu_k + \rho| = -\left(-\mu_k + \rho\right) \geqslant 0.$$

On the other hand, as $\hat{X}_p^+$ is finite-dimensional, we obtain

$$\left\||\Delta + \rho|y^+\right\|_{L^p} \leqslant C \left\||\Delta + \rho|y^+\right\|_{L^2}$$

and thus

$$\begin{aligned}
\|-(\Delta + \rho)y\|_{L^p} &\leqslant \left\|-(\Delta + \rho)y^+\right\|_{L^p} + \left\|-(\Delta + \rho)y^-\right\|_{L^p} \\
&\leqslant C \left\||\Delta + \rho|y^+\right\|_{L^p} + \left\||\Delta + \rho|y^-\right\|_{L^p} \\
&\leqslant (C+1) \left\||\Delta + \rho|y^+\right\|_{L^p} + \||\Delta + \rho|y\|_{L^p} \\
&\leqslant C \left\||\Delta + \rho|y^+\right\|_{L^2} + \||\Delta + \rho|y\|_{L^p} \\
&\leqslant C \||\Delta + \rho|y\|_{L^2} + \||\Delta + \rho|y\|_{L^p} \\
&\leqslant C \||\Delta + \rho|y\|_{L^p}.
\end{aligned}$$

This establishes the desired result. □

This auxiliary lemma forms the foundation for deriving an upper bound for the C^1-norm of the solution in terms of its C^0-norm.

Theorem 4.29. *Let $u_0 \in L^p$, $A_\delta := -\delta\Delta^2 - \Delta$, with $K_\delta \in (0,\delta)$, $\alpha \in (0,1)$, and $p > 2$, such that $\alpha p > 2$ holds. Then for each $t > 0$ we have*

$$\left\|e^{tA_\delta}u_0\right\|_{C^1} \leqslant Ce^{t\lambda_{\max}^{(\theta)}}\left[1+t^{-\frac{1}{2}}\right]^{\frac{1+\alpha}{2}}\|u_0\|_{C^0},$$

whereby $C = C_{p,\alpha,\delta,L} > 0$.

Proof. By applying Morrey's inequality (cf. Theorem A.23), Brezis–Mironescu inequality (cf. Theorem A.22) and (A.11), we can bound the analytic semigroup and thus derive

$$\begin{aligned}
&\left\|e^{tA_\delta}u_0\right\|_{C^1}\\
&\leqslant C\left\|e^{tA_\delta}u_0\right\|_{W^{1+\alpha,p}}\\
&\leqslant C\left\|e^{tA_\delta}u_0\right\|_{W^{2,p}}^{\frac{1+\alpha}{2}}\left\|e^{tA_\delta}u_0\right\|_{L^p}^{\frac{1-\alpha}{2}}\\
&\leqslant C\left[\left\|-\left(\Delta+\frac{1}{2\delta}\right)e^{tA_\delta}u_0\right\|_{L^p}+\left\|e^{tA_\delta}u_0\right\|_{L^p}\right]^{\frac{1+\alpha}{2}}\left\|e^{tA_\delta}u_0\right\|_{L^p}^{\frac{1-\alpha}{2}}\\
&\leqslant C\left[\left\|-\left(\Delta+\frac{1}{2\delta}\right)e^{-t\delta\left(\Delta+\frac{1}{2\delta}\right)^2}e^{\frac{t}{4\delta}}u_0\right\|_{L^p}+\left\|e^{tA_\delta}u_0\right\|_{L^p}\right]^{\frac{1+\alpha}{2}}\left\|e^{tA_\delta}u_0\right\|_{L^p}^{\frac{1-\alpha}{2}}\\
&\leqslant C\left[\left\|\left|\Delta+\frac{1}{2\delta}\right|e^{-t\delta\left(\Delta+\frac{1}{2\delta}\right)^2}e^{\frac{t}{4\delta}}u_0\right\|_{L^p}+\left\|e^{tA_\delta}u_0\right\|_{L^p}\right]^{\frac{1+\alpha}{2}}\left\|e^{tA_\delta}u_0\right\|_{L^p}^{\frac{1-\alpha}{2}}\\
&\leqslant C\left[e^{t\lambda_{\max}^{(\theta)}}\left(1+t^{-\frac{1}{2}}\right)\right]^{\frac{1+\alpha}{2}}\left\|e^{tA_\delta}u_0\right\|_{L^p}^{\frac{1-\alpha}{2}}\\
&\leqslant Ce^{t\lambda_{\max}^{(\theta)}}\|u_0\|_{L^p}\left[1+t^{-\frac{1}{2}}\right]^{\frac{1+\alpha}{2}}\\
&\leqslant Ce^{t\lambda_{\max}^{(\theta)}}\left[1+t^{-\frac{1}{2}}\right]^{\frac{1+\alpha}{2}}\|u_0\|_{C^0},
\end{aligned}$$

where $C > 0$ are constants depending only on L, α, p, and δ. This establishes the desired result. □

Remark 4.30. *By the assumptions of Theorem 4.29 we obtain*

$$\left\|e^{tA_\delta}u_0\right\|_{C^2} \leqslant C_{\alpha,\delta}e^{t\lambda_{\max}^{(\theta)}}\left[1+t^{-1}\right]^{\frac{2+\alpha}{4}}\|u_0\|_{C^0} \quad \textit{for each} \quad t > 0.$$

Proof. By *Morrey's inequality* (cf. Theorem A.23) and the *Brezis-Mironescu inequality* (cf. Theorem A.22), we obtain

$$\begin{aligned}
\left\|e^{tA_\delta}u_0\right\|_{C^2} &\leqslant C\left\|e^{tA_\delta}u_0\right\|_{W^{2+\alpha,p}}\\
&\leqslant C\left\|e^{tA_\delta}u_0\right\|_{W^{4,p}}^{\frac{2+\alpha}{4}}\left\|e^{tA_\delta}u_0\right\|_{L^p}^{\frac{2-\alpha}{4}}.
\end{aligned}$$

Thus, as in the proof of Theorem 4.29, we obtain the result. □

4.5.2. Lower Bound of the Stochastic Convolution

In this subsection, we analyse exponential growth conditions up to a stopping time depending on a radius r_σ with $0 \ll r_\sigma^3 \ll \sigma \ll r_\sigma \ll 1$. We aim to establish a bound for the stochastic convolution up to logarithmic time, which will help linearise the exponential growth of the solution. We begin by introducing the necessary notation.

Notation 4.31. *Let γ^+ be the smallest positive eigenvalue of $A_\delta = -\delta\Delta^2 - \Delta$ that is larger than $\frac{5}{9}\lambda_{\max}^{(\theta)}$. Formally, we have*

$$\frac{5}{9}\lambda_{\max}^{(\theta)} < \gamma^+ \leqslant \lambda_{\max} < \lambda_{\max}^{(\theta)}.$$

We define the index set $\mathcal{Z}^+ := \left\{k \in \mathcal{Z} : \lambda_k = -\delta\mu_k^2 + \mu_k > \frac{5}{9}\lambda_{\max}^{(\delta,\theta)}\right\}$ and the eigenspace of the largest eigenvalues of A_δ as

$$X_p^+ := \left\{ \sum_{k\in\mathcal{Z}} u_k e_k \in L^p\left(\mathbb{T}^2\right) : u_k = 0 \text{ for each } k \in \mathcal{Z} \setminus \mathcal{Z}^+ \right\}$$

for each $p \in [1, \infty]$. Furthermore, let $n_\delta := \dim\left(X_p^+\right)$. In particular, as in (4.6), we obtain

$$n_\delta \leqslant \sqrt{2} \sum_{l=1}^{n_*} 4l = 2\sqrt{2} n_* \left(n_* + 1\right),$$

for $n_* \in \mathbb{N}$, with $\frac{L}{2\pi\sqrt{\delta}} \in (n_*, n_* + 1]$. Note that the factor $\sqrt{2}$ arises from the change between the 1-norm and the 2-norm in $\mathcal{Z}$.

Remark 4.32. *The specific choice of $\frac{5}{9}$ is arbitrary. Any value in $(0,1)$ would suffice. However, the restriction on the parameter ρ in Corollary 4.40, Corollary 4.39, and Lemma 4.37 would differ.*

Assumption 4.33. *In this subsection, we assume $\delta > 0$ to be sufficiently small or $L > 0$ to be sufficiently large such that $\mathcal{Z}^+$ is a non-empty set.*

In the following computations we use the process

$$\hat{W}_{A_\delta}^{(\varepsilon)}(t,x) := \sum_{k\in\mathcal{Z}^+} \alpha_k^{(\varepsilon)} e_k(x) \int_0^t e^{(t-s)\lambda_k} \,\mathrm{d}\beta_k(s)$$

arising from the projection of the stochastic convolution

$$W_{A_\delta}^{(\varepsilon)}(t,x) = \int_0^t e^{(t-s)A_\delta} \,\mathrm{d}W_\varepsilon(s) = \sum_{k\in\mathcal{Z}} \alpha_k^{(\varepsilon)} e_k(x) \int_0^t e^{(t-s)\lambda_k} \,\mathrm{d}\beta_k(s)$$

onto the subspace of positive modes. The following theorem is strongly inspired by [9, Proposition 2.3].

Lemma 4.34. *Let Assumption 4.12 hold with $\sigma = r_\sigma^\rho$ and $\rho \in (1,3)$. Additionally, let $0 < a < (\rho-1)\frac{\gamma^+}{\lambda_{\max}^{(\theta)}}$ and $R_\sigma := r_\sigma^\zeta$ for*

$$1 < (1-\rho)\frac{\gamma^+}{\lambda_{\max}^{(\theta)}} + \rho + a \ < \ \zeta \ < \ \rho$$

such that we have $\sigma < R_\sigma < r_\sigma$. Then for $t_\sigma := \frac{1}{\gamma^+}\ln\left(r_\sigma^{\zeta-\rho-a}\sqrt{\gamma^+}\right)$ we have

$$\mathbb{P}\left(\sigma\left\|\hat{W}_{A_\delta}^{(\varepsilon)}(t_\sigma)\right\|_{L^2} < R_\sigma\right) \leqslant \left(\sqrt{2\gamma^+}r^a\right)^{n_\delta}\Gamma\left(\frac{n_\delta}{2}+1\right)^{-1}.$$

Proof. Let

$$\{I_j^2(t)\}_{j=1,\dots,n_\delta} := \left\{\int_0^t e^{2(t-s)\lambda_k}\,\mathrm{d}s\right\}_{k\in\mathcal{Z}^+} = \left\{\frac{1}{2\lambda_k}\left(e^{2t\lambda_k}-1\right)\right\}_{k\in\mathcal{Z}^+}.$$

For σ sufficiently small such that $e^{2t_\sigma\gamma^+} \geqslant 2$, it is easy to verify that

$$\min\{I_1^2(t_\sigma),\dots,I_{n_\delta}^2(t_\sigma)\} = \frac{1}{2\gamma^+}\left(e^{2t_\sigma\gamma^+}-1\right) \geqslant \frac{e^{2t_\sigma\gamma^+}}{4\gamma^+}.$$

Denote the centered ball in $\mathbb{R}^{n_\delta}$ with radius $a > 0$ by $B_a(0)$ and the diagonal matrix in $\mathbb{R}^{n_\delta\times n_\delta}$ with entries $I_1^2(t_\sigma),\dots,I_{n_\delta}^2(t_\sigma)$ by $\operatorname{diag}\left(I_1^2(t_\sigma),\dots,I_{n_\delta}^2(t_\sigma)\right)$. Thus, for t_σ we obtain the bound

$$\begin{aligned}
\mathbb{P}\left(\sigma\left\|\hat{W}_{A_\delta}^{(\varepsilon)}(t_\sigma)\right\|_{L^2} < R_\sigma\right) &\leqslant \frac{1}{(2\pi)^{\frac{n_\delta}{2}}\prod_{i=1}^{n_\delta} I_j(t_\sigma)}\int_{B_{\frac{R_\sigma}{\sigma}}(0)} e^{-\frac{1}{2}\sum_{i=1}^{n_\delta}\frac{y_i^2}{I_i^2}}\,\mathrm{d}\lambda^{n_\delta}(y)\\
&\leqslant (2\pi)^{-\frac{n_\delta}{2}}\int_{\operatorname{diag}\left(I_1^{-1}(t_\sigma),\dots,I_{n_\delta}^{-1}(t_\sigma)\right)B_{\frac{R_\sigma}{\sigma}}(0)} e^{-\frac{\|y\|^2}{2}}\,\mathrm{d}\lambda^{n_\delta}(y)\\
&\leqslant (2\pi)^{-\frac{n_\delta}{2}}\operatorname{vol}\left(B_{\frac{R_\sigma}{\sigma\min\{I_1(t_\sigma),\dots,I_{n_\delta}(t_\sigma)\}}}(0)\right)\\
&= \left(\frac{R_\sigma}{\sqrt{2}\sigma\min\{I_1(t_\sigma),\dots,I_{n_\delta}(t_\sigma)\}}\right)^{n_\delta}\Gamma\left(\frac{n_\delta}{2}+1\right)^{-1}\\
&\leqslant \left(\frac{R_\sigma^2\gamma^+}{\sigma^2\left(e^{2t_\sigma\gamma^+}-1\right)}\right)^{\frac{n_\delta}{2}}\Gamma\left(\frac{n_\delta}{2}+1\right)^{-1}\\
&\leqslant \left(\frac{2R_\sigma^2\gamma^+}{\sigma^2 e^{2t_\sigma\gamma^+}}\right)^{\frac{n_\delta}{2}}\Gamma\left(\frac{n_\delta}{2}+1\right)^{-1}\\
&\leqslant \left(\sqrt{2\gamma^+}r_\sigma^a\right)^{n_\delta}\Gamma\left(\frac{n_\delta}{2}+1\right)^{-1}.
\end{aligned}$$

This quantity can be made arbitrarily small by selecting r_σ appropriately, since we have $a > 0$. This completes the proof. □

Lemma 4.35. *Let Assumption 4.12 hold with $\sigma = r_\sigma^\rho$ and $\rho \in (1,3)$. Additionally, let $0 < a < (\rho-1)\frac{\gamma^+}{\lambda_{\max}^{(\theta)}}$ and $R_\sigma := r_\sigma^\zeta$ for*

$$1 < (1-\rho)\frac{\gamma^+}{\lambda_{\max}^{(\theta)}} + \rho + a \;<\; \zeta \;<\; \rho$$

such that we have $\sigma < R_\sigma < r_\sigma$. Then we have

$$0 \;<\; \frac{1}{\gamma^+}\ln\left(r_\sigma^{\zeta-\rho-a}\sqrt{\gamma^+}\right) \;<\; \frac{1}{\lambda_{\max}^{(\theta)}}\ln\left(\frac{r_\sigma}{\sigma}\right).$$

Proof. For $r_\sigma \ll 1$ we directly obtain

$$0 \;<\; \frac{1}{\gamma^+}\ln\left(r_\sigma^{\zeta-\rho-a}\sqrt{\gamma^+}\right).$$

since $\zeta-\rho-a < 0$. Furthermore, for $r_\sigma \ll 1$ the following conditions are equivalent:

- $\frac{1}{\gamma^+}\ln\left(r_\sigma^{\zeta-\rho-a}\sqrt{\gamma^+}\right) \;<\; \frac{1}{\lambda_{\max}^{(\theta)}}\ln\left(\frac{r_\sigma}{\sigma}\right)$,
- $\ln\left(r_\sigma^{\frac{\zeta-\rho-a}{\gamma^+}}(\gamma^+)^{\frac{1}{2\gamma^+}}\right) \;<\; \ln\left(r_\sigma^{\frac{1-\rho}{\lambda_{\max}^{(\theta)}}}\right)$,
- $r_\sigma^{\frac{\zeta-\rho-a}{\gamma^+}}(\gamma^+)^{\frac{1}{2\gamma^+}} \;<\; r_\sigma^{\frac{1-\rho}{\lambda_{\max}^{(\theta)}}}$,
- $(\gamma^+)^{\frac{1}{2\gamma^+}} \;<\; r_\sigma^{\frac{1}{\lambda_{\max}^{(\theta)}}(1-\rho)-\frac{1}{\gamma^+}(\zeta-\rho-a)}$,
- $\frac{1}{\lambda_{\max}^{(\theta)}}(1-\rho) - \frac{1}{\gamma^+}(\zeta-\rho-a) < 0$,
- $\frac{\gamma^+}{\lambda_{\max}^{(\theta)}}(1-\rho) + \rho + a < \zeta$,

which holds due to our choice of ζ and a. This completes the proof. □

4.5.3. Lower Bound for the Mild Solution

By applying Corollary 4.15, Lemma 4.21, and Lemma 4.34 we obtain the following lower bound on the mild solution of (4.1).

Theorem 4.36. *Let Assumption 4.12 hold and choose r_σ such that we have $\sigma = r_\sigma^\rho$ with $\rho \in (1,3)$. Then we obtain*

$$\|u(t)\|_{C^0} \geqslant \sigma > 0$$

with high probability *for each*

$$0 < t < \frac{(3-\rho)\lambda_{\max}}{\rho\big(\lambda_{\max}^{(\theta)} - \gamma^+\big)}\ln\left(\frac{1}{\sigma}\right) \wedge \tau.$$

Proof. Given that $0 < r_\sigma^3 \ll \sigma \ll r_\sigma \ll 1$, we may explicitly set $\sigma = r_\sigma^\rho$ for $\rho \in (1,3)$. Using the inverse triangle inequality, the upper bound for u_{det} from Corollary 4.15, and the lower bound for $\hat{W}_{A_\delta}^{(\varepsilon)}(t_\sigma)$ from Lemma 4.34, we derive the following inequality for each $0 < t < \frac{(3-\rho)\lambda_{\max}}{\rho\left(\lambda_{\max}^{(\theta)}-\gamma^+\right)} \ln\left(\frac{1}{\sigma}\right) \wedge \tau$ *with high probability*:

$$\begin{aligned}
\|u_\varepsilon(t)\|_{C^0} &\geqslant \left\|\sigma \hat{W}_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^0} - \left\|u_\varepsilon(t) - \sigma\hat{W}_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^0} \\
&\geqslant C\left\|\sigma \hat{W}_{A_\delta}^{(\varepsilon)}(t)\right\|_{L^2} - \left\|u_\varepsilon(t) - \sigma\hat{W}_{A_\delta}^{(\varepsilon)}(t)\right\|_{C^0} \\
&\geqslant C_L e^{t\gamma^+}\sigma - K_{\alpha,\delta} e^{t\lambda_{\max}^{(\theta)}} t^{\frac{1-\alpha}{2}} r_\sigma^3.
\end{aligned}$$

By substituting $t =: \frac{\hat{v}}{\lambda_{\max}} \ln\left(\frac{1}{\sigma}\right)$, we obtain

$$\begin{aligned}
\|u_\varepsilon(t_\sigma)\|_{C^0} &\geqslant C_L e^{t_\sigma\gamma^+}\sigma - K_{\alpha,\delta} e^{t_\sigma\lambda_{\max}^{(\theta)}} t_\sigma^{\frac{1-\alpha}{2}} r_\sigma^3 \\
&= C_L e^{\frac{\hat{v}}{\lambda_{\max}}\ln\left(\frac{1}{\sigma}\right)\gamma^+}\sigma - K_{\alpha,\delta} e^{\frac{\hat{v}\lambda_{\max}^{(\theta)}}{\lambda_{\max}}\ln\left(\frac{1}{\sigma}\right)} \left(\frac{\hat{v}}{\lambda_{\max}}\ln\left(\frac{1}{\sigma}\right)\right)^{\frac{1-\alpha}{2}} r_\sigma^3 \\
&= C_L \sigma^{-\hat{v}\frac{\gamma^+}{\lambda_{\max}}+1} - K_{\alpha,\delta}\sigma^{-\hat{v}\frac{\lambda_{\max}^{(\theta)}}{\lambda_{\max}}} \ln\left(\left(\frac{1}{\sigma}\right)^{\frac{\hat{v}}{\lambda_{\max}}}\right)^{\frac{1-\alpha}{2}} r_\sigma^3 \\
&\geqslant C_L \sigma^{-\hat{v}\frac{\gamma^+}{\lambda_{\max}}+1} - K_{\alpha,\delta}\sigma^{-\hat{v}\frac{\lambda_{\max}^{(\theta)}}{\lambda_{\max}}} \sigma^{-\eta} r_\sigma^3 \\
&= C_L r_\sigma^{-\rho\hat{v}\frac{\gamma^+}{\lambda_{\max}}+\rho} - K_{\alpha,\delta,\eta} r_\sigma^{-\rho\hat{v}\frac{\lambda_{\max}^{(\theta)}}{\lambda_{\max}}} \sigma^{-\eta} r_\sigma^3
\end{aligned}$$

with high probability, where $\eta > 0$ is sufficiently small. The fact that this expression is greater than $\sigma = r_\sigma^\rho$ corresponds to the inequality

$$1 \leqslant C_L r_\sigma^{-\rho\hat{v}\frac{\gamma^+}{\lambda_{\max}}} - K_{\alpha,\delta,\varepsilon} r_\sigma^{-\rho\hat{v}\frac{\lambda_{\max}^{(\theta)}}{\lambda_{\max}}} r_\sigma^{3-\rho-\rho\eta}$$

or equivalent to the inequality

$$1 \geqslant \frac{1}{C_L} r_\sigma^{\hat{v}\rho\frac{\gamma^+}{\lambda_{\max}}} + \frac{K_{\alpha,\delta,\varepsilon}}{C_L} r_\sigma^{3-\rho-\rho\eta+\rho\hat{v}\frac{\left(\gamma^+-\lambda_{\max}^{(\theta)}\right)}{\lambda_{\max}}}.$$

Furthermore, this inequality holds for sufficiently small r_σ provided each exponent is positive, or equivalently, if $\hat{v} > 0$ and

$$\hat{v} < \frac{(3-\rho-\rho\eta)\lambda_{\max}}{\rho\left(\lambda_{\max}^{(\theta)} - \gamma^+\right)} < \frac{2\lambda_{\max}}{\left(\lambda_{\max}^{(\theta)} - \gamma^+\right)}$$

is satisfied. Since the parameter $\eta > 0$ is arbitrary and can be chosen sufficiently small, the above expression holds for $\hat{v} \in \left(0, \frac{(3-\rho)\lambda_{\max}}{\rho\left(\lambda_{\max}^{(\theta)}-\gamma^+\right)}\right)$ and thus also for

$$t_\sigma \in \left(0, \frac{3-\rho}{\rho\left(\lambda_{\max}^{(\theta)} - \gamma^+\right)} \ln\left(\frac{1}{\sigma}\right)\right).$$

It should be noted that for $\rho \in (1, 3)$ we have

$$0 < \frac{3-\rho}{\rho\left(\lambda_{\max}^{(\theta)} - \gamma^+\right)} \ln\left(\frac{1}{\sigma}\right) < \frac{2}{\lambda_{\max}^{(\theta)} - \gamma^+} \ln\left(\frac{1}{\sigma}\right).$$

This completes the argument. □

Lemma 4.37 (Log-time scaling comparison)**.** *Assume that Assumption 4.12 holds. We further restrict to $\rho \in \left(1, \frac{31}{13}\right)$ and choose r_σ such that $\sigma = r_\sigma^\rho$. Then we have*

$$\frac{1}{\lambda_{\max}^{(\theta)}} \ln\left(\frac{r_\sigma}{\sigma}\right) \in \left(0, \frac{3-\rho}{\rho\left(\lambda_{\max}^{(\theta)} - \gamma^+\right)} \ln\left(\frac{1}{\sigma}\right)\right). \tag{4.10}$$

Proof. As $\frac{1}{\lambda_{\max}^{(\theta)}} \ln\left(\frac{r_\sigma}{\sigma}\right)$ is surely positive, we only need to show that the following holds:

$$\frac{1}{\lambda_{\max}^{(\theta)}} \ln\left(\frac{r_\sigma}{\sigma}\right) < \frac{3-\rho}{\rho\left(\lambda_{\max}^{(\theta)} - \gamma^+\right)} \ln\left(\frac{1}{\sigma}\right).$$

By the equation $\sigma = r_\sigma^\rho$, this inequality is equivalent to the following conditions:

- $\ln(r_\sigma) < \ln(\sigma)\left[1 + \frac{(\rho-3)\lambda_{\max}^{(\theta)}}{\rho\left(\lambda_{\max}^{(\theta)}-\gamma^+\right)}\right]$,
- $1 > \rho + (\rho-3)\frac{\lambda_{\max}^{(\theta)}}{\lambda_{\max}^{(\theta)}-\gamma^+}$,
- $\rho < 1 + \frac{(3-\rho)\lambda_{\max}^{(\theta)}}{\lambda_{\max}^{(\theta)}-\gamma^+}$,
- $\rho < 1 + \frac{2\lambda_{\max}^{(\theta)}}{2\lambda_{\max}^{(\theta)}-\gamma^+}$.

