

## Regularity theory for (constrained) linear growth functionals in the calculus of variations

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# Regularity Theory for (Constrained) Linear Growth Functionals in the Calculus of Variations

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*Von einem gewissen Punkt an gibt es keine Rückkehr  
mehr. Dieser Punkt ist zu erreichen.*

Franz Kafka, [Kaf31]



# Abstract

This thesis develops a regularity theory for relaxed,  $L^p$ -constrained minimisers of  $\mu$ -elliptic variational integrals with linear growth integrands depending on  $\mathbb{C}$ -elliptic operators  $\mathbb{A}$ . The class of  $\mathbb{C}$ -elliptic operators includes, for example, the full gradient  $\mathbb{A} = \nabla$ , the symmetric gradient  $\mathbb{A} = \varepsilon$ , and the deviatoric gradient  $\mathbb{A} = \text{dev } \nabla$ .

The natural strategy for proving existence of minimisers is based on the direct method in the Calculus of Variations. Since the natural domain of definition  $W^{\mathbb{A},1}$  is not weakly compact and hence existence cannot be guaranteed, we have to perform a relaxation to the space  $BV^{\mathbb{A}} \supsetneq W^{\mathbb{A},1}$ . However, we may encounter non-uniqueness due to the loss of strict convexity for the relaxed problem. To address this issue in the context of regularity, we adapt a vanishing viscosity approach of BECK and SCHMIDT, based on the Ekeland variational principle, and extend it to an abstract framework that incorporates general constraints.

For the symmetric gradient, we establish a Sobolev regularity theory parallel to that known from the full gradient setting. In particular, bounded  $BV^\varepsilon$ -minimisers of 3-elliptic problems are locally in  $W^{1,1}$ . Moreover, the regularity improves to  $W^{1,4-\mu}$  in the strict regime  $\mu \in (1,3)$ . To subsequently relax the boundedness assumption, we study  $L^p$ -bounded minimisers for  $p > 2$  and obtain  $W^{1,1}$ -regularity for  $\mu \in (1, 3 - \frac{2}{p}]$ , which again can be improved in the strict regime.

Finally, in the two-dimensional setting, we consider  $\mathbb{C}$ -elliptic operators induced by orthogonal projections as all these operators possess a common structural property. In this framework, we prove that every  $BV^{\mathbb{A}}$ -minimiser of a  $\mu$ -elliptic linear growth problem with  $\mu \in (1,2)$  is  $W^{1,q}$ -regular for all  $q \geq 1$ , generalising a result by GMEINER.



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# 1 | Introduction and Motivation

## 1.1 Outline of the Thesis and Contextualisation

Let  $\Omega \subset \mathbb{R}^n$  with  $n \geq 2$  be an open and bounded set with Lipschitz boundary  $\partial\Omega$ . In this thesis we give a contribution to the regularity theory for minimisers of the variational principle

$$\text{to minimise } F[v; \Omega] := \int_{\Omega} f(\mathbb{A}v) \, dx \quad (1.1)$$

among a suitable Dirichlet class  $\mathcal{D}_{u_0}$  of admissible functions  $v: \Omega \rightarrow V$  subject to a given boundary datum  $u_0$ . In our case,  $f: W \rightarrow \mathbb{R}$  is a real-valued convex integrand of *linear growth* depending on a first-order  $\mathbb{C}$ -elliptic differential operator  $\mathbb{A}$  between two finite-dimensional vector spaces  $V$  and  $W$ . Such operators are given by

$$\mathbb{A}v := \sum_{\alpha=1}^n \mathbb{A}_{\alpha} \partial_{\alpha} v, \quad v: \Omega \subset \mathbb{R}^n \rightarrow V, \quad (1.2)$$

with linear maps  $\mathbb{A}_{\alpha} \in \mathcal{L}(V, W)$ . We refer to Chapter 2 for precise definitions. By linear growth we mean that there exist two constants  $\gamma, \Gamma > 0$  such that

$$\gamma |z| \leq f(z) \leq \Gamma(1 + |z|) \quad \text{for all } z \in W. \quad (1.3)$$

We observe that the class of  $\mathbb{C}$ -elliptic operators includes the full and the symmetric gradient, namely,  $\mathbb{A} = \nabla$  and  $\mathbb{A} = \varepsilon$  where  $\varepsilon(w) := \frac{1}{2}(\nabla w + \nabla w^{\top})$  for  $w \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ . In case of the latter, such functionals arise naturally in the modelling of the elastic or plastic behaviour of solids or fluids. In these theories, they play the role of model-related energies in terms of the underlying displacement or velocity fields, respectively, while particular choices of the integrands  $f$  allow to model different aspects of the material or fluid. We will give a quick and concise introduction to a model stemming from plasticity in Section 1.2. For further references we mention the works of ANZELOTTI et al. [AG80, AG82, Anz84] as well as [DL76, NH81, OR83, Tem85, Lub90, FS00, DMDM06].

Usually, the existence of minimisers is proved by exploiting the direct method in the Calculus of Variations, cf. e.g. [Mor66, Giu03, Dac08, Rin18]. To this end, we first notice that the functional  $F$  is bounded from below on  $\mathcal{D}_{u_0}$ , as  $f$  satisfies (1.3) and  $\Omega$  is bounded. This ensures the existence of a minimising sequence  $(u_j)_{j \in \mathbb{N}}$ , i.e.

$$\lim_{j \rightarrow \infty} F[u_j; \Omega] = \inf_{\mathcal{D}_{u_0}} F[-; \Omega].$$

The main idea of the direct method is to first find a suitable topology allowing to extract a subsequence  $(u_{j_{\ell}})_{\ell \in \mathbb{N}}$  of  $(u_j)_{j \in \mathbb{N}}$  converging to some element  $v$ . Second, if the functional

$F$  is lower semi-continuous (with respect to the same topology), it follows that  $v$  is a minimiser of  $F$  because of

$$\inf_{\mathcal{D}_{u_0}} F[-; \Omega] \leq F[v; \Omega] \leq \liminf_{j \rightarrow \infty} F[u_j; \Omega] = \lim_{\ell \rightarrow \infty} F[u_{j_\ell}; \Omega] = \inf_{\mathcal{D}_{u_0}} F[-; \Omega].$$

However, similar to the by now well-studied full-gradient case  $\mathbb{A} = \nabla$ , there arise several difficulties when running this procedure for linear growth functionals depending on more general operators  $\mathbb{A}$ . We highlight the three main issues:

- \* Due to the linear growth condition (1.3), the functional (1.1) is well defined on the space

$$W^{\mathbb{A},1}(\Omega) := \{v \in L^1(\Omega; V) : \mathbb{A}v \in L^1(\Omega; W)\}.$$

This space is the natural replacement for the usual Sobolev space  $W^{1,1}(\Omega; \mathbb{R}^N)$ . It turns out that it is not reflexive and consequently fails to be weakly compact. Thus, we cannot expect to extract a converging subsequence from the minimising sequence  $(u_j)_{j \in \mathbb{N}}$ . The natural space for weak\*-compactness, which compensates for the lack of compactness of  $W^{\mathbb{A},1}(\Omega)$ , is

$$BV^{\mathbb{A}}(\Omega) := \{v \in L^1(\Omega; V) : \mathbb{A}v \in \text{RM}_{\text{fin}}(\Omega; W)\},$$

first introduced in [BDG20]. Moreover, there holds the strict inclusion  $W^{\mathbb{A},1}(\Omega) \subsetneq BV^{\mathbb{A}}(\Omega)$ . Since the derivatives of such functions exist only as finite Radon measures, we have to extend the functional  $F$  in a lower semi-continuous way using a suitable relaxation. This strategy requires the concept of functionals defined on measures.

- \* Another drawback is that Dirichlet classes are not preserved, since the trace operator on  $BV^{\mathbb{A}}(\Omega)$  is not continuous with respect to the weak\*-topology. In particular, whenever  $(u_j)_{j \in \mathbb{N}}$  is a sequence in  $\mathcal{D}_{u_0}$  with  $u_j \xrightarrow{*} v$  in  $BV^{\mathbb{A}}(\Omega)$ , we *cannot* conclude that  $v \in \mathcal{D}_{u_0}$ . Especially, we cannot expect that the weak\*-limit of a converging subsequence of a minimising sequence remains in  $\mathcal{D}_{u_0}$ .
- \* Finally, due to the absence of non-trivial  $L^1$ -estimates for singular integral operators, *no* version of Korn's inequality is available. This means that the estimate

$$\|\nabla u\|_{L^1(\Omega; \mathcal{L}(\mathbb{R}^n; V))} \lesssim \|\mathbb{A}u\|_{L^1(\Omega; W)} \quad \text{for } u \in C_c^\infty(\mathbb{R}^n; V)$$

fails in general, except in trivial cases. This is underpinned by Ornstein's Non-Inequality [Orn62]. As a matter of fact, we have to avoid full gradients at all stages in the estimates, which requires a very delicate analysis.

To overcome the first two issues, we extend the functional  $F$  to  $BV^{\mathbb{A}}(\Omega)$  in a lower semi-continuous way. This can be obtained by setting

$$\bar{F}_{u_0}[w; \Omega] := \inf \left\{ \liminf_{j \rightarrow \infty} F[w_j; \Omega] : (w_j)_{j \in \mathbb{N}} \text{ in } u_0 + W_0^{\mathbb{A},1}(\Omega) \text{ with } w_j \rightarrow w \text{ in } L^1(\Omega; V) \right\},$$

which is called the Lebesgue-Serrin relaxation of  $F$  to  $BV^{\mathbb{A}}$  with respect to the Dirichlet datum  $u_0$ . Following the pioneering work of [GS64, Res68, GMS79], one can derive an explicit representation formula using the Lebesgue-Radon-Nikodým decomposition for the measure  $\mathbb{A}u$ , cf. also [But89, ADM92, FM93] and the references therein. Recalling this

splitting into the absolutely continuous part  $\mathbb{A}^a u$  and the singular part  $\mathbb{A}^s u$  (with respect to the Lebesgue measure) respectively, we infer

$$\mathbb{A}u = \mathbb{A}^a u + \mathbb{A}^s u = \frac{d\mathbb{A}^a u}{d\mathcal{L}^n} \mathcal{L}^n + \frac{d\mathbb{A}^s u}{d|\mathbb{A}^s u|} |\mathbb{A}^s u|.$$

Subsequently, it can be shown that for  $u \in \text{BV}^\mathbb{A}(\Omega)$  there holds the equality

$$\begin{aligned} \overline{F}_{u_0}[u; \Omega] &= \int_{\Omega} f \left( \frac{d\mathbb{A}^a u}{d\mathcal{L}^n} \right) dx + \int_{\Omega} f^\infty \left( \frac{d\mathbb{A}^s u}{d|\mathbb{A}^s u|} \right) d|\mathbb{A}^s u| \\ &\quad + \int_{\partial\Omega} f^\infty(\text{tr}_{\partial\Omega}(u_0 - u) \otimes_{\mathbb{A}} \nu_{\partial\Omega}) d\mathcal{H}^{n-1}. \end{aligned}$$

Here,  $\nu_{\partial\Omega}$  denotes the outer unit normal field along  $\partial\Omega$  and the behaviour of the integrand of  $f$  at infinity is encoded in the *recession function*

$$f^\infty(z) := \lim_{s \rightarrow \infty} \frac{f(sz)}{s} = \lim_{t \searrow 0} t f\left(\frac{z}{t}\right) \quad \text{for } z \in W.$$

We note that if  $f$  is of linear growth, then the recession function  $f^\infty$  is a well-defined, 1-homogeneous, lower semi-continuous and convex function taking only finite values. The last condition cannot be guaranteed, if the linear growth condition from above is dropped. We now introduce the notion of minimality, which we will use in the sequel:

**Definition 1.1** ( $\text{BV}^\mathbb{A}$ - and local  $\text{BV}^\mathbb{A}$ -minimisers).

- i) Given  $u_0 \in \text{BV}^\mathbb{A}(\Omega)$ , a map  $u \in \text{BV}^\mathbb{A}(\Omega)$  is called  $\text{BV}^\mathbb{A}$ -minimiser with respect to the Dirichlet datum  $u_0$  if we have  $\overline{F}_{u_0}[u; \Omega] \leq \overline{F}_{u_0}[w; \Omega]$  for all  $w \in \text{BV}^\mathbb{A}(\Omega)$ .
- ii) A map  $u \in \text{BV}_{\text{loc}}^\mathbb{A}(\Omega)$  is called local  $\text{BV}^\mathbb{A}$ -minimiser if we have  $\overline{F}_u[u; U] \leq \overline{F}_u[w; U]$  for all  $U \Subset \Omega$  with Lipschitz boundary  $\partial U$  and all  $w \in \text{BV}^\mathbb{A}(U)$ .

It is clear from the definition, that any (global)  $\text{BV}^\mathbb{A}$ -minimiser is also a local  $\text{BV}^\mathbb{A}$ -minimiser. Using the direct method in the Calculus of Variations, it is shown in [Woz23, Theorem 3.3] or [BDG20, Theorem 5.3], that there always exists a  $\text{BV}^\mathbb{A}$ -minimiser  $u$  under our assumptions. The latter is the weak\*-limit of a minimising sequence  $(u_j)_{j \in \mathbb{N}}$  in  $\mathcal{D}_{u_0}$ , cf. also Corollary 2.15. Moreover, we do not have a relaxation gap, meaning that the *consistency relation*

$$\inf_{\mathcal{D}_{u_0}} F[-; \Omega] = \min_{\text{BV}^\mathbb{A}(\Omega)} \overline{F}_{u_0}[-; \Omega] = \overline{F}_{u_0}[u; \Omega] \tag{1.4}$$

holds, cf. [AFP00, BS13] and [BEG26, Remark 3.7]. Since the recession function is in general not strictly convex, it turns out that the whole functional  $\overline{F}_{u_0}[-; \Omega]$  is not strictly convex on  $\text{BV}^\mathbb{A}(\Omega)$ . As a consequence,  $\text{BV}^\mathbb{A}$ -minimisers may be non-unique in general. A classical example is the area integrand  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  given by  $f(z) = \sqrt{1 + |z|^2}$ , which admits the non-strictly convex recession function  $f^\infty(z) = |z|$ .

As already mentioned above, the expression  $\mathbb{A}u$  exists only as a  $W$ -valued finite Radon measure, while the full distributional gradient of a  $\text{BV}^\mathbb{A}$ -minimiser does not have to belong to  $L^1$  or the space of Radon measures due to Ornstein's Non-Inequality. As a consequence, results for linear growth functionals dealing with the full gradient cannot

be applied in our framework. In order to conclude that relaxed minimisers belong locally to  $BV(\Omega) := BV^\nabla(\Omega)$  or even to  $W^{1,1}(\Omega; \mathbb{R}^N)$ , we have to impose additional ellipticity assumptions on the integrand  $f$ . One suitable scale is that of  $\mu$ -ellipticity, namely, a  $C^2$ -integrand  $f: W \rightarrow \mathbb{R}$  is called  $\mu$ -elliptic for  $\mu > 1$ , if there exist two constants  $0 < \lambda \leq \Lambda < \infty$  such that

$$\lambda \frac{|\xi|^2}{(1 + |z|^2)^{\frac{\mu}{2}}} \leq \langle \nabla^2 f(z) \xi, \xi \rangle \leq \Lambda \frac{|\xi|^2}{(1 + |z|^2)^{\frac{1}{2}}} \quad (1.5)$$

holds for all  $z, \xi \in W$ . This is a type of non-uniform ellipticity, since the ratio between the largest and the lowest eigenvalue of  $\nabla^2 f(z)$  blows up as  $|z| \rightarrow \infty$  at most at rate  $|z|^{\mu-1}$ . This condition is inspired by works of BERNSTEIN on minimal surfaces [Ber12] and LADYZHENSKAYA & URAL'TSEVA [LU70], cf. also [GMS79]. Later, BILDHAUER, FUCHS and MINGIONE [BF02, Bil02, Bil03a, Bil03b, FM00] studied such integrands in the linear and the  $(p, q)$ -growth regime.

It is clear, that for  $\mu_1 \leq \mu_2$  any  $\mu_1$ -elliptic integrand is also  $\mu_2$ -elliptic. Note also that the case  $\mu = 1$  is excluded, since it turns out that there exist no integrands  $f$  satisfying (1.5) in this regime<sup>1</sup>. Moreover,  $\mu = 3$  displays a limit case (see below) and an example is given by the *area integrand*  $f(z) = \sqrt{1 + |z|^2}$ . The latter appears in the study of minimal graphs of codimension one. This is a very classical and well studied topic and we refer to the monographs [Giu84, MM84] for a detailed treatment and additional references. Two further examples for classes of  $\mu$ -elliptic integrands are given by

$$f_\mu(z) := \int_0^{|z|} \int_0^s (1+t)^{-\frac{\mu}{2}} dt ds \quad \text{for } z \in \mathbb{R}^{N \times n}, \mu > 1,$$

cf. [Bil03a, Example 3.9], and

$$h_{s,p} := (1 + (s^2 + |z|^2)^{\frac{p}{2}})^{\frac{1}{p}} \quad \text{for } z \in \mathbb{R}^{N \times n}, p \geq 1 \quad \text{and } s \geq 0,$$

cf. [BS13, Section 3.1]. We will now discuss several known regularity results in order to provide context for the results included in this thesis. The first result towards Sobolev regularity in case of the full gradient  $\mathbb{A} = \nabla$ , the framework of BV-minimisers, is due to BILDHAUER [Bil02]. Based on a plain vanishing viscosity approximation, it is shown that at least *one* BV-minimiser  $u$  satisfies

$$u \in W^{1,1}(\Omega; \mathbb{R}^N) \cap W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^N)$$

for some  $p > 1$  whenever  $1 < \mu < 1 + \frac{2}{n}$ . The  $W^{1,1}$ -regularity remains true for  $1 + \frac{2}{n} \leq \mu \leq 3$  when adding the additional assumption of local boundedness, which can be justified by means of maximum principles or Moser-type iterations. Clearly, the limitation of the approach is that it may not cover all minimisers. We briefly indicate why this is the case: As a first step, one defines for  $\delta \in (0, 1)$  the family of regularised functionals by

$$F_\delta[w; \Omega] := F[w; \Omega] + \frac{\delta}{2} \int_\Omega |\nabla w|^2 dx.$$

By standard methods for quadratic variational integrals, there exists a corresponding sequence of unique minimisers  $(u_\delta)_{\delta \in (0,1)}$  in  $u_0 + W_0^{1,2}(\Omega; \mathbb{R}^N)$ . As a next step, it is shown

<sup>1</sup>It follows from [Bil03a, Remark 4.2] that if  $f$  satisfies the lower bound from (1.5) with  $\mu = 1$ , then it must be of L log L-growth. Moreover, this case would correspond to a uniformly bounded ellipticity ratio.

that this sequence  $(u_\delta)_{\delta \in (0,1)}$  admits a converging minimising subsequence for the original problem and thus, corresponds to *one* specific BV-minimiser  $u$  of  $F$ . The minimality of  $u_\delta$  allows to work with an Euler-Lagrange equation to derive certain uniform a priori estimates satisfied by the sequence  $(u_\delta)_{\delta \in (0,1)}$ . In order to conclude the desired regularity result for the minimiser, one tests the Euler-Lagrange inequality with a specific test function involving the sequence and passes to the limit as  $\delta \rightarrow 0$  afterwards. Based on the Ekeland variational principle, the above result was extended by BECK and SCHMIDT in [BS13], who proved that the same result holds true for *every* relaxed minimiser in the same ellipticity range  $\mu \in (1, 3]$ . The method was later refined by SCHMIDT in [Sch15a, Section 4.4] inspired by results in [CKdN11]. In this approach, one starts with an arbitrary BV-minimiser  $u$  of  $F$ , which can be approximated by a smooth sequence  $(u_j)_{j \in \mathbb{N}}$  that is already a minimising sequence for  $F$ . However, this sequence does not satisfy the corresponding Euler-Lagrange equation and hence, no estimates can be derived. In order to overcome this issue, one applies the Ekeland variational principle to regularised and penalised functionals of the form

$$F_j[w; \Omega] = \int_{\Omega} f(\nabla w) \, dx + \frac{1}{k} \int_{\Omega} (1 + |\nabla w|^2)^n \, dx + \int_{\Omega} \mathbf{g}(w) \, dx,$$

extended by infinity, on the space  $W^{-1,1}(\Omega; \mathbb{R}^N)$ . In this way, one obtains a minimising sequence for  $F$  on  $\mathcal{D}_{u_0}$  consisting of almost-minimisers  $(v_j)_{j \in \mathbb{N}}$  of  $F_j$  close to the given minimiser  $u$ . Here,  $\mathbf{g}$  is chosen in a way to transfer the  $L^\infty$ -bounds of  $(u_j)_{j \in \mathbb{N}}$  to the new sequence  $(v_j)_{j \in \mathbb{N}}$ . The almost-minimality of  $v_j$  is quantified by a perturbation term and turns out to be sufficient to derive an Euler-Lagrange *inequality* as an adequate replacement to derive uniform estimates. This approach will be adopted in the subsequent demonstration of our primary results, with further elaboration provided in the ensuing remarks.

To proceed, we further mention [FPS25] for a Sobolev regularity result for non-autonomous linear growth<sup>2</sup> integrals in  $BV(\Omega)$ . Moreover, to the best of the authors knowledge, it is not known, whether Sobolev regularity still holds true for the Dirichlet problem in the autonomous case or if interior singularities might occur for  $\mu > 3$ . However, for non-autonomous integrands with an additional smooth  $x$ -dependence there is a counterexample presented in [Bil03a, Theorem 4.39]. This approach is based on methods from [GMS79], giving a negative answer to the question in case  $\mu > 3$ . At the contrary, a positive result holds for the Neumann problem, cf. [BBG20], where the authors provide a framework going even beyond the scale of  $\mu$ -elliptic integrands. In particular, their outcome is applicable for suitable  $\mu$ -elliptic linear growth integrands and every  $\mu \in (1, \infty)$ .

A natural generalisation deals with functionals involving the symmetric gradient  $\mathbb{A} = \varepsilon$  for  $\mu$ -elliptic variational integrands of linear growth. In this case, it is convenient to abbreviate  $BV^\varepsilon(\Omega)$  by  $BD(\Omega)$ , the space of functions of *bounded deformation*. The first contribution was given by GMEINER and KRISTENSEN [GK19], who derived a Sobolev regularity result for the non-optimal range  $\mu < 1 + \frac{1}{n}$  based on fractional methods. This result was later improved by GMEINER in [Gme20] to the range  $\mu < 1 + \frac{2}{n}$ , known from the full gradient case, using techniques heavily exploiting the specific structure of the symmetric gradient.

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<sup>2</sup>*Non-autonomous* refers to an explicit dependence on the spatial variable  $x$  in the integrand, namely,  $f(x, u(x), \nabla u(x))$ . If there is no such an explicit dependence, we speak of *autonomous* integrands. The linear growth condition is assumed with respect to the gradient variable.

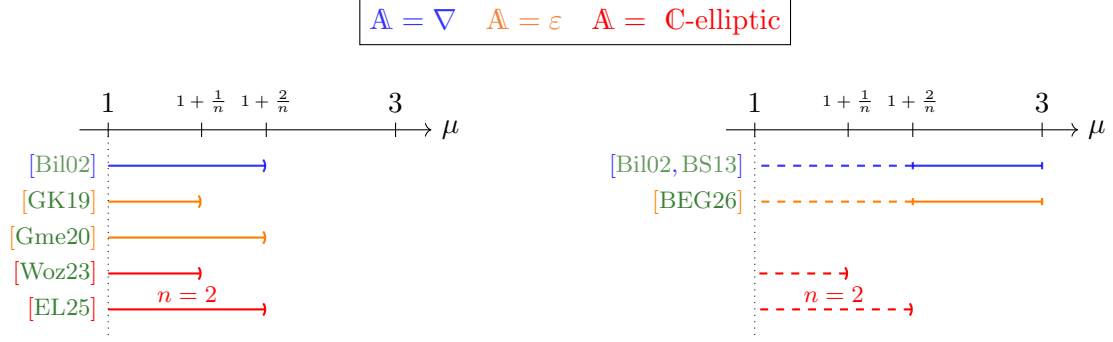


Figure 1.1: Ellipticity range for the autonomous Dirichlet problem without (left) and with  $L^\infty$ -constraint (right).

At the present stage, the only result in this area dealing with more general  $\mathbb{C}$ -elliptic operators is due to WOZNAK in [Woz23], which generalises the higher integrability and Sobolev regularity from [GK19] for the non-optimal range  $1 < \mu < 1 + \frac{1}{n}$  using that fractional methods are more flexible. For the sake of clarity, we have collected some of the described results in Figure 1.1 to give an overview. With these preparations in place, we can now present the main results of this thesis:

- \* The missing piece in obtaining a Sobolev regularity theory on  $\text{BD}(\Omega)$  that is completely parallel to the full gradient case, is the ellipticity range  $1 + \frac{2}{n} \leq \mu \leq 3$ . In order to fill this gap, we consider  $\text{BD}$ -minimisers, which are a priori bounded (even if there are no methods available to justify this in general). The upcoming result stems from [BEG26] and is presented and extended in Chapter 5 of this thesis.

**Theorem A** (Theorem 5.1). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded and consider a variational integrand  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  which satisfies (1.3) and (1.5) with  $\mu \in (1, 3]$ . Then any bounded  $\text{BD}$ -minimiser  $u \in \text{BD}(\Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$  is of class  $W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^n)$  with*

$$|\nabla u| \log(1 + |\nabla u|^2) \in L_{\text{loc}}^1(\Omega).$$

- \* Our second result partly interpolates the ellipticity range between  $\mu = 1 + \frac{2}{n}$  and  $\mu = 3$  by adding an additional super-quadratic  $L^p$ -constraint for  $p > 2$ . Without great effort, this can be done for the full and the symmetric gradient simultaneously. More precisely, we derive the following result in case of the symmetric gradient, the proof of which is included in Chapter 6.

**Theorem B** (Theorem 6.1). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded,  $p > 2$  and consider a variational integrand  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  which satisfies (1.3) and (1.5) with  $\mu \in (1, 3 - \frac{2}{p}]$ . Then any  $L^p$ -bounded  $\text{BD}$ -minimiser  $u \in \text{BD}(\Omega) \cap L^p(\Omega; \mathbb{R}^n)$  is of class  $W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^n)$  with*

$$|\nabla u| \log^{s_p-1}(1 + |\nabla u|^2) \in L_{\text{loc}}^1(\Omega),$$

where  $s_p = 2 - \frac{4}{p+2} = \frac{2p}{p+2}$ .

The restriction  $p > 2$  is of a technical nature and we note that there holds  $\mu_p \rightarrow 1 + \frac{2}{n}$  as  $p \rightarrow 1^* := \frac{n}{n-1}$ . Further details can be found in Remark 6.5 and the Outlook.

\* In Chapter 7 we deal with general  $\mathbb{C}$ -elliptic operators  $\mathbb{A}$  induced by a projection  $\mathcal{A}: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$  in two dimensions. As a speciality, all these operators share a common structure. We present the results of [EL25] and show Sobolev regularity for unconstrained  $BV^{\mathbb{A}}$ -minimisers in this setup. More precisely, since  $\mu = 1 + \frac{2}{n} = 2$  for  $n = 2$ , we derive

**Theorem C** (Theorem 7.1). *Let  $\Omega \subset \mathbb{R}^2$  be open and bounded and  $\mathbb{A}$  be  $\mathbb{C}$ -elliptic in two dimensions induced by a projection. Moreover, let  $f \in C^2(\mathcal{R}(\mathbb{A}))$  be a variational integrand which satisfies (1.3) and (1.5) with  $\mu \in (1, 2)$ . Then any  $BV^{\mathbb{A}}$ -minimiser  $u \in BV^{\mathbb{A}}(\Omega)$  is of class  $W^{\mathbb{A},1}(\Omega) \cap W_{loc}^{1,q}(\Omega; \mathbb{R}^2)$  for all  $q \in [1, \infty)$  with*

$$\nabla u \in \exp L_{loc}^{\frac{3-\mu}{2-\mu}}(\Omega).$$

We note, that the same approach also allows to derive a similar result for the deviatoric gradient  $\mathbb{A} = \text{dev } \nabla$  in all dimensions.

As previously indicated, the main strategy for proving our results will be an adaption of the vanishing viscosity method proposed by SCHMIDT in his habilitation thesis [Sch15a]. More precisely, starting with an arbitrary minimiser  $u$ , we add suitable regularising and penalising terms to our functionals in order to construct a minimising sequence  $(v_j)_{j \in \mathbb{N}}$  of almost-minimisers staying close to  $u$ . This approach is based on the Ekeland variational principle. As a consequence of the almost-minimality, every  $v_j$  satisfies an Euler-Lagrange inequality. Note that the latter also contains a contribution from the penalisation term we were adding to our functionals. The whole setup has to be implemented in a way that the  $L^p$ -constraints contained in Theorem A and Theorem B are preserved by the sequence. Although it is not necessary to treat constraints in Theorem C, the procedure will be formulated in an abstract framework in Chapter 4. Let us already mention that, in order to handle the  $L^\infty$ -bound from Theorem A, we have to allow for the penalisation term to blow up. Therefore, a very delicate analysis is necessary including *quantitative estimates* for the  $L^\infty$ -norm of the Ekeland sequence.

The key ingredient in order to show gradient integrability are weighted second-order estimates derived from the Euler-Lagrange inequality in a differentiated version. Since this Euler-Lagrange inequality is quantified in terms of perturbations, the latter must be sufficiently weak to be controllable by the a priori bounds and at the same time sufficiently strong to yield useful estimates. This can be achieved by extending the functional by infinity to the space  $W^{-2,1}$  and applying the Ekeland variational principle there. Let us briefly point out why we cannot use  $W^{-1,1}$  as the perturbation space as in [BS13], which may be surprising at first stage: To this end, we recall that for the symmetric gradient there holds

$$\partial_{ij} w^\ell = \partial_i \varepsilon(w)^{(j\ell)} - \partial_\ell \varepsilon(w)^{(ij)} + \partial_j \varepsilon(w)^{(i\ell)}$$

for all vector fields  $w \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$  and  $i, j, \ell \in \{1, \dots, n\}$ . Such a representation remains true for  $\mathbb{C}$ -elliptic differential operators induced by a projection in two dimensions, meaning that we can represent the second derivative of a vector field by a linear combination of derivatives of  $\mathbb{A}$ , cf. Lemma 2.8. In order to derive the weighted estimates, we have to test the Euler-Lagrange inequality with second-order quantities behaving approximately like  $\Delta v_j$ . At this stage, the usage of  $W^{-1,1}$  would destroy any estimates, since we only have

$$\|\Delta v_j\|_{W^{-1,1}} \lesssim \|\nabla v_j\|_{L^1}.$$

The right-hand side of the latter cannot be controlled because of Ornstein's Non-Inequality. However, the space  $W^{-2,1}$  turns out to be sufficiently weak as there holds

$$\|\Delta v_j\|_{W^{-2,1}} \lesssim \|v_j\|_{L^1},$$

cf. Section 2.2.7. In strong contrast to the full gradient, the explicit structure of the differential operator additionally comes into play. In the framework of the symmetric gradient we have to perform subtle algebraic manipulations, in order to re-introduce the symmetric gradients in the corresponding estimates. Even in the unconstrained case from [Gme20] this step is already non-trivial and becomes even more delicate in our setup. Namely, this re-introduction always comes with the price of additional pollution terms, which may be difficult to treat. Again, in view of Ornstein's Non-Inequality, the only possibility to handle such pollution terms is to use the Euler-Lagrange inequality or a differentiated version. However, these inequalities now contain additional terms coming from the penalisation, which have to be controlled as well. In particular, we can not apply the a priori bounds directly because of these additional contributions.

Without further structural assumptions on  $\mathbb{A}$  it seems at the present stage extremely challenging to apply the described method in order to obtain similar results for more general  $\mathbb{C}$ -elliptic operators in all dimensions. The main reason is the delicate interplay of all issues indicated above. However, since all two-dimensional  $\mathbb{C}$ -elliptic operators induced by projections can be classified, we were able to tackle this case successfully. We close the contextualisation by giving a brief outline of the structure of the thesis.

*Systematic Outline:* In the upcoming Section 1.2 we discuss a motivating example coming from the theory of plasticity. Subsequently, we fix notation and introduce the underlying function spaces together with the relevant theoretical background in Chapter 2. Since a concise introduction into Korn-type inequalities can hardly be found in the literature, we provide a self-contained overview in Chapter 3 including full proofs and references. In Chapter 4 we implement a constrained version of a vanishing viscosity approximation scheme based on the Ekeland variational principle in a very general framework. Although this generality is not needed in the remaining part of the thesis, it might be useful for future applications. In the main part of the thesis we focus on the special case of the symmetric gradient  $\mathbb{A} = \varepsilon$ . More precisely, in Chapter 5 we give the proof of Theorem A, namely that every  $L^\infty$ -bounded BD-minimiser of 3-elliptic variational integral has an integrable gradient. In Chapter 6 we will give the proof of Theorem B. In particular, we derive gradient integrability of BD-minimisers under an additional  $L^p$ -constraint for  $p > 2$ , partly interpolating the unconstrained and the  $L^\infty$ -constraint regime. Finally, we discuss the proof of Theorem C in Chapter 7. More precisely, we show that every  $BV^{\mathbb{A}}$ -minimiser of a  $\mu$ -elliptic variational integral is exponentially integrable whenever  $\mathbb{A}$  is a  $\mathbb{C}$ -elliptic differential operator induced by a projection  $\mathcal{A} : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$  and  $\mu \in (1, 2)$ .

We close this section and describe how the above mentioned thresholds for  $\mu$  are connected with the nowadays well-studied *non-standard*  $(p, q)$ -growth functionals to manifest our results. Because of  $\mu > 1$  we observe from (1.5) that we have different growth behaviour from above and below *on the level of second derivatives*. In this sense,  $\mu$ -elliptic integrands show a similar behaviour as  $(p, q)$ -growth integrands, the latter meaning

$$\gamma |z|^p \leq f(z) \leq \Gamma(1 + |z|^q),$$

with  $p < q$  and  $z \in W$ . A convenient ellipticity assumption in this regime is given by

$$\lambda |\xi|^2 (1 + |z|^2)^{\frac{p-2}{2}} \leq \langle \nabla^2 f(z) \xi, \xi \rangle \leq \Lambda |\xi|^2 (1 + |z|^2)^{\frac{q-2}{2}}, \quad (1.6)$$

for  $z, \xi \in W$ . Hence, computing the growth difference on the level of second derivatives in (1.6) results in  $(q - 2) - (p - 2) = q - p$ . A systematic regularity theory for non-standard growth functionals of  $(p, q)$ -type began with the seminal work of MARCELLINI [Mar89, Mar91] in the scalar case, whereas the vectorial case was first studied by MINGIONE et. al. [ELM99a, ELM99b, ELM04]. Typical bounds in this framework to obtain improved local gradient integrability read

$$\frac{q}{p} < 1 + \frac{2}{n} \quad (\text{unconstrained case}) \quad \text{and} \quad q \leq p + 2 \quad (\text{L}_{\text{loc}}^\infty\text{-constrained case}),$$

for  $2 \leq p < q$ . Phrased differently, the above inequalities can be expressed as bounds  $\frac{2p}{n}$  and 2 on the growth difference  $q - p$  for  $\nabla^2 f$ . In analogy, the growth difference for  $\nabla^2 f$  coming from the assumption of  $\mu$ -ellipticity (1.5) is given by  $\mu - 1$  and therefore, one aims for higher local gradient integrability for  $\mu < 1 + \frac{2}{n}$  in the unconstrained case and  $\mu \leq 3$  in the  $L_{\text{loc}}^\infty$ -constrained case. Here,  $q - p = (q - 2) - (p - 2)$  corresponds to  $-1 - (-\mu) = \mu - 1$ , while setting  $p = 1$  results in the bound  $\frac{2}{n}$ . For an exhaustive overview on further results we refer to the survey article of MINGIONE [Min06] including also many additional references.

## 1.2 Motivations from Continuum Mechanics

Throughout the section, let  $\Omega \subset \mathbb{R}^3$  be a physical body, i.e. an open bounded and connected domain, whose smooth boundary splits in  $\partial\Omega = \Gamma_0 \cup \Gamma_1$ , where  $\Gamma_0 \subset \partial\Omega$  is open and we have  $\Gamma_0 = \partial\Omega \setminus \bar{\Gamma}_1$  and  $\Gamma_1 = \partial\Omega \setminus \bar{\Gamma}_0$ . For a more detailed description we refer e.g. to [DL76, TM01] or [PS78] for a more physical viewpoint.

In continuum mechanics it is common to describe the evolution of mechanical properties for a continuous body  $\Omega \subset \mathbb{R}^3$  by means of a *deformation map*  $\varphi: \Omega \times [0, \infty) \rightarrow \mathbb{R}^3$ . To keep the section as simple as possible, we only treat the static case, meaning that we only consider deformation maps for fixed times, i.e.

$$\begin{cases} \varphi: \Omega \rightarrow \varphi(\Omega) \\ X \mapsto \varphi(X) := x. \end{cases}$$

We assume that  $\varphi$  is sufficiently smooth, invertible and orientation preserving, that is  $\det(\nabla\varphi) > 0$ . Moreover, it is convenient to call  $\Omega$  *reference configuration* and  $\varphi(\Omega)$  the *deformed configuration*.

Since the energy of the deformed configuration remains unchanged when translating the body, we introduce the *displacement field*  $u = \varphi - \text{id}$  as a quantity measuring only relative changes. With this notation, we define the *deformation gradient*  $F := \nabla\varphi$  as well as the *displacement gradient*  $\nabla u = \nabla\varphi - \mathbf{1}_{3 \times 3}$ , respectively. Moreover, since  $F$  is in general not rotation-invariant, we can use the polar decomposition to split  $F(X) = R(X)U(X)$  for every  $X \in \Omega$ , where  $R(X)$  denotes an orthogonal rotation. In order to get a rotation invariant quantity we define the *Cauchy-Green strain tensor field* (corresponding to  $u$ ) by

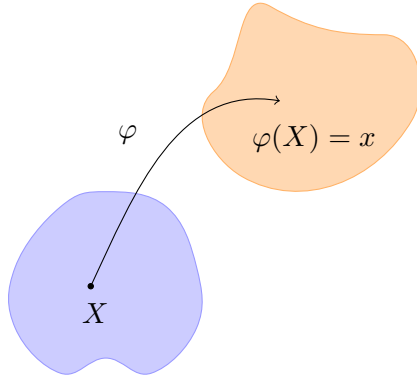


Figure 1.2: The deformation map  $\varphi$ .

$C(X) := F(X)^\top F(X) = (U(X))^2$ . In terms of the displacement field the latter reads

$$\begin{aligned} C(X) &= \nabla\varphi(X)^\top \nabla\varphi(X) \\ &= \nabla u(X)^\top \nabla u(X) + \nabla u(X)^\top + \nabla u(X) + \mathbb{1}_{3\times 3}. \end{aligned}$$

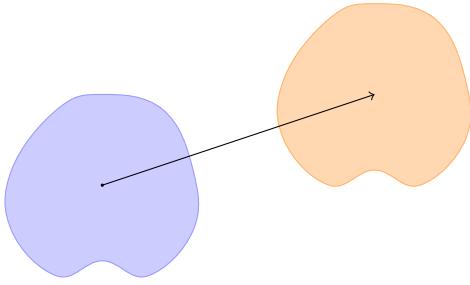


Figure 1.3: Translations

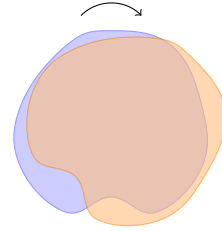


Figure 1.4: Rotations

Furthermore, it is convenient to introduce the *Green-St. Venant strain tensor field* (corresponding to  $u$ ) through

$$E(X) := \frac{1}{2}(C(X) - \mathbb{1}_{3\times 3}) = \frac{1}{2}(\nabla u(X)^\top \nabla u(X) + \nabla u(X)^\top + \nabla u(X)).$$

This tensor field describes how a given deformation differs locally from a rigid body-motion. For small strains, we can linearise this tensor field pointwise to deduce

$$E(X) \approx \frac{1}{2}(\nabla u(X) + \nabla u(X)^\top) := \varepsilon(u(X)),$$

where  $\varepsilon(u(X))$  denotes the symmetric gradient. For a better understanding, we decompose the linearised strain-tensor even further into

$$\varepsilon(u) = (\varepsilon(u) - \frac{1}{3} \operatorname{div}(u) \mathbb{1}_{3\times 3}) + \frac{1}{3} \operatorname{div}(u) \mathbb{1}_{3\times 3} := \operatorname{dev}(\varepsilon(u)) + \frac{1}{3} \operatorname{div}(\varepsilon(u)) \mathbb{1}_{3\times 3},$$

where  $\operatorname{dev}(\varepsilon(u)) := \varepsilon(u) - \frac{1}{3} \operatorname{div}(u) \mathbb{1}_{3\times 3}$  is the deviatoric symmetric gradient. This splitting is often named *volumetric-isochoric split*. From a physical viewpoint, the deviatoric part

describes the isochoric distortion of the body, i.e. changes in shape but not in volume, cf. Figure 1.5. At the contrary, the remaining part involving the divergence, encodes the volumetric dilation of the deformation illustrated in Figure 1.6.

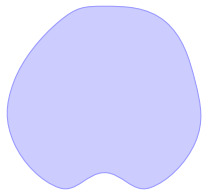


Figure 1.5: Distortion  $\rightsquigarrow \operatorname{dev}(\varepsilon(u))$

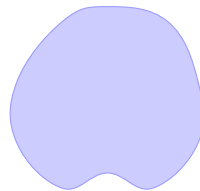
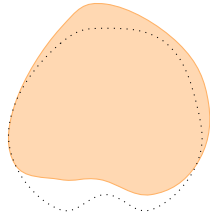


Figure 1.6: Dilation  $\rightsquigarrow \frac{1}{3}\operatorname{div}(u)\mathbb{1}_{3\times 3}$

As a next step, we assume that time-independent exterior forces act on the body  $\Omega$  in reference configuration and that the equilibrium state is the deformed configuration  $\varphi(\Omega)$ . More precisely, we consider *volume forces* described by some density  $f: \Omega \rightarrow \mathbb{R}^3$  as well as *surface forces* described by  $g: \Gamma_1 \rightarrow \mathbb{R}^3$ . Assuming the axiom of Cauchy, one can prove the existence of a *Cauchy stress tensor field*  $\sigma: \varphi(\Omega) \rightarrow \mathbb{R}^{3\times 3}$ , modelling inner forces per unit area in a given direction. Moreover, assuming the conservation of angular momentum, it can be shown that the tensor field  $\sigma$  is symmetric, i.e. there holds  $\sigma(x) = \sigma(x)^\top$  for all  $x \in \varphi(\Omega)$ .

We close the first part by mentioning that the Piola transform offers a way of transforming tensor fields from the deformed to the reference configuration, which leads to the definition of the *First Piola-Kirchhoff stress tensor field*  $P(X)$  arising as Piola-transform of the Cauchy stress tensor. In the setting of linear elasticity the two notions coincide, and we refer to [PS78, Section 2.5] for more details.

### 1.2.1 Introduction to the Theory of Linear Elasticity

In this section we assume  $f \in L^2(\Omega; \mathbb{R}^3)$  as well as  $g \in L^2(\Gamma_1; \mathbb{R}^3)$  for the force densities. For the sake of simplicity, we only deal with the static case and refer to [DL76] for the time-dependent case. In the linear elastic framework the displacement field for a hyperelastic material solves the primal or strain variational problem given by

$$\langle \text{Primal} \rangle := \inf_{u, u=u_0 \text{ on } \Gamma_0} \left\{ \int_{\Omega} W(\varepsilon(u)) \, dx - \int_{\Omega} f \cdot u \, dx - \int_{\Gamma_1} g \cdot u \, d\mathcal{H}^2 \right\}, \quad (1.7)$$

where  $W: \mathbb{R}_{\text{sym}}^{3\times 3} \rightarrow \mathbb{R}$  is the *strain-energy density*. In case of an isotropic material, the latter is given by

$$W(F) = \frac{\lambda}{2} \operatorname{tr}(F)^2 + \mu \langle F, F \rangle, \quad (1.8)$$

where the constants  $\lambda$  and  $\mu$  are called Lamé constants of the material. Moreover, if we introduce the *bulk-modulus*  $\kappa := \lambda + \frac{2\mu}{3}$ , we can rewrite the density (1.8) equivalently in terms of

$$W(F) = \frac{\kappa}{2} \operatorname{tr}(F)^2 + \mu \langle \operatorname{dev}(F), \operatorname{dev}(F) \rangle,$$

where  $\text{dev}(A) := A - \frac{1}{3}\text{tr}(A)\mathbb{1}_{3 \times 3}$  denotes the *deviatoric part of a matrix*  $A \in \mathbb{R}^{3 \times 3}$ . More generally, for non-isotropic materials the density is given by

$$W(F) = \frac{1}{2} \langle \mathbb{M}F, F \rangle \quad \text{for all } F \in \mathbb{R}_{\text{sym}}^{3 \times 3},$$

where  $\mathbb{M} = (m_{ijkl})_{ijkl}$  is a fourth-order tensor satisfying the symmetry conditions

$$m_{ijkl} = m_{ijlk} = m_{jikl} = m_{kl ij} \quad \text{for all } i, j, k, \ell \in \{1, 2, 3\}.$$

Clearly, the isotropic case is included in the non-isotropic one when choosing

$$m_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \quad \text{for } i, j, k, \ell \in \{1, 2, 3\}.$$

In both cases the natural function space for the minimisation problem (1.7) is  $W^{1,2}(\Omega; \mathbb{R}^3)$ , since the densities are of quadratic growth. Under suitable additional assumptions there exists a solution  $u$  of the minimisation problem and we can recover the stress tensor in the isotropic case through *Hooke's law*, namely

$$\begin{aligned} \sigma = \nabla W(\varepsilon(u)) &= \frac{\lambda}{2} \text{tr}(\varepsilon(u)) \mathbb{1}_{3 \times 3} + 2\mu \varepsilon(u) \\ &= \frac{\lambda}{2} \text{div}(u) \mathbb{1}_{3 \times 3} + 2\mu \varepsilon(u) = \kappa \text{div}(u) \mathbb{1}_{3 \times 3} + 2\mu \text{dev}(\varepsilon(u)). \end{aligned}$$

In the non-isotropic case this connection is known as *generalised Hooke's law* and given by  $\sigma = \mathbb{M}\varepsilon(u)$ .

