Epistemic ensembles in semantic, symbolic, and distributed environments

Alexander Knapp, Rolf Hennicker, Martin Wirsing

Angaben zur Veröffentlichung / Publication details:

Knapp, Alexander, Rolf Hennicker, and Martin Wirsing. 2026. “Epistemic ensembles in semantic, symbolic, and distributed environments.” *International Journal on Software Tools for Technology Transfer*. https://doi.org/10.1007/s10009-026-00855-z.

International Journal on Software Tools for Technology Transfer
https://doi.org/10.1007/s10009-026-00855-z

FOUNDATION FOR MASTERING CHANGE

ReoCAS 2024

Epistemic ensembles in semantic, symbolic, and distributed environments

Alexander Knapp[1] · **Rolf Hennicker**[2] · **Martin Wirsing**[2]

Accepted: 5 March 2026

Abstract

Epistemic ensembles are systems of knowledge-based agents capable of accessing, sharing, and updating information about themselves and their peers. These agents can operate on shared or local epistemic states through actions that dynamically alter the knowledge of some or all members of the ensemble. To support abstract reasoning over such systems, we introduce the notion of focus set — a selected set of logical formulæ that provide an abstraction from the underlying system state. Based on this abstraction, we define global and distributed symbolic representations of epistemic states, along with representable epistemic actions that enable efficient symbolic updates. For formal analysis, we define a generic operational semantics and develop and relate three complementary semantic frameworks: (1) a semantic environment, where system states are modelled as epistemic states, (2) a symbolic environment, where knowledge is represented as sets of logical formulæ, and (3) a distributed environment, represented by a family of local knowledge bases where each agent has its own local symbolic state. We establish a correspondence between these environments via a notion of relative elementary equivalence. Our main result demonstrates that equivalent configurations simulate each other's behaviour and satisfy the same dynamic epistemic formulæ, ensuring representational consistency across all three perspectives. This provides a robust foundation for reasoning about distributed knowledge and belief dynamics in cooperative multi-agent systems.

Keywords Ensembles · Epistemic logic · Multi-agent systems · Knowledge bases · Abstraction · Symbolic execution

1 Introduction

Ensembles [22, 33] are collective systems composed of dynamically interacting autonomic entities. In epistemic ensembles [19], these entities are epistemic agents — agents that possess knowledge and beliefs about themselves and others. These agents engage in epistemic actions to communicate and collaborate, operating in global or local knowledge contexts. An epistemic action typically involves making an agent's knowledge or beliefs known to other agents, thereby updating the collective epistemic state of the ensemble. Applications of the epistemic approach to ensembles include epistemic planning (see, e.g., [4]), gossip protocols (see, e.g., [31]), knowledge puzzles (see, e.g., [29, 30]), and the specification of behavioural properties of ensembles (see, e.g., [26]).

This paper extends our earlier work Hennicker et al. [19, 20] and, in particular, our REoCAS 2024 paper [21]. In Hennicker et al. [19], we introduced the concept of epistemic ensembles in a semantic environment characterized by a single epistemic state. Subsequently, in Hennicker et al. [20], we investigated epistemic processes and their interaction with symbolic environments, still within the context of a single semantic epistemic state. In Hennicker et al. [21], we considered an extended notion of semantic environment consisting not only of a single epistemic state but of a class of epistemic states.

In the present work, we advance this line of research by addressing also truly distributed epistemic ensembles, where individual agents maintain local symbolic states. We base our approach on a common generic operational semantics (cf. [18]) for epistemic processes and investigate the ensemble behaviour across three complementary frameworks:

✉ A. Knapp
knapp@informatik.uni-augsburg.de

R. Hennicker
hennicker@ifi.lmu.de

M. Wirsing
wirsing@ifi.lmu.de

[1] Universität Augsburg, Augsburg, Germany

[2] Ludwig-Maximilians-Universität München, Munich, Germany

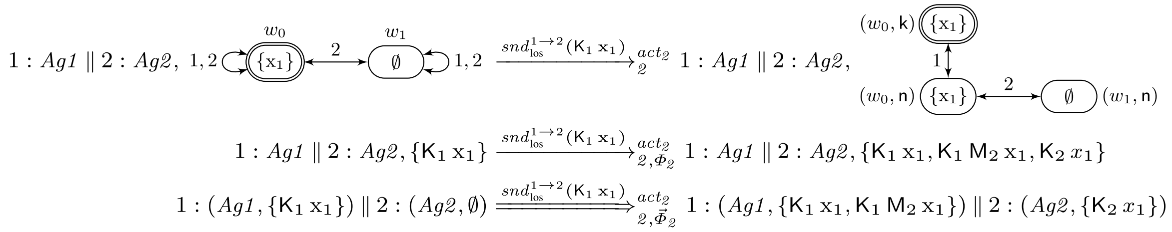


Fig. 1 Three equivalent variants of executing epistemic ensembles: global semantic, global symbolic, and distributed symbolic (see Ex. 2.1, and Figs. 2, 3 and 4)

- In a *semantic* global environment that is modelled by a global epistemic state; this environment is represented using a pointed Kripke structure, the standard formalism for describing knowledge states, for instance, in game-theoretic contexts.
- In a *symbolic* global environment that captures the collective knowledge through a global knowledge base consisting of a set of epistemic formulæ.
- In a *distributed* symbolic environment that represents decentralized knowledge using a family of local knowledge bases, where each agent maintains its own symbolic state.

Agent behaviour within an ensemble is specified using a lightweight process algebra featuring guards and recursion. Each agent has its own process and we require that it has sufficient knowledge to decide its guards and to execute its actions. The ensemble's operational semantics are generically defined via conditional transitions over uninterpreted agent actions. Both global and local properties of epistemic ensembles are specified using propositional dynamic logic, interpreted over compound ensemble actions.

While traditional Kripke structures offer a well-established means of modelling epistemic states, they tend to scale poorly due to their combinatorial complexity. Symbolic environments offer a more compact and scalable alternative. However, the complete theory of knowledge is typically infinite—and even when finite, it often consists of a prohibitively large set of formulæ. To manage this complexity, we introduce a predefined finite set of focus formulæ that selectively capture relevant global knowledge about the agents.

From a software development perspective, the combination of agent processes and the semantic environment constitutes a high-level description of the ensemble. This (constructive) specification is first implemented by transitioning from the semantic to the symbolic environment, and then further refined by introducing local environments for each agent, resulting in a truly distributed ensemble.

To relate these environments, we introduce a notion of relative elementary equivalence, whereby two semantic configurations are considered equivalent if they validate the same formulæ from the focus set of formulæ — or, in the distributed setting, from a corresponding family of such sets.

Our main contribution lies in the comparative study of the three semantic frameworks introduced above. In the semantic global environment, agent actions are interpreted via pointed action models, and their effects are captured using product updates on Kripke structures [1, 2]. In both symbolic environments, i.e., in the global and in the distributed one, we employ weakest liberal precondition reasoning [1, 14] to define the effect of actions.

We show how the equivalence between epistemic, global symbolic, and distributed local symbolic states can be maintained under action updates. To ensure this, we introduce the notions of action representability — and, in the distributed setting, local action representability and disjunction witnesses. These concepts provide sufficient conditions for updates to preserve equivalence.

Our main result demonstrates that ensemble configurations — across semantic and symbolic global environments, as well as distributed symbolic environments — can simulate one another and satisfy the same dynamic ensemble formulæ. We illustrate these equivalences by the first steps of a single ensemble in the three different environments in Fig. 1.

Related work Research on using epistemic logic in system modelling and programming began with the seminal works on Dynamic Epistemic Logic (DEL, see [2, 30]) and knowledge-based programs [15]. DEL focuses on modelling and verifying knowledge changes induced by action execution. We use DEL's reduction rules to define the weakest liberal precondition, which also underpin many foundational results of DEL, such as soundness and completeness [1, 2], as well as the definition of a sequent calculus [32]. Knowledge-based programs consider systems of concurrently running agents with local state variables. van Benthem et al. [28] examine protocols based on DEL and, as Parikh and Ramanujam [25], system properties in epistemic temporal logic.

Our work is based on a process-oriented description of dynamic system behaviour [12], in alignment with modern languages for ensembles such as SCEL [13], CARMA [7], and DEECo [9]. However, these languages rely on communication mechanisms like message passing and predicate-based communication, rather than epistemic actions. Our symbolic semantics for epistemic representation is related to the syntactic structures in Hennicker et al. [19] and belief bases in Lorini et al. [23]. While Lorini et al.'s work is also based on DEL, it lacks a process-algebraic setting and uses specialised forms of knowledge update operations. In contrast, we consider general action models and we use a distribution from global to local symbolic states.

Local epistemic states and local epistemic actions were introduced by Bolander and Andersen [5]. Our local knowledge bases can be seen as symbolic counterparts to these local epistemic states, while our locally representable actions correspond to local epistemic actions in their model. However, the focus of our work differs significantly: Whereas Bolander and Andersen [5] investigate the decidability of epistemic planning, our primary interest lies in specifying ensemble properties in terms of a propositional dynamic logic and analysing the relationships between different environmental frameworks for epistemic ensembles.

The transition from a global specification to a distributed, local implementation has been considered for several formalisms, like Message Sequence Charts [16], transition systems and process algebras [11, 27], or (multi-party) session types [10]. However, we are not aware of any symbolic approach.

Differences to conference version In the conference version [21], we investigated an abstraction from semantic epistemic states to global symbolic knowledge bases and showed that the abstraction preserves and reflects dynamic ensemble properties. Here, we add distribution as a second development step: Building on the previous global abstraction we demonstrate how a global symbolic knowledge base can be faithfully distributed into a family of local knowledge bases, one for each agent. We prove that this refinement preserves and reflects dynamic ensemble properties. We illustrate this development chain by a running example inspired by Witzel and Zvesper [34] and we include all proofs.

Structure of the paper We first introduce epistemic ensembles and their generic operational semantics in Sect. 2 and dynamic ensemble formulæ in Sect. 3. The agent actions of epistemic ensembles are interpreted by action models in Sect. 4. Section 5 and Sect. 6 present the global semantic and the global symbolic environment of epistemic ensembles. The equivalence of these semantics is shown in Sect. 7. In Sect. 8 we introduce distributed symbolic states and in Sect. 9 we show the equivalence of global symbolic semantics with local symbolic semantics and, using the results of Sect. 7, as a corollary the equivalence of all three semantics. We conclude in Sect. 10.

2 Epistemic ensembles

An epistemic ensemble is formed by a collection of agents which run concurrently to accomplish a certain task. In the epistemic context, collaboration of agents is achieved by the execution of agent actions where agents inform other agents about (parts of) their knowledge, which may concern themselves, other agents, or the environment. Each agent follows a certain protocol which is given by an epistemic process description. In this section we introduce the syntactic notions for building epistemic ensembles and we provide a generic operational semantics for epistemic ensembles which will be instantiated later on for executing ensembles in semantic, symbolic, and distributed symbolic environments.

Epistemic formulæ Epistemic formulæ provide the means to describe knowledge; see, e.g., Baltag and Renne [1], Fagin et al. [15], van Ditmarsch et al. [30]. An *epistemic signature* $\Sigma = (\Pi, A)$ consists of a set Π of (atomic) *propositions* and a set A of *agents*. The set $\mathscr{F}$ of *epistemic formulæ* φ over $\Sigma = (\Pi, A)$ is defined by the following grammar:

$$\varphi ::= p \mid \mathsf{true} \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathsf{K}_a\,\varphi \qquad \text{where } p \in \Pi \text{ and } a \in A.$$

The epistemic formula $\mathsf{K}_a\,\varphi$ is to be read as "agent *a knows* φ". We use the usual Boolean shorthand notations like false for $\neg\mathsf{true}$, $\varphi_1 \vee \varphi_2$ for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$, etc. Moreover, we write $\mathsf{M}_a\,\varphi$ for $\neg\mathsf{K}_a\,\neg\varphi$; this latter epistemic modality is dual to K_a and to be read as "agent *a deems* φ *possible*". For each $a \in A$, the set $\mathscr{F} \upharpoonright a$ of *a-epistemic formulæ* φ_a having an a-modality as their top operator, is given by the following grammar:

$$\varphi_a ::= \mathsf{true} \mid \neg\varphi_a \mid \varphi_{a,1} \wedge \varphi_{a,2} \mid \mathsf{K}_a\,\varphi \quad \text{where } \varphi \in \mathscr{F}.$$

Since in our ensembles each agent's behaviour will solely depend on its own knowledge, only formulæ from $\mathscr{F} \upharpoonright a$ are eligible for a process for agent a, as these can be decided autonomously. Thus, for any $\varphi \in \mathscr{F}$ we define the set of possible *agents* of φ by $\mathrm{ags}(\varphi) = \{a \in A \mid \varphi \in \mathscr{F} \upharpoonright a\}$. In particular, $\mathrm{ags}(\varphi)$ is either a singleton, or A (if no K_a occurs in φ), or $\emptyset$.

Example 2.1

Our running example is inspired by Witzel and Zvesper [34]. We consider a set of two agents $A_2 = \{1, 2\}$ each one holding a bit $\mathrm{x}_i \in \Pi_2 = \{\mathrm{x}_1, \mathrm{x}_2\}$. The epistemic signature is $\Sigma_2 = (\Pi_2, A_2)$. In a situation where x_1 is true, the 1-formula $\mathsf{K}_1\,\mathrm{x}_1$ expresses that agent 1 knows this. The 2-formula $\mathsf{K}_2\,\mathrm{x}_1 \vee \mathsf{K}_2\,\neg\mathrm{x}_1$ says that agent 2 knows the value of x_1 but we cannot infer its concrete value; we abbreviate this by $\mathsf{K}_2\,x_1$, using an italic font for designating that we refer to the *value*. The

1-formula $\neg \mathsf{K}_1\,\mathsf{K}_2\,x_1$ expresses that agent 1 does not know whether agent 2 knows the value of x_1. Hence, $\mathrm{ags}(\mathsf{K}_1\,\mathrm{x}_1) = \{1\} = \mathrm{ags}(\neg \mathsf{K}_1\,\mathsf{K}_2\,x_1)$. However, that agent 1 knows x_1 or agent 2 knows x_1 is not governed by a single agent, i.e., $\mathrm{ags}(\mathsf{K}_1\,\mathrm{x}_1 \vee \mathsf{K}_2\,\mathrm{x}_1) = \emptyset$.

Epistemic ensemble signatures Our formalisation of epistemic ensembles is based on the notion of an *epistemic ensemble signature* $\check{\Sigma} = (\Sigma, \vec{\mathrm{N}})$ consisting of an epistemic signature $\Sigma = (\Pi, A)$ and an A-family $\vec{\mathrm{N}} = (\mathrm{N}_a)_{a \in A}$ of pairwise disjoint sets N_a of *agent action symbols*, briefly called *agent actions*. Each set N_a determines which actions are possible for agent a. We write $\bigcup \vec{\mathrm{N}}$ for $\bigcup_{a \in A} \mathrm{N}_a$ and for each $\eta \in \bigcup \vec{\mathrm{N}}$ we set $\mathrm{ags}(\eta) = \{a\}$ if $\eta \in \mathrm{N}_a$.

Example 2.2
Continuing Ex. 2.1 we introduce an action *stop* and two actions $\mathit{snd}_{\mathrm{los}}^{1\to 2}(\mathsf{K}_1\,\mathrm{x}_1)$ and $\mathit{snd}_{\mathrm{los}}^{1\to 2}(\mathsf{K}_1\,\neg\mathrm{x}_1)$ for agent 1. The sending actions are used to express that agent 1 sends the value of x_1 to agent 2 depending whether it knows that x_1 is true or false, but, in each case, the information transfer is unreliable or lossy since agent 2 may be too far away. Thus, the action symbols for agent 1 are

$$\mathrm{N}_{2,1} = \{\mathit{stop}, \mathit{snd}_{\mathrm{los}}^{1\to 2}(\mathsf{K}_1\,\mathrm{x}_1), \mathit{snd}_{\mathrm{los}}^{1\to 2}(\mathsf{K}_1\,\neg\mathrm{x}_1)\}\ .$$

On the other hand, agent 2 (when it got the value of x_1) may acknowledge in a reliable way the reception with the action $\mathit{snd}_{\mathrm{rel}}^{2\to 1}(\mathsf{K}_2\,x_1)$, i.e.,

$$\mathrm{N}_{2,2} = \{\mathit{snd}_{\mathrm{rel}}^{2\to 1}(\mathsf{K}_2\,x_1)\}\ .$$

(Recall that $\mathsf{K}_2\,x_1$ abbreviates $\mathsf{K}_2\,\mathrm{x}_1 \vee \mathsf{K}_2\,\neg\mathrm{x}_1$.)

General assumption
In the sequel, we always assume given an epistemic ensemble signature $\check{\Sigma} = (\Sigma, \vec{\mathrm{N}})$ with underlying epistemic signature $\Sigma = (\Pi, A)$.

Epistemic processes We consider an epistemic process language for describing the behaviour of agents which participate in an epistemic ensemble. The set $\mathcal{P}$ of *epistemic processes* P over $\check{\Sigma}$ is defined by the grammar

$$P ::= \mathbf{0} \mid \eta.P \mid \varphi \supset P \mid P_1 + P_2 \mid \mu X\,.\,P \mid X$$

where $\mathbf{0}$ represents the inactive process, $\eta.P$ prefixes P with an agent action $\eta \in \bigcup \vec{\mathrm{N}}$, $\varphi \supset P$ is a guarded process with condition $\varphi \in \mathscr{F}$, $P_1 + P_2$ denotes the non-deterministic choice between processes P_1 and P_2, $\mu X\,.\,P$ is a recursive process, and X is a process variable typically used in recursive process definitions.

The *operational semantics* of epistemic processes is given by *conditional transitions* $P \xhookrightarrow{\widehat{\varphi}:\eta} P'$ relating a process P via a guard $\widehat{\varphi} \in \mathscr{F}$ and an agent action $\eta \in \bigcup \vec{\mathrm{N}}$ with another process P'. The transitions are defined inductively by the rules in Table 1, where successive guards are conjoined and true represents the empty guard. The operational semantic rules are generic and syntax-based; the evaluation of guards and applicability as well as effects of actions are instantiated through the environment in which processes run (considered later in Sects. 5, 6 and 8).