For $\gamma^+ > \frac{5}{9}\lambda_{\max}^{(\theta)}$ the last condition above holds, as we have

$$\frac{31}{13} = 1 + \frac{2\lambda_{\max}^{(\theta)}}{2\lambda_{\max}^{(\theta)} - \frac{5}{9}\lambda_{\max}^{(\theta)}} < 1 + \frac{2\lambda_{\max}^{(\theta)}}{2\lambda_{\max}^{(\theta)} - \gamma^+}.$$

This verifies the lemma. □

Remark 4.38. *Using Definition 4.9 and choosing $\gamma^+ = \lambda_{\max}$, we can extend the range for ρ to*

$$\left(1\,,\,2+\frac{1}{1+2\theta\left(\frac{1}{4K_\delta\lambda_{\max}}-1\right)}\right).$$

Note that, since $K_\delta \in (0,\delta)$ and $\theta \in (0,1)$ can be chosen arbitrarily, the upper bound of the interval $2+\frac{1}{1+2\theta\left(\frac{1}{4K_\delta\lambda_{\max}}-1\right)}$ can be chosen arbitrarily close to 3.

By applying Theorem 4.25 and Theorem 4.36, we directly obtain the following two corollaries:

Corollary 4.39. *Assume that Assumption 4.12 holds. We define $\rho \in \left(1, \frac{31}{13}\right)$ and choose r_σ such that $\sigma = r_\sigma^\rho$. Then we have*

$$0 < \sigma \leqslant \|u_\varepsilon(t)\|_{C^0} \leqslant \|u_\varepsilon(t)\|_{C^1} \ll r_\sigma \ll 1$$

with high probability *for* $0 < t \ll \frac{1}{\lambda_{\max}^{(\theta)}} \ln\left(\frac{r_\sigma}{\sigma}\right) < \frac{3-\rho}{\rho(\lambda_{\max}-\gamma^+)} \ln\left(\frac{1}{\sigma}\right)$.

The following corollary is a direct conclusion from Corollary 4.39:

Corollary 4.40. *Assume that Assumption 4.12 holds. We define $\rho \in \left(1, \frac{31}{13}\right)$ and choose r_σ such that $\sigma = r_\sigma^\rho$. Then, we have $\tau \geqslant \frac{1}{\lambda_{\max}^{(\theta)}} \ln\left(\frac{r_\sigma}{\sigma}\right)$ and*

$$0 < \sigma \leqslant \sup_{s\in[0,\tau]} \|u_\varepsilon(s)\|_{C^0} \leqslant \sup_{s\in[0,\tau]} \|u_\varepsilon(s)\|_{C^1} \leqslant r_\sigma \ll 1$$

holds with high probability*.*

Remark 4.41. *As the nonlinearity vanishes when ε approaches 0, we obtain the convergence of the mild solution (see Definition 3.1) to the solution of (3.2) (see Theorem 3.2). For initial value $u_0 = 0$ we obtain*

$$\|u_\varepsilon - \sigma Z_\varepsilon\|_{L^p(\Omega, C^0([0,T]\times\mathbb{T}^2))} \to 0 \quad \text{if} \quad \varepsilon \to 0.$$

In contrast to Section 4.5.2, if $\varepsilon > 0$ is fixed, we obtain

$$\|\sigma Z_\varepsilon\|_{C^0} \to 0 \quad \text{if } \sigma \to 0.$$

In this scenario $\|\sigma Z_\varepsilon\|_{C^0}$ remains bounded, which implies that the nonlinearity $f(\nabla u_\varepsilon)$ persists. Therefore, the limits $\sigma \to 0$ and $\varepsilon \to 0$ cannot be taken simultaneously. As a result, hill growth remains under these conditions.
However, combining a spectral Galerkin scheme in space, to regularise the white noise, with a convergent sequence of diffusion coefficients $\left(\sigma_N\right)_{N\in\mathbb{N}}$ that decays more slowly than $\ln(N)^{-1}$ yields a vanishing nonlinearity. Details of this result will be presented in Section 5.2.2.

5. Euler Scheme in Time for a Spectral Galerkin Scheme in Space

In the following numerical analysis, we present convergence rates for approximating the mild solution under rougher noise $\xi^{(\alpha)} = \partial_t W^{(\alpha)}$ for $\alpha \in (0,1)$ and the corresponding stochastic convolution $Z^{(\alpha)} \in C^0\left([0,T], \mathcal{H}^{1-\alpha-}\left(\mathbb{T}^2\right)\right)$ is given by

$$Z^{(\alpha)}(t,x) \coloneqq \sum_{k\in\mathcal{Z}} \mu_k^{\frac{\alpha}{2}} Z_k(t) e_k(x), \quad \text{for } \alpha \in (0,1) \tag{5.1}$$

for $t \in [0,T]$, $x \in \mathbb{T}^2$, with $Z_k(t) = \int_0^t e^{(t-s)A}\,\mathrm{d}W(s)$ for each $k \in \mathcal{Z}$. Furthermore, we examine the original stochastic partial differential equation (1.1) with the Bilaplace operator multiplied by the parameter $\delta = 1$. Throughout this chapter, constants denoted by C may depend on the parameters L and T. Since the focus is on analysing convergence rates with respect to the spatial and temporal discretisation, the dependence of constants on L and T will not always be tracked explicitly in this context.

This chapter employs a spectral Galerkin method for spatial discretisation and an Euler scheme for temporal integration, as detailed in the following.

Definition 5.1 (Spectral Galerkin method)**.** *Let $n \in \mathbb{N}$ and define $h \coloneqq \frac{T}{n}$. The spectral Galerkin method is then given by*

$$u_h^{(N)}(t) \coloneqq e^{tA} P_N u_0 + \int_0^t e^{(t-s)A} P_N \mathfrak{F}(u_h^{(N)}(\lfloor s \rfloor_h))\,\mathrm{d}s + P_N Z^{(\alpha)}(t), \tag{5.2}$$

where the time variable s is rounded down to the nearest grid point jh, for $j = 0, 1, \dots, \frac{T}{h}$, if $s \in [jh,\ (j+1)h)$, and we use the orthogonal projection

$$P_N : L^2\left(\mathbb{T}^2\right) \to L^2\left(\mathbb{T}^2\right), \quad P_N \sum_{k\in\mathcal{Z}} u_k e_k(x) = \sum_{k\in\mathcal{Z}, |k|\leqslant N} u_k e_k(x). \tag{5.3}$$

Definition 5.2 (Exponential Euler scheme)**.** *Evaluating the mild solution at the grid points jh yields the exponential Euler scheme for $v_j^{(N)} = u_h^{(N)}(jh)$.*

$$v_{j+1}^{(N)} = e^{hA} v_j^{(N)} + \int_0^h e^{(h-s)A} P_N \mathfrak{F}(v_j^{(N)})\,\mathrm{d}s + P_N \tilde{Z}_j, \quad j = 0, 1, \dots, \frac{T}{h}$$

with

$$\tilde{Z}_j = \int_{jh}^{(j+1)h} e^{((j+1)h-s)A} dW(s).$$

For $\alpha = 0$ and the cut-off sequence $\left(\alpha_k^{(N)}\right)_{k\in\mathcal{Z}} = \left(\mathbb{1}_{|k|\leqslant N}\right)_{k\in\mathcal{Z}}$ as a regularising sequence for the white noise, i.e.

$$P_N Z^{(\alpha)}(t) = \sum_{k\in\mathcal{Z}} \alpha_k^{(N)} Z_k(t) e_k(x),$$

we have shown

$$\begin{aligned} u^{(N)}(t) &\coloneqq e^{tA}u_0 + \int_0^t e^{(t-s)A}\mathfrak{F}\left(u^{(N)}(s)\right)\,\mathrm{d}s + \sigma P_N Z^{(\alpha)}(t) \\ &\to u(t) = e^{tA}u_0 + \sigma Z^{(\alpha)}(t) \end{aligned} \tag{5.4}$$

in $L^p\left(\Omega, C^0\left([0,T]\times\mathbb{T}^2\right)\right)$ for $p > 1$ in Theorem 3.2. In particular, the limiting function u of the sequence $\left(u^{(N)}\right)_{N\in\mathbb{N}}$ is independent of the regularisation and u is the mild solution of

$$\begin{cases} \partial_t u &= -\Delta^2 u + \sigma\partial_t W, \\ u(0) &= u_0. \end{cases} \tag{5.5}$$

Remark 5.3. *Throughout this chapter, the cut-off sequence*

$$\left(\alpha_k^{(N)}\right)_{k\in\mathcal{Z}} = \left(\mathbb{1}_{|k|\leqslant N}\right)_{k\in\mathcal{Z}}$$

can be replaced with $\left(\mathbb{1}_{|k|\leqslant\zeta(N)}\right)_{k\in\mathcal{Z}}$, *where* $(\zeta(N))_{N\in\mathbb{N}}$ *is any non-decreasing sequence that diverges to infinity as N increases. Equivalently, this can be characterised by an orthogonal projection*

$$P_{\zeta(N)} : L^2\left(\mathbb{T}^2\right) \to L^2\left(\mathbb{T}^2\right), \quad P_{\zeta(N)}\sum_{k\in\mathcal{Z}} u_k e_k(x) = \sum_{k\in\mathcal{Z},|k|\leqslant\zeta(N)} u_k e_k(x).$$

However, for the sake of simplicity, we will focus our investigation on the case from Definition 5.1 for $\alpha \in (0,1)$.

5.1. Numerical Results

This section presents the main results of the chapter, beginning with the convergence rate of the error function $e_h^{(N)} \coloneqq u^{(N)} - u_h^{(N)}$ when comparing the spectral Galerkin approximation $u^{(N)}$ to the Euler discretisation of the spectral Galerkin approximation $u_h^{(N)}$.

Theorem 5.4. *Let $T > 0$ and $u_0 \in \mathcal{H}^{1+\eta}$ with $\eta > 0$. Let $u_h^{(N)}$ and $u^{(N)}$, as defined in (5.2) and (5.4). Then, for each $p > 1$ sufficiently large, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|u_h^{(N)}(t) - u^{(N)}(t)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}}$$
$$\leqslant C\left(\|u_0\|_{\mathcal{H}^{1+\eta}}\left(h^{\frac{\eta}{4}} + N^{-\eta}\right) + N^{2+\alpha}h^{1-\frac{1}{p}-} + N^{-2} + h^{\frac{1}{2}-\frac{\gamma}{4}}N^{-\frac{\alpha}{p+2}}\right).$$

Similar to the vanishing nonlinearity result (see Theorem 3.2), we present a related theorem demonstrating that the Euler discretisation of the spectral Galerkin approximation $u_h^{(N)}$ converges. In the limit, the function does not exhibit linear instability, and the surface remains without rough hill structure.

Theorem 5.5. *Let $T > 0$ and $u_0 \in \mathcal{H}^{1+\eta}$ with $\eta > 0$. Let $u_h^{(N)}$, as defined in (5.2), and u denote the mild solution to (5.5). Then, for each $p \geqslant 1$, we have*

$$\lim_{N\to\infty}\lim_{h\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|u_h^{(N)}(t) - u(t)\right\|_{\mathcal{H}^1}^p\right] = 0. \tag{5.6}$$

Proof. By Theorem 5.4 we obtain (5.6) for $p > 1$, depending only on γ, such that $\frac{p}{p-1} \in \left(1, \frac{1}{1-\frac{\gamma}{4}}\right)$. By applying Hölder's inequality, this establishes the desired result for each $p \geqslant 1$. □

5.2. Auxiliary Results

In this section, we present auxiliary results to prove Theorem 5.4 and Theorem 5.5. Specifically, we demonstrate the divergence rate of the stochastic convolution in Lemma 5.6 to prove the convergence rate for the vanishing nonlinearity in Lemma 5.8, and the Hölder continuity in time in Lemma 5.7.

Lemma 5.6. *The stochastic convolution exhibits the following growth estimate:*

$$\mathbb{E}\left[\left\|P_N Z^{(\alpha)}(t)\right\|_{\mathcal{H}^1}^2\right] \sim \begin{cases} \ln(N), & \text{if } \alpha = 0 \\ N^\alpha, & \text{if } \alpha \in (0,1) \end{cases}$$

for each $t \in (0, T]$.

Proof. For each $t \in (0,T]$, we derive the following growth behavior by applying Itô's isometry and using the independence of the sequence $(\beta_k)_{k\in\mathbb{Z}}$ in the first equation below

$$\begin{aligned}\mathbb{E}\left[\left\|P_N Z^{(\alpha)}(t)\right\|_{\mathcal{H}^1}^2\right] &= \sum_{|k|\leqslant N} \mu_k^{1+\alpha} \int_0^t e^{-2(t-s)\mu_k^2}\,\mathrm{d}s \\ &= \sum_{|k|\leqslant N} \frac{\mu_k^\alpha}{2\mu_k}\left[1-e^{-2t\mu_k^2}\right] \\ &\sim \int_1^N \frac{x^\alpha}{2x}\left[1-e^{-2tx^2}\right]\mathrm{d}x \\ &=: I_\alpha(N).\end{aligned}$$

For $\alpha = 0$, by using the substitution $u = -2tx^2$, we obtain

$$\begin{aligned}I_\alpha(N) &= \frac{1}{2}\ln(N) - \int_1^N \frac{e^{-2tx^2}}{2x}\,\mathrm{d}x \\ &= \frac{1}{2}\ln(N) - \frac{1}{4}\int_{2t}^{2tN^2} \frac{e^{-u}}{u}\,\mathrm{d}u \\ &= \frac{1}{2}\ln(N) - \frac{1}{4}\left[\int_{2t}^{\infty} \frac{e^{-u}}{u}\,\mathrm{d}u - \int_{2tN^2}^{\infty} \frac{e^{-u}}{u}\,\mathrm{d}u\right] \\ &\sim \frac{1}{2}\ln(N)\end{aligned}$$

which holds, since

$$\int_{2tN^2}^{\infty} \frac{e^{-u}}{u}\,\mathrm{d}u$$

is bounded for each $t > 0$ and $N \in \mathbb{N}$ and converges to 0 if N goes to ∞.

For $\alpha \in (0,1)$, by using the substitution $u = -2tx^2$, we obtain

$$\begin{aligned}I_\alpha(N) &= \frac{N^\alpha - 1}{2\alpha} - \int_1^N \frac{e^{-2tx^2}}{2x^{1-\alpha}}\,\mathrm{d}x \\ &= \frac{N^\alpha - 1}{2\alpha} - \frac{1}{4}\frac{1}{(2t)^{\frac{\alpha}{2}}}\int_{2t}^{2tN^2} \frac{u^{\frac{\alpha}{2}}e^{-u}}{u}\,\mathrm{d}u \\ &= \frac{N^\alpha - 1}{2\alpha} - \frac{1}{4}\frac{1}{(2t)^{\frac{\alpha}{2}}}\left[\int_{2t}^{\infty} \frac{u^{\frac{\alpha}{2}}e^{-u}}{u}\,\mathrm{d}u - \int_{2tN^2}^{\infty} \frac{u^{\frac{\alpha}{2}}e^{-u}}{u}\,\mathrm{d}u\right] \\ &\sim N^\alpha\end{aligned}$$

which holds, since

$$\int_{2tN^2}^{\infty} \frac{u^{\frac{\alpha}{2}} e^{-u}}{u} \,\mathrm{d}u \xrightarrow{N\to\infty} 0$$

is bounded for each $t > 0$ and $N \in \mathbb{N}$ and converges to 0 if N goes to ∞. □

Lemma 5.7. *For any $p \geqslant 1$, there exists a constant C_p only depending on T, p such that for any $\eta \in (0,1)$, $\psi \in (0,1]$, and $0 \leqslant s \leqslant t \leqslant T$ the following to bounds hold*

$$\mathbb{E}\left[\|P_N Z^{(\alpha)}(t) - P_N Z^{(\alpha)}(s)\|^p_{C^0(\mathbb{T}^2)}\right]^{\frac{1}{p}} \leqslant C_p |t-s|^{\frac{\psi}{2}} N^{\frac{\alpha+\eta-1}{2}+\psi},$$
$$\mathbb{E}\left[\|P_N Z^{(\alpha)}(t) - P_N Z^{(\alpha)}(s)\|^p_{C^1(\mathbb{T}^2)}\right]^{\frac{1}{p}} \leqslant C_p |t-s|^{\frac{\psi}{2}} N^{\frac{\alpha+\eta}{2}+\psi}.$$

Proof. From the proof of Theorem 2.21, we conclude that for each $\psi \in (0,1]$ there is a constant $C_\psi > 0$ such that

$$\mathbb{E}\left[|Z_k(t) - Z_k(s)|^2\right] \leqslant C_\psi \mu_k^{2\psi-2} |t-s|^\psi$$

holds for each $t, s \in [0,T]$. Furthermore, by Theorem 2.25 we derive for $p \geqslant 1$

$$\begin{aligned}
&\mathbb{E}\left[\|P_N Z^{(\alpha)}(t) - P_N Z^{(\alpha)}(s)\|^p_{C^0}\right]^{\frac{1}{p}} \\
&\leqslant C_{p,\eta} \mathbb{E}\left[\|P_N\left(Z^{(\alpha)}(t) - Z^{(\alpha)}(s)\right)\|^2_{\mathcal{H}^\eta}\right]^{\frac{1}{2}} \\
&\leqslant C_{p,\eta} \left(\sum_{|k|\leqslant N} \mu_k^{\alpha+\eta}\, \mathbb{E}\left[|Z_k(t) - Z_k(s)|^2\right]\right)^{\frac{1}{2}} \\
&\leqslant C_{\psi,p,\eta} \left(\sum_{|k|\leqslant N} \mu_k^{\alpha+\eta+2\psi-2} |t-s|^\psi\right)^{\frac{1}{2}} \\
&\leqslant C_{\psi,p,\eta} \left(|t-s|^\psi \int_1^N \tau^{\alpha+\eta+2\psi-2}\,\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\leqslant C_{\psi,p,\eta} |t-s|^{\frac{\psi}{2}} N^{\frac{\alpha+\eta-1}{2}+\psi}.
\end{aligned}$$

Similarly as above, we directly derive

$$\begin{aligned}
&\mathbb{E}\left[\|P_N Z^{(\alpha)}(t) - P_N Z^{(\alpha)}(s)\|^p_{C^1}\right]^{\frac{1}{p}} \\
&\leqslant C_{p,\eta} \mathbb{E}\left[\|P_N\left(Z^{(\alpha)}(t) - Z^{(\alpha)}(s)\right)\|^2_{\mathcal{H}^{1+\eta}}\right]^{\frac{1}{2}} \\
&\leqslant C_{\psi,p,\eta} |t-s|^{\frac{\psi}{2}} N^{\frac{\alpha+\eta}{2}+\psi}.
\end{aligned}$$

This confirms the claim. □

5.2.1. Convergence Rate for the Rougher Noise Case

In this subsection we consider the rate for the vanishing of the nonlinearity in the case of $\alpha \in (0,1)$, i.e. noise that is rougher than the regular white noise. Recall $Z^{(\alpha)}$ from (5.1) and $u^{(N)}$ from (5.4).

Lemma 5.8. *For each $t \in (0,T]$ it holds that for $f(z) \coloneqq \frac{z}{1+|z|^2}$, $z \in \mathbb{R}^2$,*

$$\sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\left|f\left(\nabla u^{(N)}(t,x)\right)\right|^p\right]^{\frac{1}{p}} \lesssim N^{-\frac{\alpha}{p+2}}. \tag{5.7}$$

Proof. This proof builds directly on Lemma 3.10 and Theorem 3.11, using their results explicitly. By Lemma 3.10 the processes $P_N Z^{(\alpha)}$ and $\nabla P_N Z^{(\alpha)}$ are independet of the variable $x \in \mathbb{T}^2$. Let $M \geqslant \left\|u^{(N)} - P_N Z^{(\alpha)}\right\|_\infty$, $\gamma \coloneqq \frac{1}{p+2} \in \left(0, \frac{1}{2}\right)$ and

$$K_N \coloneqq N^{\alpha\gamma} \vee M.$$

Furthermore, as the covariance matrix $\Sigma_N(t,x) \coloneqq \operatorname{Cov}\left(\frac{\nabla P_N Z^{(\alpha)}(t,x)}{K_N}\right)$ does not depend on the variable $x \in \mathbb{T}^2$, we obtain

$$\begin{aligned}
&\det\left(\Sigma_N(t,x)\right) \gtrsim \frac{1}{(K_N)^4}\left[\sum_{k\in\mathcal{Z},|k|\leqslant N} \frac{\mu_k^{\frac{\alpha}{2}} k_1^2}{|k|^4}\right]^2 \\
&\gtrsim \frac{1}{(K_N)^4}\left[\int_1^N \int_1^N \frac{y^2}{(y+z)^{4-\alpha}}\,\mathrm{d}z\,\mathrm{d}y\right]^2 \\
&\gtrsim \frac{1}{N^{4\alpha\gamma}}\left[\int_1^N \frac{y^2}{(3-\alpha)(y+1)^{3-\alpha}} - \frac{y^2}{(3-\alpha)(y+N)^{3-\alpha}}\,\mathrm{d}y\right]^2 \\
&\gtrsim \frac{1}{N^{4\alpha\gamma}}\left[N^\alpha + C_\alpha\left(1+N^{\alpha-1}\right)\right]^2 \\
&\gtrsim \frac{N^{2\alpha}}{N^{4\alpha\gamma}},
\end{aligned}$$

where we used the integration by parts formula to compute the last integral above.

The nonlinearity thus obtains the following upper bound for each $t \in (0,T]$ and $x \in \mathbb{T}^2$

$$\begin{aligned}
&\sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\left|f\left(\nabla u^N(t,x)\right)\right|^p\right] \\
&\leqslant \sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\left|f\left(\nabla u^N(t,x)\right)\right|^p \middle| \; \left|\nabla P_N Z^{(\alpha)}(t,x)\right| > K_N\right] \\
&\quad + \sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\left|f\left(\nabla u^N(t,x)\right)\right|^p \middle| \; \left|\nabla P_N Z^{(\alpha)}(t,x)\right| \leqslant K_N\right] \mathbb{P}\left(\left|\nabla P_N Z^{(\alpha)}(t,x)\right| \leqslant K_N\right) \\
&\leqslant \left(\frac{2}{1+K_N-M}\right)^p + C \max_{y\in B_1(0)} \{\varphi_N(t,x,y)\} \\
&\lesssim N^{-\alpha p\gamma} + \frac{1}{\sqrt{\det\left(\Sigma_N(t,x)\right)}},
\end{aligned}$$

whereby

$$\varphi_N(t,x,y) := \frac{1}{2\pi\sqrt{\det\left(\Sigma_N(t,x)\right)}} e^{-\frac{1}{2} y^T \Sigma_N^{-1}(t,x) y}$$

is the density function of $\frac{\nabla P_N Z^{(\alpha)}(t,x)}{K_N} \sim \mathcal{N}\left(\mathbf{0}, \Sigma_N(t,x)\right)$. By the choice $\gamma = \frac{1}{p+2}$, we obtain

$$\begin{aligned}
\sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\left|f\left(\nabla u^N(t,x)\right)\right|^p\right] &\lesssim N^{-\alpha p\gamma} + N^{\alpha(2\gamma-1)} \\
&\lesssim N^{-\frac{\alpha p}{p+2}}.
\end{aligned}$$

This shows the assertion. □

Remark 5.9. *For a more general nonlinearity f satisfying*

$$f\left(\nabla u\right) \sim |\nabla u|^{-\eta}$$

for $\eta > 0$, we even get the bound

$$\sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\left|f\left(\nabla u^{(N)}(t,x)\right)\right|^p\right]^{\frac{1}{p}} \lesssim N^{-\frac{\alpha}{p\eta+2}}$$

for each $t \in (0,T]$, by choosing $\gamma := \frac{1}{p\eta+2}$ and applying the inequality

$$\sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\left|f\left(\nabla u^N(t,x)\right)\right|^p \middle| \; \left|\nabla P_N Z^{(\alpha)}(t,x)\right| > K_N\right] < |K_N - M|^{-p\eta}$$

as in the proof of Lemma 5.8.