Instead of seeking for the displacement field, we can equivalently study the dual problem in the sense of convex duality, which is a maximisation problem for the stress tensor. To this end, we need the Legendre-Fenchel conjugate  $W^*$  of  $W$ <sup>3</sup>, which can be computed in the isotropic case by

$$W^*(F) = \frac{1}{18\kappa} \text{tr}(F)^2 + \frac{1}{4\mu} \langle \text{dev}(F), \text{dev}(F) \rangle.$$

More generally in the non-isotropic case, we infer

$$W^*(F) = \frac{1}{2} \langle \mathbb{D}F, F \rangle,$$

where  $\mathbb{D} = (d_{ijkl})_{ijkl}$  denotes the fourth order ‘‘inverted’’ tensor of  $\mathbb{M}$ . We refer to [Tem85, Remark 2.4 v)] for more details on the precise definition of  $\mathbb{D}$ . Again, the isotropic case can be recovered from the non-isotropic one by writing

$$d_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \quad \text{for } i, j, k, \ell \in \{1, 2, 3\}.$$

From [Tem85, Chapter 1, Lemma 2.2], we obtain the dual maximisation problem,

$$\langle \text{dual} \rangle := \sup_{\sigma} \left\{ \frac{1}{2} \int_{\Omega} \langle \mathbb{D}\sigma, \sigma \rangle dx + \int_{\Gamma_0} (\sigma\nu) \cdot u_0 d\mathcal{H}^2 \right\}, \quad (1.9)$$

<sup>3</sup>For a function  $f: \mathbb{R}^m \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$  the *Legendre-Fenchel* or *convex conjugate* is the function  $f^*: \mathbb{R}^m \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$  defined by  $f^*(x^*) := \sup\{\langle x^*, x \rangle - f(x) : x \in \mathbb{R}^m\}$ , see e.g. [ET76].

where the supremum is taken over all weak solutions  $\sigma \in L^2(\Omega; \mathbb{R}_{\text{sym}}^{3 \times 3})$  of the problem

$$\begin{cases} -\operatorname{div}(\sigma) = f & \text{in } \Omega, \\ \sigma \nu = g & \text{on } \Gamma_0. \end{cases}$$

Finally, under suitable assumptions we have the following strong duality result as a consequence of the Fenchel duality theorem, cf. [ET76]:

**Theorem 1.2** (Strong duality, [Tem85, Chapter 1, Proposition 2.6]). *Suppose that  $\mathcal{H}^2(\Gamma_0) > 0$ . Then problem (1.7) and (1.9) are dual to each other together with*

$$\langle \text{primal} \rangle = \langle \text{dual} \rangle \in \mathbb{R}.$$

Moreover, both problems have unique solutions  $u \in W^{1,2}(\Omega; \mathbb{R}^{3 \times 3})$  and  $\sigma \in L^2(\Omega; \mathbb{R}_{\text{sym}}^{3 \times 3})$ , representing the displacement field and the stress tensor of the mechanical problem, respectively. Both solutions  $\sigma$  and  $u$  are related by the (generalised) Hooke's law  $\sigma = \mathbb{M}\varepsilon(u)$ .

## 1.2.2 The Hencky Model for Perfect Plasticity

In our description of the linear elasticity model, we were seeking for displacement fields  $u \in W^{1,2}(\Omega; \mathbb{R}^3)$  and stress tensors  $\sigma \in L^2(\Omega; \mathbb{R}_{\text{sym}}^{3 \times 3})$ . For plasticity it turns out, that the regularity requirement for  $\sigma$  is still reasonable, whereas there is no hope for a priori Sobolev regularity for  $u$ , since we expect the appearance of discontinuities. In the Hencky-model for perfect plasticity, we consider variational problems on the domain

$$K = \{\sigma \in \mathbb{R}_{\text{sym}}^{3 \times 3} : \mathcal{F}(\sigma) \leq 0\},$$

where  $\mathcal{F} : \mathbb{R}_{\text{sym}}^{3 \times 3} \rightarrow \mathbb{R}$  is called *yield function*. The boundary  $\partial K$  is called the *yield surface* or *plastic part*, whereas the interior  $K^\circ$  is the *elastic part* of the domain. Two classical choices are given by

- the von Mises yield-condition, i.e.  $\mathcal{F}^{vM}(\sigma) := \frac{1}{2} |\operatorname{dev}(\sigma)|^2 - k^2$  with  $k \in \mathbb{R}$  and
- the Tresca yield-condition, i.e.  $\mathcal{F}^T(\sigma) := \max_{i,j} |\sigma_i - \sigma_j| - \tilde{k}$ , where  $\sigma_i$  denotes the  $i$ -th eigenvalue of  $\sigma$  and  $\tilde{k} \in \mathbb{R}$ .

We focus on the derivation of a variational model for the Hencky model with the von Mises yield condition, which turns out to be a linear growth problem. For the model it is convenient to start with the Legendre-Fenchel conjugate of the strain-energy density, namely,

$$W^*(F) = \begin{cases} \frac{1}{18\kappa} \operatorname{tr}(F)^2 + \frac{1}{4\mu} \langle \operatorname{dev}(F), \operatorname{dev}(F) \rangle & \text{if } F \in K, \\ +\infty & \text{if not.} \end{cases}$$

In other words, we take the same strain density as in the linear elastic case but only if the tensors are elements of  $K$ . In contrast to the previous part, we cannot write down the Legendre-Fenchel conjugate  $W$  in general. However, this is possible, if we assume the decomposition  $K = \operatorname{dev}(K) \oplus \{\zeta \mathbf{1}_{3 \times 3} : \zeta \in \mathbb{R}\}$  with  $\operatorname{dev}(K) := \mathbb{R}_{\text{dev}}^{3 \times 3} \cap K$  and  $\mathbb{R}_{\text{dev}}^{3 \times 3} := \{A \in \mathbb{R}^{3 \times 3} : \operatorname{tr}(A) = 0\}$ . In this setting, we get the splitting

$$W(F) = \frac{\kappa}{2} \operatorname{tr}(F)^2 + W^D(\operatorname{dev}(F))$$

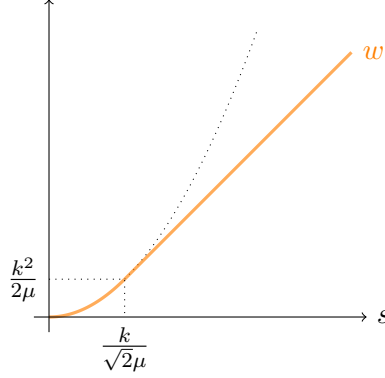


Figure 1.7: Von Mises integrand in the Hencky model of perfect plasticity.

where  $W^D$  denotes the Legendre-Fenchel conjugate of  $(W^D)^*$ , the restriction of  $W^*$  to  $\mathbb{R}_{\text{dev}}^{3 \times 3}$ . In case of the von Mises yield condition, this function can be computed explicitly, cf. [Tem85, Chapter 1, Section 4], leading to  $W^D(\text{dev}(F)) = w(|\text{dev}(F)|)$  with

$$w(s) = \begin{cases} \mu s^2 & \text{if } |s| \leq \frac{k}{\sqrt{2\mu}} \\ \sqrt{2k}|s| - \frac{k^2}{2\mu} & \text{if } |s| \geq \frac{k}{\sqrt{2\mu}}. \end{cases}$$

Clearly, this function is of linear growth at infinity, cf. Figure 1.7. In conclusion, we can formulate the variational problem for the displacement field. Using the same function spaces as before, we seek displacement fields  $v: \Omega \rightarrow \mathbb{R}^3$  among the class of competitors  $\mathcal{C}_a := \{v \in W^{1,2}(\Omega; \mathbb{R}^3) : v = u_0 \text{ on } \Gamma_0\}$ . This leads to the minimisation problem

$$\inf_{v \in \mathcal{C}_a} \{\Psi(\varepsilon(v)) - L(v)\},$$

where we have set

$$\Psi(\varepsilon(v)) = \int_{\Omega} W(\varepsilon(v)) \, dx = \frac{\kappa}{2} \int_{\Omega} (\text{div}(v))^2 \, dx + \int_{\Omega} W^D(\text{dev}(\varepsilon(v))) \, dx,$$

and

$$L(v) = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_1} g \cdot v \, d\mathcal{H}^2.$$

Since the primal variational problem is a one of linear growth, we cannot expect to find a solution  $u \in W^{1,2}(\Omega; \mathbb{R}_{\text{sym}}^{3 \times 3})$  as already indicated at the beginning. However, for the Hencky-model with the von Mises yield condition the existence of a displacement field  $u \in \{v \in \text{BD}(\Omega) : \text{div}(v) \in L^2(\Omega)\}$  is shown in [AG80] for a relaxed problem. This example serves as a strong motivation for the more general models studied in this thesis, demonstrating that the results obtained are relevant and applicable even in concrete and practically significant cases. Another class of models with linear growth integrands are related to limiting strain models and we refer to [BMRS14, BBMS17] and the references therein. For more models stemming from continuum mechanics we mention [DL76, NH81, OR83, Tem85, FS00].

## 2 | Preliminaries

### 2.1 General Notation

Throughout this thesis, we assume that the dimension  $n \in \mathbb{N}$  satisfies  $n \geq 2$ . We mainly use standard notation. Nevertheless, we briefly comment on some things to rule out any potential source of confusion. For  $n, m \in \mathbb{N}$  we equip  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$  with the usual Euclidean or Frobenius norm, respectively and denote both by  $|\cdot|$ . The latter is induced by the matrix inner product  $\langle A, B \rangle := \text{tr}(A^\top B)$  for  $A, B \in \mathbb{R}^{n \times m}$ . Moreover, we write  $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$  to denote an open ball with radius  $r > 0$  centred at  $x_0 \in \mathbb{R}^n$  as well as  $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$  for the sphere.

$\mathcal{L}^n$  and  $\mathcal{H}^{n-1}$  indicate the  $n$ -dimensional Lebesgue and  $(n-1)$ -dimensional Hausdorff measure, respectively and we simply write  $dx$ , when integrating with respect to  $\mathcal{L}^n$ . Moreover, we set  $\omega_n := \mathcal{L}^n(B_1(0))$  and denote for a (Lebesgue) measurable set  $U \subset \mathbb{R}^n$  with  $\mathcal{L}^n(U) > 0$  and  $f \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$  the integral average by

$$\int_U f(x) dx := (f)_U := \frac{1}{\mathcal{L}^n(U)} \int_U f(x) dx.$$

If  $U = B_r(x_0)$  we simply write  $(f)_{r,x_0} := (f)_{B_r(x_0)}$ . Furthermore, we denote the restriction of a measure  $\mu$  on  $\mathbb{R}^n$  to  $U$  by  $\mu \llcorner U$ . We use the abbreviation  $\text{GL}(n)$  for the general linear group of all invertible square matrices in  $n$  dimensions. For two finite dimensional inner product spaces  $V$  and  $W$  we denote the space of linear maps  $V \rightarrow W$  by  $\mathcal{L}(V, W)$  or  $\text{Hom}(V, W)$ . Moreover, for  $\ell \in \mathbb{N}$  we use  $\odot^\ell(V; W)$  to indicate the space of  $W$ -valued symmetric  $\ell$ -linear maps on  $V$ , which is the natural space for the  $\ell$ -th derivative of a function  $f \in C^\ell(V; W)$ , i.e.  $\nabla^\ell f(x) \in \odot^\ell(V, W)$  for  $x \in V$ . For more details on multilinear algebra we refer to [Fed69, Chapter 1] or [Gre67]. We use the notation  $a \otimes b := ab^\top$  for the tensor product and  $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$  for the symmetric tensor product of two vectors  $a, b \in \mathbb{R}^n$ . Throughout the thesis,  $C, c > 0$  denote generic constants whose values may vary from line to line. Their explicit dependencies are usually indicated, but their precise values are only specified when necessary. Finally, we write  $a \lesssim b$  if there exists a constant  $c > 0$  such that  $a \leq cb$ .

### 2.2 Function Spaces and Related Topics

For an open and bounded set  $\Omega \subset \mathbb{R}^n$ ,  $1 \leq p \leq \infty$  and  $k \in \mathbb{N}$  the Lebesgue and Sobolev spaces  $L^p(\Omega)$  and  $W^{k,p}(\Omega)$  are defined in the standard way. We refer to [Neč67, KJF77, GT83, AF03] or [Giu03] for their definition and properties. Without further comment, we also use vector-valued versions of these spaces, always assuming that the target space is finite dimensional. In this section, we briefly revisit selected topics that we will use later

in the thesis, assuming that the reader is familiar with the aforementioned underlying function spaces.

In the following, let  $V$  be a finite dimensional inner product space. By  $\text{RM}_{(\text{fin})}(\Omega; V)$  we denote the space of (finite)  $V$ -valued Radon measures on  $\Omega$ , short (*finite*) *Radon measures*. For  $\mu \in \text{RM}_{\text{fin}}(\Omega; V)$  we denote the total variation measure with  $|\mu|$  and the Lebesgue-Radon-Nikodým decomposition into its absolutely continuous and and singular part with respect to  $\mathcal{L}^n$  is given by

$$\mu = \mu^a + \mu^s = \frac{d\mu^a}{d\mathcal{L}^n} \mathcal{L}^n + \frac{d\mu^s}{d|\mu^s|} |\mu^s|. \quad (2.1)$$

Here,  $\mu^a$  denotes the absolutely continuous part and  $\mu^s$  the singular part of  $\mu$  with respect to the Lebesgue measure. For  $\mu \in \text{RM}_{\text{fin}}(\Omega; V)$  and  $1 \leq p \leq \infty$ , we denote  $V$ -valued  $L^p$ -space with respect to  $\mu$  by  $L^p_\mu(\Omega; V)$ , becoming relevant when dealing with weighted estimates. We refer to e.g [AFP00, EG15, HS65, Die69] for more details concerning measure theory and real analysis. Finally, the space of all  $V$ -valued distributions is denoted by  $\mathcal{D}'(\Omega; V)$ , cf. [Fed69, Sch51] for further properties.

### 2.2.1 Sobolev Spaces and Difference Quotients

At some stages of this thesis we will make use of probably well-known facts about Sobolev functions. We collect them here for the sake of completeness.

*Difference quotients.* We briefly recall difference quotients and their basic properties and refer to [Giu03, Section 8.1] or [GT83, Section 7.11] for further details. For  $h \in \mathbb{R}$  we set  $\Omega_h := \{x \in \Omega : 0 < |h| < \text{dist}(x, \partial\Omega)\}$ , allowing to define for  $s \in \{1, \dots, n\}$  and  $x \in \Omega_h$  the *difference quotient in direction  $s$*  by

$$\Delta_{s,h} f(x) := \frac{f(x + he_s) - f(x)}{h}. \quad (2.2)$$

By  $e_s$  we denote the  $s$ -th unit vector in  $\mathbb{R}^n$ . If  $f \in W^{1,p}(\Omega)$ , there holds  $\Delta_{s,h} f \in W^{1,p}(\Omega_h)$  with  $\partial_i(\Delta_{s,h} f) = \Delta_{s,h}(\partial_i f)$ . Moreover, if at least one of the two functions  $f$  or  $g$  has support contained in  $\Omega_h$ , the following *discrete integration by parts* formula holds

$$\int_{\Omega} f(x) \Delta_{s,h} g(x) \, dx = - \int_{\Omega} g(x) \Delta_{s,-h} f(x) \, dx. \quad (2.3)$$

*The product rule and consequences.* From [GT83, Section 7.3] we recall for  $u, v \in W^{1,1}(\Omega)$  with  $uv$  and  $u\nabla v + v\nabla u \in L^1_{\text{loc}}(\Omega)$  the product rule

$$\nabla(uv) = u\nabla v + v\nabla u. \quad (2.4)$$

In particular, the assumptions are satisfied if  $u \in W^{1,p}(\Omega)$  and  $v \in W^{1,p'}(\Omega)$ , where  $p' := \frac{p}{p-1}$  denotes the Hölder-conjugate of  $p \in [1, \infty]$ . As another direct consequence we obtain for  $u \in W^{1,p}(\Omega)$  and  $\zeta \in C^{0,1}(\Omega)$  that  $\zeta u \in W^{1,p}(\Omega)$ , cf. [Mor66, Theorem 3.1.4].

*Truncations:* For  $k > 0$  and  $t \geq 0$  we introduce the *truncation operator at level  $k$*  by  $T_k(t) := \max\{k, t\}$ . Obviously,  $T_k$  is Lipschitz continuous. Therefore, for  $u \in W^{1,1}_{\text{loc}}(\Omega)$ ,

we can apply [Giu03, Theorem 3.24], to obtain that the composition  $T_k \circ u$  belongs to  $W_{\text{loc}}^{1,1}(\Omega)$  together with

$$\nabla(T_k \circ u)(x) = \mathbb{1}_{\{u \leq k\}}(x) \nabla u(x). \quad (2.5)$$

Here, we have used the shorthand notation  $\{u \leq k\} := \{x \in \Omega : u(x) \leq k\}$ .

### 2.2.2 Lebesgue-Orlicz and Orlicz-Sobolev Spaces

Our main sources for this section will be [BS88, RR91]. However, we would also like to mention [KR61, KJF77] and [AF03], even though dealing with the slightly stronger concept of  $N$ -functions rather than Young functions. Now, let  $\Phi: [0, \infty) \rightarrow [0, \infty]$  be a Young function, meaning that we have the representation

$$\Phi(t) = \int_0^t \phi(s) \, ds \quad \text{for } t \geq 0,$$

where  $\phi: [0, \infty) \rightarrow [0, \infty]$  is a non-decreasing, left-continuous function, neither identical to 0 nor  $\infty$ . We note that a Young function  $\Phi$  is convex on the domain where it is finite. Moreover, we say that a Young-function belongs (globally) to the class  $\Delta_2$ , if there exists a constant  $c > 0$  such that

$$\Phi(2t) \leq c \Phi(t) \quad \text{for all } t \geq 0.$$

The function belongs (globally) to the class  $\nabla_2$ , if there exists a constant  $K > 1$  such that

$$\Phi(Kt) \geq 2K\Phi(t) \quad \text{for all } t \geq 0.$$

The  $\nabla_2$ -condition can be described equivalently using the Legendre-Fenchel conjugate  $\Phi^*$  of a Young function  $\Phi$ . Namely, there holds  $\Phi \in \Delta_2$  if and only if  $\Phi^* \in \nabla_2$ . As a meta-principle one can say that  $\Phi \in \Delta_2$  encodes that  $\Phi$  does not grow too fast, whereas  $\Phi \in \nabla_2$  means that  $\Phi$  does not grow too slow. Next, we define the *Lebesgue-Orlicz space* as the space

$$L^\Phi(\Omega; V) := \{u: \Omega \rightarrow V \text{ measurable} : \|u\|_{L^\Phi(\Omega; V)} < \infty\},$$

where  $\|\cdot\|_{L^\Phi(\Omega; V)}$  denotes the *Luxemburg norm* given by

$$\|u\|_{L^\Phi(\Omega; V)} := \left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|u(x)|}{\lambda}\right) \, dx \leq 1 \right\}.$$

It is common in the literature to denote specific Orlicz spaces differently, namely, for  $\Phi_\alpha(t) = t \log^\alpha(1 + t^2)$  with  $t \geq 0$  and  $\alpha \geq 0$  we use  $L^{\Phi_\alpha}(\Omega; V) := L \log^\alpha L(\Omega; V)$  as well as  $L^{\Psi_\beta}(\Omega; V) := \exp L^\beta(\Omega; V)$  for  $\Psi_\beta(t) := \exp(t^\beta)$  with  $t \geq 0$  and  $\beta > 0$ . We proceed and define the corresponding *Orlicz-Sobolev space*  $W^{1,\Phi}(\Omega; V)$  through

$$W^{1,\Phi}(\Omega; V) := \{u \in L^\Phi(\Omega; V) : u \text{ is weakly differentiable and } \nabla u \in L^\Phi(\Omega; \mathcal{L}(\mathbb{R}^n; V))\}.$$

The Orlicz-Sobolev space  $W^{1,\Phi}(\Omega; V)$  is a Banach space, when equipped with the norm

$$\|u\|_{W^{1,\Phi}(\Omega; V)} := \|u\|_{L^\Phi(\Omega; V)} + \|\nabla u\|_{L^\Phi(\Omega; \mathcal{L}(\mathbb{R}^n; V))}.$$

### 2.2.3 Functions of Bounded $\mathbb{A}$ -variation – General Theory

In this section, we briefly recall the notion of functions of bounded  $\mathbb{A}$ -variation first introduced by BREIT, DIENING and GMEINER in [BDG20]. To this end, we consider a first-order, linear, homogeneous, constant-coefficient differential operator  $\mathbb{A}$  on  $\mathbb{R}^n$  of the form (1.2) acting between two finite dimensional inner product spaces  $V$  and  $W$ . Although the theory extends to higher-order operators of this type, we restrict ourselves here to the first-order case. Equivalently,  $\mathbb{A}$  can be written in the form  $\mathbb{A}u = \mathcal{A}[\nabla u]$ , where  $\mathcal{A} \in \text{Hom}(V \otimes \mathbb{R}^n; W)$  is called *part map of  $\mathbb{A}$* . Moreover, we introduce the corresponding (*Fourier*) *symbol map*  $\mathbb{A}[\xi]: V \rightarrow W$  of  $\mathbb{A}$  defined through

$$\mathbb{A}[\xi]v := v \otimes_{\mathbb{A}} \xi := \sum_{\alpha=1}^n \xi^\alpha \mathbb{A}_\alpha v, \quad \xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n, \quad v \in V.$$

We call such an operator  *$\mathbb{R}$ -elliptic* or simply *elliptic*, if the symbol map  $\mathbb{A}[\xi]: V \rightarrow W$  is injective for all  $\xi \in \mathbb{R}^n \setminus \{0\}$ . Furthermore, we call it  *$\mathbb{C}$ -elliptic* if the extended symbol map  $\mathbb{A}[\xi]: V + iV \rightarrow W + iW$  is injective for all  $\xi \in \mathbb{C}^n \setminus \{0\}$ . Note that  $\otimes_{\mathbb{A}} = \otimes$  if  $\mathbb{A} = \nabla$  is the full gradient, whereas  $\otimes_{\mathbb{A}} = \odot$  if  $\mathbb{A} = \varepsilon$  is the symmetric gradient. Moreover, for  $\varphi \in C^1(\mathbb{R}^n; \mathbb{R})$  and  $v \in C^1(\mathbb{R}^n; V)$  there holds the product rule

$$\mathbb{A}(\varphi v) = \varphi \mathbb{A}v + v \otimes_{\mathbb{A}} \nabla \varphi. \quad (2.6)$$

As a consequence of [BDG20, Lemma 2.3], there exist two constants  $0 < c \leq C < \infty$  such that

$$c|v||z| \leq |v \otimes_{\mathbb{A}} z| \leq C|v||z| \quad \text{for all } v \in V \text{ and } z \in \mathbb{C}^n.$$

We will use these inequalities in the sequel with addressing it further. For an open and bounded set  $\Omega \subset \mathbb{R}^n$ , a differential operator  $\mathbb{A}$  as in (1.2) and  $1 \leq p \leq \infty$ , we introduce the function space

$$W^{\mathbb{A},p}(\Omega) := \{v \in L^p(\Omega; V) : \mathbb{A}v \in L^p(\Omega; W)\},$$

which is a Banach space with respect to the norm

$$\|u\|_{W^{\mathbb{A},p}(\Omega)} := \|u\|_{L^p(\Omega; V)} + \|\mathbb{A}u\|_{L^p(\Omega; W)}.$$

As usual, we define  $W_0^{\mathbb{A},p}(\Omega)$  to be the closure of the test functions  $C_c^\infty(\Omega; V)$  with respect to the  $\|\cdot\|_{W^{\mathbb{A},p}(\Omega)}$ -norm. Let us further note that the space  $W^{\mathbb{A},1}(\Omega)$  is not reflexive and therefore, not weakly compact; this can be seen by adapting the argument from [AF03, §3.5, p. 62]. Additionally, we introduce the function space

$$BV^{\mathbb{A}}(\Omega) := \{v \in L^1(\Omega; V) : \mathbb{A}v \in \text{RM}_{\text{fin}}(\Omega; W)\},$$

of *functions of bounded  $\mathbb{A}$ -variation*. Clearly,  $\mathbb{A} = \nabla$  corresponds to the usual BV-space of functions of bounded variation, see [VH85, AFP00], whereas for  $\mathbb{A} = \varepsilon$  it reduces to the space of functions of bounded deformation BD, cf. [TS80, Suq79]. We will examine these two examples in greater detail in Example 2.2. Moreover, we notice that  $W^{\mathbb{A},1}(\Omega) \subsetneq BV^{\mathbb{A}}(\Omega)$  as a consequence of Ornstein's Non-Inequality, see Theorem 3.8. In order to introduce a norm on  $BV^{\mathbb{A}}(\Omega)$ , we consider the *dual* or *formally adjoint operator*  $\mathbb{A}^*$ , defined as the differential operator on  $\mathbb{R}^n$  from  $W$  to  $V$  given by

$$\mathbb{A}^*v := \sum_{\alpha=1}^n \mathbb{A}_\alpha^* \partial_\alpha v, \quad v: \mathbb{R}^n \rightarrow W,$$

where  $\mathbb{A}_\alpha^* \in \text{Hom}(W, V)$  denotes the adjoint map of  $\mathbb{A}_\alpha$ . This allows to introduce the *total  $\mathbb{A}$ -variation* of  $u \in L^1_{\text{loc}}(\Omega; V)$  through

$$|\mathbb{A}u|(\Omega) := \sup \left\{ \int_{\Omega} \langle u, \mathbb{A}^* \varphi \rangle dx : \varphi \in C_c^1(\Omega; W), \|\varphi\|_{\infty} \leq 1 \right\},$$

making the space  $BV^{\mathbb{A}}$  into a Banach space with respect to the norm

$$\|u\|_{BV^{\mathbb{A}}(\Omega)} := \|u\|_{L^1(\Omega; V)} + |\mathbb{A}u|(\Omega).$$

It can be shown, that  $|\mathbb{A}u|(\Omega)$  coincides with the total variation of the  $W$ -valued Radon measure  $\mathbb{A}u$ . Furthermore, we introduce the (distributional) *null-space of a differential operator  $\mathbb{A}$*  by  $\ker(\mathbb{A}) := \{\Phi \in \mathcal{D}'(\mathbb{R}^n; V) : \mathbb{A}\Phi = 0\}$ . As a consequence of [BDG20, Theorem 2.6], it contains only polynomials and in addition, there holds

$$\mathbb{A} \text{ is } \mathbb{C}\text{-elliptic} \iff \dim \ker(\mathbb{A}) < \infty. \quad (2.7)$$

It turns out that the notion of  $\mathbb{C}$ -ellipticity is necessary (sometimes equivalent), to derive results similar to the usual BV-case:

- In [BDG20] it is shown that  $\mathbb{C}$ -ellipticity of  $\mathbb{A}$  is equivalent to have a bounded linear trace operator  $\text{tr}_{\partial\Omega} : BV^{\mathbb{A}}(\Omega) \rightarrow L^1_{\mathcal{H}^{n-1}}(\partial\Omega)$  on a Lipschitz domain  $\Omega \subset \mathbb{R}^n$ . This operator is continuous with respect to the  $\mathbb{A}$ -strict topology of  $BV^{\mathbb{A}}(\Omega)$  (see below). We note that ellipticity is in general not sufficient and the deviatoric symmetric gradient  $\text{dev}(\varepsilon(u)) := \varepsilon(u) - \frac{1}{n} \text{div}(u) \mathbb{1}_{n \times n}$  provides a counterexample for  $n = 2$ , cf. [BDG20, Example 2.2].
- The notion of  $\mathbb{C}$ -ellipticity is also equivalent to have Sobolev-type embedding theorems, which was shown by GMEINER and RAITA in [GR19a]. Especially, there holds

$$W^{\mathbb{A},1}(\Omega) \hookrightarrow L^q(\Omega; V) \quad (2.8)$$

for  $1 \leq q \leq 1^* := \frac{n}{n-1}$  and the embedding is compact if  $1 \leq q < 1^*$ .

- Moreover, since  $\mathbb{C}$ -ellipticity is equivalent to having a finite dimensional null-space, we conclude the following theorem from the main result in [GR19b]:

**Theorem 2.1.** *Let  $\mathbb{A}$  be a  $\mathbb{C}$ -elliptic differential operator of the form (1.2) and  $p \in [1, 1^*]$ . Then any function  $u \in BV^{\mathbb{A}}_{\text{loc}}(\Omega)$  is  $L^p$ -differentiable for  $\mathcal{L}^n$ -a.e.  $x \in \mathbb{R}^n$  and there holds*

$$\frac{d\mathbb{A}u}{d\mathcal{L}^n}(x) = \mathcal{A}[\nabla u](x)$$

for  $\mathcal{L}^n$ -a.e.  $x \in \mathbb{R}^n$ , where  $\nabla u$  denotes the approximate gradient<sup>1</sup> and  $\mathcal{A}$  the part map of  $\mathbb{A}$ .

This allows to rewrite the Lebesgue-Radon-Nikodým decomposition of  $\mathbb{A}u$  into

$$\mathbb{A}u = \mathbb{A}^a u + \mathbb{A}^s u = \frac{d\mathbb{A}^a u}{d\mathcal{L}^n} \mathcal{L}^n + \frac{d\mathbb{A}^s u}{d|\mathbb{A}^s u|} |\mathbb{A}^s u| = \mathcal{A}[\nabla u] \mathcal{L}^n + \frac{d\mathbb{A}^s u}{d|\mathbb{A}^s u|} |\mathbb{A}^s u|,$$

for  $u \in BV^{\mathbb{A}}(\Omega)$ . Finally, let us mention, that the above theorem was generalised by RAITA in [Rai19] using the notion of elliptic and cancelling operators of VAN SCHAFTINGEN in [VS13], cf. also [GRVS21]. The BD-case was treated in [Haj96].

<sup>1</sup>See e.g. [Fed69, Section 3.1.2] for a precise definition.

We remark that there is also an extension operator available, whose construction will be studied in further detail in the upcoming section. Moreover, similar to the BV-case there are several notions of convergence: We say that a sequence  $(v_j)_{j \in \mathbb{N}}$  in  $BV^{\mathbb{A}}(\Omega)$  converges to  $v \in BV^{\mathbb{A}}(\Omega)$  as  $j \rightarrow \infty$

- in the *weak\*-sense*, in symbols  $v_j \xrightarrow{*} v$ , if and only if  $v_j \rightarrow v$  strongly in  $L^1(\Omega; V)$  and  $\mathbb{A}v_j \xrightarrow{*} \mathbb{A}v$  in the weak\*-sense of  $W$ -valued Radon measures.
- in the  *$\mathbb{A}$ -strict sense*, if and only if  $v_j \rightarrow v$  strongly in  $L^1(\Omega; V)$  and  $|\mathbb{A}v_j|(\Omega) \rightarrow |\mathbb{A}v|(\Omega)$ .
- in the  *$\mathbb{A}$ -area-strict sense*, if  $v_j \rightarrow v$  strongly in  $L^1(\Omega; V)$  and  $\langle \mathbb{A}v_j \rangle(\Omega) \rightarrow \langle \mathbb{A}v \rangle(\Omega)$ , where we have abbreviated

$$\langle \mathbb{A}w \rangle(\Omega) := \int_{\Omega} \sqrt{1 + |\mathbb{A}w|^2} dx + |\mathbb{A}^s w|(\Omega) \quad \text{for } w \in BV^{\mathbb{A}}(\Omega).$$

Clearly,  $\mathbb{A}$ -area-strict convergence implies  $\mathbb{A}$ -strict convergence, which again implies weak\*-convergence. Next, we will briefly discuss the full, the symmetric and the deviatoric gradient, since these three operators will be encountered again later in this thesis.

**Example 2.2** (Full, symmetric and deviatoric gradient, [BDG20, Example 2.2]). *The following examples can be viewed as operators from  $V = \mathbb{R}^N$  to  $W = \mathbb{R}^{N \times n}$ :*

- In case of the full gradient  $\mathbb{A}u := \nabla u$  for  $u: \mathbb{R}^n \rightarrow \mathbb{R}^N$  it is clear that  $\ker(\nabla)$  contains only constant functions. Moreover, there holds

$$|\mathbb{A}[\xi]\eta|^2 = |\xi|^2 |\eta|^2 \quad \text{for all } \xi \in \mathbb{C}^n, \eta \in \mathbb{C}^N,$$

and therefore,  $\mathbb{A}$  is  $\mathbb{C}$ -elliptic with finite dimensional null-space.

- For  $N = n$  and a map  $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$  the symmetric gradient is defined as the tensor field  $\mathbb{A}u := \varepsilon(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$ . In this case, the part map is given by  $\mathcal{A}(z) = \frac{1}{2}(z + z^\top)$  for  $z \in \mathbb{R}^{n \times n}$ , which is the projection onto the symmetric matrices  $\mathbb{R}_{\text{sym}}^{n \times n}$ . Moreover,  $\ker(\varepsilon)$  consist of all generators of rigid deformations

$$\ker(\varepsilon) = \{x \mapsto Ax + b : A \in \mathbb{R}_{\text{skew}}^{n \times n}, b \in \mathbb{R}^n\},$$

where  $\mathbb{R}_{\text{skew}}^{n \times n} := \{A \in \mathbb{R}^{n \times n} : A = -A^\top\}$  denote the skew-symmetric matrices. Since we have

$$|\mathbb{A}[\xi]\eta|^2 = \frac{1}{2} |\xi|^2 |\eta|^2 + \frac{1}{2} |\langle \xi, \eta \rangle|^2 \quad \text{for all } \xi, \eta \in \mathbb{C}^n,$$

it follows that  $\mathbb{A}$  is  $\mathbb{C}$ -elliptic with finite dimensional kernel.

- For  $N = n$  and a map  $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$  the deviatoric or trace-free gradient is defined as the tensor field  $\mathbb{A}u := \text{dev } \nabla u := \nabla u - \frac{1}{n} \text{div}(u) \mathbb{1}_{n \times n}$ . In this case the part map is given by the projection onto the trace-free matrices  $\mathfrak{sl}(n) := \{A \in \mathbb{R}^{n \times n} : \text{tr}(A) = 0\}$ <sup>2</sup>,

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<sup>2</sup>The notation  $\mathfrak{sl}(n)$  comes from Lie theory, since the trace-free matrices form the Lie algebra of the Lie group  $SL(n) := \{A \in \mathbb{R}^{n \times n} : \det(A) = 1\}$ . For further information and details we refer to [Hum78].

namely  $\mathcal{A}(z) = z - \frac{\text{tr}(z)}{n} \mathbf{1}_{n \times n}$  for  $z \in \mathbb{R}^{n \times n}$ . Moreover,  $\ker(\text{dev } \nabla)$  consists of all generators of dilations and translations, i.e.

$$\ker(\text{dev } \nabla) = \{x \mapsto \zeta x + b : \zeta \in \mathbb{R}, b \in \mathbb{R}^n\}.$$

Since

$$|\mathbb{A}[\xi]\eta|^2 = |\xi|^2 |\eta|^2 - \frac{1}{n} \langle \xi, \bar{\eta} \rangle^2 \quad \text{for all } \xi, \eta \in \mathbb{C}^n,$$

it follows that  $\mathbb{A}$  is  $\mathbb{C}$ -elliptic with finite dimensional kernel.

Furthermore, we have the following characterisation of  $\mathbb{C}$ -elliptic operator based on the Hilbert Nullstellensatz [Hun74, VIII, Proposition 7.4] from algebraic geometry. It is essentially due to SMITH [Smi70], but we also refer to [Kal94] and [GRVS21, Proposition 3.2] for a proof.

**Lemma 2.3.** *Let  $\mathbb{A}$  be a differential operator of the form (1.2). Then  $\mathbb{A}$  is  $\mathbb{C}$ -elliptic if and only if there exists an integer  $d \in \mathbb{N}$  and a homogeneous differential operator  $\mathbb{L}$  from  $W$  to  $V \odot^{d-1} \mathbb{R}^n$  such that  $\nabla^d = \mathbb{L} \circ \mathbb{A}$ .*

We proceed with an approximation result, which is well known in the BV-setting, cf. [Bil03a, KR10, Sch15b]. As we will need a slight modification fitting into our framework, we provide a precise statement and sketch the main ideas of the proof.

**Lemma 2.4** ( $\mathbb{A}$ -area-strict approximation). *Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded set with Lipschitz boundary  $\partial\Omega$  oriented by  $\nu_{\partial\Omega} : \partial\Omega \rightarrow \mathbb{S}^{n-1}$  and  $u_0 \in W^{\mathbb{A},1}(\Omega)$ . Then for every function  $u \in \text{BV}^{\mathbb{A}}(\Omega)$  there exists a sequence  $(u_j)_{j \in \mathbb{N}}$  in  $u_0 + C_c^\infty(\Omega; V)$  such that  $u_j \rightarrow u$  in  $L^1(\Omega; V)$  and*

$$\langle \mathbb{A}u_j \rangle(\Omega) \rightarrow \langle \mathbb{A}u \rangle(\Omega) + \int_{\partial\Omega} |\text{tr}_{\partial\Omega}(u - u_0) \otimes_{\mathbb{A}} \nu_{\partial\Omega}| \, d\mathcal{H}^{n-1} \quad \text{as } j \rightarrow \infty.$$

Additionally, if  $u \in L^p(\Omega; V)$  for  $1 \leq p < \infty$ , we can choose the sequence in a way such that there holds  $u_j \in L^p(\Omega; V)$  for  $j \in \mathbb{N}$  with  $u_j \rightarrow u$  in  $L^p(\Omega; V)$  as  $j \rightarrow \infty$ . Finally, if  $u, u_0 \in L^\infty(\Omega; V)$ , then there exists a constant  $c_A = c_A(\Omega, n) > 0$  such that

$$\|u_j\|_{L^\infty(\Omega; V)} \leq c_A (\|u\|_{L^\infty(\Omega; V)} + \|u_0\|_{L^\infty(\Omega; V)}) \quad \text{for all } j \in \mathbb{N}.$$

*Sketch of the proof.* The proof is a straightforward modification of [BDG20, Proposition 4.24], using the construction of [Bil03a, Lemma B.2]. Since the proof is based on a partition of unity as well as a mollification procedure, we can use the additional assumption  $u \in L^p(\Omega; V)$  to conclude that the mollification also converges in  $L^p$  (see e.g. [GT83, Lemma 7.2]). This immediately implies the  $L^p$ -convergence of the sequence, together with the claimed estimates.  $\square$

We close this section with a Remark and an elementary Lemma on the range of  $\mathbb{A}$ .

**Remark 2.5** (Choice of  $W$ , [Ste26, Remark 2.3]). *We notice that we can view a  $W$ -valued differential operator also as a  $Z$ -valued one for every vector space  $W \hookrightarrow Z$ . We now consider the effective range  $\mathcal{R}(\mathbb{A})$  of  $\mathbb{A}$ , given by*

$$\mathcal{R}(\mathbb{A}) := \text{span}(V \otimes_{\mathbb{A}} \mathbb{R}^n) = \text{span}\{\mathbb{A}[\xi]v : \xi \in \mathbb{R}^n, v \in V\},$$

cf. [BDG20, Section 5]. It turns out that  $\mathcal{R}(\mathbb{A})$  is minimal in the following sense: We observe that pointwise there holds

$$\mathbb{A}[\xi]\widehat{u} = \sum_{\alpha=1}^n \xi^\alpha \mathbb{A}_\alpha \widehat{u} \in V \otimes_{\mathbb{A}} \mathbb{R}^n.$$

Therefore, the Fourier inversion formula, leads to  $\mathbb{A}u = (\mathbb{A}[\xi]\widehat{u})^\vee$ . Hence, we have  $\mathbb{A}u \in \mathcal{R}(\mathbb{A})$  for  $u \in C^1(\Omega; V)$ , showing that  $\mathcal{R}(\mathbb{A})$  is (up to isomorphy) the smallest space in which  $\mathbb{A}u(x)$  takes values when  $x \in \mathbb{R}^n$  and  $u \in C^1(\Omega; V)$ . Without loss of generality, it is possible to assume that

$$W = \mathcal{R}(\mathbb{A}) \hookrightarrow \mathbb{R}^{N \times n} \quad \text{if } V \cong \mathbb{R}^N,$$

which is justified in [CG22, Section 3] exploiting carefully the explicit identifications between the involved vector spaces. However, working with abstract vector spaces  $V$  and  $W$  may have some advantages, e.g. when considering exterior differential forms, cf. [CG22, Section 5.1].

**Lemma 2.6.** *Let  $\mathbb{A}$  be a  $\mathbb{C}$ -elliptic differential operator of the form (1.2). Then there holds  $\mathcal{R}(\mathbb{A}) = \text{im}(\mathcal{A})$ , where  $\mathcal{A} \in \text{Hom}(V \otimes \mathbb{R}^n; W)$  is the part map of  $\mathbb{A}$ .*

*Proof.* If  $\eta \in \text{span}(V \otimes_{\mathbb{A}} \mathbb{R}^n)$  we can find  $M \in \mathbb{N}$ ,  $\lambda_j \in \mathbb{R}$ ,  $v_j \in V$  and  $\xi_j \in \mathbb{R}^n$  for  $j \in \{1, \dots, M\}$  such that

$$\eta = \sum_{j=1}^M \lambda_j (v_j \otimes_{\mathbb{A}} \xi_j) = \sum_{j=1}^M \lambda_j \mathbb{A}[\xi_j] v_j = \sum_{j=1}^M \lambda_j \mathcal{A}[v_j \otimes \xi_j] \in \text{im}(\mathcal{A}), \quad (2.9)$$

proving one inclusion. Conversely, we notice if  $\eta \in \text{im}(\mathcal{A})$ , there exists  $P \in V \otimes \mathbb{R}^n$  with  $\eta = \mathcal{A}[P]$ . While this is trivial for  $P = 0$ , we can find for  $P \neq 0$  some  $M \in \mathbb{N}$  together with the representation (cf. [Gre67, Section 1.5])  $P = \sum_{j=1}^M v_j \otimes \xi_j$ , where  $v_j$  and  $\xi_j$  are linearly independent vectors in  $V$  and  $\mathbb{R}^n$ , respectively for all  $j \in \{1, \dots, M\}$ . Reversing the argument in (2.9) yields the second inclusion and the claim follows.  $\square$

## 2.2.4 $\mathbb{C}$ -elliptic Operators in Two Dimensions

In the previous section we have seen that the part map of the symmetric and the deviatoric gradient is given by a projection. It turns out that in two dimensions it is possible to characterise all  $\mathbb{C}$ -elliptic operators  $\mathbb{A}u = \mathcal{A}[\nabla u]$  induced by an orthogonal projection  $\mathcal{A}: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ . More precisely, all this operators share the presence of a one-dimensional almost complementary part:<sup>3</sup>

**Theorem 2.7** ([GLN23, Proposition 4.1]). *Let  $\mathbb{A}$  be  $\mathbb{C}$ -elliptic differential operator of the form (1.2) in two dimensions induced by a projection  $\mathcal{A}: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ . Then there exist a linear map  $\mathfrak{L} \in \text{Hom}(\mathbb{R}^{2 \times 2}; \mathbb{R}^{2 \times 2})$  and  $\mathfrak{G} \in \text{GL}(2)$  such that*

$$\{P - \mathfrak{L}(\mathcal{A}[P]): P \in \mathbb{R}^{2 \times 2}\} = \mathbb{R}\mathfrak{G}.$$

As a consequence, we can decompose all  $P \in \mathbb{R}^{2 \times 2}$  by means of

$$P = \mathfrak{L}(\mathcal{A}[P]) + \mathfrak{h}(P)\mathfrak{G}, \quad (2.10)$$

where  $\mathfrak{h} \in \text{Hom}(\mathbb{R}^{2 \times 2}; \mathbb{R})$  is a linear map.