Table 1 Rules for epistemic processes

$$\eta.P \xhookrightarrow{\mathrm{true}:\eta} P \qquad\qquad \frac{P \xhookrightarrow{\widehat{\varphi}:\eta} P'}{\varphi \supset P \xhookrightarrow{\widehat{\varphi}\wedge\varphi:\eta} P'}$$

$$\frac{P_\ell \xhookrightarrow{\widehat{\varphi}:\eta} P'_\ell}{P_1 + P_2 \xhookrightarrow{\widehat{\varphi}:\eta} P'_\ell} \text{ for } \ell \in \{1,2\} \qquad \frac{P\{X \mapsto \mu X\,.\,P\} \xhookrightarrow{\widehat{\varphi}:\eta} P'}{\mu X\,.\,P \xhookrightarrow{\widehat{\varphi}:\eta} P'}$$

Epistemic processes are used to describe the behaviour of single agents. But not any process expression in $\mathcal{P}$ is eligible for any agent process. For instance, a process $\eta.\mathbf{0}$ can be carried out by an agent a only if the action symbol η represents an action of a, i.e., $\eta \in \mathrm{N}_a$. Similarly, any compound process can only be executed by agents that can execute each of its parts. For any process $P \in \mathcal{P}$ the set $\mathrm{ags}(P)$ of agents that can meaningfully execute P is inductively defined by

$$\begin{aligned}
\mathrm{ags}(\mathbf{0}) &= A \\
\mathrm{ags}(\eta.P) &= \mathrm{ags}(\eta) \cap \mathrm{ags}(P) \\
\mathrm{ags}(\varphi \supset P) &= \mathrm{ags}(\varphi) \cap \mathrm{ags}(P) \\
\mathrm{ags}(P_1 + P_2) &= \mathrm{ags}(P_1) \cap \mathrm{ags}(P_2) \\
\mathrm{ags}(\mu X\,.\,P) &= \mathrm{ags}(P) \\
\mathrm{ags}(X) &= A
\end{aligned}$$

For executing a process $P \in \mathcal{P}$ the following kind of *subject reduction* holds:

Lemma 2.1
If $P \xhookrightarrow{\widehat{\varphi}:\eta} P'$*, then* $\mathrm{ags}(P) \subseteq \mathrm{ags}(\widehat{\varphi}) \cap \mathrm{ags}(\eta) \cap \mathrm{ags}(P')$.

Proof
We proceed by induction on the derivation of the conditional transition $P \xhookrightarrow{\widehat{\varphi}:\eta} P'$ by the rules in Table 1.

For $\eta.P \xhookrightarrow{\mathrm{true}:\eta} P$, we have $\mathrm{ags}(\eta.P) = \mathrm{ags}(\eta) \cap \mathrm{ags}(P) \subseteq \mathrm{ags}(\mathrm{true}) \cap \mathrm{ags}(\eta) \cap \mathrm{ags}(P)$.

For $\varphi \supset P \xhookrightarrow{\widehat{\varphi}\wedge\varphi:\eta} P'$, we have $\mathrm{ags}(P) \subseteq \mathrm{ags}(\widehat{\varphi}) \cap \mathrm{ags}(\eta) \cap \mathrm{ags}(P')$ as the induction hypothesis and thus we obtain $\mathrm{ags}(\varphi \supset P) = \mathrm{ags}(\varphi) \cap \mathrm{ags}(P) \subseteq \mathrm{ags}(\varphi) \cap \mathrm{ags}(\widehat{\varphi}) \cap \mathrm{ags}(\eta) \cap \mathrm{ags}(P') = \mathrm{ags}(\widehat{\varphi} \wedge \varphi) \cap \mathrm{ags}(\eta) \cap \mathrm{ags}(P')$.

For $P_1 + P_2 \xhookrightarrow{\widehat{\varphi}:\eta} P'_\ell$ with $\ell \in \{1,2\}$, we have $\mathrm{ags}(P_\ell) \subseteq \mathrm{ags}(\widehat{\varphi}) \cap \mathrm{ags}(\eta) \cap \mathrm{ags}(P'_\ell)$ as the induction hypothesis and thus we obtain $\mathrm{ags}(P_1 + P_2) = \mathrm{ags}(P_1) \cap \mathrm{ags}(P_2) \subseteq \mathrm{ags}(\widehat{\varphi}) \cap \mathrm{ags}(\eta) \cap \mathrm{ags}(P'_\ell)$.

For $\mu X\,.\,P \xhookrightarrow{\widehat{\varphi}:\eta} P'$, we have $\mathrm{ags}(P\{X \mapsto \mu X\,.\,P\}) \subseteq \mathrm{ags}(\widehat{\varphi}) \cap \mathrm{ags}(\eta) \cap \mathrm{ags}(P')$ as the induction hypothesis and thus $\mathrm{ags}(\mu X\,.\,P) = \mathrm{ags}(P) \subseteq \mathrm{ags}(\widehat{\varphi}) \cap \mathrm{ags}(\eta) \cap \mathrm{ags}(P')$ using that $\mathrm{ags}(P\{X \mapsto \mu X\,.\,P\}) = \mathrm{ags}(P)$. □

Epistemic ensembles Now we have all ingredients to define epistemic ensembles as families of epistemic agent processes. Formally, an *epistemic ensemble* over $\check{\Sigma}$ is given by a family $\vec{E} = (a : P_a)_{a \in A}$ such that, for each $a \in A$, P_a is an epistemic process in $\mathcal{P}$ with $a \in \mathrm{ags}(P_a)$, i.e., a is allowed to perform P_a.

For notational reasons and for defining the operational semantics of ensembles, we also consider (sub-)families $(a : P_a)_{a \in G}$ of agent processes with $G \subseteq A$; their *composition* $(a_1 : P_{a_1})_{a_1 \in G_1} \parallel (a_2 : P_{a_2})_{a_2 \in G_2}$ for disjoint $G_1, G_2 \subseteq A$ as the family $(a : P_a)_{a \in G_1 \cup G_2}$; and, for $i \in A$, the *singleton* $i : P_i$ which stands for $(a : P_a)_{a \in \{i\}}$.

Example 2.3
Relying on the epistemic ensemble signature developed in Ex. 2.1 and Ex. 2.2, we consider the following simple epistemic ensemble *Sys* with two processes *Ag1* for agent 1 and *Ag2* for agent 2. In the process descriptions we abbreviate, as above, for each $a \in \{1, 2\}$, the formula $\mathsf{K}_a\,\mathrm{x}_1 \vee \mathsf{K}_a\,\neg\mathrm{x}_1$ by $\mathsf{K}_a\,x_1$ (x_1 written in italics).

$$\begin{aligned}
\mathit{Ag1} = \mu X\,.\,(&\neg\mathsf{K}_1\,\mathsf{K}_2\,x_1 \supset \\
&\qquad (\mathsf{K}_1\,\mathrm{x}_1 \supset \mathit{snd}^{1\to 2}_{\mathrm{los}}(\mathsf{K}_1\,\mathrm{x}_1).X\ + \\
&\qquad\ \mathsf{K}_1\,\neg\mathrm{x}_1 \supset \mathit{snd}^{1\to 2}_{\mathrm{los}}(\mathsf{K}_1\,\neg\mathrm{x}_1).X)\ + \\
&\ \mathsf{K}_1\,\mathsf{K}_2\,x_1 \supset \mathit{stop}.\mathbf{0})
\end{aligned}$$

$$\mathit{Ag2} = \mathsf{K}_2\,x_1 \supset \mathit{snd}^{2\to 1}_{\mathrm{rel}}(\mathsf{K}_2\,x_1).\mathbf{0}$$

$$\mathit{Sys} = 1 : \mathit{Ag1} \parallel 2 : \mathit{Ag2}$$

By $\mathit{snd}^{1\to 2}_{\mathrm{los}}(\mathsf{K}_1\,\mathrm{x}_1)$ and $\mathit{snd}^{1\to 2}_{\mathrm{los}}(\mathsf{K}_1\,\neg\mathrm{x}_1)$ agent 1 repeatedly tells agent 2 the value of x_1 (in an unreliable way) until it is sure that agent 2 knows the value; when having indeed learnt the value of x_1, agent 2 acknowledges this fact (in a reliable way) to agent 1 with $\mathit{snd}^{2\to 1}_{\mathrm{rel}}(\mathsf{K}_2\,x_1)$.

The *generic operational semantics* of epistemic ensembles is given by *conditional transitions* $\hookrightarrow$ relating an ensemble $\vec{E}$ via a guard $\widehat{\varphi} \in \mathcal{F}$ and an agent action η with another ensemble $\vec{E}'$ according to the following rule which is based on the rules for processes:

$$a : P_a \parallel \vec{E} \xhookrightarrow{\widehat{\varphi}:\eta} a : P'_a \parallel \vec{E} \quad \text{if } P_a \xhookrightarrow{\widehat{\varphi}:\eta} P'_a \qquad (*)$$

Note that each ensemble step is well-defined, since $a \in \mathrm{ags}(P_a)$ implies $a \in \mathrm{ags}(P'_a)$ by Lem. 2.1, and thus $a : P'_a \parallel \vec{E}$ is again an ensemble. Moreover, the ordering of processes in an ensemble is irrelevant since they are families, i.e., functions mapping agents to processes.

The rule (*) is formulated in a generic way without considering evaluations of guards and effects of actions. It provides, however, a convenient basis for concrete instantiations.

3 Dynamic epistemic ensemble logic

So far we have considered a constructive approach for representing epistemic ensembles with local processes for each agent. We are now interested in a declarative language for expressing and specifying behavioural properties of epistemic ensembles from a global perspective. For this purpose we use formulæ in the style of propositional dynamic logic [17] where regular expressions of agent actions, called compound ensemble actions, are used as modalities. In contrast to local agent processes, we have now a global view where actions of different agents can be combined and properties of whole ensembles can be stated.

The set C of *compound ensemble actions* over $\check{\Sigma}$ is defined by the following grammar:

$$\lambda ::= \eta \mid \varphi? \mid \lambda_1 + \lambda_2 \mid \lambda_1;\lambda_2 \mid \lambda^* \qquad \text{where } \eta \in \bigcup\vec{\mathrm{N}} \text{ and } \varphi \in \mathcal{F}.$$

Besides agent actions the compound ensemble actions include a *test* $\varphi?$ on an epistemic formula $\varphi \in \mathcal{F}$, *non-deterministic choices* $\lambda_1 + \lambda_2$ of compound ensemble actions λ_1 and λ_2, *sequential compositions* $\lambda_1;\lambda_2$, and *sequential loops* λ^*.

Following the style of propositional dynamic logic the set $\mathcal{D}$ of *epistemic ensemble formulæ* over $\check{\Sigma}$ is defined by the following grammar where compound ensemble actions are used as modalities:

$$\psi ::= \mathrm{true} \mid \varphi \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid [\lambda]\psi \qquad \text{where } \varphi \in \mathcal{F} \text{ and } \lambda \in C.$$

The formula $[\lambda]\psi$ is to be read as "after all possible executions of the compound action λ formula ψ holds". We use the usual abbreviations like false or $\vee$ as before, and we write $\langle\lambda\rangle\psi$ for $\neg[\lambda]\neg\psi$; this latter dynamic modality is dual to $[\lambda]$ and to be read as "there is some execution of λ such that ψ holds afterwards".

Example 3.1
For our two-agents system we are interested in the following properties, in which we abbreviate the compound action $\mathit{stop} + \mathit{snd}^{1\to 2}_{\mathrm{los}}(\mathsf{K}_1\,\mathrm{x}_1) + \mathit{snd}^{1\to 2}_{\mathrm{los}}(\mathsf{K}_1\,\neg\mathrm{x}_1) + \mathit{snd}^{2\to 1}_{\mathrm{rel}}(\mathsf{K}_2\,x_1)$ by "*some*":

Table 2 Rules for witnessing compound ensemble actions

$$(\vec{E}, \widehat{\varphi} : \eta, \vec{E}') :: \eta \quad \text{if } \vec{E} \xhookrightarrow{\widehat{\varphi}:\eta} \vec{E}'$$

$$(\vec{E}, \varphi : \epsilon, \vec{E}) :: \varphi ?$$

$$\frac{(\vec{E}, \sigma, \vec{E}') :: \lambda_\ell}{(\vec{E}, \sigma, \vec{E}') :: \lambda_1 + \lambda_2} \text{ for } \ell \in \{1, 2\}$$

$$\frac{(\vec{E}, \sigma_1, \vec{E}') :: \lambda_1 \quad (\vec{E}', \sigma_2, \vec{E}'') :: \lambda_2}{(\vec{E}, \sigma_1 \cdot \sigma_2, \vec{E}'') :: \lambda_1 ; \lambda_2}$$

$$(\vec{E}, \text{true} : \epsilon, \vec{E}) :: \lambda^*$$

$$\frac{(\vec{E}, \sigma, \vec{E}') :: \lambda \quad (\vec{E}', \sigma', \vec{E}'') :: \lambda^*}{(\vec{E}, \sigma \cdot \sigma', \vec{E}'') :: \lambda^*}$$

1. "As long as agent 1 does not know that agent 2 knows the value of x_1 the ensemble will not stop":

$$[\mathit{some}^*]\neg \mathsf{K}_1\, \mathsf{K}_2\, x_1 \rightarrow \langle \mathit{some} \rangle \text{true}$$

2. "Whenever agent 1 tells the value of x_1 to agent 2, it is possible that agent 2 will eventually know the value":

$$[\mathit{some}^*; (\mathit{snd}^{1\to 2}_{\mathrm{los}}(\mathsf{K}_1\, \mathrm{x}_1) + \mathit{snd}^{1\to 2}_{\mathrm{los}}(\mathsf{K}_1\, \neg \mathrm{x}_1))] \qquad \langle \mathit{some}^* \rangle \mathsf{K}_2\, x_1$$

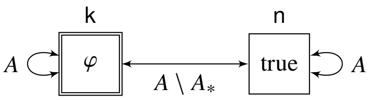
We cannot provide a formal satisfaction relation here saying when an epistemic ensemble satisfies an epistemic ensemble formula, since we cannot talk yet about satisfaction of ensemble formulæ as long as we do not have a concrete formalisation of epistemic states and of the effect of agent actions on them. But we can already now provide a definition to capture, for each compound ensemble action λ, which sequences $\sigma = (\widehat{\varphi}_1 : \eta^\epsilon_1) \dots (\widehat{\varphi}_k : \eta^\epsilon_k)$ of guard and agent action pairs can be executed when moving stepwise from an ensemble $\vec{E}$ to an ensemble $\vec{E}'$ in accordance with rule (*) from above. Thereby concatenation of such two sequences σ_1 and σ_2 is denoted by $\sigma_1 \cdot \sigma_2$ and ϵ denotes an artificial empty agent action. The inductive rules in Table 2 define when a (stepwise) move $(\vec{E}, \sigma, \vec{E}')$ is a *witness* for a compound ensemble action λ, written $(\vec{E}, \sigma, \vec{E}') :: \lambda$.

4 Action models for agent actions

Going to a concrete interpretation of ensembles a crucial step is to assign meaning to the agent actions occurring in an epistemic ensemble signature. A direct way would be to fix a domain for modelling states and to associate to each agent action η a relation (or a proper function) which models the effect of η. This has been done, for instance, by Fagin et al. [15] who associate to (joint) agent actions a so-called "global state transformer"; or by van Ditmarsch et al. [30] by epistemic learning actions and their updates. In our work we pursue a different approach and use the notion of "action model" introduced by Baltag et al. [2] to provide meaning for agent actions. The advantage is that action models still have a syntactic flavour and thus are still independent of the particular denotations used for epistemic states. Most approaches use action models in the context of Kripke structures as mathematical objects for epistemic states on which updates caused by actions are defined. But, in principle, action models allow also other domains for interpretation. For instance, in Hennicker et al. [20] we have provided an interpretation using symbolic states. Therefore, our idea is here to use action models as an intermediate step to assign an "abstract" interpretation to agent actions. Then we will consider two concrete frameworks (a Kripke style and a symbolic approach) where updates caused by action model applications are uniquely determined.

An *action model* $U = (Q, F, \mathit{pre})$ over an epistemic signature $\Sigma = (\Pi, A)$ consists of a set Q of *events*, an A-family $F = (F_a \subseteq Q \times Q)_{a \in A}$ of *action accessibility relations* F_a, and an *action precondition* function $\mathit{pre} : Q \rightarrow \mathscr{F}$. We assume that the accessibility relations F_a are equivalences.[1] For any agent $a \in A$, $(q, q') \in F_a$ models that a cannot distinguish between occurrences of events q and q'. For any event $q \in Q$, the epistemic formula $\mathit{pre}(q)$ determines a condition under which q can happen. An *epistemic action* $\mathfrak{u} = (U, q)$ is a pointed action model which selects an actual event $q \in Q$. We set $F(\mathfrak{u})_a = \{q' \in Q \mid (q, q') \in F_a\}$ for $a \in A$, and write $\mathit{pre}(\mathfrak{u})$ for $\mathit{pre}(q)$ and $\mathfrak{u} \cdot q'$ for (U, q') when $q' \in Q$. The class of epistemic actions over Σ is denoted by $\mathcal{U}$. The set of possible agents for an epistemic action $\mathfrak{u} \in \mathcal{U}$ is defined by $\mathrm{ags}(\mathfrak{u}) = \mathrm{ags}(\mathit{pre}(\mathfrak{u}))$. The idea is that an action $\eta \in \mathrm{N}_a$ of an agent a should only be interpreted by an epistemic action whose precondition is an a-formula.

Example 4.1
A *group announcement* of a formula $\varphi \in \mathscr{F}$ to a group $A_* \subseteq A$ of agents is modelled by the epistemic action $(U_{\mathit{grp}}(A_*, \varphi), \mathsf{k})$ graphically represented as



The action model $U_{\mathit{grp}}(A_*, \varphi)$ has two events k and n. Event k represents the announcement of φ which should only happen if φ holds and therefore $\mathit{pre}_{\mathit{grp},\varphi}(\mathsf{k}) = \varphi$; only agents in the group A_* can recognise this event which is indicated by the bidirectional accessibility arrows connecting k to n. All other agents consider it possible that nothing happened which is represented by n having no proper precondition, i.e., $\mathit{pre}_{\mathit{grp},\varphi}(\mathsf{n}) = \text{true}$. The possible agents of $(U_{\mathit{grp}}(A_*, \varphi), \mathsf{k})$

---

[1] This fits with our choice of Kripke structures in Sect. 5; see the "action model closure theorem" by Baltag and Renne [1].

are given by $\mathrm{ags}(\varphi)$, the possible agents of $(U_{grp}(A_*, \varphi), \mathsf{n})$ are $\mathrm{ags}(\mathrm{true}) = A$.

