Ring and module theory: To the connection between C1 and a modified form of C1 and to the significance of C2 and semi-I-regularity for exchange properties

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Ring and module theory: To the connection between C1 and a modified form of C1 and to the significance of C2 and semi-I-regularity for exchange properties

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Abstract

I introduce a stronger version of CS-modules which lies between CS- and quasi-continuous modules. Moreover I analyze modules with the property C2 that are not necessarily continuous or auto-invariant. Here some results also hold for large restricted modules, which is a weaker property compared to C2. Finally I introduce the definition of semi-I-regular modules as a generalization of the definition of semiregular modules and show that every semi-I-regular module has the finite exchange property and even the exchange property if it further is a module with LE-decomposition or a Utumi module.

Contents

1 Introduction

In this thesis we analyze different algebraic types of modules. The properties we are mostly interested in are different exchange properties. It was already shown that continuous modules do fulfill the strongest of these properties. We call a module continuous if it fulfills the properties $C1$ and $C2$. We want to research these properties on their own.

$C1$-modules are also called CS-modules and have been studied in multiple papers. Whilst researching for this thesis, it turned out that one source defined $C1$ not exactly the same as other sources. After some calculations it proved that these definitions are indeed not the same, unless the property $C3$ (a weaker version of $C2$) is fulfilled.

$C2$-modules have not been studied as much. We will see them being large restricted, a property introduced by Kasch and Mader. As a result it can be shown that they fulfill one of the exchange properties assuming some ideals coincide.

Notably there are three ideals we are interested in when studying the lesser known exchange properties. Those ideals are the singular ideal, the (Jacobson-)radical and the total.

An important connection between the radical, total and those exchange properties has already been established. On this ground we generalize the definition of semiregular modules and see further connections between those semi-I-regular modules and the exchange properties.

Using this we also get a result on the exchange property for Utumi modules, which belong to the most recent research in this field of algebraic theory. This result lets us prove the exchange property for Utumi modules such as continuous modules way faster than previously.

In the chapters 2 and 3 we will be taking a look at the basics. Chapter 2 will give some general ring- and module-theoretic definitions in its first section and introduce the aforementioned singular ideal, the radical as well as the total. Chapter 3 will focus on the exchange properties and, in its second section, reference some well-known examples where they coincide such as modules with LE-decompositions.

Chapter 4 will introduce and analyze the properties $C1$ to $C3$. In the first section we will be taking a closer look at modules with $C1$ and its connection to the radical. In the second and third section our focus will mainly shift to the property $C2$. We will first see some rings always fulfilling this property and second see its connection to automorphism-invariant and large restricted modules. In the last section we come back to $C1$ modules and introduce $C1*$ modules as a stronger version of them.

Chapter 5 contains our main theorems. In the first section we introduce semi-I-regular modules as a generalized definition of semiregular modules. We will prove that every semi-I-regular module has the finite exchange property if I is a submodule of the radical. Notably, in some cases, this even implies the exchange property. Those cases are modules with LE-decomposition as well as Utumi modules, which we will introduce in the second section. Another main theorem will be the equivalence of the $B2$-exchange property and the finite exchange property for semi-Δ-regular modules where Δ denotes the singular ideal.

In the final chapter 6 a very large example will be analyzed. In particular, it is an example of a module that fulfills $C2$, but is of no known type that would imply $C2$. Furthermore the module has the exchange property and its singular ideal is not equal to its radical, showing that their equality is no necessity for modules with any of the exchange properties. Moreover it is no Utumi module, therefore we see that $C2$ does not imply being a Utumi module.

2 Necessary definitions and theorems

First of all, we need some basis to talk about. The reader should be familiar with the notions of rings and modules. Rings specifically will be assumed to have a unit 1 and to not necessarily be commutative. As such we need to differentiate between right and left modules. For the entirety of this thesis we will be looking at right modules and thus often times omit to specify modules as right modules.

2.1 General defintions

With this in mind, let us go through the first definitions: the first notion will be that of a summand of a module, acting as a subspace or basis would for a vector space. This definition is essential for this thesis.

Definition 2.1.1.
Let M be a module and $N \subseteq M$ a submodule. Then we call N a *(direct) summand of* M if there exists another submodule $N_0 \subseteq M$ with $N + N_0 = M$ and $N \cap N_0 = 0$. We then write $N \oplus N_0 = M$ and $N \subseteq^{\oplus} M$.

If we only consider rings, we can classify summands even better. They are exactly those ideals which are generated by an idempotent (that is an element e with $e^2 = e$).

Theorem 2.1.2
Let $I \subseteq^{\oplus} R_R$ be a summand (where R is a ring). Then there exists an idempotent $e \in R$ with $I = eR$. Conversely, if e is idempotent, eR is a summand.

Proof. A stronger version of this can be found in [MaR, Thm. 7.2.3]. For the second part, we have $e(1-e) = (1-e)e = 0$ due to e being idempotent and since $1 = e + (1-e)$ always holds, we get $R = eR \oplus (1-e)R$. Notable is that if e is idempotent, so is $1-e$. □

An interesting type of modules are the square-free modules, which we will use as examples later on.

Definition 2.1.3.
Let M be a module.

- We call M a *square* if $M \simeq N \oplus N$ for some module N.
- We call M a *square-free* module if none of its submodules (except 0) is a square.

Example 2.1.4
A common example of a square-free module is $\mathbb{Q}_\mathbb{Z}$. *Assume there is a submodule* $0 \neq M \subseteq \mathbb{Q}$ *with* $0 \neq N_0, N_1 \subset M$, $N_0 \simeq N_1$ *and* $N_0 \oplus N_1 = M$. *In particular we have* $N_0 \cap N_1 = 0$ *and* $N_0, N_1 \subset \mathbb{Q}$. *Now let* $0 \neq q_0 = \frac{a_0}{b_0} \in N_0$ *and* $0 \neq q_1 = \frac{a_1}{b_1} \in N_1$. *Then we have* $a_0 a_1 = q_0 b_0 a_1 \in N_0$ *and* $a_0 a_1 = q_1 b_1 a_0 \in N_1$, *a contradicition, as this implies* $0 \neq a_0 a_1 \in N_0 \cap N_1 = 0$.

Next up, we want to take a look at essential and superfluous modules. The first one in particular will be used in quite a lot of important definitions later on.

Definition 2.1.5.
Let M be a module and $N \subseteq M$ a submodule.

- N is called *essential in* M if for every $0 \neq N_0 \subseteq M$ we have $N \cap N_0 \neq 0$. We write $N \subseteq^* M$.
- N is called *superfluous in* M if for every $M \neq N_0 \subseteq M$ we have $N + N_0 \neq M$. We write $N \subseteq^\circ M$.

If it is clear which module M we operate in, we simply say N is *essential* or *superfluous*, respectively.

The following is an immediate result, however, since it is used a lot in this thesis, we will give a formal proof once:

Lemma 2.1.6
The only essential summand of a module M *is* M *itself.*

Proof. Let N be an essential summand of M. Thus there exists N_0 with $N + N_0 = M$ and $N \cap N_0 = 0$. Since N is essential, the second equation implies $N_0 = 0$, hence the first equation becomes $M = N + N_0 = N + 0 = N$. □

For the property $C1$, which will be introduced later, we also need the notion of *complements*. Do note that we will only look at *intersection complements*, as there is also the dual notion of addition complements, which should not be confused.

Definition 2.1.7.
Let M be a module and $N \subseteq M$ a submodule. We call $A \subseteq M$ an *intersection complement* (or simply *complement*) of N if it fulfills these two properties:

- $A \cap N = 0$.
- A is maximal in the first property, that is for any $\tilde{A}$ with $A \subseteq \tilde{A}$ and $\tilde{A} \cap N = 0$ we have $A = \tilde{A}$.

We write $A := N^*$. (Do note that complements are not necessarily unique and therefore this notation may lead to the confusion of a unique complement.)

Now one very important property of intersection complements is that they always exist:

Theorem 2.1.8
Let $N \subseteq M$. Then there exists an intersection complement $N^* \subseteq M$ of N.

Proof. Follows simply by Zorn's Lemma. For a complete proof check [MaR, Lem 5.2.3]. $\square$

Due to their maximality we also get a quick result on the sum of a submodule and its complement:

Lemma 2.1.9
Let $N \subseteq M$ be a submodule and N^* a complement. Then $N + N^*$ is essential in M.

Proof. Let us assume that $N + N^*$ is not essential in M. Then there exists a $L \neq 0$ with $L \cap (N + N^*) = 0$. This implies $L \cap N = L \cap N^* = 0$ and therefore $N^* \subsetneq N^* + L$.

Let $a = a_0 + l \in N \cap (N^* + L)$ with $a_0 \in N^*$ and $l \in L$. Then $l = a - a_0 \in L \cap (N + N^*) = 0$, which implies $a = a_0 + 0 = a_0 \in N \cap N^* = 0$. In conclusion $N \cap (N^* + L) = 0$. Since N^* is a complement and $N^* \subseteq (N^* + L)$, we get $N^* = N^* + L$, a contradiction. So our assumption was wrong and $N + N^*$ has to be essential. $\square$

We also get the following lemma, which tells us that we can always choose a complement encompassing another module, as long as it does not intersect the module for which we seek a complement.

Lemma 2.1.10
Let $A, B \subseteq M$ be submodules with $A \cap B = 0$. Then there exists a complement A^* with $B \subseteq A^*$. This especially means there always exists $(N^*)^*$ with $N \subseteq (N^*)^*$. Moreover in this case $N \subseteq^* (N^*)^*$.

Proof. For the proof of the first claims check [MaR, Lem. 5.2.3]. For the last part no proof could be found, thus here is a short proof:

Let $N \subseteq (N^*)^*$ and assume $N \not\subseteq^* (N^*)^*$. Then there is $A \subset (N^*)^*$ with $A \cap N = 0$. We claim that then $(A + N^*) \cap N = 0$, leading to a contradiction.

We will show $(A + N^*) \cap N = 0$ with another contradiction. For this assume $\exists 0 \neq x \in (A + N^*) \cap N$. Then on one hand $x = a + n_0$ with $a \in A$, $n_0 \in N^*$ with $a, n_0 \neq 0$ (otherwise either $A \cap N \neq 0$ or $N^* \cap N \neq 0$) and on the other hand $x \in N \subseteq (N^*)^*$. Since $a \in A \subseteq (N^*)^*$, this implies $0 \neq n_0 = x - a \in (N^*)^*$ which further implies $N^* \cap (N^*)^* \neq 0$, a contradiction.

Thus $(A + N^*) \cap N = 0$. If $A \not\subseteq N^*$ this is a contradiction with the maximality of N^*. On the other hand if $A \subseteq N^*$ we get another immediate contradiction with $A \subseteq (N^*)^*$ and $N^* \cap (N^*)^* = 0$. $\square$

Next up, we want to quickly introduce the annihilator ideal, which we will use for a definition of the singular ideal for rings.

Definition 2.1.11.
Let R be a ring and M an R-module. Let furthermore $X \subseteq M$ be a subset (note: not necessarily submodule). Then we define

$$Ann(X) := \{r \in R | \forall x \in X : x \cdot r = 0\}.$$

This gives an ideal in R, the so called *annihilator ideal.* For any $m \in M$ we also write $Ann(m) := Ann(\{m\})$.

A very well known type of modules are injective modules. Most properties we will work with later on are fulfilled by injective modules. There are quite a few definitions, the reader may find more than the one given here in [CaD, Chap. 1 introduction].

Definition 2.1.12.
Let M be an R-module. We call M an *injective module* if for any R-module N, any submodule $X \subseteq N$ and any homomorphism $\varphi : X \to M$ there is an extension $\psi : N \to M$ with $\psi|_X = \varphi$.

We can even restrict the module A in this definition and thus generalize the definition. Still all of the properties we are interested in will be fulfilled.

Definition 2.1.13.
Let Q be a module. We say Q is *quasi-injective* if for any submodule $X \subseteq Q$ and any homomorphism $\varphi : X \to Q$ we can extend φ to $\psi : Q \to Q$ with $\psi|_X = \varphi$.

Immediately this implies injective modules to always be quasi-injective:

Lemma 2.1.14
Every injective module is quasi-injective.

Proof. Simply choose $N = M$ in the definition of injective modules to get this. $\square$

Going further in the theory of injective modules, there is also the injective hull to consider. As the name suggests this is an injective module in which we can embed our original module. We will mostly need this later on for the definition of automorphism-invariant modules.

Definition 2.1.15.
Let M, N and Q be modules.

- Let $\eta : M \to N$ be a homomorphism. We call η an *essential* homomorphism if $Im(\eta) \subseteq^* N$.
- We call a monomorphism $\eta : M \to Q$ an *injective hull* of M if Q is injective and η is essential.

(Do note that oftentimes Q is also called the *injective hull* of M.)

A very important property of the injective hull is its existence always being guaranteed:

Theorem 2.1.16
Let M be a module. Then there exists an injective hull $\eta : M \to Q$. It is unique up to isomorphisms.

Proof. For the existence check [MaR, Thm. 5.6.4]. For the uniqueness use [MaR, Thm. 5.6.3]. □

We want to conclude this section with two definitions of types of rings we will later on often times see as examples.

The first are regular rings, which we will mostly need for the definition of semiregular and semi-I-regular rings in the following section and chapter 5, respectively. Pay attention that the terminology *regular* has multiple meanings in module-theory. When talking about regular rings in this thesis we are always talking about von Neumann regular rings.

Definition 2.1.17.
Let R be a ring.

- We call $r \in R$ *regular* if there is $s \in R$ such that $r = rsr$.
- We call R *regular* (in the sense of von Neumann) if every $r \in R$ is regular.

The second type of rings we want to consider are clean rings and along with them clean modules. They are very deeply connected with the exchange property, which we will see in the next chapter.

Definition 2.1.18.
We define:

- A ring R is called *clean* if every element $r \in R$ is of the form $r = e + u$ where $e \in R$ is idempotent and $u \in R$ is a unit.
- A module M is called *clean* if $End(M)$ is a clean ring.

2.2 Important (half-)ideals in Hom

In this section we want to look at some important ideals. This is however not the usual notion of an ideal and instead the notion in the category of homomorphisms Hom_R. We assume a fixed ring R and thus simply often write Hom. Now let us first look at the definition of these ideals:

Definition 2.2.1.
Let $\{I(M,N) \subseteq Hom(M,N) | M, N\ R - \text{Modules}\}$. We call $I(\cdot,\cdot)$ a *semi-ideal* if it is closed under composition with any homomorpisms, that is

$$\forall f \in Hom(M_0, M), h \in Hom(N, N_0), g \in I(M,N) : h \circ g \circ f \in I(M_0, N_0).$$

We further call $I(\cdot,\cdot)$ an *ideal* if in addition to it being a semi-ideal, $I(M,N)$ is additively closed for every choice of M,N.

With this, we want to take a look at a very important semi-ideal, the so-called total. It is defined on the basis of partially invertible morphisms, which we first have to define:

Definition 2.2.2.
We call a mapping $f \in Hom(M,N)$ *partially invertible* if it fulfills any of the following equivalent properties:

1) There exists $g \in Hom(N,M)$ such that
$$gf = (gf)^2 \neq 0$$

2) There exists $g \in Hom(N,M)$ such that
$$fg = (fg)^2 \neq 0$$

3) There exists $g \in Hom(N,M)$ such that
$$gfg = g \neq 0$$

4) There are direct summands $0 \neq A \subseteq^{\oplus} M, B \subseteq^{\oplus} N$ such that the restriction
$$\begin{aligned} f|_A^B : A &\to B \\ a &\mapsto f(a) \end{aligned}$$
is an isomorphism.

Proof. The equivalence can be found in [RMT, Lem. II.1.1] □

Now that we have partially invertible morphisms we can define the total.

Definition 2.2.3.
We define the *total* of two modules as

$$Tot(M,N) := \{f \in Hom(M,N) | f \text{ is not partially invertible}\}.$$

Likewise we define the total of one module as $Tot(End(M)) := Tot(M,M) \subseteq End(M)$.

With this, we want to check whether the total is an ideal or at least semi-ideal.

Lemma 2.2.4
The total is a semi-ideal, however, it is generally not an ideal in Hom.

Proof. For proof of it being a semi-ideal check [RMT, Cor. II.1.10]. A well known example for it not being an ideal is $\mathbb{Z}_{\mathbb{Z}}$ as $Tot(\mathbb{Z}_{\mathbb{Z}}) = \mathbb{Z} \setminus \{1,-1\}$.

□

We will come back to the total later to compare it to the following ideals. First up we have a very important ideal, the so-called (Jacobson-)radical. There are definitions for this as a (ring-theoretic) ideal, submodule or (*Hom*-theoretic) ideal, which we will all use at one point or another.

Definition 2.2.5.
Let R be a ring.

1) Given an R-module M, we define its *radical* as

$$\begin{aligned} Rad(M) :&= \sum_{A \subseteq^\circ M} A \\ &= \bigcap \{B | B \subseteq M \text{ maximal}\} \\ &= \bigcap \{Ker(\varphi) | \varphi \in Hom(M, N) \text{ with semisimple } N\} \end{aligned}$$

2) If we consider the ring as a module itself, we also get

$$r \in Rad(R_R) \Leftrightarrow \forall s \in R : 1 - rs \text{ is invertible.}$$

3) Given two modules M, N, we define their radical as

$$\begin{aligned} Rad(M, N) :&= \{f \in Hom(M, N) | \forall g \in Hom(N, M) : gf \in Rad(End(N))\} \\ &= \{f \in Hom(M, N) | \forall g \in Hom(N, M) : fg \in Rad(End(M))\}. \end{aligned}$$

Proof. The equalities from 1) can be found in [MaR, Th. 9.1.1] (for this also recall that we call a module *semisimple* if every submodule is a summand), the proof for 2) as well as the equality from 3) in [RMT, Lem. II.2.1]. □

First up, let us check that $Rad(\cdot)$ is indeed a submodule. This is immediately shown as by the second part of its first definition it is an intersection of submodules which always is a submodule itself.

We can now analyze whether these are good definitions by checking to what extend the two definitions for $Rad(\cdot)$ and $Rad(\cdot, \cdot)$ coincide. In this context we also check this for rings:

Lemma 2.2.6
Given a ring R, we have $Rad(R) = Rad(R_R, R_R)$ when identifying $R = End_R(R_R)$.

Proof. (Stated in [RMT, II. Def. 2.2] without a proof. Let us see a short proof using only the definitions given beforehand.)

Let $f_r \in Rad(R_R, R_R); f(x) = xr$. Then

$$\begin{aligned} r \in Rad(R) &\Leftrightarrow f_r \in Rad(End(R_R)) \\ &\Leftrightarrow \forall g \in End(R_R) : gf_r \in Rad(End(R_R)) \\ &\Leftrightarrow f_r \in Rad(R_R, R_R) \end{aligned}$$

Do note that in the second row " $\Rightarrow$ " follows from $Rad(\cdot)$ being an ideal/submodule and " $\Leftarrow$ " follows from using $g = id$. □

In case of rings we are often times interested whether ideals are contained in their radical. Since they are closed under multiplication with elements of the ring, we can get the answer directly from the ring-centric definition:

Lemma 2.2.7
Let R *be a ring and* $I \subseteq R$ *a (right-)ideal. Then* $I \subseteq Rad(R)$ *if and only if for every* $a \in I$ *we have*

$$1 + a \text{ is (right-)invertible.}$$

Proof. Check [MaR, Lem. 9.3.1]. (Do note that the source proves $1 - a$ instead which is equivalent since I is multiplicatively closed and thus $a \in I \Leftrightarrow (-a) \in I$.) □

While we are on the subject of the radical of a ring, we can now give a generalization of regular rings. In chapter 5 we will expand on this definition even more.