Lemma 5.10. *For each $p > 1$ there exists a constant $C > 0$, depending only on p and T, such that*

$$\mathbb{E}\left[\left\|f\left(\nabla u^{(N)}\right)\right\|^p_{L^p([0,T]\times\mathbb{T}^2,\mathbb{R}^2)}\right]^{\frac{1}{p}} \leqslant C N^{-\frac{\alpha}{p+2}}.$$

Proof. By using Theorem A.24 and the pointwise bound on f from Lemma 5.8, we obtain the following bound

$$
\begin{aligned}
&\mathbb{E}\left[\left\|f\left(\nabla u^{(N)}\right)\right\|_{L^p([0,T]\times\mathbb{T}^2,\mathbb{R}^2)}^p\right]^{\frac{1}{p}} \\
&= \mathbb{E}\left[\int_{[0,L]^2}\int_0^T \left|f\left(\nabla u^{(N)}(t,x)\right)\right|^p \,\mathrm{d}t\,\mathrm{d}x\right]^{\frac{1}{p}} \\
&\leqslant \int_{[0,L]^2}\int_0^T \mathbb{E}\left[\left|f\left(\nabla u^{(N)}(t,x)\right)\right|^p\right]^{\frac{1}{p}} \,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant C\int_0^T \sup_{x\in\mathbb{T}^2}\mathbb{E}\left[\left|f\left(\nabla u^{(N)}(t,x)\right)\right|^p\right]^{\frac{1}{p}} \,\mathrm{d}t \\
&\leqslant C\int_0^T N^{-\frac{\alpha}{p+2}}\,\mathrm{d}t \\
&\leqslant CTN^{-\frac{\alpha}{p+2}}.
\end{aligned}
$$

This verifies the lemma. ☐

Remark 5.11. *By choosing the stochastic convolution $P_N Z := P_N Z^{(0)}$, i.e. for roughness parameter $\alpha = 0$, in*

$$\tilde{u}^N(t) := e^{tA}P_N u_0 + \int_0^t e^{(t-s)A}F\left(\tilde{u}^N(s)\right)ds + P_N Z(t)$$

we derive the following convergence rate

$$\sup_{x\in\mathbb{T}^2}\mathbb{E}\left[\left|f\left(\nabla\tilde{u}^N(t,x)\right)\right|^p\right] \lesssim \ln(N)^{-\frac{p}{p+2}}.$$

By selecting $K_N = \ln(N)^\gamma \vee M$ the proof proceeds as in the proof of Lemma 5.8.

Remark 5.12. *The application of a non-increasing taming rate $\eta : \mathbb{N} \to \mathbb{N}$ to the orthogonal projection and a non-decreasing function $\zeta : \mathbb{N} \to \mathbb{N}$ such that we have the stochastic convolution*

$$\eta(N)P_{\zeta(N)}Z(t,x) := \eta(N)\sum_{k\in\mathcal{Z},|k|\leqslant\zeta(N)}\int_0^t e^{-(t-s)\mu_k^2}\,\mathrm{d}\beta_k(s)e_k(x),$$

may improve several of the convergence rates discussed in this chapter. However, a detailed analysis of these potential improvements will be provided in our forthcoming paper [16].

5.2.2. Noise with Converging Diffusion Coefficients

Inspired by the results of Section 4.5.2, represented in Corollary 4.39, and based on the growth rate of the regularised stochastic convolution $(P_N Z)_{N\in\mathbb{N}}$ in Lemma 5.6, it is sufficient to consider a sequence of diffusion coefficients $(\sigma(N))_{N\in\mathbb{N}} \subset (0,1)$, depending only on N, that converges to 0 if $N \to \infty$. Specifically, the sequence $(\sigma(N))_{N\in\mathbb{N}}$ must converge to 0 and satisfy

$$\mathbb{E}\left[\left\|\sigma(N)P_N Z^{(\alpha)}(t)\right\|_{\mathcal{H}^1}^2\right] \xrightarrow{N\to\infty} \infty$$

which we derive by the inequality

$$\sigma(N) \geqslant \begin{cases} \ln(N)^{-\zeta}, & \text{for } \alpha = 0, \\ N^{-\zeta(1+\alpha)}, & \text{for } \alpha \in (0,1), \end{cases}$$

for almost every $N \in \mathbb{N}$ and a fixed but arbitrary $\zeta \in (0,1)$. Let

$$Z^{(\alpha)}_{\sigma(N),N}(t,x) \coloneqq \sigma(N)P_N Z^{(\alpha)}(t,x) = \sigma(N) \sum_{k\in\mathcal{Z},|k|\leqslant N} \mu_k^{\frac{\alpha}{2}} \int_0^t e^{-(t-s)\mu_k^2}\, \mathrm{d}\beta_k(s) e_k(x)$$

for a zero sequence $\sigma : \mathbb{N} \to (0,1)$ and a non-decreasing mapping $\zeta : \mathbb{N} \to \mathbb{N}$. Correspondingly, we denote

$$\tilde{u}^{(N)}_{\alpha,\sigma(N)}(t) = e^{tA}P_N u_0 + \int_0^t e^{(t-s)A} P_N \mathfrak{F}\left(\tilde{u}^{(N)}_{\sigma(N)}(s)\right) ds + Z^{(\alpha)}_{\sigma(N),N}(t).$$

Lemma 5.13. *For each $t \in (0,T]$ it holds that for $f(z) \coloneqq \frac{z}{1+|z|^2}$, $z \in \mathbb{R}^2$,*

$$\sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\left|f\left(\tilde{u}^{(N)}_{\alpha,\sigma(N)}(t,x)\right)\right|^p\right] \lesssim \begin{cases} [\sigma(N)\ln(N)]^{-\frac{p}{p+2}}, & \text{for } \alpha = 0, \\ [\sigma(N)N^{\alpha}]^{-\frac{p}{p+2}}, & \text{for } \alpha \in (0,1). \end{cases}$$

Proof. By choosing $\gamma \coloneqq \frac{1}{p+2} \in \left(0, \frac{1}{2}\right)$ and $K_{N,\sigma(N)} \coloneqq [\sigma(N)\ln(N)]^{\gamma} \vee M$ if $\alpha = 0$ holds, and by choosing $K_{N,\sigma(N)} \coloneqq [\sigma(N)N^{\alpha}]^{\gamma} \vee M$ if $\alpha \in (0,1)$ holds, the proof is analogous to the proof of Lemma 5.8 as

$$Z^N_{\alpha,\sigma(N)} \equiv \sigma(N) Z^N_{\alpha}$$

holds and $\sigma(N)$ is a constant factor that does not depend on $x \in \mathbb{T}^2$, $t \in (0,T]$ or $\omega \in \Omega$. □

Now it is straightforward to obtain the following result:

Corollary 5.14. *Let*

$$\sigma(N) := \begin{cases} \ln(N)^{-\zeta} & \text{for } \alpha = 0, \\ N^{-\zeta(1+\alpha)} & \text{for } \alpha \in (0,1), \end{cases}$$

for every $N \in \mathbb{N}$ and a fixed but arbitrary $\zeta \in (0,1)$ according to the roughness of the noise, for each $t \in (0,T]$ it holds that for $f(z) := \frac{z}{1+|z|^2}$, $z \in \mathbb{R}^2$,

$$\sup_{x \in \mathbb{T}^2} \mathbb{E}\left[\left|f\left(\nabla \tilde{u}^{(N)}_{\alpha,\sigma(N)}(t,x)\right)\right|^p\right]^{\frac{1}{p}} \lesssim \begin{cases} \ln(N)^{\frac{\zeta-1}{(p+2)}} & \text{for } \alpha = 0, \\ N^{\frac{\alpha(\zeta-1)}{(p+2)}} & \text{for } \alpha \in (0,1). \end{cases}$$

5.3. Decomposition of the Error Function

Let $e_h^{(N)} = u^{(N)} - u_h^{(N)}$ be the error function and $Q_N := I - P_N$ be an orthogonal projection. Then we have the mild formulation

$$\begin{aligned} e_h^{(N)}(t) &= e^{tA} Q_N u_0 + \int_0^t e^{(t-s)A}\left[\mathfrak{F}(u^{(N)}(s)) - P_N \mathfrak{F}(u_h^{(N)}(\lfloor s \rfloor_h))\right] \mathrm{d}s \\ &= e^{tA} Q_N u_0 + \int_0^t e^{(t-s)A} Q_N \mathfrak{F}(u^{(N)}(s)) \, \mathrm{d}s \\ &\quad + \int_0^t e^{(t-s)A}\left[P_N \mathfrak{F}(u^{(N)}(s)) - P_N \mathfrak{F}(u_h^{(N)}(s))\right] \mathrm{d}s \\ &\quad + \int_0^t e^{(t-s)A} P_N \left[\mathfrak{F}(u_h^{(N)}(s)) - \mathfrak{F}(u_h^{(N)}(\lfloor s \rfloor_h))\right] \mathrm{d}s \\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned}$$

Lemma 5.15. *Let $u_0 \in \mathcal{H}^{1+\eta}(\mathbb{T}^2)$ for $\eta > 0$. Then, for each $p \geqslant 1$, we have*

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|I_1(t)\|^p_{\mathcal{H}^1}\right]^{\frac{1}{p}} \leqslant \|u_0\|_{\mathcal{H}^{1+\eta}} \left(\frac{L}{2\pi}\right)^{\eta} N^{-\eta}.$$

Proof. For $u_0 = \sum_{k\in\mathcal{Z}} u_k e_k(0)$ we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \|I_1(t)\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} &= \sup_{t\in[0,T]} \|I_1(t)\|_{\mathcal{H}^1} \\
&\leqslant \sup_{t\in[0,T]} e^{-t\left(\frac{2\pi}{L}\right)^4} \|Q_N u_0\|_{\mathcal{H}^1} \\
&\leqslant \left(\sum_{k\in\mathcal{Z},|k|>N} u_k^2 \mu_k^{1+\eta} \mu_k^{-\eta}\right)^{\frac{1}{2}} \\
&\leqslant \|u_0\|_{\mathcal{H}^{1+\eta}} \left(\frac{L}{2\pi}\right)^{\eta} N^{-\eta},
\end{aligned}$$

where we apply $\mu_k^{-\eta} = \left(\frac{L}{2\pi|k|}\right)^{2\eta} \leqslant \left(\frac{L}{2\pi N}\right)^{2\eta}$ for $|k| > N$. □

Lemma 5.16. *For each $\gamma > 0$ and $p > 1$, depending only on γ, such that*

$$\frac{p}{p-1} \in \left(1\ ,\ \frac{1}{1-\frac{\gamma}{4}}\right)$$

we obtain

$$\mathbb{E}\left[\left\|\sup_{t\in[0,T]} I_2(t)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} \leqslant C N^{-2+\gamma-\frac{\alpha}{p+2}}.$$

Proof. By applying Hölder's inequality, Minkowski inequality, and Lemma 5.10, we conclude

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \|I_2(t)\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} \\
&= \mathbb{E}\left[\sup_{t\in[0,T]} \left\|\int_0^t e^{(t-s)A} Q_N\left[\mathfrak{F}\left(u^{(N)}(s)\right)\right] \,\mathrm{d}s\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} \\
&\leqslant \mathbb{E}\left[\sup_{t\in[0,T]} \left(\int_0^t \left\|e^{(t-s)A}\nabla\cdot\right\|_{L(\mathcal{H}^{-2+\gamma}(\mathbb{T}^2,\mathbb{R}^2),\mathcal{H}^1)} \left\|f\left(\nabla u^{(N)}(s)\right)\right\|_{L^2} \,\mathrm{d}s\right)^p\right]^{\frac{1}{p}} \\
&\quad \times \|Q_N\|_{L(\mathcal{H}^{-1},\mathcal{H}^{-3+\gamma})} \\
&\leqslant \mathbb{E}\left[\sup_{t\in[0,T]} \left(\int_0^t \left\|e^{(t-s)A}\right\|_{L(\mathcal{H}^{-3+\gamma},\mathcal{H}^1)} \left\|f\left(\nabla u^{(N)}(s)\right)\right\|_{L^2} \,\mathrm{d}s\right)^p\right]^{\frac{1}{p}} \\
&\quad \times \|Q_N\|_{L(\mathcal{H}^{-1},\mathcal{H}^{-3+\gamma})} \\
&\leqslant C_\gamma \|Q_N\|_{L(\mathcal{H}^{-1},\mathcal{H}^{-3+\gamma}))} \\
&\quad \times \left(\int_0^T s^{\left(-1+\frac{\gamma}{4}\right)\frac{p}{p-1}} \,\mathrm{d}s\right)^{\frac{p-1}{p}} \mathbb{E}\left[\int_0^T \left\|f\left(\nabla u^{(N)}(s)\right)\right\|_{L^2}^p \,\mathrm{d}s\right]^{\frac{1}{p}} \\
&\leqslant C_\gamma N^{-2+\gamma} T^{\left(-1+\frac{\gamma}{4}\right)+\frac{p-1}{p}} \int_0^T \mathbb{E}\left[\left\|f\left(\nabla u^{(N)}(s)\right)\right\|_{L^2}^p\right]^{\frac{1}{p}} \,\mathrm{d}s \\
&\leqslant C_T N^{-2+\gamma} N^{-\frac{\alpha}{p+2}}.
\end{aligned}$$

This shows the assertion. □

Lemma 5.17. *Let $u_0 \in \mathcal{H}^{1+\eta}\left(\mathbb{T}^2\right)$ for $\eta > 0$ and let $\gamma > 0$. For $p > 1$, depending only on γ, such that $\frac{p}{p-1} \in \left(1, \frac{1}{1-\frac{\gamma}{4}}\right)$ and constants $C = C_{p,T,L} > 0$ we obtain*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \left\|u_h^{(N)}(t) - u_h^{(N)}(\lfloor t\rfloor_h)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} \\
&\leqslant C\left(h^{\frac{\eta}{4}} \|u_0\|_{\mathcal{H}^{1+\eta}} + N^{2+\alpha} h^{1-\frac{1}{p}-} + h^{\frac{1}{2}-\frac{\gamma}{4}} N^{-\frac{\alpha}{p+2}} + h^{\frac{1}{2}-\frac{1}{p}} N^{-\frac{\alpha}{p+2}}\right).
\end{aligned}$$

Proof. For each $\beta \in (0, 1]$ we have from (A.12)

$$\left\|(e^{tA} - \mathrm{Id})u\right\|_{\mathcal{H}^1} \leqslant Ct^\beta \left\|A^\beta u\right\|_{\mathcal{H}^1} = Ct^\beta \|u\|_{\mathcal{H}^{4\beta+1}}$$

for each $t \in [0, T]$ (see (A.6)). Therefore, for $\gamma \in (0, 2)$ we obtain

$$\left\|(e^{tA} - \mathrm{Id})\right\|_{L(\mathcal{H}^{3-\gamma},\mathcal{H}^1)} \leqslant Ct^{\frac{1}{2}-\frac{\gamma}{4}}.$$

We now decompose the difference as follows:

$$\begin{aligned}u_h^{(N)}(t) - u_h^{(N)}(\lfloor t\rfloor_h) &= \left(e^{tA} - e^{\lfloor t\rfloor_h A}\right) P_N u_0 + P_N\left(Z^{(\alpha)}(t) - Z^{(\alpha)}\left(\lfloor t\rfloor_h\right)\right)\\ &\quad+ \int_0^{\lfloor t\rfloor_h} \left(e^{(t-s)A} - e^{(\lfloor t\rfloor_h - s)A}\right) P_N \mathfrak{F}\left(\nabla u_h^{(N)}(\lfloor s\rfloor_h)\right) \mathrm{d}s\\ &\quad+ \int_{\lfloor t\rfloor_h}^{t} e^{(t-s)A} P_N \mathfrak{F}\left(\nabla u_h^{(N)}(\lfloor s\rfloor_h)\right) \mathrm{d}s\\ &=: \hat{I}_{41}(t) + \hat{I}_{42}(t) + \hat{I}_{43}(t) + \hat{I}_{44}(t).\end{aligned}$$

Let $u_0 \in \mathcal{H}^{1+\eta}$. By applying (A.12), we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} \left\|\hat{I}_{41}(t)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} &= \sup_{t\in[0,T]} \left\|\hat{I}_{41}(t)\right\|_{\mathcal{H}^1}\\ &\leqslant \sup_{t\in[0,T]} \left\|\left(e^{tA} - e^{\lfloor t\rfloor_h A}\right) P_N u_0\right\|_{\mathcal{H}^1}\\ &\leqslant \sup_{t\in[0,T]} \left\|e^{\lfloor t\rfloor_h A}\right\|_{L(\mathcal{H}^1,\mathcal{H}^1)} \left\|\left(e^{(t-\lfloor t\rfloor_h)A} - \mathrm{Id}\right) P_N u_0\right\|_{\mathcal{H}^1}\\ &\leqslant C \sup_{t\in[0,T]} |t - \lfloor t\rfloor_h|^{\frac{\eta}{4}} \left\|\left((-\Delta)^2\right)^{\frac{\eta}{4}} u_0\right\|_{\mathcal{H}^1}\\ &\leqslant C h^{\frac{\eta}{4}} \left\|u_0\right\|_{\mathcal{H}^{1+\eta}}.\end{aligned} \tag{5.8}$$

By applying Lemma 5.7, we derive

$$\mathbb{E}\left[\left\|P_N\left(Z^{(\alpha)}(t) - Z^{(\alpha)}\left(s\right)\right)\right\|_{\mathcal{H}^1}^p\right] \leqslant C N^{p(2\psi+\alpha)} |t-s|^{p\psi}$$

for each $t, s \in [0,T]$ and $\psi \in (0,1]$. Since Z is Gaussian, Kolmogorov's continuity theorem (see [67, Theorem 2.2.3]) yields for a $\zeta \in \left(0, \psi - \frac{1}{p}\right)$ that

$$\mathbb{E}\left[\sup_{t\in[0,T], t\neq\lfloor t\rfloor_h} \frac{\left\|P_N\left(Z^{(\alpha)}(t) - Z^{(\alpha)}\left(\lfloor t\rfloor_h\right)\right)\right\|_{\mathcal{H}^1}^p}{|t - \lfloor t\rfloor_h|^{p\zeta}}\right] < C N^{p(2\psi+\alpha)}.$$

By choosing $\psi = 1$, we derive the bound

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} \left\|\hat{I}_{42}(t)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} &= \mathbb{E}\left[\sup_{t\in[0,T]} \left\|P_N\left(Z^{(\alpha)}(t) - Z^{(\alpha)}\left(\lfloor t\rfloor_h\right)\right)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}}\\ &\leqslant C N^{2+\alpha} \sup_{t\in[0,T]} |t - \lfloor t\rfloor_h|^{\zeta}\\ &\leqslant C N^{2+\alpha} h^{1-\frac{1}{p}-}.\end{aligned} \tag{5.9}$$

From (A.12) we get

$$\left\| \left(e^{hA} - \mathrm{Id}\right) y \right\|_{\mathcal{H}^1} \leqslant Ch^{\frac{1}{2}-\frac{\gamma}{4}} \left\| (-\Delta)^{\frac{1}{2}-\frac{\gamma}{4}} y \right\|_{\mathcal{H}^1} \leqslant Ch^{\frac{1}{2}-\frac{\gamma}{4}} \|y\|_{\mathcal{H}^{3-\gamma}}$$

for $y \in \mathcal{H}^{3-\gamma}$. Thus, by applying Lemma 5.10, the Minkowski inequality, and Hölder's inequality, as in the proof of Lemma 5.16, we derive

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \hat{I}_{43}(t) \right\|_{\mathcal{H}^1}^p \right]^{\frac{1}{p}} \qquad (5.10)\\
&\leqslant \mathbb{E}\left[\sup_{t\in[0,T]} \left(\int_0^{\lfloor t \rfloor_h} \left\| e^{(t-s)A} - e^{(\lfloor t \rfloor_h - s)A} \right\|_{L(\mathcal{H}^{-1},\mathcal{H}^1)} \left\| f\left(\nabla u_h^{(N)}(\lfloor s \rfloor_h)\right)\right\|_{L^2} \mathrm{d}s \right)^p \right]^{\frac{1}{p}}\\
&\leqslant \sup_{t\in[0,T]} \left\| e^{(t-\lfloor t \rfloor_h)A} - \mathrm{Id} \right\|_{L(\mathcal{H}^{3-\gamma},\mathcal{H}^1)}\\
&\quad \times \left(\int_0^{\lfloor T \rfloor_h} \left\| e^{(\lfloor T \rfloor_h - s)A} \right\|_{L(\mathcal{H}^{-1},\mathcal{H}^{3-\gamma})}^{\frac{p}{p-1}} \mathrm{d}s \right)^{\frac{p-1}{p}} \mathbb{E}\left[\int_0^T \left\| f\left(\nabla u^{(N)}\left(\lfloor s \rfloor_h\right)\right)\right\|_{L^2}^p \mathrm{d}s \right]^{\frac{1}{p}}\\
&\leqslant Ch^{\frac{1}{2}-\frac{\gamma}{4}} \left(\int_0^{\lfloor T \rfloor_h} s^{\left(-1+\frac{\gamma}{4}\right)\frac{p}{p-1}} \mathrm{d}s \right)^{\frac{p-1}{p}} \int_0^T \mathbb{E}\left[\left\| f\left(\nabla u^{(N)}\left(\lfloor s \rfloor_h\right)\right)\right\|_{L^2}^p \right]^{\frac{1}{p}} \mathrm{d}s\\
&\leqslant C_p h^{\frac{1}{2}-\frac{\gamma}{4}} T^{\frac{\gamma}{4}-\frac{1}{p}} N^{-\frac{\alpha}{p+2}}.
\end{aligned}$$

Furthermore, by the Minkowski inequality, Hölder's inequality and Lemma 5.10, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \hat{I}_{44}(t) \right\|_{\mathcal{H}^1}^p \right]^{\frac{1}{p}} \qquad (5.11)\\
&\leqslant \mathbb{E}\left[\sup_{t\in[0,T]} \left(\int_{\lfloor t \rfloor_h}^t \frac{(t-s)^{-\frac{1}{2}}}{\sqrt{2e}} \left\| f\left(\nabla u_h^{(N)}(\lfloor j \rfloor_h)\right)\right\|_{L^2} \mathrm{d}s \right)^p \right]^{\frac{1}{p}}\\
&\leqslant \mathbb{E}\left[\sup_{t\in[0,T]} \left(\int_{\lfloor t \rfloor_h}^t (t-s)^{-\frac{p}{2(p-1)}} \mathrm{d}s \right)^{p-1} \int_{\lfloor t \rfloor_h}^t \left\| f\left(\nabla u^{(N)}(\lfloor j \rfloor_h)\right)\right\|_{L^2}^p \mathrm{d}s \right]^{\frac{1}{p}}\\
&\leqslant C_p \mathbb{E}\left[\sup_{t\in[0,T]} \left(t^{\frac{p-2}{2(p-1)}} - \lfloor t \rfloor_h^{\frac{p-2}{2(p-1)}} \right)^{p-1} \int_0^T \left\| f\left(\nabla u^{(N)}(\lfloor j \rfloor_h)\right)\right\|_{L^2}^p \mathrm{d}s \right]^{\frac{1}{p}}\\
&\leqslant C_p h^{\frac{1}{2}-\frac{1}{p}} \int_0^T \mathbb{E}\left[\left\| f\left(\nabla u^{(N)}(\lfloor j \rfloor_h)\right)\right\|_{L^2}^p \right]^{\frac{1}{p}} \mathrm{d}s\\
&\leqslant C_{p,T} h^{\frac{1}{2}-\frac{1}{p}} N^{-\frac{\alpha}{p+2}}.
\end{aligned}$$

Finally, applying the triangle inequality and by combining (5.8), (5.9), (5.10) and (5.11), we conclude

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|u_h^{(N)}(t)-u_h^{(N)}(\lfloor t\rfloor_h)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}}\\&\leqslant C_{p,T}\left(h^{\frac{\eta}{4}}\left\|u_0\right\|_{\mathcal{H}^{1+\eta}}+N^{2+\alpha}h^{1-\frac{1}{p}-}+h^{\frac{1}{2}-\frac{\gamma}{4}}N^{-\frac{\alpha}{p+2}}+h^{\frac{1}{2}-\frac{1}{p}}N^{-\frac{\alpha}{p+2}}\right).\end{aligned}$$

This establishes the claim. □

Lemma 5.18. *For $p>1$, depending only on γ, such that $\frac{p}{p-1}\in\left(1,\frac{1}{1-\frac{\gamma}{4}}\right)$ we have*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|I_4(t)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}}\\&\leqslant C_p\sqrt{T}\left(h^{\frac{\eta}{4}}\left\|u_0\right\|_{\mathcal{H}^{1+\eta}}+N^{2+\alpha}h^{1-\frac{1}{p}-}+N^{-\frac{\alpha}{p+2}}\left(h^{\frac{1}{2}-\frac{\gamma}{4}}+h^{\frac{1}{2}-\frac{1}{p}}\right)\right).\end{aligned}$$

Proof. By Lemma 5.17 it holds that

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|I_4(t)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}}\\&\leqslant\int_0^T(T-s)^{-\frac{1}{2}}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|u_h^{(N)}(t)-u_h^{(N)}(\lfloor t\rfloor_h)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}}\mathrm{d}s\\&\leqslant C_p\sqrt{T}\left(h^{\frac{\eta}{4}}\left\|u_0\right\|_{\mathcal{H}^{1+\eta}}+N^{2+\alpha}h^{1-\frac{1}{p}-}+N^{-\frac{\alpha}{p+2}}\left(h^{\frac{1}{2}-\frac{\gamma}{4}}+h^{\frac{1}{2}-\frac{1}{p}}\right)\right).\end{aligned}$$

This confirms the claim. □

5.4. Proof of the Main Result

Finally, we can realise the proof of the main result of this chapter Theorem 5.4.