Epistemic choice actions We also consider non-deterministic epistemic actions, similarly to van Ditmarsch et al. [30]. They model alternatives which are not under the control of an agent but are selected by the environment. Formally, an *epistemic choice action* over Σ is a finite, non-empty set $\alpha \subseteq \mathcal{U}$ of epistemic actions. The class of epistemic choice actions over Σ is denoted by $\mathscr{A}$. The possible agents of an $\alpha \in \mathscr{A}$ are $\mathrm{ags}(\alpha) = \bigcap_{\mathfrak{u} \in \alpha} \mathrm{ags}(\mathfrak{u})$.

Example 4.2
For agents $a, a' \in A$, the epistemic choice action $\alpha_{\mathrm{los}} = \{\mathfrak{u}^{\mathsf{k}}, \mathfrak{u}^{\mathsf{n}}\}$ with $\mathfrak{u}^{\mathsf{k}} = (U_{grp}(\{a'\}, \varphi_a), \mathsf{k})$ and $\mathfrak{u}^{\mathsf{n}} = U_{grp}(\{a'\}, \varphi_a), \mathsf{n})$ models a *lossy sending* of the information $\varphi_a \in \mathscr{F} \restriction a$ from a to agent a', in the sense that afterwards a' may know φ_a if a' is aware of the announcement (modelled by $\mathfrak{u}^{\mathsf{k}}$), but need not if the announcement is lost (modelled by $\mathfrak{u}^{\mathsf{n}}$). In any case, a cannot recognise whether the announcement was successful. Similarly, a *reliable sending* is defined by the (singleton) choice action $\alpha_{\mathrm{rel}} = \{(U_{grp}(\{a, a'\}, \varphi_a), \mathsf{k})\}$ which is modelled by a group announcement of φ_a such that both agents recognise that the message did work. It holds that $\mathrm{ags}(\alpha_{\mathrm{los}}) = \mathrm{ags}(\alpha_{\mathrm{rel}}) = \{a\}$.

Let us now come back to our epistemic ensemble signature $\check{\Sigma} = (\Sigma, \vec{\mathrm{N}})$ with $\Sigma = (\Pi, A)$. An *epistemic action interpretation* for the agent actions in $\bigcup \vec{\mathrm{N}}$ is a function $act : \bigcup \vec{\mathrm{N}} \to \mathscr{A}$ such that for all $a \in A$ and $\eta \in \mathrm{N}_a$ it holds $a \in \mathrm{ags}(act(\eta))$.

Example 4.3
For the agent actions in Ex. 2.2 we use the epistemic action interpretation act_2 with

$$\begin{aligned}
act_2(snd_{\mathrm{los}}^{1\to 2}(\mathsf{K}_1\,\mathrm{x}_1)) &= \{(U_{grp}(\{2\}, \mathsf{K}_1\,\mathrm{x}_1), \mathsf{k}),\\
&\qquad (U_{grp}(\{2\}, \mathsf{K}_1\,\mathrm{x}_1), \mathsf{n})\}\\
act_2(snd_{\mathrm{los}}^{1\to 2}(\mathsf{K}_1\,\neg\mathrm{x}_1)) &= \{(U_{grp}(\{2\}, \mathsf{K}_1\,\neg\mathrm{x}_1), \mathsf{k}),\\
&\qquad (U_{grp}(\{2\}, \mathsf{K}_1\,\neg\mathrm{x}_1), \mathsf{n})\}\\
act_2(stop) &= \{(U_{grp}(\{1,2\}, \mathrm{true}), \mathsf{k})\}\\
act_2(snd_{\mathrm{rel}}^{2\to 1}(\mathsf{K}_2\,x_1)) &= \{(U_{grp}(\{1,2\}, \mathsf{K}_2\,x_1), \mathsf{k})\}
\end{aligned}$$

such that the interpretation of *stop* is the announcement of true to the agents 1 and 2.

5 Epistemic ensembles in a semantic environment

To execute an epistemic ensemble we need an environment where knowledge formulæ, in particular process guards and change of knowledge caused by agent actions, are interpreted. This section is based on the traditional possible worlds model of epistemic logic; see, e.g., Fagin et al. [15].

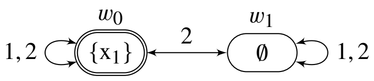
Epistemic states An *epistemic structure* $K = (W, E, L)$, also called Kripke model in Baltag and Renne [1] and van Ditmarsch et al. [30], or Kripke structure in Fagin et al. [15], over the epistemic signature $\Sigma = (\Pi, A)$ is given by a set W of *worlds*, an A-family $E = (E_a \subseteq W \times W)_{a \in A}$ of epistemic *accessibility relations*, and a *labelling* $L : W \to \wp\Pi$ which determines for each world $w \in W$ the set of atomic propositions which hold in w. As in the traditional "logic of knowledge" [15], we assume that the accessibility relations are equivalences. For any agent a, $(w, w') \in E_a$ models that agent a cannot distinguish the two worlds w and w'. An *epistemic state* $\mathfrak{K} = (K, w)$ selects a world $w \in W$ considered as the *actual* world. Thus epistemic states are pointed Kripke structures. The class of epistemic states over Σ is denoted by $\mathcal{S}$.

The *satisfaction* of an epistemic formula $\varphi \in \mathscr{F}$ by an epistemic structure $K = (W, E, L) \in \mathcal{S}$ at a world $w \in W$, written $K, w \models \varphi$, is inductively defined by:

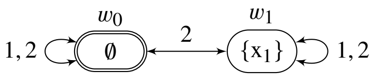
$$\begin{aligned}
&K, w \models p \iff p \in L(w)\\
&K, w \models \mathrm{true}\\
&K, w \models \neg\varphi \iff \text{not } K, w \models \varphi\\
&K, w \models \varphi_1 \wedge \varphi_2 \iff K, w \models \varphi_1 \text{ and } K, w \models \varphi_2\\
&K, w \models \mathsf{K}_a\,\varphi \iff K, w' \models \varphi \text{ for all } w' \in W\\
&\qquad\qquad\qquad\qquad\quad \text{with } (w, w') \in E_a
\end{aligned}$$

Hence, an agent a knows φ at point w if φ holds in all worlds w' which a cannot distinguish from w. For an epistemic state $\mathfrak{K} = (K, w) \in \mathcal{S}$ and for $\varphi \in \mathscr{F}$, we define $\mathfrak{K} \models \varphi$ by $K, w \models \varphi$ and for $\Phi \subseteq \mathscr{F}$ we define $\mathfrak{K} \models \Phi$ by $\mathfrak{K} \models \varphi$ for all $\varphi \in \Phi$. A formula $\varphi \in \mathscr{F}$ is a *logical consequence* of $\Phi \subseteq \mathscr{F}$, written $\Phi \models \varphi$, if $\mathfrak{K} \models \varphi$ for all $\mathfrak{K}$ with $\mathfrak{K} \models \Phi$. We write $\models \varphi$ for $\emptyset \models \varphi$, i.e., φ is a *tautology*.

Example 5.1
The following diagram represents graphically an epistemic state $\mathfrak{K}_0 = (K_0, w_0)$ in which x_1 is true and agent 1 knows this, but agent 2 does not. Indeed, agent 2 cannot distinguish between the actual world w_0 and the possible world w_1. The self-loops represent reflexivity of the accessibility relations. Note that $\mathfrak{K}_0 \models_2 \mathsf{K}_1\,\mathrm{x}_1$, $\mathfrak{K}_0 \models_2 \neg\mathsf{K}_2\,\mathrm{x}_1$, and $\mathfrak{K}_0 \models_2 \mathsf{K}_1\,\neg\mathsf{K}_2\,\mathrm{x}_1$.



A state $\mathfrak{K}_0'$ with $\mathfrak{K}_0' \models \mathsf{K}_1\,\neg\mathrm{x}_1$ and $\mathfrak{K}_0' \models \mathsf{K}_1\,\neg\mathsf{K}_2\,\neg\mathrm{x}_1$ is obtained by reversing the labelling of K_0:



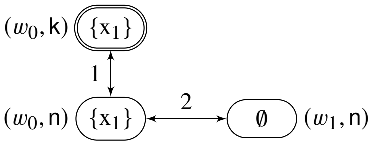
Epistemic updates The *product update* $(W, E, L) \lhd (Q, F, pre)$ of an epistemic structure $K = (W, E, L)$ and an action model $U = (Q, F, pre)$ over $\Sigma = (\Pi, A)$ yields the epistemic structure (W', E', L') with

$$\begin{aligned} W' &= \{(w,q) \in W \times Q \mid K, w \models pre(q)\} \ , \\ E'_a &= \{((w,q),(w',q')) \mid (w,w') \in E_a,\ (q,q') \in F_a\} \\ &\qquad\qquad\qquad\qquad \text{for all } a \in A \text{, and} \\ L'(w,q) &= L(w) \text{ for all } (w,q) \in W' . \end{aligned}$$

Let $\mathfrak{K} = (K, w) \in \mathcal{S}$ be an epistemic state and $\mathfrak{u} = (U, q) \in \mathcal{U}$ be an epistemic action. If $\mathfrak{K} \models pre(q)$, then the *product update* $\mathfrak{K} \lhd \mathfrak{u}$ of $\mathfrak{K}$ and $\mathfrak{u}$ is defined and given by the epistemic state

$$\mathfrak{K} \lhd \mathfrak{u} = (K \lhd U, (w,q)) \in \mathcal{S} \ .$$

Example 5.2
Applying the action $(U_{grp}(\{2\}, \mathsf{K}_1\,\mathrm{x}_1), \mathsf{k})$ to the epistemic state (K_0, w_0) in Ex. 5.1, we obtain the epistemic state $(K_1, (w_0, \mathsf{k}))$ shown, without reflexive accessibility edges, below. The world (w_1, k) does not appear since $(K_0, w_1) \not\models_2 \mathsf{K}_1\,\mathrm{x}_1$ which is the precondition of k.



It holds that $(K_1, (w_0, \mathsf{k})) \models_2 \mathsf{K}_2\,\mathsf{K}_1\,\mathrm{x}_1$ but also $(K_1, (w_0, \mathsf{k})) \models_2 \neg\mathsf{K}_1\,\mathsf{K}_2\,\mathsf{K}_1\,\mathrm{x}_1$, i.e., 2 knows that 1 knows that x_1 holds, but 1 does not know that 2 knows this. If we apply the epistemic action $(U_{grp}(\{2\}, \mathsf{K}_1\,\mathrm{x}_1), \mathsf{n})$ to (K_0, w_0), we obtain the epistemic state $(K_1, (w_0, \mathsf{n}))$. Note that there $(K_1, (w_0, \mathsf{n})) \models_2 \neg\mathsf{K}_2\,\mathsf{K}_1\,\mathrm{x}_1$.

Semantic environments As in Hennicker et al. [19, 20] we choose for our semantic environments epistemic states $\mathfrak{K} \in \mathcal{S}$. Note that this differs from the conference version [21], where we used non-empty classes of epistemic states as semantic environments; this approach, however, turned out to complicate matters without providing new insights.

The *semantics* of an epistemic choice action $\alpha \in \mathcal{A}$ is given by the relation

$$[\![\alpha]\!] = \{(\mathfrak{K}, \mathfrak{K} \lhd \mathfrak{u}) \in \mathcal{S} \times \mathcal{S} \mid \mathfrak{u} \in \alpha, \mathfrak{K} \models pre(\mathfrak{u})\} \ .$$

Ensemble configurations We are now ready to define the execution of ensembles in semantic environments. Given the epistemic ensemble signature $\check{\Sigma} = (\Sigma, \vec{\mathrm{N}})$, an epistemic *ensemble configuration* over $\check{\Sigma}$ is a pair $(\vec{E}, \mathfrak{K})$ of an ensemble $\vec{E}$ over $\check{\Sigma}$ and an epistemic state $\mathfrak{K} \in \mathcal{S}$. The *ensemble semantics* over $\check{\Sigma}$ w.r.t. an epistemic action interpretation $act : \bigcup \vec{\mathrm{N}} \to \mathcal{A}$ is the ternary relation $\to^{act}$ between configurations, agent actions and (successor) configurations defined by interpreting the generic operational ensemble semantics, given by rule (*) in Sect. 2, in the semantic environment of epistemic states:

$$(\vec{E}, \mathfrak{K}) \xrightarrow{\eta}{}^{act} (\vec{E}', \mathfrak{K}') \quad \text{if } \vec{E} \xhookrightarrow{\widehat{\varphi}:\eta} \vec{E}', \mathfrak{K} \models \widehat{\varphi}, \text{ and} \\ \qquad\qquad\qquad\qquad (\mathfrak{K}, \mathfrak{K}') \in [\![act(\eta)]\!]$$

Example 5.3
Consider again the bit transmission ensemble of Ex. 2.3 with its agent actions interpreted as in Ex. 4.3. As initial epistemic state first consider $\mathfrak{K}_0$ of Ex. 5.1 which satisfies $\mathsf{K}_1\,\mathrm{x}_1$; then only $snd_{\mathrm{los}}^{1\to2}(\mathsf{K}_1\,\mathrm{x}_1)$ can be executed. The corresponding infinite transition system starting from the ensemble configuration $(1 : Ag1 \parallel 2 : Ag2, \mathfrak{K}_0)$ is shown in Fig. 2. Analogously, if the initial epistemic state is $\mathfrak{K}'_0$ of Ex. 5.1, which satisfies $\mathsf{K}_1\,\neg\mathrm{x}_1$, then only $snd_{\mathrm{los}}^{1\to2}(\mathsf{K}_1\,\neg\mathrm{x}_1)$ is executable. Note that in both cases also the value of x_2 is fixed, though execution does not depend on it. When the ensemble reaches a configuration with an epistemic state satisfying $\mathsf{K}_1\,\mathsf{K}_2\,x_1$, e.g., $(1 : Ag1 \parallel 2 : \mathbf{0}, \mathfrak{K}_0 \lhd_2 \mathfrak{u}^{\mathsf{k}}_{\mathrm{x}_1} \lhd_2 \mathfrak{u}_2)$, then *stop* can be executed.

Satisfaction of epistemic ensemble formulæ Finally we are now also able to define when an epistemic ensemble formula (cf. Sect. 3) is satisfied by an ensemble configuration. As a preparation we define the semantics of guard-action sequences $\sigma = (\widehat{\varphi}_1 : \eta_1^{\epsilon}) \ldots (\widehat{\varphi}_k : \eta_k^{\epsilon})$ relating epistemic states inductively by

$$\begin{aligned} [\![\widehat{\varphi} : \epsilon]\!]^{act} &= \{(\mathfrak{K}, \mathfrak{K}) \in \mathcal{S} \times \mathcal{S} \mid \mathfrak{K} \models \widehat{\varphi}\} \ , \\ [\![\widehat{\varphi} : \eta]\!]^{act} &= \{(\mathfrak{K}, \mathfrak{K}') \in [\![act(\eta)]\!] \mid \mathfrak{K} \models \widehat{\varphi}\} \ , \\ [\![\sigma_1 \cdot \sigma_2]\!]^{act} &= [\![\sigma_1]\!]^{act} ; [\![\sigma_2]\!]^{act} \quad \text{(relational composition)} \end{aligned}$$

The semantics of a compound ensemble action $\lambda \in C$ relating ensemble configurations relies on the definition of witnesses in Sect. 3 and is given by the relation

$$[\![\lambda]\!]^{act} = \{((\vec{E}, \mathfrak{K}), (\vec{E}', \mathfrak{K}')) \mid (\vec{E}, \sigma, \vec{E}') :: \lambda, \\ \qquad\qquad (\mathfrak{K}, \mathfrak{K}') \in [\![\sigma]\!]^{act}\} \ .$$

The satisfaction of an epistemic ensemble formula $\psi \in \mathcal{D}$ by an ensemble configuration $(\vec{E}, \mathfrak{K})$ w.r.t. $act : \bigcup \vec{\mathrm{N}} \to \mathcal{A}$ is inductively defined along the form of ψ:

$$\begin{aligned} &(\vec{E}, \mathfrak{K}) \models^{act} \varphi \iff \mathfrak{K} \models \varphi \\ &(\vec{E}, \mathfrak{K}) \models^{act} \text{true} \\ &(\vec{E}, \mathfrak{K}) \models^{act} \neg\psi \iff \text{not } (\vec{E}, \mathfrak{K}) \models^{act} \psi \\ &(\vec{E}, \mathfrak{K}) \models^{act} \psi_1 \wedge \psi_2 \iff (\vec{E}, \mathfrak{K}) \models^{act} \psi_1 \text{ and} \\ &\qquad\qquad\qquad\qquad\qquad\quad (\vec{E}, \mathfrak{K}) \models^{act} \psi_2 \end{aligned}$$