Definition 2.2.8.
Let R be a ring. We call R a *semiregular* ring if it fulfills the following:

- $R/Rad(R)$ is regular and
- we can lift idempotents modulo $Rad(R)$ (that is, for every idempotent $\overline{e} \in R/Rad(R)$ there exists an idempotent $e_0 \in R$ with $\overline{e} = \overline{e_0}$).

We have seen that $Tot(\cdot, \cdot)$ is in general only a semi-ideal, we can however show that Rad is indeed an ideal:

Lemma 2.2.9
$Rad(\cdot, \cdot)$ *is an ideal in* Hom.

Proof. (No proof found but this has allegedly been proven before and follows directly from $Rad(\cdot)$ being a submodule. In the interest of completion a proof will be given here.)

First we have to show that $Rad(\cdot, \cdot)$ is closed under composition. Thus let $f \in Rad(M, N)$, $g \in Hom_R(X, M)$ and $h \in Hom_R(N, Y)$. We have to show (1) $fg \in Rad(X, N)$ and (2) $hf \in Rad(M, Y)$.

(1) Let $i \in Hom_R(N, X)$. We want to show $(fg)i \in Rad(End_R(N))$. Now $(fg)i = f(gi)$ and $f \in Rad(M, N)$ together with $gi \in Hom_R(N, M)$ implies $f(gi) \in Rad(End_R(N))$ directly by the first definition of this radical.

(2) Follows the same when using the second definition.

We have shown $Rad(\cdot, \cdot)$ being closed under composition, but we still have to show it is closed under addition. Hence let $f, g \in Rad(M, N)$. We want to prove $f + g \in Rad(M, N)$ which is to say for any $h \in Hom_R(N, M)$ we have $(f+g)h \in Rad(End_R(N))$. Since $f, g \in Rad(M, N)$, we have $fh \in Rad(End_R(N))$ and $gh \in Rad(End_R(N)$. As $Rad(\cdot)$ is a submodule, we get $fh + gh = (f + g)h \in Rad(End_R(N))$. □

We will need some general properties of the radical later on, therefore we follow this up with two lemmata on the radical.

The first one will be about the behaviour of the radical of a submodule.

Lemma 2.2.10
Let M be a module and $N \subseteq M$ be a submodule. Then $Rad(N) \subseteq Rad(M)$.

Proof. A more general statement can be found in [MaR, Thm. 9.1.4]. □

Our second lemma will be about how the radical interacts with direct sums.

Lemma 2.2.11
Let $M = \bigoplus_{i \in I} M_i$. Then $Rad(M) = \bigoplus_{i \in I} Rad(M_i)$.

Proof. Check [MaR, 9.1.5(c)]. □

Now that we have seen some properties of *Rad* and *Tot*, let us draw a connection between these two:

Lemma 2.2.12
For any choice of modules M, N we have

$$Rad(M, N) \subseteq Tot(M, N).$$

Proof. In [RMT, Thm. II.2.4] it is shown that $Rad(M, N) + Tot(M, N) = Tot(M, N)$. Thus the claim follows from $0 \in Tot(M, N)$ which is easily checked. □

Using these two (semi-)ideals, we can classify rings in a way which will be very helpful when we will be taking a look at different exchange properties in chapter 3.

Definition 2.2.13.
Let R be a ring.

1) We call R a *total ring* if $Tot(R_R)$ is an ideal (which is to say if it is additively closed).
2) We call R a *radicaltotal ring* if $Tot(R_R) = Rad(R)$.

Example 2.2.14
Some examples for radicaltotal rings are:

- *A module is called semisimple if every submodule is a summand. Every semisimple ring R is radicaltotal as $Tot(R) = 0$. Assuming this is not the case, any $0 \neq t \in Tot(R)$ would generate tR, which can not be a summand as otherwise t would be partially invertible.*
- *Moreover every semisimple ring is regular and every regular ring is semiregular. A common result is that every semiregular ring has the exchange property, which, as we will see later, implies that the ring is radicaltotal.*

- *Recall that we call a ring local if it has exactly one one-sided maximal ideal. Every local ring is radicaltotal:*

  Let $\mathfrak{m}$ be the maximal ideal of the local ring R. Then on the one hand we have $\mathfrak{m} = Rad(R)$ since $Rad(R)$ is the intersection of all maximal ideals of R. On the other hand for any $t \in Tot(R)$ we have $t \in tR \subseteq \mathfrak{m}$ since $tR \neq R$. (Otherwise there would be $r \in R$ with $tr = 1$, a contradiction with t not being partially invertible.)

  Thus $Tot(R) \subseteq \mathfrak{m} = Rad(R)$. This implies $Tot(R) = Rad(R)$ since $Rad(R) \subseteq Tot(R)$ is always fulfilled.

We have only seen examples for radicaltotal rings but not for total rings. Nonetheless every one of these also works as an example for a total ring. This is due to the following lemma which is an immediate consequence of the definitions:

Lemma 2.2.15
A radicaltotal ring is also a total ring.

Proof. Follows immediately from the fact that the radical is additively closed. □

On this ground we can ask whether total rings are also radicaltotal. We will later see that this is not the case. For this we first have to see the connection of those two properties and some exchange properties. With these we will give an example for a ring which is total and not radicaltotal in example 3.2.12.

But for now we will introuce another type of ring for which an equivalence to radicaltotal rings can be shown.

Definition 2.2.16.
Let R be a ring. We call R *semipotent* if it fulfills one of the following equivalent properties:

- Every left-ideal that is not contained in $Rad(R)$ contains a nonzero idempotent.
- Every right-ideal that is not contained in $Rad(R)$ contains a nonzero idempotent.
- If $a \notin Rad(R)$, then there exists a $0 \neq x \in R$ with $xax = x$.

Proof. For a proof of the equivalence check [IR, Lem. 1.1]. □

It should be noted that Nicholson used the term $I0-ring$ in [IR] instead. However, over time, the term *semipotent* prevailed and was used in works by Y. Zhou and H. Hakmi. For example Hakmi gave another equivalent definition in [ST, Thm. 2.1] where he also expanded the definition to $I-semipotent$ rings where some ideal I is used instead of $Rad(R)$.

On the other hand Zhou also expanded the definition in [SRT] in another way. In this paper he did not use ring-ideals and the radical of a ring and used ideals in Hom and the radical of two modules instead. In [SRT, Thm. 2.2] he proves a more general version of the following lemma.

Lemma 2.2.17
A ring is radicaltotal if and only if it is semipotent.

Proof. (Also check [SRT, Thm. 2.2]
Let $Rad(R) = Tot(R)$. Furthermore let $r \in R \backslash Rad(R)$. We need to show rR contains a nonzero idempotent. Since $r \in R \setminus Rad(R) = R \setminus Tot(R)$, we know that r is partially invertible, therefore there exists $s \in R$ such that $rs \neq 0$ is an idempotent.

Now let R be semipotent. We already know $Rad(R) \subseteq Tot(R)$ is always true, so we need to show $Tot(R) \subseteq Rad(R)$. For this we will show that if $x \notin Rad(R)$ we also have $x \notin Tot(R)$. Thus let $x \notin Rad(R)$, which implies $xR \nsubseteq Rad(R)$. Since R is semipotent, this implies the existence of an idempotent $0 \neq e = e^2 \in xR$. Hence there is $r \in R$ with $e = xr$ which means x is partially invertible. Finally this is equivalent to $x \notin Tot(R)$. ☐

Next we want to shift our focus from the radical to the last important ideal, the so-called *singular ideal*. Again, we will see that this is contained in the total in lemma 2.2.21, however, over the course of this thesis we will be more intersted in its connections with the radical.

Definition 2.2.18.
Let M, N be modules. Then we define the *singular ideal* of these two as

$$\Delta(M, N) := \{f \in Hom(M, N) | ker(f) \subseteq^* M\}.$$

Given only one module, we sometimes also write $\Delta(M) := \Delta(M, M)$.

If we consider a ring as a module over itself, we can also give another definition of the singular ideal using the annihilator ideal defined in the first section of this chapter.

Lemma 2.2.19
Given a ring R and the module R_R, we also have

$$\Delta(R_R) \simeq \{r \in R | Ann(r) \subseteq^* R\}.$$

Proof. Follows directly from $R \simeq End(R_R) = Hom(R_R, R_R); r \mapsto f_r$ and $ker(f_r) = Ann(r)$. ☐

Having seen the definition, let us check that the singular ideal is indeed an ideal.

Lemma 2.2.20
$\Delta(\cdot, \cdot)$ is an ideal in Hom.

Proof. Check [CaD, Lem. 3.2(1)]. ☐

We want to conclude this section with the fact that, just like the radical, the singular ideal is also contained in the total.

Lemma 2.2.21
Given two Modules M, N, we have

$$\Delta(M, N) \subseteq Tot(M, N).$$

Proof. Follows from equation (7) (p. 17) of [RMT]. ☐

3 Exchange properties

In this chapter we are going to to introduce the different exchange properties. The first section will contain their definitions, some equivalent properties as well as their connection to each other. The second section will introduce LE-decompositions and locally injective modules with the goal of referencing an example to show that two of the exchange properties are indeed not equivalent.

3.1 Three distinct exchange properties

In this section we want to take a look at the different exchange properties. The most known of these is simply called the exchange property while the other two are called the $B2-$ and $d2-$exchange property. As we will see, especially these last two are strongly intertwined with the ideals we saw in the last chapter.

Definition 3.1.1.
Let M be a module. We say M has the *exchange property* (or shortened *EP*) if for every situation

$$M \subseteq^{\oplus} A = \bigoplus_{i \in I} A_i$$

there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus \bigoplus_{i \in I} A_i'$.

If this only holds for I's of a certain cardinality, we get weaker exchange properties, most notably the *finite EP* (where $|I| < \infty$) and the $2-EP$ (where $|I| = 2$).

A ring R is called an exchange ring if R_R has the exchange property.

Example 3.1.2
There are quite a lot of modules that have the exchange property:

- *Mohamed and Müller give one possible proof in [CaD, Thm. 1.21] that every quasi-injective (and thus also every injective) module fulfills it.*
- *A commonly known type of exchange rings are local rings. A proof of this can, for example, be found in [LM, Thm. 12.2].*
- *Nicholson proves in [LI, Prop. 1.8 + Thm. 2.1] that every clean ring is an exchange ring.*
- *Nicholson also proves in [LI, Prop. 1.6 + Thm. 2.1] that every semiregular ring is an exchange ring. We will develop this furhter in chapter 5.*

There is a very important connection between a module having the finite exchange property and its endomorphism ring being an exchange ring. The following theorem can be seen as a main pillar of the theory about the exchange property.

Theorem 3.1.3
Let M be a module. Then it has the finite exchange property if and only if $End(M)$ is an exchange ring.

Proof. Check for example [LI, Cor. 2.2]. ☐

Example 3.1.4
With this and example 3.1.2 we can instantly see that every clean module has the finite exchange property.

In the more recent research papers about the exchange property one can usually only find the distinction between the 2-EP and the EP, omitting the finite EP. This is due to the following theorem, making the distinction obsolete:

Theorem 3.1.5
Let M be a module. Then M has the 2-exchange property if and only if M has the finite exchange property.

Proof. Check for example [LM, Thm. 11.11]. ☐

Until now it is neither proven nor disproven whether the 2-EP and the EP are also equivalent. However, for a lot of types of modules their equivalence was already verified. As an example we have:

Lemma 3.1.6
Let M be a finitely generated module. Then M has the 2-exchange property if and only if it has the exchange property.

Proof. Check [MEE, 1.1]. ☐

As an immediate consequence we get:

Corollary 3.1.7
Let R be a ring. Then it is an exchange ring if and only if R_R has the 2-exchange property.

Proof. Follows from the previous theorem as R_R is finitely generated by 1. ☐

For later use we will reference the following theorem on the exchange property exclusively for rings:

Theorem 3.1.8
Let R be a ring. Then R_R has the exchange property if and only if for every $r \in R$ there exists an idempotent $e \in R$ such that $e \in rR$ and $1 - e \in (1 - r)R$.

Proof. Check [TMR, Lem. II.5.1] for the dual property as well as, for example, [ERD, Cor. 2] to see that this is left-right-symmetric. □

Another common result on the exchange property is the compability with direct sums:

Lemma 3.1.9
Let $M = A \oplus B$. Then M has the exchange property if and only if both A and B have the exchange property.

Proof. Check for example [CaD, Lem. 3.20]. □

Next we come to the second exchange property, the so-called B2-exchange property, introduced by W. Schneider. The 'B' stands for "between" as it was the final exchange property found and lies in the line of implications between the EP and the d2-EP, which we will see later.

Definition 3.1.10.
Let M be a module. We say M has the *B2-exchange property* (*B2-EP*) if for any

$$A = \tilde{M} \oplus N = B \oplus C$$

(where $\tilde{M} \simeq M$) one of the following holds:

1. There exists $B' \subseteq B$ such that
$$A = \tilde{M} \oplus B' \oplus C.$$
2. There exists $C' \subseteq C$ such that
$$A = \tilde{M} \oplus B \oplus C'.$$
3. There exist $0 \neq M' \subseteq \tilde{M}$ and $0 \neq M^* \subseteq \tilde{M}$ as well as $B' \subseteq B$ and $C^* \subseteq C$ such that
$$A = M' \oplus B' \oplus C = M^* \oplus B \oplus C^*.$$

Whilst the property itself is seldom found in other works, the following equivalent property is for example studied in [SRT].

Theorem 3.1.11
A module M has the B2-EP if and only if $Rad(M, M) = Tot(M, M)$.

Proof. Check [TMR, Prop. II.3.5]. □

When remembering the notion of a radicaltotal ring, we see the implication that a module has the B2-EP if and only if its endomorphism ring is radicaltotal.

As was alluded in the introduction of the B2-EP, having the EP implies having the B2-EP as we see in the next theorem:

Theorem 3.1.12
If a module has the finite EP, it also has the $B2$-EP.

Proof. Check [TMR, Prop. II.3.4] in combination with the previous theorem or its source [TMR, Prop. II.3.5]. □

We can also show that having the $B2$-EP does not imply having the exchange property. For this we want to take a look at the following example, which is a simplified version of an example found in [TMR, example in ch. II.5]

Example 3.1.13
(of a module that has the $B2$-EP but not the EP)

Let R be the ring of all series of $\mathbb{Q}$ stagnating in some element of $\mathbb{Z}$, that is

$$R := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}} | \exists n_0 \in \mathbb{N}, z \in \mathbb{Z}\ \forall n \geq n_0 : x_n = z\}$$

with addition and multiplication on its components. (It should be noted that we could use any non-local subring $T \neq 0$ containing 1_F of any field F.)

Firstly we want to show that R fulfills the $B2$-EP by proving $Rad(R) = Tot(R)$. To achieve this we will show $Tot(R) = 0$. Therefore we have to show that for any $0 \neq x \in R$ there is a partial inverse $0 \neq y \in R$ with xy being idempotent. (Note that we usually have to show this for $x \in End(R)$, however $R \simeq End(R)$ since R is a ring.)

Now let $0 \neq x = (x_n)_{n\in\mathbb{N}} \in R$. Thus there is at least one $x_{n_0} \neq 0$. Define $y := (0, \dots, x_{n_0}^{-1}, 0, \dots)$. Then $y \in R$ since it stagnates in $0 \in \mathbb{Z}$ and $xy = (0, \dots, 0, 1, 0, \dots)$ is idempotent due to the componentwise multiplication.

Accordingly $x \notin Tot(R)$ and hence $Tot(R) = 0$, implying $Rad(R) = Tot(R)$ which further indicates R having the $B2$-EP.

Secondly we next want to show that R does not fulfill the EP. For this we will use theorem 3.1.8 which tells us that we have to find one $r \in R$ for which there is no idempotent $e \in R$ with $e \in rR$ and $1_R - e \in (1_R - r)R$.

We choose $r = (3, 3, \dots) \in R$. Then $1_R - r = (-2, -2 \dots) \in R$. Now let us assume there is an idempotent $e = (e_n)_{n\in\mathbb{N}} \in R$ with $e \in rR$ and $1 - e \in (1_R - r)R$, that is $e = r(x_n)_{n\in\mathbb{N}}$ and $1 - e = (1_R - r)(y_n)_{n\in\mathbb{N}}$ for some $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in R$.

Thus, on the one hand we get $r(x_n)_{n\in\mathbb{N}} = (3x_n)_{n\in\mathbb{N}}$ stagnating in some $3x_{n_0}$ and on the other hand by assumption we have $e = r(x_n)_{n\in\mathbb{N}}$ being idempotent, so $3x_{n_0} = (3x_{n_0})^2 = 9x_{n_0}^2$, which only $0 \in \mathbb{Z}$ fulfills. As such e has to be equal to 0.

However, then we get $1_R = 1_R - e = (1_R - r)(y_n)_{n\in\mathbb{N}} = (-2y_n)_{n\in\mathbb{N}}$, a contradiction since $-\frac{1}{2} \notin \mathbb{Z}$.

As we have seen with the EP, the B2-EP is also compatible with direct sums:

Lemma 3.1.14
Let $M = A \oplus B$. Then M has the B2-exchange property if and only if both A and B have the B2-exchange property.

Proof. Check [EPT, Thm. 2.6. 2)] □

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With this we introduce the last exchange property, the d2-EP. ('d' here stands for "direct"/"direct sum".)

Definition 3.1.15.
Let M be a module. We say it has the *d2-exchange property* (*d2-EP*) if for any

$$A = M' \oplus N = C \oplus D$$

(where $0 \neq M' \simeq M_0 \subseteq^{\oplus} M$) one of the following holds:

1. There exist $0 \neq M_1' \subseteq M'$ and $C' \subseteq C$ such that
$$A = M_1' \oplus C' \oplus D.$$
2. There exists $0 \neq M_2' \subseteq M'$ and $D' \subseteq D$ such that
$$A = M_2' \oplus C \oplus D'.$$

We have seen that the B2-EP is equivalent to the equality of the radical and the total. For the d2-EP a similar equivalence has been proven.

Theorem 3.1.16
A module M has the $d2 - EP$ if and only if $Tot(M, M)$ is additively closed, that is if $Tot(M, M)$ is an ideal.

Proof. Check [TMR, Thm. II.3.3]. □

An immediate result is the B2-EP indeed implying the d2-EP as was claimed before.

Corollary 3.1.17
If a module has the $B2$-EP, it also has the $d2-EP$.

Proof. If the module M has the $B2$-EP, as stated in Theorem 3.1.11, we have $Rad(M, M) = Tot(M, M)$. Since $Rad(M, M)$ is an ideal, it is additively closed, thus $Tot(M, M)$ is additively closed which by the previous theorem implies the $d2$-EP. □

We have seen so far that the EP as well as the B2-EP are compatible with direct sums. This has yet to be proven or disproven for the d2-EP. A result that was already achieved in this direction is the following:

Lemma 3.1.18
Let $M = A \oplus B$. If A has the B2-EP and B has the d2-EP, then M has the d2-EP.