<sup>3</sup>The complementary part of  $P \in \mathbb{R}^{2 \times 2}$  with respect to  $\mathcal{A}$  is given by  $P - \mathcal{A}[P]$ . We refer to  $P - \mathfrak{L}(\mathcal{A}[P])$  as the almost complementary part where  $\mathfrak{L} \in \text{Hom}(\mathbb{R}^{2 \times 2}; \mathbb{R}^{2 \times 2})$  is the linear map from Theorem 2.7.

Furthermore, in this explicit situation it is possible to specify the integer of Lemma 2.3:

**Lemma 2.8.** *Let  $\mathbb{A}$  be a  $\mathbb{C}$ -elliptic differential operator of the form (1.2) in two dimensions induced by a projection  $\mathcal{A}: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ . Then there exists a first-order linear homogeneous differential operator  $\mathbb{L}$  from  $W$  to  $V \odot \mathbb{R}^n$  such that  $\nabla^2 = \mathbb{L} \circ \mathbb{A}$*

*Proof.* The proof is based on the construction of the linear maps  $\mathfrak{L}$  and  $\mathfrak{h}$  as well as the matrix  $\mathfrak{G}$  from Theorem 2.7 derived in [GLN23, Proposition 4.1]. There it is shown that only  $\dim(\text{im } \mathcal{A}) \in \{3, 4\}$  is possible, ruling out dimension two. Consequently, we have to address only two cases:

- If  $\dim(\text{im } \mathcal{A}) = 4$ , we note that  $\{e_i \otimes_{\mathbb{A}} e_j : 1 \leq i, j \leq 2\}$  forms a basis for  $\text{im } \mathcal{A}$ . For any  $\mathfrak{G} \in \text{GL}(2)$  one can define then  $\mathfrak{L}$  by its action on basis vectors

$$\mathfrak{L}(e_i \otimes_{\mathbb{A}} e_j) := (e_i \otimes e_j) - \mathfrak{G},$$

cf. [GLN23, Equation (4.6)]. In particular, we can choose  $\mathfrak{G} = \mathbb{1}_{2 \times 2}$  to obtain

$$\mathfrak{L}(\mathcal{A}[\nabla u]) = \begin{pmatrix} -(\partial_1 u_2 + \partial_2 u_1 + \partial_2 u_2) & \partial_1 u_2 \\ \partial_2 u_1 & -(\partial_1 u_1 + \partial_1 u_2 + \partial_2 u_1) \end{pmatrix}$$

for  $u \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$ . Hence, already all entries of the gradient  $\nabla u$  can be expressed by a linear combination of the entries of  $\mathfrak{L}(\mathcal{A}[\nabla u])$ , e.g.

$$\partial_1 u_1 = -\mathfrak{L}(\mathcal{A}[\nabla u])^{(22)} - \mathfrak{L}(\mathcal{A}[\nabla u])^{(12)} - \mathfrak{L}(\mathcal{A}[\nabla u])^{(21)}.$$

In particular, one could even use  $d = 1$  in this case, but then also all entries of  $\nabla^2 u$  can be expressed by linear combination of the the entries of  $\nabla \mathfrak{L}(\mathcal{A}[\nabla u])$ .

- If  $\dim(\text{im } \mathcal{A}) = 3$ , we can find without loss of generality coefficients  $a_{11}, a_{12}$  and  $a_{22}$ , not all equal to zero such, that

$$e_2 \otimes_{\mathbb{A}} e_1 = a_{11} (e_1 \otimes_{\mathbb{A}} e_1) + a_{12} (e_1 \otimes_{\mathbb{A}} e_2) + a_{22} (e_2 \otimes_{\mathbb{A}} e_2)$$

and  $a_{11}a_{22} + a_{12} \neq 0$ , cf. [GLN23, Equation (4.7)f.]. In this case, we can define  $\mathfrak{L}$  through its action on basis vectors by

$$\mathfrak{L}(e_i \otimes_{\mathbb{A}} e_j) := \begin{cases} e_i \otimes e_j & \text{if } (i, j) \neq (2, 1), \\ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} & \text{if } (i, j) = (2, 1). \end{cases}$$

This leads to

$$\mathfrak{L}(\mathcal{A}[\nabla u]) := \begin{pmatrix} \partial_1 u_1 + a_{11} \partial_2 u_1 & \partial_1 u_2 + a_{12} \partial_2 u_1 \\ 0 & \partial_2 u_2 + a_{22} \partial_2 u_1 \end{pmatrix},$$

and hence, we cannot express all the entries of the gradient  $\nabla u$  by a linear combination of the entries of  $\mathfrak{L}(\mathcal{A}[\nabla u])$ . However, differentiating all the entries yields

$$\begin{pmatrix} \partial_1 \mathfrak{L}(\mathbb{A}[Du])^{(11)} \\ \partial_2 \mathfrak{L}(\mathbb{A}[Du])^{(11)} \\ \partial_1 \mathfrak{L}(\mathbb{A}[Du])^{(12)} \\ \partial_2 \mathfrak{L}(\mathbb{A}[Du])^{(12)} \\ \partial_1 \mathfrak{L}(\mathbb{A}[Du])^{(22)} \\ \partial_2 \mathfrak{L}(\mathbb{A}[Du])^{(22)} \end{pmatrix} = \begin{pmatrix} 1 & a_{11} & 0 & 0 & 0 & 0 \\ 0 & 1 & a_{11} & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 1 & 0 & 0 \\ 0 & 0 & a_{12} & 0 & 1 & 0 \\ 0 & a_{22} & 0 & 0 & 1 & 0 \\ 0 & 0 & a_{22} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_1^2 u_1 \\ \partial_1 \partial_2 u_1 \\ \partial_2^2 u_1 \\ \partial_1^2 u_2 \\ \partial_1 \partial_2 u_2 \\ \partial_2^2 u_2 \end{pmatrix}.$$

Since the determinant of the appearing  $6 \times 6$ -matrix satisfies  $-(a_{11}a_{22} + a_{12}) \neq 0$ , we can express all the entries of  $\nabla^2 u$  by a linear combination of the entries of  $\nabla \mathfrak{L}(\mathbb{A}[\nabla u])$ .

This completes the proof.  $\square$

**Remark 2.9** (Deviatoric Gradient). *The structure from Lemma 2.8 also applies to the trace-free or deviatoric gradient in all dimensions with  $\mathfrak{R}(\mathbb{A}) = \mathfrak{sl}(n)$ , cf. Example 2.2. More precisely, all the entries of the second derivative of a smooth vector field  $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$  can be expressed by a linear combination of derivatives of  $\mathbb{A} = \text{dev } \nabla$ , i.e. in the above Lemma we also have  $d = 2$ . Indeed, since this operator only applies on the diagonal elements, the off-diagonal entries remain the same, meaning*

$$(\nabla u)^{(ij)} = (\text{dev}(\nabla u))^{(ij)} \quad \text{for } i \neq j \in \{1, \dots, n\}.$$

Moreover, we note that the diagonal elements of  $\nabla u$  cannot be expressed as linear combinations of the entries of the elements of  $\text{dev}(\nabla u)$ . However, this holds true if we increase the order of differentiation, namely

$$\begin{aligned} \partial_i(\text{dev}(\nabla u))^{(ii)} &= \partial_i \left( \partial_i u_i - \frac{1}{n} \sum_{j=1}^n \partial_j u_j \right) = \frac{n-1}{n} \partial_i \partial_i u_i - \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^n \partial_i \partial_j u_j \\ &= \frac{n-1}{n} \partial_i(\nabla u)^{(ii)} - \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^n \partial_j(\text{dev}(\nabla u))^{(ij)}. \end{aligned}$$

Hence, we found a first-order differential operator  $\mathbb{L}$  from  $\mathfrak{sl}(n)$  to  $V \odot \mathbb{R}^n$  such that there holds  $\nabla^2 = \mathbb{L} \circ \text{dev } \nabla$ .

## 2.2.5 Jones-Type Extension Operator

As already announced previously, this section provides the construction of an extension operator in more detail. This operator was essentially developed by JONES [Jon81] and used by GMEINER and RAITA [GR19a] in the context of  $\mathbb{C}$ -elliptic differential operators to overcome the lack of boundedness of singular integrals on  $L^1$ .

We start by introducing a projection operator onto the finite dimensional null-space  $\ker(\mathbb{A})$  of a  $\mathbb{C}$ -elliptic differential operator  $\mathbb{A}$ . To this end, let  $U \subset \mathbb{R}^n$  be a bounded set, which will be a cube or a ball in the sequel. In order to construct a  $L^1$ -bounded projection operator  $\mathbb{P}_U: L^1(U; \mathbb{R}^N) \rightarrow \ker(\mathbb{A})$  we choose an  $L^2(U; \mathbb{R}^N)$ -orthonormal basis  $\{\pi_1, \dots, \pi_d\}$  of  $\ker(\mathbb{A})$  with respect to the averaged  $L^2$ -inner product, where  $d := \dim(\ker(\mathbb{A}))$ , and set

$$\mathbb{P}_U v := \sum_{j=1}^d \left( \int_U \pi_j v \, dx \right) \pi_j, \quad (2.11)$$

for  $v \in L^1(U; \mathbb{R}^N)$ . Since each of the functions  $\pi_1, \dots, \pi_d$  is in  $L^\infty(U; \mathbb{R}^N)$ , we conclude that the operator  $\mathbb{P}_U$  is well-defined and  $L^1$ -bounded. Next, we will rephrase and extend [GR19a, Theorem 4.1] within our framework, commenting on the changes to the proof.

**Theorem 2.10.** *Let  $p \in [1, \infty]$  and  $\Omega, \Omega_0 \subset \mathbb{R}^n$  be two open and bounded sets with  $\Omega \Subset \Omega_0$  such that  $\Omega$  has Lipschitz boundary  $\partial\Omega$ . Then there exists a (norm-)bounded linear extension operator  $\mathfrak{J}: W^{\mathbb{A},1}(\Omega) \rightarrow W^{\mathbb{A},1}(\mathbb{R}^n)$  with*

- i)  $\text{supp}(\mathfrak{J}u) \subset \Omega_0$  for all  $u \in W^{\mathbb{A},1}(\Omega)$ ,
- ii)  $\mathfrak{J}: W^{\mathbb{A},1}(\Omega) \rightarrow W_0^{\mathbb{A},1}(\Omega_0)$ ,
- iii) and there exists a constant  $c = c(\Omega, n) > 0$  such that  $\|\mathfrak{J}u\|_{L^p(\Omega; \mathbb{R}^N)} \leq c\|u\|_{L^p(\Omega; \mathbb{R}^N)}$  for all  $u \in W^{\mathbb{A},1}(\Omega) \cap L^p(\Omega; \mathbb{R}^N)$ .

In order to discuss the proof of Theorem 2.10, we briefly recall the underlying construction from [GR19a]. Towards this aim, let  $\Omega \subset \mathbb{R}^n$  be an open and bounded Lipschitz domain. For the construction of the extension operator we take a dyadic Whitney decomposition of  $\Omega$ , i.e. a countable family  $\mathscr{W}_1$  of open, dyadic cubes  $Q$  whose length is denoted by  $\ell(Q)$ , such that

(W1) the cubes from  $\mathscr{W}_1$  are pairwise disjoint with  $\bigcup_{Q \in \mathscr{W}_1} \overline{Q} = \Omega$ ,

(W2) for all  $Q \in \mathscr{W}_1$  there holds  $\sqrt{n}\ell(Q) \leq \text{dist}(Q, \partial\Omega) \leq 4\sqrt{n}\ell(Q)$ ,

(W3) for  $Q, Q' \in \mathscr{W}_1$  with  $\overline{Q} \cap \overline{Q'} \neq \emptyset$  we have

$$\frac{1}{4} \leq \frac{\ell(Q)}{\ell(Q')} \leq 4,$$

(W4) and for every  $Q \in \mathscr{W}_1$  there exist at most  $12^n$  cubes  $Q' \in \mathscr{W}_1$  with  $Q \cap Q' = \emptyset$  and  $\overline{Q} \cap \overline{Q'} \neq \emptyset$ .

Similarly, we take a dyadic Whitney decomposition  $\mathscr{W}_2$  of  $\mathbb{R}^n \setminus \overline{\Omega}$  and thereout the collection  $\mathscr{W}_3$  of all cubes  $Q \in \mathscr{W}_2$  with  $\ell(Q) \leq \frac{3 \text{diam}(\Omega)}{16n}$ . It can be shown that for all  $Q \in \mathscr{W}_3$  there exists a *reflected* cube  $Q^* \in \mathscr{W}_1$  such that for some constant  $c = c(\Omega) > 0$  there holds

$$\frac{1}{c} \leq \frac{\ell(Q)}{\ell(Q^*)} \leq c \quad \text{and} \quad \text{dist}(Q, Q^*) \leq c\ell(Q), \quad (2.12)$$

cf. [Jon81] or [DHHR11, Proposition 8.5.3, Lemma 8.5.4]. We may then blow up the cubes from  $\mathscr{W}_3$  with a suitable factor  $\theta > 1$  sufficiently close to 1 such that the family  $(\theta Q)_{Q \in \mathscr{W}_3}$  of cubes  $\theta Q$ , having the same centre as  $Q$  and  $\ell(\theta Q) = \theta\ell(Q)$ , satisfies  $\theta Q \subset \mathbb{R}^n \setminus \overline{\Omega}$  for all  $Q \in \mathscr{W}_3$ . This can be done in a way such that  $(\theta Q)_{Q \in \mathscr{W}_3}$  yields a locally uniformly finite cover of  $\bigcup_{Q \in \mathscr{W}_3} Q$  and (W2)–(W4) as well as (2.12) still hold true, possibly with worse constants. As a next step, we take a partition of unity  $(\varphi_{\theta Q})$  in  $C_c^\infty(\mathbb{R}^n \setminus \overline{\Omega}; [0, 1])$  subordinate to the covering  $(\theta Q)_{Q \in \mathscr{W}_3}$ , which allows to set

$$\overline{\mathfrak{J}}u := \begin{cases} u & \text{in } \Omega \\ \sum_{Q \in \mathscr{W}_3} \varphi_{\theta Q} \mathbb{P}_{Q^*} u & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

where  $u \in W^{\mathbb{A},1}(\Omega)$  and  $\mathbb{P}$  denotes the projection onto  $\ker(\mathbb{A})$  from (2.11). Imposing natural growth assumptions on the derivatives of  $\varphi_{\theta Q}$  it is shown in [GR19a, Section 4.1] that  $\overline{\mathfrak{J}}: W^{\mathbb{A},1}(\Omega) \rightarrow W^{\mathbb{A},1}(\mathbb{R}^n)$  is a norm bounded linear operator. In order to address the  $L^p$ -stability, we first observe that we only have to consider the cases  $1 < p < \infty$  and  $p = \infty$ , as the case  $p = 1$  is trivially satisfied. Towards the first aim, we take a cube  $Q \in \mathscr{W}_3$  and observe

$$\begin{aligned} \|\varphi_{\theta Q} \mathbb{P}_{Q^*} u\|_{L^p(\mathbb{R}^n \setminus \Omega; \mathbb{R}^N)} &\leq \|\mathbb{P}_{Q^*} u\|_{L^p(\theta Q; \mathbb{R}^N)} \\ &\leq c \|\mathbb{P}_{Q^*} u\|_{L^p(Q; \mathbb{R}^N)} \leq c \|\mathbb{P}_{Q^*} u\|_{L^p(Q^*; \mathbb{R}^N)}. \end{aligned} \quad (2.13)$$

Here, the first inequality follows by definition of the partition of unity, whereas the second is a consequence of the equivalence of all norms on a finite dimensional space with scaling. The third inequality takes into account (2.12). Since  $\ker(\mathbb{A})$  is finite dimensional, and therefore, the  $L^p$ - and  $L^1$ -norms are equivalent on  $Q$ , we can further estimate to infer

$$\|\mathbb{P}_{Q^*}u\|_{L^p(Q^*; \mathbb{R}^N)} \leq c \int_{Q^*} |\mathbb{P}_{Q^*}u| \, dx \leq c \int_{Q^*} |u| \, dx \leq c \|u\|_{L^p(\Omega; \mathbb{R}^N)} \quad (2.14)$$

for a constant  $c = c(n, \Omega, \theta) > 0$ . Here, the second inequality is a consequence of the  $L^1$ -boundedness of the projection. Finally, since the family  $(\theta Q)_{Q \in \mathscr{W}_3}$  of blown-up Whitney cubes still satisfy (W4) with worse constants, the number of overlapping cubes  $\theta Q$  with  $Q \in \mathscr{W}_3$  is uniformly bounded by a constant  $c = c(n, \theta) > 0$ . Therefore, the operator  $\tilde{\mathfrak{J}}: L^p(\Omega; \mathbb{R}^N) \rightarrow L^p(\mathbb{R}^n; \mathbb{R}^N)$  is a bounded linear operator.

In the second case, we replace the very left hand side of (2.13) by  $|(\varphi_{\theta Q} \mathbb{P}_{Q^*}u)(x)|$  for  $x \in \mathbb{R}^n \setminus \bar{\Omega}$  and repeat the above argument with  $p = \infty$ .

Finally, in order to get the precise statement of Theorem 2.10 we take an open and bounded set  $\Omega_0 \subset \mathbb{R}^n$  with  $\Omega \Subset \Omega_0$  and choose a localisation function  $\varrho \in C_c^\infty(\Omega_0; [0, 1])$  with  $\mathbf{1}_\Omega \leq \varrho \leq \mathbf{1}_{\Omega_0}$ . Setting  $\tilde{\mathfrak{J}}u := \varrho \tilde{\mathfrak{J}}u$  leads to an operator  $\mathfrak{J}: W^{\mathbb{A}, 1}(\Omega) \rightarrow W^{\mathbb{A}, 1}(\mathbb{R}^n)$  having all the desired properties of Theorem 2.10.

## 2.2.6 $\mathbb{A}$ -Orlicz-Sobolev Spaces

Let  $\mathbb{A}$  be an elliptic differential operator of the form (1.2) and  $\Phi$  be a Young function. For an open and bounded subset  $\Omega \subset \mathbb{R}^n$  we introduce the  $\mathbb{A}$ -Orlicz Sobolev space by

$$W^{\mathbb{A}, \Phi}(\Omega; V) := \{u \in L^\Phi(\Omega; V) : \mathbb{A}u \in L^\Phi(\Omega; W)\},$$

which is a Banach space with respect to the norm

$$\|u\|_{W^{\mathbb{A}, \Phi}(\Omega)} := \|u\|_{L^\Phi(\Omega; V)} + \|\mathbb{A}u\|_{L^\Phi(\Omega; W)}.$$

In the special case of the symmetric gradient  $\mathbb{A} = \varepsilon$  we use the notation  $E^{1, \Phi}(\Omega; \mathbb{R}^n) := W^{\varepsilon, \Phi}(\Omega; \mathbb{R}^n)$ , which is convenient in the literature.

## 2.2.7 Negative Sobolev Spaces

The standard way to define the space  $W^{-k, p}(\Omega)$  for  $p > 1$  and  $k \in \mathbb{N}$  is to view it as the dual space of  $W_0^{1, p'}(\Omega)$  where  $p' = \frac{p}{p-1}$  is the Hölder-conjugate of  $p$ . As a next step, one then characterise the elements of this dual spaces as certain distributions, see e.g. [KJF77, Section 5.9]. Unfortunately, we cannot follow the duality approach when dealing with  $p = 1$ . However, motivated by the aforementioned characterisation, one can define the spaces  $W^{-k, 1}(\Omega)$  as follows:

**Definition 2.11.** *Let  $k \in \mathbb{N}$ . We define the space  $W^{-k, 1}(\Omega; \mathbb{R}^N)$  as*

$$\left\{ T \in \mathscr{D}'(\Omega; \mathbb{R}^N) : T = \sum_{|\alpha| \leq k} \partial^\alpha T_\alpha, T_\alpha \in L^1(\Omega; \mathbb{R}^N), \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k \right\}.$$

*This linear space is canonically endowed with the norm*

$$\|T\|_{W^{-k, 1}(\Omega; \mathbb{R}^N)} := \inf \left\{ \sum_{|\alpha| \leq k} \|T_\alpha\|_{L^1(\Omega; \mathbb{R}^N)} : T = \sum_{|\alpha| \leq k} \partial^\alpha T_\alpha, T_\alpha \in L^1(\Omega; \mathbb{R}^N) \right\}.$$

**Lemma 2.12** (Properties of  $W^{-k,1}(\Omega; \mathbb{R}^N)$ , [Gme20, Lemma 2.5]). *Let  $\Omega \subset \mathbb{R}^n$  be open and  $k \in \mathbb{N}$ . Then the following holds:*

*i) The space  $(W^{-k,1}(\Omega; \mathbb{R}^N), \|\cdot\|_{W^{-k,1}(\Omega; \mathbb{R}^N)})$  is a Banach space.*

*ii) For every  $u \in L^1(\Omega; \mathbb{R}^N)$  and every  $\beta \in \mathbb{N}_0^n$  with  $|\beta| \leq k$  there holds*

$$\|\partial^\beta u\|_{W^{-k,1}(\Omega; \mathbb{R}^N)} \leq \|u\|_{W^{|\beta|-k,1}(\Omega; \mathbb{R}^N)}.$$

In this thesis, we will use Lemma 2.12 only in the cases  $k \in \{1, 2\}$ . In particular, we need the trivial embedding  $W^{-1,1}(\Omega; \mathbb{R}^N) \subset W^{-2,1}(\Omega; \mathbb{R}^N)$  which results in

$$\|T\|_{W^{-2,1}(\Omega; \mathbb{R}^N)} \leq \|T\|_{W^{-1,1}(\Omega; \mathbb{R}^N)} \quad \text{for all } T \in W^{-1,1}(\Omega; \mathbb{R}^N). \quad (2.15)$$

Furthermore, there holds for  $s \in \{1, \dots, n\}$ ,  $h > 0$  and  $w \in L^1(\Omega; \mathbb{R}^N)$

$$\|\partial_s w\|_{W^{-2,1}(\Omega; \mathbb{R}^N)} \leq \|w\|_{W^{-1,1}(\Omega; \mathbb{R}^N)} \leq \|w\|_{L^1(\Omega; \mathbb{R}^N)}, \quad (2.16)$$

$$\|\partial_s w\|_{W^{-1,1}(\Omega; \mathbb{R}^N)} \leq \|w\|_{L^1(\Omega; \mathbb{R}^N)}, \quad (2.17)$$

$$\|\Delta_{s,h} w\|_{W^{-1,1}(\{x \in \Omega : \text{dist}(x, \partial\Omega) > h\}; \mathbb{R}^N)} \leq \|w\|_{L^1(\Omega; \mathbb{R}^N)}. \quad (2.18)$$

The last inequality immediately follows for  $w \in C^1(\Omega; \mathbb{R}^N)$  by means of the representation formula

$$\Delta_{s,h} w(x) = \frac{\partial}{\partial x_s} \int_0^1 w(x + t e_s) dt,$$

from which the version for general  $w \in L^1(\Omega; \mathbb{R}^N)$  follows by approximation.

## 2.3 Functionals Defined on Measures

Since the expression  $\mathbb{A}u$  of a  $BV^{\mathbb{A}}$  function exists only as a finite  $W$ -valued Radon measure, rather than a function, variational integrals must be formulated in the broader framework involving measures. To this end, we say that a finite  $W$ -valued finite Radon-measure  $\mu \in \text{RM}_{\text{fin}}(\Omega; W)$  takes values in a convex cone<sup>4</sup>  $K \subset W$  if  $\mu(B) \in K$  holds for every Borel set  $B \subset \Omega$ . In this case, we briefly write  $\mu \in \text{RM}_{\text{fin}}(\Omega; K)$ . Moreover, we notice that also the Radon-Nikodým derivative  $\frac{d\mu}{d|\mu|}$  takes values in  $K$ .

Now, let  $K \subset W$  be a convex cone and  $f: K \rightarrow \mathbb{R}$  be a convex function of linear growth from below, i.e. the first inequality in (1.3) holds. For an open and bounded set  $\Omega \subset \mathbb{R}^n$  and a finite Radon measure  $\mu \in \text{RM}_{\text{fin}}(\Omega; K)$  taking values in  $K$ , we define the measures  $f(\mu)$  by

$$f(\mu)(U) := \int_U f \left( \frac{d\mu^a}{d\mathcal{L}^n} \right) dx + \int_U f^\infty \left( \frac{d\mu^s}{d|\mu^s|} \right) d|\mu^s|, \quad (2.19)$$

for Borel subsets  $U \subset \Omega$ , based on the Lebesgue-Radon-Nikodým decomposition (2.1). As mentioned in the introduction, the recession function  $f^\infty$  is defined as

$$f^\infty(z) := \lim_{s \rightarrow \infty} \frac{f(sz)}{s} = \lim_{t \searrow 0} t f \left( \frac{z}{t} \right) \quad \text{for } z \in W.$$

<sup>4</sup>A convex cone is a subset  $K \subset W$  such that for all  $x, y \in K$  and  $\alpha, \beta > 0$  there holds  $\alpha x + \beta y \in K$ .

It is a well-defined, 1-homogeneous, lower semi-continuous and convex function on  $K$ , cf. [GMS79, Proposition 1.1]. Note that, if we do not assume linear growth from above, we only have  $f^\infty: K \rightarrow \mathbb{R} \cup \{\infty\}$ .

**Remark 2.13** (Connection to convex analysis). *The recession function  $f^\infty$  can also be characterised in terms of convex analysis. Namely, for a convex function  $f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  we consider the epigraph  $\text{epi}(f) := \{(x, t) \in \mathbb{R}^n \times \mathbb{R}: f(x) \leq t\}$ . Moreover, for a non-empty convex set  $V \subset \mathbb{R}^n$  we define the recession cone of  $V$  as the set of all directions in which  $V$  is unbounded, i.e.*

$$\text{rec}(V) := \{y \in \mathbb{R}^n: \forall \lambda \geq 0, x \in V: x + \lambda y \in V\}.$$

For closed and convex sets  $V \subset \mathbb{R}^n$  we have  $\text{rec}(V) = \{0\}$  if and only if  $V$  is bounded. In terms of convex analysis, the recession function  $f^\infty$  is the function whose epigraph is the recession cone of  $\text{epi}(f)$ , i.e.  $\text{epi}(f^\infty) = \text{rec}(\text{epi}(f))$ . This characterisation provides a geometric viewpoint illustrating how  $f^\infty$  encodes the asymptotic behaviour of  $f$  in a given direction. For further details we refer to [Roc70, NGL88, AT03].

The main (lower semi-)continuity result for functionals defined on measures is due to RESHETNYAK [Res68] (see also [BS13, Theorem 2.4]). In our case the convex cone  $K$  is  $\mathcal{R}(\mathbb{A})$  and we will use the theorem in the following version:

**Theorem 2.14** (Reshetnyak (lower semi-)continuity theorem). *Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded set and let  $f: \mathcal{R}(\mathbb{A}) \rightarrow \mathbb{R}$  be a convex function of linear growth from below. For functions  $u, u_1, u_2, \dots \in \text{BV}^\mathbb{A}(\Omega)$  we have*

i) *if  $u_j \xrightarrow{*} u$  in the weak\*-sense in  $\text{BV}^\mathbb{A}(\Omega)$ , then*

$$f(\mathbb{A}u)(\Omega) \leq \liminf_{j \rightarrow \infty} f(\mathbb{A}u_j)(\Omega).$$

ii) *If in addition  $f$  is of linear growth from above and  $u_j \rightarrow u$  in the  $\mathbb{A}$ -area-strict sense in  $\text{BV}^\mathbb{A}(\Omega)$ . then*

$$f(\mathbb{A}u)(\Omega) = \lim_{j \rightarrow \infty} f(\mathbb{A}u_j)(\Omega).$$

If  $\Omega$  has a Lipschitz boundary  $\partial\Omega$  oriented by  $\nu_{\partial\Omega}: \partial\Omega \rightarrow \mathbb{S}^{n-1}$ , we can extend a function  $u \in \text{BV}^\mathbb{A}(\Omega)$  by some fixed  $u_0 \in \text{BV}^\mathbb{A}(\mathbb{R}^n)$  to the full space  $\mathbb{R}^n$ , to obtain an element of  $\text{BV}^\mathbb{A}(\mathbb{R}^n)$ . From (2.1) we get an additional boundary term and infer

**Corollary 2.15.** *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded with Lipschitz boundary  $\partial\Omega$  and  $f: \mathcal{R}(\mathbb{A}) \rightarrow \mathbb{R}$  be convex function satisfying (1.3). Then for every  $u_0 \in \text{BV}^\mathbb{A}(\Omega)$  the relaxed functional*

$$\begin{aligned} \overline{F}[u; \Omega] := & \int_{\Omega} f \left( \frac{d\mathbb{A}^a u}{d\mathcal{L}^n} \right) dx + \int_{\Omega} f^\infty \left( \frac{d\mathbb{A}^s u}{d|\mathbb{A}^s u|} \right) d|\mathbb{A}^s u| \\ & + \int_{\partial\Omega} f^\infty(\text{tr}_{\partial\Omega}(u - u_0) \otimes_{\mathbb{A}} \nu_{\partial\Omega}) d\mathcal{H}^{n-1} \end{aligned}$$

*is lower semi-continuous with respect to weak\*-convergence and continuous with respect to  $\mathbb{A}$ -area-strict-convergence in  $\text{BV}^\mathbb{A}(\Omega)$ .*

## 2.4 The Ekeland Variational Principle

The Ekeland variational principle quantifies the following meta-principle: Namely, if we have a point close to the infimum of a given functional, we can find a nearby almost-minimiser  $v$ , which is even closer in a quantified way.

**Proposition 2.16** ( $\varepsilon$ -version of the Ekeland variational principle, [Eke74]). *Let  $(X, d)$  be a complete metric space and  $F: X \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semi-continuous function with respect to the metric topology which is bounded from below and not identically  $+\infty$ . Suppose, that for some  $u \in X$  and some  $\varepsilon > 0$ , there holds  $F[u] \leq \inf_X F + \varepsilon$ . Then there exists  $v \in X$  such that*

$$i) \quad d(u, v) \leq \sqrt{\varepsilon}.$$

$$ii) \quad F[v] \leq F[u] \text{ and}$$

$$iii) \quad \text{for all } w \in X \text{ there holds } F[v] \leq F[w] + \sqrt{\varepsilon} d(v, w).$$

The Ekeland variational principle has a nice geometric interpretation in the special case that  $X$  is a Banach space with norm  $\|\cdot\|$ . If we have a point  $u \in X$  which is  $\varepsilon$ -close to the infimum, we can find a point  $v \in X$  lying in a  $\sqrt{\varepsilon}$ -neighbourhood of  $u$ , which is almost minimising. The latter means, that the shifted cone

$$\mathcal{C} := (v, F[v]) + \{(x, a) \in X \times \mathbb{R} : a + \sqrt{\varepsilon}\|x\| < 0\}$$

lies entirely under the graph of the function  $F$  in  $X \times \mathbb{R}$ , see Figure 2.1. A more general version can be found in [Eke74], whereas we refer to [AE84, Chapter 5, Section 3] for further details.

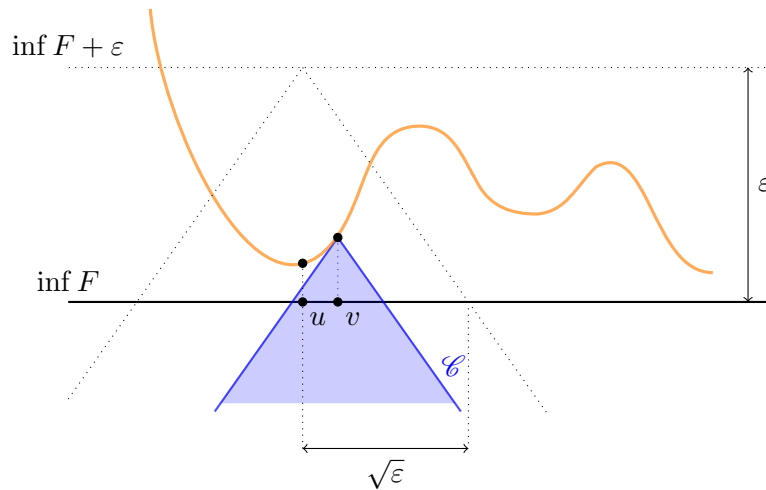


Figure 2.1: Ekeland variational principle

## 2.5 Miscellanea

In this section, we collect various auxiliary results and remarks that complement the main discussion.

\* *V-functions.* We briefly introduce some auxiliary functions  $V$ . For  $1 \leq p < \infty$  and a real normed vector space  $(X, |\cdot|)$ , we define

$$V_p(z) := (1 + |z|^2)^{\frac{p}{2}} - 1, \quad z \in X. \quad (2.20)$$

As a consequence of [AG88, Proposition 2.5], we obtain that for  $p \geq 2$ , there exist two constants  $c = c(p)$  and  $C = C(p) \geq 1$  such that

$$c|z|^p \leq V_p(z) \leq C(1 + |z|^p). \quad (2.21)$$

Moreover, if  $V = \mathbb{R}^{N \times n}$  we can compute the derivatives explicitly to deduce

$$\begin{aligned} \nabla V_p(z) &= p(1 + |z|^2)^{\frac{p}{2}-1} z \\ \nabla^2 V_p(z) &= p(1 + |z|^2)^{\frac{p}{2}-1} \left( (1 + |z|^2)^{-1} (p-2)z \otimes z + \mathbf{1}_{(N \times n) \times (N \times n)} \right). \end{aligned}$$

As a consequence, we obtain the estimates

$$|\nabla V_p(z)| \leq c(1 + |z|^2)^{\frac{p-1}{2}} \quad \text{and} \quad |\nabla^2 V_p(z)| \leq c(1 + |z|^2)^{\frac{p-2}{2}}.$$

\* *Convex Integrands.* We conclude this section by stating two well-known properties of convex integrands that provide bounds for the gradient:

**Lemma 2.17** ([Giu03, Lemma 5.2]). *Let  $m \in \mathbb{N}$  and  $f: \mathbb{R}^m \rightarrow [0, \infty)$  be a convex function with  $f(z) \leq \Gamma(1 + |z|)$  for all  $z \in \mathbb{R}^m$ . Then  $f$  is Lipschitz continuous on  $\mathbb{R}^m$ . Moreover, if  $f$  is of class  $C^1$ , then there holds*

$$|\nabla f(z)| \leq c,$$

for a constant  $c = c(m, \Gamma) > 0$ .

**Lemma 2.18** ([BS13, Lemma 2.8]). *Let  $m \in \mathbb{N}$ ,  $1 \leq p < \infty$  and  $g: \mathbb{R}^m \rightarrow [0, \infty)$  be a convex function of class  $C^1$  with  $g(0) \leq \mathfrak{b}$  and  $p$ -growth from below, i.e.  $g(x) \geq \mathfrak{a}|x|^p$ , with  $\mathfrak{a}, \mathfrak{b} > 0$ . Then*

$$\langle \nabla g(x), x \rangle \geq \mathfrak{a}|x|^p - \mathfrak{b}.$$

*Proof.* By convexity we have  $\mathfrak{b} \geq g(0) \geq g(x) - \langle \nabla g(x), x \rangle \geq \mathfrak{a}|x|^p - \langle \nabla g(x), x \rangle$ , for all  $z \in \mathbb{R}^m$ , showing the claim.  $\square$

Moreover, we need the following well-known Lemma on the identification of pointwise and weak limits in  $L^p$ -spaces.

**Lemma 2.19** ([GGZ74, Lemma 1.19], [BEG26, Lemma 3.4]). *Let  $m \in \mathbb{N}$  and consider  $\mu = \theta \mathcal{L}^n \llcorner \Omega$  for some  $\theta \in L^1_{\text{loc}}(\Omega; \mathbb{R}_{\geq 1})$ . Suppose that  $(u_j)_{j \in \mathbb{N}}$  in  $L^2_\mu(\Omega; \mathbb{R}^m)$  converges*

- i) weakly in  $L^2_\mu(\Omega; \mathbb{R}^m)$  to some function  $u \in L^2_\mu(\Omega; \mathbb{R}^m)$ , and*
- ii) pointwisely  $\mathcal{L}^n$ -a.e. to some measurable function  $v: \Omega \rightarrow \mathbb{R}^m$ .*

*Then the limits coincide, i.e we have  $u = v$ .*

\* *Elementary Bounds.* Finally, we recall some elementary, yet useful estimates, which we will use in the sequel without addressing them further. For  $z \in \mathbb{R}$  there holds

$$|z| \leq \sqrt{1 + |z|^2} \leq 1 + |z|.$$

Moreover, the logarithm grows more slowly than any root, meaning that

$$\log(1 + t) \leq C(k) t^{\frac{1}{k}}. \quad (2.22)$$

holds for all  $z \in \mathbb{R}$  and  $k \in \mathbb{N}$ . As an immediate consequence we derive the bound  $\log(1 + t^2) \leq 4\sqrt{t}$ .

\* *Extensions of Integrands.* Consider a function  $f: U \rightarrow \mathbb{R}$  defined on a subspace  $U$  of  $\mathbb{R}^{n \times n}$ . Then it is possible to introduce an inner product on  $U$  by restricting the Hilbert-Schmidt inner product  $\langle \cdot, \cdot \rangle$  from the ambient space  $\mathbb{R}^{n \times n}$  to  $U$ . It would be natural to work with this inner product on  $U$  as we have  $\nabla f(u) \in U$  and  $\nabla^2 f(u)v \in U$  for all  $u, v \in U$ . However, at later stages we have to treat expressions of the form

$$\langle \nabla f(u), A \rangle \quad \text{and} \quad \langle \nabla^2 f(u)A, B \rangle$$

for  $u \in U$  and general matrices  $A, B \in \mathbb{R}^{n \times n}$ . In order to do so, we extend our given function  $f$  from  $U$  to  $\mathbb{R}^{n \times n}$  by setting

$$\bar{f}(z) := f(\pi(z)),$$

where  $\pi: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  denotes the orthogonal projection onto the subspace  $U$ . Clearly, if  $f$  is a convex function on  $U$ , then so is its extension  $\bar{f}$ . Moreover, as an immediate consequence of the chain rule, we conclude from  $f \in C^2(U)$  that  $\bar{f} \in C^2(\mathbb{R}^n)$  together with  $\nabla \bar{f}(z) = \nabla f(\pi(z)) \in U$ . Since the orthogonal projection is self-adjoint, we also have

$$\langle \nabla \bar{f}(z), A \rangle = \langle \nabla f(\pi(z)), \pi(A) \rangle = \langle \nabla f(\pi(z)), A \rangle$$

and

$$\langle \nabla^2 \bar{f}(z)A, B \rangle = \langle \nabla^2 f(\pi(z))\pi(A), \pi(B) \rangle = \langle \nabla^2 f(\pi(z))\pi(A), B \rangle$$

for all  $z, A, B \in \mathbb{R}^{n \times n}$ . Both quantities follow from the abstract (second-order) chain rule for the (Fréchet) derivative, namely we have

$$d\bar{f}(z)(A) = d(f \circ \pi)(z)(A) = (df(\pi(z))d\pi(z))(A) = df(\pi(z))(\pi(A))$$

and

$$\begin{aligned} d^2 \bar{f}(z)[A, B] &= d^2 f(\pi(z))[d\pi(z)A, d\pi(z)B] + df(\pi(z))(d^2 \pi(x)[A, B]) \\ &= d^2 f(\pi(z))[\pi(A), \pi(B)], \end{aligned}$$

using  $d\pi(z) = \pi$  and  $d^2 \pi(z) = 0$  for all  $z \in \mathbb{R}^{n \times n}$ , as  $\pi$  is a linear map. As a word of caution, we note that if  $f$  is  $\mu$ -elliptic on  $U$ , we *cannot* conclude that  $\bar{f}$  is  $\mu$ -elliptic on all matrices. Indeed, taking directions in  $U^\top$  immediately destroys any lower bound of the type given in (1.5). Throughout this thesis, we will not address this detail explicitly and consider, if necessary, always the extension to make sense of all

appearing expressions. When dealing with symmetric gradients at later passages, we will use the above construction for  $U = \mathbb{R}_{\text{sym}}^{n \times n}$ . Especially, using the orthogonal sum decomposition

$$\mathbb{R}^{n \times n} = \mathbb{R}_{\text{sym}}^{n \times n} \oplus \mathbb{R}_{\text{skew}}^{n \times n}, \quad (2.23)$$

into symmetric and skew-symmetric matrices, we recover the well-known identity

$$\langle A, B \rangle = \langle A, B^{\text{sym}} \rangle + \langle A, B^{\text{skew}} \rangle = \langle A, B^{\text{sym}} \rangle$$

for  $A \in \mathbb{R}_{\text{sym}}^{n \times n}$  and  $B \in \mathbb{R}^{n \times n}$ . Here,  $B^{\text{sym}} := \frac{1}{2}(B + B^\top)$  and  $B^{\text{skew}} := \frac{1}{2}(B - B^\top)$  denote the symmetric and skew-symmetric parts of  $B$ , respectively.

### 3 | Korn-type Inequalities

This chapter presents sharp and generalised versions of Korn’s inequality. We begin with a discussion of the Calderón-Zygmund theory and the theory of Fourier multipliers, which are powerful tools for establishing such inequalities. These approaches rely on singular integral estimates that are valid in  $L^p$ -spaces for  $1 < p < \infty$ . However, it is crucial to note that these methods fail at the endpoints due to the breakdown of boundedness properties of singular integral operators in that setting.

*Structure of the chapter:* In Section 3.1, we introduce the main results from the Calderón-Zygmund theory of singular integral operators, along with key concepts from the theory of Fourier multipliers. In Section 3.2, we focus on Korn-type inequalities in  $L^p$  spaces for  $1 < p < \infty$ , outlining the classical results and emphasising the role of singular integral estimates. We also discuss Ornstein’s Non-Inequality, which demonstrates the failure of such estimates in the limiting case  $p = 1$ . Finally, Section 3.3 is devoted to Korn-type inequalities in Orlicz spaces.

#### 3.1 Calderón-Zygmund Theory and Fourier Multipliers

In this section, we briefly collect some facts from harmonic analysis. We will use the following normalisation for the Fourier transform of a Schwartz function  $f \in \mathcal{S}(\mathbb{R}^n; \mathbb{C})$ :

$$\mathcal{F}f(\xi) := \widehat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x)e^{-ix \cdot \xi} dx \quad \text{for } \xi \in \mathbb{R}^n. \tag{3.1}$$

More precisely, we consider for  $m \in L^\infty(\mathbb{R}^n; \mathbb{C})$  operators of the form

$$T_m : \mathcal{S}(\mathbb{R}^n; \mathbb{C}) \rightarrow \mathcal{S}(\mathbb{R}^n; \mathbb{C}), \quad T_m f := \mathcal{F}^{-1}[m\widehat{f}], \tag{3.2}$$

where  $\mathcal{F}^{-1}$  denotes the inverse Fourier transform<sup>1</sup>. Such operators are called *Fourier multiplier operators*. More specifically, we say that  $T_m$  is a *kernel operator*, if there exists a kernel  $\mathfrak{K} \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}; \mathbb{C})$  such that  $\widehat{\mathfrak{K}} \in L^\infty(\mathbb{R}^n; \mathbb{C})$  and

$$T_m(f)(x) := (\mathfrak{K} * f)(x) := \int_{\mathbb{R}^n} \mathfrak{K}(x - y)f(y) dy \tag{3.3}$$

holds for all  $x \notin \text{supp}(f)$ . Such a kernel satisfies the *Hörmander condition*, if the map

$$\mathbb{R}^n \ni y \mapsto \int_{\{|x|>2|y|\}} |\mathfrak{K}(x - y) - \mathfrak{K}(x)| dx \tag{3.4}$$

---

<sup>1</sup>Recall, that the Fourier transform is a linear isomorphism on the Schwartz space  $\mathcal{S}(\mathbb{R}^n; \mathbb{C})$ , cf. [SW71].

is uniformly bounded. By a famous and by now well-known theorem of CALDERÓN and ZYGMUND [CZ52], later improved by HÖRMANDER [Hör60], such operators can be extended to the scale of  $L^p$ -spaces for  $1 < p < \infty$ . More precisely, we have

**Theorem 3.1** (Calderón-Zygmund, [Duo01, Theorem 5.1]). *Let  $m \in L^\infty(\mathbb{R}^n; \mathbb{C})$  and  $T_m$  be a kernel operator as in (3.3) satisfying the Hörmander condition (3.4). Then  $T_m$  extends to a bounded linear operator  $T_m: L^p(\mathbb{R}^n; \mathbb{C}) \rightarrow L^p(\mathbb{R}^n; \mathbb{C})$  for all  $1 < p < \infty$ , i.e. there exists a constant  $c > 0$  such that*

$$\|T_m f\|_{L^p(\mathbb{R}^n; \mathbb{C})} \leq c \|f\|_{L^p(\mathbb{R}^n; \mathbb{C})}.$$

Moreover,  $T_m$  is of weak-type  $(1, 1)$ , i.e. there exists a constant  $c > 0$  such that

$$\mathcal{L}^n(\{x \in \mathbb{R}^n: |(\mathfrak{K} * f)(x)| > \lambda\}) \leq \frac{c}{\lambda} \|f\|_{L^1(\mathbb{R}^n; \mathbb{C})}.$$

for all  $\lambda > 0$ .