Proof of Theorem 5.4. We define the auxiliary functions for each $s \in [0, T]$

$$h(s) := \mathbb{E}\left[\sup_{t\in[0,s]} \left\|e_h^{(N)}(t)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}}$$

and

$$\begin{aligned} a := C &\left(\|u_0\|_{\mathcal{H}^{1+\eta}} N^{-\eta} + N^{-2+\gamma-\frac{\alpha}{p+2}}\right) \\ &+ C\left(h^{\frac{\eta}{4}}\|u_0\|_{\mathcal{H}^{1+\eta}} + N^{2+\alpha} h^{1-\frac{1}{p}-} + N^{-\frac{\alpha}{p+2}}\left(h^{\frac{1}{2}-\frac{\gamma}{4}} + h^{\frac{1}{2}-\frac{1}{p}}\right)\right). \end{aligned}$$

By combining the results of Lemma 5.15, Lemma 5.16, and Lemma 5.18, we obtain

$$\begin{aligned} h(T) :=& \mathbb{E}\left[\sup_{t\in[0,T]} \left\|e_h^{(N)}(t)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} \\ \leqslant& \mathbb{E}\left[\sup_{t\in[0,T]} \|I_1(t)\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} + \mathbb{E}\left[\sup_{t\in[0,T]} \|I_2(t)\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} + \mathbb{E}\left[\sup_{t\in[0,T]} \|I_4(t)\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} \\ &+ \mathbb{E}\left[\sup_{t\in[0,T]} \|I_3(t)\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} \\ \leqslant& C\left(\|u_0\|_{\mathcal{H}^{1+\eta}} N^{-\eta} + N^{-2+\gamma-\frac{\alpha}{p+2}}\right) \\ &+ C\left(h^{\frac{\eta}{4}}\|u_0\|_{\mathcal{H}^{1+\eta}} + N^{2+\alpha} h^{1-\frac{1}{p}-} + N^{-\frac{\alpha}{p+2}}\left(h^{\frac{1}{2}-\frac{\gamma}{4}} + h^{\frac{1}{2}-\frac{1}{p}}\right)\right) \\ &+ \frac{1}{\sqrt{2e}}\int_0^T (T-s)^{-\frac{1}{2}} \mathbb{E}\left[\sup_{\tau\in[0,s]} \|e_h^{(N)}(\tau)\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} \mathrm{d}s \\ =& a + \frac{1}{\sqrt{2e}}\int_0^T (T-s)^{-\frac{1}{2}} h(s)\,\mathrm{d}s. \end{aligned}$$

Thus, by applying Lemma A.19, we derive

$$
\begin{aligned}
h(T) &= \mathbb{E}\left[\sup_{t\in[0,T]}\left\|e_h^{(N)}(t)\right\|_{\mathcal{H}^1}^p\right]^{\frac{1}{p}} \\
&\leqslant C\left(\|u_0\|_{\mathcal{H}^{1+\eta}}N^{-\eta}+N^{-2+\gamma-\frac{\alpha}{p+2}}\right) \\
&\quad + C\left(h^{\frac{\eta}{4}}\|u_0\|_{\mathcal{H}^{1+\eta}}+N^{2+\alpha}h^{1-\frac{1}{p}-}+N^{-\frac{\alpha}{p+2}}\left(h^{\frac{1}{2}-\frac{\gamma}{4}}+h^{\frac{1}{2}-\frac{1}{p}}\right)\right) \\
&\quad + \frac{\pi}{2e}\int_0^T E'_{\frac{1}{2}}\left(\frac{\pi}{2e}(T-s)\right)a\,\mathrm{d}s \\
&\leqslant C\left(\|u_0\|_{\mathcal{H}^{1+\eta}}N^{-\eta}+N^{-2+\gamma-\frac{\alpha}{p+2}}\right) \\
&\quad + C\left(h^{\frac{\eta}{4}}\|u_0\|_{\mathcal{H}^{1+\eta}}+N^{2+\alpha}h^{1-\frac{1}{p}-}+N^{-\frac{\alpha}{p+2}}\left(h^{\frac{1}{2}-\frac{\gamma}{4}}+h^{\frac{1}{2}-\frac{1}{p}}\right)\right) \\
&\quad + \frac{\pi}{2e}\int_0^T\left[e^{\frac{\pi}{2e}(T-s)}\left(1+\sqrt{s}\right)+\frac{1}{\sqrt{\pi s}}\right]a\,\mathrm{d}s \\
&\leqslant Ca \\
&= C\left(\|u_0\|_{\mathcal{H}^{1+\eta}}N^{-\eta}+N^{-2+\gamma-\frac{\alpha}{p+2}}\right) \\
&\quad + C\left(h^{\frac{\eta}{4}}\|u_0\|_{\mathcal{H}^{1+\eta}}+N^{2+\alpha}h^{1-\frac{1}{p}-}+N^{-\frac{\alpha}{p+2}}\left(h^{\frac{1}{2}-\frac{\gamma}{4}}+h^{\frac{1}{2}-\frac{1}{p}}\right)\right) \\
&\leqslant C\left(\|u_0\|_{\mathcal{H}^{1+\eta}}\left(h^{\frac{\eta}{4}}+N^{-\eta}\right)+N^{2+\alpha}h^{1-\frac{1}{p}-}+N^{-2+\gamma-\frac{\alpha}{p+2}}\right) \\
&\quad + C\left(N^{-\frac{\alpha}{p+2}}\left(h^{\frac{1}{2}-\frac{\gamma}{4}}+h^{\frac{1}{2}-\frac{1}{p}}\right)\right) \\
&\leqslant C\left(\|u_0\|_{\mathcal{H}^{1+\eta}}\left(h^{\frac{\eta}{4}}+N^{-\eta}\right)+N^{2+\alpha}h^{1-\frac{1}{p}-}+N^{-2}+h^{\frac{1}{2}-\frac{\gamma}{4}}N^{-\frac{\alpha}{p+2}}\right).
\end{aligned}
$$

This establishes the desired result. □

5.5. Numerical Simulations

The numerical simulations presented in this section were created by David Buchberger as part of our joint work [13]:
Simulations are conducted on the domain $[0,1]^2$ up to time $T = 10$ using a step size of $h = 0.001$ and parameters $\sigma = 0.11$ and $\psi = 0.02$. The initial condition u_0 is set to the constant value 0. The values presented in the tables below represent averages over 100 samples at time $T = 10$. The exponential Euler scheme in Fourier space with regularity weights is formulated as follows:

$$\hat{v}_{j+1}^{(N)} = e^{hA}\hat{v}_j^{(N)} + \frac{1-e^{hA}}{A}\mathfrak{F}(\hat{v}_j^{(N)}) + \sigma\frac{1-e^{2hA}}{2A}(-\Delta)^{\frac{\alpha}{2}}N^2\,\mathrm{Re}(\mathcal{N}(0,1)+i\mathcal{N}(0,1)),$$

where A contains the eigenvalues corresponding to the eigenfunctions e_k of $-\Delta^2$ with $|k| \leqslant N$. Similarly, the exponential Euler scheme for the stochastic convolution is expressed as follows:

$$\hat{z}_{j+1}^{(N)} = e^{hA}\hat{z}_j^{(N)} + \sigma\frac{1-e^{2hA}}{2A}(-\Delta)^{\frac{\alpha}{2}}N^2\,\mathrm{Re}(\mathcal{N}(0,1)+i\mathcal{N}(0,1)).$$

The error is defined as $e_h^{(N)} := v^{(N)} - z^{(N)}$.

5.5.1. Numerical Result Regarding Rougher Noise

For $\alpha = 0.4$, we observe that the inf-norm of the gradient of $v^{(N)}$ does not grow with more Fourier modes.

Number of fourier modes	$\lVert v^{(N)}\rVert_\infty$	$\lVert \nabla v^{(N)}\rVert_\infty$	$\lVert v^{(N)}\rVert_{C^1}$
$(2^2)^2 = 16$	0.130713	0.614351	0.745064
$(2^3)^2 = 64$	0.122107	0.670314	0.792421
$(2^4)^2 = 256$	0.133159	0.746035	0.879194
$(2^5)^2 = 1024$	0.139909	0.760671	0.900580
$(2^6)^2 = 4096$	0.141632	0.761279	0.902910
$(2^7)^2 = 16384$	0.143825	0.760267	0.904092
$(2^8)^2 = 65536$	0.144930	0.751583	0.896513
$(2^9)^2 = 262144$	0.146088	0.738105	0.884193
$(2^{10})^2 = 1048576$	0.147054	0.721802	0.868856

Table 5.1.: Norms of the Exponential Euler Scheme in Fourier Space with Respect to the Number of Fourier Modes for $\alpha = 0.4$.

Here, convergence of $v^{(N)}$ to the stochastic convolution Z is not expected due to the persistent nonlinearity. The stochastic convolution Z displays only noise, which is not visible in this case. The figure below demonstrates the characteristic hill-growth observed in the simulations.

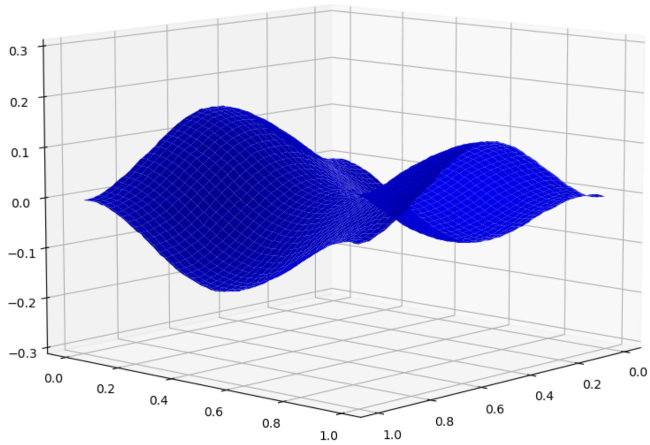

Figure 5.1.: Simulation of $v^{(N)}$ at time $T = 10$ for $\alpha = 0.4$ and $N = 2^7$. (Figure created by David Buchberger using Python.)

The table below demonstrates that in the numerical simulations, the error $\nabla e_h^{(N)}$ stabilises at approximately 0.45 and thus does not converge to zero, as anticipated.

Number of fourier modes	$\left\|e_h^{(N)}\right\|_\infty$	$\left\|\nabla e_h^{(N)}\right\|_\infty$	$\left\|e_h^{(N)}\right\|_{C^1}$
$(2^2)^2 = 16$	0.102194	0.436299	0.538493
$(2^3)^2 = 64$	0.108146	0.464480	0.572626
$(2^4)^2 = 256$	0.124719	0.474413	0.599132
$(2^5)^2 = 1024$	0.135589	0.459246	0.594835
$(2^6)^2 = 4096$	0.139585	0.451026	0.590611
$(2^7)^2 = 16384$	0.142452	0.452216	0.594668
$(2^8)^2 = 65536$	0.144310	0.453707	0.598017
$(2^9)^2 = 262144$	0.145731	0.456384	0.602115
$(2^{10})^2 = 1048576$	0.146847	0.458761	0.605608

Table 5.2.: Norms of the error function with respect to the number of Fourier modes with $\alpha = 0.4$.

Table 5.2 summarizes the numerical values of the error norms for $\alpha = 0.4$ and increasing numbers of Fourier modes. The corresponding convergence behaviour is illustrated in Figure 5.2.

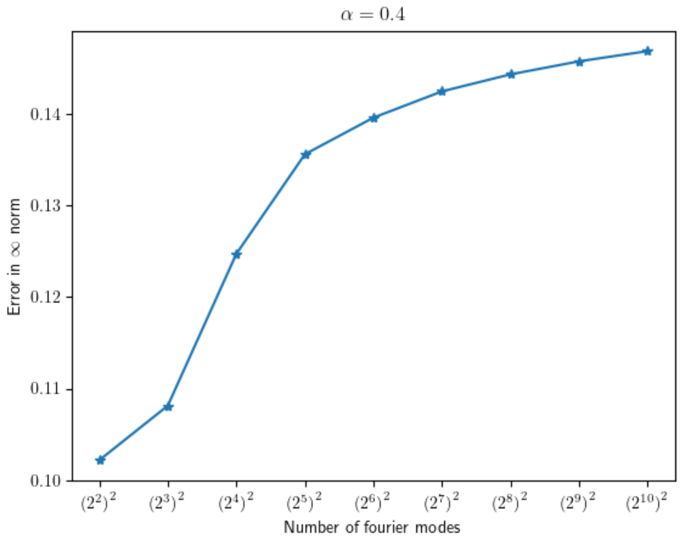

Figure 5.2.: Simulation of $\left\| e_h^{(N)} \right\|_\infty$ for $\alpha = 0.4$ and an increasing number of fourier modes N. (Figure created by David Buchberger using Python.)

5.5.2. Numerical Result for Higher Roughness Parameter

Increasing α is expected to cause a significant increase in the gradient of $v^{(N)}$. Simulations were conducted for $\alpha = 0.7$ and $\alpha = 0.9$. In both cases, the error $e_h^{(N)}$ converged to zero. The following presents the data generated for $v^{(N)}$ with $\alpha = 0.7$:

Number of fourier modes	$\left\|v^{(N)}\right\|_\infty$	$\left\|\nabla v^{(N)}\right\|_\infty$	$\left\|v^{(N)}\right\|_{C^1}$
$(2^2)^2 = 16$	0.256342	1.485884	1.742225
$(2^3)^2 = 64$	0.194115	2.131889	2.326005
$(2^4)^2 = 256$	0.147894	3.378832	3.526726
$(2^5)^2 = 1024$	0.114401	5.150998	5.265400
$(2^6)^2 = 4096$	0.083890	7.595636	7.679525
$(2^7)^2 = 16384$	0.060421	11.047536	11.107957
$(2^8)^2 = 65536$	0.043793	15.591852	15.635645
$(2^9)^2 = 262144$	0.031083	22.357294	22.388378
$(2^{10})^2 = 1048576$	0.021860	31.011626	31.033486

Table 5.3.: Norms of the exponential Euler scheme in Fourier space with respect to the number of Fourier modes for $\alpha = 0.7$.

The norm of $\nabla v^{(N)}$ increases gradually as the number of Fourier modes increases. Consequently, the nonlinearity diminishes. The graph below no longer exhibits hill-like growth and instead only displays noise.

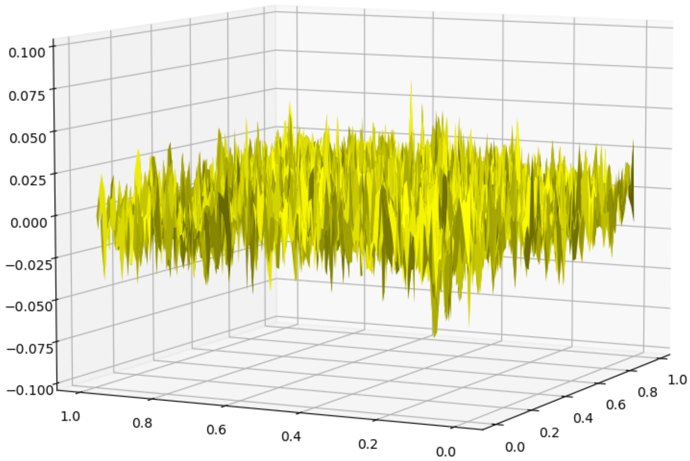

Figure 5.3.: Simulation of $v^{(N)}$ at time $T = 10$ for $\alpha = 0.7$ and $N = 2^7$. (Figure created by David Buchberger using Python.)

As we might expect from Figure 5.3, we also see convergence of $e_h^{(N)}$ to 0 in the numerical data:

Number of fourier modes	$\left\|e_h^{(N)}\right\|_\infty$	$\left\|\nabla e_h^{(N)}\right\|_\infty$	$\left\|e_h^{(N)}\right\|_{C^1}$
$(2^2)^2 = 16$	0.065330	0.320001	0.385331
$(2^3)^2 = 64$	0.025951	0.149824	0.175775
$(2^4)^2 = 256$	0.009055	0.064278	0.073333
$(2^5)^2 = 1024$	0.002977	0.025431	0.028408
$(2^6)^2 = 4096$	0.001025	0.009572	0.010598
$(2^7)^2 = 16384$	0.000367	0.003833	0.004200
$(2^8)^2 = 65536$	0.000146	0.001659	0.001805
$(2^9)^2 = 262144$	0.000059	0.000690	0.000749
$(2^{10})^2 = 1048576$	0.000024	0.000286	0.000310

Table 5.4.: Norms of the error function with respect to the number of Fourier modes with $\alpha = 0.7$.

Table 5.4 summarizes the numerical values of the error norms for $\alpha = 0.7$ and increasing numbers of Fourier modes. The corresponding convergence behaviour is illustrated in Figure 5.4.

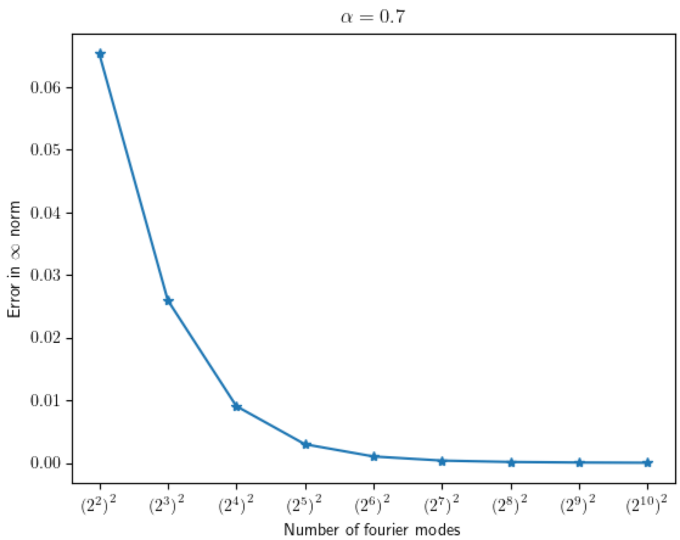

Figure 5.4.: Simulation of $\left\|e_h^{(N)}\right\|_\infty$ for $\alpha = 0.7$ and an increasing number of Fourier modes N. (Figure created by David Buchberger using Python.)

With $\alpha = 0.9$, we observe the same effects as for $\alpha = 0.7$, but the process occurs significantly faster, as shown by the values of $\nabla v^{(N)}$.

Number of fourier modes	$\left\|v^{(N)}\right\|_\infty$	$\left\|\nabla v^{(N)}\right\|_\infty$	$\left\|v^{(N)}\right\|_{C^1}$
$(2^2)^2 = 16$	0.527842	3.426971	3.954813
$(2^3)^2 = 64$	0.562849	6.971630	7.534479
$(2^4)^2 = 256$	0.551854	14.496256	15.048111
$(2^5)^2 = 1024$	0.548000	30.283880	30.831880
$(2^6)^2 = 4096$	0.537246	59.610059	60.147305
$(2^7)^2 = 16384$	0.507359	113.373272	113.880632
$(2^8)^2 = 65536$	0.476628	212.051834	212.528462
$(2^9)^2 = 262144$	0.441572	398.740028	399.181600
$(2^{10})^2 = 1048576$	0.405952	739.753947	740.159899

Table 5.5.: Norms of the exponential Euler scheme in Fourier space with respect to the number of Fourier modes for $\alpha = 0.9$.

The results for errors with $\alpha = 0.9$ are as follows:

Number of fourier modes	$\left\Vert e_h^{(N)} \right\Vert_\infty$	$\left\Vert \nabla e_h^{(N)} \right\Vert_\infty$	$\left\Vert e_h^{(N)} \right\Vert_{C^1}$
$(2^2)^2 = 16$	0.036118	0.179939	0.216057
$(2^3)^2 = 64$	0.008669	0.060559	0.069228
$(2^4)^2 = 256$	0.001975	0.020830	0.022805
$(2^5)^2 = 1024$	0.000615	0.007402	0.008017
$(2^6)^2 = 4096$	0.000197	0.002494	0.002691
$(2^7)^2 = 16384$	0.000065	0.000831	0.000896
$(2^8)^2 = 65536$	0.000021	0.000273	0.000294
$(2^9)^2 = 262144$	0.000006	0.000084	0.000090
$(2^{10})^2 = 1048576$	0.000002	0.000026	0.000028

Table 5.6.: Norms of the error function with respect to the number of Fourier modes with $\alpha = 0.9$.

6. Numerical approximation of nonlinear fourth-order SPDE with additive space-time white noise

This chapter is based on collaborative work with Dirk Blömker and Chengcheng Ling, as published in [15]. Although subsequent work has led to improvements, this chapter is concerned solely with the findings presented in the version of the paper posted on arXiv on January 30, 2025:

We consider the strong numerical approximation for a nonlinear stochastic partial differential equation (SPDE) of fourth order driven by space-time white noise on the two-dimensional torus. We consider its full discretisation with a splitting scheme: a spectral Galerkin scheme in space and Euler scheme in time. We show the convergence with almost spatial rate $\frac{1}{2}$ and almost temporal rate 1 obtained mainly via *stochastic sewing* technique.

6.1. Introduction

The Cahn–Hilliard equation was first developed by the work of Cahn and Hilliard on phase separation in binary alloys, where they introduced a free energy functional that accounts for both bulk and interfacial contributions [24, 25]. In its deterministic form, this fourth-order partial differential equation models how a concentration field evolves to form distinct regions (phases) over time. The addition of random fluctuations or thermal noise gives rise to the stochastic Cahn–Hilliard equation, providing a more accurate portrayal of microscopic uncertainties present in real materials [4, 27, 30, 71].
As a very active field of research [25, 39], there has been a long list of work on numerical approximation of various types of SPDEs. [8, 10] consider stochastic Burgers and [6, 20, 23, 36, 65] study stochastic Allen–Cahn equations, moreover [50, 52, 53, 64] provide broader framework. For further interests on this topic, we refer to the references within the aforementioned works.