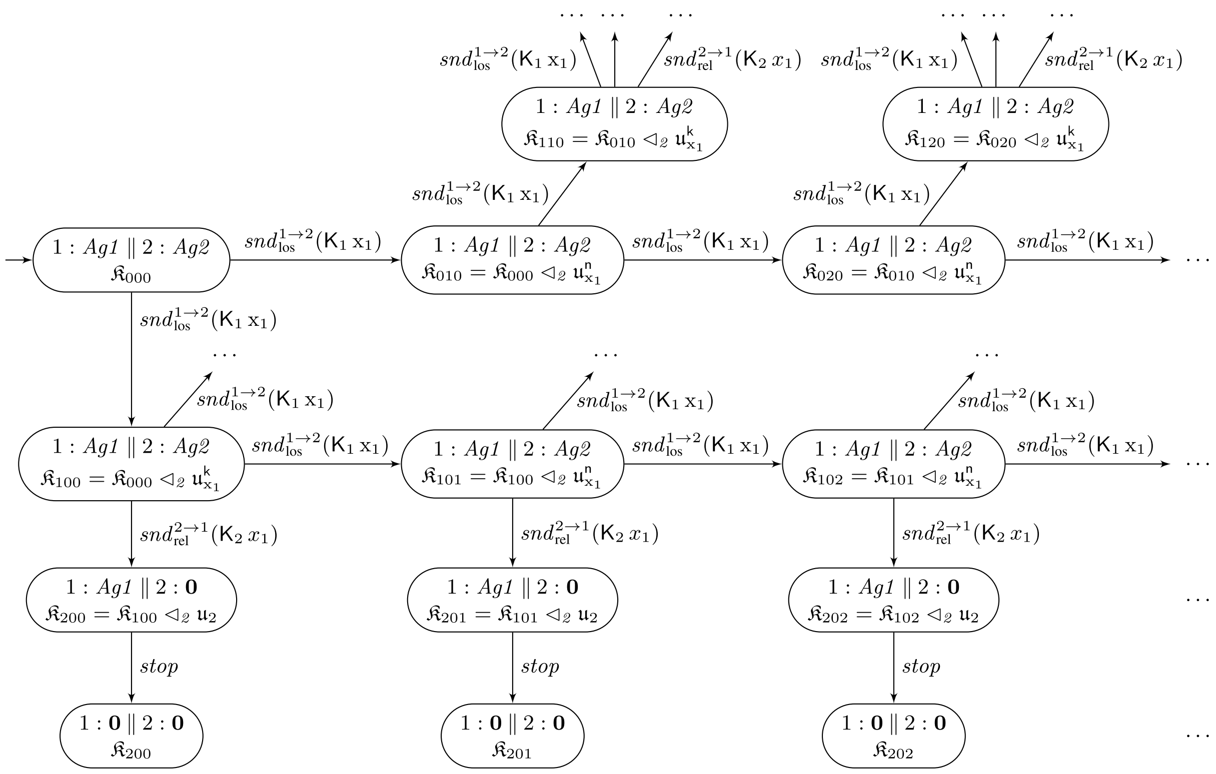


Fig. 2 Semantic global realisation: Excerpt of the infinite transition system for the semantic bit transmission ensemble started in $\mathfrak{K}_{000} = \mathfrak{K}_0$ with $\mathfrak{K}_0 \models_2 \mathsf{K}_1\,\mathrm{x}_1$ (see Ex. 5.1), $\mathfrak{u}^{\mathsf{k}}_{\mathrm{x}_1} = (U_{grp}(\{2\}, \mathsf{K}_1\,\mathrm{x}_1), \mathsf{k})$, $\mathfrak{u}^{\mathsf{n}}_{\mathrm{x}_1} = (U_{grp}(\{2\}, \mathsf{K}_1\,\mathrm{x}_1), \mathsf{n})$, and $\mathfrak{u}_2 = (U_{grp}(\{1,2\}, \mathsf{K}_2\,x_1), \mathsf{k})$ (see Ex. 4.3). The index scheme is $\mathfrak{K}_{0i0} = \mathfrak{K}_{000} \lhd_2 (\mathfrak{u}^{\mathsf{n}}_{\mathrm{x}_1})^i$, $\mathfrak{K}_{1ij} = \mathfrak{K}_{0i0} \lhd_2 \mathfrak{u}^{\mathsf{k}}_{\mathrm{x}_1} \lhd_2 (\mathfrak{u}^{\mathsf{n}}_{\mathrm{x}_1})^j$, and $\mathfrak{K}_{2ij} = \mathfrak{K}_{1ij} \lhd_2 \mathfrak{u}_2$. The upward transitions from the configurations with $\mathfrak{K}_{10j}$ choose $\mathfrak{u}^{\mathsf{k}}_{\mathrm{x}_1}$; the annotation $snd^{1\to2}_{\mathrm{los}}(\mathsf{K}_1\,\mathrm{x}_1)$ pertains to both transitions outgoing from $\mathfrak{K}_{1i0}$, one for each choice

$$(\vec{E}, \mathfrak{K}) \models^{act} [\lambda]\psi \iff (\vec{E}', \mathfrak{K}') \models^{act} \psi \text{ for all } (\vec{E}', \mathfrak{K}') \text{ with } ((\vec{E}, \mathfrak{K}), (\vec{E}', \mathfrak{K}')) \in [\![\lambda]\!]^{act}$$

Example 5.4

The ensemble configuration $(Sys, \mathfrak{K}_0)$ for the bit transmission ensemble Sys from Ex. 2.3 and the epistemic state $\mathfrak{K}_0$ from Ex. 5.1 satisfies both dynamic ensemble formulæ of Ex. 3.1; the same is true for $(Sys, \mathfrak{K}'_0)$.

6 Epistemic ensembles in a symbolic environment

We are now going to execute ensembles in a symbolic environment which allows for a more compact representation of epistemic states represented by sets of epistemic formulæ, i.e., knowledge bases. For doing this we extend the approach of Hennicker et al. [20] to deal with ensembles. A *symbolic epistemic signature* (Σ, Φ) extends the epistemic signature $\Sigma = (\Pi, A)$ by a finite set of epistemic formulæ $\Phi \subseteq \mathscr{F}$ which are in the *focus* of evaluation; we require that each $\varphi \in \Phi$ is neither a tautology (i.e., $\models \varphi \leftrightarrow \mathrm{true}$) nor a contradiction (i.e., $\models \varphi \leftrightarrow \mathrm{false}$). Focusing on a finite set allows for effective computations and decisions; including tautologies or contradictions would not lead to any additional knowledge.

Example 6.1

For the scenario of Ex. 2.3 a possible set of focus formulæ is $\{\mathsf{K}_1\,\mathrm{x}_1,\ \mathsf{K}_1\,\neg\mathrm{x}_1,\ \mathsf{K}_2\,x_1,\ \mathsf{K}_1\,\mathsf{K}_2\,x_1,\ \neg\mathsf{K}_1\,\mathsf{K}_2\,x_1\}$, which directly reflects the conditions used in the ensemble (as before we write $\mathsf{K}_a\,x_1$ for $\mathsf{K}_a\,\mathrm{x}_1 \vee \mathsf{K}_a\,\neg\mathrm{x}_1$).

Symbolic epistemic states A *symbolic epistemic state* over (Σ, Φ) is a subset $\Gamma \subseteq \Phi$. The set of symbolic epistemic states over (Σ, Φ) is denoted by $\mathcal{S}^{\Phi}$.

Any set of focus formulæ Φ can be enlarged by constructing the *Boolean closure* $bcl(\Phi)$ of Φ consisting of the epistemic formulæ ϕ defined by

$$\phi ::= \varphi \mid \mathrm{true} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \qquad \text{where } \varphi \in \Phi.$$

For example, in Ex. 6.1, $\neg\mathsf{K}_1\,\mathsf{K}_2\,x_1$ can be omitted.

The *symbolic satisfaction relation* $\Gamma \models^{\Phi} \phi$ between symbolic states $\Gamma \in \mathcal{S}^{\Phi}$ and formulæ $\phi \in bcl(\Phi)$ is defined as:

$$\text{if } \varphi \in \Phi\text{: } \Gamma \models^{\Phi} \varphi \iff \varphi \in \Gamma$$

$$\Gamma \models^{\Phi} \mathrm{true}$$

$$\text{if } \neg\phi \notin \Phi\colon \Gamma \models^{\Phi} \neg\phi \iff \text{not } \Gamma \models^{\Phi} \phi$$

$$\begin{aligned}\text{if } \phi_1 \wedge \phi_2 \notin \Phi\colon \Gamma \models^{\Phi} \phi_1 \wedge \phi_2 \iff{}& \Gamma \models^{\Phi} \phi_1 \text{ and}\\ & \Gamma \models^{\Phi} \phi_2\end{aligned}$$

Symbolic epistemic updates To define the symbolic update of a symbolic epistemic state $\Gamma \in \mathcal{S}^{\Phi}$ w.r.t. an epistemic action $\mathfrak{u} \in \mathcal{U}$, we utilise the notion of weakest liberal precondition [14]. Let $\mathfrak{u} \in \mathcal{U}$ be an epistemic action and $\varphi \in \mathscr{F}$. A formula $\rho \in \mathscr{F}$ is a *weakest liberal precondition* of $\mathfrak{u}$ for φ if the following holds for all $\mathfrak{K} \in \mathcal{S}$:

$$\mathfrak{K} \models \rho \iff (\mathfrak{K} \models \mathit{pre}(\mathfrak{u}) \text{ implies } \mathfrak{K} \lhd \mathfrak{u} \models \varphi) \qquad \text{(wlp)}$$

The set of the weakest liberal precondition formulæ of $\mathfrak{u}$ for φ is denoted by $\mathit{Wlp}(\mathfrak{u}, \varphi)$. Obviously, if $\rho, \rho' \in \mathit{Wlp}(\mathfrak{u}, \varphi)$, then $\models \rho \leftrightarrow \rho'$. There is indeed, for any $\mathfrak{u} \in \mathcal{U}$ and any $\varphi \in \mathscr{F}$, a formula $\mathit{wlp}(\mathfrak{u}, \varphi) \in \mathit{Wlp}(\mathfrak{u}, \varphi)$ that can be recursively computed by the function $\mathit{wlp} : \mathcal{U} \times \mathscr{F} \to \mathscr{F}$ defined in accordance with the reduction rules originally stated in the context of dynamic epistemic logic (DEL) in van Ditmarsch et al. [30, pp. 162sqq.] and Baltag and Renne [1, p. 37]:

$$\begin{aligned}
&\mathit{wlp}(\mathfrak{u}, p) = \mathit{pre}(\mathfrak{u}) \to p\\
&\mathit{wlp}(\mathfrak{u}, \mathrm{true}) = \mathrm{true}\\
&\mathit{wlp}(\mathfrak{u}, \neg\varphi) = \mathit{pre}(\mathfrak{u}) \to \neg \mathit{wlp}(\mathfrak{u}, \varphi)\\
&\mathit{wlp}(\mathfrak{u}, \varphi_1 \wedge \varphi_2) = \mathit{wlp}(\mathfrak{u}, \varphi_1) \wedge \mathit{wlp}(\mathfrak{u}, \varphi_2)\\
&\mathit{wlp}(\mathfrak{u}, \mathsf{K}_a\, \varphi) = \mathit{pre}(\mathfrak{u}) \to \textstyle\bigwedge_{q \in F(\mathfrak{u})_a} \mathsf{K}_a\, \mathit{wlp}(\mathfrak{u} \cdot q, \varphi)
\end{aligned}$$

For a $\mathfrak{u} \in \mathcal{U}$ and a $\varphi \in \mathscr{F}$, define the set of weakest liberal preconditions of $\mathfrak{u}$ for φ *given that* $\mathit{pre}(\mathfrak{u})$ *holds* by

$$\begin{aligned}&\mathit{Wlp}(\mathfrak{u}, \varphi)/\mathit{pre}(\mathfrak{u}) =\\ &\qquad \{\rho \in \mathscr{F} \mid\ \models \mathit{pre}(\mathfrak{u}) \to (\mathit{wlp}(\mathfrak{u}, \varphi) \leftrightarrow \rho)\}\ .\end{aligned}$$

We are now ready to define symbolic epistemic updates of symbolic states Γ. The idea is to consider all focus formulæ having a weakest liberal precondition $\rho \in \mathit{bcl}(\Phi)$ such that $\Gamma \models^{\Phi} \rho$ given that $\mathit{pre}(\mathfrak{u})$ holds. The *symbolic epistemic update* $\Gamma \lhd^{\Phi} \mathfrak{u}$ of a symbolic epistemic state $\Gamma \in \mathcal{S}^{\Phi}$ by an epistemic action $\mathfrak{u} \in \mathcal{U}$ is defined as

$$\begin{aligned}\Gamma \lhd^{\Phi} \mathfrak{u} = \{\varphi' \in \Phi \mid \text{ex. } \rho \in{}& \mathit{Wlp}(\mathfrak{u}, \varphi')/\mathit{pre}(\mathfrak{u}) \cap \mathit{bcl}(\Phi)\\ & \text{s.t. } \Gamma \models^{\Phi} \rho\}\ .\end{aligned}$$

Symbolic environments A key notion to obtain compatibility of epistemic actions in semantic and symbolic environments is the notion of Φ-representability. We call an epistemic action $\mathfrak{u} \in \mathcal{U}$ representable w.r.t. a set Φ of focus formulæ if, on the one hand, its precondition is equivalent to some epistemic formula in the Boolean closure of Φ and, on the other hand, for each $\varphi \in \Phi$, the weakest liberal precondition formula $\mathit{wlp}(\mathfrak{u}, \varphi)$ is also equivalent to some formula in the Boolean closure of Φ, but now $\mathit{pre}(\mathfrak{u})$ can be assumed. More formally, an $\mathfrak{u} \in \mathcal{U}$ is Φ-*representable* if

1. $\mathit{Pre}(\mathfrak{u}) \cap \mathit{bcl}(\Phi) \neq \emptyset$, where $\mathit{Pre}(\mathfrak{u}) = \{\rho \in \mathscr{F} \mid\ \models \mathit{pre}(\mathfrak{u}) \leftrightarrow \rho\}$;
2. $\mathit{Wlp}(\mathfrak{u}, \varphi)/\mathit{pre}(\mathfrak{u}) \cap \mathit{bcl}(\Phi) \neq \emptyset$ for all $\varphi \in \Phi$.

The class of such epistemic actions is denoted by $\mathcal{U}^{\Phi}$. An epistemic choice action $\alpha \in \mathcal{A}$ is Φ-*representable* if all $\mathfrak{u} \in \alpha$ are Φ-representable; the class of such epistemic choice actions is denoted by $\mathcal{A}^{\Phi}$.

Example 6.2
For the scenario of Ex. 2.3 consider the preliminary set of focus formulæ $\Phi_2^{(0)} = \{\mathsf{K}_1\, \mathrm{x}_1, \mathsf{K}_1\, \neg\mathrm{x}_1, \mathsf{K}_2\, x_1, \mathsf{K}_1\, \mathsf{K}_2\, x_1\}$ (as before we write $\mathsf{K}_a\, x_1$ for $\mathsf{K}_a\, \mathrm{x}_1 \vee \mathsf{K}_a\, \neg\mathrm{x}_1$). The epistemic actions occurring in the act_2-interpretations of $\mathit{snd}_{\mathrm{los}}^{1\to 2}(\mathsf{K}_1\, \mathrm{x}_1)$, $\mathit{snd}_{\mathrm{los}}^{1\to 2}(\mathsf{K}_1\, \neg\mathrm{x}_1)$, and $\mathit{snd}_{\mathrm{rel}}^{2\to 1}(\mathsf{K}_2\, x_1)$ (see Ex. 4.3) are $\mathfrak{u}_{\varphi}^{q} = (U_{grp}(\{2\}, \mathsf{K}_1\, \varphi), q)$ for $\varphi \in \{\mathrm{x}_1, \neg\mathrm{x}_1\}$, $q \in \{\mathsf{k}, \mathsf{n}\}$ and $\mathfrak{u}_2 = (U_{grp}(\{1, 2\}, \mathsf{K}_2\, x_1), \mathsf{k})$. All these actions have a precondition that is expressible over $\Phi_2^{(0)}$. Table 3 shows possible representatives ρ satisfying $\models_2 \mathit{pre}(\mathfrak{u}) \to (\mathit{wlp}_2(\mathfrak{u}, \varphi) \leftrightarrow \rho)$.[2]

Indeed, $\mathfrak{u}_2$ is not $\Phi_2^{(0)}$-representable, but becomes Φ_2-representable for $\Phi_2 = \Phi_2^{(0)} \cup \{\mathsf{K}_1\, \mathsf{M}_2\, \mathrm{x}_1, \mathsf{K}_1\, \mathsf{M}_2\, \neg\mathrm{x}_1\}$.

The *symbolic semantics* of a Φ-representable epistemic choice action $\alpha \in \mathcal{A}^{\Phi}$ is given by the relation

$$\begin{aligned}[\![\alpha]\!]^{\Phi} = \{(\Gamma, \Gamma \lhd^{\Phi} \mathfrak{u}) \in \mathcal{S}^{\Phi} \times \mathcal{S}^{\Phi} \mid{}& \mathfrak{u} \in \alpha,\\ & \text{ex. } \rho \in \mathit{Pre}(\mathfrak{u}) \cap \mathit{bcl}(\Phi) \text{ s.t. } \Gamma \models^{\Phi} \rho\}\ .\end{aligned}$$

A *symbolic epistemic ensemble signature* $(\check{\Sigma}, \Phi)$ consists of an epistemic ensemble signature $\check{\Sigma} = (\Sigma, \vec{\mathrm{N}})$ and a set of focus formulæ Φ such that (Σ, Φ) is a symbolic epistemic signature. An ensemble $\vec{E}$ over $\check{\Sigma}$ is an ensemble over $(\check{\Sigma}, \Phi)$ if all guards occurring in $\vec{E}$ are in $\mathit{bcl}(\Phi)$. An epistemic action interpretation $\mathit{act} : \bigcup \vec{\mathrm{N}} \to \mathcal{A}$ is a Φ-*interpretation* if $\mathit{act}(\eta) \in \mathcal{A}^{\Phi}$ for all $\eta \in \bigcup \vec{\mathrm{N}}$. A symbolic epistemic *ensemble configuration* over $(\check{\Sigma}, \Phi)$ is a pair $(\vec{E}, \Gamma)$ of an ensemble $\vec{E}$ over $(\check{\Sigma}, \Phi)$ and a symbolic epistemic state $\Gamma \in \mathcal{S}^{\Phi}$. The *symbolic ensemble semantics* over $(\check{\Sigma}, \Phi)$ w.r.t. a Φ-interpretation $\mathit{act} : \bigcup \vec{\mathrm{N}} \to \mathcal{A}^{\Phi}$ is the ternary relation $\to_{\Phi}^{\mathit{act}}$ between symbolic configurations, agent actions and (successor) configurations defined by interpreting the operational ensemble semantics, given by rule (*) in Sect. 2, in the environment of symbolic epistemic states:

$$\begin{aligned}(\vec{E}, \Gamma) \overset{\eta}{\to}{}_{\Phi}^{\mathit{act}} (\vec{E}', \Gamma') \quad \text{if } & \vec{E} \xhookrightarrow{\widehat{\phi}:\eta} \vec{E}', \Gamma \models^{\Phi} \widehat{\phi}, \text{ and}\\ & (\Gamma, \Gamma') \in [\![\mathit{act}(\eta)]\!]^{\Phi}\end{aligned}$$

---

[2] Computed with a small Maude tool available at https://github.com/AlexanderKnapp/epistemic.git.