Proof. Check [EPT, Thm. 2.7]. □

Later on in the second section we will also see an example where a module has the $d2$-EP whilst not having the $B2$-EP. For this check example 3.2.12 in combination with the theorems 3.2.5 ($d2$-EP) and 3.2.2 ($B2$-EP equal to EP for this type of module).

3.2 LE-decompositions and locally injective modules

In this section we want to take a look at modules which have LE-decompositions, a type of module which has been thoroughly researched on the topic of exchange properties. We also want to introduce locally injective modules as defined by F. Kasch, which are, as of now, barely researched beyond his and Maders work. Some further research was done by Zhou in [SRT].

We will start the section with LE-modules and modules with LE-decompositions.

Definition 3.2.1.
Let M be a module. Then we call M a

- *LE-module* if $End(M)$ is a local ring.
- *module with LE-decomposition* if $M = \bigoplus_{i \in I} M_i$ where every M_i is a LE-module.

As was already said for modules with LE-decompositions, the question on the exchange property has already been researched quite a lot. A quite impactful result is the following theorem.

Theorem 3.2.2
Let M be a module with an LE-decomposition. Then the following are equivalent:

1. *M has the exchange property.*
2. *M has the finite exchange property.*
3. *M has the $B2$-exchange property.*

Proof. Check [LM, Thm. 14.22] for the equivalence of 1, 2 and the fact that $Tot(M, M) = Rad(M, M)$, which, by theorem 3.1.11, is equivalent to having the $B2$-exchange property. $\square$

Let us now introduce locally injective modules, which we will use together with LE-decompositions to generate an example of a module with d2-EP and without B2-EP.

Definition 3.2.3.
A module M is called *locally injective* if for every submodule $A \subseteq M$ that is not essential in M, there exists a non-zero, injective submodule $Q \subseteq M$ with $A \cap Q = 0$.

Whilst this definition validates the term locally injective way better than the following equivalent property proven by Kasch and Mader, we actually prefer working with the equivalent property due to the significance of the singular ideal and the total.

Theorem 3.2.4
A module M is locally injective if and only if for any other R-rightmodule N

$$\Delta(M, N) = Tot(M, N).$$

Proof. Check [RMT, Thm. III.2.2]. ☐

Therefore we immediately see that every locally injective module fulfills the $d2$-EP and thus we do not need to prove it in our example later on.

Corollary 3.2.5
Every locally injective module M fulfills the $d2$-exchange property.

Proof. By the previous theorem 3.2.4 we instantly get $\Delta(M,M) = Tot(M,M)$, which implies that $Tot(M)$ is an ideal. Thus, by theorem 3.1.16, M fulfills the $d2$-EP. ☐

We have seen that the question whether the sum of two modules which have the d2-EP also has the d2-EP is still open. At least for locally injective modules this holds due to the following theorem.

Theorem 3.2.6
In general we have:

- *Let M, N be locally injective. Then $M \oplus N$ is locally injective as well.*
- *Let $M = A \oplus B$ be locally injective. Then A and B are also locally injective.*

Proof. Check [RMT, Thm. III.4.2]. ☐

In general an infinite sum of injective modules does not need to be injective. Kasch and Mader did, however, prove that the sum is at least locally injective. We will use this later when we have some decompositions into injective modules.

Theorem 3.2.7
Let $M = \bigoplus_{i\in I} M_i$ where every M_i is injective. Then M is locally injective.

Proof. Check [RMT, Prop. III.3.1]. ☐

Whilst the following could be proven from the definition alone, we want to simply formulate it as a corollary to the previous theorem.

Corollary 3.2.8
Every injective module is locally injective.

Proof. Follows instantly when we choose I as 1-elemental set. ☐

Kasch and Mader checked in [MaR] under which cirumstances $\Delta = Rad = Tot$ hold. Inspired by that we instantly get the following:

Corollary 3.2.9
Let $M = \bigoplus_{i\in I} M_i$ be a decomposition where every M_i is injective. Then if M has the $B2-$exchange property, it also fulfills $\Delta(M,M) = Tot(M,M) = Rad(M,M)$.

Proof. By our previous theorem 3.2.7, M is locally injective and thus by theorem 3.2.4, we have $\Delta(M,M) = Tot(M,M)$. Furthermore, by theorem 3.1.11, we have $Rad(M,M) = Tot(M,M)$ as M has the B2-EP. Consequently $Rad(M,M) = Tot(M,M) = \Delta(M,M)$. □

As we have seen in corollary 3.2.5 every locally injective module fulfills the $d2$-EP. In general this is not true for the $B2$- or the finite exchange property. To see this, we want to look at a well known example introduced by B. Osofsky and extended upon by K.Yamagata. To understand it we have to first introduce locally semi-T-nilpotent families and their connection to modules with LE-decompositions.

Definition 3.2.10.
Let $\{M_i | i \in I\}$ be a family of modules. We call this family *locally semi-T-nilpotent* if for any countable set $\{i_n | n \in \mathbb{N}\} \subseteq I$ with $i_n \neq i_m$ for $n \neq m$ and any family $\{f_n : M_{i_n} \to M_{i_{n+1}}\}$ where no f_n is an isomorphism as well as any $x \in M_{i_1}$ there is a $k \in \mathbb{N}$ such that

$$(f_k \circ f_{k-1} \circ \ldots \circ f_1)(x) = 0.$$

With this definition theorem 3.2.2 can be extended to:

Theorem 3.2.11
Let M be a module with an LE-decomposition $M = \bigoplus_{i \in I} M_i$. Then the following are equivalent:

1. *M has the exchange property.*
2. *M has the $B2$-exchange property.*
3. *The family $\{M_i | i \in I\}$ is locally semi-T-nilpotent.*

Proof. The equivalence of 1) and 2) was already shown in 3.2.2. For the equivalence of 1) and 3) check [LM, Thm. 14.22]. □

Having all the prerequisites, we can take a look at the example. Large parts of it are taken from [MSE, example on page 249] where K. Yamagata proved the module is not locally semi-T-nilpotent. We simply extend from there. (Do note that Zhou also used this example to find a locally injective module with $Rad \neq Tot$ which is the same result as this when using our previous theorems.)

Example 3.2.12
Let $p \in \mathbb{N}$ be prime and consider the p-adic integers

$$\mathbb{Z}_p = \{(z_1 + p\mathbb{Z}, z_2 + p^2\mathbb{Z}, \ldots) | z_i \in \mathbb{Z}/p^i\mathbb{Z} \text{ and } z_i \cong z_j \text{ mod } p^i\mathbb{Z} \text{ for } i \leq j\}.$$

As is well known these form a local ring with the maximal ideal being $p\mathbb{Z}_p = \{(0, z_2, \ldots) | z_i \in \mathbb{Z}/p^i\mathbb{Z}$ and $z_i = z_j$ mod $p^i\mathbb{Z}$ for $i \leq j\}\}$.

Furthermore let $A_p := \{\frac{m}{p^k} | m \in \mathbb{Z}, k \in \mathbb{N}_0\}$ and $\mathbb{Z}_{p^\infty} := A_p/\mathbb{Z} = \{\frac{m}{p^k} | m \in \mathbb{Z}, k \in \mathbb{N}_0\}$.

With these we now construct the ring $R := \mathbb{Z}_p \times \mathbb{Z}_{p^\infty}$ with

- $(\lambda, x) + (\mu, y) = (\lambda + \mu, x + y)$ *and*
- $(\lambda, x) \cdot (\mu, y) = (\lambda\mu, \lambda y + \mu x)$.

(Also found in [GQF, ex. 1].) It is easily checked that this does indeed form a commutative ring with multiplicative neutral element $(1, 0)$. *Notably the ring is also local with maximal ideal* $(p, 0)R$ *and the module* R_R *is injective, which can both be found in [GQF, ex. 1].*

Next let us define our module:

$$M := \bigoplus_{i=1}^{\infty} M_i \text{ with } M_i = R$$

This is a sum of injective modules and thus a locally injective module by theorem 3.2.7. Moreover it is a sum of local rings whose endomorphism rings are themselves and thus M *also has an LE-decomposition. By our previous theorem this means it has the exchange property if and only if* $\{M_i\}_{i\in I}$ *is locally semi-T-nilpotent. We use the counter-example also found in [MSE] to show that it is not locally semi-T-nilpotent:*

Consider the family $f_i : M_i \to M_{i+1}$ *of homomorphisms where*

$$\begin{aligned} f_i : &M_i \to M_{i+1} \\ &x \mapsto x \cdot (p, 0). \end{aligned}$$

Neither of those is an isomorphism as each image is the maximal ideal of the next copy of R. *Moreover we have for any* $n \in \mathbb{N}$

$$\begin{aligned} (f_n \circ \ldots \circ f_1)((1, 0)) &= (1, 0)(p, 0)^n \\ &= (p^n, 0) \neq 0. \end{aligned}$$

Thus $(M_i)_{i\in I}$ *is not locally semi-T-nilpotent and as such does not have the exchange property.*

4 C-modules

In this chapter we want to introduce the properties $C1$ to $C3$. These come from the notion of a *continuous* (or *quasi-continuous*) module. Whilst $C1$-modules have been researched quite a lot both stand alone as well as in combination with $C2$ or $C3$, the same can not be said about $C2$-modules. Hence we want to see some examples of $C2$-modules without $C1$ as well some properties $C2$-modules have. In the last section we will also define the $C1*$ property as a stronger version of $C1$.

4.1 C1-modules

Let us start the first section of this chapter +with the definition of $C1$-modules.

Definition 4.1.1.
A module M is said to be a $C1$- *module* (or *CS-module*) if for every submodule $N \subseteq M$ there exists a summand of M such that N is essential in the summand.

Example 4.1.2
Recall the definition of a semisimple module where every submodule is a summand. Clearly every semisimple module fulfills $C1$ *since every submodule is essential in itself.*

Often times one can also see the following equivalent description of $C1$-modules. It will be important when we introduce $C1*$-modules later on.

Lemma 4.1.3
A module fulfills $C1$ *if and only if every compliment is a direct summand.*

Proof. (This is for example claimed in [CaD, Prop. 2.4], however, no proof is given. For this reason here is a short one.)

Let M fulfill $C1$. Let further $N^* \subseteq M$ be a complement of $N \subseteq M$. Assume N^* is no summand of M, then by $C1$ $N^* \subsetneq^* K \subseteq^{\oplus} M$. Now we get two possibilities for K, both being a contradiction:

- $K \cap N = 0$ is a contradction to the maximality of N^*.
- $K \cap N \neq 0$ implies $N \cap N^* = (K \cap N) \cap N^* \neq 0$ since $N^* \subset^* K$, a contradiction with $N \cap N^* = 0$.

Thus our assumption was wrong and N^* has to be a summand.

Now let every complement be a summand. We have to show that every submodule N is essential in a summand. This is however instantly proven as $N \subseteq^* (N^*)^*$ for some double summand $(N^*)^*$ by lemma 2.1.10. □

In the next section we will introduce (quasi-)continuous modules which are defined by the property $C1$ as well as some other properties. Right now we want to see some result we can show without these other properties.

As the radical is a very important submodule, we want to take a look at how $C1$ impacts it. Due to $C1$ being a property that produces sums, we will use it to separate a module into a sum with summands having good properties for their radicals. Unfortunately we have to restrict ourself a little bit and can only prove this for superfluous radicals. Firstly let us take a look at a more general theorem after which we will include $C1$ in a corollary.

Theorem 4.1.4
Let M be an R module and let $Rad(M)$ be superfluous. Furthermore let $Rad(M) \subseteq L$ where $M = L \oplus N$. Then $Rad(L) = Rad(M)$ and $Rad(N) = 0$.

Proof. As $Rad(M)$ is superfluous in M, we can show that it is also superfluous in L. To see this, let us assume a submodule $U \subseteq L$ with $U + Rad(M) = L$, we want to show $U = L$. By $U + Rad(M) = L$ and $M = L \oplus N$ we can see

$$M = Rad(M) + U + N.$$

Due to $Rad(M)$ being superfluous in M, this further implies $M = U + N$. Now we use $N \cap L = 0$ and $U \subseteq L$ to see $N \cap U = 0$, implying $M = U \oplus N$. Then we have $M = U \oplus N = L \oplus N$ with $U \subseteq L$, proving $U = L$.

As $L \subseteq M$, we also have $Rad(L) \subseteq Rad(M)$ by 2.2.10.

On the other hand we can now easily show $Rad(M) \subseteq Rad(L)$: we have just seen that $Rad(M)$ is superfluous in L and as $Rad(L)$ is the sum of all superfluous submodules of L, this implies $Rad(M) \subseteq Rad(L)$. As such we have $Rad(L) = Rad(M)$.

We still have to show $Rad(N) = 0$. We know by 2.2.11 that $M = L \oplus N$ implies $Rad(M) = Rad(L) \oplus Rad(N)$ which, as we have seen, implies $Rad(M) = Rad(M) \oplus Rad(N)$. This being a sum implies $Rad(N) \subseteq Rad(M)$ and, as it is a direct sum, $Rad(N) \cap Rad(M) = 0$. Both of these together can only be true if $Rad(N) = 0$.

□

With this we will next add $C1$. This allows us to guarantee the second condition in the theorem being fulfilled.

Corollary 4.1.5
Let M be an R-module with superfluous radical. Furthermore let M fulfill $C1$. Then there exists a sum $M = L \oplus N$ with $Rad(M) = Rad(L)$ and $Rad(N) = 0$. Furthermore $Rad(M)$ is essential in L.

Proof. As M fulfills $C1$ and $Rad(M)$ is a submodule of M, there exists a summand $L \subseteq^{\oplus} M$ in which $Rad(M)$ is essential. We can then use the previous lemma 4.1.4 to show $Rad(M) = Rad(L)$ and $Rad(N) = 0$ (for $M = L \oplus N$).

□

Now the question remains which modules actually fulfill this. One important group will be seen in the following example:

Example 4.1.6
By [MaR, Thm. 9.2.1(c)] the radical of finitely generated modules is superfluous. This especially means for every ring R the module R_R has superfluous radical. As such any ring that fulfills $C1$ can be split into a sum where the radical of one summand is essential and the radical of the other summand is 0.

4.2 C2, C3 and continuous modules

In this next section we will look at the properties $C2$ and $C3$. With these also comes the notion of *(quasi-)continuous* modules. Accordingly the first part of the section will be about properties of (quasi-) continuous modules, whilst the later part will focus solely on modules with $C2$.

Definition 4.2.1.
Let M be a module. Then we define:

1) M is said to fulfill $C2$ if every submodule of M that is isomorphic to a summand of M is a summand itself.

2) M is said to fulfill $C3$ if for any two summands $M_0, M_1 \subseteq^{\oplus} M$ with $M_0 \cap M_1 = 0$ their sum $M_0 \oplus M_1$ is also a summand of M.

If M also fulfills $C1$ in addition to $C2$ or $C3$, we call it a *continuous* or *quasi-continuous* module, respectively.

A general result for these is $C2$ implying $C3$, hence implying that every continuous module is quasi-continuous.

Lemma 4.2.2
If a module fulfills $C2$, it also fulfills $C3$.

Proof. Check [CaD, Prop 2.2].

□

We can furthermore see that (quasi-)continuous modules are generalizations of (quasi-)injective modules and thus (quasi-)injective modules also fulfill $C1$, $C2$ and $C3$:

Theorem 4.2.3
Every (quasi-)injective module is continuous.

Proof. Check [CaD, Prop. 2.1]. □

Another important general result on the properties C1 to C3 is that they are inherited by summands.

Lemma 4.2.4
Let $N \subseteq^{\oplus} M$. If M has $C1$, $C2$ or $C3$, so does N. In particular this means that if M is (quasi-)continuous, so is N.

Proof. Check [CaD, Prop. 2.7]. ☐

A very impactful theorem Müller and Mohamed used in [CaD] to show that continuous modules have the exchange property was the following one about the radical and singular ideal. We will also use this later to prove the same result in a faster way.

Theorem 4.2.5
Let M be a continuous module, $\Delta(M, M)$ be its singular ideal and $Rad(M, M)$ the radical of its endomorphism ring. Then $\Delta(M, M) = Rad(M, M)$, $End(M)/\Delta(M, M)$ is regular and idempotents modulo $\Delta(M, M)$ can be lifted.

Proof. For $\Delta(M, M) = Rad(M, M)$ and $End(M)/\Delta(M, M)$ being regular check [CaD, Prop. 3.5] and for lifting idempotents check [CaD, Lem. 3.7]. ☐

As we said before, Müller and Mohamed proved continuous modules always having the exchange property, a result we can not omit.

Theorem 4.2.6
Every continuous (and thus every (quasi-)injective) module has the exchange property.

Proof. Check [CaD, Thm. 3.24]. ☐

We see that the properties $C1$ and $C2$ together imply the exchange property. The same can, however, not be said if we weaken $C2$ to $C3$, as can be seen in the following example:

Example 4.2.7
A quasi-continuous module does not necessarily have the exchange property.

Proof. A commonly known counter-example is $\mathbb{Z}_{\mathbb{Z}}$. Since any submodule is of the form $x\mathbb{Z}$ and any intersection is of the form $x\mathbb{Z} \cap y\mathbb{Z} = LCM(x; y)\mathbb{Z}$, we get as a first property that the only summands are $\mathbb{Z}$ and 0 and as a second property that all submodules (except 0) are essential.

On one hand, by the first property, we easily see that $C2$ is not fulfilled as we have $\mathbb{Z} \simeq x\mathbb{Z}$ for all $x \neq 0$. Likewise $C3$ is instantly fulfilled.

On the other hand the second property implies that all submodules are essential in the summand $\mathbb{Z}$ (or in the case of 0, essential in itself). Therefore $C1$ is fulfilled and it is a quasi-continous module.

Let us now consider its total. For this we need all endomorphisms which are not partially invertible. Every endomorphism has the form $f_n : \mathbb{Z} \to \mathbb{Z}; x \mapsto nx$ with $n \in \mathbb{Z}$. For each of these we have $Im(f_n) = n\mathbb{Z}$, which is no summand according to the first property when $n \neq 0$ and $n \neq \pm 1$. Neither can we constrict the pre-image to a summand, as by the first property there is no non-trivial summand. Thus no f_n, except f_{-1} and f_1, is partially invertible.

As such $Tot(\mathbb{Z}) \simeq \mathbb{Z} \setminus \{-1; 1\}$ (when identifying $f_n \simeq n$). This is not additively closed, therefore $\mathbb{Z}$ does not have the $d2$-exchange property and hence neither has the exchange property. ☐

In conclusion we have seen that on one hand there are quasi-continuous modules that do not have the exchange property. On the other hand we will see a quasi-continuous module, which is not continuous, and has the exchange property in the next example. In particular this shows having the exchange property does not imply having $C2$.

Example 4.2.8
There are quasi-continuous modules that are not continuous and fulfill the exchange property.

Proof. As noted by Mohamed and Müller in [CaD, Ex. 3.26] any local commutative domain R, which is not a field, fulfills these properties. Let us prove this:

By example 3.1.2 we know that R has the exchange property since it is local.

Let us first prove that R is quasi-continuous. For $C1$ let $A \subseteq R$ be an ideal. If $A = 0$, it is essential in the summand 0, so let us assume $A \neq 0$. Since R is local, the only other summand is R itself, so we have to show that A is essential in R. For this let $0 \neq B \subset R$ be another ideal. Since $A, B \neq 0$ there are $a \in A$, $b \in B$ with $a, b \neq 0$. Then we also have $ab \in A$ and $ab = ba \in B$ since R is commutative. As R is also a domain, we have $ab \neq 0$ and thus $A \cap B \neq 0$.