Stochastic Cahn–Hilliard equations extend the classical Cahn–Hilliard models with noise, requiring careful numerical methods. Early simulations employed an unconditionally gradient-stable scheme [34], focusing on stable conservation of mass [30]. Gradually finite-element, spectral, and hybrid strategies address various geometries and stability needs [54, 69], driving ongoing improvements in efficiency and consistency.

The focus of this chapter is the numerical approximation of a nonlinear fourth-order stochastic partial differential equation (SPDE) driven by space-time white noise on the two-dimensional torus, defined as:

$$\partial_t u = -\delta \Delta^2 u - \mathbf{G}(u) + \sigma \partial_t W \tag{6.1}$$

where $\mathbf{G} : \mathbb{R} \to \mathbb{R}$ is the nonlinear term satisfying $\|\mathbf{G}\|_\infty, \|\partial \mathbf{G}\|_\infty < \infty$ and the space-time white noise is denoted by $\xi := \partial_t W$ representing random fluctuations, scaled by a constant diffusion coefficient $\sigma > 0$, which governs the intensity of the stochastic effects.

We further propose the property of u to be in a moving frame (see (A.3)), i.e. $\int_{\mathbb{T}^2} u(t,x)\,\mathrm{d}x = 0$ for any $t \geqslant 0$. The scheme we consider is the full discretisation of the equation which was first addressed by [39]. More precisely, (see also (6.5)) a spectral Galerkin scheme in space and Euler scheme in time based on sampling rectangular increments of W on a grid with meshsize n^{-1} in time and N^{-1} in space.

Inspired by the works [23, 31] and our forthcoming paper [16], we finally obtain a convergence rate in L^p of order almost $N^{-\frac{1}{2}} + n^{-1}$. Evidently, we also overcome the order barrier $\frac{1}{4}$ with respect to the temporal step size which has been addressed in [51, 31]. The crucial idea within our analysis is to combine the *regularity estimates on the semigroup generated by* $-\Delta^2$ (see Lemma 6.6 and Lemma 6.7) together with *Stochastic sewing lemma* [58, 31] (see Lemma 6.4) so that we can tune the spatial and temporal regularity of the mild solution and noise, which in the end yields the desired quantitative rate. This idea was originally invented to study the numerical approximation for singular SDEs [22, 59, 35]. In this instance, we are able to apply it to stochastic Cahn–Hilliard equations (6.1).

Remark 6.1. *In our forthcoming work [16], we apply the stochastic sewing technique to the surface growth model* (1.1)*, with the nonlinear term*

$$\mathfrak{F}(u) = \nabla \cdot f(\nabla u) = \nabla \cdot \frac{\nabla u}{1 + |\nabla u|^2},$$

which, in contrast to the setting of (6.1)*, leads to additional ill-posedness problems. Our aim is to establish a convergence rate of order* $n^{-1/2}$ *by choosing the spatial resolution according to* $N \approx n^\alpha$ *with* $\alpha \in (0,1)$*.*

In contrast to Equation (6.1), Equation (1.1) exhibits the vanishing nonlinearity effect *(see Chapter 3 for the groundwork, Chapter 4 for a linearisation approach, and Lemma 5.8, and Remark 5.11 for the explicit convergences rates), which avoids such regularity issues. Moreover, Equation (1.1) features an energy barrier, i.e. the* Schwoebel barrier *[72], which prevents adatoms from crossing step edges. This effect is encoded in the structure of $\mathfrak{F}$, which tends to zero as $|\nabla u| \to \infty$.*
For the analysis of (6.1), we employ the stochastic sewing lemma *(cf. Lemma 6.4), which requires a Grönwall-type estimate in either L^2 or $\mathcal{H}^1$ (cf. Lemma 6.5). This approach relies crucially on the Lipschitz continuity of the nonlinearity $\mathbf{G}$. Although $f : \mathbb{R}^2 \to \mathbb{R}^2$ is Lipschitz continuous, this property does not extend to the induced maps*

$$\mathfrak{F} : \mathcal{H}^{\alpha+2}(\mathbb{T}^2) \to \mathcal{H}^{\alpha}(\mathbb{T}^2), \qquad \text{for } \alpha < 0,$$

since Lipschitz continuity is generally not preserved when passing to distribution spaces. This constitutes the main analytical difficulty in extending the convergence analysis to this class of nonlinearities.

6.2. Preliminary and Main Results

Note that we use the complex setting $L^2(\mathbb{T}^2, \mathbb{C})$, as introduced in Appendix A.4. Here we introduce the solution and its corresponding approximation scheme that we consider. Let Z be the stochastic convolution (see (2.1)) and $P_N Z$ be the regularised stochastic convolution regularised by the orthogonal projection P_N (see (2.3),(5.3)).

Definition 6.2. *For $t \in [0, T]$, for $k = 0, 1, \ldots, n$, $h = \frac{T}{n}$, $t_k := kh$, we define the mild solution v of (6.1), its spectral Galerkin approximation u^N and its Euler-spectral Galerkin approximation $u^{N,n}$ as follows*

$$v(t) := e^{tA} u_0 + \int_0^t e^{(t-s)A} \mathbf{G}\left(v(s)\right) \mathrm{d}s + Z(t), \tag{6.2}$$

$$u^N(t) := e^{tA} P_N u_0 + \int_0^t e^{(t-s)A} P_N \mathbf{G}\left(u^N(s)\right) \mathrm{d}s + P_N Z(t), \tag{6.3}$$

$$u^{N,n}(t_{k+1}) := e^{hA} P_N u^{N,n}(t_k) + e^{tA} P_N \mathbf{G}\left(u^{N,n}(t_k)\right) + P_N Z(t_{k+1}) - e^{hA} P_N Z(t_k). \tag{6.4}$$

Note that (6.4) is equivalent to

$$u^{N,n}(t) = e^{tA} P_N u_0 + \int_0^t e^{(t-s)A} P_N \mathbf{G}\left(u^{N,n}(k_n(s))\right) \mathrm{d}s + P_N Z(t), \quad t \geqslant 0, \tag{6.5}$$

where $k_n(s) := \frac{\lfloor ns \rfloor}{n}$, for $n \in \mathbb{N}$.

Our main result can be stated as follows.

Theorem 6.3. *Suppose $u_0 \in C^{\frac{1}{2}}$ and $\|\mathbf{G}\|_\infty, \|\partial\mathbf{G}\|_\infty < \infty$. For the solution $u^{N,n}$ to (6.4) and the solution v to (6.2) we have for some sufficiently small $\varepsilon > 0$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\|u^{N,n}(t) - v(t)\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \leqslant C(N^{-\frac{1}{2}+\varepsilon} + n^{-1+\varepsilon}) \tag{6.6}$$

where C depends on T, p, ε.

6.3. Tools and Auxiliary Estimates

We denote

$$[S,T]_\leqslant := \{(s,t)|S \leqslant s < t \leqslant T\}$$
$$[S,T]^*_\leqslant := \{(s,t)|S \leqslant s < t \leqslant T, t-s \leqslant T-t\}.$$

For a function A of one variable and $s \leqslant t$, we write $A_{s,t} = A_t - A_s$ and for functions A of two variables and $s \leqslant u \leqslant t$, we denote $\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}$. Furthermore, denote by $\mathbb{E}_s$ the conditional expectation with respect to $\mathcal{F}_s$.

Lemma 6.4 (Stochastic sewing lemma, [31, Lemma 3.2])**.** *Fix $p \geqslant 2$ and $0 \leqslant S < T \leqslant 1$. Let $A : [S,T]_\leqslant \to L^p(\Omega)$ be such that $A_{s,t}$ is $\mathcal{F}_t$-measurable for all $(s,t) \in [S,T]_\leqslant$. Suppose that there exist $\varepsilon_1, \varepsilon_2 > 0$, $\delta_1, \delta_2 \geqslant 0$ and $C_1, C_2 < \infty$ satisfying $\frac{1}{2} + \varepsilon_1 - \delta_1 > 0$, $1 + \varepsilon_2 - \delta_2 > 0$ and such that for all $(s,t) \in [S,T]^*_\leqslant$, $u \in [s,t]$ the following bounds hold:*

$$\|A_{s,t}\|_{L^p(\Omega)} \leqslant C_1|T-t|^{-\delta_1}|t-s|^{\frac{1}{2}+\varepsilon_1}, \tag{6.7}$$
$$\|\mathbb{E}_s\delta A_{s,u,t}\|_{L^p(\Omega)} \leqslant C_2|T-t|^{-\delta_2}|t-s|^{1+\varepsilon_2}. \tag{6.8}$$

Then there exists a unique $(\mathcal{F}_t)_{t\in[S,T]}$-adapted process $\mathcal{A} : [S,T] \to L^p(\Omega)$ such that $\mathcal{A}_S = 0$ and that there exist K_1, K_2 such that for all $(s,t) \in [S,T]^*_\leqslant$ one has

$$\|\mathcal{A}_{s,t} - A_{s,t}\|_{L^p(\Omega)} \leqslant K_1|T-t|^{-\delta_1}|t-s|^{\frac{1}{2}+\varepsilon_1} + K_2|T-t|^{-\delta_2}|t-s|^{1+\varepsilon_2}, \tag{6.9}$$
$$\|\mathbb{E}_s(\mathcal{A}_{s,t} - A_{s,t})\|_{L^p(\Omega)} \leqslant K_2|T-t|^{-\delta_2}|t-s|^{1+\varepsilon_2}. \tag{6.10}$$

Furthermore, there exists a constant C depending only on $p, \varepsilon_1, \varepsilon_2$, such that the above bounds hold with $K_1 = CC_1, K_2 = CC_2$. Finally, there exists a constant $\tilde{C}$ depending only on $p, \varepsilon_1, \varepsilon_2, \delta_1, \delta_2$, such that for all $(s,t) \in [S,T]_\leqslant$ one has

$$\|\mathcal{A}_{s,t}\|_{L^p(\Omega)} \leqslant \tilde{C}(C_1|t-s|^{\frac{1}{2}+\varepsilon_1-\delta_1} + C_2|t-s|^{\frac{1}{2}+\varepsilon_2-\delta_2}). \tag{6.11}$$

We state a version of Grönwall lemma that we use quite often in the later analysis.

Lemma 6.5 (Grönwall lemma, [31, Proposition 3.4])**.** *Let V be a Banach space. Let $X, Y, Z \in L^p(\Omega, C^0([0,T], V))$ for $p \geqslant 1$. Assume that there exists a Lipschitz continuous function F on V with Lipschitz constant L_1, a family $(\mathcal{S}(s,t))_{0\leqslant s\leqslant t\leqslant T}$ of uniformly bounded linear operators on V with uniform bound L_2 and such that the mapping $(s,t) \mapsto \mathcal{S}(s,t)v$ is measurable for any $v \in V$, and a measurable mapping $\tau : [0,T] \to [0,T]$ such that $\tau(s) \leqslant s$ and that the following equality holds for all $0 \leqslant t \leqslant T$:*

$$X_t - Y_t = Z_t + \int_0^t \mathcal{S}(s,t)(F(X_{\tau(s)}) - F(Y_{\tau(s)}))\,\mathrm{d}s. \tag{6.12}$$

Then there exists a constant $C = C(p, L_1, L_2, T)$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t - Y_t\|^p\right] \leqslant C\mathbb{E}\left[\sup_{t\in[0,T]} \|Z_t\|^p\right]. \tag{6.13}$$

Lemma 6.6. *For the identity matrix $\mathbb{I}_2$ in $\mathbb{R}^{2\times 2}$ and for any $\alpha, \beta \in \mathbb{R}$ with $\alpha > \beta$ we have*

$$\left\|e^{tA} f\right\|_{C^\alpha} \lesssim t^{-\frac{\alpha-\beta}{4}} \|f\|_{C^\beta}, \quad \left\|(\mathbb{I}_2 - e^{tA})f\right\|_{C^\alpha} \lesssim t^{-\frac{\alpha-\beta}{4}} \|f\|_{C^\beta}. \tag{6.14}$$

Proof. In [38, Lemma A.6, Lemma A.7] the author prove the case when the semigroup is generated by the negative Laplacian $-\Delta$. For $-\Delta^2$ we obtain (6.14) via the same proof except changing the ratial from $t^{-\frac{\alpha-\beta}{2}}$ of [38, Lemma A.6, Lemma A.7] to $t^{-\frac{\alpha-\beta}{4}}$ as in (6.14). Therefore, it is clear enough to omit the duplicate of the proof of [38, Lemma A.6, Lemma A.7] here. □

Lemma 6.7. *For $p \in [1,\infty)$, $\lambda \in (0,2)$ and $\varepsilon \in (0,1)$, there exist constants $C = C(\varepsilon, \lambda, T, p)$ and $\tilde{C} = C(\varepsilon, \lambda, T, p)$ such that for any $0 \leqslant s \leqslant t \leqslant T$ the following holds:*

$$\mathbb{E}\left[\|Z(t) - Z(s)\|^p_{C^{1-\lambda-\varepsilon}}\right] \leqslant C|t-s|^{\frac{\lambda p}{4}}, \tag{6.15}$$

$$\mathbb{E}\left[\|Z\|_{C^{\frac{\lambda}{4}}\left([0,T],C^{1-\lambda-\varepsilon}\right)}\right] \leqslant C, \tag{6.16}$$

$$\mathbb{E}\left[\left\|\sup_{r\in[0,T]} (e^{(t+s)A} - e^{tA})Z(r)\right\|^p_{C^{-1+\varepsilon}}\right]^{\frac{1}{p}} \leqslant \tilde{C} t^{-\frac{\lambda}{4}-\frac{\varepsilon}{2}} s^{\frac{2+\lambda}{4}}. \tag{6.17}$$

Proof. Notice that (6.16) can be directly obtained by (6.15) and Kolmogorov's continuity theorem. If (6.16) holds, after applying both bounds of (6.14) to (6.16) we obtain (6.17). Therefore, we only need to show (6.15).

We first assume p to be sufficiently large. Considering $C^\alpha \simeq \mathcal{B}^\alpha_{\infty,\infty}$ and by the definition of $\mathcal{B}^\alpha_{\infty,\infty}$, for $\alpha < 1$, in order to verify (6.15) it is sufficient to show

$$\mathbb{E}\left[|\Delta_j Z(t,x) - \Delta_j Z(s,x)|^p\right]^{\frac{1}{p}} \lesssim 2^{-(1-\lambda)}|t-s|^{\frac{\lambda}{4}}. \tag{6.18}$$

Indeed, by (6.18), raising p-th power and multiplying by $2^{jp(1-\lambda-\varepsilon)}$ leads to

$$\mathbb{E}\left[\|Z(t,x) - Z(s,x)\|^p_{\mathcal{B}^{1-\lambda-\varepsilon}_{p,p}}\right] \lesssim |t-s|^{\frac{p\lambda}{4}}.$$

Then via the Sobolev embedding $\mathcal{B}^{1-\lambda-\varepsilon}_{p,p} \hookrightarrow C^{1-\lambda-\varepsilon}$, we obtain (6.15).
In order to show (6.18), we use Corollary 2.5 and Itô's isometry to obtain

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_j Z(t) - \Delta_j Z(s)|^p\right]^{\frac{2}{p}} \\
&\lesssim \mathbb{E}\left[|\Delta_j Z(t) - \Delta_j Z(s)|^2\right] \\
&\lesssim \mathbb{E}\left[\left|\int_s^t \int_{\mathbb{T}^2} \Delta_j p_{t-r}(x-y)\,\mathrm{d}W(r,y)\right|^2\right] \\
&\quad + \mathbb{E}\left[\left|\int_0^s \int_{\mathbb{T}^2} \Delta_j (p_{t-r} - p_{s-r})(x-y)\,\mathrm{d}W(r,y)\right|^2\right] \\
&= \int_s^t \|\Delta_j p_{t-r}\|^2_{L^2(\mathbb{T}^2)}\,\mathrm{d}r + \int_0^s \|\Delta_j (p_{t-r} - p_{s-r})\|^2_{L^2(\mathbb{T}^2)}\,\mathrm{d}r.
\end{aligned}$$

From Parseval's identity and (A.10) we continue the above calculation and obtain

$$\begin{aligned}
&(\mathbb{E}|\Delta_j Z(t) - \Delta_j Z(s)|^p)^{\frac{2}{p}} \\
&\lesssim \int_s^t \sum_{k\in\mathcal{Z}} \phi_j^2(k) e^{-2(t-r)\mu_k^2}\,\mathrm{d}r + \int_0^s \sum_{k\in\mathcal{Z}} \phi_j^2(k) e^{-2(s-r)\mu_k^2}(1 - e^{-(t-s)\mu_k^2})^2\,\mathrm{d}r \\
&\lesssim \sum_{k\in\mathcal{Z}} \phi_j^2(k) \min(|t-s|, \mu_k^{-2}) + \sum_{k\in\mathcal{Z}} \phi_j^2(k) \min(|t-s|\mu_k^{-2}, \mu_k^{-2})
\end{aligned}$$

since $e^{-x} \leqslant \min(x^{-1}, 1)$ and $1 - e^{-x} \lesssim \min(x, x^{\frac{1}{2}}, 1)$, for any $x \geqslant 0$. Since we consider the 2-dimensional standard torus of length $L = 1$, we obtain from (A.2) that $\mu_k = 4\pi^2|k|^2$ holds for each $k \in \mathcal{Z}$, hence for any $\theta \in [0,1]$ we obtain

$$\begin{aligned}
(\mathbb{E}|\Delta_j Z(t) - \Delta_j Z(s)|^p)^{\frac{2}{p}} &\lesssim 2^{2j} \min(2^{-4j}, |t-s|) \\
&\lesssim 2^{2j} 2^{-4j(1-\theta)} |t-s|^\theta \\
&= |t-s|^\theta 2^{(4\theta-2)j}.
\end{aligned}$$

This implies

$$\mathbb{E}\left[|\Delta_j Z(t) - \Delta_j Z(s)|^p\right]^{\frac{1}{p}} \lesssim |t-s|^{\frac{\lambda}{4}} 2^{(\lambda-1)j}$$

by taking $\lambda = 2\theta \in (0,2)$. Therefore, we obtain (6.18) and the proof is complete. □

6.4. Proof of the Main Results

According to the different types' estimate desired in the proof of the main results, we organise the auxiliary lemmas in the following way and the proof of Theorem 6.3 is given at the end of this section. First, we denote

$$\hat{u}^N(t) := e^{tA}P_N u_0 + \int_0^t e^{(t-s)A}P_N \mathbf{G}\left(\hat{u}^N(s)\right)\,\mathrm{d}s + Z(t), \tag{6.19}$$

$$\hat{u}^{N,n}(t) := e^{tA}P_N u_0 + \int_0^t e^{(t-s)A}P_N \mathbf{G}\left(\hat{u}^{N,n}(k_n(s))\right)\,\mathrm{d}s + Z(t), \quad t \geqslant 0, \tag{6.20}$$

where we do not use a cut-off for the white noise. It reads

$$u^{N,n} - v = u^{N,n} - \hat{u}^{N,n} + \hat{u}^{N,n} - \hat{u}^N + \hat{u}^N - u^N + u^N - v =: I_1 + I_2 + I_3 + I_4. \tag{6.21}$$

The following result shows that all of the terms that are introduced above are well-defined.

Lemma 6.8. *Suppose $u_0 \in C^{\frac{1}{2}}$ and $\|\mathbf{G}\|_\infty, \|\partial\mathbf{G}\|_\infty < \infty$. Then there exist unique mild solution u^N to (6.3) and $\hat{u}^N$ to (6.19). Moreover, for $\varepsilon \in (0, \frac{1}{2})$ and $p \geqslant 1$, there exists a constant $C(p, T, \|\mathbf{G}\|_\infty, \|\partial\mathbf{G}\|_\infty < \infty)$ such that*

$$\begin{aligned}&\sup_{N\in\mathbb{N}} \mathbb{E}\left[\|P_N Z\|^p_{C^{\frac{\lambda}{4}}\left([0,T],C^{1-\lambda-\varepsilon}\right)}\right] + \sup_{N\in\mathbb{N}} \mathbb{E}\left[\left\|\hat{P}_N Z\right\|^p_{C^{\frac{\lambda}{4}}\left([0,T],C^{1-\lambda-\varepsilon}\right)}\right] \\ &\leqslant C\left(1 + \mathbb{E}\left[\|u_0\|^p_{C^{\frac{1}{2}}}\right]\right).\end{aligned}$$

Proof. For the existence and uniqueness we apply fixed point argument (see also [31, Proposition 4.1]). By the global Lipschitz bound on $\mathbf{G}$ and the fact that $\sup_N \left\|e^{tA}P_N u\right\|_\infty \leqslant \|u\|_\infty$ one finds a unique fixed point of the mild formulation of u^N to (6.3) and $\hat{u}^N$ to (6.19) in $C^0([0,T], L^\infty)$ for T chosen small enough. For any arbitrary time horizon $T > 0$, we can glue the solutions on subintervals together yielding a solution globally. Then plugging the solution back in the mild formulation and using the semigroup estimates for P^N instead of P, one obtains the claimed regularity bounds by a similar calculation Lemma 6.7. □

6.4.1. First Estimate

Lemma 6.9. *Suppose $u_0 \in C^{\frac{1}{2}}$ and $\|\mathbf{G}\|_\infty, \|\partial\mathbf{G}\|_\infty < \infty$. For $u^{N,n}$ from (6.5) and $\hat{u}^{N,n}$ from (6.20), we have for sufficiently small $\varepsilon > 0$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\|u^{N,n}(t) - \hat{u}^{N,n}(t)\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \lesssim N^{-\frac{1}{2}+\varepsilon}. \tag{6.22}$$

Proof. By (6.5) and (6.20) we obtain

$$u^{N,n}(t) - \hat{u}^{N,n}(t)$$
$$= [Z(t) - P_N Z(t)] + \int_0^t e^{(t-s)A} P_N \left[\mathbf{G}\left(u^{N,n}(k_n(s))\right) - \mathbf{G}\left(\hat{u}^{N,n}(k_n(s))\right)\right] \, \mathrm{d}s.$$

Thus, using the Lipschitz continuity of $\mathbf{G}$, we obtain from Lemma 6.5 the following bound:

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} \left\|u^{N,n}(t) - \hat{u}^{N,n}(t)\right\|^p_{L^2(\mathbb{T}^2)}\right] &\lesssim \mathbb{E}\left[\sup_{t\in[0,T]} \|Z(t) - P_N Z(t)\|^p_{L^2(\mathbb{T}^2)}\right] \\ &\lesssim \left(\sum_{k\in\mathcal{Z},|k|>N} \mu_k^{4\delta-2}\right)^{\frac{p}{2}} \\ &\lesssim \left(\int_N^\infty x^{4\delta-2} \, \mathrm{d}x\right)^{\frac{p}{2}} \\ &\lesssim N^{-\frac{p}{2}} N^{2\delta p}\end{aligned} \tag{6.23}$$

whereby $\delta \in \left(0, \frac{1}{4}\right)$ can be chosen arbitrarily small. This establishes the desired result (6.22). □

6.4.2. Second Estimate

Lemma 6.10. *Suppose $u_0 \in C^{\frac{1}{2}}$ and $\|\mathbf{G}\|_\infty, \|\partial\mathbf{G}\|_\infty < \infty$. For $\hat{u}^{N,n}$ from (6.20) and $\hat{u}^N$ from (6.19), for $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $p \geqslant 1$, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\|\hat{u}^N(t) - \hat{u}^{N,n}(t)\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \leqslant C n^{-1+\varepsilon} \tag{6.24}$$

where $C = C(T, p, \varepsilon, u_0)$.