Table 3 Representatives ρ satisfying $\models_2 pre(\mathfrak{u}) \rightarrow (wlp_2(\mathfrak{u}, \varphi') \leftrightarrow \rho)$

φ'	$\mathfrak{u}$				
	$\mathfrak{u}^{\mathsf{k}}_{\mathrm{x}_1}$	$\mathfrak{u}^{\mathsf{k}}_{\neg\mathrm{x}_1}$	$\mathfrak{u}^{\mathsf{n}}_{\mathrm{x}_1}$	$\mathfrak{u}^{\mathsf{n}}_{\neg\mathrm{x}_1}$	$\mathfrak{u}_2$
$\mathsf{K}_1\,\mathrm{x}_1$	true	false	$\mathsf{K}_1\,\mathrm{x}_1$	$\mathsf{K}_1\,\mathrm{x}_1$	$\mathsf{K}_1\,\mathsf{M}_2\,\mathrm{x}_1$
$\mathsf{K}_1\,\neg\mathrm{x}_1$	false	true	$\mathsf{K}_1\,\neg\mathrm{x}_1$	$\mathsf{K}_1\,\neg\mathrm{x}_1$	$\mathsf{K}_1\,\mathsf{M}_2\,\neg\mathrm{x}_1$
$\mathsf{K}_2\,x_1$	true	true	$\mathsf{K}_2\,x_1$	$\mathsf{K}_2\,x_1$	true
$\mathsf{K}_1\,\mathsf{K}_2\,x_1$	$\mathsf{K}_1\,\mathsf{K}_2\,x_1$	$\mathsf{K}_1\,\mathsf{K}_2\,x_1$	$\mathsf{K}_1\,\mathsf{K}_2\,x_1$	$\mathsf{K}_1\,\mathsf{K}_2\,x_1$	true
$\mathsf{K}_1\,\mathsf{M}_2\,\mathrm{x}_1$	true	false	$\mathsf{K}_1\,\mathsf{M}_2\,\mathrm{x}_1$	$\neg\mathsf{K}_1\,\neg\mathrm{x}_1 \land \mathsf{K}_1\,\mathsf{M}_2\,\mathrm{x}_1$	$\mathsf{K}_1\,\mathsf{M}_2\,\mathrm{x}_1$
$\mathsf{K}_1\,\mathsf{M}_2\,\neg\mathrm{x}_1$	false	true	$\neg\mathsf{K}_1\,\mathrm{x}_1 \land \mathsf{K}_1\,\mathsf{M}_2\,\neg\mathrm{x}_1$	$\mathsf{K}_1\,\mathsf{M}_2\,\neg\mathrm{x}_1$	$\mathsf{K}_1\,\mathsf{M}_2\,\neg\mathrm{x}_1$

Example 6.3
Fig. 3 shows the transition system for the bit transmission ensemble over Φ_2 when started in a (global) symbolic state that contains $\mathsf{K}_1\,\mathrm{x}_1$ but not $\mathsf{K}_1\,\mathsf{K}_2\,x_1$. This start ensemble configuration is $1 : \mathit{Ag1} \parallel 2 : \mathit{Ag2}, \{\mathsf{K}_1\,\mathrm{x}_1\}$.

Satisfaction of epistemic ensemble formulæ The *symbolic semantics* of a guard-agent action sequence $\sigma = (\widehat{\phi}_1 : \eta_1^{\epsilon}) \ldots (\widehat{\phi}_k : \eta_k^{\epsilon})$ with all guards $\widehat{\phi}_i$ formulæ in $bcl(\Phi)$ w.r.t. the Φ-interpretation *act* is inductively given by

$$\begin{aligned}
[\![\widehat{\phi} : \epsilon]\!]^{act}_{\Phi} &= \{(\Gamma, \Gamma) \in (\mathcal{S}^{\Phi})^2 \mid \Gamma \models^{\Phi} \widehat{\phi}\}\ , \\
[\![\widehat{\phi} : \eta]\!]^{act}_{\Phi} &= \{(\Gamma, \Gamma') \in [\![act(\eta)]\!]^{\Phi} \mid \Gamma \models^{\Phi} \widehat{\phi}\}\ , \\
[\![\sigma_1 \cdot \sigma_2]\!]^{act}_{\Phi} &= [\![\sigma_1]\!]^{act}_{\Phi} ; [\![\sigma_2]\!]^{act}_{\Phi} \quad \text{(rel. comp.)}
\end{aligned}$$

The set of compound ensemble actions $\lambda \in C$ with all tests in λ formulæ in $bcl(\Phi)$ is denoted C^{Φ}. The *semantics* of a $\lambda \in C^{\Phi}$ w.r.t. *act* is given by the relation

$$\begin{aligned}
[\![\lambda]\!]^{act}_{\Phi} = \{((\vec{E}, \Gamma), (\vec{E}', \Gamma')) \mid {}& (\vec{E}, \sigma, \vec{E}') :: \lambda, \\
& (\Gamma, \Gamma') \in [\![\sigma]\!]^{act}_{\Phi}\}\ .
\end{aligned}$$

The epistemic ensemble formulæ $\mathscr{D}^{\Phi}$ over $(\check{\Sigma}, \Phi)$ are the epistemic ensemble formulæ over $\check{\Sigma}$ that only contain basic, non-dynamic formulæ $\phi \in \mathscr{F}$ that are in $bcl(\Phi)$. The satisfaction of a $\psi \in \mathscr{D}^{\Phi}$ by a symbolic epistemic ensemble configuration $(\vec{E}, \Gamma)$ w.r.t. Φ-interpretation $act : \bigcup \check{\mathrm{N}} \rightarrow \mathscr{A}^{\Phi}$ is inductively defined along the structure of ψ:

$$\begin{aligned}
&(\vec{E}, \Gamma) \models^{act}_{\Phi} \phi \iff \Gamma \models^{\Phi} \phi \\
&(\vec{E}, \Gamma) \models^{act}_{\Phi} \text{true} \\
&(\vec{E}, \Gamma) \models^{act}_{\Phi} \neg\psi \iff \text{not } (\vec{E}, \Gamma) \models^{act}_{\Phi} \psi \\
&(\vec{E}, \Gamma) \models^{act}_{\Phi} \psi_1 \land \psi_2 \iff (\vec{E}, \Gamma) \models^{act}_{\Phi} \psi_1 \text{ and} \\
&\qquad\qquad\qquad\qquad\qquad\quad (\vec{E}, \Gamma) \models^{act}_{\Phi} \psi_2 \\
&(\vec{E}, \Gamma) \models^{act}_{\Phi} [\lambda]\psi \iff (\vec{E}', \Gamma') \models^{act}_{\Phi} \psi \text{ for all } (\vec{E}', \Gamma') \\
&\qquad\qquad\qquad\qquad\qquad\quad \text{with } ((\vec{E}, \Gamma), (\vec{E}', \Gamma')) \in [\![\lambda]\!]^{act}_{\Phi}
\end{aligned}$$

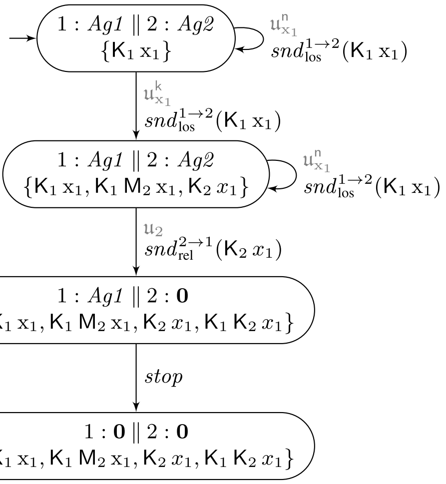


Fig. 3 Symbolic global realisation: Finite transition system for the symbolic bit transmission ensemble started in $\{\mathsf{K}_1\,\mathrm{x}_1\}$ with $\mathfrak{u}^{\mathsf{k}}_{\mathrm{x}_1} = (U_{grp}(\{2\}, \mathsf{K}_1\,\mathrm{x}_1), \mathsf{k})$, $\mathfrak{u}^{\mathsf{n}}_{\mathrm{x}_1} = (U_{grp}(\{2\}, \mathsf{K}_1\,\mathrm{x}_1), \mathsf{n})$, and $\mathfrak{u}_2 = (U_{grp}(\{1, 2\}, \mathsf{K}_2\,x_1), \mathsf{k})$ (see Ex. 4.3). The chosen epistemic action from a composite action is indicated in gray

7 Relating ensembles in semantic and symbolic environments

We are now interested in relating ensembles which are executed in a semantic environment and in a symbolic environment. First, we introduce a notion of Φ-equivalence between semantic and symbolic states and we lift this equivalence to a mutual simulation of ensemble configurations. Our main results show that ensembles running in equivalent semantic and symbolic environments mutually simulate each other and hence satisfy the same dynamic epistemic ensemble logic formulæ.

For achieving such a simulation, we move from a concrete semantic state $\mathfrak{K}$ to its abstract symbolic state representation given by the (finite) Φ-theory of $\mathfrak{K}$, i.e., $\mathrm{Th}^{\Phi}(\mathfrak{K}) = \{\varphi \in \Phi \mid \mathfrak{K} \models \varphi\}$. Let us mention that, in contrast to symbolic updates, update operations on semantic epistemic states tend to become large quickly due to the product update construction which may duplicate the set of possible worlds (see [20]).

Epistemic state equivalence We say that an epistemic state $\mathfrak{K} \in \mathcal{S}$ and a symbolic epistemic state $\Gamma \in \mathcal{S}^{\Phi}$ are Φ-*equivalent*, written $\mathfrak{K} \equiv^{\Phi} \Gamma$, if for all $\varphi \in \Phi$ it holds that $\mathfrak{K} \models \varphi$ if, and only if, $\varphi \in \Gamma$.

Example 7.1

The initial epistemic state $\mathfrak{K}_0$ for the bit transmission ensemble in the semantic environment, see Ex. 5.3, is Φ_2-equivalent to $\{\mathsf{K}_1\,\mathsf{x}_1, \mathsf{K}_1\,\mathsf{M}_2\,\mathsf{x}_1\}$. Note that the initial symbolic epistemic state $\{\mathsf{K}_1\,\mathsf{x}_1\}$ used in Ex. 6.3 does not contain $\mathsf{K}_1\,\mathsf{M}_2\,\mathsf{x}_1$; indeed, the execution of $snd^{1\to 2}_{los}(\mathsf{K}_1\,\mathsf{x}_1)$ does not depend on this formula, and it is added after the successful transmission of $\mathsf{K}_1\,\mathsf{x}_1$.

The next lemma shows that state equivalence w.r.t. Φ can be lifted to the infinite Boolean closure of Φ.

Lemma 7.1

Let $\mathfrak{K} \in \mathcal{S}$ and $\Gamma \in \mathcal{S}^{\Phi}$. Then $\mathfrak{K} \equiv^{\Phi} \Gamma$ if, and only if, for all $\phi \in bcl(\Phi)$ it holds that $\mathfrak{K} \models \phi$ iff $\Gamma \models^{\Phi} \phi$.

Proof

"$\Rightarrow$": Assume $\mathfrak{K} \equiv^{\Phi} \Gamma$. The proof is performed by structural induction on the form of ϕ.

Case $\phi = \varphi \in \Phi$: Then $\mathfrak{K} \models \varphi$ iff (since $\mathfrak{K} \equiv^{\Phi} \Gamma$) $\varphi \in \Gamma$ iff $\Gamma \models^{\Phi} \varphi$.

Case true: $\mathfrak{K} \models \mathrm{true}$ and $\Gamma \models^{\Phi} \mathrm{true}$ hold.

Case $\neg\phi$ with $\neg\phi \notin \Phi$: $\mathfrak{K} \models \neg\phi$ iff not $\mathfrak{K} \models \phi$ iff (by induction hypothesis) not $\Gamma \models^{\Phi} \phi$ iff (since $\neg\phi \notin \Phi$) $\Gamma \models^{\Phi} \neg\phi$.

Case $\phi_1 \wedge \phi_2$ with $\phi_1 \wedge \phi_2 \notin \Phi$: $\mathfrak{K} \models \phi_1 \wedge \phi_2$ iff $\mathfrak{K} \models \phi_1$ and $\mathfrak{K} \models \phi_2$ iff (by induction hypothesis) $\Gamma \models^{\Phi} \phi_1$ and $\Gamma \models^{\Phi} \phi_2$ iff (since $\phi_1 \wedge \phi_2 \notin \Phi$) $\Gamma \models^{\Phi} \phi_1 \wedge \phi_2$.

"$\Leftarrow$": Since $\Phi \subseteq bcl(\Phi)$, the assumption implies $\mathfrak{K} \models \varphi \iff \Gamma \models^{\Phi} \varphi \iff \varphi \in \Gamma$ for all $\varphi \in \Phi$; hence, $\mathfrak{K} \equiv^{\Phi} \Gamma$. □

The following lemma shows that semantic and symbolic state equivalence is preserved by performing updates w.r.t. the Φ-representable actions $\mathcal{U}^{\Phi}$. This is a crucial fact for our subsequent results.

Lemma 7.2

Let $\mathfrak{K} \in \mathcal{S}$ and $\Gamma \in \mathcal{S}^{\Phi}$ with $\mathfrak{K} \equiv^{\Phi} \Gamma$. Let $\mathfrak{u} \in \mathcal{U}^{\Phi}$ such that $\mathfrak{K} \models pre(\mathfrak{u})$. Then $\mathfrak{K} \lhd \mathfrak{u} \equiv^{\Phi} \Gamma \lhd^{\Phi} \mathfrak{u}$.

Proof

Let $\varphi \in \Phi$ and let first $\mathfrak{K} \lhd \mathfrak{u} \models \varphi$ hold, i.e., $\mathfrak{K} \models wlp(\mathfrak{u}, \varphi)$. Since $\mathfrak{u}$ is Φ-representable, there is a $\rho \in Wlp(\mathfrak{u}, \varphi)/pre(\mathfrak{u}) \cap bcl(\Phi)$. By $\mathfrak{K} \models pre(\mathfrak{u})$ it follows that $\mathfrak{K} \models \rho$, and by $\mathfrak{K} \equiv^{\Phi} \Gamma$ and Lem. 7.1 that $\Gamma \models^{\Phi} \rho$, i.e., $\varphi \in \Gamma \lhd^{\Phi} \mathfrak{u}$.

Now let conversely $\varphi \in \Gamma \lhd^{\Phi} \mathfrak{u}$ hold, i.e., $\Gamma \models^{\Phi} \rho$ for some $\rho \in Wlp(\mathfrak{u}, \varphi)/pre(\mathfrak{u}) \cap bcl(\Phi)$. By $\mathfrak{K} \equiv^{\Phi} \Gamma$ and Lem. 7.1 it follows that $\mathfrak{K} \models \rho$. By $\mathfrak{K} \models pre(\mathfrak{u})$ we obtain that $\mathfrak{K} \models wlp(\mathfrak{u}, \varphi)$, i.e., $\mathfrak{K} \lhd \mathfrak{u} \models \varphi$. □

Lemma 7.2 is extended to epistemic choice actions as follows.

Lemma 7.3

Let $\mathfrak{K} \in \mathcal{S}$ and $\Gamma \in \mathcal{S}^{\Phi}$ with $\mathfrak{K} \equiv^{\Phi} \Gamma$. Let $\alpha \in \mathcal{A}^{\Phi}$. Then the following holds:

1. *If $(\mathfrak{K}, \mathfrak{K}') \in [\![\alpha]\!]$, then there is a $\Gamma' \in \mathcal{S}^{\Phi}$ such that $(\Gamma, \Gamma') \in [\![\alpha]\!]^{\Phi}$ and $\mathfrak{K}' \equiv^{\Phi} \Gamma'$.*
2. *If $(\Gamma, \Gamma') \in [\![\alpha]\!]^{\Phi}$, then there is a $\mathfrak{K}' \in \mathcal{S}$ such that $(\mathfrak{K}, \mathfrak{K}') \in [\![\alpha]\!]$ and $\mathfrak{K}' \equiv^{\Phi} \Gamma'$.*

Proof

(1) Let $(\mathfrak{K}, \mathfrak{K}') \in [\![\alpha]\!]$. Then there is a $\mathfrak{u} \in \alpha$ such that $\mathfrak{K}' = \mathfrak{K} \lhd \mathfrak{u}$ and $\mathfrak{K} \models pre(\mathfrak{u})$. By Lem. 7.1 and $\mathfrak{u} \in \mathcal{U}^{\Phi}$ it holds that there is a $\rho \in Pre(\mathfrak{u}) \cap bcl(\Phi)$ such that $\Gamma \models^{\Phi} \rho$ and hence $(\Gamma, \Gamma') \in [\![\alpha]\!]^{\Phi}$ with $\Gamma' = \Gamma \lhd^{\Phi} \mathfrak{u}$. By Lem. 7.2, $\mathfrak{K} \lhd \mathfrak{u} \equiv^{\Phi} \Gamma \lhd^{\Phi} \mathfrak{u}$, i.e., $\mathfrak{K}' \equiv^{\Phi} \Gamma'$.

(2) Let $(\Gamma, \Gamma') \in [\![\alpha]\!]^{\Phi}$. Then there is a $\mathfrak{u} \in \alpha$ such that $\Gamma' = \Gamma \lhd^{\Phi} \mathfrak{u}$ and $\Gamma \models^{\Phi} \rho$ for some $\rho \in Pre(\mathfrak{u}) \cap bcl(\Phi)$. By Lem. 7.1 and $\mathfrak{u} \in \mathcal{U}^{\Phi}$ it holds that $\mathfrak{K} \models pre(\mathfrak{u})$ and hence $(\mathfrak{K}, \mathfrak{K}') \in [\![\alpha]\!]$ with $\mathfrak{K}' = \mathfrak{K} \lhd \mathfrak{u}$. By Lem. 7.2, $\mathfrak{K} \lhd \mathfrak{u} \equiv^{\Phi} \Gamma \lhd^{\Phi} \mathfrak{u}$, i.e., $\mathfrak{K}' \equiv^{\Phi} \Gamma'$. □

Ensemble configuration equivalence As a consequence of Lem. 7.3 we can prove that semantic and symbolic ensemble configurations mutually simulate action execution while preserving Φ-equivalence.

Proposition 7.1

Let $(\vec{E}, \mathfrak{K})$ be an epistemic ensemble configuration over $\check{\Sigma}$ and $(\vec{E}, \Gamma)$ a symbolic epistemic ensemble configuration over $(\check{\Sigma}, \Phi)$. Let $\eta \in \bigcup \vec{\mathrm{N}}$ be an agent action, and let $\mathfrak{K} \equiv^{\Phi} \Gamma$ hold.