We notice that the above theorem excludes kernel operators of principal value type, which can be treated under additional assumptions on the kernel. We refer to [Ste70, Jou83, Duo01, Abe12] for further details. A sufficient condition for a function  $m$  to be a Fourier multiplier is provided by the Mihlin multiplier theorem:

**Theorem 3.2** (Mihlin multiplier theorem, [Abe12, Theorem 4.23]). *Let  $m: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}$  be an  $(n + 2)$ -times continuous differentiable function such that there holds*

$$|\partial_\xi^\alpha m(\xi)| \leq c |\xi|^{-|\alpha|} \quad \text{for all } \xi \neq 0 \text{ and } \alpha \in \mathbb{N}^n, |\alpha| \leq n + 2 \quad (3.5)$$

for a constant  $c > 0$ . Then for  $1 < p < \infty$  the Fourier multiplier operator  $T_m$  from (3.2) is a kernel operator. More precisely, there exist a continuously differentiable kernel  $\mathfrak{K}: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}$  such that (3.3) and

$$|\partial_z^\alpha \mathfrak{K}(z)| \leq C |z|^{-n-|\alpha|} \quad \text{for all } z \neq 0 \text{ and } |\alpha| \leq 1, \quad (3.6)$$

holds. Especially,  $T_m$  extends to a bounded linear operator  $T_m: L^p(\mathbb{R}^n; \mathbb{C}) \rightarrow L^p(\mathbb{R}^n; \mathbb{C})$ .

**Remark 3.3.**

- i) The Mihlin multiplier theorem still holds true in a vector-valued framework for  $|\alpha| \leq \lfloor \frac{n}{2} \rfloor + 1$ , see [BL76, Theorem 6.1.6]. Moreover, (3.5) can be weakened, leading to the Mihlin-Hörmander multiplier theorem, see [Duo01, Corollary 8.11]. For a good overview regarding different versions of multiplier theorems we recommend [Gra21].
- ii) It can be shown that (3.6) implies the Hörmander condition (3.4), see e.g. [Duo01, Proposition 5.2].
- iii) The Mihlin condition (3.5) is satisfied if  $m: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}$  is homogeneous of degree 0, i.e. there holds  $m(\lambda x) = m(x)$  for all  $\lambda > 0$  and  $x \in \mathbb{R}^n \setminus \{0\}$ . In particular,  $m$  is constant along rays emanating from the origin and hence,  $m$  is completely determined by the behaviour of its restriction to the sphere  $\mathbb{S}^{n-1}$ . More precisely, let  $m|_{\mathbb{S}^{n-1}}$  be of class  $C^k$  and notice that the derivatives  $\partial^\alpha m$  are homogeneous of degree  $-|\alpha|$  because of

$$|\partial^\alpha m(x)| = |\lambda^{|\alpha|} (\partial^\alpha m)(\lambda x)| \quad \text{for all } \lambda > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\}.$$

Choosing  $\lambda := |x|^{-1}$  and setting  $c_\alpha := \sup_{|\theta|=1} |\partial^\alpha m(\theta)|$  results in

$$|\partial^\alpha m(x)| = |x|^{-|\alpha|} \left| \partial^\alpha m \left( \frac{x}{|x|} \right) \right| \leq c_\alpha |x|^{-|\alpha|},$$

showing the validity of (3.5). Finally, we notice that in this specific situation the kernel representation can be specified further and refer to [Duo01, Theorem 4.13].

## 3.2 Korn-type Inequalities in $L^p$ -spaces and Ornstein's Non-Inequality

This section is devoted to Korn-type inequalities on the scale of  $L^p$ -spaces, which turn out to hold for  $1 < p < \infty$ . The latter is exactly the range for which it is known that singular integrals as well as Fourier multipliers are bounded. Moreover, we discuss the failure of such an inequality at the endpoints  $p = 1$  and  $p = \infty$  by stating known counterexamples.

The following inequality is essentially a consequence of the methods given in [CZ52]; however, since we were unable to locate a precise reference, we provide a proof based on the Mihlin multiplier theorem.

**Theorem 3.4** (Korn-type-inequality). *Let  $\mathbb{A}$  be an elliptic operator of the form (1.2) and  $1 < p < \infty$ . Then there exists a constant  $c = c(n, p) > 0$  such that*

$$\|\nabla v\|_{L^p(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; V))} \leq c \|\mathbb{A}v\|_{L^p(\mathbb{R}^n; W)} \quad (3.7)$$

for all  $v \in C_c^\infty(\mathbb{R}^n; V)$ .

*Proof.* Let  $v \in C_c^\infty(\mathbb{R}^n; V)$  be arbitrary. Recalling standard properties for the Fourier transform, cf. [SW71], gives

$$\widehat{(\partial_\alpha v)}(\xi) = i \xi_\alpha \widehat{v}(\xi) \quad \text{and} \quad \widehat{(\mathbb{A}v)}(\xi) = \mathbb{A}[\xi](\widehat{v}(\xi))$$

for  $\alpha \in \{1, \dots, n\}$ . Moreover, since  $\mathbb{A}$  is elliptic, the Fourier symbol map  $\mathbb{A}[\xi]: V \rightarrow W$  is injective and therefore, the composition  $\mathbb{A}^*[\xi]\mathbb{A}[\xi]: V \rightarrow V$  is invertible. This allows to rewrite

$$\begin{aligned} \widehat{(\partial_\alpha v)}(\xi) &= i \xi_\alpha \widehat{v}(\xi) = i \xi_\alpha ((\mathbb{A}^*[\xi]\mathbb{A}[\xi])^{-1}(\mathbb{A}^*[\xi]\mathbb{A}[\xi])\widehat{v}(\xi)) \\ &= i \xi_\alpha ((\mathbb{A}^*[\xi]\mathbb{A}[\xi])^{-1}\mathbb{A}^*[\xi]\widehat{(\mathbb{A}v)}(\xi)) \\ &=: i m_\alpha(\xi)\widehat{(\mathbb{A}v)}(\xi), \end{aligned}$$

where  $m_\alpha(\xi) = \xi_\alpha(\mathbb{A}^*[\xi]\mathbb{A}[\xi])^{-1}\mathbb{A}^*[\xi]$ . By means of the Fourier inversion formula we obtain

$$\partial_\alpha v = c_\alpha \mathcal{F}^{-1}(m_\alpha(\xi)\widehat{(\mathbb{A}v)}(\xi)),$$

with a suitable constant  $c_\alpha \in \mathbb{C}$ . Since  $\mathbb{A}[\xi]$  is linear, it follows that  $m_\alpha$  is homogeneous of degree 0, namely for  $\lambda > 0$  we have

$$m_\alpha(\lambda\xi) = \lambda\xi_\alpha(\mathbb{A}^*[\lambda\xi]\mathbb{A}[\lambda\xi])^{-1}\mathbb{A}[\lambda\xi] = \lambda\xi_\alpha\lambda^{-2}(\mathbb{A}^*[\xi]\mathbb{A}[\xi])\lambda\mathbb{A}[\xi] = m_\alpha(\xi).$$

In view of Remark 3.3 iii), we conclude that  $m \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(W; V))$  satisfies assumption (3.5) of the Mihlin multiplier Theorem 3.2. Therefore,  $m_\alpha$  is a  $L^p$ -Fourier multiplier for every  $\alpha \in \{1, \dots, n\}$  and  $1 < p < \infty$ . Finally, summing over all  $\alpha = 1, \dots, n$  results in

$$\|\nabla v\|_{L^p(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; V))} \leq c \|\mathbb{A}v\|_{L^p(\mathbb{R}^n; W)},$$

for a constant  $c = c(n, p) > 0$ . This finishes the proof.  $\square$

The following Korn-type inequality for  $\mathbb{C}$ -elliptic operators on the scale of  $L^p$ -spaces is well-known by results of ARONSZAJN [Aro55], NEČAS [Neč66, HN70] and SMITH [Smi70], cf. also [Res70, Kał94]. For a recent proof including more irregular John domains we refer to [DG24]. However, for the sake of completeness, we give a quick proof on Lipschitz domains based on the Jones-type extension operator from [GR19a] in the context of  $\mathbb{C}$ -elliptic differential operators:

**Theorem 3.5.** *Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded Lipschitz set,  $\mathbb{A}$  be a  $\mathbb{C}$ -elliptic differential operator of the form (1.2) and  $1 < p < \infty$ . Then there exists a constant  $c = c(n, p, \Omega) > 0$  such that for all  $v \in W^{1,p}(\Omega; V)$  there holds*

$$\|v\|_{W^{1,p}(\Omega; V)} \leq c(\|v\|_{L^p(\Omega; V)} + \|\mathbb{A}v\|_{L^p(\Omega; W)}). \quad (3.8)$$

*Proof.* Let  $1 < p < \infty$ . Since  $\mathbb{A}$  is  $\mathbb{C}$ -elliptic, which is, because of (2.7), equivalent to have finite dimensional null-space there exists a bounded linear extension operator

$$\bar{E}_\Omega: W^{\mathbb{A},p}(\Omega) \rightarrow W^{\mathbb{A},p}(\mathbb{R}^n),$$

cf. [GR19a, Theorem 4.1]. Moreover, we take an open and bounded set  $\Omega_0 \subset \mathbb{R}^n$  with  $\Omega \Subset \Omega_0$ , together with a cut-off function  $\varrho \in C_c^\infty(\mathbb{R}^n; [0, 1])$  satisfying  $\mathbf{1}_\Omega \leq \varrho \leq \mathbf{1}_{\Omega_0}$ . To obtain a compactly supported extension operator, we define  $E_\Omega w := \varrho \bar{E}_\Omega w$  for  $w \in W^{\mathbb{A},p}(\Omega)$ , which yields  $E_\Omega: W^{\mathbb{A},p}(\Omega) \rightarrow W_0^{\mathbb{A},p}(\Omega_0)$ , with  $\text{supp}(E_\Omega w) \subset \Omega_0$ . We now take an arbitrary  $v \in C_c^\infty(\mathbb{R}^n; V)$  with  $\text{supp}(v) \subset \Omega$ . Using (3.7) together with the norm boundedness of the extension operator, we infer

$$\begin{aligned} \|\nabla v\|_{L^p(\Omega; \mathcal{L}(\mathbb{R}^n; V))} &= \|\nabla(E_\Omega v)\|_{L^p(\Omega; \mathcal{L}(\mathbb{R}^n; V))} \\ &\leq c \|\nabla(E_\Omega v)\|_{L^p(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; V))} \\ &\leq c \|\mathbb{A}(E_\Omega v)\|_{L^p(\mathbb{R}^n; W)} \\ &\leq c \|E_\Omega v\|_{W^{\mathbb{A},p}(\mathbb{R}^n)} \leq c \|v\|_{W^{\mathbb{A},p}(\Omega)}. \end{aligned}$$

Finally, the previous inequality remains true in  $W^{1,p}(\Omega; V)$  using a standard approximation argument. Therefore, (3.8) follows and the proof is finished.  $\square$

Since many parts of this thesis are dealing with the symmetric gradient, we explicitly state the classical Korn inequality in this framework:

**Corollary 3.6** (Korn inequality in  $L^p$ ). *Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded Lipschitz set and  $1 < p < \infty$ . Then there exists a constant  $c = c(n, p) > 0$  such that*

$$\|\nabla v\|_{L^p(\mathbb{R}^n; \mathbb{R}^{n \times n})} \leq c \|\varepsilon(v)\|_{L^p(\mathbb{R}^n; \mathbb{R}_{\text{sym}}^{n \times n})},$$

for all  $v \in W_0^{1,p}(\Omega; \mathbb{R}^n)$ . Moreover, for any  $v \in W^{1,p}(\Omega; \mathbb{R}^n)$  there exists a constant  $c = c(n, p, \Omega) > 0$  such that

$$\|v\|_{W^{1,p}(\Omega; \mathbb{R}^n)} \leq c(\|v\|_{L^p(\Omega; \mathbb{R}^n)} + \|\varepsilon(v)\|_{L^p(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})}).$$

*Proof.* Choose  $\mathbb{A} = \varepsilon$ ,  $V = \mathbb{R}^n$  and  $W = \mathbb{R}_{\text{sym}}^{n \times n}$  in the previous theorems.  $\square$

The failure of  $L^1$ -estimates was demonstrated by ORNSTEIN, not just by giving a specific counterexample, even by proving the impossibility of non-trivial estimates. This underscores the necessity for alternative approaches in low-regularity settings. The original statement reads as follows:

**Theorem 3.7** (Ornstein Non–Inequality, [Orn62, Theorem 1]). *Let  $\{\mathbb{B}, \mathbb{L}_1, \dots, \mathbb{L}_\ell\}$  be a family of linearly independent linear, homogeneous differential operators in  $n$ -variables of order  $k \in \mathbb{N}$ . Then for any  $K > 0$  there exists  $f \in C_c^\infty(\mathbb{R}^n)$  with  $\text{supp}(f) \subset (0, 1)^n$  such that*

$$\int_{\mathbb{R}^n} |\mathbb{B}f| \, dx \geq K \quad \text{and} \quad \int_{\mathbb{R}^n} |\mathbb{L}_j f| \, dx < 1$$

holds for all  $j \in \{1, \dots, \ell\}$ .

Nowadays, various proofs and generalisations are available, which we aim to discuss. We will begin by referencing [CFM05] for an elegant proof in the case of the symmetric gradient using laminates. Moreover, [FG22] provides a concise proof of ORNSTEIN’S result in two dimensions. KIRCHHEIM and KRISTENSEN derived a more modern proof resulting in a more general statement in [KK11, KK16], allowing for an additional  $x$ -dependence.

**Theorem 3.8** (Generalised Ornstein Non–Inequality [KK16, Theorem 1.3]). *Let  $V, W_1, W_2$  be three finite dimensional inner product spaces and consider two  $k$ -th order linear partial differential operators with locally integrable coefficients  $a_\alpha^{(i)} \in L_{\text{loc}}^1(\mathbb{R}^n; \mathcal{L}(V, W_i))$ ,  $i = 1, 2$  defined by*

$$A_i(x, D)\varphi := \sum_{|\alpha|=k} a_\alpha^{(i)}(x) \partial^\alpha \varphi$$

for  $\varphi \in C_c^\infty(\mathbb{R}^n; V)$ . Then there exists a constant  $c > 0$  such that

$$\|A_2(x, D)\varphi\|_{L^1(\mathbb{R}^n; W)} \leq c \|A_1(x, D)\varphi\|_{L^1(\mathbb{R}^n; W)} \quad \forall \varphi \in C_c^\infty(\Omega; V)$$

if and only if there exists  $T \in L^\infty(\mathbb{R}^n; \mathcal{L}(W_1, W_2))$  with  $\|T\|_{L^\infty(\mathbb{R}^n; \mathcal{L}(W_1, W_2))} \leq c$  such that

$$a_\alpha^{(2)}(x) = T(x) a_\alpha^{(1)}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n,$$

and all multi-indices  $\alpha \in \mathbb{N}^n$  with  $|\alpha| = k$ .

We also want to mention [KSW17] for an anisotropic version of the above result. In summary, we emphasise that  $L^p$ -estimates for differential operators derived via Calderón–Zygmund estimates and Fourier multipliers can be obtained for all  $1 < p < \infty$  and even for the weak  $L^1$ -space. However, they fail for the limit case  $p = 1$  except for trivial cases. We close this section by pointing out that the other limit case  $p = \infty$  also fails due to the results in [Mit58] and [dLM62].

### 3.3 Korn-type Inequalities in Orlicz Spaces

In this section we investigate Korn-type inequalities on the scale of Orlicz spaces. We begin by mentioning several known results. In [BD12] BREIT and DIENING prove that an inequality of the form

$$\|\nabla v\|_{L^\Phi(\Omega; \mathbb{R}^{n \times n})} \lesssim \|\varepsilon(v)\|_{L^\Phi(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})} \quad (3.9)$$

can only hold for all  $v \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$  if and only if  $\Phi \in \nabla_2 \cap \Delta_2$ . The same inequality holds true if the symmetric gradient is replaced by a more general elliptic operator as shown by CONTI and GMEINER in [CG22]. We point that the result is somehow natural as  $\Phi \in \Delta_2$  and  $\Phi \in \nabla_2$  loosely speaking mean that the norm  $\|\cdot\|_{L^\Phi}$  is not *too close* to  $\|\cdot\|_{L^\infty}$  and  $\|\cdot\|_{L^1}$  respectively, for which the Korn inequality fails. Moreover, it is possible to weaken the assumption  $\Phi \in \nabla_2 \cap \Delta_2$  by replacing the norm on the right-hand side of (3.9) by a slightly weaker Orlicz norm due to results of CIANCHI [Cia14] for the symmetric gradient. The following statement is a generalisation of [Cia14, Theorem 3.1] and can be found in [Ste26, Theorem 5.1].

**Theorem 3.9** (Korn-type inequality in Orlicz spaces). *Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded set and  $\mathbb{A}$  be an elliptic differential operator of the form (1.2). Moreover, let  $\Psi$  and  $\Phi$  be Young functions such that*

$$t \int_0^t \frac{\Phi(s)}{s^2} ds \leq \Psi(ct) \quad \text{and} \quad t \int_0^t \frac{\Psi^*(s)}{s^2} ds \leq \Phi^*(ct), \quad (3.10)$$

hold for all  $t \geq 0$  and some  $c > 0$ . Then there exist a constant  $C > 0$  such that

$$\int_\Omega \Phi(|\nabla v|) dx \lesssim \int_\Omega \Psi(C |\mathbb{A}v|) dx \quad \text{for all } v \in W_0^{1,\Psi}(\Omega; V). \quad (3.11)$$

We observe that, as a consequence of [Cia14, Proposition 3.5], the conditions in (3.10) imply that  $\Psi$  dominates  $\Phi$  globally, i.e. there exists a constant  $c > 0$  such that

$$\Psi(t) \leq \Phi(ct) \quad \text{holds for all } t \geq 0.$$

Moreover, as a matter of fact, modular estimates in Orlicz spaces are always stronger than Luxemburg norm one. Therefore, in view of (3.11), we obtain for  $v \in W_0^{1,\Psi}(\Omega; V)$  the estimate

$$\int_\Omega \Phi \left( \frac{|\nabla v|}{C \|\mathbb{A}v\|_{L^\Psi(\Omega; W)}} \right) dx \lesssim \int_\Omega \Psi \left( \frac{|\mathbb{A}v|}{\|\mathbb{A}v\|_{L^\Psi(\Omega; W)}} \right) dx \leq 1$$

and hence, by the definition of the Luxemburg norm, also

$$\|\nabla v\|_{L^\Phi(\Omega; \mathcal{L}(\mathbb{R}^n; V))} \leq C \|\mathbb{A}v\|_{L^\Psi(\Omega; W)}. \quad (3.12)$$

We now aim to prove a Korn-type inequality for balls on the scale of Orlicz spaces. To this end, we first need to derive a Poincaré-type inequality in Orlicz spaces, which is the content of the following theorem:

**Theorem 3.10** (Poincaré-type inequality). *Let  $r > 0$ ,  $x_0 \in \mathbb{R}^n$  and  $B_r(x_0) \subset \mathbb{R}^n$  be a ball,  $\mathbb{A}$  be a  $\mathbb{C}$ -elliptic differential operator of the form (1.2) and  $\Psi$  be a Young function. Then for every  $v \in W^{\mathbb{A},\Psi}(B_r(x_0))$  there exists a constant  $c = c(B_r(x_0), \mathbb{A}) > 0$  such that*

$$\inf_{\mathbf{p} \in \ker(\mathbb{A})} \|v - \mathbf{p}\|_{L^\Psi(B_r(x_0); V)} \leq c \omega_n r \|\mathbb{A}v\|_{L^\Psi(B_r(x_0); W)}. \quad (3.13)$$

*Proof.* For the sake of simplicity we abbreviate  $B := B_r(x_0)$ . The result is an immediate consequence of [DG24, Proposition 3.8.]. Namely, there exists a constant  $c = c(\mathbb{A}, B) > 0$  such that for  $v \in W^{\mathbb{A},1}(B)$  there holds the pointwise estimate

$$|(v - \Pi_{\mathbb{A}}^B v)(x)| \leq c \int_B |x - y|^{1-n} |\mathbb{A}v(y)| dy \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in B, \quad (3.14)$$

where  $\Pi_{\mathbb{A}}^{\mathbb{B}}$  denotes the projection onto the kernel of  $\mathbb{A}$  as constructed in [DG24], cf. also Remark 4.5. As a next step, we define for  $x \in \mathbb{B}$  a Borel measure  $\mu_x: \mathcal{B}(\mathbb{B}) \rightarrow [0, \infty)$  via

$$\mu_x(A) := c \int_{\mathbb{B}} |x - y|^{1-n} dy \quad \text{for } A \in \mathcal{B}(\mathbb{B}),$$

and set  $m_x := \mu_x(\mathbb{B})$ . Now, since  $|x - y| \leq \text{diam}(\mathbb{B}) = 2r$  for all  $x, y \in \mathbb{B}$ , we can estimate

$$m_x = c \int_{\mathbb{B}} \frac{dy}{|x - y|^{n-1}} \geq c \frac{\mathcal{L}^n(\mathbb{B})}{(\text{diam}(\mathbb{B}))^{n-1}} = c \omega_n r \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega. \quad (3.15)$$

Dividing (3.14) by  $m_x$  and  $\lambda := \|\mathbb{A}v\|_{L^\Psi(\mathbb{B}; W)}$  and applying the Jensen inequality, we obtain for  $\mathcal{L}^n$ -a.e.  $x \in \Omega$

$$\Psi \left( \frac{|(v - \Pi_{\mathbb{A}}^{\mathbb{B}}v)(x)|}{m_x \lambda} \right) \leq \Psi \left( \int_{\mathbb{B}} \frac{|\mathbb{A}v(y)|}{\lambda} d\mu_x(y) \right) \leq \int_{\mathbb{B}} \Psi \left( \frac{|\mathbb{A}v(y)|}{\lambda} \right) d\mu_x(y)$$

Therefore, integrating the above inequality over  $\mathbb{B}$  with respect to the  $x$ -variable, in conjunction with the Tonelli theorem, results in

$$\begin{aligned} \int_{\mathbb{B}} \Psi \left( \frac{|(v - \Pi_{\mathbb{A}}^{\mathbb{B}}v)(x)|}{m_x \lambda} \right) dx &\leq \int_{\mathbb{B}} \int_{\mathbb{B}} \Psi \left( \frac{|\mathbb{A}v(y)|}{\lambda} \right) d\mu_x(y) dx \\ &= \int_{\mathbb{B}} \Psi \left( \frac{|\mathbb{A}v(y)|}{\lambda} \right) \int_{\mathbb{B}} \frac{c}{|x - y|^{n-1}} \frac{1}{m_x} dy dx \\ &= \int_{\mathbb{B}} \Psi \left( \frac{|\mathbb{A}v(y)|}{\|\mathbb{A}v\|_{L^\Psi(\mathbb{B}; W)}} \right) dy \\ &\leq 1. \end{aligned}$$

The last inequality follows from the fact that  $u \in W^{\mathbb{A}, \Psi}(\mathbb{B})$ . Using the lower bound (3.15) together with the definition of the Luxemburg norm gives

$$\|v - \Pi_{\mathbb{A}}^{\mathbb{B}}v\|_{L^\Psi(\mathbb{B}; V)} \leq c \omega_n r \|\mathbb{A}v\|_{L^\Psi(\mathbb{B}; W)}, \quad (3.16)$$

where  $c = c(\mathbb{A}, \mathbb{B}) > 0$ . This leads to the desired inequality (3.13), since we can always bound the left hand side from below by the infimum taken over all  $\mathfrak{p} \in \ker(\mathbb{A})$ . This finishes the proof.  $\square$

As a side remark, we refer to [GR19a, Proposition 4.2] for Poincaré-type inequalities in  $L^p$ -spaces on star-shaped domains, and to [DG24, Proposition 3.7] for those on John domains. We proceed by giving the proof of the already announced Korn-type inequality in Orlicz spaces.

**Theorem 3.11** (Korn-type inequality). *Let  $r > 0$ ,  $x_0 \in \mathbb{R}^n$  and  $B_r(x_0) \subset \mathbb{R}^n$  be a ball. Moreover, let  $\Psi$  and  $\Phi$  be Young functions such that (3.10) is satisfied. Then for each  $\beta > 0$  there exists a constant  $c = c(\Phi, \Psi, B_r(x_0), \mathbb{A}) > 0$  such that*

$$\inf_{\mathfrak{p} \in \ker(\mathbb{A})} \|\nabla(v - \mathfrak{p})\|_{L^\Phi(B_r(x_0); \mathcal{L}(\mathbb{R}^n; V))} \leq c \left(1 + \frac{1}{r}\right) \|\mathbb{A}v\|_{L^\Psi(B_{2r}(x_0); W)} \quad (3.17)$$

for all  $v \in W^{\mathbb{A}, \Psi}(B_{2r}(x_0))$ , where  $\mathbb{A}$  is a  $\mathbb{C}$ -elliptic differential operator of the form (1.2).

*Proof.* Let  $\varrho \in C_c^\infty(\mathbb{R}^n; [0, 1])$  be a cut-off function satisfying  $\mathbf{1}_{B_r(x_0)} \leq \varrho \leq \mathbf{1}_{B_{2r}(x_0)}$  and  $|\nabla \varrho| \leq \frac{2}{r}$ . Moreover, we recall from (3.16) that the Poincaré-type inequality

$$\|v - \Pi_{\mathbb{A}}^{\mathbb{B}} v\|_{L^\Psi(B_{2r}(x_0); V)} \leq c \omega_n r \|\mathbb{A}v\|_{L^\Psi(B_{2r}(x_0); W)} \quad (3.18)$$

holds for a constant  $c = c(B_r(x_0), \mathbb{A}) > 0$ , where we have abbreviated  $\Pi_{\mathbb{A}}^{\mathbb{B}} := \Pi_{\mathbb{A}}^{\mathbb{B}_{2r}(x_0)}$  for the projection onto the kernel. Therefore, taking into account (3.12) on  $\Omega = B_{5r}(x_0)$  applied for the function  $\varrho(v - \Pi_{\mathbb{A}}^{\mathbb{B}} v) \in W_0^{1, \Psi}(B_{5r}(x_0); V)$  in conjunction with (3.18), leads to

$$\begin{aligned} & \|\nabla(v - \Pi_{\mathbb{A}}^{\mathbb{B}} v)\|_{L^\Phi(B_r(x_0); \mathcal{L}(\mathbb{R}^n; V))} \\ & \leq \|\nabla(\varrho(v - \Pi_{\mathbb{A}}^{\mathbb{B}} v))\|_{L^\Phi(B_{5r}(x_0); \mathcal{L}(\mathbb{R}^n; V))} \\ & \leq c \|\mathbb{A}(\varrho(v - \Pi_{\mathbb{A}}^{\mathbb{B}} v))\|_{L^\Psi(B_{5r}(x_0); W)} \\ & \leq c \left( \|\varrho \mathbb{A}(v - \Pi_{\mathbb{A}}^{\mathbb{B}} v)\|_{L^\Psi(B_{5r}(x_0); W)} + \|(v - \Pi_{\mathbb{A}}^{\mathbb{B}} v) \otimes_{\mathbb{A}} \nabla \varrho\|_{L^\Psi(B_{5r}(x_0); W)} \right) \\ & \leq c \left( \|\mathbb{A}(v - \Pi_{\mathbb{A}}^{\mathbb{B}} v)\|_{L^\Psi(B_{2r}(x_0); W)} + \|\nabla \varrho\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \|v - \Pi_{\mathbb{A}}^{\mathbb{B}} v\|_{L^\Psi(B_{5r}(x_0); W)} \right) \\ & \leq c \left( \|\mathbb{A}v\|_{L^\Psi(B_{2r}(x_0); W)} + \frac{1}{r} \|v - \Pi_{\mathbb{A}}^{\mathbb{B}} v\|_{L^\Psi(B_{2r}(x_0); V)} \right) \\ & \leq c \left( 1 + \frac{1}{r} \right) \|\mathbb{A}v\|_{L^\Psi(B_{2r}(x_0); W)}. \end{aligned}$$

This yields the desired inequality (3.17), as we can always bound the left-hand side from below by the infimum taken over all  $\mathfrak{p} \in \ker(\mathbb{A})$ , finishing the proof.  $\square$

As an immediate consequence, cf. also [Cia14, Example 3.11], we observe

**Corollary 3.12.** *Let  $r > 0$ ,  $x_0 \in \mathbb{R}^n$  and  $B_r(x_0) \subset \mathbb{R}^n$  be a ball. Then for all  $\beta > 0$  there exists a constant  $c = c(\beta, B_r(x_0), \mathbb{A}) > 0$  such that*

$$\inf_{\mathfrak{p} \in \ker(\mathbb{A})} \|\nabla(v - \mathfrak{p})\|_{\exp L^{\frac{\beta}{\beta+1}}(B_r(x_0); \mathcal{L}(\mathbb{R}^n; V))} \leq c \left( 1 + \frac{1}{r} \right) \|\mathbb{A}v\|_{\exp L^\beta(B_{2r}(x_0); W)} \quad (3.19)$$

for all  $v \in W^{\mathbb{A}, \exp L^\beta}(B_{2r}(x_0))$ , where  $\mathbb{A}$  is a  $\mathbb{C}$ -elliptic operator of the form (1.2).

**Remark 3.13** (Korn-Type inequality on Lipschitz domains). *It is possible to establish a version of (3.17) without increasing the domains of integration on the right-hand side, i.e. for a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^n$  there holds*

$$\inf_{\mathfrak{p} \in \ker(\mathbb{A})} \|v - \mathfrak{p}\|_{L^\Psi(\Omega; V)} \leq c \|\mathbb{A}v\|_{L^\Psi(\Omega; W)} \quad (3.20)$$

if  $\Phi$  and  $\Psi$  are Young functions satisfying (3.10). To this end, there are several strategies: One can either copy the arguments from [Cia14, Theorem 3.3] in the more general framework of  $\mathbb{C}$ -elliptic operators or construct an extension operator in Orlicz-Sobolev spaces. For the latter one can use abstract interpolation theory since the extension operator constructed in [GR19a, Section 4.1] is bounded for  $p = 1$  and  $p = \infty$ . Since we do not need (3.20) in this generality, we will not go into further detail. Finally, we refer to [EL25, Remark 3.14], which provides a step-by-step outline for proving the inequality in the case of  $\mathbb{C}$ -elliptic operators in two dimensions induced by a projection.

Now we recall the abbreviations  $\Phi_\alpha(t) := t \log^\alpha(1 + t^2)$  for  $t \geq 0$  and  $\alpha > 0$  and set  $\mathcal{K} := \ker(\varepsilon)$ . We conclude this section with a Korn-type inequality in logarithmic Orlicz spaces, which falls beyond the scope of the above theorems since  $\Phi_\alpha \in \Delta_2 \setminus \nabla_2$ . Moreover, (3.10) is not satisfied for  $\Phi_\alpha$  and  $\Phi_{\alpha-1}$  and therefore has to be refined. We refer to [Cia14] for further details, noting that the subsequent inequality is essentially a combination of results therein.

**Lemma 3.14** (Scaled logarithmic Korn-type inequality). *Let  $x_0 \in \mathbb{R}^n$  and  $r > 0$ . For every  $\alpha \geq 1$  there exists a constant  $c = c(n, \alpha)$  such that for every  $v \in E^{1, \Phi_\alpha}(B_r(x_0))$  there holds  $v \in W^{1, \Phi_\alpha}(B_r(x_0); \mathbb{R}^n)$  with*

$$\int_{B_r(x_0)} \Phi_{\alpha-1}(|\nabla v|) \, dx \leq c \left( 1 + \int_{B_r(x_0)} \Phi_{\alpha-1} \left( \frac{|v|}{r} \right) \, dx + \int_{B_r(x_0)} \Phi_\alpha(|\varepsilon(v)|) \, dx \right). \quad (3.21)$$

*Proof.* By scaling, there is no loss of generality to  $x_0 = 0$  and  $r = 1$ . From [Cia14, Theorem 3.13, Example 3.15], we conclude for all  $u \in E^{1, \Phi_\alpha}(B_1(0))$  the inequality

$$\inf_{\mathfrak{p} \in \mathcal{K}} \int_{B_1(0)} \Phi_{\alpha-1}(|\nabla(v - \mathfrak{p})|) \, dx \leq c \left( 1 + \int_{B_1(0)} \Phi_\alpha(|\varepsilon(u)|) \, dx \right), \quad (3.22)$$

with a constant  $c = c(n, \alpha) > 0$ . In the following, let  $\mathbb{P}$  be the projection on  $B_1(0)$  onto the finite-dimensional space  $\mathcal{K}$ , cf. (2.11). Since all norms are equivalent on finite dimensional spaces, there exists a constant such that

$$\sup_{x \in B_1(0)} |\nabla \mathbb{P}w(x)| \leq c \int_{B_1(0)} |w| \, dx,$$

and consequently, using the monotonicity of  $\Phi_{\alpha-1}$ , also

$$\int_{B_1(0)} \Phi_{\alpha-1}(|\nabla \mathbb{P}w|) \, dx \leq c \Phi_{\alpha-1} \left( \int_{B_1(0)} |w| \, dx \right) \quad (3.23)$$

for all  $w \in L^1(B_1(0); \mathbb{R}^n)$ . As  $\Phi_{\alpha-1}$  is convex and doubling, i.e.  $\Phi_{\alpha-1} \in \Delta_2$ , in the first two steps, we have for any  $\mathfrak{p} \in \mathcal{K}$  with  $(u - \mathfrak{p})_{B_1(0)} = 0$  the chain of inequalities

$$\begin{aligned} & \int_{B_1(0)} \Phi_{\alpha-1}(|\nabla v|) \, dx \\ & \leq c \int_{B_1(0)} \Phi_{\alpha-1}(|\nabla(v - \mathfrak{p})|) \, dx + c \int_{B_1(0)} \Phi_{\alpha-1}(|\nabla(\mathbb{P}(v - \mathfrak{p}))|) \, dx \\ & \quad + c \int_{B_1(0)} \Phi_{\alpha-1}(|\nabla \mathbb{P}v|) \, dx \\ & \stackrel{(3.23)}{\leq} c \int_{B_1(0)} \Phi_{\alpha-1}(|\nabla(v - \mathfrak{p})|) \, dx + c \Phi_{\alpha-1} \left( \int_{B_1(0)} |v - \mathfrak{p}| \, dx \right) \\ & \quad + c \Phi_{\alpha-1} \left( \int_{B_1(0)} |v| \, dx \right) \\ & \leq c \int_{B_1(0)} \Phi_{\alpha-1}(|\nabla(v - \mathfrak{p})|) \, dx + c \left( \int_{B_1(0)} \Phi_{\alpha-1}(|v|) \, dx \right), \end{aligned}$$

where the last inequality is true because of the classical Poincare-inequality and the convexity of  $\Phi_{\alpha-1}$ . Moreover, we observe that it does not matter whether we take the infimum in (3.22) over all  $\mathbf{p} \in \mathcal{K}$  or over all  $\mathbf{p} \in \mathcal{K}$  with  $(v - \mathbf{p})_{B_1(0)} = 0$ . Therefore, taking the infimum over all  $\mathbf{p} \in \mathcal{K}$  with  $(v - \mathbf{p})_{B_1(0)} = 0$  in the above chain of inequalities yield the desired inequality (3.21) for  $x_0 = 0$  and  $r = 1$ . Finally, since every function  $v \in L^{\Phi_\alpha}(B_1(0); \mathbb{R}^n)$  automatically belongs to  $L^{\Phi_{\alpha-1}}(B_1(0); \mathbb{R}^n)$  the proof is finished.  $\square$

## 4 | General Ekeland-type Approximation

In this section, we present a generalised approximation strategy based on the Ekeland variational principle, which allows us to account for an additional constraint. As already mentioned in the introduction, this construction is crucial for overcoming the potential non-uniqueness for  $BV^{\mathbb{A}}$ -minimisers. It enables us to construct a suitable minimising sequence staying close to a given  $BV^{\mathbb{A}}$ -minimiser, which satisfies this constraint. This procedure will form the basis for deriving regularity results for all  $BV^{\mathbb{A}}$ -minimisers in the subsequent chapters.

We restrict ourselves to the case  $V = \mathbb{R}^N$ ,  $N \geq 2$  and  $W = \mathcal{R}(\mathbb{A}) \subseteq \mathbb{R}^{N \times n}$ , turning out to be sufficient for our purposes. As already mentioned before, this can be done without loss of generality and we refer to Remark 2.5 for further details and references.

In the following,  $L^p(\Omega; \mathbb{R}^N)$  with  $p \in [1, \infty]$  will play the role of the constraint space, whereas we employ a regularising procedure in  $W^{1,q}(\Omega; \mathbb{R}^N)$  for  $q \geq 2$ . Finally,  $k \in \mathbb{N}$  will be the order of the negative Sobolev space  $W^{-k,1}(\Omega; \mathbb{R}^N)$  in which we aim to apply the Ekeland variational principle.

*Structure of the chapter:* In Section 4.1 we construct a regular minimising sequence of  $F$  subject to smooth boundary, associated with a fixed  $BV^{\mathbb{A}}$ -minimiser  $u$ . Subsequent, in Section 4.2 we derive initial (non-uniform) estimates for this sequence as well as an Euler-Lagrange inequality and a differentiated version.

### 4.1 The Ekeland-type Viscosity Approximation

At the beginning, we recall the general framework: We consider the variational principle

$$\text{to minimise } F[v; \Omega] = \int_{\Omega} f(\mathbb{A}v) \, dx$$

among the Dirichlet class  $\mathcal{D}_{u_0} = u_0 + W_0^{\mathbb{A},1}(\Omega)$ . In our framework,  $f: W \rightarrow \mathbb{R}$  is a convex integrand of linear growth (1.3) and  $\mathbb{A}$  is a  $\mathbb{C}$ -elliptic differential operator of the form (1.2). The main objective of this section is to construct a suitable minimising sequence of  $F$  which stays close to a given  $BV^{\mathbb{A}}$ -minimiser subject to an additional  $L^p$ -constraint. We observe that whenever  $u \in BV_{\text{loc}}^{\mathbb{A}}(\Omega)$  is a local  $BV^{\mathbb{A}}$ -minimiser of  $F$ , the restriction  $u|_U \in BV^{\mathbb{A}}(U)$  to a relatively compact set  $U \Subset \Omega$  with Lipschitz boundary  $\partial U$ , is a  $BV^{\mathbb{A}}$ -minimiser with respect to its own boundary values. Therefore, it is not too restrictive to assume

$$u = u_0 \in BV^{\mathbb{A}}(\Omega) \cap L^p(\Omega; \mathbb{R}^N) \quad \text{with} \quad m := \|u\|_{L^p(\Omega; \mathbb{R}^N)}, \quad (4.1)$$

for some fixed  $p \in [1, \infty]$ . We now begin the realisation of the vanishing viscosity method based on the Ekeland variational principle. To simplify for the reader, we split the construction into three steps. To keep track of the  $L^p$ -constraint within the approximation

scheme, it is necessary to deploy a very careful approximation procedure displayed in Step 1. Subsequent in Step 2, we introduce suitable stabilised and penalised integrands to define new functionals  $F_j$ . In the last Step 3, we apply the Ekeland variational principle to  $F_j$  and draw certain conclusions for the outcoming sequence of almost minimisers.

*Step 1: Construction of a regular minimising sequence.* Let  $p \in [1, \infty]$  and  $u$  be an arbitrary fixed  $L^p$ -bounded  $BV^{\mathbb{A}}$ -minimiser of  $F$ . In view of Lemma 2.4, we obtain a sequence  $(u_j)_{j \in \mathbb{N}}$  in  $(u_0 + C_c^\infty(\Omega; \mathbb{R}^N)) \cap L^p(\Omega; \mathbb{R}^N)$  satisfying

$$u_j \rightarrow u \quad \text{in the } \mathbb{A}\text{-area-strict topology of } BV^{\mathbb{A}}(\Omega), \quad (4.2)$$

as  $j \rightarrow \infty$ . For future reference, we record the estimates

$$\|u_j - u\|_{L^p(\Omega; \mathbb{R}^N)} \leq (C_A + 1)m, \quad (4.3)$$

where  $C_A = C_A(\Omega, n) > 0$ , cf. Lemma 2.4, and  $m$  is specified in (4.1). Moreover, from Corollary 2.15 we know that  $\bar{F}_{u_0}[-; \Omega]$  is continuous with respect to the  $\mathbb{A}$ -area-strict topology and therefore using the consistency relation (1.4), we obtain

$$\lim_{j \rightarrow \infty} F[u_j; \Omega] = \lim_{j \rightarrow \infty} \bar{F}_{u_0}[u_j; \Omega] = \bar{F}_{u_0}[u; \Omega] = \min_{BV^{\mathbb{A}}(\Omega)} \bar{F}_{u_0}[-; \Omega] = \inf_{\mathcal{D}_{u_0}} F[-; \Omega].$$

This means that the sequence  $(u_j)_{j \in \mathbb{N}}$  is a minimising sequence for  $F[-; \Omega]$  in the Dirichlet class  $\mathcal{D}_{u_0}$ . If we extract a non-relabelled subsequence if necessary, we can ensure

$$F[u_j; \Omega] \leq \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{1}{8j^2} \quad \text{for all } j \in \mathbb{N}. \quad (4.4)$$

As a next step, we want to pass over to another sequence, which enjoys the same properties as  $(u_j)_{j \in \mathbb{N}}$  but has more regular boundary-values. Towards this aim, we recall for an open and bounded domain  $\Omega_0 \subset \mathbb{R}^n$  with  $\Omega \Subset \Omega_0$  the extension operator  $\mathfrak{J}: W^{\mathbb{A},1}(\Omega) \rightarrow W_0^{\mathbb{A},1}(\Omega_0)$  from Theorem 2.10. It preserves  $L^p$ -bounds in the sense that

$$\mathfrak{J}: W^{\mathbb{A},1}(\Omega) \cap L^p(\Omega; \mathbb{R}^N) \rightarrow L^p(\Omega_0; \mathbb{R}^N)$$

is a bounded linear operator with respect to the  $L^p$ -norm, and its operator norm is

$$\|\mathfrak{J}\| := \sup\{\|\mathfrak{J}v\|_{L^p(\Omega_0; \mathbb{R}^N)} : v \in W^{\mathbb{A},1}(\Omega) \cap L^p(\Omega; \mathbb{R}^N)\}.$$

We proceed by defining  $\bar{u} := \mathfrak{J}u_0$  and consider for  $j \in \mathbb{N}$  the mollification  $u_j^{\partial\Omega} := (\varrho_{\varepsilon_j} * \bar{u}_0)|_{\Omega}$  with a scaled standard mollifier  $\varrho_{\varepsilon_j}$ , such that

$$\|u_j^{\partial\Omega} - u_0\|_{W^{\mathbb{A},1}(\Omega)} \leq \frac{1}{8\text{Lip}(f)j^2}, \quad (4.5)$$

$$\|u_j^{\partial\Omega}\|_{L^p(\Omega; \mathbb{R}^N)} \leq \|\mathfrak{J}\| \|u_0\|_{L^p(\Omega; \mathbb{R}^N)} = \|\mathfrak{J}\|m. \quad (4.6)$$

Having constructed a sequence  $(u_j^{\partial\Omega})_{j \in \mathbb{N}}$  in  $W^{1,q}(\Omega; \mathbb{R}^N) \cap L^p(\Omega; \mathbb{R}^N)$ , we can introduce approximate Dirichlet classes via

$$\mathcal{D}_j := u_j^{\partial\Omega} + W_0^{\mathbb{A},1}(\Omega).$$

Now, setting  $\tilde{u}_j := u_j - u_0 + u_j^{\partial\Omega} \in \mathcal{D}_{u_0}$ , we then conclude from (4.5) that

$$\|\tilde{u}_j - u_j\|_{W^{\mathbb{A},1}(\Omega)} = \|u_j^{\partial\Omega} - u_0\|_{W^{\mathbb{A},1}(\Omega)} \leq \frac{1}{8\text{Lip}(f)j^2}. \quad (4.7)$$

Moreover, from (4.1), (4.3) and (4.6) we infer that the  $L^p$ -bounds are preserved when passing to the sequence  $(\tilde{u}_j)_{j \in \mathbb{N}}$ . Namely, for all  $j \in \mathbb{N}$  there holds

$$\|\tilde{u}_j\|_{L^p(\Omega; \mathbb{R}^N)} \leq \|u_j - u_0\|_{L^p(\Omega; \mathbb{R}^N)} + \|u_j^{\partial\Omega}\|_{L^p(\Omega; \mathbb{R}^N)} \leq (1 + C_A + \|\mathfrak{J}\|)m =: M. \quad (4.8)$$

As a last step, we show that the infimum of  $F[-; \Omega]$  over  $\mathcal{D}_{u_0}$  can be approximated by the corresponding infima taken over  $\mathcal{D}_j$ , which in turn is almost attained for  $\tilde{u}_j$ . To this end, we notice that for all  $v, w \in W^{\mathbb{A},1}(\Omega)$  we have

$$|F[v; \Omega] - F[w; \Omega]| \leq \text{Lip}(f) \|\mathbb{A}v - \mathbb{A}w\|_{L^1(\Omega; \mathcal{R}(\mathbb{A}))},$$

and thus, we obtain

$$\begin{aligned} F[u_0 + \varphi; \Omega] &\leq F[u_j^{\partial\Omega} + \varphi; \Omega] + \text{Lip}(f) \|\mathbb{A}u_0 - \mathbb{A}u_j^{\partial\Omega}\|_{L^1(\Omega; \mathcal{R}(\mathbb{A}))} \\ F[u_j^{\partial\Omega} + \varphi; \Omega] &\leq F[u_0 + \varphi; \Omega] + \text{Lip}(f) \|\mathbb{A}u_0 - \mathbb{A}u_j^{\partial\Omega}\|_{L^1(\Omega; \mathcal{R}(\mathbb{A}))} \end{aligned}$$

for every perturbation  $\varphi \in W^{\mathbb{A},1}(\Omega)$ . Hence, using (4.7), we can infimise on the right-hand side over all  $\varphi \in W^{\mathbb{A},1}(\Omega)$  to infer the approximation property of the infima, namely

$$\left| \inf_{\mathcal{D}_{u_0}} F[-; \Omega] - \inf_{\mathcal{D}_j} F[-; \Omega] \right| \leq \frac{1}{8j^2}. \quad (4.9)$$

Using the specific choice  $\varphi = u_j - u_0$  in combination with (4.4), we see that the infimum of  $F[-; \Omega]$  on  $\mathcal{D}_{u_0}$  is almost attained by  $\tilde{u}_j$ , meaning

$$F[\tilde{u}_j; \Omega] \leq F[u_j; \Omega] + \frac{1}{8j^2} \leq \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{1}{4j^2}, \quad (4.10)$$

for all  $j \in \mathbb{N}$ . In conclusion, we have constructed a minimising sequence  $(\tilde{u}_j)_{j \in \mathbb{N}}$  that retains the essential properties: smoothness in  $\Omega$ , uniform  $L^p$ -bounds, and convergence to  $u$  in the  $\mathbb{A}$ -area-strict topology, while additionally possessing more regular boundary values.