$C3$ is quickly shown as the only summands of R are 0 and R itself, hence any possible sum (that does not intersect) is again either R or 0.

Now let us show that R does not fulfill $C2$. For this we will show $R \simeq aR$ for any element $0 \neq a \in R$. To do this consider

$$\begin{aligned} f : R &\to aR \\ r &\mapsto ar. \end{aligned}$$

f is a well-defined epimorphism. It is, however, also a monomorphism. To show this, assume $ar = 0$ for some $r \in R$. Then, since $a \neq 0$ and R is a domain, we instantly get $r = 0$. ☐

For a lot of modules, including Utumi modules, which we will introduce later on and which are a generalization of continuous modules, having the exchange property and being clean is closely linked. As such we want to take a quick look at the main result of [CMC].

Theorem 4.2.9
Every continuous module is clean.

Proof. Check [CMC, Thm. 3.9]. ☐

As was already said, a main goal of this thesis was analyzing the properties $C1$ to $C3$ on their own. Herefore having examples is very helpful. Finding examples for modules fulfilling $C1$ but not $C2$ is rather simple, any true quasi-continuous module fulfills this.

However, it is rather difficult to find an example for a $C2$-module that is not continuous. Hence here is one such example:

Example 4.2.10

(A module that is C2 but not C1)

Consider the ring $R := \mathbb{Z}[X]/(4, 2X, X^2)$. *Its structrue of ideals is as follows:*

$$\begin{tikzcd}
 & R & \\
 & (2,X) \arrow[u, hook] & \\
(2) \arrow[ur, hook] & (X) \arrow[u, hook] & (X+2) \arrow[ul, hook] \\
 & 0 \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

We can instantly see that it is a local ring and as such has no non-trivial summands. One one hand, since it is also finite, no submodule can be isomorphic to R *and therefore* $C2$) *is trivially fulfilled. On the other hand the three ideals* $(2), (X), (X+2)$ *are not essential in* R *(since either intersection of two of these is* 0*) and thus there is no direct summand for them to be essential in. Hence the ring does not fulfill* $C1$.

As we have seen, being local made proving $C2$ very quick. It was, however, not even needed in this case. We can prove that any finite, commutative ring fulfills $C2$. For this we will first look at a much more general theorem where we do not need any element excluding idempotents to commute.

Theorem 4.2.11

Let R *be a ring where every idempotent is central. Moreover let* $M \subseteq^{\oplus} R$ *be a summand and* $M \simeq N \subseteq R$. *Then* $N \subseteq M$.

Proof. Let $f : M \overset{\sim}{\to} N$ and let $M = eR$ for some idempotent e (which exists by 2.1.2). Then for any $x \in N$ we have $x = f(y) = f(er) = f(e)r$ and moreover for any $r \in R$ we have $er \in eR = M$, thus $f(e)r = f(er) \in f(M) = N$. In conclusion $N = f(e)R$.

Furthermore we have $e = e^2$, thus $f(e) = f(e^2) = f(e)e = ef(e)$, where the last part uses the fact that all idempotents are central by our prerequisite. Then we have $f(e) = ef(e) \in eR$ which implies $f(e)R \subseteq eR$. We have already established $N = f(e)R$ and $M = eR$, proving $N \subseteq M$. $\square$

This leads to the question which rings even fulfill the property of every idempotent being central. A few examples can be seen in the following.

Example 4.2.12

Rings where every idempotent is central:

- *Rings where every element is central or in other words commutative rings.*
- *Every reduced ring, as stated and proven in [RoQ, Lem 12.2, p.40]. (A ring is called reduced if there are no non-zero nilpotent elements.)*
- *Let M be a square-free module. Then all idempotents of $End(M)/\Delta(M)$ are central by [CaD, Lem. 3.4].*

If we further restrict ourself to finite rings, we can find an even stronger conclusion:

Corollary 4.2.13
Let R be a finite ring where every idempotent is central. Furthermore let $M \subseteq^{\oplus} R$ be a summand and $M \simeq N \subseteq R$. Then $M = N$.

Proof. By the previous lemma 4.2.11 we know that $N \subseteq M$. Since now M and N have the same (finite) number of elements, this immediatly implies $N = M$. ☐

We will now come to the interesting result concerning the property $C2$:

Corollary 4.2.14
Let R be a finite ring where every idempotent is central. Then R_R does fulfill $C2$.

Proof. For this we need some summand $M \subseteq^{\oplus} R$ and a module $N \subseteq R$ with $M \simeq N$. We need to show that N also is a summand. By 4.2.13 we have $N = M$, so it is a summand. ☐

This does also show why we can very quickly check $C2$ for finite, commutative rings:

Corollary 4.2.15
Let R be a finite, commutative ring. Then R_R does fulfill $C2$.

Proof. Follows immediatly from 4.2.14 as any element including any idempotent in a commutative ring is central. ☐

Now we come back to our previous chapter. In particular we want to prove some results on the exchange properties of modules that fulfill $C2$ and are locally injective. In order to do this, we want to extract the following lemma on the radical and singular ideal from another lemma Mohamed and Müller proved in [CaD].

Lemma 4.2.16
Let M be a module, $Rad(M, M)$ the radical of its endomorphism ring and $\Delta(M, M)$ its singular ideal. If M fulfills $C2$, then $\Delta(M, M) \subseteq Rad(M, M)$.

Proof. [CaD, parts of Prop. 3.5]

Let $f : M \to M \in \Delta(M, M)$. Consider $m \in ker(f) \cap ker(Id - f)$. Then $0 = f(m) = Id(m) - f(m)$ implies $m = 0$ and thus $ker(f) \cap ker(Id - f) = 0$. Furthermore $ker(f) \subseteq^{*} M$ holds due to $f \in \Delta(M, M)$. As such we get $ker(Id - f) = 0$.

Moreover $M \simeq Im(Id - f) \oplus ker(Id - f)$ generally holds, so we get $M \simeq Im(Id - f)$. Since $M \subseteq^{\oplus} M$ and M fulfills $C2$, this implies $Im(Id - f) \subseteq^{\oplus} M$.

Next let us prove that $ker(f) \subseteq Im(Id - f)$. This is, however, immediate due to

$$m \in ker(f) \Rightarrow f(m) = 0 \Rightarrow m = Id(m) + f(m) \Rightarrow m \in Im(Id - f).$$

This and $ker(f) \subseteq^* M$ implies $Im(Id - f) \subseteq^* M$. Combined with our previous result we get $Im(Id - f) = M$, as M is the only essential summand in M. As such $Id - f$ is an automorphism, which is to say a unit in $End(M)$.

Since $\Delta(M, M)$ is an ideal, we can analogously prove $Id - fg$ for any $g \in End(M)$. Thus $f \in Rad(End(M)) = Rad(M, M)$.

In conclusion $\Delta(M, M) \subseteq Rad(M, M)$.

□

Following this, we instantly get the next theorem.

Theorem 4.2.17
If a module M fulfills $C2$ and $\Delta(M, M) = Tot(M, M)$, it also has the $B2$-exchange property.

Proof. By lemma 4.2.16 we have $\Delta(M, M) \subseteq Rad(M, M)$, as M fulfills $C2$. Since in general $Rad(M, M) \subseteq Tot(M, M)$, we get

$$\begin{aligned} \Delta(M, M) &\subseteq Rad(M, M) \\ &\subseteq Tot(M, M) \\ &= \Delta(M, M). \end{aligned}$$

In conclusion $Rad(M, M) = Tot(M, M)$, which is equivalent to M having the $B2$-EP.

□

We recall the definition of a locally injective module M and, more importantly, theorem 3.2.4 which implies $\Delta(M, M) = Tot(M, M)$.

Corollary 4.2.18
If a module M is both locally injective and it fulfills $C2$, it also fulfills the $B2$-exchange property.

Proof. By theorem 3.2.4 we have $\Delta(M, M) = Tot(M, M)$ as M is locally injective. Thus the claim follows by the previous theorem 4.2.17. □

As we have seen in example 3.2.12, this is not true if we omit C2. As we know a module that is a sum of injective modules is locally injective, we can improve the previous result if those injective summands also have local endomorphism rings.

Corollary 4.2.19
If $M = \bigoplus_{i \in I} M_i$ is an LE-decomposition where every M_i is injective and M fulfills $C2$, it also fulfills the exchange property.

Proof. As M has a decomposition into injective modules, it is locally injective by theorem 3.2.7. Thus it has the $B2$-EP by our previous corollary 4.2.18. Finally, since M has an LE-decomposition, we know that the $B2$-EP is fulfilled if and only if the exchange property is fulfilled by theorem 3.2.2. □

4.3 Automorphism-invariant and large restricted modules

In this section we want to see two more types of modules that are connected to the C2 property. The first type are so-called automorphism-invariant modules (recently often abbreviated to auto-invariant modules), which also fulfill C2. The second type are large restricted modules, which were studied by Kasch. We will see that every module with C2 is large restricted.

For the automorphism-invariant modules we first want to take a look at the following theorem, which gives a connection between quasi-continuous modules and the injective hull:

Theorem 4.3.1
Let M be a module and $\eta : M \to Q$ be an injective hull. (Since η is a monomorphism, we will assume $M \subseteq Q$.) Then M is quasi-continuous if and only if $f(M) \subseteq M$ for any idempotent $f \in End(Q)$.

Proof. Check [CaD, Thm. 2.8]. □

Now we want to introduce automorphism-invariant modules. As their name suggests, these are modules which are invariant under certain automorphisms, namely those of their injective hull.

Definition 4.3.2.
Let M be a module and $\eta : M \to Q$ be an injective hull. (Since η is a monomorphism, we will assume $M \subseteq Q$.) We call M *automorphism-invariant* if for any automorphism $\phi : Q \to Q$ we have $\phi(M) \subseteq M$.

Automorphism-invariant modules fulfill two very important properties we are interested in. The first of is that they always fulfill C2.

Theorem 4.3.3
Every automorphism-invariant module fulfills $C2$.

Proof. [MSA, Comment on last page]. □

The second property we are interested in is that automorphism-invariant modules always have the exchange property.

Theorem 4.3.4
Every automorphism-invariant module has the exchange property.

Proof. Check [AEP, Thm. 3]. □

On this ground Asensio and Srivastava were able to prove the following, which yet again shows a connection between clean modules and those with the exchange property.

Theorem 4.3.5
Every automorphism-invariant module is clean.

Proof. Check [AEP, Cor. 4]. □

We want to end the part on automorphism-invariant modules with another definition for which it has been shown that it is equivalent to that of automorphism-invariance.

Definition 4.3.6.
Let M be a module. Then we call M *pseudo-injective* if for every $A \subseteq M$ and every monomorphism $\varphi : A \to M$ there is an extension $\psi : M \to M$ with $\psi|_A = \varphi$.

We can clearly see the connection between pseudo-injective and quasi-injective modules. The only difference is that for quasi-injective modules any homomorphism, instead of every monomorphism, needs to be extendable. Thus the following is immediate:

Lemma 4.3.7
Every quasi-injective (and thus every injective) module is pseudo-injective.

Proof. Follows immediately from the definitions. □

This also implies that every (quasi-)injective module is automorphism-invariant through the next theorem.

Theorem 4.3.8
A module is automorphism-invariant if and only if it is pseudo-injective.

Proof. Check [MSA, Thm. 16]. □

Now let us come to the second type of module we want to discuss in this section. These so-called large restricted modules were first introduced by Kasch and Mader in [RMT] and have not been further researched a lot since. However the property can for example be found in [CaD] without a designated name.

Definition 4.3.9.
Let M be a module. We then call M *restricted for large submodules* or simply *large restricted* if and only if every monomorphism $f : M \to M$ with essential image is already an automorphism, that is $Im(f) = M$.

A quick example for large restricted modules is the following:

Example 4.3.10
Let M be a finite module. Then it is large restricted, as any monomorphism between finite modules is an automorphism.

Mohamed and Müller have proven the following in [CaD], which gives a connection between quasi-continuous, continuous and large restriced modules.

Lemma 4.3.11
Let M be a quasi-continuous module. Then it is continuous if and only if it is large restricted.

Proof. Check [CaD, Lem. 3.14]. $\square$

In the proof of Lem. 3.14 the authors write that the "only if" part is obvious, we can however actually specify the statement (as it does not rely on M having C1). Since it is a rather interesting result, we want to give a proof.

Lemma 4.3.12
Let M fulfill C2. Then M is large restricted.

Proof. To show this let $f : M \to M$ be any monomorphism with $Im(f) \subseteq^* M$. We want to show $Im(f) = M$.

Since $M \subseteq^{\oplus} M$ and $f|^{Im(f)} : M \to Im(f)$ is an isomorphism (as f was already a monomorphism), we get by C2 that $Im(f) \subseteq^{\oplus} M$. As such $Im(f)$ is an essential summand, which can only be M itself. Thus $Im(f) = M$. $\square$

As an immediate consequence a lot of the modules we have seen ao far are large restricted:

Corollary 4.3.13
Every module of the following types is large restricted:

- *(quasi-)injective modules*
- *continuous modules*
- *automorphism-invariant modules.*

Proof. Follows immediately as each of these fulfills C2. $\square$

Furthermore we can ask ourselves how important the quasi-continuous part of lemma 4.3.11 is and whether we could show that C2 and large restricted are equivalent. However, we can show that this is not the case with the following counter-example, which can partially be found in [CaD, Ex. 2.9].

Example 4.3.14
(of a large restricted module that is not C2)

We have already seen in example 4.3.10 that every finite module is large restricted. As such any finite module not fulfilling C2 can be seen as an example. We will see later in 4.4.3 that the ring

$$R := \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$$

is not fulfilling $C3$ for any field F. This especially implies that for $F = \mathbb{F}_2$ the module R_R is a finite module that does not fulfill $C2$.

We want to close this section with the following theorem from [RMT] where Kasch and Mader proved a more general result of lemma 4.2.16 for large restricted modules, which by the above is thus true for modules fulfilling C2, as well as some corollaries.

Theorem 4.3.15
Let M be large restricted. Then we have for any R-module N

$$\Delta(M,N) \subseteq Rad(M,N).$$

Proof. Check [RMT, Thm. III.1.2]. ☐

As Kasch and Mader also proved in [RMT], this gives us an immediate result for large restricted locally injective modules.

Corollary 4.3.16
Let M be large restricted and locally injective. Then

$$\Delta(M,M) = Rad(M,M) = Tot(M,M).$$

In particular this also means that M has the B2-EP.

Proof. Follows from the theorems 4.3.15 and 3.2.4. ☐

We can use this to get a result for modules with LE-decompositions that also fulfill these properties.

Corollary 4.3.17
Let $M = \bigoplus_{i\in I} M_i$ be a LE-decomposition. Furthermore let M be locally injective (for example because every M_i is injective) and large restricted. Then M has the EP.

Proof. Follows instantly from theorem 3.2.2, which says that for a module with LE-decomposition the EP and B2-EP are equivalent. ☐

4.4 C1*-modules

We return to the definition of $C1$. As we have seen in 4.1.3, we can describe it as "every complement is a direct summand". One may make the assumption that the sum would be with the usual double-complement. Indeed this is the definition for $C1$ found in [RMT]. We will give this the name of $C1*$ to differentiate between the two definitions. As we will see later on these are indeed not the same.

Definition 4.4.1.
Let M be a module. Then we say M fulfills $C1*$ if for every submodule $N \subseteq M$ and its complement N^* and double-complement N^{**} (with $N \subseteq N^{**}$) we have $M = N^* \oplus N^{**}$.

In the case of (quasi-)continuous modules these two definitions are interchangable, so we can visualize a chain of

$$C1-\text{modules} \supsetneq C1*-\text{modules} \supset \text{quasi-contionuous modules}.$$

Lemma 4.4.2
Let M fulfill $C3$. Then it fulfills $C1$ if and only if it fulfills $C1*$.

Proof. Since $C1*$ immediately implies $C1$ by lemma 4.1.3, we only have to show $C1$ implying $C1*$. As such let M fulfill both $C1$ and $C3$. Now given any submodule $N \subseteq M$ and its (double)-complements N^* and N^{**} (with $N \subseteq N^{**}$), we have $N^* \cap N^{**} = 0$ by definition of them being complements. Since each of them is a complement of some submodule, both of them are summands by $C1$. As such we can use $C3$ to obtain $N^* \oplus N^{**} \subseteq^{\oplus} M$. However by 2.1.9 we also have $N^* \oplus N^{**} \subseteq^* M$. As the only essential summand is M itself, we have $N^* \oplus N^{**} = M$, proving $C1*$. $\square$

Having shown this, the question still remains whether the two definitions could be equivalent. The next example shows this not being the case. Considering our previous lemma, this means it doubles as an example of a module that is $C1$ but not $C3$.

Example 4.4.3
(of a C1-module that is not C1; parts of this also found in [CaD, Ex. 2.9])*
Given a field F, we consider the ring

$$R := \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$$

where we use the usual addition and multiplication for matrices. Then we can compute its submodules (that is its ideals):

First we consider the determinant of a matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, which is ad. If it is not 0, the matrix is invertible and thus any ideal containing any such matrix can only be R itself. Therefore we can consider only matrices whose determinant is 0, that is only matrices where either a or d is 0 as F is a field.

So let $A := \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ be a matrix in an ideal I. In the first case where $d = 0$ and either $a \neq 0$ or $b \neq 0$ we can compute I to be the ideal $\begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$.

In the second case where $a = 0$ and $d \neq 0$ we see that the ideal generated by A is the ideal $\begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ if we choose $b = 0$. In the other subcase where $b \neq 0$ we get the ideal generated by A to be $I_k := F \begin{bmatrix} 0 & 1 \\ 0 & k \end{bmatrix}$ for some $k \in F$. Unlike the first case we get more than one possible ideal and as such can also add any of these up (that is take an ideal generated by more than one matrix) to get the last possible non-trivial ideal $\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$.