Proof. It writes

$$\begin{aligned}\hat{u}^N(t) - \hat{u}^{N,n}(t) &= \int_0^t e^{(t-s)A} P_N \mathbf{G}\left(\hat{u}^N(s)\right) - e^{(t-s)A} P_N \mathbf{G}\left(\hat{u}^{N,n}(k_n(s))\right) \, \mathrm{d}s \\ &= \int_0^t e^{(t-s)A} P_N \left(\mathbf{G}(\hat{u}^N(s)) - \mathbf{G}\left(\hat{u}^N(k_n(s))\right)\right) \, \mathrm{d}s \\ &\quad + \int_0^t e^{(t-s)A} P_N \left(\mathbf{G}\left(\hat{u}^N(k_n(s))\right) - \mathbf{G}\left(\hat{u}^{N,n}(k_n(s))\right)\right) \, \mathrm{d}s \\ &=: I_{21}(t) + I_{22}(t).\end{aligned} \tag{6.25}$$

Our idea is to bound $\hat{u}^N(t) - \hat{u}^{N,n}(t)$ from above via Grönwall inequality, see Lemma 6.5, in which the second term I_{22} can be buckled. In order to obtain the desired estimate for I_{21}, we divide the whole proof into the following steps:

1. Reduction via Girsanov theorem:

We follow the idea of [31, Corollary 4.7]. Denote the probability measure $\mathbb{Q} := \rho\, \mathrm{d}\mathbb{P}$ where ρ is given by

$$\rho := \exp\left(-\int_0^T \int_{\mathbb{T}^2} \mathbf{G}\left(\hat{u}^N(s,y)\right) \mathrm{d}W(s,y) - \frac{1}{2}\int_0^T \int_{\mathbb{T}^2} \left|\mathbf{G}\left(\hat{u}^N(s,y)\right)\right|^2 \mathrm{d}y\, \mathrm{d}s \right).$$

From Girsanov's theorem (see [27, Theorem 10.14]) we know that $\mathbb{Q}$ here indeed is a probability measure since $\mathrm{d}W(s,y) + \mathbf{G}\left(\hat{u}^N(s,y)\right) \mathrm{d}y\, \mathrm{d}s$ defines a space-time white noise measure under $\mathbb{Q}$ independent of $\mathcal{F}_0$ by the bound $\mathbb{E}\left[\rho^{-1}\right] \leqslant C(T) < \infty$. Thus, we obtain that $\hat{u}^N$ has the same distribution under the probability measure $\mathbb{Q}$ as $Z(t) + e^{tA}P_N u_0$ under the probability measure $\mathbb{P}$, formally, we have the equality of the pushforward measures

$$\mathbb{Q} \circ \left(\hat{u}^N\right)^{-1} = \mathbb{P} \circ \left(Z(t) + e^{tA}P_N u_0\right)^{-1}.$$

Define the functional on the process $\mathcal{Z}$

$$\mathcal{B}(Z) := \sup_{t\in[0,T]} \left\| \int_0^t e^{(t-s)A} P_N \left(\mathbf{G}\left(\mathcal{Z}(s)\right) - \mathbf{G}\left(\mathcal{Z}(k_n(s))\right)\right) \mathrm{d}s \right\|.$$

Then, via Girsanov theorem and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}\left[|\mathcal{B}(\hat{u}^N)|^p\right] &= \mathbb{E}^{\mathbb{P}}\left[\rho\rho^{-1}|\mathcal{B}(\hat{u}^N)|^p\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\rho^{-1}|\mathcal{B}(\hat{u}^N)|^p\right] \\
&\lesssim \mathbb{E}^{\mathbb{Q}}\left[\rho^{-2}\right]^{\frac{1}{2}} \mathbb{E}^{\mathbb{Q}}\left[|\mathcal{B}(\hat{u}^N)|^{2p}\right]^{\frac{1}{2}} \\
&= \mathbb{E}^{\mathbb{Q}}\left[\rho^{-2}\right]^{\frac{1}{2}} \mathbb{E}^{\mathbb{P}}\left[|\mathcal{B}(Z(t) + e^{tA}P_N u_0)|^{2p}\right]^{\frac{1}{2}} \\
&\lesssim \mathbb{E}^{\mathbb{P}}\left[|\mathcal{B}(Z(t) + e^{tA}P_N u_0)|^{2p}\right]^{\frac{1}{2}}.
\end{aligned}$$

Therefore, if we denote

$$\hat{I}_{21} := \int_0^t e^{(t-s)A} P_N \left(\mathbf{G}\left(Z(s) + e^{sA}P_N u_0\right) - \mathbf{G}\left(Z(k_n(s)) + e^{k_n(s)A}P_N u_0\right)\right) \mathrm{d}s, \tag{6.26}$$

then the estimate of I_{21} is reduced to

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|I_{21}(t)\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \lesssim \mathbb{E}\left[\sup_{t\in[0,T]} \left\|\hat{I}_{21}(t)\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}}. \tag{6.27}$$

2. Estimates of $\hat{I}_{21}$ via *stochastic sewing*:

For convenience, we denote $\tilde{Z}(t) := Z(t) + e^{tA}u_0$. In this step, in order to obtain estimates of (6.27), we first show the following for any $(s,t) \in [0,1]_{\leqslant}$:

$$\begin{aligned}
&\mathbb{E}\left[\left\|\hat{I}_{21}(t) - \hat{I}_{21}(s)\right\|^p_{L^\infty(\mathbb{T}^2)}\right]^{\frac{1}{p}} \\
&= \mathbb{E}\left[\left\|\int_s^t e^{(t-r)A} P_N \left(\mathbf{G}\left(\tilde{Z}(r)\right) - \mathbf{G}\left(Z(k_n(r))\right)\right) \mathrm{d}r\right\|^p_{L^\infty(\mathbb{T}^2)}\right]^{\frac{1}{p}} \\
&\lesssim (t-s)^{\frac{1}{4}+\frac{\varepsilon}{2}} n^{-1+\varepsilon}.
\end{aligned} \tag{6.28}$$

According to the property of $\mathcal{B}^\alpha_{p,q}$ and Sobolev embedding, it is sufficient to show (6.28) via showing

$$\mathbb{E}\left[\left|\int_s^t \Delta_j e^{(t-r)A} P_N \left(\mathbf{G}\left(\tilde{Z}(r)\right) - \mathbf{G}\left(Z(k_n(r))\right)\right) \mathrm{d}r\right|^p\right]^{\frac{1}{p}} \lesssim 2^{-j\varepsilon}(t-s)^{\frac{1}{4}+\frac{\varepsilon}{2}} n^{-1+\varepsilon}. \tag{6.29}$$

More precisely, (6.29) implies that

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_s^t \Delta_j e^{(t-r)A} P_N \left(\mathbf{G}\left(\tilde{Z}(r)\right) - \mathbf{G}\left(Z(k_n(r))\right)\right) \mathrm{d}r\right\|^p_{\mathcal{B}^\alpha_{p,p}}\right]^{\frac{1}{p}} \\
&\lesssim 2^{-j\varepsilon}(t-s)^{\frac{1}{4}+\frac{\varepsilon}{2}} n^{-1+\varepsilon}
\end{aligned}$$

by the Sobolev embedding $\mathcal{B}^\alpha_{p,p} \hookrightarrow C^{\alpha-\frac{2}{p}} \hookrightarrow L^\infty$ for sufficiently large p. Therefore, in the following, we only need to show (6.29). To do so, the idea is to apply Lemma 6.4. Let $t^* \leqslant T$ and define

$$A_{s,t} := \mathbb{E}_s\left[\int_s^t e^{(t^*-r)A} P_N \left(\mathbf{G}\left(\tilde{Z}(r)\right) - \mathbf{G}\left(Z(k_n(r))\right)\right) \mathrm{d}r\right] \tag{6.30}$$

for $(s,t) \in [0,T]^*_{\leqslant}$, $t \leqslant t^*$. We first verify the condition (6.7) of Lemma 6.4. To do this, we consider the following two cases.

1. $|t-s| \leqslant \frac{3}{n}$.

Following from (6.16) and (6.17), we have for $\lambda \in [0,2)$

$$\left\|e^{(t+s)A}u_0 - e^{tA}u_0\right\|_{C^{-1+\varepsilon}} \lesssim s^{\frac{2+\lambda}{4}} t^{-\frac{\varepsilon}{2}-\frac{\lambda}{4}} \left\|u_0\right\|_{C^{1-\varepsilon}},$$
$$\left\|e^{(t+s)A}u_0 - e^{tA}u_0\right\|_{C^{-1+\varepsilon}} \lesssim s^{\frac{2-\lambda}{4}} \left\|u_0\right\|_{C^{1-\varepsilon}},$$

which implies

$$\mathbb{E}\left[\left\|\tilde{Z}(t) - \tilde{Z}(s)\right\|^p_{C^{1-\lambda-\varepsilon}}\right] \leqslant C|t-s|^{\frac{\lambda p}{4}}, \tag{6.31}$$

$$\mathbb{E}\left[\left\|\tilde{Z}\right\|_{C^{\frac{\lambda}{4}}\left([0,T],C^{1-\lambda-\varepsilon}\right)}\right] \leqslant C, \tag{6.32}$$

$$\mathbb{E}\left[\left\|\sup_{r\in[0,T]} (e^{(t+s)A} - e^{tA})\tilde{Z}(r)\right\|^p_{C^{-1+\varepsilon}}\right]^{\frac{1}{p}} \leqslant \tilde{C} t^{-\frac{\lambda}{4}-\frac{\varepsilon}{2}} s^{\frac{2+\lambda}{4}}. \tag{6.33}$$

Observe that for Lipschitz continuous $\mathbf{G}$ we have

$$\left\|\mathbf{G}(u_1) - \mathbf{G}(u_2)\right\|_{C^{-1-\varepsilon}} \lesssim \left\|u_1 - u_2\right\|_{C^{-1+\varepsilon}}. \tag{6.34}$$

To be more precise, for $u_1, u_2 \in C^{1-\varepsilon}$, by the fundamental theorem of calculus we have

$$\mathbf{G}(u_1) - \mathbf{G}(u_2) = \int_0^1 \nabla\mathbf{G}\left(\theta u_1 + (1-\theta)u_2\right) \mathrm{d}\theta \cdot (u_1 - u_2),$$

which yields

$$\begin{aligned}\left\|\mathbf{G}(u_1) - \mathbf{G}(u_2)\right\|_{C^{-1-\varepsilon}} &= \left\|\int_0^1 \nabla\mathbf{G}\left(\theta u_1 + (1-\theta)u_2\right) \mathrm{d}\theta \cdot (u_1 - u_2)\right\|_{C^{-1-\varepsilon}} \\ &\leqslant \sup_{\theta\in[0,1]} \left\|\nabla\mathbf{G}\left(\theta u_1 + (1-\theta)u_2\right) \cdot (u_1 - u_2)\right\|_{C^{-1-\varepsilon}}.\end{aligned}$$

Furthermore, from the rule for the product of two distributions [3, Section 2], we have

$$\left\|g \cdot h\right\|_{C^{\min(\theta,\beta)}} \leqslant C(\theta,\beta) \left\|g\right\|_{C^\theta} \left\|h\right\|_{C^\beta} \tag{6.35}$$

for any $g \in C^\theta, h \in C^\beta$ with $\beta, \theta \in \mathbb{R}$ and $\beta + \theta > 0$

and thus by applying (6.35) and (A.16) we obtain

$$\begin{aligned}\left\|\mathbf{G}\left(u_1\right)-\mathbf{G}\left(u_2\right)\right\|_{C^{-1-\varepsilon}} &\lesssim \sup_{\theta\in[0,1]}\left\|\nabla\mathbf{G}\left(\theta u_1+(1-\theta)u_2\right)\right\|_{C^{1-\frac{\varepsilon}{2}}}\left\|u_1-u_2\right\|_{C^{-1+\varepsilon}} \\ &\approx \sup_{\theta\in[0,1]}\left\|\mathbf{G}\left(\theta u_1+(1-\theta)u_2\right)\right\|_{C^{2-\frac{\varepsilon}{2}}}\left\|u_1-u_2\right\|_{C^{-1+\varepsilon}} \\ &\lesssim \|\mathbf{G}\|_{C^2}\left\|u_1-u_2\right\|_{C^{-1+\varepsilon}}.\end{aligned} \tag{6.36}$$

By using (6.31), (6.33),(6.34), and (6.36), we obtain

$$\begin{aligned}&\|A_{s,t}\|_{L^p(\Omega)} \\ &\lesssim \mathbb{E}\left[\left|\int_s^t \Delta_j e^{(t^*-r)A}P_N\left(\mathbf{G}\left(\tilde{Z}(r)\right)-\mathbf{G}\left(Z(k_n(r))\right)\right)\,\mathrm{d}r\right|^p\right]^{\frac{1}{p}} \\ &\lesssim 2^{-j\varepsilon}\mathbb{E}\left[\left\|\int_s^t e^{(t^*-r)A}P_N\left(\mathbf{G}\left(\tilde{Z}(r)\right)-\mathbf{G}\left(Z(k_n(r))\right)\right)\,\mathrm{d}r\right\|_{C^\varepsilon}^p\right]^{\frac{1}{p}} \\ &\lesssim 2^{-j\varepsilon}\int_s^t (t^*-r)^{-\frac{1+\varepsilon}{4}}\mathbb{E}\left[\left\|\left(\mathbf{G}\left(\tilde{Z}(r)\right)-\mathbf{G}\left(Z(k_n(r))\right)\right)\right\|_{C^{-1}}^p\right]^{\frac{1}{p}}\,\mathrm{d}r \\ &\lesssim 2^{-j\varepsilon}\int_s^t (t^*-r)^{-\frac{(1+\varepsilon)}{4}} \\ &\qquad\times\mathbb{E}\left[\left\|\int_0^1 \nabla\mathbf{G}\Big((1-\theta)\tilde{Z}(r)+(1-\theta)Z(k_n(r))\Big)\right.\right. \\ &\qquad\qquad\left.\left.\times\left(\tilde{Z}(r)-Z(k_n(r))\right)\mathrm{d}\theta\right\|_{C^{-1}}^p\right]^{\frac{1}{p}}\mathrm{d}r \\ &\lesssim 2^{-j\varepsilon}\|\mathbf{G}\|_{C^2}(t^*-t)^{-\frac{(1+\varepsilon)}{4}}\int_s^t (r-k_n(r))^{-\frac{2-\varepsilon}{4}}\,\mathrm{d}r \\ &\lesssim 2^{-j\varepsilon}(t^*-t)^{-\frac{(1+2\varepsilon)}{4}}(t-s)(r-k_n(r))^{\frac{2-\varepsilon}{4}} \\ &\lesssim 2^{-j\varepsilon}(t^*-t)^{-\frac{(1+2\varepsilon)}{4}}(t-s)^{\frac{1}{2}+\varepsilon}n^{-1+\varepsilon}.\end{aligned} \tag{6.37}$$

The last inequality holds due to $|t-s|\leqslant 3n^{-1}$.

2. $|t-s| > \frac{3}{n}$.

Denote $\tilde{k}_n(s) = k_n(s) + \frac{2}{n}$ such that $\frac{r-s}{2} \geqslant n^{-1}$ for $r \in [\tilde{k}_n(s), t]$, which implies that $k_n(r) - s \geqslant \frac{r-s}{2}$ for $r \in [\tilde{k}_n(s), t]$.

We write

$$\begin{aligned} &\int_s^t e^{(t^*-r)A} P_N \left(\mathbf{G}\left(\tilde{Z}(r)\right) - \mathbf{G}\left(Z(k_n(r))\right)\right) \mathrm{d}r \\ &= \int_s^{\tilde{k}_n(s)} e^{(t^*-r)A} P_N \left(\mathbf{G}\left(\tilde{Z}(r)\right) - \mathbf{G}\left(Z(k_n(r))\right)\right) \mathrm{d}r \\ &\quad + \int_{\tilde{k}_n(s)}^t e^{(t^*-r)A} P_N \left(\mathbf{G}\left(\tilde{Z}(r)\right) - \mathbf{G}\left(Z(k_n(r))\right)\right) \mathrm{d}r =: S_1 + S_2. \end{aligned} \tag{6.38}$$

Since $\tilde{k}_n(s) - s \leqslant 2n^{-1}$, by the same idea as in (6.37), we can obtain

$$\|S_1\|_{L^p(\Omega)} \lesssim 2^{-j\varepsilon}(t^*-t)^{-\frac{(1+2\varepsilon)}{4}}(t-s)^{\frac{1}{2}+\varepsilon} n^{-1+\varepsilon}. \tag{6.39}$$

For S_2, we use that $\tilde{Z}(r,x) = e^{(r-s)A}\tilde{Z}(s,x) + \int_s^r \int_{\mathbb{T}^2} p_{r-v}(x-y)\,\mathrm{d}W(v,y)$ and $e^{(r-s)A}\tilde{Z}(s,x) \in \mathcal{F}_s$ as well as $\int_s^r \int_{\mathbb{T}^2} p_{r-v}(x-y)\,\mathrm{d}W(v,y)$ is independent of $\mathcal{F}_s$. Moreover, we know that $\int_s^r \int_{\mathbb{T}^2} p_{r-v}(x-y)\,\mathrm{d}W(v,y)$ follows Gaussian distribution with 0 mean and variance

$$Q(r-s) := \mathbb{E}\left[\left(\int_s^r \int_{\mathbb{T}^2} p_{r-v}(x-y)\,\mathrm{d}W(v,y)\right)^2\right]. \tag{6.40}$$

If we denote $\mathcal{P}_{Q(r-s)}$ as the heat semigroup on $\mathbb{R}$ with variance $Q(r-s)$ (see Definition A.18), we can write

$$\begin{aligned} &\mathbb{E}_s\left[S_2(x)\right] \\ &= \int_{\tilde{k}_n(s)}^t \Delta_j e^{(t^*-r)A} \\ &\qquad P_N[(\mathcal{P}_{Q(r-s)}\mathbf{G})(e^{(r-s)A}\tilde{Z}(s)) - (\mathcal{P}_{Q(k_n(r)-s)}\mathbf{G})(e^{(t^*-r)A}P_N\tilde{Z}(s))](x)\,\mathrm{d}r \\ &= \int_{\tilde{k}_n(s)}^t \Delta_j e^{(t^*-r)A} \\ &\qquad P_N[(\mathcal{P}_{Q(r-s)}\mathbf{G})(e^{(r-s)A}\tilde{Z}(s)) - (\mathcal{P}_{Q(r-s)}\mathbf{G})(e^{(t^*-r)A}P_N\tilde{Z}(s))](x)\,\mathrm{d}r \\ &\quad + \int_{\tilde{k}_n(s)}^t \Delta_j e^{(t^*-r)A} \\ &\qquad P_N[\left((\mathcal{P}_{Q(r-s)}\mathbf{G}) - (\mathcal{P}_{Q(k_n(r)-s)}\mathbf{G})\right)(e^{(t^*-r)A}P_N\tilde{Z}(s))](x)\,\mathrm{d}r \\ &=: S_{21} + S_{22}. \end{aligned} \tag{6.41}$$

We first estimate S_{21}. Observe that, similar to (6.34), for $u_1, u_2 \in C^{-1+\varepsilon}$, we have

$$\left\|\mathcal{P}_t\mathbf{G}\left(u_1\right)-\mathcal{P}_t\mathbf{G}\left(u_2\right)\right\|_{C^{-1-\varepsilon}(\mathbb{T}^2)} \lesssim \left\|u_1-u_2\right\|_{C^{-1+\varepsilon}(\mathbb{T}^2)}. \tag{6.42}$$

Indeed, again by the fundamental theorem of calculus, (6.35) and (A.16) we have

$$\begin{aligned}
&\left\|\mathcal{P}_t\mathbf{G}\left(u_1\right)-\mathcal{P}_t\mathbf{G}\left(u_2\right)\right\|_{C^{-1-\varepsilon}(\mathbb{T}^2)}\\
&=\left\|\int_0^1 \nabla\mathcal{P}_t\mathbf{G}\left(\theta u_1+(1-\theta)u_2\right)\,\mathrm{d}\theta\cdot(u_1-u_2)\right\|_{C^{-1-\varepsilon}}\\
&\leqslant \sup_{\theta\in[0,1]}\left\|\nabla\mathcal{P}_t\mathbf{G}\left(\theta u_1+(1-\theta)u_2\right)\cdot(u_1-u_2)\right\|_{C^{-1-\varepsilon}}.\\
&\approx \sup_{\theta\in[0,1]}\left\|\mathcal{P}_t\mathbf{G}\left(\theta u_1+(1-\theta)u_2\right)\right\|_{C^{2-\frac{\varepsilon}{2}}}\left\|u_1-u_2\right\|_{C^{-1+\varepsilon}}\\
&\lesssim \left\|\mathbf{G}\right\|_{C^2}\left\|u_1-u_2\right\|_{C^{-1+\varepsilon}}.
\end{aligned}$$

By applying above with $u_1=e^{(r-s)A}\bar{Z}(s), u_2=e^{(t^*-r)A}P_N\bar{Z}(s)$ we obtain

$$\begin{aligned}
&\mathbb{E}\left[|S_{21}|^p\right]^{\frac{1}{p}} \\
&\lesssim 2^{-j\varepsilon}\,\mathbb{E}\Bigg[\Bigg\|\int_{\tilde{k}_n(s)}^t e^{(t^*-r)A}P_N\Big[\left(\mathcal{P}_{Q(r-s)}\mathbf{G}\right)\left(e^{(r-s)A}\tilde{Z}(s)\right)\\
&\qquad -\left(\mathcal{P}_{Q(r-s)}\mathbf{G}\right)\left(e^{(t^*-r)A}P_N\tilde{Z}(s)\right)\Big]\,\mathrm{d}r\Bigg\|_{C^{\varepsilon}}^p\Bigg]^{1/p}\\
&\lesssim 2^{-j\varepsilon}\int_{\tilde{k}_n(s)}^t (t^*-r)^{-\frac{1+\varepsilon}{4}}\\
&\quad\times\mathbb{E}\left[\left\|(\mathcal{P}_{Q(r-s)}\mathbf{G})(e^{(r-s)A}\tilde{Z}(s))-(\mathcal{P}_{Q(r-s)}\mathbf{G})(e^{(t^*-r)A}P_N\tilde{Z}(s))\right\|_{C^{-1}}^p\right]^{\frac{1}{p}}\,\mathrm{d}r\\
&\lesssim 2^{-j\varepsilon}\int_{\tilde{k}_n(s)}^t (t^*-r)^{-\frac{(1+2\varepsilon)}{4}}\\
&\quad\times\mathbb{E}\left[\sup_{r\in[\tilde{k}_n(s),t]}\sup_{\theta\in[0,1]}\left\|\theta\left(e^{(r-s)A}\tilde{Z}(s)\right)+(1-\theta)\left(e^{(t^*-r)A}P_N\tilde{Z}(s)\right)\right\|_{C^{1-\frac{\varepsilon}{2}}}^p\right]^{\frac{1}{p}}\\
&\quad\times\mathbb{E}\left[\left\|e^{(r-s)A}\tilde{Z}(s)-(e^{(t^*-r)A}P_N\tilde{Z}(s))\right\|_{C^{-1+\varepsilon}}^p\right]^{\frac{1}{p}}\,\mathrm{d}r.
\end{aligned} \tag{6.43}$$

Following from (6.32) and (6.14) together with (6.17), we obtain that

$$\mathbb{E}\left[\sup_{r\in[\tilde{k}_n(s),t]}\sup_{\theta\in[0,1]}\left\|\left(\theta(e^{(r-s)A}\tilde{Z}(s))+(1-\theta)(e^{(t^*-r)A}P_N\tilde{Z}(s))\right)\right\|_{C^{1-\frac{\varepsilon}{2}}}^{2p}\right]^{\frac{1}{2p}}\lesssim 1.$$

Moreover, from (6.33) by taking $\lambda = 2 - 4\varepsilon$ we know that

$$
\begin{aligned}
&\mathbb{E}\left[\left\| e^{(r-s)A}\tilde{Z}(s) - (e^{(t^*-r)A}P_N\tilde{Z}(s))\right\|_{C^{-1+\varepsilon}}^{2p}\right]^{\frac{1}{2p}} \\
&\lesssim |(r-s)-(k_n(r)-s)|^{1-\varepsilon}(k_n(r)-s)^{-\frac{2-4\varepsilon}{4}-\frac{\varepsilon}{2}} \\
&\lesssim n^{-1+\varepsilon}(k_n(r)-s)^{-\frac{1}{2}+\varepsilon},
\end{aligned}
$$

which combining with (6.43) and $k_n(r) - s \geqslant \frac{r-s}{2}$, for $r \in [\tilde{k}_n(s), t]$, implies that

$$
\begin{aligned}
\mathbb{E}\left[|S_{21}|^p\right]^{\frac{1}{p}} &\lesssim 2^{-j\varepsilon}\int_{\tilde{k}_n(s)}^{t}(t^*-r)^{-\frac{(1+2\varepsilon)}{4}}n^{-1+\varepsilon}(k_n(r)-s)^{-\frac{1}{2}+\varepsilon}\,\mathrm{d}r \\
&\lesssim 2^{-j\varepsilon}n^{-1+\varepsilon}(t^*-t)^{-\frac{(1+2\varepsilon)}{4}}\int_{\tilde{k}_n(s)}^{t}(r-s)^{-\frac{1}{2}+\varepsilon}\,\mathrm{d}r \\
&\lesssim 2^{-j\varepsilon}n^{-1+\varepsilon}(t^*-t)^{-\frac{(1+2\varepsilon)}{4}}(t-s)^{\frac{1}{2}+\varepsilon}.
\end{aligned} \tag{6.44}
$$