1. *If $(\vec{E}, \mathfrak{K}) \xrightarrow{\eta}{}^{act} (\vec{E}', \mathfrak{K}')$, then there is a $\Gamma' \in \mathcal{S}^{\Phi}$ such that $(\vec{E}, \Gamma) \xrightarrow{\eta}{}^{act}_{\Phi} (\vec{E}', \Gamma')$ and $\mathfrak{K}' \equiv^{\Phi} \Gamma'$.*
2. *If $(\vec{E}, \Gamma) \xrightarrow{\eta}{}^{act}_{\Phi} (\vec{E}', \Gamma')$, then there is a $\mathfrak{K}' \in \mathcal{S}$ such that $(\vec{E}, \mathfrak{K}) \xrightarrow{\eta}{}^{act} (\vec{E}', \mathfrak{K}')$ and $\mathfrak{K}' \equiv^{\Phi} \Gamma'$.*

Proof

(1) Let $(\vec{E}, \mathfrak{K}) \xrightarrow{\eta}{}^{act} (\vec{E}', \mathfrak{K}')$ be given. Then there is a $\vec{E} \xhookrightarrow{\widehat{\varphi}:\eta} \vec{E}'$ with $\mathfrak{K} \models \widehat{\varphi}$ and $(\mathfrak{K}, \mathfrak{K}') \in [\![act(\eta)]\!]$. Since $\mathfrak{K} \equiv^{\Phi} \Gamma$ and $\widehat{\varphi} \in bcl(\Phi)$ we have, by Lem. 7.1, $\Gamma \models^{\Phi} \widehat{\varphi}$ and, by Lem. 7.3(1), there exists a $\Gamma' \in \mathcal{S}^{\Phi}$ such that $(\Gamma, \Gamma') \in [\![act(\eta)]\!]^{\Phi}$ and $\mathfrak{K}' \equiv^{\Phi} \Gamma'$. Therefore $(\vec{E}, \Gamma) \xrightarrow{\eta}{}^{act}_{\Phi} (\vec{E}', \Gamma')$ and $\mathfrak{K}' \equiv^{\Phi} \Gamma'$.

(2) Let $(\vec{E}, \Gamma) \xrightarrow{\eta}{}^{act}_{\Phi} (\vec{E}', \Gamma')$ be given. Then there is a $\vec{E} \xhookrightarrow{\widehat{\phi}:\eta} \vec{E}'$ with $\Gamma \models^{\Phi} \widehat{\phi}$ and $(\Gamma, \Gamma') \in [\![act(\eta)]\!]^{\Phi}$. Since $\mathfrak{K} \equiv^{\Phi} \Gamma$ we have, by Lem. 7.1, $\mathfrak{K} \models \widehat{\phi}$ and, by Lem. 7.3(2), there exists $\mathfrak{K}' \in \mathcal{S}$ such that $(\mathfrak{K}, \mathfrak{K}') \in [\![act(\eta)]\!]$ and $\mathfrak{K}' \equiv^{\Phi} \Gamma'$. Therefore $(\vec{E}, \mathfrak{K}) \xrightarrow{\eta}{}^{act} (\vec{E}', \mathfrak{K}')$ and $\mathfrak{K}' \equiv^{\Phi} \Gamma'$. □

The Φ-equivalence $\mathfrak{K} \equiv^{\Phi} \Gamma$ between semantic and symbolic epistemic states induces a (Φ, act)-equivalence between semantic and symbolic epistemic ensemble configurations defined as $(\vec{E}, \mathfrak{K}) \equiv_{\Phi}^{act} (\vec{E}, \Gamma)$ if $\mathfrak{K} \equiv^{\Phi} \Gamma$. The next proposition lifts Prop. 7.1 to compound epistemic actions:

Proposition 7.2
Let $\lambda \in C^{\Phi}$ and let $(\vec{E}, \mathfrak{K}) \equiv_{\Phi}^{act} (\vec{E}, \Gamma)$ hold.

1. *If $((\vec{E}, \mathfrak{K}), (\vec{E}', \mathfrak{K}')) \in [\![\lambda]\!]^{act}$, then there is a $\Gamma' \in \mathcal{S}^{\Phi}$ such that $((\vec{E}, \Gamma), (\vec{E}', \Gamma')) \in [\![\lambda]\!]_{\Phi}^{act}$ and $(\vec{E}', \mathfrak{K}') \equiv_{\Phi}^{act} (\vec{E}', \Gamma')$.*
2. *If $((\vec{E}, \Gamma), (\vec{E}', \Gamma')) \in [\![\lambda]\!]_{\Phi}^{act}$, then there is a $\mathfrak{K}' \in \mathcal{S}$ such that $((\vec{E}, \mathfrak{K}), (\vec{E}', \mathfrak{K}')) \in [\![\lambda]\!]^{act}$ and $(\vec{E}', \mathfrak{K}') \equiv_{\Phi}^{act} (\vec{E}', \Gamma')$.*

Proof
This follows by structural induction on the form of $\lambda \in C^{\Phi}$, where $\eta \in \bigcup \vec{N}$ is covered by Prop. 7.1, $\phi?$ with $\phi \in bcl(\Phi)$ by Lem. 7.1, and all other cases follow directly from the induction hypothesis. □

Finally, (Φ, act)-equivalent ensemble configurations satisfy the same dynamic ensemble logic formulæ. Thus symbolic epistemic ensemble configurations can be considered as correct realisations of semantic epistemic ensemble configurations.

Theorem 7.1
Let $(\vec{E}, \mathfrak{K}) \equiv_{\Phi}^{act} (\vec{E}, \Gamma)$ hold. Then for all $\psi \in \mathscr{D}^{\Phi}$, it holds that $(\vec{E}, \mathfrak{K}) \models^{act} \psi \iff (\vec{E}, \Gamma) \models_{\Phi}^{act} \psi$.

Proof
This follows by structural induction on the form of $\psi \in \mathscr{D}^{\Phi}$, where $[\lambda]\psi$ is a consequence of Prop. 7.2. □

8 Distributed symbolic epistemic ensembles

In the last section we have considered a refinement step which allows us to realise epistemic ensemble configurations $(\vec{E}, \mathfrak{K})$, using semantic epistemic states $\mathfrak{K} \in \mathcal{S}$, by epistemic ensemble configurations $(\vec{E}, \Gamma)$ relying on symbolic epistemic states $\Gamma \in \mathcal{S}^{\Phi}$. Such Γs can be considered as knowledge bases giving a global view on the current epistemic state of the whole ensemble. It is, however, more realistic and natural to distribute the global knowledge base into a family of local knowledge bases such that our ensemble becomes a truly distributed system where any agent has its own local symbolic state. Thus we will end up in distributed symbolic epistemic ensemble configurations $(a : (P_a, \Lambda_a))_{a \in A}$ with local symbolic epistemic states Λ_a, one for each agent $a \in A$.

A *distributed symbolic epistemic signature* $(\Sigma, \vec{\Phi})$ is given by an epistemic signature $\Sigma = (\Pi, A)$ and an A-family $\vec{\Phi} = (\Phi_a)_{a \in A}$ of finite sets of local *focus formulæ* such that $\Phi_a \subseteq \mathscr{F} \upharpoonright a$ and no $\varphi \in \bigcup \vec{\Phi} = \bigcup_{a \in A} \Phi_a$ is a tautology or a contradiction.

Distributed symbolic epistemic states A *distributed symbolic epistemic state* over $(\Sigma, \vec{\Phi})$ is an A-family $\vec{\Lambda} = (\Lambda_a)_{a \in A}$ with $\Lambda_a \subseteq \Phi_a$ for all $a \in A$. The set of such states is denoted by $\vec{\mathcal{S}}^{\vec{\Phi}}$. For a $\vec{\Lambda} \in \vec{\mathcal{S}}^{\vec{\Phi}}$ and a $\phi \in \bigcup bcl(\vec{\Phi}) = \bigcup_{a \in A} bcl(\Phi_a)$ we define $\vec{\Lambda} \models^{\vec{\Phi}} \phi$ if, and only if, $\Lambda_a \models^{\Phi_a} \phi$ for all $a \in \mathrm{ags}(\phi)$. Note that, in general, $\bigcup bcl(\vec{\Phi}) \subsetneq bcl(\bigcup \vec{\Phi})$.

Distributed symbolic epistemic updates The notion of symbolic update is extended pointwise to the distributed case. The *distributed symbolic epistemic update* $\vec{\Lambda} \lhd^{\vec{\Phi}} \mathfrak{u}$ of a distributed symbolic epistemic state $\vec{\Lambda} \in \vec{\mathcal{S}}^{\vec{\Phi}}$ by an epistemic action $\mathfrak{u} \in \mathcal{U}$ is defined as the A-family

$$\vec{\Lambda} \lhd^{\vec{\Phi}} \mathfrak{u} = (\Lambda_a \lhd^{\Phi_a} \mathfrak{u})_{a \in A} .$$

Distributed symbolic environments We adjust the notion of representability of epistemic actions in Sect. 6 to the distributed case and say that an $\mathfrak{u} \in \mathcal{U}$ is *locally $\vec{\Phi}$-representable* if

1. $Pre(\mathfrak{u}) \cap \bigcup bcl(\vec{\Phi}) \neq \emptyset$;
2. $Wlp(\mathfrak{u}, \varphi'_a)/pre(\mathfrak{u}) \cap bcl(\Phi_a) \neq \emptyset$ for all $\varphi'_a \in \Phi_a$, $a \in A$.

This means that the precondition of $\mathfrak{u}$ is equivalent to some local agent formula in the Boolean closure $\bigcup_{a \in A} bcl(\Phi_a)$ and the weakest liberal precondition preserves locality. The class of such epistemic actions is denoted by $\mathcal{U}^{\vec{\Phi}}$. An epistemic choice action $\alpha \in \mathcal{A}$ is *locally $\vec{\Phi}$-representable* if all $\mathfrak{u} \in \alpha$ are locally $\vec{\Phi}$-representable; the class of such epistemic choice actions is denoted by $\mathcal{A}^{\vec{\Phi}}$.

The *distributed symbolic semantics* of a locally $\vec{\Phi}$-representable epistemic choice action $\alpha \in \mathcal{A}^{\vec{\Phi}}$ is given by the relation

$$\begin{aligned}[\![\alpha]\!]^{\vec{\Phi}} = \{(\vec{\Lambda}, \vec{\Lambda} \lhd^{\vec{\Phi}} \mathfrak{u}) \in \vec{\mathcal{S}}^{\vec{\Phi}} \times \vec{\mathcal{S}}^{\vec{\Phi}} \mid{}& \mathfrak{u} \in \alpha, \\ & \text{ex. } \rho \in Pre(\mathfrak{u}) \cap \textstyle\bigcup bcl(\vec{\Phi}) \text{ s. t. } \vec{\Lambda} \models^{\vec{\Phi}} \rho\} .\end{aligned}$$

The notions of the next paragraph are direct extensions of their non-distributed counterparts:

A *distributed symbolic epistemic ensemble signature* $(\check{\Sigma}, \vec{\Phi})$ consists of an epistemic ensemble signature $\check{\Sigma} = (\Sigma, \vec{N})$ and an A-family of focus formulæ $\vec{\Phi} = (\Phi_a)_{a \in A}$ such that $(\Sigma, \vec{\Phi})$ is a distributed symbolic epistemic signature. An ensemble $\vec{E}$ over $\check{\Sigma}$ is an ensemble over $(\check{\Sigma}, \vec{\Phi})$ if all guards occurring in $\vec{E}$ are in $\bigcup bcl(\vec{\Phi})$. An epistemic action interpretation $act : \bigcup \vec{N} \to \mathcal{A}$ is a *$\vec{\Phi}$-interpretation* if $act(\eta) \in \mathcal{A}^{\vec{\Phi}}$ for all $\eta \in \bigcup \vec{N}$.

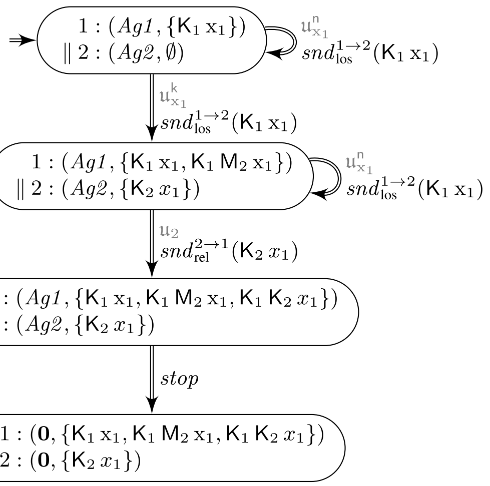


Fig. 4 Symbolic distributed realisation: Finite transition system for the distributed symbolic bit transmission ensemble started in $\Lambda_{2,1} = \{\mathsf{K}_1\, \mathrm{x}_1\}$ and $\Lambda_{2,2} = \emptyset$ with $\mathfrak{u}^{\mathsf{k}}_{\mathrm{x}_1} = (U_{grp}(\{2\}, \mathsf{K}_1\, \mathrm{x}_1), \mathsf{k})$, $\mathfrak{u}^{\mathsf{n}}_{\mathrm{x}_1} = (U_{grp}(\{2\}, \mathsf{K}_1\, \mathrm{x}_1), \mathsf{n})$, and $\mathfrak{u}_2 = (U_{grp}(\{1, 2\}, \mathsf{K}_2\, x_1), \mathsf{k})$ (see Ex. 4.3)

Distributed symbolic epistemic ensemble configurations over $(\Sigma, \vec{\Phi})$ add a local knowledge base to each agent. For defining such configurations, we consider families $(a : (P_a, \Lambda_a))_{a \in G}$, now of pairs of agent processes and local symbolic epistemic states with $G \subseteq A$ such that $a \in \mathrm{ags}(P_a)$ and $\Lambda_a \subseteq \Phi_a$ for each $a \in G$. The *composition* $(a_1 : (P_{a_1}, \Lambda_{a_1}))_{a_1 \in G_1} \parallel (a_2 : (P_{a_2}, \Lambda_{a_2}))_{a_2 \in G_2}$ of sub-families for disjoint $G_1, G_2 \subseteq A$ is given by the family $(a : (P_a, \Lambda_a))_{a \in G_1 \cup G_2}$; and the *singleton* $i : (P_i, \Lambda_i)$ stands for $(a : (P_a, \Lambda_a))_{a \in \{i\}}$. A *distributed symbolic epistemic ensemble configuration* over $(\Sigma, \vec{\Phi})$ is a family $\vec{D} = (a : (P_a, \Lambda_a))_{a \in A}$ such that $(a : P_a)_{a \in A}$ is an ensemble over $(\check{\Sigma}, \vec{\Phi})$ and $(\Lambda_a)_{a \in A}$ a distributed symbolic epistemic state in $\vec{\mathcal{S}}^{\vec{\Phi}}$. The *distributed symbolic ensemble semantics* over $(\check{\Sigma}, \vec{\Phi})$ w.r.t. a $\vec{\Phi}$-interpretation $act : \bigcup \vec{\mathrm{N}} \to \mathcal{A}^{\vec{\Phi}}$ is the ternary relation $\Rightarrow^{act}_{\vec{\Phi}}$ between distributed configurations, agent actions and (successor) configurations defined by

$$(a : (P_a, \Lambda_a))_{a \in A} \overset{\eta}{\Rightarrow}{}^{act}_{\vec{\Phi}} (a : (P'_a, \Lambda'_a))_{a \in A}$$

$$\text{if } (a : P_a)_{a \in A} \xhookrightarrow{\widehat{\phi} : \eta} (a : P'_a)_{a \in A},\ \vec{\Lambda} \models^{\vec{\Phi}} \widehat{\phi}, \text{ and}$$
$$((\Lambda_a)_{a \in A}, (\Lambda'_a)_{a \in A}) \in [\![act(\eta)]\!]^{\vec{\Phi}}$$

Example 8.1
How we continue with Ex. 6.2 for the distributed case is shown in Fig. 4. The start ensemble configuration is $1 : (Ag1, \{\mathsf{K}_1\, \mathrm{x}_1\}) \parallel 2 : (Ag2, \emptyset)$.