*Step 2. Definition of regularised and penalised functionals.* In this section we will introduce suitable stabilised and penalised variational integrals defined in terms of  $\tilde{u}_j$ . To the latter we want to apply the Ekeland variational principle from Proposition 2.16 at  $\tilde{u}_j$ . Towards this aim, we define for  $j \in \mathbb{N}$  and  $q > 1$  the regularised and penalised functionals by

$$F_j[v; \Omega] := \begin{cases} \int_{\Omega} \mathfrak{f}_{j,q}(\mathbb{A}u) \, dx + \Upsilon \left( \int_{\Omega} \mathfrak{g}(u) \, dx \right) & \text{if } u \in \mathcal{D}_j \\ +\infty & \text{if } u \in W^{-k,1}(\Omega; \mathbb{R}^N) \setminus \mathcal{D}_j. \end{cases}$$

Here,  $\mathfrak{f}_{j,q}: \mathcal{R}(\mathbb{A}) \rightarrow \mathbb{R}$  denotes the regularised integrand given by

$$\mathfrak{f}_{j,q}(z) := f(z) + \frac{1}{2A_{j,q}j^2} (1 + |z|^2)^{\frac{q}{2}} \quad \text{with} \quad A_{j,q} := 1 + \int_{\Omega} (1 + |\mathbb{A}\tilde{u}_j|^2)^{\frac{q}{2}} \, dx, \quad (4.11)$$

whereas for the second term, which plays the role of a penalisation, we consider

- a convex and increasing function  $\Upsilon: \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}$  which is  $C^1$  on the *effective domain*  $\text{dom}(\Upsilon) := \{t \in \mathbb{R}: \Upsilon(t) < \infty\}$  and satisfies  $\Upsilon(+\infty) = +\infty$  and hence, also  $\lim_{t \rightarrow \infty} \Upsilon(t) = \infty$ .
- a lower semi-continuous function  $\mathbf{g}: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{\infty\}$  such that the restriction  $\mathbf{g}|_{\text{dom}(\mathbf{g})}$  is of class  $C^2$ , where  $\text{dom}(\mathbf{g}) := \{x \in \mathbb{R}^N: \mathbf{g}(x) < \infty\}$ .

A natural assumption for a penalisation term is to require that

$$\Upsilon \left( \int_{\Omega} \mathbf{g}(v) \, dx \right) \geq 0 \quad (4.12)$$

holds for all measurable functions  $v: \Omega \rightarrow \mathbb{R}^N$  whenever the integral on the left-hand is well-defined and finite. Furthermore, a penalisation term should have no impact when plugging in the original function  $\tilde{u}_j$ . Therefore, we additionally assume that

$$\Upsilon \left( \int_{\Omega} \mathbf{g}(\tilde{u}_j) \, dx \right) = 0 \quad \text{for all } j \in \mathbb{N}. \quad (4.13)$$

For further reference, we compute

$$\nabla \mathfrak{f}_{j,q}(z) = \nabla f(z) + \frac{q}{2A_{j,q}j^2} (1 + |z|^2)^{\frac{q-2}{2}} z \quad (4.14)$$

$$\nabla^2 \mathfrak{f}_{j,q}(z) = \nabla^2 f(z) + \frac{q}{2A_{j,q}j^2} (1 + |z|^2)^{\frac{q-4}{2}} ((q-2)z \otimes z + (1 + |z|^2)\mathbb{1}), \quad (4.15)$$

where  $\mathbb{1} := \mathbb{1}_{(N \times n) \times (N \times n)}$  abbreviates the identity tensor. In conjunction with Lemma 2.17 this leads to

$$|\nabla \mathfrak{f}_{j,q}(z)| \leq \Gamma + \frac{q}{2A_{j,q}j^2} (1 + |z|^2)^{\frac{q-1}{2}} \leq c(q, \Gamma) (1 + |z|^2)^{\frac{q-1}{2}} \quad (4.16)$$

$$|\nabla^2 \mathfrak{f}_{j,q}(z)| \leq |\nabla^2 f(z)| + \frac{c(q)}{A_{j,q}j^2} (1 + |z|^2)^{\frac{q-2}{2}} \leq c(q, \Lambda) (1 + |z|^2)^{\frac{q-2}{2}}. \quad (4.17)$$

In order to proceed we need a tailor-made lower semi-continuity result for functionals extended from Dirichlet classes to negative Sobolev spaces. This is in the spirit of [BS13, Lemma 2.6], [Gme20, Lemma 2.6] or [BEG26, Lemma 3.8], but for the readers convenience we provide a full proof.

**Lemma 4.1.** *Let  $q \in (1, \infty)$  and  $k \in \mathbb{N}$  be given. Moreover, let*

- $\mathfrak{K}: \mathcal{R}(\mathbb{A}) \rightarrow \mathbb{R}_{\geq 0}$  be a convex function that satisfies  $\gamma|z|^q \leq \mathfrak{K}(z) \leq \Gamma(1 + |z|^q)$  for some constants  $\gamma, \Gamma > 0$  and all  $z \in \mathcal{R}(\mathbb{A})$ ,
- $\Upsilon: \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}$  be a convex and increasing function,
- and  $\mathbf{g}: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semi-continuous function.

Then, for every  $u_0 \in W^{1,q}(\Omega; \mathbb{R}^N)$ , the functional

$$\mathfrak{F}[u] := \begin{cases} \int_{\Omega} \mathfrak{K}(Au) \, dx + \Upsilon \left( \int_{\Omega} \mathbf{g}(u) \, dx \right) & \text{if } u \in \mathcal{D}_{u_0} := u_0 + W_0^{1,q}(\Omega; \mathbb{R}^N), \\ +\infty & \text{if } u \in W^{-k,1}(\Omega; \mathbb{R}^N) \setminus \mathcal{D}_{u_0} \end{cases}$$

is lower semi-continuous in the norm topology on  $W^{-k,1}(\Omega; \mathbb{R}^N)$ .

*Proof.* Throughout the proof we do not relabel subsequences and denote them with the same index. Since we want to prove lower semi-continuity in the norm-topology of  $W^{-k,1}$ , we start with a sequence  $(w_j)_{j \in \mathbb{N}}$  in  $W^{-k,1}(\Omega; \mathbb{R}^N)$  converging to  $w$  in  $W^{-k,1}(\Omega; \mathbb{R}^N)$  as  $j \rightarrow \infty$ . Without loss of generality, we can assume that, up to a subsequence, we have

$$\lim_{j \rightarrow \infty} \mathfrak{F}[w_j] = \liminf_{j \rightarrow \infty} \mathfrak{F}[w_j] < \infty,$$

since otherwise there is nothing to prove. First, by necessity, it follows  $w_j \in \mathcal{D}_{u_0}$  for all sufficiently large  $j$ . Using the lower bound  $\gamma |z|^q \leq \mathfrak{K}(z)$  and (4.12), we obtain that the sequence  $(\mathbb{A}w_j)_{j \in \mathbb{N}}$  is uniformly bounded in  $L^q(\Omega; \mathbb{R}^N)$ . Recalling  $w_j \in \mathcal{D}_{u_0}$ , we can use an approximation argument, to apply the Korn-type inequality from Theorem 3.4 in the space  $W_0^{1,q}(\Omega; \mathbb{R}^N)$ , to conclude that  $(w_j)_{j \in \mathbb{N}}$  is also uniformly bounded in  $W^{1,q}(\Omega; \mathbb{R}^N)$ . Now, standard weak compactness in Sobolev spaces allows to extract a subsequence converging weakly in  $W^{1,q}$  to some limit  $\tilde{w} \in \mathcal{D}_{u_0}$ , recalling that  $\mathcal{D}_{u_0}$  is weakly closed, as the trace operator is continuous in the weak topology. Extracting a further subsequence if necessary, the Rellich-Kondrachov compactness theorem improves weak  $W^{1,q}$ -convergence to strong  $L^q$ -convergence. Using the embedding  $L^q(\Omega; \mathbb{R}^N) \hookrightarrow W^{-k,1}(\Omega; \mathbb{R}^N)$ , cf. Lemma 2.12 *ii*), we infer

$$\|w - \tilde{w}\|_{W^{-k,1}(\Omega; \mathbb{R}^N)} \leq \|w - w_j\|_{W^{-k,1}(\Omega; \mathbb{R}^N)} + \|w_j - \tilde{w}\|_{L^1(\Omega; \mathbb{R}^N)} \rightarrow 0$$

as  $j \rightarrow \infty$ . This identifies the limit, namely there holds  $w = \tilde{w}$ . Passing to a further subsequence if necessary, we may assume that  $w_j(x) \rightarrow w(x)$  for  $\mathcal{L}^n$ -a.e.  $x \in \Omega$ . Hence, we infer  $\liminf_{j \rightarrow \infty} \mathfrak{g}(w_j(x)) \geq \mathfrak{g}(w(x))$ , as  $\mathfrak{g}$  is lower semi-continuous. A direct application of Fatou's Lemma leads to

$$\liminf_{j \rightarrow \infty} \int_{\Omega} \mathfrak{g}(w_j) \, dx \geq \int_{\Omega} \liminf_{j \rightarrow \infty} \mathfrak{g}(w_j) \, dx \geq \int_{\Omega} \mathfrak{g}(w) \, dx.$$

Using the fact that  $\Upsilon$  is increasing and convex, we obtain

$$\begin{aligned} \liminf_{j \rightarrow \infty} \Upsilon \left( \int_{\Omega} \mathfrak{g}(w_j) \, dx \right) &\geq \Upsilon \left( \liminf_{j \rightarrow \infty} \int_{\Omega} \mathfrak{g}(w_j) \, dx \right) \\ &\geq \Upsilon \left( \int_{\Omega} \liminf_{j \rightarrow \infty} \mathfrak{g}(w_j) \, dx \right) \geq \Upsilon \left( \int_{\Omega} \mathfrak{g}(w) \, dx \right). \end{aligned} \quad (4.18)$$

Finally, invoking standard results on weak lower semi-continuity for convex variational integrals of superlinear growth, cf. e.g. [Dac08, Chapter 3], we obtain

$$\liminf_{j \rightarrow \infty} \int_{\Omega} \mathfrak{K}(\mathbb{A}w_j) \, dx \geq \int_{\Omega} \mathfrak{K}(\mathbb{A}w) \, dx. \quad (4.19)$$

Therefore, combining (4.18) and (4.19), the proof of the Lemma is finished.  $\square$

As a next step, we apply the Ekeland variational principle from Proposition 2.16 to the functionals  $F_j[-; \Omega]$  at the point  $\tilde{u}_j$  whenever  $j \in \mathbb{N}$  is fixed.

*Step 3: Application of the Ekeland-variational principle.* As a direct consequence of Lemma 4.1, we conclude the lower semi-continuity of  $F_j$ . Moreover, it is clear that  $F_j[-; \Omega]$  is not identically  $+\infty$  on the Banach space  $W^{-k,1}(\Omega; \mathbb{R}^N)$ , cf. Lemma 2.12. Exploiting the definition of  $A_{j,q}$  in combination with (4.13), we obtain

$$\begin{aligned}
F_j[\tilde{u}_j; \Omega] &\leq F[\tilde{u}_j; \Omega] + \frac{1}{2j^2} \stackrel{(4.10)}{\leq} \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{3}{4j^2} \\
&\stackrel{(4.9)}{\leq} \inf_{\mathcal{D}_j} F[-; \Omega] + \frac{1}{j^2} \leq \inf_{W^{-k,1}(\Omega; \mathbb{R}^N)} F_j[-; \Omega] + \frac{1}{j^2},
\end{aligned} \tag{4.20}$$

which means, that  $\tilde{u}_j$  is  $j^{-1}$ -close to the infimum of  $F_j[-; \Omega]$  in  $W^{-k,1}(\Omega; \mathbb{R}^N)$ . Hence, we can apply the Ekeland variational principle from Proposition 2.16 to find an almost minimiser  $v_j \in W^{-k,1}(\Omega; \mathbb{R}^N)$  satisfying

$$\|v_j - \tilde{u}_j\|_{W^{-k,1}(\Omega; \mathbb{R}^N)} \leq \frac{1}{j} \quad \text{and} \tag{4.21}$$

$$F_j[v_j; \Omega] \leq F_j[w; \Omega] + \frac{1}{j} \|v_j - w\|_{W^{-k,1}(\Omega; \mathbb{R}^N)} \quad \text{for all } w \in W^{-k,1}(\Omega; \mathbb{R}^N). \tag{4.22}$$

In the next proposition, we derive several conclusions from (4.21) and (4.22) and list some immediate estimates for the Ekeland-type approximation sequence  $(v_j)_{j \in \mathbb{N}}$ .

**Proposition 4.2** (Properties of the Ekeland sequence). *Let  $(v_j)_{j \in \mathbb{N}}$  be the Ekeland-type approximation sequence from above. Then for every  $j \in \mathbb{N}$  we have the following estimates:*

$$\int_{\Omega} |\mathbb{A}v_j| \, dx \leq \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) \tag{4.23}$$

$$\frac{1}{2A_{j,q}j^2} \int_{\Omega} (1 + |\mathbb{A}v_j|^2)^{\frac{q}{2}} \, dx \leq \frac{2}{j^2}, \tag{4.24}$$

$$\Upsilon \left( \int_{\Omega} \mathbf{g}(v_j) \, dx \right) \leq \frac{2}{j^2}. \tag{4.25}$$

*Proof.* Let  $j \in \mathbb{N}$  be arbitrary. We test (4.22) with  $w = \tilde{u}_j$ , which leads together with (4.21) and (4.20), to

$$\begin{aligned}
F_j[v_j; \Omega] &\leq F_j[\tilde{u}_j; \Omega] + \frac{1}{j} \|v_j - \tilde{u}_j\|_{W^{-k,1}(\Omega; \mathbb{R}^N)} \\
&\leq F_j[\tilde{u}_j; \Omega] + \frac{1}{j^2} \leq \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{7}{4j^2}.
\end{aligned} \tag{4.26}$$

In particular, this implies that  $F_j[v_j; \Omega]$  is finite and therefore, by definition of the functionals, it follows  $v_j \in \mathcal{D}_j \cap W^{1,q}(\Omega; \mathbb{R}^N) \cap L^p(\Omega; \mathbb{R}^N)$ . Especially, there holds  $\text{im}(v_j) \subseteq \text{dom}(\mathbf{g})$ . Using the linear growth condition (1.3), we immediately obtain the first bound (4.23) because of

$$\gamma \int_{\Omega} |\mathbb{A}v_j| \, dx \leq F[v_j; \Omega] \leq F_j[v_j; \Omega] \leq \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{7}{4j^2}.$$

Finally, (4.24) and (4.25) follow at once using  $v_j \in \mathcal{D}_j$ , (4.26) and (4.9). More precisely, we have

$$\begin{aligned}
\frac{1}{2A_{j,q}j^2} \int_{\Omega} (1 + |\mathbb{A}v_j|^2)^{\frac{q}{2}} \, dx + \Upsilon \left( \int_{\Omega} \mathbf{g}(v_j) \, dx \right) &= F_j[v_j; \Omega] - F[v_j; \Omega] \\
&\leq \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{7}{4j^2} - F[v_j; \Omega] \\
&\leq \inf_{\mathcal{D}_j} F[-; \Omega] - F[v_j; \Omega] + \frac{2}{j^2} \leq \frac{2}{j^2}.
\end{aligned}$$

This finishes the proof.  $\square$

Using a Poincaré-type inequality, we obtain the following  $L^1$ -bound:

**Corollary 4.3.** *Let  $(v_j)_{j \in \mathbb{N}}$  be the Ekeland-type approximation sequence from above. Then for every  $j \in \mathbb{N}$  we have*

$$\|v_j\|_{L^1(\Omega; \mathbb{R}^N)} \leq c(\Omega, \Gamma) \left[ \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) + \frac{1}{j^2} \right]. \quad (4.27)$$

*Proof.* By construction, we know that  $v_j \in \mathcal{D}_j = u_j^{\partial\Omega} + W_0^{\mathbb{A},1}(\Omega; \mathbb{R}^2)$ . Moreover, taking into account (4.5) and (4.23), we can apply [GR19a, Lemma 2.12] to estimate

$$\begin{aligned} \|v_j\|_{L^1(\Omega; \mathbb{R}^N)} &\leq \|v_j - u_j^{\partial\Omega}\|_{L^1(\Omega; \mathbb{R}^N)} + \|u_j^{\partial\Omega}\|_{L^1(\Omega; \mathbb{R}^N)} \\ &\leq c(\Omega) \|\mathbb{A}v_j - \mathbb{A}u_j^{\partial\Omega}\|_{L^1(\Omega; \mathcal{R}(\mathbb{A}))} + \|u_j^{\partial\Omega}\|_{L^1(\Omega; \mathbb{R}^N)} \\ &\leq c(\Omega) \left( \|\mathbb{A}v_j\|_{L^1(\Omega; \mathcal{R}(\mathbb{A}))} + \|u_j^{\partial\Omega}\|_{W^{\mathbb{A},1}(\Omega)} \right) \\ &\leq c(\Omega, \Gamma) \left[ \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) + \frac{1}{j^2} \right]. \end{aligned} \quad (4.28)$$

This gives the desired estimate and finishes the proof.  $\square$

**Corollary 4.4.** *Let  $q \geq 2$  and  $(v_j)_{j \in \mathbb{N}}$  be the Ekeland-type approximation sequence from above. Then for every  $j \in \mathbb{N}$  the following estimate holds*

$$\frac{1}{2A_{j,q}j^2} \int_{\Omega} |v_j|^q \, dx \leq c \left[ \frac{2}{j^2} + \left( \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) + \frac{1}{j^2} \right)^q \right] \quad (4.29)$$

for a constant  $c = c(q, \Omega, \Gamma, \mathbb{A}) > 0$ .

*Proof.* Since  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain and therefore also a John domain, it satisfies the emanating chain condition with centre ball  $B$ , cf. [DG24]. Moreover, the Poincaré-type inequality from [DG24, Theorem 3.7] (see also [GR19a, Proposition 4.2]) implies

$$\int_{\Omega} |v_j - \Pi_{\mathbb{A}}^B v_j|^q \, dx \leq c(\mathbb{A}, q) \operatorname{diam}(\Omega)^q \int_{\Omega} |\mathbb{A}v_j|^q \, dx,$$

where  $\Pi_{\mathbb{A}}^B$  denotes the projection onto the kernel of  $\mathbb{A}$  constructed in [DG24]. Furthermore, we observe that the inverse estimates from [DG24, Lemma 3.5] remain true for our domains leading to

$$\int_{\Omega} |\Pi_{\mathbb{A}}^B v_j|^q \, dx \leq \mathcal{L}^n(\Omega) \|\Pi_{\mathbb{A}}^B v_j\|_{L^\infty(\Omega; \mathbb{R}^N)} \leq c \mathcal{L}^n(\Omega) \left( \int_B |v_j| \, dx \right)^q.$$

Hence, we obtain

$$\begin{aligned} \frac{1}{2A_{j,q}j^2} \int_{\Omega} |v_j|^q \, dx &\leq \frac{1}{2A_{j,q}j^2} \left[ \int_{\Omega} |v_j - \Pi_{\mathbb{A}}^B v_j|^q \, dx + \int_{\Omega} |\Pi_{\mathbb{A}}^B v_j|^q \, dx \right] \\ &\leq \frac{c(q, \Omega, \mathbb{A})}{2A_{j,q}j^2} \left[ \int_{\Omega} |\mathbb{A}v_j|^q \, dx + \left( \int_B |v_j| \, dx \right)^q \right] \\ &\leq \frac{c(q, \Omega, \mathbb{A})}{2A_{j,q}j^2} \left[ \int_{\Omega} (1 + |\mathbb{A}v_j|^2)^{\frac{q}{2}} \, dx + \left( \int_B |v_j| \, dx \right)^q \right] \end{aligned}$$

and therefore, taking into account (4.24) in combination with (4.27), we obtain

$$\frac{1}{2A_{j,q}j^2} \int_{\Omega} |v_j|^q \, dx \leq c \left[ \frac{2}{j^2} + \left( \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) + \frac{1}{j^2} \right)^q \right],$$

for a constant  $c = c(q, \Omega, \Gamma, \mathbb{A}) > 0$ . This yields the claimed estimate and the proof is therefore complete.  $\square$

**Remark 4.5** (Different projections). *It is also possible to use the projection  $\mathbb{P}_{\mathbb{B}}$  onto  $\ker(\mathbb{A})$  as constructed in (2.11) instead of the inverse estimates for  $\Pi_{\mathbb{A}}^{\mathbb{B}}$ . Namely, using  $\mathbb{P}_{\mathbb{B}}(\Pi_{\mathbb{A}}^{\mathbb{B}} v_j) = \Pi_{\mathbb{A}}^{\mathbb{B}} v_j$  and (2.14), we deduce*

$$\begin{aligned} \|v_j\|_{L^q(\Omega; \mathbb{R}^N)} &\leq \|v_j - \mathbb{P}_{\mathbb{B}} v_j\|_{L^q(\Omega; \mathbb{R}^N)} + \|\mathbb{P}_{\mathbb{B}} v_j\|_{L^q(\Omega; \mathbb{R}^N)} \\ &\leq \|v_j - \Pi_{\mathbb{A}}^{\mathbb{B}} v_j\|_{L^q(\Omega; \mathbb{R}^N)} + \|\mathbb{P}_{\mathbb{B}}(\Pi_{\mathbb{A}}^{\mathbb{B}} v_j - v_j)\|_{L^q(\Omega; \mathbb{R}^N)} + \|\mathbb{P}_{\mathbb{B}} v_j\|_{L^q(\Omega; \mathbb{R}^N)} \\ &\leq c \left[ \|v_j - \Pi_{\mathbb{A}}^{\mathbb{B}} v_j\|_{L^q(\Omega; \mathbb{R}^N)} + \int_{\mathbb{B}} |v_j| \, dx \right]. \end{aligned}$$

Hence, we can proceed as in the proof above.

With the help of the almost minimality condition (4.22) we can derive an Euler-Lagrange inequality assuming a suitable finiteness condition. We already want to emphasise that the estimates from Proposition 4.2 may not be sufficient to guarantee this condition and may require refinement in certain applications.

**Theorem 4.6** (Euler-Lagrange inequality). *Let  $(v_j)_{j \in \mathbb{N}}$  be the Ekeland-type approximation sequence from above. Moreover, let  $j \in \mathbb{N}$  such that we have  $F_j[v_j \pm \theta \varphi; \Omega] < \infty$  whenever  $\theta \in (0, 1)$  and  $\varphi \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ . Then the following Euler-Lagrange inequality is valid*

$$\left| \int_{\Omega} \langle \nabla f_{j,q}(\mathbb{A} v_j), \mathbb{A} \varphi \rangle \, dx + \Upsilon' \left( \int_{\Omega} \mathbf{g}(v_j) \, dx \right) \int_{\Omega} \langle \nabla \mathbf{g}(v_j), \varphi \rangle \, dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-k,1}(\Omega; \mathbb{R}^N)}. \quad (4.30)$$

*Proof.* Let  $j \in \mathbb{N}$  such that  $F_j[v_j \pm \theta \varphi; \Omega] < \infty$  for  $\varphi \in W_0^{1,q}(\Omega; \mathbb{R}^N)$  and  $\theta \in (0, 1)$ . In this case, we can test (4.22) with  $w = v_j \pm \theta \varphi$ . Dividing the resulting inequality by  $\theta > 0$  results in

$$\frac{F_j[v_j; \Omega] - F_j[v_j \pm \theta \varphi; \Omega]}{\theta} \leq \frac{1}{j} \|\varphi\|_{W^{-k,1}(\Omega; \mathbb{R}^N)}.$$

Passing to the limit  $\theta \searrow 0$ , we obtain the two one-sided Gâteaux derivatives of  $F_j$  in direction  $\pm \varphi$ , which can be calculated explicitly by

$$\begin{aligned} \lim_{\theta \searrow 0} \frac{F_j[v_j; \Omega] - F_j[v_j \pm \theta \varphi; \Omega]}{\theta} &= \frac{d}{d\theta} \Big|_{\theta=0} F_j[v_j \pm \theta \varphi; \Omega] \\ &= \pm \left[ \int_{\Omega} \langle \nabla f_{j,q}(\mathbb{A} v_j), \mathbb{A} \varphi \rangle \, dx + \Upsilon' \left( \int_{\Omega} \mathbf{g}(v_j) \, dx \right) \int_{\Omega} \langle \nabla \mathbf{g}(v_j), \varphi \rangle \, dx \right]. \end{aligned}$$

In particular, all quantities on the right-hand side are well-defined and finite because of  $\text{im}(v_j) \in \text{dom}(\mathbf{g})$ . Therefore, we infer the validity of (4.30), which completes the proof.  $\square$

**Remark 4.7** (Finiteness condition). *Notice that the finiteness condition is trivially satisfied if  $\mathbf{g}$  only takes finite values. However, in Chapter 5 we will need a penalisation function taking the value infinity. In order to fulfil the finiteness condition we have to improve the estimates from Proposition 4.2 in a strict and quantified way.*

## 4.2 Non-Uniform Estimates

As a next step, we derive higher differentiability whenever the Euler-Lagrange inequality (4.30) holds. Namely, we obtain the following result:

**Lemma 4.8** (Non-uniform second-order estimates). *Let  $(v_j)_{j \in \mathbb{N}}$  be the Ekeland-type approximation sequence from above. Moreover, let  $q \geq 2$  and  $f \in C^2(\mathcal{R}(\mathbb{A}))$  satisfy (1.3) as well as the estimate*

$$0 \leq \langle \nabla^2 f(z) \xi, \xi \rangle \leq \Lambda \frac{|\xi|^2}{(1 + |z|^2)^{\frac{1}{2}}} \quad \text{for all } z, \xi \in \mathcal{R}(\mathbb{A}) \quad (4.31)$$

for some  $\Lambda \in (0, \infty)$ . Then for every  $j \in \mathbb{N}$  for which (4.30) holds, we have  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^N)$  with

$$(1 + |\mathbb{A}v_j|^2)^{\frac{q-2}{2}} |\partial_s \mathbb{A}v_j|^2 \in L_{\text{loc}}^1(\Omega) \quad \text{for all } s \in \{1, \dots, n\}. \quad (4.32)$$

*Proof.* For the sake of readability we divide the proof into two steps. Let  $x_0 \in \Omega$  be a point,  $r, R$  two radii with  $0 < r < R < \text{dist}(x_0, \partial\Omega)$  and  $\varrho \in C_c^\infty(\Omega; [0, 1])$  be a cut-off function with  $\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_R(x_0)}$ . Moreover, we consider an arbitrary direction  $s \in \{1, \dots, n\}$  and a step size  $0 < h < \frac{1}{2}(\text{dist}(x_0, \partial\Omega) - R)$ .

*Step 1: Existence of second derivatives.* By construction of the Ekeland-type approximation sequence, we have  $v_j \in W^{1,q}(\Omega; \mathbb{R}^N)$ . Therefore, the function  $\varphi = \Delta_{s,-h}(\varrho^2 \Delta_{s,h} v_j)$  is an admissible test function for the Euler-Lagrange inequality (4.30) from Theorem 4.6, where  $\Delta_{s,h}$  denotes the difference quotient in direction  $s \in \{1, \dots, n\}$  with step size  $0 < h < \frac{1}{2}(\text{dist}(x_0, \partial\Omega) - R)$ , cf. (2.2). Next, using the product rule from (2.6), we obtain

$$\mathbb{A}(\varrho^2 \Delta_{s,h} v_j) = \varrho^2 \Delta_{s,h} \mathbb{A}v_j + \Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\varrho \nabla \varrho,$$

Performing a discrete integration by parts, cf. (2.3), and inserting the above formula yields

$$\begin{aligned} & \left| \int_{\Omega} \langle \Delta_{s,h} \nabla f_{j,q}(\mathbb{A}v_j), \varrho^2 \Delta_{s,h} \mathbb{A}v_j + \Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\varrho \nabla \varrho \rangle dx \right. \\ & \quad \left. + \Upsilon' \left( \int_{\Omega} \mathfrak{g}(v_j) dx \right) \int_{\Omega} \langle \Delta_{s,h} \nabla \mathfrak{g}(v_j), \varrho^2 \Delta_{s,h} v_j \rangle dx \right| \\ & \leq \frac{1}{j} \|\Delta_{s,-h}(\varrho^2 \Delta_{s,h} v_j)\|_{W^{-k,1}(\Omega; \mathbb{R}^N)}. \end{aligned} \quad (4.33)$$

Therefore, switching the order of the terms leads to

$$\begin{aligned} \text{I} + \text{II} & := \int_{\Omega} \langle \Delta_{s,h} \nabla f_{j,q}(\mathbb{A}v_j), \varrho^2 \Delta_{s,h} \mathbb{A}v_j \rangle dx \\ & \quad + \Upsilon' \left( \int_{\Omega} \mathfrak{g}(v_j) dx \right) \int_{\Omega} \langle \Delta_{s,h} \nabla \mathfrak{g}(v_j), \varrho^2 \Delta_{s,h} v_j \rangle dx \\ & \leq \frac{1}{j} \|\Delta_{s,-h}(\varrho^2 \Delta_{s,h} v_j)\|_{W^{-k,1}(\Omega; \mathbb{R}^N)} - \int_{\Omega} \langle \Delta_{s,h} \nabla f_{j,q}(\mathbb{A}v_j), \Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\varrho \nabla \varrho \rangle dx \\ & =: \text{III} + \text{IV}. \end{aligned}$$

With the help of the fundamental theorem of calculus in combination with the chain rule, we rewrite

$$\begin{aligned}\Delta_{s,h}\nabla\mathfrak{f}_{j,q}(\mathbb{A}v_j)(x) &= \frac{1}{h}[\nabla\mathfrak{f}_{j,q}(\mathbb{A}v_j)(x+he_s) - \nabla\mathfrak{f}_{j,q}(\mathbb{A}v_j)(x)] \\ &= \int_0^1 \nabla^2\mathfrak{f}_{j,q}(\mathbb{A}v_j(x) + t[\varepsilon(v_j)(x+he_s) - \mathbb{A}v_j(x)]) dt \Delta_{s,h}\mathbb{A}v_j(x) \\ &= \int_0^1 \nabla^2\mathfrak{f}_{j,q}(\mathbb{A}v_j(x) + th\Delta_{s,h}\mathbb{A}v_j(x)) dt \Delta_{s,h}\mathbb{A}v_j(x).\end{aligned}\tag{4.34}$$

Motivated by the above representation formula, we introduce for  $\mathcal{L}^n$ -a.e.  $x \in B_r(x_0)$  the family of symmetric bilinear forms  $\mathfrak{B}_{j,s,h}(x): \mathcal{R}(\mathbb{A}) \times \mathcal{R}(\mathbb{A}) \rightarrow \mathbb{R}$  via

$$\mathfrak{B}_{j,s,h}(x)[\eta, \xi] := \int_0^1 \langle \nabla^2\mathfrak{f}_{j,q}(\mathbb{A}v_j(x) + th\Delta_{s,h}\mathbb{A}v_j(x))\eta, \xi \rangle dt \quad \text{for all } \eta, \xi \in \mathcal{R}(\mathbb{A}).$$

As a next step, we show that each of these bilinear forms is positive definite together with suitable upper and lower bounds. This in turn, allows to use a version of the Cauchy-Schwarz inequality in the sequel. To this end, we recall the following inequality

$$c(q)(1 + |z_1|^2 + |z_2|^2)^{\frac{q-2}{2}} \leq \int_0^1 (1 + |z_1 + tz_2|^2)^{\frac{q-2}{2}} dt \tag{4.35}$$

for all  $z_1, z_2 \in \mathcal{R}(\mathbb{A})$ , following from [Cam82, Lemma 2.VI]. Moreover, we observe

$$\mathbb{A}v_j(x) + th\Delta_{s,h}\mathbb{A}v_j(x) = (1-t)\mathbb{A}v_j(x) + t\mathbb{A}v_j(x+he_s).$$

To obtain a lower bound, we recall the explicit formula (4.15) which leads, using (4.31) and  $\langle (z \otimes z)\eta, \eta \rangle = \langle z, \eta \rangle^2 \geq 0$  for  $z, \eta \in \mathbb{R}^{N \times n}$ , to

$$\begin{aligned}\mathfrak{B}_{j,s,h}(x)[\xi, \xi] &\geq \frac{q}{2A_{j,q}j^2} \int_0^1 (1 + |\mathbb{A}v_j(x) + th\Delta_{s,h}\mathbb{A}v_j(x)|^2)^{\frac{q-2}{2}} |\xi|^2 dt \\ &\geq \frac{c(q)}{A_{j,q}j^2} (1 + |\mathbb{A}v_j(x)|^2 + |h\Delta_{s,h}\mathbb{A}v_j(x)|^2)^{\frac{q-2}{2}} |\xi|^2 \\ &\geq \frac{c(q)}{A_{j,q}j^2} (1 + |\mathbb{A}v_j(x)|^2 + |\mathbb{A}v_j(x+he_s)|^2)^{\frac{q-2}{2}} |\xi|^2,\end{aligned}\tag{4.36}$$

where we have used (4.35) in the second inequality. Towards an upper bound, we utilise (4.17) in combination with (4.31) to infer the estimate

$$\begin{aligned}\mathfrak{B}_{j,s,h}(x)[\xi, \xi] &\leq \Lambda \int_0^1 \frac{|\xi|^2}{(1 + |\mathbb{A}v_j(x) + th\Delta_{s,h}\mathbb{A}v_j(x)|^2)^{\frac{1}{2}}} dt \\ &\quad + \frac{c(q)}{2A_{j,q}j^2} \int_0^1 (1 + |\mathbb{A}v_j(x) + th\Delta_{s,h}\mathbb{A}v_j(x)|^2)^{\frac{q-2}{2}} |\xi|^2 dt \\ &\leq c(q, \Lambda) (1 + |\mathbb{A}v_j(x)|^2 + |\mathbb{A}v_j(x+he_s)|^2)^{\frac{q-2}{2}} |\xi|^2.\end{aligned}\tag{4.37}$$

As a direct consequence of the lower bound, we can estimate I by means of

$$\begin{aligned}\text{I} &= \int_{\Omega} \langle \Delta_{s,h}\nabla\mathfrak{f}_{j,q}(\mathbb{A}v_j), \varrho^2\mathbb{A}(\Delta_{s,h}v_j) \rangle dx \\ &= \int_{\Omega} \mathfrak{B}_{j,s,h}(x)[\varrho\mathbb{A}(\Delta_{s,h}v_j), \varrho\mathbb{A}(\Delta_{s,h}v_j)] dx \\ &\geq \frac{c(q)}{A_{j,q}j^2} \int_{\Omega} (1 + |\mathbb{A}v_j(x)|^2)^{\frac{q-2}{2}} |\varrho\mathbb{A}(\Delta_{s,h}v_j)|^2 dx.\end{aligned}\tag{4.38}$$

An argument similar to that in (4.34) motivates the introduction of the bilinear form  $\mathfrak{C}_{j,s,h}(x): \mathcal{R}(\mathbb{A}) \times \mathcal{R}(\mathbb{A}) \rightarrow \mathbb{R}$ , which is defined for any two elements  $\eta, \xi \in \mathcal{R}(\mathbb{A})$  by

$$\mathfrak{C}_{j,s,h}(x)[\eta; \xi] := \int_0^1 \langle \nabla^2 \mathfrak{g}(v_j(x) + th\Delta_{s,h}v_j(x))\eta, \xi \rangle dt,$$

As the function  $\mathfrak{g}$  is convex and of class  $C^2$ , the above bilinear form is positive semi-definite. Therefore, II admits a positive sign namely

$$\text{II} := \int_{\Omega} \langle \Delta_{s,h} \nabla \mathfrak{g}(v_j), \varrho^2 \Delta_{s,h} v_j \rangle dx = \int_{\Omega} \mathfrak{C}_{j,s,h}(x)[\varrho \Delta_{s,h} v_j, \varrho \Delta_{s,h} v_j] dx \geq 0.$$

Now, using inequality (2.18) for negative Sobolev spaces, we have

$$\text{III} \leq \frac{1}{j} \|\varrho^2 \Delta_{s,h} v_j\|_{L^1(\Omega; \mathbb{R}^N)} \leq \|\partial_s v_j\|_{L^1(\Omega; \mathbb{R}^N)} \leq C(q, \Omega) \|v_j\|_{W^{1,q}(\Omega; \mathbb{R}^N)}.$$

In order to estimate IV, we take into account (4.33) to infer

$$\begin{aligned} \text{IV} &= - \int_{\Omega} \mathfrak{B}_{j,s,h}(x)[\varrho \mathbb{A}(\Delta_{s,h} v_j), \Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\nabla \varrho] dx \\ &\leq \frac{1}{2} \int_{\Omega} \mathfrak{B}_{j,s,h}(x)[\varrho \mathbb{A}(\Delta_{s,h} v_j), \varrho \mathbb{A}(\Delta_{s,h} v_j)] dx \\ &\quad + \frac{1}{2} \int_{\Omega} \mathfrak{B}_{j,s,h}(x)[\Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\nabla \varrho, \Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\nabla \varrho] dx \\ &= \frac{1}{2} \text{I} + \frac{1}{2} \int_{\Omega} \mathfrak{B}_{j,s,h}(x)[\Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\nabla \varrho, \Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\nabla \varrho] dx. \end{aligned} \tag{4.39}$$

While the first can be absorbed into I, we have to estimate the second term. For  $q > 2$ , we apply Hölder's inequality with exponents  $\frac{q}{q-2}$  and  $\frac{q}{2}$ , together with the upper bound (4.37), and [Giu03, Lemma 8.1] to deduce

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} \mathfrak{B}_{j,s,h}(x)[\Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\nabla \varrho, \Delta_{s,h} v_j \otimes_{\mathbb{A}} 2\nabla \varrho] dx \\ &\leq c(q, \Lambda) \int_{\Omega} (1 + |\mathbb{A}v_j(x)|^2 + |\mathbb{A}v_j(x + he_s)|^2)^{\frac{q-2}{2}} |\Delta_{s,h} v_j \otimes_{\mathbb{A}} \nabla \varrho|^2 dx \\ &\leq c(q, \Lambda) \|\nabla \varrho\|_{L^\infty(\Omega; \mathbb{R}^N)}^2 \|1 + |\mathbb{A}v_j|\|_{L^q(\Omega)}^{q-2} \|\Delta_{s,h} v_j\|_{L^q(\Omega; \mathbb{R}^N)}^2 \\ &\leq c(q, \Lambda) \|\nabla \varrho\|_{L^\infty(\Omega; \mathbb{R}^N)}^2 \|1 + |\mathbb{A}v_j|\|_{L^q(\Omega)}^{q-2} \|\partial_s v_j\|_{L^q(\Omega; \mathbb{R}^N)}^2. \end{aligned}$$

In the same spirit, it follows immediately for  $q = 2$  that

$$\frac{1}{2} \int_{\Omega} \mathfrak{B}_{j,s,h}(x)[2\nabla \varrho \otimes_{\mathbb{A}} \Delta_{s,h} v_j, 2\nabla \varrho \otimes_{\mathbb{A}} \Delta_{s,h} v_j] dx \leq c(\Lambda) \|\nabla \varrho\|_{L^\infty(\Omega; \mathbb{R}^N)}^2 \|\partial_s v_j\|_{L^2(\Omega; \mathbb{R}^N)}^2.$$

Collecting and combining all the previous estimates yields

$$\begin{aligned} \int_{\Omega} |\varrho \Delta_{s,h} \mathbb{A}v_j|^2 dx &\leq \int_{\Omega} (1 + |\mathbb{A}v_j|^2)^{\frac{q-2}{2}} |\varrho \Delta_{s,h} \mathbb{A}v_j|^2 dx \\ &\leq c(n, \Omega, \Lambda, j, A_j, \|v_j\|_{W^{1,q}(\Omega; \mathbb{R}^N)}), \end{aligned} \tag{4.40}$$

where the constant on the right-hand side does not depend on  $h$ .

Hence, taking into account (4.40), the sequence  $(\Delta_{s,h}\mathbb{A}v_j)_h$  is uniformly bounded in  $L^2(B_r(x_0); \mathcal{R}(\mathbb{A}))$  and therefore  $\partial_s \mathbb{A}(v_j)$  exists in  $L^2(B_r(x_0); \mathcal{R}(\mathbb{A}))$  by [Giu03, Lemma 8.2]. Since  $s \in \{1, \dots, n\}$ ,  $x_0 \in \Omega$  and  $0 < r < R < \text{dist}(x_0, \partial\Omega)$  were arbitrarily chosen, we can apply the Korn-type inequality (3.8) from Theorem 3.5 to conclude  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^N)$ .

*Step 2: Weighted estimate.* As another consequence of [Giu03, Lemma 8.2], we have the strong convergence  $\Delta_{s,h}\mathbb{A}v_j \rightarrow \partial_s \mathbb{A}v_j$  in  $L^2(K; \mathcal{R}(\mathbb{A}))$  for  $K \Subset \Omega$ . Therefore, we can select a suitable sequence  $(h_i)_{i \in \mathbb{N}}$  in  $\mathbb{R}_{>0}$  with  $h_i \searrow 0$  and  $\Delta_{s,h_i}\mathbb{A}v_j \rightarrow \partial_s \mathbb{A}v_j$  pointwise  $\mathcal{L}^n$ -a.e. in  $K$ . Applying Fatou's Lemma then implies

$$\int_K (1 + |\mathbb{A}v_j|^2)^{\frac{q-2}{2}} |\partial_s \mathbb{A}v_j|^2 \, dx \leq \liminf_{i \rightarrow \infty} \int_K (1 + |\mathbb{A}v_j|^2)^{\frac{q-2}{2}} |\Delta_{s,h_i}\mathbb{A}v_j|^2 \, dx.$$

Since the right-hand side is bounded by the previous reasoning of Step 1, we have also proved the second claim  $(1 + |\mathbb{A}v_j|^2)^{\frac{q-2}{2}} |\partial_s \mathbb{A}v_j|^2 \in L^1_{\text{loc}}(\Omega)$ , finishing the proof.  $\square$

Invoking the second-order estimates from (5.17), we are allowed to differentiate the Euler-Lagrange inequality from (5.16) to obtain a suitable differentiated version. This will be the starting point for deriving weighted second-order estimates in the subsequent chapters. To this end, it is convenient to abbreviate

$$\sigma_{j,q} := \nabla \mathfrak{f}_{j,q}(\mathbb{A}v_j),$$

where  $\mathfrak{f}_{j,q}$  is given by (4.11) and  $v_j \in \mathcal{D}_j \cap W^{1,q}(\Omega; \mathbb{R}^N) \cap L^p(\Omega; \mathbb{R}^N)$  is the Ekeland-type approximation function. Moreover, we briefly recall from our construction the inclusion  $\text{im}(v_j) \Subset \text{dom}(\mathfrak{g})$ .