To better understand the structure, we can also visualize it:

$$\begin{tikzcd}[ampersand replacement=\&]
 \& \& R \& \& \\
\begin{bmatrix}F & F\\0 & 0\end{bmatrix} \arrow[urr, hook] \& \& \begin{bmatrix}0 & F\\0 & F\end{bmatrix} \arrow[u, hook] \& \& \\
I_0 = \begin{bmatrix}0 & F\\0 & 0\end{bmatrix} \arrow[u, hook] \arrow[urr, hook] \& \& \begin{bmatrix}0 & 0\\0 & F\end{bmatrix} \arrow[u, hook] \& \& I_k = F\begin{bmatrix}0 & 1\\0 & k\end{bmatrix} (k \neq 0) \arrow[ull, hook] \\
 \& \& 0 \arrow[ull, hook] \arrow[u, hook] \arrow[urr, hook] \& \&
\end{tikzcd}$$

Looking at these, we see that $R = \begin{bmatrix}F & F\\0 & 0\end{bmatrix} \oplus \begin{bmatrix}0 & 0\\0 & F\end{bmatrix} = \begin{bmatrix}F & F\\0 & 0\end{bmatrix} \oplus I_k$ *(with* $k \neq 0$*) are the only non-trivial possibilites for direct sums. We also instantly get*

$$\begin{bmatrix}0 & F\\0 & F\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & F\end{bmatrix} \oplus I_k$$

($k \neq 0$*) being a sum of two non-intersecting summands and still not being a summand itself, proving* R *does not fulfill* $C3$*.*

Proving $C1$ *is rather quick as well. We use the definition where we have to show that every submodule is essential in a summand. With the exception of* $\begin{bmatrix}0 & F\\0 & F\end{bmatrix}$ *and* $I_0 = \begin{bmatrix}0 & F\\0 & 0\end{bmatrix}$ *every ideal is already a summand, thus essential in itself. The first exception is essential in* R *and the second one is essential in* $\begin{bmatrix}F & F\\0 & 0\end{bmatrix}$*.*

Now we want to show R *does not fulfill* $C1*$*. Consider the ideal* I_k *for any* $k \neq 0$*. Then one possible complement is* $I_k^* = \begin{bmatrix}0 & 0\\0 & F\end{bmatrix}$*. Now we want a complement of this that also contains* I_k*. Since* $\begin{bmatrix}0 & F\\0 & F\end{bmatrix}$ *is the only non-trivial ideal containing* I_k*, which however is not a complement for* I_k^**, we see that the only possibility is* $(I_k^*)^* = I_k$*. However* $I_k \oplus I_k^* = \begin{bmatrix}0 & F\\0 & F\end{bmatrix} \neq R$ *and thus* $C1*$ *is not fulfilled.*

5 Utumi modules

In this chapter we want to take a look at some of the most recent research on this topic, the so-called Utumi modules. At first we, however, have to go way back to some of the older literature on the exchange property.

5.1 Semi-I-regular and suitable modules

In this section we want to form a basis for the main theorem in the next section on Utumi modules. For this we will take a closer look at W.K. Nicholsons work [LI] on suitable rings, which proved the exchange property for clean rings among other things.

First, let us define suitable rings.

Definition 5.1.1.
Let R be a ring.

- An element $x \in R$ is called *suitable* if it fulfills one of the following four equivalent conditions:
  1. There exists an idempotent $e \in R$ with $e - x \in (x - x^2)R$.
  2. There exists an idempotent $e \in xR$ and $c \in R$ with $(1-e)-(1-x)c \in Rad(R)$.
  3. There exists an idempotent $e \in xR$ with $R = eR + (1 - x)R$.
  4. There exists an idempotent $e \in xR$ with $1 - e \in (1 - x)R$.
- R is called *suitable* if every $x \in R$ is suitable.

Proof. (of equivalence): Check [LI, Prop. 1.1]. □

We will skip a little ahead in Nicholsons work to see why these rings are of interest for us: He proved that suitable endomorphism rings have a very important property for their corresponding module, namely proving the exchange property.

Moreover we want to note that Nicholson uses left-modules whilst we are using right-modules. As stated by Nicholson, the following theorem implies that all of his results are also correct for right-modules.

Theorem 5.1.2
Let M be a module. Then the following are equivalent:

1. *$End(M)$ is right-suitable.*
2. *$End(M)$ is left-suitable.*

3. M has the finite exchange property.

(where right-suitable is our definition for suitable and left-suitable is the obvious dual notion. This also proves that suitable is a notion in which we do not need to differentiate between left- and right-suitable.)

Proof. Check [LI, Thm.2.1]. □

Corollary 5.1.3

A ring is suitable if and only if it is an exchange ring.

Let us take a step back and check which rings we can prove to be suitable. The following lemma was formulated by Nicholson only for $\mathit{Rad}(R)$, but he noted it was possible to prove it for any $I \subseteq \mathit{Rad}(R)$. Since we are also very interested in some of those possibilities for I, we will give this extended prove.

Lemma 5.1.4

Let R be a ring and $I \subseteq \mathit{Rad}(R)$ be any ideal contained in its radical.

Then R is suitable if and only if R/I is suitable and idempotents can be lifted modulo I.

Proof. (taken from [LI, Prop. 1.5].)

First let R/I be suitable and idempotents liftable. We want to show R is suitable.

Let $x \in R$. Write $\overline{x} = x+I \in \overline{R} = R/I$. Since $\overline{R}$ is suitable, there exists an idempotent $\overline{a} \in \overline{x}\overline{R}$ and an element $\overline{c} \in \overline{R}$ such that

$$\overline{1} - \overline{a} = (\overline{1} - \overline{x})\overline{c}.$$

Due to $\overline{a} \in \overline{x}\overline{R}$, we will assume $a \in xR$. As we can lift idempotents, there also exists an idempotent $f \in R$ with $\overline{f} = \overline{a}$, which is to say $f - a \in I$.

Since $I \subseteq \mathit{Rad}(R)$, we get $f - a \in \mathit{Rad}(R)$, which implies that $u := 1 - (f - a)$ is a unit in R. Then we also get

$$\begin{aligned} \overline{u} &= \overline{1} - \overline{(f-a)} = \overline{1} - \overline{0} \\ &= \overline{1} \end{aligned}$$

and thus also $\overline{u^{-1}} = \overline{1} \in \overline{R}$.

Next let us define $e := ufu^{-1} \in R$. First this gives us $e^2 = e$ since f is idempotent and second this implies

$$\begin{aligned} e &= (1 - f + a)fu^{-1} = (f - f^2 + af)u^{-1} \\ &= afu^{-1}. \end{aligned}$$

As $a \in xR$, we get $e \in xR$. Furthermore we have $\overline{e} = \overline{f} = \overline{a}$ due to $\overline{u} = \overline{u^{-1}} = \overline{1}$ and $e = ufu^{-1}$. Going back to our original equation, we get

$$\overline{1} - \overline{e} = \overline{1} - \overline{a} = (\overline{1} - \overline{x})\overline{c}.$$

This implies $(1-e)-(1-x)c \in I \subseteq Rad(R)$. Hence, by definition, x is suitable and thus also R is suitable.

Now let R be suitable. Then by [LI, Cor. 1.3] we can lift idempotents modulo every ideal including I. Moreover by [LI, Prop. 1.4] every homomorphic image of a suitable ring is suitable. As R/I is the homomorphic image of the canonical morphism, it is suitable. □

Next Nicholson proves that every semiregular ring is suitable. We want to introduce a new notion, that of a semi-I-regular ring, for which this also holds.

Definition 5.1.5.
Let R be a ring and I be an ideal. Then we call R *semi-I-regular* if it fulfills the following two conditions:

- R/I is regular and
- we can lift idempotents modulo I.

Moreover we say a module M is *semi-I-regular* for some $I \subset End(M)$ if $End(M)$ is semi-I-regular.

We want to split Nicholsons next proposition into two different theorems.

Theorem 5.1.6
Every regular ring is suitable.

Proof. Check [LI, Proof of Prop. 1.6]. □

Furthermore let us give a more general result of Nicholsons proposition:

Theorem 5.1.7
Let R be a ring and $I \subseteq Rad(R)$ be an ideal. If R is semi-I-regular, it is suitable.

Proof. (analogue to the proof of [LI, Prop. 1.6]) R/I is regular since R is semi-I-regular. Thus it is suitable by our previous theorem. Also, since R is semi-I-regular, we can lift idempotents modulo I. And finally, since $I \subseteq Rad(R)$, we can use lemma 5.1.4 to show that R is suitable. □

We can combine these results to the following theorem which we will use in our next section:

Theorem 5.1.8
Let R be a ring, $I \subseteq Rad(R)$ and R be semi-I-regular. Then R is an exchange ring.

Proof. From our previous theorem we immediately get that R is suitable. Then corollary 5.1.3 tells us that R is an exchange ring. □

Now we can apply this result on the endomorphism ring of a module to get the following.

Corollary 5.1.9
Let M be a module, $I \subseteq Rad(End(M))$ and M be semi-I-regular. Then M has the finite exchange property.

Proof. By our previous theorem 5.1.8 we know that $End(M)$ is an exchange ring and thus by theorem 3.1.3 M has the finite exchange property. □

As a main result of this thesis we can now formulate the following implication for modules with LE-decomposition that are semi-I-regular.

Theorem 5.1.10
Let M be a module with LE-decomposition. Furthermore let $I \subseteq Rad(End(M))$ and M be semi-I-regular. Then M has the exchange property.

Proof. By our previous corollary 5.1.9 we know that M has the finite exchange property and by theorem 3.2.2 we know that for modules with LE-decomposition this is equivalent to having the exchange property. □

In the second section we will see a similar result to the above for another type of module. First, however, we want to prove another main result connecting different exchange properties and the singular ideal with semi-I-regularity.

Theorem 5.1.11
Let M be semi-$\Delta(M, M)$-regular. Then M has the finite exchange property if and only if M has the B2-exchange property.

Proof. Since the finite exchange property always implies the B2-exchange property let us assume that M is semi-$\Delta(M, M)$-regular and has the B2-exchange property. By theorem 5.1.2 M has the exchange property if $End(M)$ is suitable. Due to M having the B2-exchange property, we have $Rad(M, M) = Tot(M, M)$. As such $\Delta(M, M) \subseteq Tot(M, M) = Rad(M, M)$. Hence we can use theorem 5.1.4, which implies $End(M)$ is suitable.

□

5.2 Utumi modules

Recently the notion of *Utumi* modules was introduced, which are generalizations of both continuous and automorphism-invariant modules. As such we will also take a look at them and their connection with the exchange property. First let us introduce them:

Definition 5.2.1.
We call M a *Utumi module* if for any $A, B \subseteq M$ with $A \simeq B$ and $A \cap B = 0$ we have summands $K, L \subseteq^{\oplus} M$ with $A \subseteq^{*} K$, $B \subseteq^{*} L$ and $K \oplus L \subseteq^{\oplus} M$.

Recall that square-free modules are modules that do not contain a submodule isomorphic to a square $A \oplus A$. As noted in [UM, Ex. 2.2], square-free modules are Utumi modules.

Example 5.2.2
Let M be square-free. Then M is a Utumi module, as the existence of $A, B \subseteq M$ with $A \simeq B$, $A, B \neq 0$ and $A \cap B = 0$ would imply the existence of the square $A \oplus B \simeq A \oplus A$. Since $A \cap B = 0$ we get $A \oplus A \simeq A \oplus B = 0$, which implies $A = B = 0$, we then choose $K := L := 0 \subseteq^{\oplus} M$.

Conversely, we have already seen an example of a module that is not a Utumi module.

Example 5.2.3
As an example for an R, that is no Utumi module, we look at the ring $\mathbb{Z}[X]/(4, 2X, X^2)$, which was our example of a module with $C2$ but without $C1$ (check example 4.2.10). We have $(2) \simeq (X)$ with $(2) \cap (X) = 0$, however (2) is not essential R, which is the only summand containing (2).

Also recently N. Ding, Y. Ibrahim, M. Yousif and Y.Zhou found a generalization of C3-modules and aptly called them C4-modules in [C4]. These are of interest, as later on Ibrahim and Zhou also showed that every Utumi module is C4. First let us give the definition they found.

Definition 5.2.4.
A module is said to fulfill $C4$ if for every decomposition $M = A_1 \oplus A_2$ and every homomorphism $f : A_1 \to A_2$ with $ker(f) \subseteq^{\oplus} A_1$ we have $Im(f) \subseteq^{\oplus} A_2$.

We call a module *pseudo-continuous* if it fulfills both C1 and C4.

The basis for their work on C4-modules was [C3, prop. 2.3] where Amin, Ibrahim and Yousif showed that C3-modules fulfill the property they would later call C4. The proof for the following lemma can thus be found in both [C3] and [C4].

Lemma 5.2.5
If a module fulfills $C3$, it also fulfills $C4$. Thus every quasi-continuous module is also pseudo-continuous.

Proof. Check [C4, Lem. 2.1]. □

We come back to the Utumi modules and their connection to C4-modules.

Lemma 5.2.6
Every Utumi module fulfills $C4$.

Proof. Check [UM, Lem. 2.8]. □

The next theorem connects our previous chapters to this one as it shows that Utumi modules are a generalization of types of modules we have already seen:

Theorem 5.2.7
Every automorphism-invariant module as well as every pseudo-continuous module (and thus also every (quasi-)continuous module) is a Utumi module.

Proof. For automorphism-invariant modules check [UM, ex. 2.6] and for pseudo-continuous modules check [UM, Cor. 2.15]. ☐

Example 5.2.8
With this theorem we can give a lot of examples for Utumi modules. However, we can also ask the question whether there are Utumi modules that do not fall in any of those other categories. Fortunately this was already answered confirmedly by Yousif and Ibrahim:

- *Every (quasi-)injective module is a Utumi module.*
- $\mathbb{Z}_{\mathbb{Z}}$ *is a Utumi module as it is quasi-continuous.*
- *In the same way every local commutative domain is a Utumi module as we have seen they are quasi-continuous.*
- *It is, however, possible to give an example of a module that is Utumi whilst neither being automorpism-invariant, square-free nor a* $C1$*-module (and thus not (quasi-/pseudo-)continuous).*

  We will refer to [UM, ex. 2.19] where Yousif and Ibrahim show that the two $\mathbb{Z}$*-modules*

  $$M_1 := \mathbb{Q} \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/p\mathbb{Z} \qquad (k \in \mathbb{N} \cup \{\infty\}, k > 1;\ p \text{ prime})$$

  and

  $$M_2 := \mathbb{Q} \oplus \bigoplus_{i \in I} \mathbb{Z}/p_i\mathbb{Z} \oplus L \quad (\{p_i | i \in I\} \text{ distinct primes };0 \neq\ L \subseteq^{\oplus} \bigoplus_{i \in I} \mathbb{Z}/p_i\mathbb{Z})\}$$

  are Utumi modules that do not fulfill any of the properties previously mentioned.

As Yousif and Ibrahim note, several types of modules are clean if and only if they fulfill the (finite) exchange property. They reference this result for quasi-continuous as well as for square-free modules. Since Utumi modules are generalizations of these, the question whether this result also holds for them is reasonable. They were able to answer this affirmatively:

Theorem 5.2.9
Let M *be a Utumi module. Then the following are equivalent:*

1. M *is clean.*
2. M *has the finite exchange property.*
3. M *has the exchange property.*

Proof. Check [UM, Thm. 5.2]. ☐

Next, let us give a new result on Utumi modules and the exchange property. This is where the last section, where we expanded upon Nicholsons work, is necessary.

Theorem 5.2.10
Let M be a Utumi module. Furthermore let $I \subseteq Rad(End(M))$ and $End(M)$ be semi-I-regular. Then M has the exchange property.

Proof. Since $End(M)$ is semi-I-regular for $I \subseteq Rad(End(M))$, it is an exchange ring by theorem 5.1.8. Thus M has the finite exchange property by theorem 3.1.3. Finally by the previous theorem 5.2.9 M has the exchange property. $\square$

Example 5.2.11
This gives us a much easier method for showing that certain Utumi modules have the exchange property. We have already seen that the property was proven for continuous and automorphism-invariant modules. As we know they are Utumi modules, we can give shorter proofs now:

- *Let M be continuous. By theorem 4.2.5 we see that $End(M)$ is semi-$\Delta(M,M)$-regular and $\Delta(M,M) \subseteq Rad(M,M)$. Since continuous modules are Utumi modules by theorem 5.2.7, we can use the previous theorem 5.2.10 to see that M has the exchange property.*
- *Let M be automorphism-invariant. By [AEP, Prop. 1] M is semi-$\Delta(M,M)$-regular (and $\Delta(M,M) = Rad(M,M)$). Thus we see again that M has the exchange property.*

(It should be noted that [AEP] uses the same decomposition as [UM] to show that a module fulfills the EP and thus the above should only be seen as an abbreviation of their proof.)

6 An extensive example

In this last chapter we want to take a closer look at an example of a module which does fulfill $C2$ but not $C1$. The module in question is a ring seen as a right-module over itself. Parts of this example were also published in [CI, Ex. 4.8], where it was shown that R_R does fulfill $C3$ but ${}_RR$ does not.

Furthermore it was also shown in [CI] that the module has the exchange property. The author of the source keeps himself rather short on the explanation, as such most of our following first two sections can be seen as an extension of what was already written there.

In the third section we want to take a closer look at the radical, total and singular ideal of the module. The penultimate section will contain quite a lot of calculations. In this section we will take a look at the properties $C1$ and $C2$, since, until now, we only know about $C3$. Sadly, especially proving $C2$ is quite a computational task and as such we can only show it in a very specific subcase.

The last section will be a concluding summarization.

6.1 Setting up the example

Consider the (non-commutative) ring

$$R := F < x, y, z > /(x^2 - x, y^2 - y, xy - y, yx - x, yz - xz, z^2, zxz)$$

where F is any field. (Do note that the notation $F < \cdots >$ is meant to imply a polynomial extension where the variables do not commute with each other.) By construction R has the following properties:

- $x^2 = x$ and $y^2 = y$
- $xy = y$ and $yx = x$
- $yz = xz$
- $z^2 = 0$ and $zxz = 0$.

First of all, let us take a look at how elements of the ring even look like. By construction we only have the monomes $1, x, y, z, xz, zx, zy, xzx, xzy$ left and as such any element $r \in R$ is of the form

$$r = a + bx + cy + dz + exz + fzx + gzy + hxzx + ixzy$$

with $a, \dots, i \in F$. When using concrete values, this can be used to calculate products very well, however for general purposes it becomes cluttered very fast. As such we will often instead use 9-tupels to calculate products. For the readers convenience here is how a general product thus looks like:

$$\begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ e_0 \\ f_0 \\ g_0 \\ h_0 \\ i_0 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ e_1 \\ f_1 \\ g_1 \\ h_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} a_0a_1 \\ a_0b_1 + b_0b_1 + b_0a_1 + c_0b_1 \\ a_0c_1 + b_0c_1 + c_0a_1 + c_0c_1 \\ a_0d_1 + d_0a_1 \\ a_0e_1 + b_0d_1 + b_0e_1 + c_0d_1 + c_0e_1 + e_0a_1 \\ a_0f_1 + d_0b_1 + f_0a_1 + f_0b_1 + g_0b_1 \\ a_0g_1 + d_0c_1 + f_0c_1 + g_0a_1 + g_0c_1 \\ a_0h_1 + b_0f_1 + b_0h_1 + c_0f_1 + c_0h_1 + e_0b_1 + h_0a_1 + h_0b_1 + i_0b_1 \\ a_0i_1 + b_0g_1 + b_0i_1 + c_0g_1 + c_0i_1 + e_0c_1 + h_0c_1 + i_0a_1 + i_0c_1 \end{bmatrix}$$

Now we want to consider the right-module R_R. By [CI] it is a $C3$-module, we want to study it for more properties.

6.2 An important ideal

As x and y are idempotents, we already know quite a lot about the ideals they generate respectively, notably that they are summands. But what about the ideal generated by z? We will find that it will help us study the module on multiple occasions. Therefore let us consider $I := RzR$. Then we have for any $t \in I$:

$$\begin{aligned} t &= r_1zr_2 \\ &= (a_1 + b_1x + c_1y + d_1z + e_1xz + f_1zx + g_1zy + h_1xzx + i_1xzy)zr_2 \\ &= (a_1z + (b_1 + c_1)xz + 0)(a_2 + b_2x + c_2y + d_2z + e_2xz + f_2zx + g_2zy + h_2xzx + i_2xzy) \\ &= a_1a_2z + (b_1 + c_1)a_2xz + a_1b_2zx + (b_1 + c_1)b_2xzx + a_1c_2zy + (b_1 + c_1)c_2xzy + 0 \\ &= \tilde{d}z + \tilde{e}xz + \tilde{f}zx + \tilde{g}zy + \tilde{h}xzx + \tilde{i}xzy. \end{aligned}$$

Conversely any $s \in R$ that has this form also is an element of I as $z,\ xz,\ zx,\ zy,\ xzx,\ xzy \in RzR = I$ and thus any multiple and sum of these also is an element of I.