Satisfaction of epistemic ensemble formulæ The *distributed symbolic semantics* of a guard-agent action sequence $\sigma = (\widehat{\phi}_1 : \eta^{\epsilon}_1) \ldots (\widehat{\phi}_k : \eta^{\epsilon}_k)$ with all guards $\widehat{\phi}_i$ formulæ in $\bigcup bcl(\vec{\Phi})$ w.r.t. the $\vec{\Phi}$-interpretation act is inductively given by

$$[\![\widehat{\phi} : \epsilon]\!]^{act}_{\vec{\Phi}} = \{(\vec{\Lambda}, \vec{\Lambda}) \in (\vec{\mathcal{S}}^{\vec{\Phi}})^2 \mid \vec{\Lambda} \models^{\vec{\Phi}} \widehat{\phi}\}\ ,$$

$$[\![\widehat{\phi} : \eta]\!]^{act}_{\vec{\Phi}} = \{(\vec{\Lambda}, \vec{\Lambda}') \in [\![act(\eta)]\!]^{\vec{\Phi}} \mid \vec{\Lambda} \models^{\vec{\Phi}} \widehat{\phi}\}\ ,$$

$$[\![\sigma_1 \cdot \sigma_2]\!]^{act}_{\vec{\Phi}} = [\![\sigma_1]\!]^{act}_{\vec{\Phi}} ; [\![\sigma_2]\!]^{act}_{\vec{\Phi}} \quad \text{(rel. comp.)}$$

The set of compound ensemble actions $\lambda \in C$ with all tests in λ formulæ in $\bigcup bcl(\vec{\Phi})$ is denoted $C^{\vec{\Phi}}$. The *semantics* of a $\lambda \in C^{\vec{\Phi}}$ w.r.t. act is given by the relation

$$\begin{aligned}[\![\lambda]\!]^{act}_{\vec{\Phi}} = \{((a : (P_a, \Lambda_a))_{a \in A}, (a : (P'_a, \Lambda'_a))_{a \in A}) \mid \\ ((a : P_a)_{a \in A}, \sigma, (a : P'_a)_{a \in A}) :: \lambda, \\ ((\Lambda_a)_{a \in A}, (\Lambda'_a)_{a \in A}) \in [\![\sigma]\!]^{act}_{\vec{\Phi}}\}\ .\end{aligned}$$

The epistemic ensemble formulæ $\mathscr{D}^{\vec{\Phi}}$ over $(\check{\Sigma}, \vec{\Phi})$ are the epistemic ensemble formulæ over $\check{\Sigma}$ that only contain sub-formulæ $\phi \in \bigcup bcl(\vec{\Phi})$. The satisfaction of a $\psi \in \mathscr{D}^{\vec{\Phi}}$ by a distributed symbolic epistemic ensemble configuration $\vec{D} = (a : (P_a, \Lambda_a))_{a \in A}$ w.r.t. the $\vec{\Phi}$-interpretation $act : \bigcup \vec{\mathrm{N}} \to \mathcal{A}^{\vec{\Phi}}$ is inductively defined along the structure of ψ:

$$(a : (P_a, \Lambda_a))_{a \in A} \models^{act}_{\vec{\Phi}} \phi \iff (\Lambda_a)_{a \in A} \models^{\vec{\Phi}} \phi$$

$$\vec{D} \models^{act}_{\vec{\Phi}} \mathrm{true}$$

$$\vec{D} \models^{act}_{\vec{\Phi}} \neg\psi \iff \text{not } \vec{D} \models^{act}_{\vec{\Phi}} \psi$$

$$\vec{D} \models^{act}_{\vec{\Phi}} \psi_1 \wedge \psi_2 \iff \vec{D} \models^{act}_{\vec{\Phi}} \psi_1 \text{ and } \vec{D} \models^{act}_{\vec{\Phi}} \psi_2$$

$$\vec{D} \models^{act}_{\vec{\Phi}} [\lambda]\psi \iff \vec{D}' \models^{act}_{\vec{\Phi}} \psi \text{ for all } \vec{D}' \text{ with } (\vec{D}, \vec{D}') \in [\![\lambda]\!]^{act}_{\vec{\Phi}}$$

9 Relating ensembles in symbolic and distributed environments

We are now interested in relating ensembles which are executed in a (global) symbolic environment and in a distributed symbolic environment. After some preliminary technical statements, we introduce a notion of local $\vec{\Phi}$-equivalence between symbolic and distributed symbolic states and we lift this equivalence to a mutual simulation of ensemble configurations. Our main results show that ensembles running in equivalent symbolic and distributed symbolic environments mutually simulate each other and hence satisfy the same dynamic epistemic ensemble logic formulæ.

The following lemma shows that each formula that has more than one agent is either a tautology or a contradiction. Such formulæ are excluded from the set of focus formulæ.

Consequently, we deal only with agents whose formulæ do not overlap.

Lemma 9.1
If $|\mathrm{ags}(\varphi)| > 1$ *for some* $\varphi \in \mathscr{F}$*, then* $\models \varphi \leftrightarrow \mathsf{true}$ *or* $\models \varphi \leftrightarrow \mathsf{false}$.

Proof
By induction on the structure of $\varphi \in \mathscr{F}$. $\square$

A symbolic epistemic state $\Gamma \in \mathcal{S}^{\Phi}$ is *satisfiable* if there is a $\mathfrak{R} \in \mathcal{S}$ with $\mathfrak{R} \models \Gamma$; it has Φ*-disjunction witnesses* if for all subsets $\Phi_* \subseteq \Phi$, $\Gamma \models \bigvee \Phi_*$ implies that there is a (witness) $\varphi \in \Phi_*$ with $\varphi \in \Gamma$. Being satisfiable and providing disjunction witnesses guarantees that there is a semantic epistemic state satisfying exactly the focus formulæ in Γ.

Example 9.1
For our focus formulæ Φ_2 of Ex. 6.2 the symbolic epistemic state $\Gamma_0 = \{\mathsf{K}_2 x_1\}$ is satisfiable and has Φ_2-disjunction witnesses (recall that $\mathsf{K}_2 x_1$ abbreviates $\mathsf{K}_2\,\mathsf{x}_1 \vee \mathsf{K}_2\,\neg\mathsf{x}_1$). But if we add $\varphi_1 = \mathsf{K}_2\,\mathsf{x}_1$ and $\varphi_2 = \mathsf{K}_2\,\neg\mathsf{x}_1$ to obtain the focus formulæ Φ_2', then Γ_0 does not have Φ_2'-disjunction witnesses, since $\{\mathsf{K}_2 x_1\} \models \varphi_1 \vee \varphi_2$, but neither $\mathsf{K}_2\,\mathsf{x}_1 \in \{\mathsf{K}_2 x_1\}$ nor $\mathsf{K}_2\,\neg\mathsf{x}_1 \in \{\mathsf{K}_2 x_1\}$.

Similarly, for the focus formulæ $\Phi' = \{\mathsf{K}_1\,\mathsf{x}_1, \neg\mathsf{K}_1\,\mathsf{x}_1\}$ and the symbolic epistemic state $\Gamma_1 = \emptyset$ it holds that $\Gamma_1 \models \mathsf{K}_1\,\mathsf{x}_1 \vee \neg\mathsf{K}_1\,\mathsf{x}_1$, but $\mathsf{K}_1\,\mathsf{x}_1 \notin \emptyset$ and $\neg\mathsf{K}_1\,\mathsf{x}_1 \notin \emptyset$; thus Γ_1 does not have Φ'-disjunction witnesses.

Lemma 9.2
Let $\Gamma \in \mathcal{S}^{\Phi}$. Γ *is satisfiable and has* Φ*-disjunction witnesses if, and only if, there is a* $\mathfrak{R} \in \mathcal{S}$ *with* $\mathfrak{R} \equiv^{\Phi} \Gamma$.

Proof
Let first Γ be satisfiable and have Φ-disjunction witnesses. As Γ is satisfiable there is a $\mathfrak{R} \in \mathcal{S}$ with $\mathfrak{R} \models \Gamma$. Assume for a contradiction that for the finite set $\Phi \setminus \Gamma$ each $\mathfrak{R} \in \mathcal{S}$ with $\mathfrak{R} \models \Gamma$ satisfies $\mathfrak{R} \models \bigvee(\Phi \setminus \Gamma)$, i.e., $\Gamma \models \bigvee(\Phi \setminus \Gamma)$. Then $\varphi \in \Gamma$ for some $\varphi \in \Phi \setminus \Gamma$ by Γ having Φ-disjunction witnesses, which contradicts $\varphi \notin \Gamma$. Hence there is a $\mathfrak{R} \in \mathcal{S}$ with $\mathfrak{R} \models \varphi$ for each $\varphi \in \Gamma$ and $\mathfrak{R} \not\models \varphi$ for each $\varphi \in \Phi \setminus \Gamma$, i.e., $\mathfrak{R} \equiv^{\Phi} \Gamma$.

Let conversely $\mathfrak{R} \equiv^{\Phi} \Gamma$ hold for some $\mathfrak{R} \in \mathcal{S}$. Then Γ is satisfiable. If $\Gamma \models \bigvee \Phi_*$ for some $\Phi_* \subseteq \Phi$, then $\mathfrak{R} \models \bigvee \Phi_*$ as $\mathfrak{R} \models \Gamma$, and hence $\mathfrak{R} \models \varphi$ for some $\varphi \in \Phi_*$; since $\varphi \in \Phi$ and $\mathfrak{R} \equiv^{\Phi} \Gamma$ it holds that $\varphi \in \Gamma$, and thus Γ has Φ-disjunction witnesses. $\square$

Corollary 9.1
Let $\Gamma \in \mathcal{S}^{\Phi}$ *be satisfiable and have* Φ*-disjunction witnesses. Let* $\mathfrak{u} \in \mathcal{U}^{\Phi}$ *such that* $\Gamma \models^{\Phi} \rho_{\mathfrak{u}}$ *for some* $\rho_{\mathfrak{u}} \in \mathit{Pre}(\mathfrak{u}) \cap \mathit{bcl}(\Phi)$*. Then* $\Gamma \lhd^{\Phi} \mathfrak{u}$ *is satisfiable and has* Φ*-disjunction witnesses.*

Proof
By Lem. 9.2, there is a $\mathfrak{R} \in \mathcal{S}$ with $\mathfrak{R} \equiv^{\Phi} \Gamma$. By Lem. 7.1 and $\rho_{\mathfrak{u}} \in \mathit{Pre}(\mathfrak{u})$, $\mathfrak{R} \models \mathit{pre}(\mathfrak{u})$. Thus $\mathfrak{R} \lhd \mathfrak{u} \equiv^{\Phi} \Gamma \lhd^{\Phi} \mathfrak{u}$ by Lem. 7.2 and hence $\Gamma \lhd^{\Phi} \mathfrak{u}$ is satisfiable and has Φ-disjunction witnesses by Lem. 9.2. $\square$

To establish a relationship between global symbolic states and families of local symbolic states we introduce the notion of $\vec{\Phi}$-equivalence. For doing this we must assume that the global set Φ is distributable.

A $\Phi \subseteq \mathscr{F}$ is A*-distributable* and the A-family $\vec{\Phi} = (\Phi_a)_{a \in A}$ is its *distribution* if $\Phi = \bigcup_{a \in A} \Phi_a$ and $\Phi_a = \Phi \cap \mathscr{F} \restriction a$ for all $a \in A$. Let Φ be A-distributable with distribution $\vec{\Phi}$. We say that a symbolic epistemic state $\Gamma \in \mathcal{S}^{\Phi}$ and a distributed symbolic epistemic state $\vec{\Lambda} \in \vec{\mathcal{S}}^{\vec{\Phi}}$ are $\vec{\Phi}$*-equivalent*, written $\Gamma \equiv^{\vec{\Phi}} \vec{\Lambda}$, if for all $a \in A$ and $\varphi_a \in \Phi_a$ it holds that $\varphi_a \in \Gamma$ if, and only if, $\varphi_a \in \Lambda_a$.

Lemma 9.3
Let $\vec{\Lambda} \in \vec{\mathcal{S}}^{\vec{\Phi}}$ *and* $\varphi_a \in \Phi_a$ *for some* $a \in A$*. Then* $\vec{\Lambda} \models^{\vec{\Phi}} \varphi_a$ *if, and only if,* $\Lambda_a \models^{\Phi_a} \varphi_a$.

Proof
By definition, $\vec{\Lambda} \models^{\vec{\Phi}} \varphi_a$ if, and only if, $\Lambda_{a'} \models^{\Phi_{a'}} \varphi_a$ for all $a' \in \mathrm{ags}(\varphi_a)$. But $\mathrm{ags}(\varphi_a) = \{a\}$ by Lem. 9.1, since Φ contains neither tautologies nor contradictions. $\square$

Lemma 9.4
Let $\Gamma \in \mathcal{S}^{\Phi}$ *and* $\vec{\Lambda} \in \vec{\mathcal{S}}^{\vec{\Phi}}$*. Then* $\Gamma \equiv^{\vec{\Phi}} \vec{\Lambda}$ *if, and only if, for all* $\phi \in \bigcup \mathit{bcl}(\vec{\Phi})$ *it holds that* $\Gamma \models^{\Phi} \phi$ *iff* $\vec{\Lambda} \models^{\vec{\Phi}} \phi$.

Proof
"$\Rightarrow$": Assume $\Gamma \equiv^{\vec{\Phi}} \vec{\Lambda}$. The proof is performed by structural induction on the form of ϕ.

Case $\phi = \varphi_a \in \Phi_a$: Then $\Gamma \models^{\Phi} \varphi_a$ iff (by the definition of symbolic satisfaction in Sect. 6) $\varphi_a \in \Gamma$ iff (since $\Gamma \equiv^{\vec{\Phi}} \vec{\Lambda}$) $\varphi_a \in \Lambda_a$ iff (by the definition of symbolic satisfaction in Sect. 6) $\Lambda_a \models^{\Phi_a} \varphi_a$ iff (by Lem. 9.3) $\vec{\Lambda} \models^{\vec{\Phi}} \varphi_a$.

Case true: $\Gamma \models^{\Phi} \mathsf{true}$ and $\vec{\Lambda} \models^{\vec{\Phi}} \mathsf{true}$ hold.

Case $\neg\phi$ with $\neg\phi \notin \Phi$: $\Gamma \models^{\Phi} \neg\phi$ iff not $\Gamma \models^{\Phi} \phi$ iff (by induction hypothesis) not $\vec{\Lambda} \models^{\vec{\Phi}} \phi$ iff (since $\neg\phi \notin \Phi$) $\vec{\Lambda} \models^{\vec{\Phi}} \neg\phi$.

Case $\phi_1 \wedge \phi_2$ with $\phi_1 \wedge \phi_2 \notin \Phi$: $\Gamma \models^{\Phi} \phi_1 \wedge \phi_2$ iff $\Gamma \models^{\Phi} \phi_1$ and $\Gamma \models^{\Phi} \phi_2$ iff (by induction hypothesis) $\vec{\Lambda} \models^{\vec{\Phi}} \phi_1$ and $\vec{\Lambda} \models^{\vec{\Phi}} \phi_2$ iff (since $\phi_1 \wedge \phi_2 \notin \Phi$) $\vec{\Lambda} \models^{\vec{\Phi}} \phi_1 \wedge \phi_2$.

"$\Leftarrow$": Since $\Phi \subseteq \bigcup \mathit{bcl}(\vec{\Phi})$, the assumption implies $\varphi_a \in \Gamma \iff \Gamma \models^{\Phi} \varphi_a \iff \vec{\Lambda} \models^{\vec{\Phi}} \varphi_a \iff \varphi_a \in \Lambda_a$ for all $a \in A$ and $\varphi_a \in \Phi_a$; hence, $\Gamma \equiv^{\vec{\Phi}} \vec{\Lambda}$. $\square$

The following lemma is crucial for establishing the desired relationship between configurations starting in global symbolic and distributed symbolic environments.

Lemma 9.5
Let $\Gamma \in \mathcal{S}^{\Phi}$ *be satisfiable and have* Φ*-disjunction witnesses. Let* $\vec{\Lambda} \in \vec{\mathcal{S}}^{\vec{\Phi}}$ *with* $\Gamma \equiv^{\vec{\Phi}} \vec{\Lambda}$*. Let* $\mathfrak{u} \in \mathcal{U}^{\vec{\Phi}}$ *such that* $\Gamma \models^{\Phi} \rho_{\mathfrak{u}}$ *for some* $\rho_{\mathfrak{u}} \in Pre(\mathfrak{u}) \cap bcl(\Phi)$*. Then* $\Gamma \lhd^{\Phi} \mathfrak{u} \equiv^{\vec{\Phi}} \vec{\Lambda} \lhd^{\vec{\Phi}} \mathfrak{u}$*.*

Proof
Let $\varphi'_a \in \Phi_a$ and let first $\varphi'_a \in \Gamma \lhd^{\Phi} \mathfrak{u}$ hold, i.e., let $\rho \in Wlp(\mathfrak{u}, \varphi'_a)/pre(\mathfrak{u}) \cap bcl(\Phi)$ such that $\Gamma \models^{\Phi} \rho$. Since $\mathfrak{u}$ is locally $\vec{\Phi}$-representable there is some $\rho_a \in Wlp(\mathfrak{u}, \varphi'_a)/pre(\mathfrak{u}) \cap bcl(\Phi_a)$. We show that $\Gamma \models^{\Phi} \rho_a$; then using $\Gamma \equiv^{\vec{\Phi}} \vec{\Lambda}$ and Lem. 9.4 we obtain $\vec{\Lambda} \models^{\vec{\Phi}} \rho_a$ and hence $\Lambda_a \models^{\Phi_a} \rho_a$, i.e., $\varphi'_a \in \Lambda_a \lhd^{\Phi_a} \mathfrak{u}$. Now by Lem. 9.2 and Γ's satisfiability and provision of Φ-disjunction witnesses, there is a $\mathfrak{K} \in \mathcal{S}$ such that $\mathfrak{K} \equiv^{\Phi} \Gamma$. From $\Gamma \models^{\Phi} \rho_{\mathfrak{u}}$ and $\Gamma \models^{\Phi} \rho$ it follows $\mathfrak{K} \models pre(\mathfrak{u})$ and $\mathfrak{K} \models \rho$ by Lem. 7.1, hence $\mathfrak{K} \models \rho_a$, as $\models pre(\mathfrak{u}) \rightarrow (\rho \leftrightarrow \rho_a)$, and thus $\Gamma \models^{\Phi} \rho_a$ again by Lem. 7.1.

Now let conversely $\varphi'_a \in \Lambda_a \lhd^{\Phi_a} \mathfrak{u}$ hold, i.e., $\Lambda_a \models^{\Phi_a} \rho_a$ for some $\rho_a \in Wlp(\mathfrak{u}, \varphi'_a)/pre(\mathfrak{u}) \cap bcl(\Phi_a)$. By $\Gamma \equiv^{\Phi} \vec{\Lambda}$ and Lem. 9.4 it follows that $\Gamma \models^{\Phi} \rho_a$, i.e., $\varphi'_a \in \Gamma \lhd^{\Phi} \mathfrak{u}$ since $\rho_a \in bcl(\Phi_a) \subseteq bcl(\Phi)$. □

Lemma 9.5 is extended to epistemic choice actions as follows.

Lemma 9.6
Let $\Gamma \in \mathcal{S}^{\Phi}$ *and* $\vec{\Lambda} \in \vec{\mathcal{S}}^{\vec{\Phi}}$ *with* $\Gamma \equiv^{\vec{\Phi}} \vec{\Lambda}$*. Let* $\alpha \in \mathcal{A}^{\vec{\Phi}}$*. Then the following holds:*

1. *If* $(\Gamma, \Gamma') \in \llbracket \alpha \rrbracket^{\Phi}$*, then there is a* $\vec{\Lambda}' \in \vec{\mathcal{S}}^{\vec{\Phi}}$ *such that* $(\vec{\Lambda}, \vec{\Lambda}') \in \llbracket \alpha \rrbracket^{\vec{\Phi}}$ *and* $\Gamma' \equiv^{\vec{\Phi}} \vec{\Lambda}'$*.*
2. *If* $(\vec{\Lambda}, \vec{\Lambda}') \in \llbracket \alpha \rrbracket^{\vec{\Phi}}$*, then there is a* $\Gamma' \in \mathcal{S}^{\Phi}$ *such that* $(\Gamma, \Gamma') \in \llbracket \alpha \rrbracket^{\Phi}$ *and* $\Gamma' \equiv^{\Phi} \vec{\Lambda}'$*.*

Proof
Analogous to Lem. 7.3. □

Ensemble configuration simulation The next proposition extends Lem. 9.6 to epistemic ensembles running in global symbolic and in distributed symbolic environments.