**Lemma 4.9** (Differentiated Euler-Lagrange inequality). *Let  $q \geq 2$  and  $f \in C^2(\mathcal{R}(\mathbb{A}))$  satisfy (1.3) and for some  $\Lambda \in (0, \infty)$  the bound (4.31). Then, for every  $j \in \mathbb{N}$  for which the Euler-Lagrange inequality (4.30) holds, we have  $\sigma_{j,q} \in W_{\text{loc}}^{1,q'}(\Omega; \mathcal{R}(\mathbb{A}))$  with  $q' = \frac{q}{q-1}$ . Moreover, for all  $s \in \{1, \dots, n\}$  and  $\varphi \in W_0^{1,q}(\Omega; \mathbb{R}^n)$  there holds*

$$\left| \int_{\Omega} \langle \partial_s \sigma_{j,q}, \mathbb{A}\varphi \rangle \, dx + \Upsilon' \left( \int_{\Omega} \mathfrak{g}(v_j) \, dx \right) \int_{\Omega} \langle \partial_s \nabla \mathfrak{g}(v_j), \varphi \rangle \, dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-1,1}(\Omega; \mathbb{R}^n)}. \quad (4.41)$$

*Proof.* We first prove that  $\sigma_{j,q} \in W_{\text{loc}}^{1,q'}(\Omega; \mathcal{R}(\mathbb{A}))$  for  $q' = \frac{q}{q-1}$ , showing that the first integral of (4.41) is well-defined. Recalling  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^N)$  from Lemma 4.8, we deduce that  $\sigma_j$  is weakly differentiable with

$$\partial_s \sigma_{j,q} = \nabla^2 f_j(\mathbb{A}v_j) \partial_s \mathbb{A}v_j \quad \text{for } s \in \{1, \dots, n\}.$$

Consequently, for  $q = 2$ , it follows  $\sigma_{j,q} \in W_{\text{loc}}^{1,2}(\Omega; \mathcal{R}(\mathbb{A}))$ . For  $q > 2$  and  $U \Subset \Omega$  we apply Hölder's inequality with exponents  $\frac{2q-2}{q-2}$  and  $\frac{2q-2}{q}$  to infer

$$\begin{aligned} & \|\partial_s \sigma_{j,q}\|_{L^{\frac{q}{q-1}}(U; \mathcal{R}(\mathbb{A}))}^{\frac{q}{q-1}} \\ & \leq \int_U (1 + |\mathbb{A}v_j|^2)^{\frac{q-2}{2} \frac{q}{q-1}} |\partial_s \mathbb{A}v_j|^{\frac{q}{q-1}} \, dx \\ & = \int_U (1 + |\mathbb{A}v_j|^2)^{\frac{q-2}{4} \frac{q}{q-1}} (1 + |\mathbb{A}v_j|^2)^{\frac{q-2}{4} \frac{q}{q-1}} |\partial_s \mathbb{A}v_j|^{\frac{q}{q-1}} \, dx \\ & \leq \left( \int_U (1 + |\mathbb{A}v_j|^2)^{\frac{q}{2}} \, dx \right)^{\frac{q-2}{2q-2}} \left( \int_U (1 + |\mathbb{A}v_j|^2)^{\frac{q-2}{2}} |\partial_s \mathbb{A}v_j|^2 \, dx \right)^{\frac{q}{2q-2}}. \end{aligned}$$

To obtain the differentiated version of the Euler-Lagrange inequality (4.41), we test the Euler-Lagrange inequality (4.30) with  $\partial_s \varphi$  for some arbitrary  $\varphi \in C_c^\infty(\Omega; \mathbb{R}^N)$ . Using integration by parts in combination with (2.17), we then find

$$\begin{aligned}
& \left| \int_{\Omega} \langle \partial_s \sigma_{j,q}, \mathbb{A} \varphi \rangle dx + \Upsilon' \left( \int_{\Omega} \mathbf{g}(v_j) dx \right) \int_{\Omega} \langle \partial_s \nabla \mathbf{g}(v_j), \varphi \rangle dx \right| \\
&= \left| \int_{\Omega} \langle \nabla \mathbf{f}_{j,q}(\mathbb{A} v_j), \mathbb{A}(\partial_s \varphi) \rangle dx + \Upsilon' \left( \int_{\Omega} \mathbf{g}(v_j) dx \right) \int_{\Omega} \langle \nabla \mathbf{g}(v_j), \partial_s \varphi \rangle dx \right| \\
&\leq \frac{1}{j} \|\partial_s \varphi\|_{W^{-2,1}(\Omega; \mathbb{R}^N)} \\
&\leq \frac{1}{j} \|\varphi\|_{W^{-1,1}(\Omega; \mathbb{R}^N)}.
\end{aligned} \tag{4.42}$$

Finally, we could make sense of the above inequality for arbitrary  $\varphi \in W_0^{1,q}(\Omega; \mathbb{R}^N)$  using an approximation argument. To this end, let  $(\varphi_k)_{k \in \mathbb{N}}$  be a smooth sequence in  $C_c^\infty(\Omega; \mathbb{R}^N)$  such that  $\varphi_k \rightarrow \varphi$  in  $W^{1,q}(\Omega; \mathbb{R}^N)$  as  $k \rightarrow \infty$ . Because of the embedding  $W^{1,q}(\Omega; \mathbb{R}^N) \hookrightarrow W^{-1,1}(\Omega; \mathbb{R}^N)$ , we can pass to the limit in the right-hand side, whereas for the left-hand side, we make use of  $\sigma_{j,q} \in W_{\text{loc}}^{1,q'}(\Omega; \mathcal{R}(\mathbb{A}))$  to deduce

$$\left| \int_{\Omega} \langle \partial_s \sigma_{j,q}, \mathbb{A} \varphi_k - \mathbb{A} \varphi \rangle dx \right| \leq \|\partial_s \sigma_{j,q}\|_{L^{q'}(\Omega; \mathcal{R}(\mathbb{A}))} \|\mathbb{A} \varphi_k - \mathbb{A} \varphi\|_{L^q(\Omega; \mathcal{R}(\mathbb{A}))}.$$

Moreover, since  $\mathbf{g}$  is of class  $C^2$  on  $\text{dom}(\mathbf{g})$ , we infer

$$\|\partial_s \nabla \mathbf{g}\|_{L^\infty(U; \mathbb{R}^N)} < \infty \quad \text{for all } U \Subset \text{dom}(\mathbf{g})$$

which leads, together with  $\text{im}(v_j) \Subset \text{dom}(\mathbf{g})$ , to

$$\left| \int_{\Omega} \langle \partial_s \nabla \mathbf{g}(v_j), \varphi_k - \varphi \rangle dx \right| \leq C \|\varphi_k - \varphi\|_{L^1(\Omega; \mathbb{R}^N)}.$$

Therefore, we can pass to the limit on both sides of (4.42) completing the proof.  $\square$

In later passages, we will require estimates of the same type multiple times; to shorten the arguments, we summarise them in the following lemma.

**Lemma 4.10.** *Let  $f \in C^2(\mathcal{R}(\mathbb{A}))$  satisfy, for some  $\Lambda \in (0, \infty)$ , the bound (4.31). Then, for vectors  $w, e \in \mathbb{R}^n$ , a matrix  $\mathfrak{N} \in \mathbb{R}^{N \times n}$  and every  $z \in \mathcal{R}(\mathbb{A})$  we have the estimates*

$$\begin{aligned}
& \langle \nabla^2 \mathbf{f}_{j,q}(z)(e \otimes_{\mathbb{A}} (\mathfrak{N}w)), (e \otimes_{\mathbb{A}} (\mathfrak{N}w)) \rangle \\
& \leq |\mathfrak{N}|^2 \left[ \Lambda \frac{|w|^2 |e|^2}{(1 + |z|^2)^{\frac{1}{2}}} + \frac{c}{A_{j,q} j^2} |w|^2 |e|^2 \right]
\end{aligned} \tag{4.43}$$

if  $q = 2$  as well as

$$\begin{aligned}
& \langle \nabla^2 \mathbf{f}_{j,q}(z)(e \otimes_{\mathbb{A}} (\mathfrak{N}w)), (e \otimes_{\mathbb{A}} (\mathfrak{N}w)) \rangle \\
& \leq |\mathfrak{N}|^2 \left[ \Lambda \frac{|w|^2 |e|^2}{(1 + |z|^2)^{\frac{1}{2}}} + \frac{c(q)}{A_{j,q} j^2} \left( (1 + |z|^2)^{\frac{q}{2}} + |w|^q |e|^q \right) \right]
\end{aligned} \tag{4.44}$$

for  $q > 2$ .

*Proof.* By combining the explicit expression for  $\nabla^2 \mathfrak{f}_{j,q}(z)$  given in (4.15) with the estimate from (4.31), a straightforward computation shows that

$$\begin{aligned} & \langle \nabla^2 \mathfrak{f}_{j,q}(z)(e \otimes_{\mathbb{A}} (\mathfrak{N}w)), (e \otimes_{\mathbb{A}} (\mathfrak{N}w)) \rangle \\ & \leq \langle \nabla^2 f(z)(e \otimes_{\mathbb{A}} (\mathfrak{N}w)), (e \otimes_{\mathbb{A}} (\mathfrak{N}w)) \rangle + \frac{c(q)}{A_{j,q} j^2} (1 + |z|^2)^{\frac{q-2}{2}} |\mathfrak{N}|^2 |w|^2 |e|^2 \\ & \leq \Lambda \frac{|\mathfrak{N}|^2 |w|^2 |e|^2}{(1 + |z|^2)^{\frac{1}{2}}} + \frac{c(q)}{A_{j,q} j^2} (1 + |z|^2)^{\frac{q-2}{2}} |\mathfrak{N}|^2 |w|^2 |e|^2. \end{aligned}$$

Now, the claimed estimate follows directly when applying in addition Young's inequality with exponents  $\frac{q}{q-2}$  and  $\frac{q}{2}$  in the case  $q > 2$ . This finishes the proof.  $\square$

We close this section and comment on a more general constraint space:

**Remark 4.11** (General constraints). *It is possible to use a more general function space for the constraint instead of  $L^p$ . We believe that it is possible to take any rearrangement-invariant Banach function space  $\mathfrak{X}(\Omega; V)$  with norm  $\|\cdot\|_{\mathfrak{X}}$ , cf. [BS88], if the following two assumptions are satisfied:*

- *The statement of Lemma 2.4 holds true with  $\mathfrak{X}$ -constraint, i.e. for  $u_0 \in W^{\mathbb{A},1}(\Omega)$  and  $u \in BV^{\mathbb{A}}(\Omega) \cap \mathfrak{X}(\Omega; V)$  there exists a sequence  $(u_j)_{j \in \mathbb{N}}$  in  $(u_0 + C_c^\infty(\Omega; V)) \cap \mathfrak{X}(\Omega; V)$  with  $u_j \rightarrow u$   $\mathbb{A}$ -area-strict and  $\|u_j\|_{\mathfrak{X}} \leq c_A \|u\|_{\mathfrak{X}}$ .*
- *The statement of Theorem 2.10 holds true with  $\mathfrak{X}$ -constraint: Let  $\Omega, \Omega_0 \subset \mathbb{R}^n$  be two open and bounded sets with  $\Omega \Subset \Omega_0$  such that  $\Omega$  has Lipschitz boundary  $\partial\Omega$ . Then there exists a (norm-)bounded linear extension operator  $\mathfrak{J}: W^{\mathbb{A},1}(\Omega) \rightarrow W^{\mathbb{A},1}(\mathbb{R}^n)$  such that*

- i)  $\text{supp}(\mathfrak{J}u) \subset \Omega_0$  for all  $u \in BV^{\mathbb{A}}(\Omega)$ ,*
- ii)  $\mathfrak{J}: W^{\mathbb{A},1}(\Omega) \rightarrow W_0^{\mathbb{A},1}(\Omega_0)$ ,*
- iii) and there exists a constant  $c = c(\Omega, n) > 0$  such that  $\|\mathfrak{J}u\|_{\mathfrak{X}} \leq c \|u\|_{\mathfrak{X}}$  for all  $u \in BV^{\mathbb{A}}(\Omega) \cap \mathfrak{X}(\Omega; V)$ .*

## 5 | Gradient Integrability for Bounded BD-Minimisers

In this section, we present and extend the results from [BEG26]. More precisely, we show for the whole ellipticity range  $\mu \in (1, 3]$  that every *bounded* relaxed minimiser  $u \in \text{BD}_{\text{loc}}(\Omega) \cap L_{\text{loc}}^\infty(\Omega; \mathbb{R}^n)$  of the variational principle

$$\text{to minimise } F[v; \Omega] := \int_{\Omega} f(\varepsilon(v)) \, dx \quad \text{over } v \in \mathcal{D}_{u_0} := u_0 + \text{LD}_0(\Omega),$$

has a integrable gradient. The main result will be the following:

**Theorem 5.1** (Gradient integrability -  $L^\infty$ -constraint). *Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded set and consider a variational integrand  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  which satisfies (1.3) and (1.5) with  $\mu \in (1, 3]$ . Then any bounded local BD-minimiser  $u \in \text{BD}_{\text{loc}}(\Omega) \cap L_{\text{loc}}^\infty(\Omega; \mathbb{R}^n)$  of  $F$  is of class  $W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^n)$ . More specifically, for every subset  $U \Subset \Omega$  there exists a positive constant  $c = c(n, \Omega, \lambda, \Lambda, \gamma, \Gamma, \|u\|_{L^\infty(U; \mathbb{R}^n)}) > 0$  such that whenever  $B_{2r}(x_0) \Subset U$  we have the bound*

$$\int_{B_r(x_0)} |\nabla u| \log(1 + |\nabla u|^2) \, dx \leq c \left(1 + \frac{1}{r^2}\right) \left(r^n + r^{n-2} + |Eu|(\overline{B}_{2r}(x_0))\right). \quad (5.1)$$

*Structure of the chapter.* In Section 5.1 we specify the correct application of the approximation scheme from Chapter 4. To do so, we have to improve certain estimates following from the explicit construction. In Section 5.2 we prove weighted uniform estimates in order to justify the limit passage in the proof of Theorem 5.1. Moreover, Section 5.3 is devoted to the main theorem and the corresponding proof. Finally, we can improve the results, if we restrict ourselves to a stricter ellipticity range namely  $\mu \in (1, 3)$ . This is the content of Theorem 5.11, the proof of which will be given in Section 5.4.

### 5.1 The Ekeland-type Viscosity Approximation

In this section, we apply the results from Chapter 4 to the symmetric gradient  $\mathbb{A} = \varepsilon$  under a local  $L^\infty$ -constraint, focusing on the case  $q = n + 1$ , where the higher-order regularisation is essential for handling the  $L^\infty$ -constraint. Of course, the latter corresponds to  $p = \infty$  in the framework of Chapter 4. Moreover, as discussed in Example 2.2 we obtain  $\mathcal{R}(\mathbb{A}) = \mathbb{R}_{\text{sym}}^{n \times n}$  as the natural domain for the integrand  $f$ . Let us recall further the constant  $M > 0$ , which was characterised in (4.8).

We begin the construction by taking a constant  $C_M > 0$  for the Morrey embedding  $W^{1,n+1}(\Omega; \mathbb{R}^n) \hookrightarrow C^{0, \frac{1}{n+1}}(\bar{\Omega}; \mathbb{R}^n)$ , cf. [AF03], such that every  $w \in W^{1,n+1}(\Omega; \mathbb{R}^n)$  satisfies

$$|w(x) - w(y)| \leq C_M \|w\|_{W^{1,n+1}(\Omega; \mathbb{R}^n)} |x - y|^{\frac{1}{n+1}} \quad \text{for all } x, y \in \bar{\Omega}. \quad (5.2)$$

In the same spirit, we let  $C_{K,n+1} = C_{K,n+1}(n, \Omega) \geq 1$  be a constant for the Korn-type inequality from Corollary 3.6, i.e. for every  $w \in W^{1,n+1}(\Omega; \mathbb{R}^n)$  there holds

$$\|w\|_{W^{1,n+1}(\Omega; \mathbb{R}^n)} \leq C_{K,n+1} (\|w\|_{L^{n+1}(\Omega; \mathbb{R}^n)} + \|\varepsilon(w)\|_{L^{n+1}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})}). \quad (5.3)$$

Moreover, for the blow-up rate of the norms  $\|\varepsilon(\tilde{u}_j)\|_{L^{n+1}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})}$  we may assume the estimate

$$\|\varepsilon(\tilde{u}_j)\|_{L^{n+1}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})} \leq \mathfrak{q}(j), \quad (5.4)$$

where  $\mathfrak{q}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a convex function of class  $C^2$  with  $\mathfrak{q}(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . We then consider a convex and increasing function  $h: [0, 2) \rightarrow \mathbb{R}_{\geq 0}$  which satisfies  $h \equiv 0$  on  $[0, 1]$ , and is strictly increasing for  $\frac{3}{2} \leq t < 2$  with

$$\frac{1}{\omega_n} \left( \frac{4C_M}{M(2-t)^4} \left( 1 + 2^{\frac{n-1}{2}} \left( \mathcal{L}^n(\Omega) + \left( \mathfrak{q} \left( \frac{1}{2-t} \right) \right)^{n+1} \right) \right) \right)^{n(n+1)} < h(t). \quad (5.5)$$

The construction of such two functions is elementary and we briefly address to it in the Appendix 5.5. As a next step, let  $g: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  be given by  $g(\cdot) = \tilde{g}(|\cdot|)$ , where

$$\tilde{g}(t) := \begin{cases} h(t) & \text{if } 0 \leq t < 2, \\ +\infty & \text{if } t \geq 2, \end{cases}$$

cf. Figure 5.1. We record that the function  $g$  is convex, lower semi-continuous, and its restriction to  $B_2(0)$  is of class  $C^2$ . In particular, we can define  $\mathfrak{g}: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$  through  $\mathfrak{g}(y) = g\left(\frac{y}{M}\right)$  to obtain a convex lower semi-continuous function, whose restriction to  $B_2(0)$  is also of class  $C^2$ . Clearly,  $\mathfrak{g}$  satisfies all relevant properties of Chapter 4, especially (4.12) with  $\Upsilon = \text{id}$ .

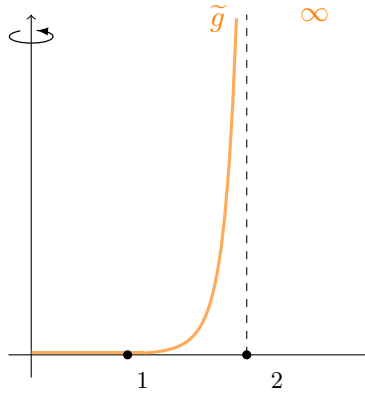


Figure 5.1: The profile  $\tilde{g}$  of the penalisation function  $g$

With this preparations we can now introduce regularised integrands  $f_j: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$  through  $f_j := \mathfrak{f}_{j,n+1}$ , meaning

$$f_j(z) := f(z) + \frac{1}{2A_j j^2} (1 + |z|^2)^{\frac{n+1}{2}} \quad \text{with} \quad A_j := A_{j,n+1} = 1 + \int_{\Omega} (1 + |\varepsilon(\tilde{u}_j)|^2)^{\frac{n+1}{2}} dx.$$

Note, that we omit the  $q$ -dependence in the notation of  $f_j$  as  $q = n + 1$  stays fixed until the end of the chapter. Finally, we introduce regularised and penalised functionals by

$$F_j[w; \Omega] := \begin{cases} \int_{\Omega} f_j(\varepsilon(w)) \, dx + \int_{\Omega} g\left(\frac{w}{M}\right) \, dx & \text{if } w \in \mathcal{D}_j \\ +\infty & \text{if } w \in W^{-2,1}(\Omega; \mathbb{R}^n) \setminus \mathcal{D}_j. \end{cases} \quad (5.6)$$

This functionals fit into the framework of Chapter 4 when choosing  $f_j = \mathbf{f}_{j,n+1}$ ,  $\mathbf{g} = g(\frac{\cdot}{M})$  and  $\Upsilon = \text{id}$ . Hence, we obtain a Ekeland-type approximation sequence  $(v_j)_{j \in \mathbb{N}}$  of almost-minimisers satisfying all the estimates from Proposition 4.2. However, because of the finite blow-up of the function  $g$ , we have to sharpen these estimates in a quantified manner in order to fulfil the finiteness condition from Theorem 4.6. We collect all a priori estimates in the following proposition and give the proof of the remaining arguments:

**Proposition 5.2** (Properties of the approximation sequence). *Let  $(v_j)_{j \in \mathbb{N}}$  be the Ekeland-type approximation sequence from above. Then for every  $j \in \mathbb{N}$  we have*

$$\int_{\Omega} |\varepsilon(v_j)| \, dx \leq \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right), \quad (5.7)$$

$$\frac{1}{2A_j j^2} \int_{\Omega} (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} \, dx \leq \frac{2}{j^2}, \quad (5.8)$$

$$\sup_{j \in \mathbb{N}} \|v_j\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 2M, \quad (5.9)$$

where  $\gamma$  is as in the linear growth condition (1.3) and  $M$  as in (4.8). Moreover, there exist an index  $j_0 \in \mathbb{N}$  with  $j_0 = j_0(n, \Omega, M) \geq 2$  such that there holds

$$\|v_j\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq \left(1 - \frac{2}{j^2}\right) M < 2M \quad \text{for every } j \geq j_0. \quad (5.10)$$

*Proof.* It is obvious that (4.23) and (4.24) correspond to (5.7) and (5.8). Moreover, since  $v_j \in \mathcal{D}_j \cap W^{1,n+1}(\Omega; \mathbb{R}^n)$ , and thus  $v_j \in C(\bar{\Omega}; \mathbb{R}^n)$  using the Morrey embedding, it follows

$$\sup_{\Omega} |v_j| \leq 2M \quad \text{and} \quad \mathcal{L}^n(\{x \in \Omega : |v_j(x)| \geq 2M\}) = 0.$$

This proves (5.9) when taking the supremum over all  $j \in \mathbb{N}$ . Recalling the Korn-type inequality from (5.3), we may now choose a sufficiently large index  $j_0 \in \mathbb{N}_{\geq 2}$  depending only on  $n$ ,  $\Omega$  and  $M$  such that the inequalities

$$C_{K,n+1} \leq j_0^2 \quad \text{and} \quad C_{K,n+1} M \mathcal{L}^n(\Omega)^{\frac{1}{n+1}} \leq j_0^2$$

are simultaneously satisfied. Clearly, (5.8) implies

$$\|\varepsilon(v_j)\|_{L^{n+1}(\Omega; \mathbb{R}^{n \times n})} \leq 4^{\frac{1}{n+1}} A_j^{\frac{1}{n+1}}.$$

Since  $n \geq 2$  and  $A_j \geq 1$  there holds  $4^{\frac{1}{n+1}} \leq 2$  and  $A_j^{\frac{1}{n+1}} \leq A_j$  which yields the estimate  $\|\varepsilon(v_j)\|_{L^{n+1}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})} \leq 2A_j$ . Therefore, in view of (5.3) and (5.9) we then derive

$$\begin{aligned} \|v_j\|_{W^{1,n+1}(\Omega; \mathbb{R}^n)} &\leq C_{K,n+1} (\|v_j\|_{L^{n+1}(\Omega; \mathbb{R}^n)} + \|\varepsilon(v_j)\|_{L^{n+1}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})}) \\ &\leq C_{K,n+1} (2M \mathcal{L}^n(\Omega)^{\frac{1}{n+1}} + 2A_j) \leq 4A_j j^2 \end{aligned}$$

for all  $j \geq j_0$ . Thus, as a consequence of the Morrey inequality (5.2), we find

$$|v_j(x) - v_j(y)| \leq 4C_M A_j j^2 |x - y|^{\frac{1}{n+1}} \quad \text{for all } x, y \in \overline{\Omega}. \quad (5.11)$$

Rephrasing (4.25) in our framework, we obtain the estimate

$$\int_{\Omega} g\left(\frac{v_j}{M}\right) dx \leq \frac{2}{j^2} \leq 1 \quad \text{for } j \geq j_0 \geq 2. \quad (5.12)$$

Moreover, recalling the basic inequality  $(a + b)^m \leq 2^{m-1}(a^m + b^m)$  for  $a, b \in \mathbb{R}_{\geq 0}$  and  $m \in \mathbb{N}$  in combination with (5.4), we infer

$$A_j \leq 1 + 2^{\frac{n-1}{2}} (\mathcal{L}^n(\Omega) + (\mathfrak{q}(j))^{n+1}).$$

Therefore, choosing  $t = 2 - \frac{1}{j}$  in (5.5) leads to the estimate

$$\frac{1}{\omega_n} \left( \frac{4C_M A_j j^4}{M} \right)^{n(n+1)} < \tilde{g} \left( 2 - \frac{1}{j} \right). \quad (5.13)$$

Hence, the strict monotonicity of  $\tilde{g}$  on  $[\frac{3}{2}, 2)$  gives

$$\begin{aligned} \mathcal{L}^n \left( \left\{ x \in \Omega : \frac{|v_j(x)|}{M} > 2 - \frac{1}{j} \right\} \right) &= \mathcal{L}^n \left( \left\{ x \in \Omega : \tilde{g} \left( \frac{|v_j(x)|}{M} \right) > \tilde{g} \left( 2 - \frac{1}{j} \right) \right\} \right) \\ &\leq \frac{1}{\tilde{g} \left( 2 - \frac{1}{j} \right)} \int_{\Omega} g \left( \frac{v_j}{M} \right) dx \\ &\stackrel{(5.12)}{\leq} \frac{1}{\tilde{g} \left( 2 - \frac{1}{j} \right)} \stackrel{(5.13)}{<} \omega_n \left( \frac{M}{4C_M A_j j^4} \right)^{n(n+1)}. \end{aligned}$$

In particular, this inequality implies that the set  $\Omega_j^M := \{x \in \Omega : \frac{|v_j(x)|}{M} > 2 - \frac{1}{j}\}$  cannot contain a ball of radius  $r_j := (M/(4C_M A_j j^4))^{n+1}$ . We now argue that  $|v_j(x)|$  is strictly below  $2M$  with a quantified ( $j$ -dependent) estimate. For this purpose, we distinguish points away from and close to the boundary, cf. Figure 5.2.

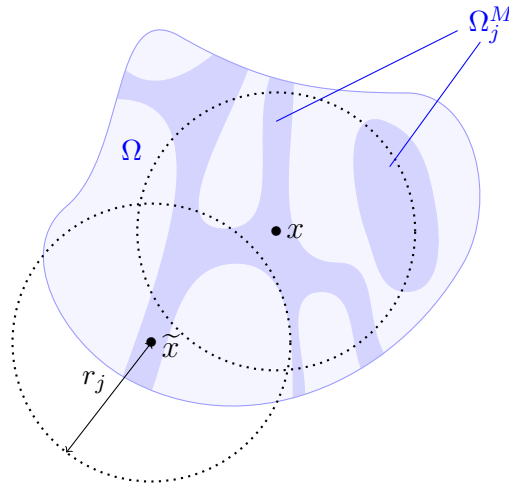


Figure 5.2: The geometric situation from the proof of the quantified  $L^\infty$ -bounds.

Firstly, for  $x \in \Omega$  with  $\text{dist}(x, \partial\Omega) \geq r_j$  (implying  $B_{r_j}(x) \subset \Omega$ ) and  $\frac{|v_j(x)|}{M} > 2 - \frac{1}{j}$ , there exists  $y \in \Omega$  with  $|x - y| < r_j$  and  $\frac{|v_j(y)|}{M} \leq 2 - \frac{1}{j}$ . Since  $v_j \in C^{0, \frac{1}{n+1}}(\bar{\Omega}; \mathbb{R}^n)$  with (5.11), the definition of  $r_j$  leads to

$$\frac{|v_j(x)|}{M} \leq \frac{|v_j(y)|}{M} + \frac{4C_M A_j j^2}{M} |x - y|^{\frac{1}{n+1}} \leq 2 - \frac{1}{j} + \frac{1}{j^2} < 2. \quad (5.14)$$

Secondly, for  $x \in \bar{\Omega}$  with  $\text{dist}(x, \partial\Omega) < r_j$ , there exists a boundary point  $y \in \partial\Omega$  with  $|x - y| < r_j$ . Taking now advantage of  $v_j(y) = u_j^{\partial\Omega}(y)$  combined with the bound (4.6), we find in a similar way

$$\frac{|v_j(x)|}{M} \leq \frac{|u_j^{\partial\Omega}(y)|}{M} + \frac{4C_M A_j j^2}{M} |x - y|^{\frac{1}{n+1}} \leq 1 + \frac{1}{j^2} < 2. \quad (5.15)$$

This case distinction implies

$$\|v_j\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq \left(2 - \frac{1}{j^2}\right)M,$$

and therefore the final claim (5.10) follows. This completes the proof of the Proposition.  $\square$

We point out that by construction and Korn's inequality from Corollary 3.6, the value of the extended functional  $F_j[w; \Omega]$  can only be finite if  $w \in W_{\text{loc}}^{1, n+1}(\Omega; \mathbb{R}^n) \cap L_{\leq 2M}^\infty(\Omega; \mathbb{R}^n)$ . In particular, this is the case if  $w \in C(\bar{\Omega}; \mathbb{R}^n)$  by virtue of the classical Morrey embedding theorem.

As already mentioned, the strict  $j$ -dependent bound (5.10) is necessary to derive the Euler-Lagrange inequality (4.30) from Theorem 4.6. Towards this aim, let  $\varphi \in W_0^{1, n+1}(\Omega; \mathbb{R}^n)$  be arbitrary for which another application of Morrey's inequality yields  $\varphi \in C(\bar{\Omega}; \mathbb{R}^n)$ . Moreover, we infer from (5.10) that there exists  $\theta_0 > 0$  sufficiently small such that  $|v_j \pm \theta\varphi| < 2M$  whenever  $\theta \in (0, \theta_0)$ . In particular, the finiteness condition

$$F_j[v_j \pm \theta\varphi; \Omega] < \infty$$

from Theorem 4.6 is fulfilled for all  $\theta \in (0, \min\{\theta_0, 1\})$  and therefore we infer

$$\left| \int_{\Omega} \langle \nabla f_j(\varepsilon(v_j)), \varepsilon(\varphi) \rangle dx + \int_{\Omega} \left\langle \nabla g\left(\frac{v_j}{M}\right), \frac{\varphi}{M} \right\rangle dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-2,1}(\Omega; \mathbb{R}^n)} \quad (5.16)$$

for all  $j \geq j_0$ . Assuming the growth condition (4.31), we obtain from Lemma 4.8 the higher regularity

$$v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^n) \quad \text{and} \quad (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} |\partial_s \varepsilon(v_j)|^2 \in L_{\text{loc}}^1(\Omega) \quad (5.17)$$

whenever  $j \geq j_0$  and  $s \in \{1, \dots, n\}$ . Omitting again the dependence on  $q = n + 1$ , and denoting  $\sigma_j := \nabla f_j(\varepsilon(v_j))$ , we conclude from Lemma 4.9 the regularity  $\sigma_j \in W_{\text{loc}}^{1, \frac{n+1}{n}}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$  as well as the differentiated Euler-Lagrange inequality

$$\left| \int_{\Omega} \langle \partial_s \sigma_j, \varepsilon(\varphi) \rangle dx + \int_{\Omega} \left\langle \partial_s \left( g\left(\frac{v_j}{M}\right) \right), \frac{\varphi}{M} \right\rangle dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-1,1}(\Omega; \mathbb{R}^n)}, \quad (5.18)$$

for all  $s \in \{1, \dots, n\}$ ,  $\varphi \in W_0^{1, n+1}(\Omega; \mathbb{R}^n)$  and  $j \geq j_0$ . This will be the starting point to derive uniform weighted second-order estimates.

**Remark 5.3** ( $L^\infty$ -threshold). *The specific strategy as displayed above is necessary in order to satisfy the finiteness condition from Remark 4.7. The latter is crucial to conclude the Euler-Lagrange-inequality (5.16). To this end, the key point is the quantified  $L^\infty$ -bound (5.10), whereas the slightly weaker estimate  $\|v_j\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 2M$  turns out not to be sufficient to derive the Euler-Lagrange inequality in the requisite form. Indeed, even for  $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$ ,  $F_j[v_j + t\varphi; \Omega] = \infty$  then could potentially happen howsoever small  $|t| > 0$  might be. Compared with the BV-setting considered in [BS13, Sch15a], the bound (5.10) moreover controls the minimal distance of  $\|v_j\|_{L^\infty(\Omega; \mathbb{R}^n)}$  to the threshold  $2M$  in a precisely quantified way. This is also the chief reason for considering  $W^{1, n+1}$ -stabilisations (instead of  $W^{1, 2}$ -stabilisations considered in [Gme20]), letting us work with bounded and continuous approximations.*

**Remark 5.4** (On bounded minimising sequences). *In the BV-setting of [BS13], the specific minimising sequence  $(u_j)_{j \in \mathbb{N}}$  as chosen at the very beginning of the approximation scheme of Chapter 4 can a priori be taken to belong to  $L_{\leq m}^\infty(\Omega; \mathbb{R}^n)$ . In the situation of [BS13], this is achieved by possibly passing to the truncations*

$$\tilde{u}_j(x) := \begin{cases} u_j(x) & \text{if } |u_j(x)| \leq m, \\ \frac{u_j(x)}{|u_j(x)|} m & \text{if } |u_j(x)| > m, \end{cases}$$

for  $j \in \mathbb{N}$  and  $x \in \Omega$ . The truncated sequence then satisfies  $|\nabla \tilde{u}_j| \leq |\nabla u_j|$ , but it is not clear to us how to show that the same function satisfies a similar pointwise bound when dealing with symmetric gradients. Hence, we use Lemma 2.4.

## 5.2 Weighted Uniform Estimates

Now we will make use of the differentiated Euler-Lagrange inequality (5.18) to derive weighted second-order estimates that are uniform in  $j \in \mathbb{N}$ . These bounds will be the key ingredient in order to prove gradient integrability in the next section.

**Theorem 5.5** (Uniform second-order estimates). *Let  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  satisfy (1.3) and, for some  $\Lambda \in (0, \infty)$ , the bound (4.31). Then there exists a constant  $c = c(n, \Gamma, \Lambda, M) > 0$  such that for every ball  $B_{2r}(x_0) \Subset \Omega$  and every localisation function  $\varrho \in C_c^\infty(\Omega; [0, 1])$  with  $\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq (\frac{2}{r})^s$  for  $s \in \{1, 2, 3\}$  there holds*

$$\begin{aligned} & \sum_{k=1}^n \int_{B_{2r}(x_0)} \varrho^4 \langle \nabla^2 f_j(\varepsilon(v_j)) \partial_k \varepsilon(v_j), \partial_k \varepsilon(v_j) \rangle \, dx \\ & \leq \frac{c}{r^2} \left[ \int_{B_{2r}(x_0)} \left( \frac{1}{r} + \frac{1}{r^2} + \frac{r^2}{j} + |\varepsilon(v_j)| \right) \, dx \right. \\ & \quad \left. + \frac{1}{A_j j^2} \int_{B_{2r}(x_0)} \left( 1 + \frac{1}{r^2} + |\varepsilon(v_j)|^2 \right)^{\frac{n+1}{2}} \, dx \right] \end{aligned} \tag{5.19}$$

for all  $j \geq j_0$ , with  $j_0 \in \mathbb{N}$  as in Proposition 5.2.

**Remark 5.6** (Uniform weighted second-order estimate). *Recalling the estimates (5.7) and (5.8), we see that all quantities at the right-hand side of (5.19) are uniformly bounded in  $j \geq j_0$ . Hence, assuming in addition that the integrand  $f$  satisfies (1.5), we can bound*

(5.19) further from below using the explicit formula for  $\nabla^2 f_j(z)$  from (4.15). This results in

$$\begin{aligned}
& \int_{B_{2r}(x_0)} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{-\frac{\mu}{2}} |\nabla \varepsilon(v_j)|^2 \, dx + \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} |\nabla \varepsilon(v_j)|^2 \, dx \\
& \leq \frac{c(n, \Gamma, \lambda, \Lambda, M)}{r^2} \left[ \int_{B_{2r}(x_0)} |\varepsilon(v_j)| \, dx + \left( \frac{1}{r} + \frac{1}{r^2} + \frac{r^2}{j} \right) \mathcal{L}^n(B_{2r}(x_0)) \right. \\
& \qquad \qquad \qquad \left. + \frac{1}{A_j j^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} \, dx + \frac{1}{j^2} \frac{1}{r} \right] \\
& \leq \frac{c(n, \gamma, \Gamma, \lambda, \Lambda, M)}{r^2} \left[ \inf_{\mathcal{G}u_0} F[-; \Omega] + \left( 1 + \frac{r^2}{j} \right) \mathcal{L}^n(\Omega) + 1 + \frac{1}{j^2} \left( 1 + \frac{1}{r} \right) \right].
\end{aligned}$$

**Remark 5.7** (Weighted estimates for the minimiser). *In Theorem 5.14, we will extract similar estimates for every BD-minimiser  $u$ , which are crucial for obtaining higher-order Sobolev regularity in the strict ellipticity regime  $\mu \in (1, 3)$ , see Section 5.4.*

With the help of Lemma 4.10, we can now address the proof of Theorem 5.5.

*Proof of Theorem 5.5.* Given a point  $x_0 \in \Omega$  and a radius  $r > 0$  such that  $B_{2r}(x_0) \Subset \Omega$  we consider a localisation function  $\varrho \in C_c^\infty(\Omega; [0, 1])$  with  $\mathbf{1}_{B_r(x_0)} \leq \varrho \leq \mathbf{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq \left(\frac{2}{r}\right)^s$  for  $s \in \{1, 2, 3\}$ . For the sake of readability, we divide the proof into several steps.

*Step 1:* For  $k \in \{1, \dots, n\}$  we use the product rule to compute

$$\varrho^4 \partial_k \varepsilon(v_j) = \varrho^4 \varepsilon(\partial_k v_j) = -\nabla \varrho^4 \odot \partial_k v_j + \varepsilon(\varrho^4 \partial_k v_j).$$

Therefore, we can rewrite and rephrase the left-hand side of (5.19) to deduce

$$\begin{aligned}
& \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_k \varepsilon(v_j), \varrho^2 \partial_k \varepsilon(v_j) \rangle \, dx \\
& = \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \varrho^4 \partial_k \varepsilon^{(im)}(v_j) \, dx \\
& = -\frac{1}{2} \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \left[ \partial_i \varrho^4 \partial_k v_j^{(m)} + \partial_m \varrho^4 \partial_k v_j^{(i)} \right] \, dx \\
& \qquad \qquad \qquad + \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \varepsilon^{(im)}(\varrho^4 \partial_k v_j) \, dx \\
& =: \text{A} + \text{B}.
\end{aligned} \tag{5.20}$$

All integrals appearing are well-defined, recalling that  $\partial_k \sigma_j = \nabla^2 f_j(\varepsilon(v_j)) \partial_k \varepsilon(v_j)$  in conjunction with (4.17), and the weighted estimate from (5.17).

*Step 2: Estimate for A.* Since Proposition 5.2 provides only estimates involving the symmetric gradient, we have to smuggle in some terms to create symmetric derivatives.