For S_{22}, observe from (6.40)

$$
\begin{aligned}
&Q(r-s) - Q(k_n(r)-s) \\
&= \mathbb{E}\left[\left(\int_s^r\int_{\mathbb{T}^2}p_{r-v}(x-y)\,\mathrm{d}W(v,y)\right)^2\right] \\
&\quad - \mathbb{E}\left[\left(\int_s^{k_n(r)}\int_{\mathbb{T}^2}p_{k_n(r)-v}(x-y)\,\mathrm{d}W(v,y)\right)^2\right] \\
&= \int_s^r\int_{\mathbb{T}^2}|p_{r-v}(x-y)|^2\,\mathrm{d}y\,\mathrm{d}v - \int_s^{k_n(r)}\int_{\mathbb{T}^2}|p_{k_n(r)-v}(x-y)|^2\,\mathrm{d}y\,\mathrm{d}v \\
&= \int_{k_n(r)-s}^{r-s}\int_{\mathbb{T}^2}|p_v(x-y)|^2\,\mathrm{d}y\,\mathrm{d}v \\
&\lesssim \int_{k_n(r)-s}^{r-s}\|p_v(x-y)\|_\infty\int_{\mathbb{T}^2}|p_v(x-y)|\,\mathrm{d}y\,\mathrm{d}v \\
&\lesssim \int_{k_n(r)-s}^{r-s}v^{-1}\,\mathrm{d}v \lesssim (r-k_n(r))^{1-\varepsilon'}|k_n(r)-s|^{\varepsilon'}
\end{aligned}
$$

for each $\varepsilon' \in (0,1)$. Combining it with the property of the heat semigroup

$$
\|(\mathcal{P}_t - \mathcal{P}_s)\mathbf{G}\,(u)\|_{C^{-2-\varepsilon}} \lesssim |t-s|^{\frac{1}{2}}\,\|\mathbf{G}\,(u)\|_{C^{-1-\varepsilon}} \lesssim |t-s|^{\frac{1}{2}}\,\|u\|_{C^{1-\varepsilon}}
$$

shows that, taking $\varepsilon' = 2\varepsilon$

$$\begin{aligned}
&\left\|(\mathcal{P}_{Q(r-s)}\mathbf{G}) - (\mathcal{P}_{Q(k_n(r)-s)}\mathbf{G})(u)\right\|_{C^{-2-\frac{\varepsilon}{2}}} \\
&\lesssim |Q(r-s) - Q(k_n(r)-s)| \, \|u\|_{C^{1-\varepsilon}} \\
&\lesssim (r-k_n(r))^{1-\varepsilon}|k_n(r)-s|^{\varepsilon} \, \|u\|_{C^{1-\varepsilon}} \\
&\lesssim n^{-1+\varepsilon}|k_n(r)-s|^{\varepsilon} \, \|u\|_{C^{1-\varepsilon}} \\
&\lesssim n^{-1+\varepsilon}|r-s|^{\varepsilon} \, \|u\|_{C^{1-\varepsilon}}
\end{aligned}$$

for $r \in [\tilde{k}_n(s), t]$, where $k_n(r) - s \geqslant \frac{r-s}{2}$.

Applying the above estimate to (6.41), by taking $u = e^{(t^*-r)A}P_N\tilde{Z}(s)$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[|S_{22}|^p\right]^{\frac{1}{p}} \qquad (6.45)\\
&\lesssim 2^{-j\varepsilon} \\
&\times \int_{\tilde{k}_n(s)}^{t} \left\|e^{(t^*-r)A}P_N[\left((\mathcal{P}_{Q(r-s)}\mathbf{G}) - (\mathcal{P}_{Q(k_n(r)-s)}\mathbf{G})\right)(e^{(t^*-r)A}P_N\tilde{Z}(s))]\right\|_{C^{\varepsilon}} \mathrm{d}r \\
&\lesssim 2^{-j\varepsilon}\int_{\tilde{k}_n(s)}^{t} |t^*-r|^{-\frac{1+\varepsilon}{2}} \\
&\qquad \times \left\|\left[\left((\mathcal{P}_{Q(r-s)}\mathbf{G}) - (\mathcal{P}_{Q(k_n(r)-s)}\mathbf{G})\right)(e^{(t^*-r)A}P_N\tilde{Z}(s))\right]\right\|_{C^{-2-\varepsilon}} \mathrm{d}r \\
&\lesssim 2^{-j\varepsilon}\int_{\tilde{k}_n(s)}^{t} |t^*-r|^{-\frac{1+\varepsilon}{2}} n^{-1+\varepsilon}|r-s|^{\varepsilon}\left\|e^{(t^*-r)A}P_N\tilde{Z}(s)\right\|_{C^{1-\varepsilon}} \mathrm{d}r \\
&\lesssim 2^{-j\varepsilon}|t^*-r|^{-\frac{1+\varepsilon}{2}}(t-s)^{1+\varepsilon}n^{-1+\varepsilon}.
\end{aligned}$$

In the end, we put (6.41), (6.44), (6.45) together with (6.39) and (6.38) and then obtain

$$\|A_{s,t}\|_{L^p(\Omega)} \lesssim 2^{-j\varepsilon}|t^*-r|^{-\frac{1+\varepsilon}{2}}(t-s)^{1+\varepsilon}n^{-1+\varepsilon}. \qquad (6.46)$$

By combining the estimates of these two cases, i.e. (6.37) together with (6.46), we have

$$\|A_{s,t}\|_{L^p(\Omega)} \lesssim 2^{-j\varepsilon}|t^*-r|^{-\frac{1+\varepsilon}{2}}(t-s)^{1+\varepsilon}n^{-1+\varepsilon}.$$

Hence, the first condition of Lemma 6.4 is verified by taking $C_1 := n^{-1+\varepsilon}$ and $\delta_1 = -\frac{1+\varepsilon}{2}$.

For the second condition of Lemma 6.4, it is evident that it holds since we have

$$\mathbb{E}_s\left[\delta A_{s,u,t}\right] = 0.$$

Indeed, for any $(s,t) \in [0,T]^*_{\leqslant}$, $s \leqslant u \leqslant t$, by (6.30), we have

$$\begin{aligned}\delta A_{s,u,t} &= A_{s,t} - A_{s,u} - A_{u,t}\\ &= \mathbb{E}_s\left[\int_u^t e^{(t^*-r)A} P_N\left(\mathbf{G}(\tilde{U}_r) - \mathbf{G}(\tilde{U}_{k_n(r)})\right)\,\mathrm{d}r\right]\\ &\quad - \mathbb{E}_u\left[\int_u^t e^{(t^*-r)A} P_N\left(\mathbf{G}(\tilde{U}_r) - \mathbf{G}(\tilde{U}_{k_n(r)})\right)\,\mathrm{d}r\right].\end{aligned}$$

By applying the tower property of conditional expectations, we see that

$$\mathbb{E}_s \delta A_{s,u,t} = 0.$$

Therefore, (6.8) holds for $A_{s,t}$ defined in (6.30). Up to this point, we are ready to apply Lemma 6.4. Let

$$\mathcal{A}_t := \int_0^t e^{(t^*-r)A} P_N\left(\mathbf{G}\left(\tilde{Z}(r)\right) - \mathbf{G}\left(Z(k_n(r))\right)\right)\,\mathrm{d}r.$$

Note that

$$\mathbb{E}_s\left[\mathcal{A}_{s,t} - A_{s,t}\right] = 0, \quad |\mathcal{A}_{s,t} - A_{s,t}| \leqslant |t-s| \sup_{r\in[s,t]} \left\|e^{(t^*-r)A} P_N \mathbf{G}\left(\tilde{Z}(r)\right)\right\|_\infty$$

which shows (6.29) by (6.11). Finally, we obtain the result (6.28).

3. Uniform bounds via Kolmogorov continuity theorem:

In this step, our aim is to show the following equality:

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\hat{I}_{21}(t)\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \qquad (6.47)\\ &= \mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t e^{(t-r)A} P_N\left(\mathbf{G}(\tilde{Z}(r)) - \mathbf{G}\left(Z(k_n(r))\right)\right)\,\mathrm{d}r\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}}.\end{aligned}$$

Following from a version of Kolmogorov's continuity theorem (see also [31, Proof of Corollary 4.6]), which says that for a continuous process starting from 0 with values in a Banach space V and a semigroup $(S_t)_{t\geqslant 0}$ of bounded linear operators on V, if for some $p>0, \alpha>0$ one has

$$\mathbb{E}\left\|X_t - S_{t-s}X_s\right\|^p \leqslant C_1|t-s|^{1+\alpha}, \tag{6.48}$$

then there exists $C_2 = C(T,p,d,\alpha)$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t\|^p \leqslant C_1C_2. \tag{6.49}$$

Now we choose

$$X_t = \int_0^t e^{(t-r)A} P_N \left(\mathbf{G}\left(\tilde{Z}(r)\right) - \mathbf{G}\left(Z(k_n(r))\right)\right) \mathrm{d}r.$$

Then from (6.28), we know that (6.48) holds with $S = P^N$, $V = L^2(\mathbb{T}^2)$ and $\alpha = 2\varepsilon$, $p \geqslant 4$, $C_1 = (Cn^{-1+\varepsilon})^p$. Hence, (6.49) yields (6.47).

4. Buckling via Grönwall lemma:

Putting (6.47), (6.27) and (6.25) together, we can see that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|I_{21}(t)\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \leqslant Cn^{-1+\varepsilon}. \tag{6.50}$$

With (6.50) at hand, observing from (6.25), we are ready to apply Lemma 6.5 by taking $X = \hat{u}^N, Y = \hat{u}^{N,n}$ and $Z = I_{21}, \mathcal{S}(s,t) = P^N_{t-s}, \tau(s) = k_n(s)$ for (6.12). In the end, the desired estimate (6.24) is obtained via (6.50) and (6.13). The proof is complete. □

6.4.3. Third Estimate

Lemma 6.11. *Suppose $u_0 \in C^{\frac{1}{2}}$ and $\|\mathbf{G}\|_\infty, \|\partial\mathbf{G}\|_\infty < \infty$. For u^N from (6.3) and $\hat{u}^N$ from (6.19) we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\|u^N(t) - \hat{u}^N(t)\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \lesssim N^{-\frac{1}{2}+\varepsilon} \tag{6.51}$$

for sufficiently small $\varepsilon > 0$.

Proof. By (6.3) and (6.19) we obtain

$$u^N(t) - \hat{u}^N(t) = [Z(t) - P_N Z(t)] + \int_0^t e^{(t-s)A} P_N \left[\mathbf{G}\left(u^N(s)\right) - \mathbf{G}\left(\hat{u}^N(s)\right)\right] \mathrm{d}s$$

for each $t \in [0,T]$. As in the proof of Lemma 6.9, we obtain by the Lipschitz continuity of $\mathbf{G}$ and Lemma 6.5 the following bound:

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\|u^N(t) - \hat{u}^N(t)\right\|^p_{L^2(\mathbb{T}^2)}\right] \lesssim \mathbb{E}\left[\sup_{t\in[0,T]} \|Z(t) - P_N Z(t)\|^p_{L^2(\mathbb{T}^2)}\right] \lesssim N^{-\frac{p}{2}} N^{2\delta p} \tag{6.52}$$

whereby $\delta \in \left(0, \frac{1}{4}\right)$ can be chosen arbitrarily small. This establishes the desired result (6.22). □

6.4.4. Fourth Estimate

Lemma 6.12. *Suppose $u_0 \in C^{\frac{1}{2}}$ and $\|\mathbf{G}\|_\infty, \|\partial\mathbf{G}\|_\infty < \infty$. For u^N from (6.3) and v from (6.2) we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\|u^N(t) - v(t)\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \lesssim N^{-\frac{1}{2}} \|u_0\|_{C^{\frac{1}{2}}}. \tag{6.53}$$

Proof. For each $t \in [0,T]$ we have

$$\begin{aligned} u^N(t) - v(t) &= \left(e^{tA} - e^{tA}P_N\right)u_0 + Z(t) - P_N Z(t) \\ &\quad + \int_0^t e^{(t-s)A}P_N\mathbf{G}\left(u^N(s)\right) - e^{(t-s)A}\mathbf{G}\left(v(s)\right) ds \\ &= \left(e^{tA} - e^{tA}P_N\right)u_0 + Z(t) - P_N Z(t) \\ &\quad + \int_0^t \left(e^{(t-s)A}P_N - e^{(t-s)A}\right)\mathbf{G}\left(u^N(s)\right) ds \\ &\quad + \int_0^t e^{(t-s)A}\left(\mathbf{G}\left(u^N(s)\right) - \mathbf{G}\left(v(s)\right)\right) \mathrm{d}s. \end{aligned}$$

By applying the triangle inequality and Lemma 6.5, we obtain

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]} \left\|u^N(t) - v(t)\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \\ &\lesssim \mathbb{E}\left[\sup_{t\in[0,T]} \left\|\left(e^{tA} - e^{tA}P_N\right)u_0\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \\ &\quad + \mathbb{E}\left[\sup_{t\in[0,T]} \left\|\int_0^t \left(e^{(t-s)A}P_N - e^{(t-s)A}\right)\mathbf{G}\left(u^N(s)\right) ds\right\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \\ &\quad + \mathbb{E}\left[\sup_{t\in[0,T]} \|Z(t) - P_N Z(t)\|^p_{L^2(\mathbb{T}^2)}\right]^{\frac{1}{p}} \\ &=: I_{41} + I_{42} + I_{43}. \end{aligned}$$

Following from (6.52), we obtain

$$I_{43} \lesssim N^{-\frac{1}{2}+\varepsilon}. \tag{6.54}$$

For I_{41}, we use Π_N, the orthogonal projection from $L^2(\mathbb{T}^2, \mathbb{C})$ to its subspace $\operatorname{span}\left((e_k)_{|k|\leqslant N}\right)$,

$$I_{41} \lesssim \sup_{t\in[0,T]} \left\|(e^{tA} - e^{tA}P_N)u_0\right\|_{L^\infty(\mathbb{T}^2)} \lesssim \|u_0 - \Pi_N u_0\|_{L^\infty(\mathbb{T}^2)} \lesssim N^{-\frac{1}{2}+\varepsilon} \|u_0\|_{C^{\frac{1}{2}}}, \tag{6.55}$$

for $\varepsilon > 0$ being sufficiently small. By a similar idea as used for (6.55) and the result from Lemma 6.8 we have

$$
\begin{aligned}
I_{42} &= \mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t \left(e^{(t-s)A}P_N - e^{(t-s)A}\right)\mathbf{G}\left(u^N(s)\right)\,\mathrm{d}s\right\|_{L^2(\mathbb{T}^2)}^p\right]^{\frac{1}{p}} \\
&\leqslant \mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t \left\|\mathbf{G}\left(u^N(s)\right) - \Pi_N\mathbf{G}\left(u^N(s)\right)\right\|_{L^\infty(\mathbb{T}^2)}\,\mathrm{d}s\right)^p\right]^{\frac{1}{p}} \\
&\leqslant CN^{-\frac{1}{2}+\varepsilon}\sup_{t\in[0,T]}\left\|u^N(t)\right\|_{C^{\frac{1}{2}}} \\
&\lesssim N^{-\frac{1}{2}+\varepsilon}.
\end{aligned}
\tag{6.56}
$$

Collecting (6.54), (6.55) and (6.56) together shows the desired result (6.53). □

6.4.5. Proof of the Main Result

With estimates (6.22), (6.24), (6.51), (6.53) at hand, we are ready to prove the main result.

Proof of Theorem 6.3. Putting together the estimates (6.22), (6.24), (6.51), (6.53) and recalling the decomposition

$$
u^{N,n} - v = u^{N,n} - \hat{u}^{N,n} + \hat{u}^{N,n} - \hat{u}^N + \hat{u}^N - u^N + u^N - v =: I_1 + I_2 + I_3 + I_4
$$

(cf. (6.21)), we obtain

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\left\|u^{N,n}(t) - v(t)\right\|_{L^2(\mathbb{T}^2)}^p\right]^{\frac{1}{p}} \leqslant C(N^{-\frac{1}{2}+\varepsilon} + n^{-1+\varepsilon})
$$

where C depends only on T, p, ε.
This completes the proof of Theorem 6.3. □

7. Conclusion

This thesis started by describing the physical and mathematical framework of the stochastic partial differential equation

$$\partial_t u_\varepsilon(t,x) = -\delta\Delta^2 u_\varepsilon(t,x) - \nabla \cdot \frac{\nabla u_\varepsilon(t,x)}{1+|\nabla u_\varepsilon(t,x)|^2} + \sigma\, \partial_t W_\varepsilon(t,x),$$

for each $t \geqslant 0$ and $x \in \mathbb{T}^2$ in Chapter 1 and introducing essential elements of the theory of stochastic processes in Chapter 2. We established an approach using regularisation by space-time white noise for an ill-posed problem in Chapter 3 which gave us the result of vanishing nonlinearity in the limit. However, we even derived uniform boundedness of $v_\varepsilon := u_\varepsilon - Z_\varepsilon - e^{tA}u_0$ (cf. Definition 3.1).
In Chapter 4, we used a decomposition approach for a linearisation of the equation above (see (4.4)) to derive an upper bound for the $C^1(\mathbb{T}^2)$-norm and a lower bound for the $C^0(\mathbb{T}^2)$-norm of the mild solution u_ε up to a stopping time τ. In particular, we showed

$$0 < \sigma \leqslant \sup_{s\in[0,\tau]} \|u_\varepsilon(s)\|_{C^0} \leqslant \sup_{s\in[0,\tau]} \|u_\varepsilon(s)\|_{C^1} \leqslant r_\sigma \ll 1$$

with high probability.
In Chapter 5, we presented the spectral Galerkin method in space for the Euler scheme in time and established for the error function $e^{(N)} := u^{(N)} - u_h^{(N)}$ the convergence rate

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\| u_h^{(N)}(t) - u^{(N)}(t)\right\|_{H^1}^p\right]^{\frac{1}{p}}$$
$$\leqslant C\left(\|u_0\|_{H^{1+\eta}}\left(h^{\frac{\eta}{4}} + N^{-\eta}\right) + N^{2+\alpha}h^{1-\frac{1}{p}-} + N^{-2} + h^{\frac{1}{2}-\frac{1}{p}}N^{-\frac{\alpha}{p^2+2}}\right)$$

for $u_0 \in H^{1+\eta}$ with $\eta > 0$ and u_h^N and u^N, as defined in (5.2) and (5.4). We analysed a nonlinear fourth-order stochastic partial differential equation (SPDE) driven by space-time white noise on the two-dimensional torus, in Chapter 6, defined as

$$\partial_t u = -\delta\Delta^2 u - \mathbf{G}(u) + \sigma\partial_t W$$

where $\mathbf{G} : \mathbb{R} \to \mathbb{R}$ is the nonlinear term satisfying $\|\mathbf{G}\|_\infty, \|\partial\mathbf{G}\|_\infty < \infty$. In this examination we applied a numerical approximation to establish convergence with almost spatial rate $\frac{1}{2}$ and almost temporal rate 1 obtained mainly via *stochastic sewing* technique.

A. Appendix

We begin with standard material on Fourier series for the two-dimensional torus, fractional Sobolev spaces, and analytic semigroups generated by the Bilaplace operator. These classical results are detailed in [38] and [68].

A.1. Fourier Series on the Two-Dimensional Torus

Let $\mathbb{T}^2 = [0, L]^2$ be the two-dimensional torus of length L with periodic boundary conditions. For $u \in L^2([0,T] \times \mathbb{T}^2)$, we consider its Fourier series in space

$$u(t,x) = \sum_{k \in \mathbb{Z}^2} u_k(t)\, e_k(x),$$

for each $x = (x_1, x_2) \in \mathbb{T}^2$, where $\{e_k\}_{k \in \mathbb{Z}^2}$ is the standard orthonormal Fourier basis of $L^2(\mathbb{T}^2)$ defined by

$$e_k(x) := \omega_{k_1}(x_1)\, \omega_{k_2}(x_2), \qquad \omega_{k_i}(z) := \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi k_i}{L} z\right), & k_i > 0, \\ \frac{1}{\sqrt{L}}, & k_i = 0, \\ \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi k_i}{L} z\right), & k_i < 0. \end{cases} \tag{A.1}$$

The Fourier coefficients are given by

$$u_k(t) = \langle u(t), e_k \rangle_{L^2(\mathbb{T}^2)}.$$

Eigenfunctions of the Bilaplace Operator: For each $k = (k_1, k_2) \in \mathbb{Z}^2$, the functions e_k satisfy

$$-\Delta e_k = \mu_k\, e_k, \qquad \mu_k := \left(\frac{2\pi|k|}{L}\right)^2. \tag{A.2}$$

The Bilaplace operator $A := -\Delta^2$ acts diagonally with eigenvalues $-\mu_k^2$:

$$Ae_k = -\Delta^2 e_k = -\mu_k^2\, e_k. \tag{A.3}$$

A.2. Fractional Sobolev Spaces

We recall the standard characterisation of Sobolev spaces and fractional Sobolev spaces on a Lipschitz domain $\Omega \subset \mathbb{R}^d$.

Definition A.1 (Sobolev space). *For $m \in \mathbb{N}$ and $p \geqslant 1$ we define*

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\beta u \in L^p(\Omega) \text{ for each } |\beta| \leqslant m\},$$

endowed with the norm

$$\|u\|_{W^{m,p}} := \sum_{|\beta|\leqslant m} \|D^\beta u\|_{L^p}.$$

Definition A.2 (Fractional Sobolev space). *For $\alpha = m+s$, with $m \in \mathbb{N}$, $s \in (0,1)$ and $p \geqslant 1$ we define*

$$W^{\alpha,p}(\Omega) := \{u \in W^{m,p}(\Omega) : D^\beta u \in W^{s,p}(\Omega) \text{ for each } |\beta| = m\},$$

endowed with the norm

$$\|u\|_{W^{\alpha,p}(\Omega)} := \sum_{|\beta|\leqslant m} \|D^\beta u\|_{L^p(\Omega)} + \sum_{|\beta|=m} [D^\beta u]_{W^{s,p}(\Omega)},$$

whereby we have

$$[u]^p_{W^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y.$$

Assumption A.3 (Moving frame). *We assume all functions have zero spatial mean, i.e.*

$$\int_{\mathbb{T}^2} u(x)\,\mathrm{d}x = 0. \tag{A.4}$$

This corresponds to working in a moving frame and allows us to avoid the constant Fourier mode.

For simplicity of notation, we set $\mathcal{Z} := \mathbb{Z}^2 \setminus \{(0,0)\}$.