Proposition 9.1
Let $((a : P_a)_{a \in A}, \Gamma)$ *be a symbolic epistemic ensemble configuration over* $(\check{\Sigma}, \Phi)$ *and let* $(a : (P_a, \Lambda_a))_{a \in A}$ *be a distributed symbolic epistemic ensemble configuration over* $(\check{\Sigma}, \vec{\Phi})$*. Let* $\eta \in \bigcup \vec{\mathrm{N}}$ *be an agent action, and let* $\Gamma \equiv^{\vec{\Phi}} (\Lambda_a)_{a \in A}$ *hold.*

1. *If* $((a : P_a)_{a \in A}, \Gamma) \xrightarrow{\eta}{}^{act}_{\Phi} ((a : P'_a)_{a \in A}, \Gamma')$*, then there is a* $(\Lambda'_a)_{a \in A} \in \vec{\mathcal{S}}^{\vec{\Phi}}$ *such that* $(a : (P_a, \Lambda_a))_{a \in A} \xRightarrow{\eta}{}^{act}_{\vec{\Phi}} (a : (P'_a, \Lambda'_a))_{a \in A}$ *and* $\Gamma' \equiv^{\vec{\Phi}} (\Lambda'_a)_{a \in A}$*.*
2. *If* $(a : (P_a, \Lambda_a))_{a \in A} \xRightarrow{\eta}{}^{act}_{\vec{\Phi}} (a : (P'_a, \Lambda'_a))_{a \in A}$*, then there is a* $\Gamma' \in \mathcal{S}^{\Phi}$ *such that* $((a : P_a)_{a \in A}, \Gamma) \xrightarrow{\eta}{}^{act}_{\Phi} ((a : P'_a)_{a \in A}, \Gamma')$ *and* $\Gamma' \equiv^{\vec{\Phi}} (\Lambda'_a)_{a \in A}$*.*

Proof
Analogous to Prop. 7.1. □

A symbolic ensemble configuration $((a : P_a)_{a \in A}, \Gamma)$ and a distributed symbolic ensemble configuration $(a : (P_a, \Lambda_a))_{a \in A}$ are $(\vec{\Phi}, act)$*-equivalent*, written $((a : P_a)_{a \in A}, \Gamma) \equiv^{act}_{\vec{\Phi}} (a : (P_a, \Lambda_a))_{a \in A}$ if $\Gamma \equiv^{\vec{\Phi}} (\Lambda_a)_{a \in A}$ are $\vec{\Phi}$-equivalent. The following Prop. 9.2 provides the preparation for our final Thm. 9.1 which shows that $(\vec{\Phi}, act)$-equivalent ensemble configurations with global symbolic and with distributed symbolic states satisfy the same formulæ of our dynamic epistemic ensemble logic. Thus epistemic ensemble configurations with distributed symbolic states can be considered as correct realisations of ensemble configurations with global symbolic states.

Proposition 9.2
Let $\lambda \in C^{\vec{\Phi}}$ *and let* $(\vec{E}, \Gamma) \equiv^{act}_{\vec{\Phi}} \vec{D}$ *hold.*

1. *If* $((\vec{E}, \Gamma), (\vec{E}', \Gamma')) \in \llbracket \lambda \rrbracket^{act}_{\Phi}$*, then there is a* $\vec{D}'$ *such that* $(\vec{D}, \vec{D}') \in \llbracket \lambda \rrbracket^{act}_{\vec{\Phi}}$ *and* $(\vec{E}', \Gamma') \equiv^{act}_{\vec{\Phi}} \vec{D}'$*.*
2. *If* $(\vec{D}, \vec{D}') \in \llbracket \lambda \rrbracket^{act}_{\vec{\Phi}}$*, then there are an* $\vec{E}'$ *and a* $\Gamma' \in \mathcal{S}^{\Phi}$ *such that* $((\vec{E}, \Gamma), (\vec{E}', \Gamma')) \in \llbracket \lambda \rrbracket^{act}_{\Phi}$ *and* $(\vec{E}', \Gamma') \equiv^{act}_{\Phi} \vec{D}'$*.*

Proof
Analogous to Prop. 7.2. □

Theorem 9.1
Let $(\vec{E}, \Gamma) \equiv^{act}_{\vec{\Phi}} \vec{D}$ *hold. Then for all* $\psi \in \mathscr{D}^{\vec{\Phi}}$*, it holds that* $(\vec{E}, \Gamma) \models^{act}_{\Phi} \psi \iff \vec{D} \models^{act}_{\vec{\Phi}} \psi$*.*

Proof
Analogous to Thm. 7.1. □

The combination of our simulations from Sect. 7 and this section completes our two step development methodology: An ensemble configuration with a concrete epistemic state can be faithfully realised by the same ensemble but with a global symbolic epistemic state; this can in turn be represented by a distributed ensemble with local knowledge bases.

Corollary 9.2
Let the equivalences

$$\begin{aligned} &((a : P_a)_{a \in A}, \mathfrak{K}) \\ &\quad \equiv^{act}_{\Phi} ((a : P_a)_{a \in A}, \Gamma) \\ &\quad \equiv^{act}_{\vec{\Phi}} (a : (P_a, \Lambda_a))_{a \in A} \end{aligned}$$

hold. Then for all $\psi \in \mathscr{D}^{\vec{\mathbf{\Phi}}}$, *it holds that*

$$\begin{aligned} &((a : P_a)_{a \in A}, \mathfrak{K}) \models^{act} \psi \\ &\quad \iff ((a : P_a)_{a \in A}, \Gamma) \models^{act}_{\mathbf{\Phi}} \psi \\ &\quad \iff (a : (P_a, \Lambda_a))_{a \in A} \models^{act}_{\vec{\mathbf{\Phi}}} \psi \ . \end{aligned}$$

10 Conclusions

Epistemic ensembles are families of interacting knowledge-based processes. Based on a common generic operational semantics, we presented three complementary mathematical semantics for such ensembles: one in a semantic environment defined by global epistemic states, one in a symbolic environment defined by a global knowledge base, and a distributed symbolic semantics with local knowledge bases. Our main result is a two-step development methodology for epistemic ensembles: Starting from the traditional Kripke structure model of epistemic states we move to global symbolic epistemic states (w.r.t. to a finite set of focus formulæ) which in turn are realised by distributed symbolic knowledge bases. As a direct consequence we get that ensemble descriptions in equivalent environments satisfy the same dynamic logic properties.

In this paper, we presented only small-scale examples; in future work, we aim to address larger case studies across various application domains, such as multi-robot systems (see, e.g., [6]), autonomic clouds (see, e.g., [24]), and e-mobility (see, e.g., [8]). These could form the basis for a general implementation framework for distributed epistemic ensembles. We also intend to extend our approach to ensembles with distributed local states (see, e.g., [5]) and to LLM-assisted software development [3].

Acknowledgements We would like to thank the anonymous reviewers for valuable and constructive comments.

Funding information Open Access funding enabled and organized by Projekt DEAL.

References

1. Baltag, A., Renne, B.: Dynamic epistemic logic. In: Zalta, E.N., Nodelman, U., Allen, C., Anderson, R.L. (eds.) Stanford Encyclopedia of Philosophy. The Metaphysics Research Lab, Stanford University (2016).
2. Baltag, A., Moss, L.S., Solecki, S.: The logic of public announcements and common knowledge and private suspicions. In: Gilboa, I. (ed.) Proc. 7th Conf. Theoretical Aspects of Rationality and Knowledge (TARK), pp. 43–56. Morgan Kaufmann, San Mateo (1998). https://doi.org/10.1007/978-3-319-20451-2_38
3. Belzner, L., Gabor, T., Wirsing, M.: Large language model assisted software engineering: prospects, challenges, and a case study. In: Steffen, B. (ed.) Proc. 1st Intl. Conf. Bridging the Gap Between AI and Reality (AISoLA). Lect. Notes Comp. Sci., vol. 14380, pp. 355–374. Springer, Berlin (2023). https://doi.org/10.1007/978-3-031-46002-9_23
4. Bolander, T.: A gentle introduction to epistemic planning: the DEL approach. In: Ghosh, S., Ramanujam, R. (eds.) Proc. 9th Ws. Methods for Modalities (M4M@ICLA), EPTCS, vol. 243, pp. 1–22 (2017). https://doi.org/10.4204/EPTCS.243.1
5. Bolander, T., Andersen, M.B.: Epistemic planning for single and multi-agent systems. J. Appl. Non-Class. Log. **21**(1), 9–34 (2011). https://doi.org/10.3166/JANCL.21.9-34
6. Bolander, T., Dissing, L., Herrmann, N.: DEL-based epistemic planning for human-robot collaboration: theory and implementation. In: Bienvenu, M., Lakemeyer, G., Erdem, E. (eds.) Proc. 18th Intl. Conf. Principles of Knowledge Representation and Reasoning (KR), pp. 120–129 (2021). https://doi.org/10.24963/KR.2021/12
7. Bortolussi, L., De Nicola, R., Galpin, V., Gilmore, S., Hillston, J., Latella, D., Loreti, M., Massink, M.: CARMA: collective adaptive resource-sharing Markovian agents. In: Bertrand, N., Tribastone, M. (eds.) Proc. 13th Ws. Quantitative Aspects of Programming Languages and Systems (QAPL), EPTCS, vol. 194, pp. 16–31 (2015). https://doi.org/10.4204/EPTCS.194.2
8. Bures, T., De Nicola, R., Gerostathopoulos, I., Hoch, N., Kit, M., Koch, N., Monreale, G.V., Montanari, U., Pugliese, R., Serbedzija, N.B., Wirsing, M., Zambonelli, F.: A life cycle for the development of autonomic systems: the e-mobility showcase. In: Proc. 7th IEEE Intl. Conf. Self-Adaptation and Self-Organizing Systems (SASO) Ws., pp. 71–76. IEEE (2013). https://doi.org/10.1109/SASOW.2013.23
9. Bures, T., Gerostathopoulos, I., Hnetynka, P., Keznikl, J., Kit, M., Plasil, F.: DEECO: an ensemble-based component system. In: Proc. 16th ACM SIGSOFT Symp. Component Based Software Engineering (CBSE), pp. 81–90. ACM (2013). https://doi.org/10.1145/2465449.2465462
10. Castagna, G., Dezani-Ciancaglini, M., Padovani, L.: On global types and multi-party session. Log. Methods Comput. Sci. **8**(1) (2012). https://doi.org/10.2168/LMCS-8(1:24)2012
11. Castellani, I., Mukund, M., Thiagarajan, P.S.: Synthesizing distributed transition systems from global specification. In: Rangan, C.P., Raman, V., Ramanujam, R. (eds.) Proc. 19th Conf. Foundations of Software Technology and Theoretical Computer Science (FSTTCS). Lect. Notes Comp. Sci., vol. 1738, pp. 219–231. Springer, Berlin (1999). https://doi.org/10.1007/3-540-46691-6_17
12. De Nicola, R.: Process algebras. In: Padua, D. (ed.) Encyclopedia of Parallel Computing, pp. 1624–1636. Springer, Berlin (2011). https://doi.org/10.1007/978-0-387-09766-4_450
13. De Nicola, R., Latella, D., Lluch-Lafuente, A., Loreti, M., Margheri, A., Massink, M., Morichetta, A., Pugliese, R., Tiezzi, F., Vandin, A.: The SCEL language: design, implementation, verification. In: [33] (2015). https://doi.org/10.1007/978-3-319-16310-9
14. Dijkstra, E.W., Scholten, C.: Predicate Calculus and Program Semantics. Springer, Berlin (1990). https://doi.org/10.1007/978-1-4612-3228-5

15. Fagin, R., Halpern, J.Y., Moses, Y., Vardi, M.Y.: Reasoning About Knowledge. MIT Press, Cambridge (2003). https://doi.org/10.7551/mitpress/5803.001.0001
16. Harel, D., Thiagarajan, P.S.: Message sequence charts. In: Lavagno, L., Martin, G., Selic, B. (eds.) UML for Real - Design of Embedded Real-Time Systems, pp. 77–105. Kluwer (2003). https://doi.org/10.1007/0-306-48738-1_4
17. Harel, D., Kozen, D., Tiuryn, J.: Dynamic Logic. MIT Press, Cambridge (2000). https://doi.org/10.7551/mitpress/2516.001.0001
18. Havelund, K., Larsen, K.G.: The fork calculus. In: Lingas, A., Karlsson, R.G., Carlsson, S. (eds.) Proc. 20th Intl. Conf. Automata, Languages and Programming (ICALP). Lect. Notes Comp. Sci., vol. 700, pp. 544–557. Springer, Berlin (1993). https://doi.org/10.1007/3-540-56939-1_101
19. Hennicker, R., Knapp, A., Wirsing, M.: Epistemic ensembles. In: Margaria, T., Steffen, B. (eds.) Proc. 11th Intl. Symp. Leveraging Applications of Formal Methods, Verification and Validation (ISoLA). Part III: Adaptation and Learning. Lect. Notes Comp. Sci., vol. 13703, pp. 110–126. Springer, Berlin (2022). https://doi.org/10.1007/978-3-031-19759-8_8
20. Hennicker, R., Knapp, A., Wirsing, M.: Symbolic realisation of epistemic processes. In: Bjørner, N., Heule, M., Voronkov, A. (eds.) Proc. 25th Intl. Conf. Logic for Programming, Artificial Intelligence and Reasoning (LPAR). EPiC Series in Comp., vol. 100, pp. 390–407 (2024). https://doi.org/10.29007/H8H3
21. Hennicker, R., Knapp, A., Wirsing, M.: Epistemic ensembles in semantic and symbolic environments. In: Margaria, T., Steffen, B. (eds.) Proc. 12th Intl. Symp. Leveraging Applications of Formal Methods, Verification and Validation (ISoLA). Part II. Lect. Notes Comp. Sci., vol. 15220, pp. 69–87. Springer, Berlin (2024). https://doi.org/10.1007/978-3-031-75107-3_5
22. Hölzl, M.M., Rauschmayer, A., Wirsing, M.: Software engineering for ensembles. In: Wirsing, M., Banâtre, J., Hölzl, M.M., Rauschmayer, A. (eds.) Software-Intensive Systems and New Computing Paradigms — Challenges and Visions. Lect. Notes Comp. Sci., vol. 5380, pp. 45–63. Springer, Berlin (2008). https://doi.org/10.1007/978-3-540-89437-7_2
23. Lorini, E., Perrotin, E., Schwarzentruber, F.: Epistemic actions: comparing multi-agent belief bases with action models. In: Kern-Isberner, G., Lakemeyer, G., Meyer, T. (eds.) Proc. 19th Intl. Conf. Principles of Knowledge Representation and Reasoning (KR) (2022). https://proceedings.kr.org/2022/24/
24. Mayer, P., Velasco, J., Klarl, A., Hennicker, R., Puviani, M., Tiezzi, F., Pugliese, R., Keznikl, J., Bures, T.: The autonomic cloud. In: [33] (2015). https://doi.org/10.1007/978-3-319-16310-9_16
25. Parikh, R., Ramanujam, R.: A knowledge based semantics of messages. J. Log. Lang. Inf. **12**, 453–467 (2003). https://doi.org/10.1023/A:1025007018583
26. Sürmeli, J.: Epistemic logic in ensemble specification. In: Margaria, T., Steffen, B. (eds.) Proc. 9th Intl. Symp. Leveraging Applications of Formal Methods, Verification and Validation. Distributed Systems (ISoLA 2020), Part II. Lect. Notes Comp. Sci., vol. 12477, pp. 329–343. Springer, Berlin (2020). https://doi.org/10.1007/978-3-030-61470-6_20
27. ter Beek, M.H., Hennicker, R., Proença, J.: Realisability of global models of interaction. In: Ábrahám, E., Dubslaff, C., Tarifa, S.L.T. (eds.) Prof. 20th Intl. Coll. Theoretical Aspects of Computing (ICTAC). Lect. Notes Comp. Sci., vol. 14446, pp. 236–255. Springer, Berlin (2023). https://doi.org/10.1007/978-3-031-47963-2_15
28. van Benthem, J., Gerbrandy, J., Hoshi, T., Pacuit, E.: Merging frameworks for interaction. J. Philos. Log. **38**(5), 491–526 (2009). https://doi.org/10.1007/S10992-008-9099-X
29. van Ditmarsch, H., van der Hoek, W., Kooi, B.P.: Dynamic epistemic logic and knowledge puzzles. In: Priss, U., Polovina, S., Hill, R. (eds.) Proc. 15th Intl. Conf. Conceptual Structures (ICCS). Lect. Notes Comp. Sci., vol. 4604, pp. 45–58. Springer, Berlin (2007). https://doi.org/10.1007/978-3-540-73681-3_4
30. van Ditmarsch, H., van der Hoek, W., Kooi, B.: Dynamic Epistemic Logic, Synthese Library, vol. 337. Springer, Berlin (2008). https://doi.org/10.1007/978-1-4020-5839-4
31. van Ditmarsch, H., van Eijck, J., Pardo, P., Ramezanian, R., Schwarzentruber, F.: Epistemic protocols for dynamic gossip. J. Appl. Log. **20**, 1–31 (2017). https://doi.org/10.1016/J.JAL.2016.12.001
32. Wirsing, M., Knapp, A.: A reduction-based cut-free Gentzen calculus for dynamic epistemic logic. Log. J. IGPL **31**(6), 1047–1068 (2023). https://doi.org/10.1093/JIGPAL/JZAC078
33. Wirsing, M., Hölzl, M.M., Koch, N., Mayer, P. (eds.) Software Engineering for Collective Autonomic Systems — the ASCENS Approach. Lect. Notes Comp. Sci., vol. 8998. Springer, Berlin (2015). https://doi.org/10.1007/978-3-319-16310-9
34. Witzel, A., Zvesper, J.A.: Epistemic logic and explicit knowledge in distributed programming. In: Padgham, L., Parkes, D.C., Müller, J.P., Parsons, S. (eds.) Proc. 7th Intl. Conf. Autonomous Agents and Multiagent Systems (AAMAS), vol. 3, pp. 1463–1466. IFAAMAS (2008). https://doi.org/10.5555/1402821.1402899