This gives rise to the following decomposition:

$$\begin{aligned}
\Lambda &= -\frac{1}{2} \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \left[ \partial_i \varrho^4 \partial_k v_j^{(m)} + \partial_i \varrho^4 \partial_m v_j^{(k)} \right] dx \\
&\quad + \frac{1}{2} \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \left[ \partial_i \varrho^4 \partial_m v_j^{(k)} + \partial_m \varrho^4 \partial_i v_j^{(k)} \right] dx \\
&\quad - \frac{1}{2} \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \left[ \partial_m \varrho^4 \partial_i v_j^{(k)} + \partial_m \varrho^4 \partial_k v_j^{(i)} \right] dx \\
&=: \text{I} + \text{II} + \text{III}.
\end{aligned} \tag{5.21}$$

Next, we proceed by estimating the three appearing terms I, II and III gradually. Exploiting the symmetry of  $\sigma_j$ , i.e.  $\sigma_j(x) \in \mathbb{R}_{\text{sym}}^{n \times n}$  for all  $x \in \Omega$  (compare the discussion in Section 2.5), yields the equality I = III. Moreover, we can rewrite the term in square brackets in I as follows:

$$\frac{1}{2} \left[ \partial_i \varrho^4 \partial_k v_j^{(m)} + \partial_i \varrho^4 \partial_m v_j^{(k)} \right] = 4\varrho^3 (\nabla \varrho \otimes (\varepsilon(v_j) e_k))^{(im)} \quad \text{for all } k, i, m \in \{1, \dots, n\}.$$

Once again, we use the symmetry of  $\sigma_j$  in combination with the orthogonal sum decomposition (2.23), the definition of  $\sigma_j$  and the Cauchy-Schwarz inequality for the associated bilinear form of  $\nabla^2 f_j(\varepsilon(v_j))$  as in the proof of Lemma 4.8. This leads to

$$\begin{aligned}
|\text{I} + \text{III}| &\leq 2 |\text{I}| = 8 \left| \sum_{k=1}^n \int_{\Omega} \langle \partial_k \sigma_j, \varrho^3 \nabla \varrho \otimes (\varepsilon(v_j) e_k) \rangle dx \right| \\
&= 8 \left| \sum_{k=1}^n \int_{\Omega} \langle \partial_k \sigma_j, \varrho^3 \nabla \varrho \odot (\varepsilon(v_j) e_k) \rangle dx \right| \\
&= 8 \left| \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_k \varepsilon(v_j), \varrho \nabla \varrho \odot (\varepsilon(v_j) e_k) \rangle dx \right| \\
&\leq \frac{1}{4} \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_k \varepsilon(v_j), \varrho^2 \partial_k \varepsilon(v_j) \rangle dx \\
&\quad + 64 \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho \nabla \varrho \odot (\varepsilon(v_j) e_k), \varrho \nabla \varrho \odot (\varepsilon(v_j) e_k) \rangle dx.
\end{aligned} \tag{5.22}$$

The first term on the right-hand side can be absorbed into the left-hand side of (5.20), whereas the second is bounded thanks to the estimate  $|\nabla \varrho| \leq \frac{2}{r}$  and (4.44) from Lemma 4.10 applied with  $\mathfrak{N} = r^{-1} \varrho \mathbb{1}_{n \times n}$ ,  $w = r \nabla \varrho$  and  $e = \varepsilon(v_j) e_k$  for  $k \in \{1, \dots, n\}$ . More precisely, this results in

$$\begin{aligned}
&64 \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho \nabla \varrho \odot (\varepsilon(v_j) e_k), \varrho \nabla \varrho \odot (\varepsilon(v_j) e_k) \rangle dx \\
&\leq \frac{c\Lambda}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \frac{c(n)}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} dx.
\end{aligned} \tag{5.23}$$

We observe that  $\Pi$  is the most difficult term, since we cannot produce quantities involving the symmetric gradient directly. In order to overcome this issue, we first observe that by symmetry of  $\sigma_j$ , the components satisfy  $\sigma_j^{(im)} = \sigma_j^{(mi)}$  for  $i, m \in \{1, \dots, n\}$ , which yields

$$\Pi = \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \partial_i \varrho^4 \partial_m v_j^{(k)} \, dx.$$

As the appearing derivatives  $\partial_m v_j^{(k)}$  cannot be handled directly, we integrate by parts twice, noting that all computations are justified due to the regularity estimate for  $v_j$  in (5.17). In this way, we obtain

$$\begin{aligned} \Pi &= - \sum_{k,i,m=1}^n \int_{\Omega} \sigma_j^{(im)} \left[ \partial_{ik} \varrho^4 \partial_m v_j^{(k)} + \partial_i \varrho^4 \partial_{mk} v_j^{(k)} \right] \, dx \\ &= \sum_{k,i,m=1}^n \int_{\Omega} \left( \partial_m \left[ \sigma_j^{(im)} \partial_{ik} \varrho^4 \right] v_j^{(k)} + \partial_m \left[ \sigma_j^{(im)} \partial_i \varrho^4 \right] \partial_k v_j^{(k)} \right) \, dx \\ &= \sum_{k,i,m=1}^n \int_{\Omega} \left( \partial_m \sigma_j^{(im)} \partial_{ik} \varrho^4 v_j^{(k)} + \sigma_j^{(im)} \partial_{ikm} \varrho^4 v_j^{(k)} \right. \\ &\quad \left. + \partial_m \sigma_j^{(im)} \partial_i \varrho^4 \partial_k v_j^{(k)} + \sigma_j^{(im)} \partial_{im} \varrho^4 \partial_k v_j^{(k)} \right) \, dx \\ &=: \Pi_1 + \dots + \Pi_4, \end{aligned} \tag{5.24}$$

where now the only derivatives of  $v_j$  appearing in  $\Pi_1, \dots, \Pi_4$  are of the form  $\partial_k v_j^{(k)}$ . After summation over  $k \in \{1, \dots, n\}$  this gives rise to the divergence of  $v_j$ , which can be estimated pointwisely by  $|\varepsilon(v_j)|$ . We proceed by estimating  $\Pi_1, \dots, \Pi_4$  separately:

- On  $\Pi_1$ . Summing over  $k, i, m \in \{1, \dots, n\}$ , we first rewrite the integrand of  $\Pi_1$  using the symmetry of  $\sigma_j$  as

$$\begin{aligned} \sum_{k,i,m=1}^n \partial_m \sigma_j^{(im)} \partial_{ik} \varrho^4 v_j^{(k)} &= \sum_{k,i,m,\ell=1}^n \partial_{\ell} \sigma_j^{(im)} \partial_{ik} \varrho^4 v_j^{(k)} \delta_{\ell m} \\ &= \sum_{\ell=1}^n \langle \partial_{\ell} \sigma_j, (\nabla^2 \varrho^4 v_j) \otimes e_{\ell} \rangle \\ &= \sum_{\ell=1}^n \langle \partial_{\ell} \sigma_j, (\nabla^2 \varrho^4 v_j) \odot e_{\ell} \rangle \\ &= \sum_{\ell=1}^n \langle \nabla^2 f_j(\varepsilon(v_j)) \partial_{\ell} \varepsilon(v_j), (\nabla^2 \varrho^4 v_j) \odot e_{\ell} \rangle. \end{aligned} \tag{5.25}$$

By the product rule we next observe

$$\nabla^2 \varrho^4 = \nabla (4\varrho^3 \nabla \varrho) = 12\varrho^2 \nabla \varrho \otimes \nabla \varrho + 4\varrho^3 \nabla^2 \varrho,$$

and via the Cauchy–Schwarz inequality for the associated bilinear form of  $\nabla^2 f_j(\varepsilon(v_j))$

we then find for  $\text{II}_1$  the estimate<sup>1</sup>

$$\begin{aligned}
|\text{II}_1| &= 12 \left| \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), (\nabla \varrho \cdot v_j) \nabla \varrho \odot e_{\ell} \rangle dx \right. \\
&\quad \left. + 4 \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho (\nabla^2 \varrho v_j) \odot e_{\ell} \rangle dx \right| \\
&\leq \frac{1}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho^2 \partial_{\ell} \varepsilon(v_j) \rangle dx \\
&\quad + 160 \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) (\nabla \varrho \cdot v_j) \nabla \varrho \odot e_{\ell}, (\nabla \varrho \cdot v_j) \nabla \varrho \odot e_{\ell} \rangle dx \\
&\quad + 160 \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho (\nabla^2 \varrho v_j) \odot e_{\ell}, \varrho (\nabla^2 \varrho v_j) \odot e_{\ell} \rangle dx.
\end{aligned} \tag{5.26}$$

For the second and the third integral on the right-hand side, we apply (4.44) from Lemma 4.10, for the choices  $\mathfrak{N} = r \nabla \varrho \otimes \nabla \varrho$ ,  $\mathfrak{N} = r \varrho \nabla^2 \varrho$ ,  $w = r^{-1} v_j$  and  $e = e_{\ell}$  for  $\ell \in \{1, \dots, n\}$ . With  $|\mathfrak{N}| \leq \frac{4}{r}$  (and dropping the factor  $(1 + |z|^2)^{-1/2}$  in the first term) we thus arrive at

$$\begin{aligned}
|\text{II}_1| &\leq \frac{1}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho^2 \partial_{\ell} \varepsilon(v_j) \rangle dx \\
&\quad + \frac{c(n, \Lambda)}{r^2} \int_{B_{2r}(x_0)} \frac{|v_j|^2}{r^2} dx \\
&\quad + \frac{c(n)}{A_j j^2 r^2} \int_{B_{2r}(x_0)} \left( (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} + \frac{|v_j|^{n+1}}{r^{n+1}} \right) dx.
\end{aligned} \tag{5.27}$$

- *On  $\text{II}_3$ .* We initially proceed exactly as for  $\text{II}_1$  and estimate by the Cauchy–Schwarz inequality

$$\begin{aligned}
|\text{II}_3| &= \left| \sum_{\ell=1}^n \int_{\Omega} \langle \partial_{\ell} \sigma_j, (\nabla \varrho^4 \text{div}(v_j)) \odot e_{\ell} \rangle dx \right| \\
&= 4 \left| \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho (\nabla \varrho \text{div}(v_j)) \odot e_{\ell} \rangle dx \right| \\
&\leq \frac{1}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho^2 \partial_{\ell} \varepsilon(v_j) \rangle dx \\
&\quad + 16 \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho (\nabla \varrho \text{div}(v_j)) \odot e_{\ell}, \varrho (\nabla \varrho \text{div}(v_j)) \odot e_{\ell} \rangle dx.
\end{aligned} \tag{5.28}$$

We again use (4.43) from Lemma 4.10 with  $\mathfrak{N} = r^{-1} \mathbf{1}_{n \times n}$ ,  $w = r \varrho \nabla \varrho \text{div}(v_j)$  and  $e = e_{\ell}$  for  $\ell \in \{1, \dots, n\}$ . Together with the pointwise estimate  $|\text{div}(v_j)| \leq |\varepsilon(v_j)|$

<sup>1</sup>Observe that  $(\nabla \varrho \otimes \nabla \varrho) v_j \odot e_{\ell} = (\nabla \varrho \cdot v_j) \nabla \varrho \odot e_{\ell}$ .

this results in

$$|\mathbb{I}_3| \leq \frac{1}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho^2 \partial_{\ell} \varepsilon(v_j) \rangle dx \quad (5.29)$$

$$+ \frac{c(n, \Lambda)}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \frac{c(n)}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} dx.$$

- *On  $\mathbb{I}_2$ .* By means of (4.17) together with  $|\nabla^3 \varrho| \leq \frac{8}{r^3}$  and Young's inequality applied with exponents  $\frac{n+1}{n}$  and  $n+1$  we can estimate

$$|\mathbb{I}_2| = \left| \sum_{k,i,m=1}^n \int_{\Omega} \nabla f_j(\varepsilon(v_j))^{(im)} (\partial_{ikm} \varrho^4) v_j^{(k)} dx \right| \quad (5.30)$$

$$\leq \frac{c(n, \Gamma)}{r^3} \int_{B_{2r}(x_0)} |v_j| dx + \frac{c(n)}{A_j j^2 r^3} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n}{2}} |v_j| dx$$

$$\leq \frac{c(n, \Gamma)}{r^2} \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx + \frac{c(n)}{A_j j^2 r^2} \int_{B_{2r}(x_0)} \left( (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} + \frac{|v_j|^{n+1}}{r^{n+1}} \right) dx.$$

- *On  $\mathbb{I}_4$ .* We proceed similarly as in the estimation of  $\mathbb{I}_2$ , now additionally using the pointwise estimate  $|\operatorname{div}(v_j)| \leq |\varepsilon(v_j)|$ . In view of Young's inequality with exponents  $\frac{n+1}{n-1}$  and  $\frac{n+1}{2}$  we infer

$$|\mathbb{I}_4| = \left| \sum_{k,i,m=1}^n \int_{\Omega} \nabla f_j(\varepsilon(v_j))^{(im)} \partial_{im} \varrho^4 \partial_k v_j^{(k)} dx \right| \quad (5.31)$$

$$\leq \frac{c(n, \Gamma)}{r^2} \int_{B_{2r}(x_0)} |\operatorname{div}(v_j)| dx + \frac{c(n)}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n}{2}} |\operatorname{div}(v_j)| dx$$

$$\leq \frac{c(n, \Gamma)}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \frac{c(n)}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} dx.$$

Combining all the estimates form Step 2, we arrive at

$$A \leq \frac{3}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho^2 \partial_{\ell} \varepsilon(v_j) \rangle dx$$

$$+ \frac{c(n, \Gamma, \Lambda)}{r^2} \int_{B_{2r}(x_0)} \left( |\varepsilon(v_j)| + \frac{|v_j|}{r} + \frac{|v_j|^2}{r^2} \right) dx \quad (5.32)$$

$$+ \frac{c(n)}{A_j j^2 r^2} \int_{B_{2r}(x_0)} \left( (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} + \frac{|v_j|^{n+1}}{r^{n+1}} \right) dx.$$

*Step 3: Estimating B.* The estimate for the second term B, which is given as

$$B = \sum_{k=1}^n \int_{\Omega} \langle \partial_k \sigma_j, \varepsilon(\varrho^4 \partial_k v_j) \rangle dx,$$

can be obtained by exploiting the differentiated Euler-Lagrange inequality stated in (5.18), with the test function  $\varphi = \varrho^4 \partial_k v_j$ . However, we have to justify that this choice of  $\varphi$ , even

though it does not belong to  $W^{1,n+1}(\Omega; \mathbb{R}^n)$ , is admissible in (5.18) for every fixed  $j \geq j_0$ . We recall that we have  $v_j \in W^{1,n+1}(\Omega; \mathbb{R}^n) \cap W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^n)$  from (5.17) meaning that we can choose a sequence  $(h_i)_{i \in \mathbb{N}}$  in  $\mathbb{R}_{>0}$  with  $h_i \searrow 0$  such that for  $k \in \{1, \dots, n\}$  there holds

$$\varrho^4 \Delta_{k,h_i} v_j \rightarrow \varrho^4 \partial_k v_j \quad \text{strongly in } L^{n+1}(\Omega; \mathbb{R}^n), \quad (5.33)$$

$$\varepsilon(\varrho^4 \Delta_{k,h_i} v_j) \rightarrow \varepsilon(\varrho^4 \partial_k v_j) \quad \mathcal{L}^n\text{-a.e. in } \Omega, \quad (5.34)$$

as  $i \rightarrow \infty$ . Next, we consider the Hilbert space  $\mathcal{H} := L_{\mu_j}^2(B_{2r}(x_0); \mathbb{R}_{\text{sym}}^{n \times n})$  with respect to the weighted measure

$$\mu_j := (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} \mathcal{L}^n \llcorner B_{2r}(x_0).$$

Rephrasing (4.40) in our framework yields that  $(\varepsilon(\varrho^4 \Delta_{k,h_i} v_j))_{i \in \mathbb{N}}$  is uniformly bounded in  $\mathcal{H}$ . Therefore, we can extract a non-labeled subsequence converging weakly in  $\mathcal{H}$  whose limit can be identified by the pointwise convergence (5.34) in combination with Lemma 2.19 as  $\varepsilon(\varrho^4 \partial_k v_j)$ . In view of (5.17), the latter also belongs to  $\mathcal{H}$ .

Recalling the estimate  $|\nabla^2 f_j(z)| \leq C(n, \Lambda)(1 + |z|^2)^{\frac{n-1}{2}}$  for all  $z \in \mathbb{R}_{\text{sym}}^{n \times n}$  from (4.17) combined with (5.17) and applying the Cauchy-Schwarz inequality, we notice that the linear functional

$$\Psi_k: \mathcal{H} \ni \psi \mapsto \int_{B_{2r}(x_0)} \langle \partial_k \sigma_j, \psi \rangle dx = \int_{B_{2r}(x_0)} \langle \nabla^2 f_j(\varepsilon(v_j)) \partial_k \varepsilon(v_j), \psi \rangle dx \in \mathbb{R}$$

is well-defined and belongs to the dual space  $\mathcal{H}'$ . Hence, due to the weak convergence in  $\mathcal{H}$ , we have that  $\Psi_k(\varepsilon(\varrho^4 \Delta_{k,h_i} v_j)) \rightarrow \Psi_k(\varepsilon(\varrho^4 \partial_k v_j))$ , i.e.

$$\int_{\Omega} \langle \partial_k \sigma_j, \varepsilon(\varrho^4 \Delta_{k,h_i} v_j) \rangle dx \rightarrow \int_{\Omega} \langle \partial_k \sigma_j, \varepsilon(\varrho^4 \partial_k v_j) \rangle dx \quad \text{as } i \rightarrow \infty.$$

Concerning the second term in the Euler-Lagrange inequality (5.18), we notice that  $|\nabla^2 g(\frac{v_j}{M})|$  is uniformly bounded in  $B_{2r}(x_0)$  by the strict inequality (5.10) together with  $g \in C^2(B_2(0))$ . Taking into account  $\partial_k v_j \in L^{n+1}(\Omega; \mathbb{R}^n) \subset L^{(n+1)/n}(\Omega; \mathbb{R}^n)$ , and the convergence (5.33), we then infer

$$\int_{\Omega} \left\langle \partial_k \left( \nabla g \left( \frac{v_j}{M} \right) \right), \frac{\varrho^4 \Delta_{k,h_i} v_j}{M} \right\rangle dx \rightarrow \int_{\Omega} \left\langle \partial_k \left( \nabla g \left( \frac{v_j}{M} \right) \right), \frac{\varrho^4 \partial_k v_j}{M} \right\rangle dx \quad \text{as } i \rightarrow \infty.$$

Noticing that the limit on the right-hand side is non-negative by the strict convexity of  $g$  on  $B_2(0)$ , we can now take advantage of the differentiated Euler-Lagrange inequality (5.18), applied with the test function  $\varphi = \varrho^4 \Delta_{k,h_i} v_j \in W_0^{1,n+1}(\Omega; \mathbb{R}^n)$  for every  $k \in \{1, \dots, n\}$ . With the limit of the two integrals as established above, this yields

$$\begin{aligned} B &\leq \sum_{k=1}^n \left( \int_{\Omega} \langle \partial_k \sigma_j, \varepsilon(\varrho^4 \partial_k v_j) \rangle dx + \int_{\Omega} \left\langle \partial_k \left( \nabla g \left( \frac{v_j}{M} \right) \right), \frac{\varrho^4 \partial_k v_j}{M} \right\rangle dx \right) \\ &= \lim_{i \rightarrow \infty} \sum_{k=1}^n \int_{\Omega} \langle \partial_k \sigma_j, \varepsilon(\varrho^4 \Delta_{k,h_i} v_j) \rangle dx + \int_{\Omega} \left\langle \partial_k \left( \nabla g \left( \frac{v_j}{M} \right) \right), \frac{\varrho^4 \Delta_{k,h_i} v_j}{M} \right\rangle dx \\ &\leq \frac{1}{j} \lim_{i \rightarrow \infty} \sum_{k=1}^n \|\varrho^4 \Delta_{k,h_i} v_j\|_{W^{-1,1}(\Omega; \mathbb{R}^n)}. \end{aligned}$$

Moreover, with the convergence (5.33) combined with the embedding  $L^{n+1}(\Omega; \mathbb{R}^n) \hookrightarrow W^{-1,1}(\Omega; \mathbb{R}^n)$ , the product rule and the inequalities in (2.16) and (2.17), we obtain for the left-hand side of the previous inequality

$$\lim_{i \rightarrow \infty} \|\varrho^4 \Delta_{k,h_i} v_j\|_{W^{-1,1}(\Omega; \mathbb{R}^n)} \leq \|\varrho^4 \partial_k v_j\|_{W^{-1,1}(\Omega; \mathbb{R}^n)} \leq \left(1 + \frac{4}{r}\right) \|v_j\|_{L^1(B_{2r}(x_0); \mathbb{R}^n)}$$

for  $k \in \{1, \dots, n\}$ . Therefore, we end up with

$$B \leq \frac{c(n)}{j} \left(1 + \frac{1}{r}\right) \|v_j\|_{L^1(B_{2r}(x_0); \mathbb{R}^n)}. \quad (5.35)$$

*Step 4: Conclusion.* We now return to the estimate (5.20) from Step 1. Taking advantage of the estimate (5.32) for A from Step 2, where we can absorb the first term on the left-hand side of (5.20), and of the estimate (5.35) for B from Step 3, we arrive at

$$\begin{aligned} & \sum_{k=1}^n \int_{B_{2r}(x_0)} \varrho^4 \langle \nabla^2 f_j(\varepsilon(v_j)) \partial_k \varepsilon(v_j), \partial_k \varepsilon(v_j) \rangle dx \\ & \leq \frac{c(n, \Gamma, \Lambda)}{r^2} \int_{B_{2r}(x_0)} \left( |\varepsilon(v_j)| + \frac{|v_j|}{r} + \frac{|v_j|^2}{r^2} + \frac{r^2 |v_j|}{j} + \frac{r |v_j|}{j} \right) dx \\ & \quad + \frac{c(n)}{A_j j^2 r^2} \int_{B_{2r}(x_0)} \left( (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} + \frac{|v_j|^{n+1}}{r^{n+1}} \right) dx. \end{aligned}$$

We have employed the fact that the localisation function  $\varrho$  is supported only inside of the ball  $B_{2r}(x_0)$ . At this stage, we utilise the unicorn bound (5.9) on the  $L^\infty$ -norm of  $v_j$  on the right-hand side. Finally, taking into account the elementary inequality  $\frac{r}{j} \leq \frac{1}{r} + \frac{r^2}{j}$ , we then arrive at the precise estimate stated in the theorem, with the claimed dependence of the constant  $c = c(n, \Gamma, \Lambda, M) > 0$ .  $\square$

We conclude this subsection by commenting on a detail of the structure of the above proof that may be non-standard.

**Remark 5.8.** *For the estimation of the terms  $\Pi_1$  and  $\Pi_3$  in the above proof, it might seem more natural and conceptually easier to employ the non-differentiated Euler-Lagrange inequality (5.16) rather than its differentiated analogue (5.18). To be more specific, one might be tempted to write*

$$\Pi_1 = \sum_{k,i,m=1}^n \int_{\Omega} \partial_m \sigma_j^{(im)} \partial_{ik} \varrho^4 v_j^{(k)} dx = \int_{\Omega} \langle \operatorname{div}(\sigma_j), \tau_j \rangle dx = - \int_{\Omega} \langle \sigma_j, \varepsilon(\tau_j) \rangle dx, \quad (5.36)$$

where  $\tau_j$  is defined in the obvious way. Then, if we aim to use (5.16), we are forced to re-introduce the corresponding  $L^\infty$ -penalisation term, leading us to

$$|\Pi_1| \leq \left| \int_{\Omega} \langle \sigma_j, \varepsilon(\tau_j) \rangle dx + \int_{\Omega} \left\langle \nabla g\left(\frac{v_j}{M}\right), \frac{\tau_j}{M} \right\rangle dx \right| + \left| \int_{\Omega} \left\langle \nabla g\left(\frac{v_j}{M}\right), \frac{\tau_j}{M} \right\rangle dx \right| \quad (5.37)$$

Whereas the first term is conveniently controlled by (5.16), the available a priori bounds do not exclude a potential blow-up of the second term as  $j \rightarrow \infty$ . Especially the strict bound in (5.10) is not sufficient to rule out the possibility that

$$\limsup_{j \rightarrow \infty} \|v_j\|_{L^\infty(U; \mathbb{R}^n)} = 2M$$

holds on an open subset  $U \Subset \Omega$ . In this case, uniform  $L^1$ -bounds on  $\varepsilon(\tau_j)$  turn out to be useless. This is the main reason for the more involved algebraic manipulations employed in the above proof. However, the principal advantage is only usable through the passage to the weighted second-order estimates: Proceeding in this way, the difficulties inherent in (5.36)–(5.37) do not vanish completely but rather transfer to the appearance of second-order quantities and two pollution terms in (5.26). It is thus a key point that our approach in the above proof of Theorem 5.5 results in terms which are either absorbable or conveniently controllable, which would not be the case for (5.36) and (5.37).

### 5.3 $L \log L$ -Regularity in the Case $\mu \leq 3$

In this section we give the proof of the main Theorem 5.1 based on the weighted second order estimates from the previous section. Namely we will give the

*Proof of Theorem 5.1.* We split the proof into four steps. To this end, let  $x_0 \in \Omega$  and  $r > 0$  be a radius such that  $B_{2r}(x_0) \Subset \Omega$  and  $\varrho \in C_c^\infty(\Omega; [0, 1])$  be a localisation function with  $\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq \left(\frac{2}{r}\right)^s$  for  $s \in \{1, 2, 3\}$ . Moreover, we will need a special case of (2.22), namely,

$$\log^2(1 + t^2) \leq 16t \quad \text{for all } t \geq 0. \quad (5.38)$$

*Step 1: Derivation of a partially integrated Euler-Lagrange inequality.* To weaken the regularity requirements we make use of a partially integrated version of the Euler-Lagrange inequality (5.16), cf. also Remark 5.10. Recalling  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^n)$  from (5.17), we conclude  $\nabla f(\varepsilon(v_j)) \in W_{\text{loc}}^{1,2}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$ . Moreover, applying Lemma 4.9 for  $q = n + 1$  we obtain  $\nabla f_j(\varepsilon(v_j)) \in W_{\text{loc}}^{1,s}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$  for  $s = \frac{n+1}{n}$  and therefore,

$$\frac{n+1}{2A_j j^2} (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} \varepsilon(v_j) = (\nabla f_j - \nabla f)(\varepsilon(v_j)) \in W_{\text{loc}}^{1, \frac{n+1}{n}}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n}). \quad (5.39)$$

Thus, we are allowed to perform an integration by parts in the regularisation term of the Euler-Lagrange inequality (5.16). Exploiting the symmetry of  $\nabla f_j$  we infer that

$$\sum_{i,m=1}^n \int_{\Omega} (\nabla f_j - \nabla f)(\varepsilon(v_j))^{(im)} \partial_m \varphi^{(i)} \, dx = - \sum_{i,m=1}^n \int_{\Omega} \partial_m ((\nabla f_j - \nabla f)(\varepsilon(v_j))^{(im)}) \varphi^{(i)} \, dx$$

holds for all  $\varphi \in W_0^{1,n+1}(\Omega; \mathbb{R}^n)$ . Therefore, the new partially integrated Euler-Lagrange inequality reads

$$\begin{aligned} & \left| \int_{\Omega} \langle \nabla f(\varepsilon(v_j)), \varepsilon(\varphi) \rangle \, dx - \sum_{i,m=1}^n \int_{\Omega} \partial_m ((\nabla f_j - \nabla f)(\varepsilon(v_j))^{(im)}) \varphi^{(i)} \, dx \right. \\ & \quad \left. + \int_{\Omega} \left\langle \nabla g \left( \frac{v_j}{M} \right), \frac{\varphi}{M} \right\rangle \, dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-2,1}(\Omega; \mathbb{R}^n)}, \end{aligned} \quad (5.40)$$

for all  $\varphi \in W_0^{1,n+1}(\Omega; \mathbb{R}^n)$ . However, because of  $\nabla f(\varepsilon(v_j)) \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$ , (5.39) and

$$\left\| g \left( \frac{v_j}{M} \right) \right\|_{L^\infty(\Omega; \mathbb{R}^n)} < \infty,$$

cf. (5.10), we can approximate maps  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^n) \cap L^{n+1}(\Omega; \mathbb{R}^n)$  by  $C_c^\infty$ -functions in the norm-topology of  $W^{1,1}(\Omega; \mathbb{R}^n) \cap L^{n+1}(\Omega; \mathbb{R}^n)$  to find that (5.40) holds for all competitors  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^n) \cap L^{n+1}(\Omega; \mathbb{R}^n)$ . Hence, the integration by parts in the regularisation term allows to weaken the regularity requirement for the test functions.

*Step 2: Admissibility of the test function.* We want to take the logarithmic test function  $\varphi := \varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) v_j$  in the Euler-Lagrange inequality (5.40) and argue shortly that it belongs to the space  $W_0^{1,1}(\Omega; \mathbb{R}^n) \cap L^{n+1}(\Omega; \mathbb{R}^n)$ . To simplify the notation, we suppress the  $j$ -dependence in the notation of  $\varphi$ , as it is clear from the context. Since  $v_j \in L^\infty(\Omega; \mathbb{R}^n)$  cf. (5.6), and that the logarithm grows more slowly than any root, cf. (2.22), we obtain  $\varphi \in L^{n+1}(\Omega; \mathbb{R}^n)$ . Moreover, both terms of

$$\log^2(1 + |\varepsilon(v_j)|^2) \nabla v_j + v_j \otimes \nabla (\log^2(1 + |\varepsilon(v_j)|^2))$$

are elements in  $L_{\text{loc}}^1(\Omega; \mathbb{R}^n)$ , which can be seen as follows: We immediately observe the bound

$$|\nabla (\log^2(1 + |\varepsilon(v_j)|^2))| \leq 4(1 + |\varepsilon(v_j)|^2)^{-1} |\varepsilon(v_j)| |\nabla \varepsilon(v_j)| \log(1 + |\varepsilon(v_j)|^2), \quad (5.41)$$

and therefore, using (5.38) and  $\log(1 + t^2) \leq 1 + |t|$  for  $t \in \mathbb{R}$ , the claim follows for both terms by means of  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^n)$  from (5.17). Applying the product rule (2.4) and its consequences, we obtain in conclusion that  $\varphi$  is of class  $W_0^{1,1}(\Omega; \mathbb{R}^n) \cap L^{n+1}(\Omega; \mathbb{R}^n)$ . Next, we compute the symmetric gradient of  $\varphi$  directly, namely

$$\begin{aligned} \varepsilon(\varphi) &= \varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) \varepsilon(v_j) \\ &\quad + 4\varrho^3 \nabla \varrho \odot \log^2(1 + |\varepsilon(v_j)|^2) v_j + \varrho^4 \nabla (\log^2(1 + |\varepsilon(v_j)|^2)) \odot v_j. \end{aligned}$$

Substituting this expression into (5.40) and rearranging the terms, we obtain

$$\begin{aligned} \text{I} &:= \int_{\Omega} \varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) \langle \nabla f(\varepsilon(v_j)), \varepsilon(v_j) \rangle dx \\ &\leq - \int_{\Omega} \langle \nabla f(\varepsilon(v_j)), 4\varrho^3 \nabla \varrho \odot \log^2(1 + |\varepsilon(v_j)|^2) v_j \rangle dx \\ &\quad - \int_{\Omega} \langle \nabla f(\varepsilon(v_j)), \varrho^4 \nabla (\log^2(1 + |\varepsilon(v_j)|^2)) \odot v_j \rangle dx \\ &\quad + \sum_{i,m=1}^n \int_{\Omega} \partial_m ((\nabla f_j - \nabla f)(\varepsilon(v_j))^{(im)}) \varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) v_j^{(i)} dx \quad (5.42) \\ &\quad - \int_{\Omega} \varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) \left\langle \nabla g \left( \frac{v_j}{M} \right), \frac{v_j}{M} \right\rangle dx \\ &\quad + \frac{1}{j} \|\varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) v_j\|_{W^{-2,1}(\Omega; \mathbb{R}^n)} \\ &=: \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI}. \end{aligned}$$

*Step 3: Estimating the appearing terms.* From Lemma 2.18 and the elementary logarithmic bound (5.38), we observe

$$\begin{aligned} \text{I} &\geq \gamma \int_{\Omega} \varrho^4 |\varepsilon(v_j)| \log^2(1 + |\varepsilon(v_j)|^2) dx - \Gamma \int_{\Omega} \varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) dx \\ &\geq \gamma \int_{\Omega} \varrho^4 |\varepsilon(v_j)| \log^2(1 + |\varepsilon(v_j)|^2) dx - c(\Gamma) \int_{\Omega} \varrho^4 |\varepsilon(v_j)| dx. \end{aligned} \quad (5.43)$$

The first term of (5.43) is the crucial one for the  $L \log^2 L$ -regularity and therefore, after restructuring the terms in (5.42), we have to estimate the right hand side of

$$\begin{aligned} \gamma \int_{\Omega} \varrho^4 |\varepsilon(v_j)| \log(1 + |\varepsilon(v_j)|^2) dx &\leq \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI} + c(\Gamma) \int_{\Omega} \varrho^4 |\varepsilon(v_j)| dx \\ &=: \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI} + \text{VII}. \end{aligned} \quad (5.44)$$

We argue the boundedness of the appearing terms successively, starting with II together with VII. Towards this aim, we recall from Lemma 2.17 that  $f$  is Lipschitz continuous with bound  $\Gamma$  and that  $v_j$  satisfies (5.9). Therefore, using again the elementary logarithmic bound (5.38), we infer

$$\begin{aligned} \text{II} + \text{VII} &\leq 4 \int_{\Omega} \varrho^3 |\nabla f(\varepsilon(v_j))| |\nabla \varrho| \log^2(1 + |\varepsilon(v_j)|^2) |v_j| dx + c(\Gamma) \int_{\Omega} \varrho^4 |\varepsilon(v_j)| dx \\ &\leq c(\Gamma, M) \left(1 + \frac{1}{r}\right) \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx. \end{aligned}$$

Recalling (5.41), we can apply Young's inequality to treat III by means of

$$\begin{aligned} \text{III} &\leq c(\Gamma, M) \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{-1} |\varepsilon(v_j)| |\nabla \varepsilon(v_j)| \log(1 + |\varepsilon(v_j)|^2) dx \\ &\leq \frac{\gamma}{2} \int_{\Omega} \varrho^4 |\varepsilon(v_j)| \log^2(1 + |\varepsilon(v_j)|^2) dx \\ &\quad + c(\gamma, \Gamma, M) \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{-\frac{\mu}{2}} |\nabla \varepsilon(v_j)|^2 dx. \end{aligned} \quad (5.45)$$

In this way, we have produced a term that can be absorbed into the right-hand side of (5.44) together with a controllable term, using the weighted estimates from Remark 5.6. In a similar fashion, we treat IV, taking into account the estimate

$$|\partial_m(\nabla f_j - \nabla f)(\varepsilon(v_j))| \leq \frac{c(n)}{A_j j^2} (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} |\nabla \varepsilon(v_j)|,$$

which follows immediately from the explicit formula (5.39). Again, by Young's inequality, we proceed to deduce

$$\begin{aligned} \text{IV} &\leq \frac{c(n, M)}{A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} |\nabla \varepsilon(v_j)| \log^2(1 + |\varepsilon(v_j)|^2) dx \\ &\leq \frac{c(n, M)}{A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} |\nabla \varepsilon(v_j)|^2 dx \\ &\quad + \frac{c(n, M)}{A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} \log^4(1 + |\varepsilon(v_j)|^2) dx. \end{aligned} \quad (5.46)$$

Combining the estimates from (5.45) and (5.46) and taking into consideration  $\frac{\mu}{2} - 1 \leq \frac{3}{2} - 1 \leq \frac{1}{2}$ , we can use the weighted second-order estimates from Theorem 5.5 as stated in Remark 5.6, leading to

$$\begin{aligned} \text{III} + \text{IV} &\leq \frac{\gamma}{2} \int_{\Omega} \varrho^4 |\varepsilon(v_j)| \log^2(1 + |\varepsilon(v_j)|^2) dx \\ &\quad + \frac{c(n, \gamma, \Gamma, \lambda, \Lambda, M)}{r^2} \left[ \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \left(\frac{1}{r} + \frac{1}{r^2} + \frac{r^2}{j}\right) \mathcal{L}^n(B_{2r}(x_0)) \right. \\ &\quad \left. + \frac{1}{A_j j^2} (1 + r^2) \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} dx + \frac{1}{j^2} \frac{1}{r} \right]. \end{aligned}$$

We next focus on the treatment of V. By construction the restriction  $g|_{B_2(0)}$  is finite, of class  $C^2$  and convex, meaning that we have the monotonicity inequality

$$0 \leq \langle \nabla g(y_1) - \nabla g(y_2), y_1 - y_2 \rangle < \infty \quad \text{for all } y_1, y_2 \in B_2(0).$$

Recalling  $\nabla g(0) = 0$  in conjunction with  $\|v_j\|_{L^\infty(\Omega; \mathbb{R}^n)} < 2M$ , cf. (5.10), we apply the above monotonicity inequality with  $y_1 = v_j(x)$  and  $y_2 = 0$  to obtain

$$\left\langle \nabla g\left(\frac{v_j}{M}\right), \frac{v_j}{M} \right\rangle \geq 0 \quad \mathcal{L}^n\text{-a.e in } \Omega.$$

Hence, term V has a negative sign, namely,

$$V = - \int_{\Omega} \varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) \left\langle \nabla g\left(\frac{v_j}{M}\right), \frac{v_j}{M} \right\rangle dx \leq 0. \quad (5.47)$$

Finally, VI can be estimated by taking into account (2.15), (2.16), and, once more, the logarithmic inequality (5.38), resulting

$$VI = \frac{1}{j} \|\varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) v_j\|_{W^{-2,1}(\Omega; \mathbb{R}^n)} \leq \frac{c(M)}{j} \|\varepsilon(v_j)\|_{L^1(B_{2r}(x_0); \mathbb{R}_{\text{sym}}^{n \times n})}.$$

This closes the estimation of all appearing terms and thus, collecting all the estimates from above, (5.44) leads to

$$\begin{aligned} & \int_{B_r(x_0)} |\varepsilon(v_j)| \log^2(1 + |\varepsilon(v_j)|^2) dx \\ & \leq \int_{\Omega} \varrho^4 |\varepsilon(v_j)| \log^2(1 + |\varepsilon(v_j)|^2) dx \\ & \leq c \left( 1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{j} \right) \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \left( \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{j} \right) \mathcal{L}^n(B_{2r}(x_0)) \\ & \quad + \frac{c}{A_j j^2} \left( 1 + \frac{1}{r^2} \right) \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} dx + \frac{1}{j^2} \frac{1}{r^3} \\ & \leq c \left( 1 + \frac{1}{r^2} + \frac{1}{j} \right) \int_{B_r(x_0)} |\varepsilon(v_j)| dx + \left( \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{j} \right) \mathcal{L}^n(B_{2r}(x_0)) \\ & \quad + \frac{c}{j^2} \left( 1 + \frac{1}{r^2} \right) + \frac{1}{j} \frac{1}{r^3}, \end{aligned} \quad (5.48)$$

with a constant  $c = c(n, \gamma, \Gamma, \lambda, \Lambda, M) > 0$ .

*Step 3: LD<sub>loc</sub>-regularity.* Our goal is now to use the Reshetnyak (lower semi-)continuity Theorem 2.14 to show that  $E^s u \equiv 0$  in  $B_r(x_0)$ . To this end, we notice that (4.2), (4.7), and (4.21) imply that

$$v_j \rightarrow u \quad \text{as } j \rightarrow \infty \quad \text{in } W^{-2,1}(\Omega; \mathbb{R}^n).$$

Moreover, (5.7) in combination with the compact embedding  $LD(\Omega) \hookrightarrow L^1(\Omega; \mathbb{R}^n)$ , cf. (2.8), gives us a non-reabeled subsequence of  $(v_j)_{j \in \mathbb{N}}$ , such that  $v_j \overset{*}{\rightharpoonup} u$  in  $BD(\Omega)$ . In particular, we infer from (5.48) that

$$\liminf_{j \rightarrow \infty} \int_{B_r(x_0)} |\varepsilon(v_j)| \log^2(1 + |\varepsilon(v_j)|^2) dx \leq c \left( 1 + \frac{1}{r^2} \right) \left( r^{n-2} + |Eu|(\overline{B}_{2r}(x_0)) \right)$$

for a constant  $c = c(n, \gamma, \Gamma, \lambda, \Lambda, M) > 0$ . Recalling the notation  $\Phi_\alpha(t) := t \log^\alpha(1 + t^2)$  from Section 2.2.2, we next consider the function  $\phi : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow [0, \infty)$  by  $\phi(z) := \Phi_2(|z|)$  for  $z \in \mathbb{R}_{\text{sym}}^{n \times n}$ . The corresponding recession function can be computed explicitly, namely, we infer

$$\phi^\infty(z) := \lim_{t \rightarrow \infty} \frac{\phi(tz)}{t} = \lim_{t \rightarrow \infty} |z| \log^2(1 + t^2 |z|^2) = \begin{cases} 0 & \text{if } |z| = 0 \\ +\infty & \text{if } |z| > 0. \end{cases} \quad (5.49)$$

Using  $v_j \xrightarrow{*} u$  in  $\text{BD}(\mathbb{B}_r(x_0))$  in combination with the lower semi-continuity part of Theorem 2.14, we obtain

$$\begin{aligned} \int_{\mathbb{B}_r(x_0)} \phi(\mathcal{E}u) \, dx + \int_{\mathbb{B}_r(x_0)} \phi^\infty \left( \frac{d\mathbb{E}^s u}{d|\mathbb{E}^s u|} \right) d|\mathbb{E}^s u| &= \phi(\mathbb{E}u)(\mathbb{B}_r(x_0)) \\ &\leq \liminf_{j \rightarrow \infty} \phi(\mathbb{E}v_j)(\mathbb{B}_r(x_0)) \leq c \left( 1 + \frac{1}{r^2} \right) \left( r^{n-2} + |\mathbb{E}u|(\overline{\mathbb{B}}_{2r}(x_0)) \right), \end{aligned} \quad (5.50)$$

with the constant  $c = c(n, \gamma, \Gamma, \lambda, \Lambda, M) > 0$  from above. In particular, the right-hand side and thus, also the left-hand side of (5.50), is finite. Therefore, recalling the definition of  $\phi^\infty$  from (5.49), we must have  $\mathbb{E}^s u \equiv 0$  on  $\mathbb{B}_r(x_0)$ . Since  $\mathbb{B}_{2r}(x_0) \Subset \Omega$  was arbitrary we conclude  $\mathbb{E}^s u \equiv 0$  in  $\Omega$ . This gives  $u \in \text{LD}_{\text{loc}}(\Omega; \mathbb{R}^n)$  with  $\mathcal{E}u = \varepsilon(u)$ , whereas (5.50) encodes  $u \in \text{E}^{1, \Phi_2}(\mathbb{B}_r(x_0))$  with the corresponding estimate.

*Step 4: L log L-regularity.* Taking into account (5.50) in combination with the Korn-type inequality from Lemma 3.14 we are able to conclude the claimed L log L-regularity for the gradient. More precisely, we have

$$\begin{aligned} \int_{\mathbb{B}_r(x_0)} |\nabla u| \log(1 + |\nabla u|^2) \, dx &= \int_{\mathbb{B}_r(x_0)} \Phi_1(|\nabla u|) \, dx \\ &\leq c \left( r^n + \int_{\mathbb{B}_r(x_0)} \Phi_1 \left( \frac{|u|}{r} \right) \, dx + \int_{\mathbb{B}_r(x_0)} \Phi_2(|\varepsilon(u)|) \, dx \right) \\ &\leq c \left( 1 + \frac{1}{r^2} \right) \left( r^n + r^{n-2} + |\mathbb{E}u|(\overline{\mathbb{B}}_{2r}(x_0)) \right) \end{aligned}$$

for every ball  $\mathbb{B}_r(x_0)$  with  $\mathbb{B}_{2r}(x_0) \Subset \Omega$  and a constant  $c = c(n, \gamma, \Gamma, \lambda, \Lambda, M) > 0$ . Finally, from Theorem 2.10 and (4.8) we infer that  $M$  only depends on  $\Omega, n$  and  $m$  and therefore, we get the desired estimate (5.1) completing the proof.  $\square$

We conclude this subsection with two remarks on the above proof, both concerning potential improvements and its application to full gradient scenarios.

**Remark 5.9** ( $\text{W}^{1, \text{L log L}}$ -regularity for  $\mu = 3$ ). *The above proof yields that every locally bounded BD-minimiser  $u$  of  $F$  belongs to  $\text{W}_{\text{loc}}^{1, \text{L log L}}(\Omega; \mathbb{R}^n)$  with  $\varepsilon(u) \in \text{L log}^2 \text{L}_{\text{loc}}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$  provided that  $\mu \leq 3$ . It is not clear to us whether the above strategy can be improved for  $\mu = 3$  to obtain*

$$\varepsilon(u) \in \text{L log}^q \text{L}_{\text{loc}}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n}) \quad \text{and in turn} \quad u \in \text{W}_{\text{loc}}^{1, \text{L log}^{q-1} \text{L}}(\Omega; \mathbb{R}^n)$$

by Lemma 3.14 (which is sharp) for some  $q > 2$ . To arrive at this conclusion, one might be inclined to employ the test function  $\varphi = \varrho^4 \log^q(1 + |\varepsilon(v_j)|^2) v_j$  in the partially integrated

Euler-Lagrange inequality (5.40). When estimating term III by means of Young's inequality as above, we then have to control

$$\frac{\gamma}{2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{1}{2}} \log^{2(q-1)} (1 + |\varepsilon(v_j)|^2) dx.$$

By the uniform  $L \log^2 L_{\text{loc}}$ -integrability of  $\varepsilon(v_j)$  or by absorption, this is possible only for  $q \leq 2$ . In the full gradient case and subject to additional structure conditions on the integrands  $f$ , this can be overcome by use of stronger weights in Theorem 5.5 (see e.g. [BS13, Lemma 4.2], [Bil02, Lemma 3.2]). Here, however, the appearance of the symmetric gradients seems to destroy any benefits of such additional assumptions on  $f$ , whereby the local  $L \log^2 L$ -integrability of  $\varepsilon(u)$  might be optimal. We refer to the upcoming Section 5.4 for improvements for  $\mu$ -elliptic integrands with  $\mu \in (1, 3)$ .

**Remark 5.10** (Admissibility of competitors). The integration by parts in Step 1 of the preceding proof is motivated by the fact that the Euler-Lagrange inequality (5.16) requires competitors  $\varphi \in W_0^{1,n+1}(\Omega; \mathbb{R}^n)$ . Aiming to test with  $\varphi = \varrho^4 \log^2(1 + |\varepsilon(v_j)|^2) v_j$ , we however cannot argue by analogous means as in (5.33) ff.. More precisely, we put  $\mathcal{H} = L_{\mu_j}^2(B_{2r}(x_0); \mathbb{R}_{\text{sym}}^{n \times n})$ , where  $\mu_j = (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} \mathcal{L}^n$  is the weighted measure as in Step 3 of the proof of Theorem 5.5. Considering approximations

$$\varphi_h := \varrho^4 \log^\kappa(1 + \beta_j^h) v_j \quad \text{with} \quad \beta_j^h := \frac{1}{4} \sum_{i,m=1}^n |\Delta_{i,h} v_j^{(m)} + \Delta_{m,h} v_j^{(i)}|^2 \quad \text{and} \quad \kappa = 2, \quad (5.51)$$

the desired Euler-Lagrange inequality satisfied by  $\varphi$  then would follow from Lemma 2.19 and

$$\Psi \in \mathcal{H}', \quad \text{where} \quad \Psi: \mathcal{H} \ni \psi \mapsto \int_{B_{2r}(x_0)} \langle \nabla f_j(\varepsilon(v_j)), \psi \rangle dx$$

provided that  $(\varphi_h)_{h>0}$  is bounded in  $\mathcal{H}$ . Expanding the symmetric gradients  $\varepsilon(\varphi_h)$  as

$$\varrho^4 \log^2(1 + \beta_j^h) \varepsilon(v_j) + 2\varrho^4 v_j \odot \frac{\log(1 + \beta_j^h)}{1 + \beta_j^h} \nabla \beta_j^h + (\nabla \varrho^4) \odot \log^2(1 + \beta_j^h) v_j =: J_1^h + J_2^h + J_3^h,$$

The corresponding version of Lemma 4.8, cf. also (5.17), is too weak to conclude that  $(J_2^h)_{h>0}$ , and so  $(\varepsilon(\varphi_h))_{h>0}$ , is bounded in  $\mathcal{H}$ . This could be resolved by setting  $\kappa = 1$  in (5.51), but then comes at the cost of the substantially weaker regularity conclusion  $u \in \text{BV}_{\text{loc}}(\Omega; \mathbb{R}^n)$  by the above proof of Theorem 5.1. The integration by parts circumvents this issue, and also provides a simplification in the full gradient case, see [BS13, Lemma 5.3], where the  $L \log^2 L_{\text{loc}}$ -gradient integrability of local BV-minimisers only follows by use of a two-step argument based on the analogue of (5.51) with  $\kappa = 1$ .

## 5.4 Higher Gradient Integrability in the Case $\mu < 3$ .

Assuming the slightly stronger  $\mu$ -ellipticity condition  $\mu \in (1, 3)$  allows us to derive higher integrability of the gradient, which in turn implies improved Sobolev regularity. The goal of this section is to provide a proof of the following theorem:

**Theorem 5.11** (Higher integrability for  $\mu \in (1, 3)$  – full result). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded and consider a variational integrand  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  which satisfies (1.3) and (1.5) with  $\mu \in (1, 3)$ . Then any bounded local BD-minimiser  $u \in \text{BD}_{\text{loc}}(\Omega) \cap L_{\text{loc}}^\infty(\Omega; \mathbb{R}^n)$  of  $F$  is of class  $W_{\text{loc}}^{1,4-\mu}(\Omega; \mathbb{R}^n)$ .*

In order to prove Theorem 5.11 we proceed in two steps. First, we will work at the level of the minimising sequence to conclude local  $W^{1,1+\alpha}$ -regularity for  $\alpha = \min\{1, 3 - \mu\}$  after passing to the limit, cf. Theorem 5.12. As a second step, we will improve this result for  $\mu \in (1, 2)$ , for which we currently only have  $\alpha = 1$ . Towards this claim, we will work at the level of the minimiser. As a consequence of the weighted second-order estimates for the sequence  $(v_j)_{j \in \mathbb{N}}$ , we derive a corresponding estimate for the minimiser  $u$ , together with higher local integrability, cf. Theorem 5.14. This in turn, allows to use the standard *Euler-Lagrange equation*, which holds for the minimiser  $u$  and allows to conclude also local  $W^{1,4-\mu}$ -regularity in this case, cf. Theorem 5.16. The entire procedure is illustrated in Figure 5.3: The first step is shown in blue, and the second one in orange.