Hence we have

$$\begin{aligned} I &= \{dz + exz + fzx + gzy + hxzx + ixzy | d, \dots, i \in F\} \\ &= \{[0, 0, 0, d, e, f, g, h, i] | d, \dots, i \in F\}. \end{aligned}$$

Now that we have constructed the ideal, we can take a look at some of its properties. Our goal will be to show that R is an exchange-ring. For this we want to prove that R/I is clean and we can lift idempotents. In order to do that we need the following lemma which shows a needed connection between I and $Rad(R)$.

Lemma 6.2.1
Let $I = RzR$ be the ideal generated by z in our module and let $Rad(R)$ be its radical. Then we have $I \subset Rad(R)$.

Proof. To show this we will simply prove that for every $t \in I$ the element $1 + t$ is right-invertible and thus $t \in Rad(R)$ by lemma 2.2.7.

As such let $t = [0, 0, 0, d, e, f, g, h, i] \in I$. We claim that $[1, 0, 0, -d, -e, -f, -g, -h, -i]$ is the right-inverse of $1 + t$. Let us calculate this:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ -d \\ -e \\ -f \\ -g \\ -h \\ -i \end{bmatrix} = \begin{bmatrix} 1 * 1 \\ 0+0+0+0 \\ 0+0+0+0 \\ -1*d+d*1 \\ -1*e+0+0+0+0+e*1 \\ -1*f+0+f*1+0+0 \\ -1*g+0+0+g*1+0 \\ -1*h+0+0+0+0+0+h*1+0+0 \\ -1*i+0+0+0+0+0+0+i*1+0 \end{bmatrix} = 1.$$

$\square$

An interesting remark when looking at the calculation in the proof above is that for any two $s, t \in I$ we get $s \cdot t = 0$ and thus $I^2 = 0$. (A result that, in general, is not true for ideals in non-commutative ring that are generated by a nilpotent element):

Lemma 6.2.2
Let $I = RzR$ be the ideal generated by z. Then we have $I^2 = 0$, thus I is nilpotent.

Proof. Follows from $ztz = 0$ for any $t \in R$, which can easily be checked from the equations $z^2 = 0$, $zxz = 0$ and $yz = xz$. $\square$

Next up we want to consider the ring R/I and see that we can lift idempotents. By construction of I we instantly see $R/I = \{[a, b, c, 0, 0, 0, 0, 0, 0] | a, b, c \in F\}$. To avoid those zeroes for better readability we will simply use $R/I \simeq \{[a, b, c] | a, b, c \in F\}$ with the same multiplication R uses on its first three components.

In order to show the lifting property we will show it more generally for any nilpotent ideal.

Lemma 6.2.3
Let R_0 be any unitary ring and I_0 be a nilpotent ideal. Then we can lift idempotents from R_0/I_0 to R_0, that is for every idempotent $\overline{e} \in R_0/I_0$ there is an idempotent $f \in R_0$ with $\overline{f} \equiv \overline{e}$.

Proof. (Taken from [SE].) Let $\overline{e}$ be an idempotent, then we have $e \in R_0$ not necessarily idempotent. We will define $r := 1 - e$, then on one hand we have $e + r = 1$ and especially $\overline{e} + \overline{r} = \overline{1}$ and on the other hand we have $\overline{r}$ being an idempotent in R_0/I_0 (due to $(\overline{1} - \overline{e})^2 = \overline{1} - 2\overline{e} + \overline{e}^2 \equiv \overline{1} - 2\overline{e} + \overline{e} = \overline{1} - \overline{e}$).

Next up we consider $er = e(1-e) = e - e^2 = (1-e)e = re$, thus e and r commute, and $e - e^2 \equiv \overline{0} \; mod \; I_0$, that is $er \in I_0$. Next up we use the prerequisite that I_0 is nilpotent, resulting in $(er)^k = 0$ for some k.

Now we consider $1 \equiv 1^k \equiv (e+r)^k \equiv e^k + r^k \; mod \; I_0$ where the latter equation simply follows from $er \in I_0$ and thus any mixed terms vanishing. Now let $x := 1 - (e^k + r^k)$. Due to our previous observation we have $x \equiv 0 \; mod \; I_0$, or in other words $x \in I_0$. Also we will note that by construction of x as well as e and r commuting, x also commutes with both e and r.

Next let $u := (1-x)^{-1} \in R_0$. This exists due to x being nilpotent (which follows from $x \in I_0$ and I_0 being nilpontent) and the difference of 1 and any nilpotent element always being invertible with concrete value $u = 1 + x + \ldots + x^{l-1}$ where $x^l = 0$. Moreover $u \equiv 1 + x + \ldots + x^{l-1} \equiv 1 + 0 + \ldots + 0^{l-1} \equiv 1 \; mod \; I_0$. Finally u also commutes with e and r due to being a sum of values commuting with these.

Using these properties we can conclude the following: $ue^k + ur^k = u(e^k + r^k) = u(1-x) = 1$. We then multiply both sides by ue^k to get $(ue^k)^2 + u^2 e^k r^k = 1 \cdot ur^k$ using the the fact that all of these commute. However, since $e^k r^k = 0$, this leaves $(ue^k)^2 = ue^k$, so $f := ue^k$ is an idempotent in R_0. Lastly $\overline{f} = \overline{ue^k} = \overline{1}\overline{e}^k = \overline{e}^k = \overline{e}$ completes the claim. $\square$

Using these two we can now conclude that we can lift modulo I:

Corollary 6.2.4
Let $I = RzR$ be the ideal generated by z. Then idempotents of R/I can be lifted.

Proof. Follows directly from 6.2.2 and 6.2.3. $\square$

Having shown that we can lift idempotents, our next step will be showing that R/I is a clean ring. For this we want to remember the connection $R/I \simeq \{[a,b,c] | a,b,c \in F\}$ with the inherited multiplication. Using this, we split the proof in three simple lemmata.

Lemma 6.2.5
Let $r := [a,b,c] \in R/I$ with $a+b+c \neq 0$ and $a \neq 0$. Then r is invertible with $r^{-1} = s := [a^{-1}, -ba^{-1}(a+b+c)^{-1}, -ca^{-1}(a+b+c)^{-1}] := [\tilde{a}, \tilde{b}, \tilde{c}]$.

Proof. First note that s is well-defined as we operate in a field and as such any value (except 0) does have an inverse. Then we simply calculate the products rs and sr to be equal to 1:

$$
\begin{aligned}
r \cdot s = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \end{bmatrix} &= \begin{bmatrix} a\tilde{a} \\ a\tilde{b} + b\tilde{b} + c\tilde{b} + b\tilde{a} \\ a\tilde{c} + b\tilde{c} + c\tilde{c} + c\tilde{a} \end{bmatrix} \\
&= \begin{bmatrix} aa^{-1} \\ (a+b+c)(-ba^{-1}(a+b+c)^{-1}) + ba^{-1} \\ (a+b+c)(-ca^{-1}(a+b+c)^{-1}) + ca^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ -ba^{-1} + ba^{-1} \\ -ca^{-1} + ca^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1
\end{aligned}
$$

and

$$
\begin{aligned}
s \cdot r = \begin{bmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} \tilde{a}a \\ \tilde{a}b + \tilde{b}b + \tilde{c}b + \tilde{b}a \\ \tilde{a}c + \tilde{b}c + \tilde{c}c + \tilde{c}a \end{bmatrix} \\
&= \begin{bmatrix} a^{-1}a \\ b(a^{-1} + (-ba^{-1}(a+b+c)^{-1}) + (-ca^{-1}(a+b+c)^{-1})) + (-ba^{-1}(a+b+c)^{-1})a \\ c(a^{-1} + (-ba^{-1}(a+b+c)^{-1}) + (-ca^{-1}(a+b+c)^{-1})) + (-ca^{-1}(a+b+c)^{-1})a \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ a^{-1}b + b(-ba^{-1}(a+b+c)^{-1}) + c(-ba^{-1}(a+b+c)^{-1}) + a(-ba^{-1}(a+b+c)^{-1}) \\ a^{-1}c + b(-ca^{-1}(a+b+c)^{-1}) + c(-ca^{-1}(a+b+c)^{-1}) + a(-ca^{-1}(a+b+c)^{-1}) \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ a^{-1}b - (b+c+a)ba^{-1}(a+b+c)^{-1} \\ a^{-1}c - (b+c+a)ca^{-1}(a+b+c)^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1.
\end{aligned}
$$

□

Having found some units, we next need to compute some idempotents:

Lemma 6.2.6
Let $r_k := [0, 1-k, k] \in R/I$ and $s_k = 1 - r_k = [1, -1+k, -k] \in R/I$. Then both of these are idempotents.

Proof. We again simply compute this:

$$
r_k^2 = \begin{bmatrix} 0 \\ 1-k \\ k \end{bmatrix}^2 = \begin{bmatrix} 0 \cdot 0 \\ 0(1-k) + (1-k)^2 + (1-k)k + (1-k)0 \\ 0k + (1-k)k + k^2 + k \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - 2k + k^2 + k - k^2 \\ k - k^2 + k^2 \end{bmatrix} = r_k
$$

and

$$
\begin{aligned}
s_k^2 = \begin{bmatrix} 1 \\ -1+k \\ -k \end{bmatrix}^2 &= \begin{bmatrix} 1 \cdot 1 \\ 1(-1+k) + (-1+k)^2 + (-1+k)(-k) + (-1+k)1 \\ 1(-k) + (-1+k)(-k) + (-k)^2 + (-k) \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ k - 1 + 1 - 2k + k^2 + k - k^2 + k - 1 \\ -k + k - k^2 + k^2 - k \end{bmatrix} = s_k.
\end{aligned}
$$

□

Now we have the necessary idempotents and units to prove R/I is clean.

Lemma 6.2.7
R/I is a clean ring.

Proof. We have to show that every element is a sum of a unit and an idempotent, so let $r = [a, b, c] \in R/I$. We will look at different cases and prove each individually:

$a+b+c \neq 0$ and $a \neq 0$: This case is fairly easy, as by 6.2.5 r is a unit and as such is a sum of itself and the idempotent 0.

$a+b+c \neq 0$ and $a = 0$: We consider $[0, b, c] = r := s_1 + s_2 = [1, -1, 0] + [-1, b+1, c]$. By 6.2.6 s_1 is idempotent (when choosing $s_1 := s_k$ with $k = 0$) and since $-1 \neq 0$ and $(-1) + (b+1) + c = a + b + c \neq 0$, we can use 6.2.5 to show s_2 is a unit.

$a+b+c = 0$ and $a \neq 0$: We will consider $r = s_1 + s_2 = [0, 1, 0] + (a, b-1, c)$. s_1 again is idempotent by 6.2.6 (this time choosing $s_1 = r_k$ with $k = 0$) and since $a \neq 0$ and $a + (b-1) + c = 0 - 1 = -1 \neq 0$, 6.2.5 shows s_2 is a unit.

$a+b+c = 0$ and $a = 0$: Lastly we will use $r = s_1 + s_2 = 1 + [-1, b, c]$, where 1 is obviously idempotent and s_2 is again a unit as $-1 \neq 0$ and $-1 + b + c = -1 \neq 0$ also fulfills the requirements of 6.2.5.

□

Now we have shown that $I \subseteq Rad(R)$, that we can lift idempotents mod I and that R/I is clean. The following theorem then implies R being an exchange ring.

Let us remember the notion of a suitable ring. We have proven in 5.1.4 that a ring is an exchange ring if a quotient ring is suitable and we can lift idempotents. In chapter 5 we focused on regular quotient rings, we can, however, also prove the same for clean quotient rings:

Lemma 6.2.8
Let R be any ring and $A \subseteq Rad(R)$ be an ideal contained in its radical. Also let R/A be clean and let idempotents be liftable modulo A. Then R is an exchange ring.

Proof. Since R/A is clean, it is also suitable by [LI, Prop. 1.8]. Then we can use lemma 5.1.4 to show that R is suitable as we can lift idempotents, R/A is suitable and $A \subseteq Rad(R)$. Finally, by corollary 5.1.3, this is equivalent to R being an exchange ring. □

Having proven the above, we can finally show that R is an exchange ring.

Theorem 6.2.9
R is an exchange ring.

Proof. This follows from 6.2.8 as by 6.2.1 I is a subset of $Rad(R)$, by 6.2.7 R/I is clean and by 6.2.4 we can lift idempotents modulo I.

$\square$

Before moving on to the next section, we also want to show some small fact about I which will be necessary later on.

Lemma 6.2.10
I is not essential in R.

Proof. To show this we simply need to find some ideal $0 \neq U \subseteq R$ with $U \cap I = 0$. For this let us take a look at $U := (x - y)R$. Then we have for some $u \in U$:

$$u = (x-y)r = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \\ \vdots \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ b+a-b \\ c-a-c \\ 0 \\ d+e-d-e \\ 0 \\ 0 \\ f+h-f-h \\ g+i-g-i \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ -a \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

thus showing $U = \{[0, a, -a, 0, \dots, 0] | a \in K\}$. We can immediately see $U \cap I = 0$.

$\square$

6.3 On the radical, total and singular ideal of R

Next up we want to determine the important ideals of R. First we know that radical and total are the same as, by the previous section, R is an exchange ring. As such the singular ideal is contained in the radical, as it is always contained in the total. The question remains whether they are identical.

To do this we first calculate the radical in two steps:

Lemma 6.3.1
Let $r = [a, b, c, \dots] \in R$ with $a \neq 0$. Then $r \notin Rad(R)$.

Proof. We know that if $r \in Rad(R)$ was true, then $1 - rs$ would be a unit for any $s \in R$. Hence, in order to show that r is not an element of the radical, we will simply find some s which does not fulfill this. An example would be $s := [a^{-1}, 0, 0, \dots, 0]$. (This is well defined as $a \neq 0$ and we are operating over a field.) Therefore we have:

$$1 - rs = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} - \begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} a^{-1} \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 - aa^{-1} \\ -ba^{-1} \\ -ca^{-1} \\ \vdots \end{bmatrix} = - \begin{bmatrix} 0 \\ ba^{-1} \\ ca^{-1} \\ \vdots \end{bmatrix}.$$

Thus $1 - rs$ has a zero in its first entry. As such also any product $(1 - rs) \cdot t$ will have a zero in its first entry. However, then it cannot be 1, as $1 = [1, 0, 0, \ldots]$. Thus $1 - rs$ is not a unit and therefore $r \notin \mathit{Rad}(R)$.

□

Using this lemma we can calculate the radical in the second step.

Lemma 6.3.2
We have $\mathit{Rad}(R) = \{[0, b, -b, d, e, f, g, h, i] | b, d, \ldots, i \in K\}$.

Proof. By the previous lemma 6.3.1 we know that the first entry of any element of $\mathit{Rad}(R)$ has to be zero. Next up we use the fact that in 6.2.1 we have shown $I \subseteq \mathit{Rad}(R)$ where $I = RzR = \{[0, 0, 0, d, e, \ldots, i] | d, \ldots i \in F\}$. Therefore we can write any $r \in R$ as $r = r_1 + r_2 = [a, b, c, 0 \ldots 0] + [0, 0, 0, d, e, \ldots, i]$ with $r_2 \in I \subseteq \mathit{Rad}(R)$. Since $\mathit{Rad}(R)$ is additively closed as an ideal, this means $r \in \mathit{Rad}(R) \Leftrightarrow r_1 \in \mathit{Rad}(R)$.

Now we choose any $r = [0, b, c, 0, \ldots, 0] \in \mathit{Rad}(R)$ and want to show $b = -c$. Let us assume this not being the case, that is $b + c \neq 0$. Since $r \in \mathit{Rad}(R)$, we know $1 - rs$ is a unit for any $s \in R$, so we choose $s = [0, (b+c)^{-1}, 0 \ldots, 0]$ (this exists by our assumption) which gives $rs = [0, 1, 0 \ldots, 0]$ and thus $1 - rs = [1, -1, 0 \ldots, 0]$. We get

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} = (1 - rs)t = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} t_0 \\ t_1 - t_1 - t_0 \\ t_2 - t_2 \\ t_3 \\ t_4 - t_3 - t_4 \\ t_5 \\ t_6 \\ t_7 - t_5 - t_7 \\ t_8 - t_6 - t_8 \end{bmatrix} = \begin{bmatrix} t_0 \\ -t_0 \\ 0 \\ t_3 \\ t_3 \\ t_5 \\ t_6 \\ t_5 \\ t_6 \end{bmatrix}$$

where t is the inverse of $1 - rs$. We instantly see $t_0 = 1$ and $t_0 = 0$, a contradiction. Thus the premise must have been wrong and we can conclude $b + c = 0$ if $r \in \mathit{Rad}(R)$.

Conversely, we still want to show that $b + c = 0$ also implies $r \in \mathit{Rad}(R)$. For this let $r := [0, b, -b, 0, \ldots] \in R$ and $s \in R$, we again consider $1 - rs$ and want to show it is invertible:

$$1 - rs = 1 - \begin{bmatrix} 0 \\ b \\ -b \\ 0 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \end{bmatrix} = 1 - \begin{bmatrix} 0 \\ bs_1 + bs_0 - bs_1 \\ bs_2 - bs_1 - bs_2 \\ 0 \\ bs_3 + bs_4 - bs_3 - bs_4 \\ 0 \\ 0 \\ bs_6 + bs_8 - bs_6 - bs_8 \\ bs_7 + bs_9 - bs_7 - bs_9 \end{bmatrix} = \begin{bmatrix} 1 \\ -bs_0 \\ bs_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We claim that $t = [1, bs_0, -bs_0, 0 \dots, 0]$ is its inverse:

$$(1-rs)t = \begin{bmatrix} 1 \\ -bs_0 \\ bs_0 \\ 0 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} 1 \\ bs_0 \\ -bs_0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 \\ 1 \cdot bs_0 - bs_0 \cdot bs_0 - bs_0 \cdot 1 + bs_0 \cdot bs_0 \\ 1 \cdot (-bs_0) - bs_0 \cdot bs_0 + bs_0 \cdot 1 + bs_0 \cdot (-bs_0) \\ 0 \\ \vdots \end{bmatrix} = 1.$$

In conclusion $1 - rs$ is invertible for any s, thus showing $r \in Rad(R)$. □

Therefore we also know the total,

Corollary 6.3.3
We also have $Rad(R) = Tot(R,R) = \{[0, b-b, d, e, f, g, h, i] | b, d \dots, i \in K\}$.

Proof. By 6.2.9 we know that R is an exchange ring. Thus we know $Rad(R,R) = Tot(R,R)$, as exchange rings have the B2-EP by theorem 3.1.12 and the B2-EP is equivalent to $Rad(R,R) = Tot(R,R)$ by theorem 3.1.11. Hence the claim follows from the previous lemma 6.3.2 and lemma 2.2.6, which says $Rad(R) = Rad(R,R)$.

□

Next up we want to show that while the singular ideal is a subset of the radical by the previous corollary, it is not the same set.

Lemma 6.3.4
We have $\Delta(R,R) \subsetneq Rad(R)$.