Definition A.4. *For $\alpha \in \mathbb{R}$ we define*

$$\mathcal{H}^\alpha := \mathcal{H}^\alpha(\mathbb{T}^2) := \left\{ u = \sum_{k\in\mathcal{Z}} u_k e_k : \sum_{k\in\mathcal{Z}} \mu_k^\alpha\, |u_k|^2 < \infty,\ \int_{\mathbb{T}^2} u(x)\,\mathrm{d}x = 0 \right\}, \tag{A.5}$$

equipped with norm

$$\|u\|_{\mathcal{H}^\alpha} := \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^2} = \left(\sum_{k\in\mathcal{Z}} \mu_k^\alpha |u_k|^2\right)^{\frac{1}{2}}. \tag{A.6}$$

Remark A.5 (Poincaré's inequality)**.** *Due to* (A.4)*, Poincaré's inequality holds on* $\mathbb{T}^2$*, i.e. for each* $p \in [1,\infty)$ *and* $u \in W^{1,p}(\mathbb{T}^2)$ *we have*

$$\|u\|_{L^p(\mathbb{T}^2)} \leqslant C\|\nabla u\|_{L^p(\mathbb{T}^2,\mathbb{R}^2)}.$$

In particular, the canonical Sobolev norm $\|\cdot\|_{H^1}$ *is equivalent to* $\|\cdot\|_{\mathcal{H}^1}$*.*

Definition A.6 (Vector valued fractional Sobolev spaces)**.** *For* $\alpha \in \mathbb{R}$*,* $p \geqslant 1$ *we define the vector-valued fractional Sobolev spaces*

$$\mathcal{H}^\alpha\left(\mathbb{T}^2,\mathbb{R}^2\right) := \mathcal{H}^\alpha\left(\mathbb{T}^2\right) \times \mathcal{H}^\alpha\left(\mathbb{T}^2\right)$$
$$W^{\alpha,p}\left(\mathbb{T}^2,\mathbb{R}^2\right) := W^{\alpha,p}\left(\mathbb{T}^2\right) \times W^{\alpha,p}\left(\mathbb{T}^2\right)$$

equipped with the norms

$$\|g\|_{\mathcal{H}^\alpha(\mathbb{T}^2,\mathbb{R}^2)} := \|g_1\|_{\mathcal{H}^\alpha(\mathbb{T}^2)} + \|g_2\|_{\mathcal{H}^\alpha(\mathbb{T}^2)}$$
$$\|h\|_{W^{\alpha,p}(\mathbb{T}^2,\mathbb{R}^2)} := \|h_1\|_{W^{\alpha,p}(\mathbb{T}^2)} + \|h_2\|_{W^{\alpha,p}(\mathbb{T}^2)}$$

for $g = \left(g_1, g_2\right) \in \mathcal{H}^\alpha\left(\mathbb{T}^2,\mathbb{R}^2\right)$ *and* $h = \left(h_1, h_2\right) \in W^{\alpha,p}\left(\mathbb{T}^2,\mathbb{R}^2\right)$*.*

Due to our choice of the Sobolev-norm in $\mathcal{H}^\alpha$ it holds that the operator

$$\nabla : \mathcal{H}^\alpha(\mathbb{T}^2) \to \mathcal{H}^{\alpha-1}\left(\mathbb{T}^2,\mathbb{R}^2\right)$$

is an isometric bounded linear operator. This is a direct consequence of integration by parts formula

$$\|\nabla y\|^2_{\mathcal{H}^{\alpha-1}(\mathbb{T}^2,\mathbb{R}^2)} = \langle(-\Delta)y, y\rangle_{\mathcal{H}^{\alpha-1}} = \|(-\Delta)^{\frac{1}{2}}y\|^2_{\mathcal{H}^{\alpha-1}} = \|y\|^2_{\mathcal{H}^\alpha}$$

for $y \in \mathcal{H}^\alpha$. In particular, for the divergence we obtain

$$\|\nabla\cdot g\|_{L^2} = \|\partial_x g_1 + \partial_y g_2\|_{L^2} \leqslant C\|g\|_{\mathcal{H}^1(\mathbb{T}^2,\mathbb{R}^2)},$$

where $g = \left(g_1, g_2\right) \in \mathcal{H}^1\left(\mathbb{T}^2,\mathbb{R}^2\right)$.

A.3. Analytic Semigroup Generated by the Bilaplace Operator

The negative Bilaplace operator $A = -\Delta^2$ with periodic boundary conditions generates — due to its self-adjointness — an analytic semigroup $\left(e^{tA}\right)_{t\geqslant 0}$ on $L^p(\mathbb{T}^2)$ for every $p \in (1,\infty)$, see [68, Section 2.6.]. Using the Fourier representation,

$$e^{tA}u(x) = \sum_{k\in\mathcal{Z}} e^{-t\mu_k^2}u_k e_k(x), \qquad t \geqslant 0,\ x \in \mathbb{T}^2. \tag{A.7}$$

Lemma A.7. *For $\beta > \alpha$ and $t > 0$, the following estimate holds:*

$$\left\| e^{tA} \right\|_{L\left(\mathcal{H}^{\alpha}, \mathcal{H}^{\beta}\right)} \leqslant \left(\frac{\beta - \alpha}{4e} \right)^{\frac{\beta-\alpha}{4}} t^{\frac{\alpha-\beta}{4}}. \tag{A.8}$$

This result follows directly from basic calculus applied to the Fourier expression and the eigenvalues of $-\Delta^2$. For a more detailed overview, see [38, Lemma A.7].

Remark A.8. *In order to achieve integrability at time $t = 0$ in convolution integrals, it is necessary for the condition $\alpha < \beta < \alpha + 4$ to be satisfied.*

Thus,

$$e^{tA} \nabla \cdot : \mathcal{H}^{\alpha-1}\left(\mathbb{T}^2, \mathbb{R}^2\right) \to \mathcal{H}^{\alpha}\left(\mathbb{T}^2\right)$$

is a bounded linear operator with

$$\left\| e^{tA} \nabla \cdot \right\|_{L\left(\mathcal{H}^{\alpha-1}(\mathbb{T}^2, \mathbb{R}^2), \mathcal{H}^{\alpha}\right)} \leqslant \left\| e^{tA} \right\|_{L\left(\mathcal{H}^{\alpha-2}, \mathcal{H}^{\alpha}\right)} \left\| \nabla \cdot \right\|_{L\left(\mathcal{H}^{\alpha-1}(\mathbb{T}^2, \mathbb{R}^2), \mathcal{H}^{\alpha-2}\right)} \leqslant C t^{-\frac{1}{2}}.$$

A.4. Complex Fourier Series on the Two-Dimensional Torus

The notations of this subsection are only used in Chapter 6.

For $u \in L^2(\mathbb{T}^2, \mathbb{C})$, we consider its spatial Fourier series in space

$$u(t,x) = \sum_{k \in \mathbb{Z}^2} u_k(t)\, e_k(x),$$

for each $x = (x_1, x_2) \in \mathbb{T}^2$, where $\{e_k\}_{k \in \mathbb{Z}^2}$ is the standard orthonormal Fourier basis of $L^2(\mathbb{T}^2, \mathbb{C})$ defined by

$$e_k(x) := \begin{cases} C_k & \text{if} \quad k = 0, \\ C_k e^{i\pi x \cdot k} & \text{if} \quad |k| > 0, \end{cases} \quad C_k := \begin{cases} \sqrt{2} & \text{if} \quad k_1 k_2 = 0, \\ 2 & \quad \text{otherwise}, \end{cases}$$

for $k \in \mathbb{Z}^2, x \in \mathbb{T}^2$. Then $(e_k)_{k \in \mathbb{Z}^2}$ forms an orthonormal basis of $L^2(\mathbb{T}^2, \mathbb{C})$.
For $f \in L^1(\mathbb{T}^2, \mathbb{C})$ and $k \in \mathbb{Z}^2$ we define Fourier transform $\mathscr{F}$ by

$$\mathscr{F} f(k) := \hat{f}(k) := \int_{\mathbb{T}^2} e^{-2\pi i k \cdot x} f(x) dx,$$

For the semigroup $(e^{tA})_{t \geq 0}$ with kernel $(p_t)_{t \geq 0}$ and its projection $(e^{tA} P_N)_{t \geq 0}$ we define

$$e^{tA} u(x) := \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} e^{-t\mu_k^2} u_k e_k(x), \quad t \geqslant 0, \quad u_k := \langle u, e_k \rangle_{L^2(\mathbb{T}^2)}. \tag{A.9}$$

One can equivalently write $e^{tA}u = p_t * u$, where

$$p_t(x) := \mathscr{F}^{-1}\left(e^{-\frac{t}{2}|\pi\cdot|^4}\right)(x) = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} e^{-t\mu_k^2} e_k(x). \tag{A.10}$$

A.5. Operator Theory

This section follows [27, Appendix C].

Definition A.9 (Hilbert-Schmidt operator)**.** *A linear operator $\phi : \mathcal{H} \to \mathcal{K}$ between separable Hilbert spaces is called* Hilbert-Schmidt*, if for any basis $(e_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ one has*

$$\|\phi\|^2_{\mathrm{HS}(\mathcal{H},\mathcal{K})} := \sum_{k\in\mathbb{N}} \|\phi e_k\|^2_{\mathcal{K}} = \mathrm{tr}\left(\phi^*\phi\right)$$

We denote the space of Hilbert-Schmidt operators by $\mathrm{HS}(\mathcal{H},\mathcal{K})$*.*

Remark A.10 (Properties of Hilbert-Schmidt operators)**.** *A Hilbert-Schmidt operator $\phi : \mathcal{H} \to \mathcal{K}$ between separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ fulfills the following properties:*

- *$\phi^*\phi$ is always a nonnegative self-adjoint operator.*
- $\mathrm{HS}(\mathcal{H},\mathcal{K}) \subset L(\mathcal{H},\mathcal{K})$*, i.e. ϕ is continuous as we have*

  $$\|\phi u\|_{\mathcal{K}} \leqslant \|\phi\|_{L(\mathcal{H},\mathcal{K})}\|u\|_{\mathcal{H}} \leqslant \|\phi\|_{\mathrm{HS}(\mathcal{H},\mathcal{K})}\|u\|_{\mathcal{H}}$$

  for each $u \in \mathcal{H}$.
- $(\mathcal{H}, \langle . \rangle_{\mathrm{HS}})$ *is a Hilbert space, where the scalarproduct is defined as follows*

  $$\langle \phi_1, \phi_2\rangle_{\mathrm{HS}} = \sum_{k\in\mathbb{N}} \langle \phi_1 e_k, \phi_2 e_k\rangle_{\mathcal{K}}.$$

- *Hilbert-Schmidt operators are compact, i.e. for each $T \in \mathrm{HS}(\mathcal{H},\mathcal{K})$ there is a sequence of operators $(T_n)_{n\in\mathbb{N}} \subset \mathrm{HS}(\mathcal{H},\mathcal{K})$ of finite-dimensional range and finite-dimensional domain such that $\|T_n - T\|_{\mathrm{HS}} \xrightarrow[n\to\infty]{} 0$ holds.*

A.6. Semigroups of Linear Operators

A more comprehensive introduction to semigroups can be found in [68], which serves as a primary source of inspiration for this chapter. Throughout this section X will be a Banach space.

Definition A.11 (Strongly continuous semigroup). *A semigroup $(T(t))_{t\in[0,\infty)}$ of bounded linear operators on the Banach space X is called* strongly continuous semigroup *or simply* C_0-semigroup *if*

- $T(0) = \mathrm{Id}$,
- $T(t+s) = T(t)T(s)$ *for each* $t, s \geqslant 0$,
- $\lim\limits_{t\searrow 0} \|T(t)x - x\|_X = 0$ *holds for each* $x \in X$.

Theorem A.12 (Hille–Yosida). *A linear (unbounded) operator A in X is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ that satisfies $\|T(t)\| \leqslant Me^{\omega t}$ for $M > 0$ and $\omega \in \mathbb{R}$ if and only if*

- *A is closed and $\overline{D(A)} = X$.*
- *Every real $\lambda > \omega$ belongs to the resolvent set $\rho(A)$ of A and*
$$\left\|(\lambda \,\mathrm{Id} - A)^{-1}\right\| \leqslant \frac{M}{\lambda - \omega}$$
holds for each $\lambda > \omega$.
- $\lim\limits_{t\searrow 0} \frac{T(t)-I}{t} = A$.

Definition A.13 (Uniformly continuous semigroup). *A semigroup $(T(t))_{t\in[0,\infty)}$ of bounded linear operators on X is called* uniformly continuous semigroup *if*

$$\lim_{t\searrow 0} \|T(t) - \mathrm{Id}\,\|_{L(X)} = 0$$

Theorem A.14. *Let $(T(t))_{t\in[0,\infty)}$ be a C_0-semigroup and let A be its infinitesimal generator. Then*

(a) For each $x \in X$ we have $\lim\limits_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)x \,\mathrm{d}s = T(t)x$.

(b) For each $x \in D(A)$ we have $T(t)x \in D(A)$ and $\frac{\partial}{\partial t}T(t)x = AT(t)x = T(t)Ax$

(c) For each $x \in D(A)$ we have $T(t)x - T(s)x = \int_s^t T(\tau)Ax \,\mathrm{d}\tau = \int_s^t AT(\tau)x \,\mathrm{d}\tau$.

The following theorem covers a few important results from [68, Chapter 2].

Theorem A.15. *Let $-A$ be the infinitesimal generator of an analytic semigroup $\left(e^{tA}\right)_{t\geqslant 0}$ on a Banach space X. If $0 \in \rho(A)$ holds, then we obtain the following properties.*

- *There is a constant $\delta > 0$ such that $-A+\delta$ is still an infinitesimal generator of an analytic semigroup.*
- *$e^{tA} : X \to D(A^\alpha)$ for every $t > 0$ and $\alpha \geqslant 0$.*
- *For every $x \in D(A^\alpha)$ we have $e^{tA}A^\alpha x = A^\alpha e^{tA}x$.*
- *For every $t > 0$ and $\alpha \geqslant 0$ the operator $A^\alpha e^{tA}$ is bounded and*
$$\left\|A^\alpha e^{tA}\right\|_X \leqslant M_\alpha t^{-\alpha} e^{-\delta t}. \tag{A.11}$$
- *Let $0 < \alpha \leqslant 1$ and $x \in D\left(A^\alpha\right)$ then*
$$\left\|e^{tA}x - x\right\|_X \leqslant C_\alpha t^\alpha \left\|A^\alpha x\right\|_X. \tag{A.12}$$

Definition A.16 (Dissipativity)**.** *A linear operator A is called* dissipative *if for every $x \in D(A)$ there is a $x^* \in F(x)$ such that* $\operatorname{Re}\left(\langle x^*, Ax\rangle_{X^*,X}\right) \leqslant 0$*, whereby*
$$F(x) := \left\{x^* \in X^* \mid \langle x^*, x\rangle_{X^*,X} = \|x\|_X^2 = \|x^*\|_{X^*}^2\right\}.$$

Remark A.17. *By the integration by parts formula for $u \in \mathcal{H}^1$ we obtain*
$$\langle \Delta u, u\rangle_{L^2} = -\langle(-\Delta)u, u\rangle_{L^2} = -\|\nabla u\|_{L^2(\mathbb{T}^2,\mathbb{R}^2)}^2 \leqslant 0$$
and obtain therefore that the Laplace operator is in fact dissipative. In particular, the operators Δ and $-\Delta^2$ are energy absorbing.

The following definition can be found in [33, Section 2.2.]:

Definition A.18 (Heat semigroup)**.** *We denote by $(\mathcal{P}_t)_{t\geq 0}$ the heat semigroup on $\mathbb{R}$ acting on bounded Lebesgue measurable functions $\phi \in L^\infty(\mathbb{R})$, i.e.*
$$(\mathcal{P}_t\phi)(z) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4t}} \phi(y)\,\mathrm{d}y,$$
for each $z \in \mathbb{R}$ and $t > 0$.

A.7. Inequalities, Sobolev Embeddings and Integral Transformations

The following lemma from [47, Lemma 7.1.1.] provides an explicit upper bound for the growth rate of solutions, even in the presence of a singularity at time t within the integrand.

Lemma A.19 (Henry–Grönwall lemma)**.** *Suppose $b \geqslant 0, \beta > 0$ and a is a nonnegative function locally integrable on $[0,T]$ (some $T \leqslant +\infty$), and suppose f is nonnegative and locally integrable on $0 \leqslant t \leqslant T$ with*

$$f(t) \leqslant a(t) + b\int_0^t (t-s)^{\beta-1} f(s)ds$$

for each $0 \leqslant t < T$. Then

$$f(t) \leqslant a(t) + \theta \int_0^t E'_\beta(\theta(t-s))a(s)ds, \tag{A.13}$$

holds for each $0 \leqslant t < T$, where

$$\theta = b(\Gamma(\beta))^{\frac{1}{\beta}}, \quad E_\beta = \sum_{n=0}^{\infty} \frac{z^{n\beta}}{\Gamma(n\beta+1)}, \quad E'_\beta = \frac{\partial}{\partial z}E_\beta(z).$$

Particularly, if $a(t) \equiv a$ is constant, we obtain

$$f(t) \leqslant aE_\beta(\theta t).$$

Remark A.20. *In order to obtain Lemma A.19 on $[0,T]$ instead of $[0,T)$, we must require that f is continuous at T.*

The following two Sobolev inequality theorems can be found in [21, (1.1)] and [21, Theorem 1.].

Theorem A.21 (Gagliardo–Nirenberg interpolation inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be either the whole space, a half space or a bounded Lipschitz domain. Let $1 \leqslant q \leqslant \infty$, $j, m \in \mathbb{N}$ such that $j < m$. Furthermore, let $1 \leqslant r \leqslant \infty$, $p \geqslant 1$ and $\theta \in [0,1]$ such that*

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\theta}{q}, \qquad \frac{j}{m} \leqslant \theta \leqslant 1$$

hold. Then there is a constant $C = C\left(j,m,n,q,r,\theta,\Omega\right) > 0$ such that for every $u \in L^q\left(\Omega\right)$ such that $D^m u \in L^r\left(\Omega\right)$ and arbitrary $\sigma \geqslant 1$

$$\left\|D^j u\right\|_{L^p(\Omega)} \leqslant C\left\|D^m u\right\|^{\theta}_{L^r(\Omega)} \left\|u\right\|^{1-\theta}_{L^q(\Omega)} + C\left\|u\right\|_{L^\sigma(\Omega)}$$

with the exceptional case that if $r > 1$ and $m - j - \frac{n}{r} \in \mathbb{N}$ holds, then the additional assumption $\frac{j}{m} \leqslant \theta < 1$ is needed.

Theorem A.22 (Brezis-Mironescu inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be either the whole space, a half space or a bounded Lipschitz domain. Let $1 \leqslant p, p_1, p_2 \leqslant \infty$, $s, s_1, s_2 \geqslant 0$. Furthermore, let $\theta \in (0,1)$ such that*

$$s_1 \leqslant s_2, \quad s = \theta s_1 + (1-\theta) s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{(1-\theta)}{p_2}$$

holds. Then the following two conditions are equivalent:

- *There is a constant $C = C(p, p_1, p_2, s, s_1, s_2, \theta, \Omega) > 0$ such that for each $u \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega)$ we have*

$$\|u\|_{W^{s,p}(\Omega)} \leqslant C \|u\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|u\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}.$$

- *At least one of the following conditions is not fulfilled*

$$\begin{cases} s_2 & \in \mathbb{N}^*, \\ p_2 & = 1, \\ 0 & < s_2 - s_1 \leqslant 1 - \frac{1}{p_1}. \end{cases}$$

The Sobolev embedding given below is presented in [33, p.283].

Theorem A.23 (Morrey's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1-boundary $\partial\Omega$, $\alpha \in (0,1]$ and $1 \leqslant p < \infty$, with $\alpha p > n$. Then, for $\gamma := \alpha - \frac{n}{p} > 0$, every $u \in W^{\alpha,p}(\Omega)$ has a representative $u^* \in C^{0,\gamma}(\overline{\Omega})$.*
Then there exists a constant $C = C_{\alpha,p,n,\Omega} > 0$ such that

$$\|u^*\|_{C^{0,\gamma}(\overline{\Omega})} \leqslant C \|u\|_{W^{\alpha,p}(\Omega)}.$$

The following key inequality originates from [73, A.1].

Theorem A.24 (Minkowski's integral inequality)**.** *Let (S_1, μ_1) and (S_2, μ_2) be two σ-finite measure spaces and $f : S_1 \times S_2 \to \mathbb{R}$ be measurable. Then for each $p \in [1, \infty)$ we have*

$$\left[\int_{S_2} \left| \int_{S_1} f(x,y) d\mu_1(x) \right|^p d\mu_2(x) \right]^{\frac{1}{p}} \leqslant \int_{S_1} \left[\int_{S_2} |f(x,y)|^p \, d\mu_2(x) \right]^{\frac{1}{p}} d\mu_1(x).$$

Theorem A.25 (Stochastic Fubini theorem, [27, Theorem 4.18])**.** *Let $(E, \mathcal{E}, \mu)$ be a measurable space with finite positive measure μ and W be a Q-Wiener process (see Definition 2.6). For $T > 0$ let*

$$\begin{aligned} \phi : ([0,T] \times \Omega \times E\,, \mathcal{B}([0,T] \times \Omega) \times \mathcal{B}(E)) \to \left(L_0^2\,, \mathcal{B}\left(L_0^2\right)\right), \\ (t, \omega, x) \mapsto \phi(t, \omega, x) \end{aligned}$$

be a measurable mapping. In particular, for each $x \in E$, $\phi(\cdot, \cdot, x)$ is predictable L_0^2-valued process. Moreover, assume that

$$\int_E \mathbb{E}\left[\int_0^T \operatorname{tr}\left(\phi(t,\omega,x)^* \phi(t,\omega,x)\right) \mathrm{d}t\right]^{\frac{1}{2}} \mathrm{d}\mu(x) < \infty.$$

Then μ-almost surely

$$\int_E \left[\int_0^T \phi(t,x)\,\mathrm{d}W(t)\right] \mathrm{d}\mu(x) = \int_0^T \left[\int_E \phi(t,x)\,\mathrm{d}\mu(x)\right] \mathrm{d}W(t).$$

A.8. Besov Spaces

The following brief introduction to Besov spaces can be found in [3] in more detail.

Definition A.26 (Dyadic partition of unity in $\mathbb{R}^2$)**.** *Let $(\phi_j)_{j \geqslant -1}$ be the standard smooth dyadic partition of unity in $\mathbb{R}^2$. That is, a family of functions $\phi_j \in C^\infty(\mathbb{R}^2)$ for $j \geqslant -1$, where ϕ_{-1} and ϕ_0 are non-negative even functions. Furthermore, the support satisies*

$$\begin{aligned} \operatorname{supp}(\phi_{-1}) &\subset B_{1/2}(0) \\ \operatorname{supp}(\phi_0) &\subset B_1(0) \setminus B_{1/4}(0) \\ \operatorname{supp}(\phi_j) \cap \operatorname{supp}(\phi_j) &= \emptyset, \end{aligned}$$

for $|j - i| > 1$. Moreover, $\phi_j(x) = \phi_0(2^{-j}x)$, $x \in \mathbb{R}^2$ for $j \geqslant 0$ and $\sum_{j \geqslant -1}^{\infty} \phi_j = 1$ for every $x \in \mathbb{R}^2$.

Definition A.27 (Besov space)**.** *Let $\mathcal{S}'(\mathbb{T}^2)$ denote the space of Schwartz distributions on $\mathbb{T}^2$ and $\mathcal{S}(\mathbb{T}^2) := C^\infty(\mathbb{T}^2)$. For $u \in \mathcal{S}'(\mathbb{T}^2)$, we define*

$$\Delta_j : \mathcal{S}'(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2), \quad \Delta_j u := \mathscr{F}^{-1}\left(k \mapsto \phi_j(k) \mathscr{F}(u)(k)\right). \tag{A.14}$$

Then the Besov space $\mathcal{B}_{p,q}^\alpha$ on $\mathbb{T}^2$ for $p, q \in [1, \infty]$, $\alpha \in \mathbb{R}$ is defined as

$$\mathcal{B}_{p,q}^\alpha := \{f \in \mathcal{S}'(\mathbb{T}^2) : \|f\|_{\mathcal{B}_{p,q}^\alpha} := \|(2^{j\alpha} \|\Delta_j f\|_{L^p(\mathbb{T}^2)})_{j \geqslant -1}\|_{l^q} < \infty\}. \tag{A.15}$$

We let $C^\alpha := \mathcal{B}_{\infty,\infty}^\alpha$, for $\alpha \in \mathbb{R}$. Following [3, Chapter 2], we recall that C^α coincides with the classical Hölder continuous function space, for $\alpha \in (0,1)$.

Corollary A.28. *For any $g \in \mathcal{B}_{p,q}^\alpha$, we have*

$$\|\partial^n g\|_{B_{p,q}^{\alpha - n}} \lesssim \|u\|_{B_{p,q}^\alpha}. \tag{A.16}$$

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