Proof. By 6.3.3 we know $Tot(R,R) = Rad(R)$ and by 2.2.21 we generally know $\Delta(R,R) \subseteq Tot(R,R)$, together showing $\Delta(R,R) \subseteq Rad(R)$.

Next up we want to find an $r \in Rad(R) \setminus \Delta(R,R)$. For this we use lemma 2.2.19 which tells us $\Delta(R,R) = \{r \in R | Ann(r) \subseteq^* R\}$. In other words we want to find some $r \in Rad(R)$ whose annihilator is not essential. We can look at our ideal I which we used in the last chapter, of which quite a lot of elements fulfill this. As an example we will consider $r = xz + zx = [0,0,0,0,1,1,0,0,0]$. To determine its annihilator we calculate

$$r \cdot s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ a+b \\ c \\ b \\ c \end{bmatrix}.$$

As such $s = [a, \dots, i] \in Ann(r) \Leftrightarrow rs = 0 \Leftrightarrow a = b = c = 0$ shows $Ann(r) = \{[0,0,0,d,\dots,i] | d, \dots, i \in K\} = I$. By 6.2.10 we know that I is not essential, thus concluding the proof.

□

6.4 On C1 and C2 in the case of a specific field

Next we want to check the ring on the properties of a continuous module. As we have seen in the last section $\Delta(R,R) \neq Rad(R)$ and thus R can not be continuous, but it could still fulfill either $C1$ or $C2$. We will check it for $C2$, however, as this property needs us to check all of the summands, this is only feasible for a fixed field F. We will choose $F = \mathbb{F}_2$ which will still require some serious calculations.

Definition 6.4.1.
For the remainder of this chapter we will use the ring $R_2 := \{[a, \dots, i] | a, \dots i \in \mathbb{F}_2\}$ with the multiplication we already worked with up to this point.

First up we want to calculate all of the idempotents in order to find the summands. We will also sort them into 4 groups as they behave quite similar.

Lemma 6.4.2
Given the ring R_2, there are a total of 32 idempotents, sorted into the following 4 groups:

- $\{[0,1,0,0,e,f,0,f+e+i,i] | e,f,i \in \mathbb{F}_2\}$
- $\{[1,1,0,0,e,f,0,f+e+i,i] | e,f,i \in \mathbb{F}_2\}$
- $\{[0,0,1,0,e,0,g,h,g+e+h] | e,g,h \in \mathbb{F}_2\}$
- $\{[1,0,1,0,e,0,g,h,g+e+h] | e,g,h \in \mathbb{F}_2\}$.

(Note that we additionally have the trivial idempotents 0 *and* 1*.)*

Proof. Let $s = s^2 \in R$ be idempotent, then we get:

$$\begin{bmatrix} a \\ \vdots \\ i \end{bmatrix} = s = s^2 = \begin{bmatrix} a \\ \vdots \\ i \end{bmatrix} \cdot \begin{bmatrix} a \\ \vdots \\ i \end{bmatrix} = \begin{bmatrix} a^2 \\ ab + b^2 + ba + cb \\ ac + bc + ca + c^2 \\ ad + da \\ ae + bd + be + cd + ce + ea \\ af + db + fa + fb + gb \\ ag + dc + fc + ga + gc \\ ah + bf + bh + cf + ch + eb + ha + hb + ib \\ ai + bg + bi + cg + ci + ec + hc + ia + ic \end{bmatrix}$$

$$= \begin{bmatrix} a^2 \\ 2ab + b^2 + cb \\ 2ac + bc + c^2 \\ 2ad \\ 2ae + bd + be + cd + ce \\ 2af + db + fb + gb \\ 2ag + dc + fc + gc \\ 2ah + bf + 2bh + cf + ch + eb + ib \\ 2ai + bg + bi + cg + 2ci + ec + hc \end{bmatrix} \overset{*}{=} \begin{bmatrix} a^2 \\ b^2 + cb \\ bc + c^2 \\ 0 \\ bd + be + cd + ce \\ db + fb + gb \\ dc + fc + gc \\ bf + cf + ch + eb + ib \\ bg + bi + cg + ec + hc \end{bmatrix},$$

where the last equation follows from operating on $\mathbb{F}_2$. As such we instantly get $a^2 = a$, which in $\mathbb{F}_2$ simply means a is a free variable, furthermore we instantly see $d = 0$. Next up both the second line $b = b^2 + cb$ and the third line $c = bc + c^2$ imply $1 = b + c$, which in $\mathbb{F}_2$ means exactly one of them is equal to 1 and the other is 0, or $0 = b = c$, which then implies $s = 1$ or $s = 0$, depending on a. There are no more idempotents to be found with the second possibility, hence we will from here on assume the first possibility $1 = b + c$ to hold for the rest of this proof.

The fifth line $e = bd + be + cd + ce = (b + c)e = e$ gives no new information. The sixth line $f = db + fb + gb = (f + g)b$ and the seventh line $g = dc + fc + gc = (f + g)c$ are dependent on whether b or c is 1 making either f free and $g = 0$ or the other way around. The last two lines $h = bf + cf + ch + eb + ib = (b + c)f + ch + (e + i)b = f + ch + (e + i)b$ and $i = bg + bi + cg + ec + hc = g + bi + (e + h)c$ are dependent on the previous choices, making one of them free and the other completely dependent.

Choosing $b = 1$ gets us idempotents of the form of $[a, 1, 0, 0, e, f, 0, f + e + i, i]$ and choosing $c = 1$ gives idempotents of the form $[a, 0, 1, 0, e, 0, g, h, g + e + h]$. The four groups of idempotents result from choosing a. □

As we have seen in the proof there is, at first, no separation depending on the first entry when calculating the groups. This separation is, however, very useful as the idempotents with $a = 0$ generate summands with 2^5 elements whilst the other idempotents with $a = 1$ generate summands with 2^4 elements.

Lemma 6.4.3
The idempotents of R_2 with 0 as their first entry generate two distinct ideals with 2^5 elements, whereas the idempotents with 1 as the first entry generate eight distinct ideals with 2^4 elements. (Excluding the trivial idempotents 1 and 0 which generate R_2 and 0, respectively)

Proof. We will simply calculate each of these. For better readability we will denote the idempotents in the normal notation and every calculation as well as the notation for our ideals in vector-notation. As a reminder we have $[a, b, \dots, i] = a + bx + cy + dz + exz + fzx + gzy + hxzx + ixzy$. For the calculation of e_0R, given an idempotent e_0, we will also use a general element $r = [a, b, \dots, i] \in R$. With this, let us see the ideals:

The ideal $\{[0, a, b, 0, c, 0, 0, d, e] | a, \dots, e \in \mathbb{F}_2\}$ is generated by

- x
- $x + xz + xzx$
- $x + xzx + xzy$
- $x + xz + xzy$
- y
- $y + xz + xzy$

- $y + xzx + xzy$ and
- $y + xz + xzx$:

$$x \cdot r = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ \vdots \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ a+b \\ c \\ 0 \\ d+e \\ 0 \\ 0 \\ f+h \\ g+i \end{bmatrix} := \begin{bmatrix} 0 \\ \tilde{a} \\ \tilde{b} \\ 0 \\ \tilde{c} \\ 0 \\ 0 \\ \tilde{d} \\ \tilde{e} \end{bmatrix}$$

$$(x + xz + xzx) \cdot r = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ a+b \\ c \\ 0 \\ a+d+e \\ 0 \\ 0 \\ a+f+h \\ g+i \end{bmatrix} := \begin{bmatrix} 0 \\ \tilde{a} \\ \tilde{b} \\ 0 \\ \tilde{c} \\ 0 \\ 0 \\ \tilde{d} \\ \tilde{e} \end{bmatrix}$$

$$(x + xzx + xzy) \cdot r = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ a+b \\ c \\ 0 \\ d+e \\ 0 \\ 0 \\ a+f+h \\ a+g+i \end{bmatrix} := \begin{bmatrix} 0 \\ \tilde{a} \\ \tilde{b} \\ 0 \\ \tilde{c} \\ 0 \\ 0 \\ \tilde{d} \\ \tilde{e} \end{bmatrix}$$

$$(x + xz + xzy) \cdot r = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ a+b \\ c \\ 0 \\ a+d+e \\ 0 \\ 0 \\ f+h \\ a+g+i \end{bmatrix} := \begin{bmatrix} 0 \\ \tilde{a} \\ \tilde{b} \\ 0 \\ \tilde{c} \\ 0 \\ 0 \\ \tilde{d} \\ \tilde{e} \end{bmatrix} .$$

To shorten this, we also get in the same manner:

$$y \cdot r = \begin{bmatrix} 0 \\ b \\ a+c \\ 0 \\ d+e \\ 0 \\ 0 \\ f+h \\ g+i \end{bmatrix} \qquad (y + xz + xzy) \cdot r = \begin{bmatrix} 0 \\ b \\ a+c \\ 0 \\ a+d+e \\ 0 \\ 0 \\ f+h \\ a+g+i \end{bmatrix}$$

$$(y + xzx + xzy) \cdot r = \begin{bmatrix} 0 \\ b \\ a+c \\ 0 \\ d+e \\ 0 \\ 0 \\ a+f+h \\ a+g+i \end{bmatrix} \qquad (y + xz + xzx) \cdot r = \begin{bmatrix} 0 \\ b \\ a+c \\ 0 \\ a+d+e \\ 0 \\ 0 \\ a+f+h \\ g+i \end{bmatrix}.$$

Next up we want to calculate the other ideal with 2^5 Elements, which is generated by the remaining eight idempotents which do not have 1 as their first entry. This is the ideal $\{[0, a, b, 0, c, a, b, d, e] | a, \ldots, e \in \mathbb{F}_2\}$ generated by $x + zx + xzx$, $x + xz + zx$, $x + zx + xzy$, $x + xz + zx + xzx + xyz$, $y + zy + xzy$, $y + xz + xy$, $y + zy + xzx$ and $y + xz + zy + xzx + xzy$. Notably these are exactly the idempotents containing zx or zy.

$$(x + zx + xzx) \cdot r = \begin{bmatrix} 0 \\ a+b \\ c \\ 0 \\ d+e \\ a+b \\ c \\ a+b+f+h \\ c+g+i \end{bmatrix} \qquad (x + xz + zx) \cdot r = \begin{bmatrix} 0 \\ a+b \\ c \\ 0 \\ a+d+e \\ a+b \\ c \\ b+f+h \\ c+g+i \end{bmatrix}$$

$$(x+zx+xzy)\cdot r = \begin{bmatrix} 0 \\ a+b \\ c \\ 0 \\ d+e \\ a+b \\ c \\ b+f+h \\ a+c+g+i \end{bmatrix} \quad (x+xz+zx+xzx+xzy)\cdot r = \begin{bmatrix} 0 \\ a+b \\ c \\ 0 \\ a+d+e \\ a+b \\ c \\ a+b+f+h \\ a+c+g+i \end{bmatrix}$$

$$(y+zy+xzy)\cdot r = \begin{bmatrix} 0 \\ b \\ a+c \\ 0 \\ d+e \\ b \\ a+c \\ b+f+h \\ a+c+g+i \end{bmatrix} \quad (y+xz+zy)\cdot r = \begin{bmatrix} 0 \\ b \\ a+c \\ 0 \\ a+d+e \\ b \\ a+c \\ b+f+h \\ c+g+i \end{bmatrix}$$

$$(y+zy+xzx)\cdot r = \begin{bmatrix} 0 \\ b \\ a+c \\ 0 \\ d+e \\ b \\ a+c \\ a+b+f+h \\ c+g+i \end{bmatrix} \quad (y+xz+zy+xzx+xzy)\cdot r = \begin{bmatrix} 0 \\ b \\ a+c \\ 0 \\ a+d+e \\ b \\ a+c \\ a+b+f+h \\ a+c+g+i \end{bmatrix}.$$

Here we have seen all of the idempotents with 0 being their first entry, so let us now consider those with 1 as their first entry. These will generate eight different ideals, each generated by two idempotents. We will start with $\{[a,a,0,b,b,c,d,c,d] | a,\ldots,d \in \mathbb{F}_2\}$, generated by $1+x$ and $1+x+zx+xzx$:

$$(1+x)r = \begin{bmatrix}1\\1\\0\\\dots\\0\end{bmatrix} \cdot \begin{bmatrix}a\\b\\\dots\\i\end{bmatrix} = \begin{bmatrix}a\\b+b+a\\c+c\\d\\e+d+e\\f\\g\\h+f+h\\i+g+i\end{bmatrix} = \begin{bmatrix}a\\a\\0\\d\\d\\f\\g\\f\\g\end{bmatrix}$$

$$(1+x+zx+xzx)r = \begin{bmatrix}1\\1\\0\\0\\0\\1\\0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}a\\b\\c\\d\\e\\f\\g\\h\\i\end{bmatrix} = \begin{bmatrix}a\\b+b+a\\c+c\\d\\e+d+e\\f+a+b\\g+c\\h+f+h+a+b\\i+g+i+c\end{bmatrix} = \begin{bmatrix}a\\a\\0\\d\\d\\a+b+f\\c+g\\a+b+f\\c+g\end{bmatrix}.$$

As we can see here, it is also important that we operate on $\mathbb{F}_2$ so $2=0$ cancels out. Keeping this in mind, as this is going to be important for the calculation of all the remaining idempotents, we will omit the calculations.

Next up we have the ideal $\{[a,a,0,b,a+b,c,d,a+c,d] | a,\dots,d \in \mathbb{F}_2\}$, generated by $1+x+xz+xzx$ and $1+x+xz+zx$:

$$(1+x+xz+xzx)r = \begin{bmatrix}a\\a\\0\\d\\a+d\\f\\g\\a+f\\g\end{bmatrix} \qquad (1+x+xz+zx)r = \begin{bmatrix}a\\a\\0\\d\\a+d\\a+b+f\\c+g\\b+f\\c+g\end{bmatrix}.$$

Do note that in the ideal the sixth and eight entry have to add up to a. Thus the second equation also gives an element of this ideal. Next we can look at the third ideal $\{[a,a,0,b,b,c,d,a+c,a+d] | a,\dots, \in \mathbb{F}_2\}$ which is generated by $1+x+xzx+xzy$ and $1+x+zx+xzy$:

$$(1+x+xzx+xzy)r = \begin{bmatrix} a \\ a \\ 0 \\ d \\ d \\ f \\ g \\ a+f \\ a+g \end{bmatrix} \qquad (1+x+zx+xzy)r = \begin{bmatrix} a \\ a \\ 0 \\ d \\ d \\ a+b+f \\ c+g \\ b+f \\ a+c+g \end{bmatrix}.$$

This leaves us with one ideal of those whose second entry is 1, the ideal $\{[a,a,0,b,a+b,c,d,c,a+d] | a,\ldots,d \in \mathbb{F}_2\}$, generated by $1+x+xz+xzy$ and $1+x+xz+zx+xzx+xzy$:

$$(1+x+xz+xzy)r = \begin{bmatrix} a \\ a \\ 0 \\ d \\ a+d \\ f \\ g \\ f \\ a+g \end{bmatrix} \quad (1+x+xz+zx+xzx+xzy)r = \begin{bmatrix} a \\ a \\ 0 \\ d \\ a+d \\ a+b+f \\ c+g \\ a+b+f \\ a+c+g \end{bmatrix}.$$

We have seen all ideals generated by idempotents of the form $1+x+\ldots$, so let us now consider the ones with $1+y+\ldots$. As we have seen, all the ideals of the former type had elements looking like $[1,1,0,\ldots]$, but now we will see elements of the form $[1,0,1,\ldots]$. Just as before, we will start with the ideal generated by $1+y$, which is also generated by $1+y+zy+xzy$, that being the ideal $\{[a,0,a,b,b,c,d,c,d] | a,\ldots,d \in \mathbb{F}_2\}$:

$$(1+y)r = \begin{bmatrix} a \\ 0 \\ a \\ d \\ d \\ f \\ g \\ f \\ g \end{bmatrix} \qquad (1+y+zy+xzy)r = \begin{bmatrix} a \\ 0 \\ a \\ d \\ d \\ b+f \\ a+c+g \\ b+f \\ a+c+g \end{bmatrix}.$$

The next ideal we want to look at is $\{[a,0,a,b,a+b,c,d,c,a+d] | a,\ldots,d \in \mathbb{F}_2\}$, generated by $1+y+xz+xzy$ and $1+y+xz+zy$:

$$(1+y+xz+xzy)r = \begin{bmatrix} a \\ 0 \\ a \\ d \\ a+d \\ g \\ f \\ g \\ a+f \end{bmatrix} \qquad (1+y+xz+zy)r = \begin{bmatrix} a \\ 0 \\ a \\ d \\ a+d \\ b+g \\ a+c+f \\ b+g \\ c+f \end{bmatrix}.$$

Our third ideal of this form will be $\{[a,0,a,b,b,c,d,a+c,a+d] | a,\ldots,d \in \mathbb{F}_2\}$, generated by $1+y+xzx+xzy$ and $1+y+zy+xzx$:

$$(1+y+xzx+xzy)r = \begin{bmatrix} a \\ 0 \\ a \\ d \\ d \\ f \\ g \\ a+f \\ a+g \end{bmatrix} \qquad (1+y+zy+xzx)r = \begin{bmatrix} a \\ 0 \\ a \\ d \\ d \\ b+f \\ a+c+g \\ a+b+f \\ c+g \end{bmatrix}.$$

The last ideal is $\{[a,0,a,b,a+b,c,d,a+c,d] | a,\ldots,d \in \mathbb{F}_2\}$, which is generated by $1+y+xz+xzx$ and $1+y+xz+zy+xzx+xzy$:

$$(1+y+xz+xzx)r = \begin{bmatrix} a \\ 0 \\ a \\ d \\ a+d \\ f \\ g \\ a+f \\ g \end{bmatrix} \quad (1+y+xz+zy+xzx+xzy)r = \begin{bmatrix} a \\ 0 \\ a \\ d \\ a+d \\ b+f \\ a+c+g \\ a+b+f \\ a+c+g \end{bmatrix}.$$

□

Having calculated all of the summands, we can now show that R_2 does indeed fulfill $C2$. For this we will reference two important theorems:

Theorem 6.4.4
Let $M \neq 0$ be a directly indecomposable module with finite length. Then $End(M)$ is local.

Proof. Check [MaR, Thm. 7.2.7]. $\square$

And secondly the Krull-Remak-Schmidt-Theorem.

Theorem 6.4.5
Let $M = \bigoplus_{i \in I} M_i$ *with* $End(M_i)$ *local for all* $i \in I$. *Furthermore, let* $M = \bigoplus_{j \in J} N_j$ *where* N_j *is directly indecomposable for all* $j \in J$. *Then there exists a bijection* $\beta : I \to J$ *with* $M_i \simeq N_{\beta(i)}$ *for all* $i \in I$.

Proof. Check [MaR, 7.3.1]. $\square$

With these two we can finally come to the main theorem of this section.

Theorem 6.4.6
R_2 *does fulfill* $C2$. *More precisely all of the summands with* 2^4 *or* 2^5 *elements are isomorphic to each other and only to each other, respectively.*

Proof. We have seen all of the summands in lemma 6.4.3. We will choose one of each group and calculate from there on.

First, for the summands with 2^5 elements we will choose x as the idempotent generating $xR = \{[0, a, b, 0, c, 0, 0, d, e] | a, \ldots, e \in \mathbb{F}_2\}$ and have to show this is only isomorphic to $\{[0, a, b, 0, c, a, b, d, e] | a, \ldots, e \in \mathbb{F}_2\}$. So let us assume there is some isomorphism $\lambda : xR \xrightarrow{\sim} M; xr \mapsto \lambda(xr)$ for some $M \subseteq R_2$. Then, due to $\lambda(xr) = \lambda(x)r$, this isomorphism is only determined by $\lambda(x)$. As such we will take a closer look at $\lambda(x) := [a, \ldots, i]$. We know $x = x^2$ and λ is a homomorphism, thus

$$\lambda(x) = \lambda(x^2) = \lambda(x) \cdot x = \begin{bmatrix} a \\ b \\ c \\ \vdots \\ i \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a+b+c \\ 0 \\ 0 \\ 0 \\ d+f+g \\ 0 \\ e+h+i \\ 0 \end{bmatrix} := \begin{bmatrix} 0 \\ \tilde{a} \\ 0 \\ 0 \\ 0 \\ \tilde{b} \\ 0 \\ \tilde{c} \\ 0 \end{bmatrix}.$$

We see that there are six possibilities for $\lambda(x)$. First of all let us assume $\tilde{a} = 0$ and let us look at $\lambda(x)R$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \tilde{b} \\ 0 \\ \tilde{c} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ \vdots \\ i_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \tilde{b}a_0 + \tilde{b}b_0 \\ \tilde{b}c_0 \\ \tilde{c}a_0 + \tilde{c}b_0 \\ \tilde{c}c_0 \end{bmatrix}.$$

As we can see, with this $\lambda(x)$ the module $\lambda(x)R$ has at most 2^4 elements and thus can not be isomorphic to xR which has 2^5 elements. Hence we need $\tilde{a} = 1$. With this we get

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \tilde{b} \\ 0 \\ \tilde{c} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ \vdots \\ i_0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_0 + b_0 \\ c_0 \\ 0 \\ d_0 + e_0 \\ \tilde{b}a_0 + \tilde{b}b_0 \\ fc_0 \\ \tilde{c}a_0 + \tilde{c}b_0 + f_0 + h_0 \\ \tilde{c}c_0 + g_0 + i_0 \end{bmatrix}.$$

We find that $\tilde{c}$ does not actually contribute anything here and that if $\tilde{b} = 1$ we get the summand $\{[0, a, b, 0, c, a, b, d, e] | a, \dots, e \in \mathbb{F}_2\}$ and if $\tilde{b} = 0$ we instead have the summand $\{[0, a, b, 0, c, 0, 0, d, e] | a, \dots, e \in \mathbb{F}_2\} = xR$. Therefore any module isomorphic to xR has to be either xR itself or our other summand with 2^5 elements. (Do note, however, that this does not show they are isomorphic yet. This will be shown later on.)

Next up we will have a look at the idempotent $1 + x$ and possibilities for $\lambda(1 + x) := [a, \dots, i]$. Here we will have quite a lot more computing to do. We will again use $\lambda(1 + x) = \lambda((1 + x)^2)$ to see

$$\lambda(1+x) = \lambda((1+x)(1+x)) = (\lambda(1+x))(1+x) = \begin{bmatrix} a \\ b \\ c \\ \vdots \\ i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ a + c \\ c \\ d \\ e \\ d + g \\ g \\ e + i \\ i \end{bmatrix}.$$

(Do note that we again did use $2 = 0$ due to operating on $\mathbb{F}_2$.) Hence we further get for $\lambda(1 + x)R$:

$$\begin{aligned}
\lambda(1+x)r &= \begin{bmatrix} a \\ a+c \\ c \\ d \\ e \\ d+g \\ g \\ e+i \\ i \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ e_0 \\ f_0 \\ g_0 \\ h_0 \\ i_0 \end{bmatrix} \\
&= \begin{bmatrix} aa_0 \\ ab_0 + (a+c)b_0 + (a+c)a_0 + cb_0 \\ ac_0 + (a+c)c_0 + ca_0 + cc_0 \\ ad_0 + da_0 \\ ae_0 + (a+c)d_0 + (a+c)e_0 + cd_0 + ce_0 + ea_0 \\ af_0 + db_0 + (d+g)a_0 + (d+g)b_0 + gb_0 \\ ag_0 + dc_0 + (d+g)c_0 + ga_0 + gc_0 \\ ah_0 + (a+c)f_0 + (a+c)h_0 + cf_0 + ch_0 + eb_0 + (e+i)a_0 + (e+i)b_0 + ib_0 \\ ai_0 + (a+c)g_0 + (a+c)i_0 + cg_0 + ci_0 + ec_0 + (e+i)c_0 + ia_0 + ic_0 \end{bmatrix} \\
&= \begin{bmatrix} aa_0 \\ 2ab_0 + 2cb_0 + aa_0 + ca_0 \\ 2ac_0 + 2cc_0 + ca_0 \\ ad_0 + da_0 \\ 2ae_0 + ad_0 + 2cd_0 + 2ce_0 + ea_0 \\ af_0 + 2db_0 + da_0 + ga_0 + 2gb_0 \\ ag_0 + 2dc_0 + 2gc_0 + ga_0 \\ 2ah_0 + af_0 + 2cf_0 + 2ch_0 + 2eb_0 + ea_0 + ia_0 + 2ib0 \\ 2ai_0 + ag_0 + 2cg_0 + 2ci_0 + 2ec_0 + 2ic_0 + ia_0 \end{bmatrix} \\
&= \begin{bmatrix} aa_0 \\ aa_0 + ca_0 \\ ca_0 \\ ad_0 + da_0 \\ ad_0 + ea_0 \\ af_0 + da_0 + ga_0 \\ ag_0 + ga_0 \\ af_0 + ea_0 + ia_0 \\ ag_0 + ia_0 \end{bmatrix}.
\end{aligned}$$

We see that if $a = 0$ we have a module with only two elements (or one if also $b = c = \ldots = 0$), which is clearly not isomorphic to the 2^4-elemental $(a+x)R$. Thus $a = 1$ is required, which gives us

$$\lambda(1+x)r = \begin{bmatrix} a_0 \\ (1+c)a_0 \\ ca_0 \\ d_0 + da_0 \\ d_0 + ea_0 \\ f_0 + da_0 + ga_0 \\ g_0 + ga_0 \\ f_0 + ea_0 + ia_0 \\ g_0 + ia_0 \end{bmatrix}.$$

We see that for any choice of $b, \ldots, i$ this has the free variables a_0, d_0, f_0 and g_0 and hence any of these possibilities for $\lambda(1+x)$ generate modules with 2^4 elements. As we can see there is no $b,\ f$ and h left in $\lambda(1+x)r$ and thus we can omit looking at these, leaving us still having to check the possibilities of $c,\ d,\ e,\ g$ and i being 1 or 0. We can, however, shorten this as we only need to distinct whether $d = e$ and $g = i$ holds, as only these cases generate different modules. With this we are looking at eight possibilities:

1. $c = 0, d = e, g = i$: We get the module $\{[\tilde{a}, \tilde{a}, 0, \tilde{b}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{c}, \tilde{d}]|\tilde{a}, \ldots, \tilde{d} \in \mathbb{F}_2\}$ if we denote $\tilde{a} = a_0,\ \tilde{b} = d_0 + da_0 = d_0 + e_0,\ \tilde{c} = f_0 + da_0 + ga_0 = f_0 + ea_0 + ia_0$ and $\tilde{d} = g_0 + ga_0 = g_0 + ia_0$. This module is a summand, generated by $1 + x$.

2. $c = 0, d = e, g \neq i$: We get the module $\{[\tilde{a}, \tilde{a}, 0, \tilde{b}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{a} + \tilde{c}, \tilde{a} + \tilde{d}|\tilde{a} \ldots, \tilde{d} \in \mathbb{F}_2\}$ as $g \neq i \Leftrightarrow g = 1 + i$ due to calculating in $\mathbb{F}_2$. This module is the summand $(1 + x + xzx + xzy)R$.

3. $c = 0, d \neq e, g = i$: We get the module $\{[\tilde{a}, \tilde{a}, 0, \tilde{b}, \tilde{a} + \tilde{b}, \tilde{c}, \tilde{d}, \tilde{a} + \tilde{c}, \tilde{d}|\tilde{a} \ldots, \tilde{d} \in \mathbb{F}_2\}$. This module is the summand $(1 + x + xz + xzx)R$.

4. $c = 0, d \neq e, g \neq i$: We get the module $\{[\tilde{a}, \tilde{a}, 0, \tilde{b}, \tilde{a} + \tilde{b}, \tilde{c}, \tilde{d}, \tilde{c}, \tilde{a} + \tilde{d}|\tilde{a} \ldots, \tilde{d} \in \mathbb{F}_2\}$. Do note that in the penultimate entry we have $f_0 + ea_0 + ia_0 = f_0 + (1 + d)a_0 + (1 + g)a_0 = f_0 + da_0 + ga_0 = \tilde{c}$. This module is the summand $(1 + x + xz + xzy)R$.

5. $c = 1, d = e, g = i$: We get the module $\{[\tilde{a}, 0, \tilde{a}, \tilde{b}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{c}, \tilde{d}]|\tilde{a}, \ldots, \tilde{d} \in \mathbb{F}_2\}$, which is generated by $1 + y$.

6. $c = 1, d = e, g \neq i$: We get the module $\{[\tilde{a}, 0, \tilde{a}, \tilde{b}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{a} + \tilde{c}, \tilde{a} + \tilde{d}]|\tilde{a}, \ldots, \tilde{d} \in \mathbb{F}_2\}$, which is generated by $1 + y + xzx + xzy$.

7. $c = 1, d \neq e, g = i$: We get the module $\{[\tilde{a}, 0, \tilde{a}, \tilde{b}, \tilde{a} + \tilde{b}, \tilde{c}, \tilde{d}, \tilde{a} + \tilde{c}, \tilde{d}]|\tilde{a}, \ldots, \tilde{d} \in \mathbb{F}_2\}$, which is generated by $1 + y + xz + xzx$.

8. $c = 1, d \neq e, g \neq i$: Lastly, we get the module $\{[\tilde{a}, 0, \tilde{a}, \tilde{b}, \tilde{a} + \tilde{b}, \tilde{c}, \tilde{d}, \tilde{c}, \tilde{a} + \tilde{d}]|\tilde{a}, \ldots, \tilde{d} \in \mathbb{F}_2\}$, which is generated by $1 + y + xzx + xzy$.

As we can see any possible combination leads to one of the other summands, thus $(1+x)R$ can only be isomorphic to those.

Up until now we shown that xR and $(1+x)R$ can at most be isomorphic to other summands, the same was not shown for any other summand however. Now we could show the same thing for every summand the same way, which would be quite tedious. So instead we will show that all 2^4 and 2^5 elemental summands are indeed isomorphic to each other respectively. Then we can use the transitive property of isomorphisms to show that any module isomorphic to any summand must also be a summand as it would also be isomorphic to either xR or $(1+x)R$, for which we have already shown this.

In order to show that all of the summands with the same number of elements are indeed isomorphic to each other, we will use the Krull-Remak-Schmidt-Theorem.

To use it we will first have to decompose R_2 into some decomposition of modules with local endomorphism ring. This is where our first theorem comes into play. As any decomposition of any of our summands would also give new summands of R_2, there can be no non-trivial one, as then we would have a non-zero summand with less than 2^4 elements, which we know does not exist. Thus all of our summands are directly indecomposable. Furthermore, as they are finite, they are also of finite length. Thus the prerequisites of 6.4.4 hold and we get that all of our summands have local endomorphism ring.

With this we can use the Krull-Remak-Schmidt-Theorem (6.4.5). For this let $R_2 = M_1 \oplus M_2$ be a non-trivial fixed direct sum. Furthermore, let $R_2 = N_1 \oplus N_2$ be another non-trivial direct sum. As we have seen, all summands have either 2^4 or 2^5 elements, hence we will assume M_1 and N_1 each have 2^4 and M_2 and N_2 have 2^5 elements. We have already established M_1 and M_2 having local endomorphism rings and N_1 and N_2 being directly indecomposable. Therefore, by the Krull-Remak-Schmidt-Theorem, M_1 and N_1 as well as M_2 and N_2 are isomorphic, respectively. Finally, this implies all summands with 2^4 elements being isomorphic to M_1 and all summands with 2^5 elements being isomorphic to M_2.

□

As a furhter result we can now also show that R_2 does not fulfill $C1$. As such we have found another module that is $C2$ but not $C1$.

Theorem 6.4.7

R_2 does not fulfill $C1$.

Proof. We can show this in two different ways. First, we do know that a contionuous module M fulfills $\Delta(M) = Rad(M)$ by theorem 4.2.5. We have, however, seen in 6.3.4 that $\Delta(R_2) \neq Rad(R_2)$ (where we have not required operating on $\mathbb{F}_2$ yet). Thus R_2 can not be continuous. As it fulfills $C2$, it can not fulfill $C1$.

The other way to prove this would be to simply give a counter-example. We have already considered the ideal $I := R_2 z R_2 = \{[0,0,0,d,\dots,i] | d,\dots,i \in \mathbb{F}_2\}$. We can easily see that it is not contained in any of our summands (as all of them require some entries to be the same, whereas I does not have this restriction.) As such, if R_2 would fulfill $C1$, the ideal I would nessecarily need to be essential in the summand R_2 itself.

We have, however, seen in 6.2.10 that this is not the case. As such there is no summand in which I is essential, thus R_2 does not fulfill $C1$.

$\square$

6.5 The ring is no Utumi module

Now we want to check if the ring is a Utumi module. For this we will once again only focus on $F = \mathbb{F}_2$. To show that R_2 is no Utumi module, we will simply construct two submodules which will not fulfill the necessary property.

Lemma 6.5.1
Let $A = (x+y)R_2$ *and* $B = (xzx + xzy)R_2$. *Then* $A = \{(0, a, a, 0, \ldots, 0) | a \in \mathbb{F}_2\}$ *and* $B = \{0, \ldots, 0, b, b) | b \in \mathbb{F}_2\}$. *Thus* $A \cap B = 0$.

Proof. For a general element $r = \{a_0, \ldots, i_0\} \in R$ we get

$$(x+y)r = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ \vdots \\ i_0 \end{bmatrix} = \begin{bmatrix} 0 \\ b_0 + a_0 + b_0 \\ c_0 + a_0 + c_0 \\ 0 \\ d_0 + e_0 + d_0 + e_0 \\ 0 \\ 0 \\ f_0 + h_0 + f_0 + h_0 \\ g_0 + i_0 + g_0 + i_0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_0 \\ a_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$(xzx + xzy)r = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ \vdots \\ i_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_0 + b_0 + b_0 \\ c_0 + a_0 + c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_0 \\ a_0 \end{bmatrix}.$$

$\square$

Now that we have our two modules, we want to show that they are indeed isomorphic:

Lemma 6.5.2
Let $A, B \subset R$ *be the two ideals from lemma 6.5.1. Then* $\lambda : A \to B, (0, a, a, 0, \ldots, 0) \mapsto (0, \ldots, 0, a, a)$ *is an* $R-$*isomorphism.*

Proof. Since λ is well defined and preserves addition, we only have to check scalar multiplication. The multiplication of elements works exactly the same as in the proof of lemma 6.5.1 and is thus omitted. We get

- $\lambda((0, a, a, 0 \ldots, 0) \cdot (a_0, \ldots, i_0)) = \ldots = \lambda(0, aa_0, aa_0, 0, \ldots, 0) = (0, \ldots, 0, aa_0, aa_0)$ and
- $\lambda((0, a, a, 0 \ldots, 0)) \cdot (a_0, \ldots, i_0) = (0, \ldots, 0, a, a) \cdot (a_0, \ldots, i_0) = \ldots = (0 \ldots, 0, aa_0, aa_0)$.

Thus λ preserves the module-structure. Clearly it is a bijection. □

With this we can show that R_2 is no Utumi module:

Theorem 6.5.3
R_2 is no Utumi module.

Proof. Let $A, B \subset R_2$ be the modules from lemma 6.5.1. By that lemma we have $A \cap B = 0$ and by lemma 6.5.2 they are isomorphic. Thus, if R_2 was a Utumi module, there would be summands $K, L \subseteq^{\oplus} R_2$ with $A \subseteq^* K$, $B \subseteq^* L$ and $K \oplus L \subseteq^{\oplus} M$.

By lemma 6.4.2 we know all of the summands. Checking all of those, we see that A is contained in exactly the two summands xR_2 and $(x + zx + xzx)R_2$, which have 2^5 elements, and R_2 itself. We want to show that A is not essential in any of those. For this we will, yet again, consider the ideal $I = R_2zR_2 = \{(0, 0, 0, d, e, f, g, h, i) | d, \ldots, i \in \mathbb{F}_2\}$. We get the submodule

$$I \cap xR_2 = I \cap (x + zx + xzx)R_2 = \{(0, 0, 0, 0, e, 0, 0, h, i) | e, h, i \in \mathbb{F}_2\}$$

contained in both summands. Clearly $A \cap (I \cap xR_2) = 0$ and thus A is not essential in either of them. Likewise it is also not essential in R_2. Thus we can not find any summand K in which A is essential, proving R_2 is not a Utumi module. □

With this we also have another proof that R_2 does not fulfill $C1$, as then it would be continuous and thus a Utumi module. In the same way we can prove the following:

Corollary 6.5.4
R_2 is not automorphism-invariant.

Proof. Assuming R_2 was automorphism-invariant, it would also be a Utumi module by theorem 5.2.7, contradicting the previous theorem. □

6.6 Conclusion

Finally let us summarize our results for the ring $R = F < x, y, z > /(x^2 - x, y^2 - y, xy - y, yx - x, yz - xz, z^2, zxz)$:

- For the ideal $I = RzR$ generated by z, we have for any field F:
    - $I \subseteq Rad(R)$. (Check 6.2.1)

  - R/I is clean. (Check 6.2.7)

- R is an exchange ring for any field F. (Check 6.2.9)
- $\Delta(R, R) \subsetneq Rad(R)$ for any field F. (Check 6.3.4)
- R does fulfill $C2$ for $F = \mathbb{F}_2$. (Check 6.4.6)
- R does not fulfill $C1$ for $F = \mathbb{F}_2$. (Check 6.4.7)
- R is no Utumi module for $F = \mathbb{F}_2$. (Check 6.5.3)
- R is not automorphism-invariant for $F = \mathbb{F}_2$. (Check 6.5.4)

Hence, at least in the special case $F = \mathbb{F}_2$, the module R_R is an example of a module that does fulfill $C2$ whilst neither being continuous nor being automorphism-invariant. Furthermore it is an example of a module that does fulfill $C2$ whilst not being a Utumi module.

Also, due to R fulfilling $C2$ and $\Delta(R, R) \neq Rad(R)$, we see that we can not improve lemma 4.2.16, which says that having $C2$ implies $\Delta(M) \subseteq Rad(M)$, to an equation. This is notable, as both types of modules, which imply $C2$ (namely continuous and automorphism-invariant modules), do fulfill this equation.

In the same manner we see that having the exchange property does not imply $\Delta(M, M) = Rad(M)$ either. Here we also only have the weaker $\Delta(M, M) \subseteq Rad(M)$, which follows from the $B2-$exchange property.

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[SE] The proof of Lemma 6.2.3 is taken from an answer of the user rschwieb on https://math.stackexchange.com/questions/2023115/lifting-idempotents-modulo-a-nilpotent-ideal . The user does not cite a source but says they have seen the proof elsewhere. No better source could be found.

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