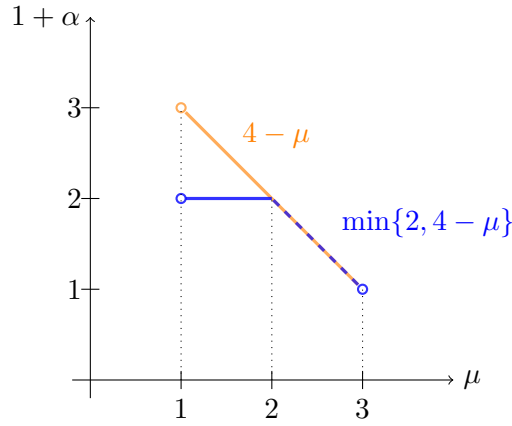


Figure 5.3: Regularity improvement

**Theorem 5.12** (Higher Integrability for  $\mu \in (1, 3)$  – first result). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded and consider a variational integrand  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  which satisfies (1.3) and (1.5) with  $\mu \in (1, 3)$ . Then any bounded local BD-minimiser  $u \in \text{BD}_{\text{loc}}(\Omega) \cap L_{\text{loc}}^\infty(\Omega; \mathbb{R}^n)$  of  $F$  is of class  $W_{\text{loc}}^{1,1+\alpha}(\Omega; \mathbb{R}^n)$  where  $\alpha = \min\{1, 3 - \mu\}$ . More precisely, for every subset  $U \Subset \Omega$  there exist a constant  $c = c(n, \Omega, \gamma, \Gamma, \lambda, \Lambda, \|u\|_{L^\infty(U; \mathbb{R}^n)}) > 0$  such that whenever  $B_{2r}(x_0) \Subset U$  we have*

$$\int_{B_r(x_0)} (1 + |\varepsilon(u)|^2)^{\frac{1+\alpha}{2}} dx \leq c \left(1 + \frac{1}{r^2}\right) \left(r^{n-2} + |\text{Eu}|(\overline{B}_{2r}(x_0))\right) \quad (5.52)$$

**Remark 5.13.** *This result is consistent with that known from the BV-case derived by BILDHAUER, cf. [Bil02, Remark 5.3], worked out explicitly by GMEINER in [Gme16, Theorem 1.4], after noting the equality  $4 - \max\{2, \mu\} = \min\{2, 4 - \mu\}$  for  $\mu \in (1, 3)$ .*

*Proof.* We split the proof into three steps. As usual, let  $x_0 \in \Omega$  and  $r > 0$  be a radius such that  $B_{2r}(x_0) \Subset \Omega$ . Moreover, let  $\varrho \in C_c^\infty(\Omega; [0, 1])$  be a localisation function with

$\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq \left(\frac{2}{r}\right)^s$  for  $s \in \{1, 2, 3\}$ . Since  $\alpha \leq 1$  we will use the elementary estimate

$$t \leq (1 + t^2)^{\frac{\alpha}{2}} \leq 1 + t \quad \text{for all } t \geq 0. \quad (5.53)$$

In the following, let  $j \geq j_0$  where  $j_0 \in \mathbb{N}$  denotes the index from Proposition 5.2, and set  $\alpha := \min\{1, 3 - \mu\}$ . Here, we suppress the  $\mu$ -dependence, as it is clear from the context.

*Step 1: Choice of the test function and admissibility.* We want to use

$$\varphi := \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} v_j$$

as a test function in the partially integrated Euler-Lagrange inequality (5.40). Therefore, we argue quickly that  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^n) \cap L^{n+1}(\Omega; \mathbb{R}^n)$ . Similar to the proof of Theorem 5.1, the argument relies on the product rule (2.4) and its consequences. To this end, we note that the product  $v_j (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}}$  is a well-defined element of  $L_{\text{loc}}^1(\Omega; \mathbb{R}^n)$  as  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^n)$  and  $\alpha \leq 1$ . Moreover, both terms of

$$\nabla v_j (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} + v_j \otimes (\nabla (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}})$$

are in  $L_{\text{loc}}^1(\Omega; \mathbb{R}^n)$ . For the first term, we use again  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^n)$  together with  $\alpha \leq 1$ , whereas for the second we observe the estimate

$$|\nabla (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}}| \leq \alpha (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2} - 1} |\varepsilon(v_j)| |\nabla \varepsilon(v_j)|. \quad (5.54)$$

As the exponent  $\frac{\alpha}{2} - 1$  is negative for  $\alpha \leq 1$  we can drop this term completely and argue once more with  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^n)$ . Finally, as  $v_j \in W^{1,n+1}(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$  we gain that  $\varphi \in W_c^{1,1}(\Omega; \mathbb{R}^n) \cap L^{n+1}(\Omega; \mathbb{R}^n)$  and hence is admissible. Since we may now use  $\varphi$  as a test function, we compute its symmetric gradient which is given by

$$\varepsilon(\varphi) = \varepsilon(v_j) (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} \varrho^4 + 4\varrho^3 \nabla \varrho \odot (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} v_j + \varrho^4 \nabla ((1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}}) \odot v_j.$$

Plugging this expression into the partially integrated Euler-Lagrange inequality (5.40) and rearranging terms leads to

$$\begin{aligned} \text{I} &:= \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} \langle \nabla f(\varepsilon(v_j)), \varepsilon(v_j) \rangle dx \\ &\leq - \int_{\Omega} \langle \nabla f(\varepsilon(v_j)), 4\varrho^3 \nabla \varrho \odot (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} v_j \rangle dx \\ &\quad - \int_{\Omega} \langle \nabla f(\varepsilon(v_j)), \varrho^4 \nabla ((1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}}) \odot v_j \rangle dx \\ &\quad + \sum_{i,m=1}^n \int_{\Omega} \partial_m (\nabla f_j - \nabla f)(\varepsilon(v_j))^{(im)} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} v_j^{(i)} dx \quad (5.55) \\ &\quad - \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} \left\langle \nabla g \left( \frac{v_j}{\Gamma} \right), \frac{v_j}{\Gamma} \right\rangle dx \\ &\quad + \frac{1}{j} \|\varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} v_j\|_{W^{-2,1}(\Omega; \mathbb{R}^n)} \\ &=: \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI}. \end{aligned}$$

*Step 2: Estimating the appearing terms.* Using the lower bound from Lemma 2.18, we deduce the lower estimate

$$\begin{aligned} \text{I} &= \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} \langle \nabla f(\varepsilon(v_j)), \varepsilon(v_j) \rangle \, dx \\ &\geq \gamma \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha+1}{2}} \, dx - \Gamma \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} \, dx \\ &\geq \gamma \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha+1}{2}} \, dx - c(\Gamma) \int_{\Omega} \varrho^4 |\varepsilon(v_j)| \, dx, \end{aligned}$$

where we have used (5.53). Restructuring the terms in (5.55) leads then to

$$\begin{aligned} \gamma \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha+1}{2}} \, dx &\leq \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI} + c(\Gamma) \int_{\Omega} \varrho^4 |\varepsilon(v_j)| \, dx \\ &=: \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI} + \text{VII}. \end{aligned} \quad (5.56)$$

We proceed by controlling all appearing terms at the right-hand side of (5.56) gradually. Considering II and VII in conjunction with the upper bound from Lemma 2.17 and (5.53), we obtain

$$\begin{aligned} \text{II} + \text{VII} &\leq c(M, \Gamma) \int_{\Omega} \varrho^3 |\nabla \varrho| (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} \, dx + c(\Gamma) \int_{\Omega} \varrho^4 |\varepsilon(v_j)| \, dx \\ &\leq c(M, \Gamma) \left(1 + \frac{1}{r}\right) \int_{B_{2r}(x_0)} |\varepsilon(v_j)| \, dx. \end{aligned}$$

Now estimate (5.54) leads to a situation in which we can treat III via Young's inequality. More precisely, we can produce a term which can be absorbed into the right-hand side of (5.56) together with a pollution term, namely,

$$\begin{aligned} \text{III} &\leq c(M, \Gamma) \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}-1} |\varepsilon(v_j)| |\nabla \varepsilon(v_j)| \, dx \\ &\leq \frac{\gamma}{2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{1+\alpha}{2}} \, dx + c(\gamma, \Gamma, M) \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha-3}{2}} |\nabla \varepsilon(v_j)|^2 \, dx \\ &\leq \frac{\gamma}{2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{1+\alpha}{2}} \, dx + c(\gamma, \Gamma, M) \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{-\frac{\mu}{2}} |\nabla \varepsilon(v_j)|^2 \, dx. \end{aligned}$$

In the last inequality, we have used  $\alpha \leq 3 - \mu$ , which is trivially satisfied by our choice of  $\alpha$ . For VI, we again apply Young's inequality together with  $\alpha \leq 1$  to deduce

$$\begin{aligned} \text{IV} &\leq \frac{1}{2A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} |\nabla \varepsilon(v_j)| (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} |v_j| \, dx \\ &\leq \frac{c(n, M)}{2A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} |\nabla \varepsilon(v_j)|^2 \, dx + \frac{c(n, M)}{2A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2} + \alpha} \, dx \\ &\leq \frac{c(n, M)}{2A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} |\nabla \varepsilon(v_j)|^2 \, dx + \frac{c(n, M)}{2A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} \, dx. \end{aligned}$$

Furthermore, we can bound the two terms III + IV by means of the weighted second-order

estimates from Remark 5.6 leading to

$$\begin{aligned}
\text{III} + \text{IV} &\leq \frac{\gamma}{2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{1+\alpha}{2}} dx \\
&\quad + c(n, \gamma, \Gamma, M) \left[ \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{-\frac{\mu}{2}} |\nabla \varepsilon(v_j)|^2 dx \right. \\
&\quad \left. + \frac{1}{2A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n-1}{2}} |\nabla \varepsilon(v_j)|^2 dx \right] \\
&\quad + \frac{c(n, \Gamma)}{2A_j j^2} \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} dx \\
&\leq \frac{c(n, \gamma, \Gamma, \lambda, \Lambda, M)}{r^2} \left[ \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \left( \frac{1}{r} + \frac{1}{r^2} + \frac{r^2}{j} \right) \mathcal{L}^n(B_{2r}(x_0)) + \frac{1}{j^2} \frac{1}{r} \right. \\
&\quad \left. + \frac{1}{A_j j^2} (1 + r^2) \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} dx \right].
\end{aligned}$$

Next, we observe that term V can be handled completely analogous to the corresponding term (5.47) in the proof of Theorem 5.1. It has no impact due to the negative sign. Finally, by combining the negative Sobolev norm estimates from (2.15) and (2.16) with (5.53), we conclude

$$\begin{aligned}
\text{VI} &= \frac{1}{j} \|\varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} v_j\|_{W^{-2,1}(\Omega; \mathbb{R}^n)} \leq \frac{1}{j} \|\varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha}{2}} v_j\|_{L^1(\Omega; \mathbb{R}^n)} \\
&\leq \frac{c(M)}{j} \|\varepsilon(v_j)\|_{L^1(B_{2r}(x_0); \mathbb{R}^n)}.
\end{aligned}$$

This completes the estimation of all terms, and thus, by collecting the above bounds, (5.56) together with (5.8) yields

$$\begin{aligned}
&\int_{B_r(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha+1}{2}} dx \\
&\leq \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{\alpha+1}{2}} dx \\
&\leq c \left( 1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{j} \right) \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \left( \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{j} \right) \mathcal{L}^n(B_{2r}(x_0)) \quad (5.57) \\
&\quad + \frac{c}{A_j j^2} \left( 1 + \frac{1}{r^2} \right) \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}} dx + \frac{1}{j} \frac{1}{r^3} \\
&\leq c \left( 1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{j} \right) \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \left( \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{j} \right) \mathcal{L}^n(B_{2r}(x_0)) \\
&\quad + \frac{c}{j^2} \left( 1 + \frac{1}{r^2} \right) + \frac{1}{j} \frac{1}{r^3},
\end{aligned}$$

where  $c = c(n, \gamma, \Gamma, \lambda, \Lambda, M) > 0$ .

*Step 3: Sobolev regularity.* As an immediate consequence of (5.57) we obtain that the sequence  $(\varepsilon(v_j))_{j \geq j_0}$  is uniformly bounded in  $L^{1+\alpha}(B_r(x_0); \mathbb{R}^n)$ . Therefore, since  $\mu < 3$ ,

the space  $L^{1+\alpha}(B_r(x_0); \mathbb{R}^n)$  is reflexive and we can apply the Banach-Alaoglu theorem to extract a non-relabeled subsequence of  $(v_j)_{j \in \mathbb{N}}$  such that

$$v_j \xrightarrow{*} u \quad \text{in} \quad \text{BD}(\Omega) \quad \text{and} \quad \varepsilon(v_j) \rightharpoonup w \in L_{\text{loc}}^{1+\alpha}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$$

as  $j \rightarrow \infty$ . Note that the weak\*-convergence in  $\text{BD}(\Omega)$  follows from the same argument as in Step 3 of the proof of Theorem 5.1. As a final step, we identify the limit by arguing  $\varepsilon(u) = w$ . Towards this aim, we recall from the proof of Theorem 5.1 that  $E^s u \equiv 0$  on  $\Omega$  and consider some arbitrary  $\psi \in C_0(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$ . Exploiting the Lebesgue-Radon-Nikodým decomposition  $Eu = \mathcal{E}(u)\mathcal{L}^n + E^s u = \varepsilon(u)\mathcal{L}^n$ , we infer

$$\begin{aligned} \langle Eu, \psi \rangle_{\text{RM}_{\text{fin}}, C_0} &= \langle \varepsilon(u)\mathcal{L}^n, \psi \rangle_{\text{RM}_{\text{fin}}, C_0} = \int_{\Omega} \langle \varepsilon(u), \psi \rangle dx \\ &= \lim_{j \rightarrow \infty} \int_{\Omega} \langle \varepsilon(v_j), \psi \rangle dx = \int_{\Omega} \langle w, \psi \rangle dx, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle_{\text{RM}_{\text{fin}}, C_0}$  denotes the dual pairing in the space of matrix-valued Radon measures. Since  $\psi$  was arbitrarily chosen, it follows  $\varepsilon(u) = w$  in  $\Omega$ . Thus, passing to the limit in (5.57) as  $j \rightarrow \infty$  we obtain

$$\int_{B_r(x_0)} (1 + |\varepsilon(u)|^2)^{\frac{1+\alpha}{2}} dx \leq c \left(1 + \frac{1}{r^2}\right) \left(r^{n-2} + |Eu|(\overline{B}_{2r}(x_0))\right)$$

for a constant  $c = c(n, \gamma, \Gamma, \lambda, \Lambda, M) > 0$ . Finally, from Theorem 2.10 and (4.8) we infer that  $M$  only depends on  $\Omega, n$  and  $m$  and therefore we get the desired estimate (5.52) completing the proof.  $\square$

As announced at the beginning of the subsection, we aim to eliminate the minimum from  $\alpha$  in the previous theorem. We will now work at the level of the minimiser and derive weighted second-order estimates for them:

**Theorem 5.14** (Second-order estimates for minimisers). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded and consider a variational integrand  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  which satisfies (1.3) and (1.5) with  $\mu \in (1, 2)$ . Then any bounded local BD-minimiser  $u \in \text{BD}_{\text{loc}}(\Omega) \cap L_{\text{loc}}^{\infty}(\Omega; \mathbb{R}^n)$  of  $F$  is of class  $W_{\text{loc}}^{2,s}(\Omega, \mathbb{R}^n)$  for  $s = \frac{4}{2+\mu}$ . Moreover, for every subset  $U \Subset \Omega$  there exist a constant  $c = c(n, \Omega, \gamma, \Gamma, \lambda, \Lambda, \|u\|_{L^{\infty}(U; \mathbb{R}^n)}) > 0$  such that whenever  $B_{2r}(x_0) \Subset U$  we have the weighted second-order estimates*

$$\begin{aligned} &\int_{B_{2r}(x_0)} (1 + |\varepsilon(u)|^2)^{\frac{2-\mu}{2}-2} |\nabla |\varepsilon(u)|^2|^2 dx \\ &\leq \frac{c}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(u)| dx + \left(\frac{1}{r} + \frac{1}{r^2}\right) \mathcal{L}^n(B_{2r}(x_0)). \end{aligned} \tag{5.58}$$

*Proof.* Let  $U \Subset \Omega$  such that  $B_{2r}(x_0) \Subset U$ , and  $\mu \in (1, 2)$ . From the proof of Theorem 5.12 we know that the sequence  $(\varepsilon(v_j))_{j \geq j_0}$  is uniformly bounded in  $L_{\text{loc}}^2(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$ . Moreover, we have the weighted second-order estimates from Remark 5.6. Next, we use Young's inequality with exponents  $\frac{2+\mu}{2}$  and  $\frac{2+\mu}{\mu}$  to conclude the estimate

$$\begin{aligned} \int_{B_{2r}(x_0)} |\nabla \varepsilon(v_j)|^{\frac{4}{2+\mu}} dx &= \int_{B_{2r}(x_0)} |\nabla \varepsilon(v_j)|^{\frac{4}{2+\mu}} (1 + |\varepsilon(v_j)|^2)^{\frac{\mu}{2+\mu}} (1 + |\varepsilon(v_j)|^2)^{-\frac{\mu}{2+\mu}} dx \\ &\leq \int_{B_{2r}(x_0)} |\nabla \varepsilon(v_j)|^2 (1 + |\varepsilon(v_j)|^2)^{-\frac{\mu}{2}} dx + \int_{B_{2r}(x_0)} |\varepsilon(v_j)|^2 dx. \end{aligned}$$

Abbreviating  $s = \frac{4}{2+\mu}$ , we infer that the sequence  $(\varepsilon(v_j))_{j \geq j_0}$  is uniformly bounded in the space  $W^{1,s}(B_{2r}(x_0); \mathbb{R}^n)$ . Especially, we can pass to a non-relabeled subsequence satisfying

$$\nabla \varepsilon(v_j) \rightharpoonup \nabla \varepsilon(u) \quad \text{in } L^s(B_{2r}(x_0); \odot^2 \mathbb{R}_{\text{sym}}^{n \times n}) \quad \text{as } j \rightarrow \infty.$$

The claimed regularity  $u \in W_{\text{loc}}^{2,s}(\Omega; \mathbb{R}^n)$  follows immediately using the weak lower semi-continuity of the norm in conjunction with Korn's inequality from Corollary 3.6. Exploiting the Sobolev embedding  $W^{1,s}(B_{2r}(x_0); \mathbb{R}^n) \hookrightarrow L^{s^*}(B_{2r}(x_0); \mathbb{R}^n)$ , where  $s^* = \frac{ns}{n-s}$  is the Sobolev conjugate, we can extract a further non-relabeled subsequence to guarantee  $\varepsilon(v_j) \rightarrow \varepsilon(v)$  pointwise  $\mathcal{L}^n$ -a.e. in  $B_{2r}(x_0)$ . Recalling the notion from (2.20), we consider

$$V(z) := V_{\frac{2-\mu}{4}}(z) = (1 + |z|^2)^{\frac{2-\mu}{4}} - 1 \quad \text{for } z \in \mathbb{R}_{\text{sym}}^{n \times n}.$$

Now, an explicit computation shows  $\nabla V(\varepsilon(v_j)) = \frac{2-\mu}{4}(1 + |\varepsilon(v_j)|^2)^{\frac{2-\mu}{4}-1} \nabla |\varepsilon(v_j)|^2$  and hence, there holds

$$|\nabla V(\varepsilon(v_j))|^2 = \left( \frac{2-\mu}{2} \right)^2 (1 + |\varepsilon(v_j)|^2)^{\frac{2-\mu}{2}-2} |\nabla |\varepsilon(v_j)|^2|^2.$$

In particular, we conclude

$$\int_{B_{2r}(x_0)} |\nabla V(\varepsilon(v_j))|^2 dx \leq C \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{-\frac{\mu}{2}} |\nabla \varepsilon(v_j)|^2 dx. \quad (5.59)$$

and thus, using the weighted second-order estimates from Remark 5.6, also that the sequence  $(V(\varepsilon(v_j)))_{j \in \mathbb{N}}$  is uniformly bounded in  $W^{1,2}(B_{2r}(x_0); \mathbb{R}^n)$ . Therefore, we can extract another non-relabeled subsequence such that  $V(\varepsilon(v_j)) \rightharpoonup g$  weakly as  $j \rightarrow \infty$  for some  $g \in W^{1,2}(B_{2r}(x_0); \mathbb{R}^n)$ . Since  $\varepsilon(v_j) \rightarrow \varepsilon(u)$  pointwise  $\mathcal{L}^n$ -a.e. we can conclude that

$$V(\varepsilon(v_j)) \rightarrow V(\varepsilon(u)) \quad \mathcal{L}^n\text{-a.e. in } B_{2r}(x_0),$$

and consequently, Lemma 2.19 yields  $g = V(\varepsilon(u))$ . Using again the weak lower semi-continuity of the norm, we infer

$$\begin{aligned} \int_{B_{2r}(x_0)} (1 + |\varepsilon(u)|^2)^{\frac{2-\mu}{2}-2} |\nabla |\varepsilon(u)|^2|^2 dx &= \frac{1}{2} \int_{B_{2r}(x_0)} |\nabla V(\varepsilon(u))|^2 dx \\ &\leq \liminf_{j \rightarrow \infty} \frac{1}{2} \int_{B_{2r}(x_0)} |\nabla V(\varepsilon(v_j))|^2 dx. \end{aligned}$$

Combining the previous inequality with (5.59) we deduce, after passing to the limit  $j \rightarrow \infty$  in the weighted second-order estimate from Remark 5.6,

$$\begin{aligned} &\int_{B_{2r}(x_0)} (1 + |\varepsilon(u)|^2)^{\frac{2-\mu}{2}-2} |\nabla |\varepsilon(u)|^2|^2 dx \\ &\leq \frac{c}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(u)| dx + \left( \frac{1}{r} + \frac{1}{r^2} \right) \mathcal{L}^n(B_{2r}(x_0)). \end{aligned}$$

This is the desired inequality and finishes the proof.  $\square$

**Remark 5.15.** Comparing the outcome of Theorem 5.14 with the corresponding bound given in Remark 5.6, we observe, that they differ slightly. However, this is only a technicality, as we have the same scaling in both estimates, which is sufficient for our purposes.

Now, that we have the weighted second-order estimates for  $u$ , we can continue our regularity studies. Unlike in Theorem 5.12 we can now utilise the Euler-Lagrange inequality, since we know already  $u \in W_{\text{loc}}^{1,2}(\Omega; \mathbb{R}^n)$  for  $\mu \in (1, 2)$ . This significantly shortens the proof, as we do not have to consider the regularisation terms when working at the level of the minimiser.

**Theorem 5.16** (Higher integrability for  $\mu \in (1, 2)$  – Improvement). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded, and let  $f \in C^2(\mathbb{R}_{\text{sym}}^{n \times n})$  be a variational integrand which satisfies (1.3) and (1.5) with  $\mu \in (1, 2)$ . Then any bounded local BD-minimiser  $u \in \text{BD}_{\text{loc}}(\Omega) \cap L_{\text{loc}}^\infty(\Omega, \mathbb{R}^n)$  of  $F$  is of class  $W_{\text{loc}}^{1,4-\mu}(\Omega; \mathbb{R}^n)$ . More precisely, for every subset  $U \Subset \Omega$  there exists a constant  $c = c(n, \Omega, \gamma, \Gamma, \lambda, \Lambda, \|u\|_{L^\infty(U; \mathbb{R}^n)}) > 0$  such that whenever  $B_{2r}(x_0) \Subset U$  we have*

$$\begin{aligned} & \int_{B_r(x_0)} (1 + |\varepsilon(u)|^2)^{\frac{4-\mu}{2}} dx \\ & \leq c \left[ \left(1 + \frac{1}{r}\right) \int_{B_{2r}(x_0)} |\varepsilon(u)|^2 dx + \frac{1}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(u)| dx + \left(\frac{1}{r} + \frac{1}{r^2}\right) \mathcal{L}^n(B_{2r}(x_0)) \right], \end{aligned}$$

*Proof.* As before, we take a point  $x_0 \in \Omega$ , a radius  $r > 0$  such that  $B_{2r}(x_0) \Subset \Omega$  and a cut-off function  $\varrho \in C_c^\infty(\Omega; [0, 1])$  with  $\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq \left(\frac{2}{r}\right)^s$  for  $s \in \{1, 2, 3\}$ . We recall from Theorem 5.12 that there holds  $u \in W_{\text{loc}}^{1,2}(\Omega; \mathbb{R}^n)$  in case  $\mu \in (1, 2)$ . Therefore,  $u$  is a weak solution of the Euler-Lagrange equation

$$\int_{\Omega} \langle \nabla f(\varepsilon(u)), \varepsilon(\varphi) \rangle dx = 0 \quad \text{for all } \varphi \in W_0^{1,1}(\Omega; \mathbb{R}^n), \quad (5.60)$$

which can be seen as follows: For an arbitrary  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^n)$  we consider the perturbation  $u + \theta\varphi$  for  $\theta > 0$ . Since  $u$  is a minimiser, we have  $F[u + \theta\varphi; \Omega] \geq F[u; \Omega]$  and the first variation of  $F$  vanishes, i.e.

$$0 = \delta F(u, \varphi) := \frac{d}{d\theta} \Big|_{\theta=0} F[u + \theta\varphi; \Omega] = \lim_{\theta \rightarrow 0} \frac{1}{\theta} \left( F[u + \theta\varphi; \Omega] - F[u; \Omega] \right).$$

In order to compute this limit, we exploit the dominated convergence theorem. More precisely, we consider

$$\frac{1}{\theta} \left( F[u + \theta\varphi; \Omega] - F[u; \Omega] \right) = \frac{1}{\theta} \int_{\Omega} \int_0^1 \frac{d}{dt} f(\varepsilon(u) + t\theta\varepsilon(\varphi)) dt dx = \int_{\Omega} g(x, \theta) dx$$

with

$$g(x, \theta) := \frac{1}{\theta} \int_0^1 \frac{d}{dt} f(\varepsilon(u) + t\theta\varepsilon(\varphi)) dt = \int_0^1 \langle \nabla f(\varepsilon(u) + t\theta\varepsilon(\varphi)), \varepsilon(\varphi) \rangle dt.$$

Since  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^n)$  and Lemma 2.17 provides an integrable majorant for  $g$ , we can pass to the limit  $\theta \rightarrow 0$  to infer (5.60).

*Step 1: Admissibility of the test function.* We want to use  $\varphi := \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}} u$  as a test function in this Euler-Lagrange equation (5.60), where  $\mathbb{T}_k$  denotes the truncation

operator at level  $k > 0$ , cf. (2.5). We briefly comment on the admissibility, arguing  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^n)$  based on the product rule (2.4). Since  $u \in L_{\text{loc}}^\infty(\Omega; \mathbb{R}^n)$  and the truncation is bounded by definition, we conclude that the product  $u(\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}}$  is an element of  $L_{\text{loc}}^1(\Omega; \mathbb{R}^n)$ . Moreover, both terms of

$$\nabla u(\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}} + \mathbb{1}_{\{1+|\varepsilon(u)|^2 \leq k\}}(\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{1-\mu}{2}} u \otimes \nabla |\varepsilon(u)|^2,$$

are elements of  $L_{\text{loc}}^1(\Omega; \mathbb{R}^n)$ , which can be justified as follows: For the first term we use  $u \in W_{\text{loc}}^{1,2}(\Omega; \mathbb{R}^n)$  in combination with the boundedness of the truncation, whereas the second has to be treated by means of the weighted-order estimates from Theorem 5.14. More precisely, there holds

$$\begin{aligned} & \int_{\Omega} \varrho^4 \mathbb{1}_{\{1+|\varepsilon(u)|^2 \leq k\}} (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{1-\mu}{2}} |u \otimes \nabla |\varepsilon(u)|^2| \, dx \\ & \leq c(M, k) \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{1-\mu}{2}} |\nabla |\varepsilon(u)|^2| \, dx \\ & = c(M, k) \int_{\Omega} \left( \varrho^2 |\nabla |\varepsilon(u)|^2| (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{1}{2}(\frac{2-\mu}{2}-2)} \right) \\ & \quad \left( \varrho^2 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{1-\mu}{2} - \frac{1}{2}(\frac{2-\mu}{2}-2)} \right) \, dx \tag{5.61} \\ & \leq c(M, k) \left( \int_{\Omega} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{\frac{2-\mu}{2}-2} |\nabla |\varepsilon(u)|^2|^2 \, dx \right. \\ & \quad \left. + \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{4-\mu}{2}} \, dx \right), \end{aligned}$$

where all appearing quantities on the right hand side are bounded, noting that the appearing constant may be  $k$ -dependent. Finally, if we multiply the above product with the localisation function  $\varrho^4$  we gain  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^n)$ , closing the argument.

*Step 2: Testing the Euler-Lagrange equation.* Since  $\varphi$  is admissible by Step 1, we can plug it into the Euler-Lagrange equation (5.60). Therefore, we need to compute the symmetric gradient  $\varepsilon(\varphi)$  which is given by

$$\begin{aligned} \varepsilon(\varphi) &= \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}} \varepsilon(u) + (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}} 4\varrho^3 \nabla \varrho \odot u \\ & \quad + \varrho^4 \mathbb{1}_{\{1+|\varepsilon(u)|^2 \leq k\}} (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{1-\mu}{2}} \nabla |\varepsilon(u)|^2 \odot u. \end{aligned}$$

Plugging this expression into (5.60), and rearranging terms we obtain

$$\begin{aligned} \text{I} &:= \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}} \langle \nabla f(\varepsilon(u)), \varepsilon(u) \rangle \, dx \\ &= - \int_{\Omega} (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}} \langle \nabla f(\varepsilon(u)), 4\varrho^3 \nabla \varrho \odot u \rangle \, dx \\ & \quad - \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{1-\mu}{2}} \langle \nabla f(\varepsilon(u)), \nabla |\varepsilon(u)|^2 \odot u \rangle \mathbb{1}_{\{1+|\varepsilon(u)|^2 \leq k\}} \, dx \\ &=: \text{II} + \text{III}. \end{aligned}$$

Similarly to the proof of Theorem 5.12 we now have to estimate all the appearing terms. Commemorating Lemma 2.18, we can estimate I from below in terms of

$$I \geq \gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{4-\mu}{2}} dx - \Gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}} dx,$$

leading to

$$\begin{aligned} \gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{4-\mu}{2}} dx &\leq \text{II} + \text{III} + \Gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}} dx \\ &=: \text{II} + \text{III} + \text{IV}. \end{aligned} \quad (5.62)$$

Since all the terms in II and IV are of the same scaling, we may treat them together. As  $u$  is a bounded minimiser, we can apply Lemma 2.17 to infer

$$\begin{aligned} \text{II} + \text{IV} &\leq c(\Gamma, M) \int_{\Omega} (\varrho^3 |\nabla \varrho| + \varrho^4) (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{3-\mu}{2}} dx \\ &\leq c(\Gamma, M) \int_{\Omega} (\varrho^3 |\nabla \varrho| + \varrho^4) (1 + |\varepsilon(u)|^2)^{\frac{3-\mu}{2}} dx \\ &\leq c(\Gamma, M) \left(1 + \frac{1}{r}\right) \int_{B_{2r}(x_0)} |\varepsilon(u)|^2 dx. \end{aligned}$$

The latter term is bounded independent of the level  $k$ , because of  $u \in W_{\text{loc}}^{1,2}(\Omega; \mathbb{R}^n)$  and noting that  $\frac{3-\mu}{2} \leq 1$  is always fulfilled for  $\mu > 1$  as in our framework. For III, proceeding as in (5.61) produces an additional, absorbable term. In detail, this yields

$$\begin{aligned} \text{III} &\leq c(\Gamma, M) \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{1-\mu}{2}} |\nabla |\varepsilon(u)|^2| \mathbf{1}_{\{1+|\varepsilon(u)|^2 \leq k\}} dx \\ &\leq \frac{\gamma}{2} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{4-\mu}{2}} \mathbf{1}_{\{1+|\varepsilon(u)|^2 \leq k\}} dx \\ &\quad + c(\gamma, \Gamma, M) \int_{\Omega} \varrho^4 |\nabla |\varepsilon(u)|^2|^2 (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{2-\mu}{2}-2} dx. \end{aligned}$$

As already announced, we are able to absorb the first appearing term into the right hand side of (5.62), while the rest can be bounded again by means of the uniform weighted second-order estimates from Theorem 5.14 as stated in Remark 5.6. Gathering all the above estimates results in

$$\begin{aligned} &\int_{B_r(x_0)} (\mathbb{T}_k(1 + |\varepsilon(u)|^2))^{\frac{4-\mu}{2}} dx \\ &\leq c \left[ \left(1 + \frac{1}{r}\right) \int_{B_{2r}(x_0)} |\varepsilon(u)|^2 dx + \frac{1}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(u)| dx + \left(\frac{1}{r} + \frac{1}{r^2}\right) \mathcal{L}^n(B_{2r}(x_0)) \right], \end{aligned}$$

for a constant  $c = c(n, \Omega, \gamma, \Gamma, \lambda, \Lambda, \|u\|_{L^\infty(U; \mathbb{R}^n)}) > 0$ , recalling that  $M$  depends on  $\Omega, n$  and  $m$ . Since the right-hand side is independent of the level  $k \in \mathbb{R}_+$ , we can apply the monotone convergence theorem to pass to the limit  $k \rightarrow \infty$ . This yields the stated estimate and finishes the proof.  $\square$

Finally, the proof of Theorem 5.11 is an immediate consequence of Theorem 5.12 and Theorem 5.16, closing this subsection.

## 5.5 Appendix

We briefly discuss the elementary construction of the auxiliary functions  $\mathfrak{q}$  and  $h$  which entered the proof of existence of the Ekeland-type approximation sequence in Section 5.1, cf. (5.4) and (5.5).

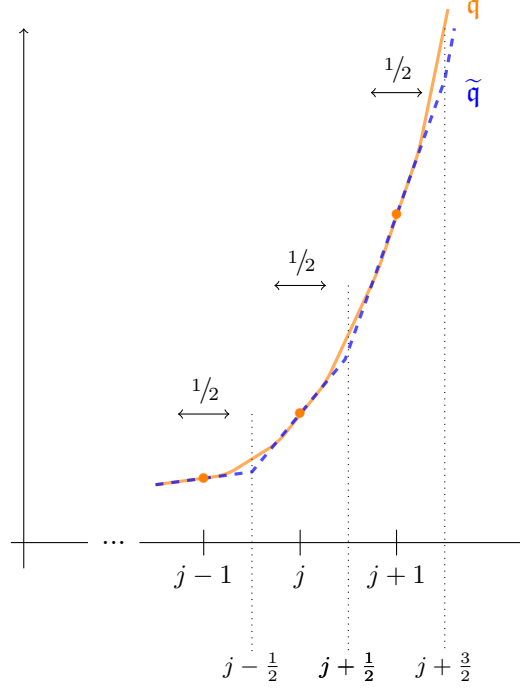


Figure 5.4: Construction of  $\mathfrak{q}$

Concerning the function  $\mathfrak{q}$ , we set  $\tilde{\mathfrak{q}}(0) = 0$  and

$$\tilde{\mathfrak{q}}(j) := \sum_{\ell=1}^j (j+1-\ell) \|\varepsilon(\tilde{u}_j)\|_{L^{n+1}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})} \quad \text{for } j \in \mathbb{N}.$$

By construction,  $\tilde{\mathfrak{q}}$  is an increasing function on  $\mathbb{N}_0$ , and the differences

$$\tilde{\mathfrak{q}}(j) - \tilde{\mathfrak{q}}(j-1) = \sum_{\ell=1}^j \|\varepsilon(\tilde{u}_j)\|_{L^{n+1}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})} \quad \text{for } j \in \mathbb{N}$$

are increasing as well. By use of an affine-linear interpolation of  $\tilde{\mathfrak{q}}$  at the points  $j + \frac{1}{2}$ ,  $j \in \mathbb{N}$  (blue dotted line) as indicated in Figure 5.4, we may extend  $\tilde{\mathfrak{q}}$  to the entire  $\mathbb{R}_{\geq 0}$ . Setting  $\mathfrak{q} := \eta_\delta * \tilde{\mathfrak{q}}$ , where  $*$  denotes convolution with a standard mollifier  $\eta_\delta$  of radius  $\delta = \frac{1}{4}$ , leaves  $\tilde{\mathfrak{q}}$  unchanged on intervals of length  $\frac{1}{2}$  around each  $j \in \mathbb{N}$  (orange line). Moreover,  $\mathfrak{q}$  is convex, increasing and of class  $C^\infty$  on  $\mathbb{R}_{\geq 0}$  with

$$\mathfrak{q}(j) \geq \|\varepsilon(\tilde{u}_j)\|_{L^{n+1}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})} \quad \text{for each } j \in \mathbb{N}.$$

Concerning the function  $h$ , we start by writing the left-hand side of (5.5) as  $\mathfrak{H}(t)$  for  $t \in [\frac{3}{2}, 2)$ . Based on the observation that the resulting function  $\mathfrak{H}: [\frac{3}{2}, 2) \rightarrow (0, \infty)$  is given

in terms of concatenations and products of non-negative, non-decreasing and convex  $C^2$ -functions, it is not difficult to see that  $\mathfrak{H}$  itself is a non-negative, (even strictly) increasing and convex  $C^2$ -function. We then define the desired function  $h$  on  $[0, 1]$  as  $h \equiv 0$  and on  $[\frac{3}{2}, 2)$  as

$$h(t) := \beta \mathfrak{H}(t) + (1 - \beta) \mathfrak{H}(\frac{3}{2}) \geq \mathfrak{H}(t) \quad \text{for } t \in [\frac{3}{2}, 2)$$

for some constant  $\beta > 1$  with  $\beta \mathfrak{H}'(\frac{3}{2}) > 2 \mathfrak{H}(\frac{3}{2})$ . Based on this choice, we then may perform a similar extension as above to obtain an increasing, convex  $C^2$ -function on  $[0, 2)$  with the requisite properties.

## 6 | Gradient Integrability for $L^p$ -Bounded BV- and BD-Minimisers

In this section, we interpolate the results of [Gme20] and Chapter 5, namely, we derive gradient integrability for  $L^p$ -bounded,  $p > 2$ , relaxed minimisers  $u \in \text{BV}_{\text{loc}}^{\mathbb{A}}(\Omega) \cap L_{\text{loc}}^p(\Omega; \mathbb{R}^N)$  of the variational principle

$$\text{to minimise } F[v, \Omega] := \int_{\Omega} f(\mathbb{A}v) \, dx \quad \text{over } v \in \mathcal{D}_{u_0} := u_0 + W_0^{\mathbb{A},1}(\Omega),$$

where  $\mathbb{A} \in \{\nabla, \varepsilon\}$ . This result is particularly interesting in the case of the symmetric gradient, since no a priori methods are available to guarantee  $L^\infty$ -bounds, unlike maximum principles or Moser-type iterations. Since many arguments coincide with the full gradient case, we treat both settings together, separating steps only where required. We also emphasise that we adopt the convention  $N = n$  whenever  $\mathbb{A} = \varepsilon$ , as otherwise the symmetric gradient is not well-defined. The main result will be the following theorem:

**Theorem 6.1** (Gradient integrability -  $L^p$ -constraint). *Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded set,  $p > 2$  and consider a variational integrand  $f \in C^2(\mathcal{R}(\mathbb{A}))$  which satisfies (1.3) and (1.5) for  $\mu \in (1, 3 - \frac{2}{p}]$ . Then any local  $\text{BV}^{\mathbb{A}}$ -minimiser  $u \in \text{BV}_{\text{loc}}^{\mathbb{A}}(\Omega) \cap L_{\text{loc}}^p(\Omega; \mathbb{R}^N)$  of  $F$  is of class  $W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^N)$ . More specifically, for every subset  $U \Subset \Omega$  there exists a constant  $c = c(n, p, \Omega, \gamma, \Gamma, \lambda, \Lambda, \|u\|_{L^p(U; \mathbb{R}^N)}) > 0$  such that whenever  $B_{2r}(x_0) \Subset U$  we have*

- for the full gradient  $\mathbb{A} = \nabla$

$$\int_{B_r(x_0)} |\nabla u| \log^{s_p}(1 + |\nabla u|^2) \, dx \leq c \left[ \left(1 + \frac{1}{r^2}\right) |Du|(\overline{B}_{2r}(x_0)) + 1 \right],$$

- and for the symmetric gradient  $\mathbb{A} = \varepsilon$

$$\begin{aligned} & \int_{B_r(x_0)} |\nabla u| \log^{s_p-1}(1 + |\nabla u|^2) \, dx \\ & \leq c \left[ \left(1 + \frac{1}{r^2}\right) \left(r^n + |Eu|(\overline{B}_{2r}(x_0))\right) + \frac{1}{r^2} \int_{B_{2r}(x_0)} \frac{|u|}{r} \, dx + \left(1 + \frac{1}{r^4}\right) \right], \end{aligned}$$

where  $s_p = 2 - \frac{4}{p+2} = \frac{2p}{p+2}$ .

*Structure of the chapter.* In Section 6.1 we implement a quadratic vanishing viscosity approach keeping track of the  $L^p$ -bounds based on Chapter 4. Subsequently, in Section 6.2 we derive uniform weighted second-order estimates for the Ekeland-approximation sequence stemming from the previous section. Moreover, Section 6.3 includes the proof of Theorem 6.1 in conjunction with a discussion concerning the restriction  $p > 2$ . Finally, we can improve, unfortunately non-optimal, the integrability, of the gradient when assuming the strict ellipticity range  $\mu \in (1, 3 - \frac{2}{p})$ , which is the content of Section 6.4.

## 6.1 The Ekeland-type Viscosity Approximation

Similar to the previous chapter we aim to use the results from Chapter 4. Throughout our presentation we assume  $p > 2$ . In order to keep control of the  $L^p$ -bounds, we have to introduce a suitable penalisation term. To this end, we consider the linear growth function  $\Upsilon: \mathbb{R} \rightarrow \mathbb{R}$  defined through

$$\Upsilon(t) := \begin{cases} 0 & \text{if } t < 0 \\ \sqrt{1+t^2} - 1 & \text{if } t \geq 0, \end{cases} \quad (6.1)$$

cf. Figure 6.1. We set up a quadratic regularisation, meaning that we consider perturbed functionals on the Sobolev space  $W^{1,2}(\Omega; \mathbb{R}^N)$ . Towards this aim, we define  $f_j: \mathcal{R}(\mathbb{A}) \rightarrow \mathbb{R}$  through  $f_j := \mathfrak{f}_{j,2}$ , i.e.

$$f_j(z) = f(z) + \frac{1}{2A_j j^2} (1 + |z|^2) \quad \text{with} \quad A_j := A_{j,2} = 1 + \int_{\Omega} (1 + |\mathbb{A} \tilde{u}_j|^2) dx.$$

Clearly, we have  $\mathcal{R}(\mathbb{A}) = \mathbb{R}^{N \times n}$  for  $\mathbb{A} = \nabla$ , whereas there holds  $\mathcal{R}(\mathbb{A}) = \mathbb{R}_{\text{sym}}^{n \times n}$  for  $\mathbb{A} = \varepsilon$ , cf. Example 2.2.

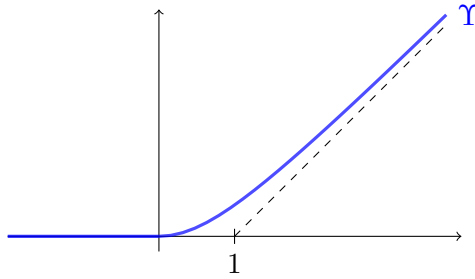


Figure 6.1: The penalisation function  $\Upsilon$

As a next step, we introduce regularised and penalised functionals on the negative Sobolev space  $W^{-2,1}(\Omega; \mathbb{R}^N)$  by

$$F_j[w; \Omega] := \begin{cases} \int_{\Omega} f_j(\mathbb{A}w) dx + \Upsilon \left( \int_{\Omega} V_p(w) dx - M \right) & \text{if } w \in \mathcal{D}_j \\ +\infty & \text{if } w \in W^{-2,1}(\Omega; \mathbb{R}^N) \setminus \mathcal{D}_j, \end{cases}$$

where  $V_p$  denotes the  $V$ -function from (2.20). This perfectly fits into the framework of Chapter 4 with  $\Upsilon$  as defined in (6.1) and  $\mathfrak{g}: \mathbb{R}^N \rightarrow \mathbb{R}$  given by

$$\mathfrak{g}(z) := V_p(z) - \frac{M}{\mathcal{L}^n(\Omega)}, \quad z \in \mathbb{R}^N$$

with  $M > 0$  from (4.8). Especially, (4.12) is satisfied since the function  $\Upsilon$  only takes non-negative values. Therefore, the results from Chapter 4 are applicable and provide us with a Ekeland-type approximation sequence  $(v_j)_{j \in \mathbb{N}}$  of almost-minimisers satisfying the estimates from Proposition 4.2. In our framework the latter read as follows:

**Proposition 6.2** (Properties of the Ekeland sequence -  $L^p$ -case). *Let  $(v_j)_{j \in \mathbb{N}}$  be the Ekeland-type approximation sequence. Then for every  $j \in \mathbb{N}$  we have the following estimates*

$$\int_{\Omega} |\mathbb{A}v_j| \, dx \leq \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right), \quad (6.2)$$

$$\frac{1}{2A_j j^2} \int_{\Omega} (1 + |\mathbb{A}v_j|^2) \, dx \leq \frac{2}{j^2}, \quad (6.3)$$

$$\sup_{j \in \mathbb{N}} \|v_j\|_{L^p(\Omega; \mathbb{R}^N)}^p \leq c(p) (3 + M). \quad (6.4)$$

*Proof.* It is obvious that (4.23) and (4.24) correspond to (6.2) and (6.3). To conclude the precise estimate (6.4) from (4.25) we observe the elementary bound  $t - 1 \leq \Upsilon(t)$  for all  $t \in \mathbb{R}$ , leading to

$$\int_{\Omega} V_p(v_j) \, dx - (M + 1) \leq \Upsilon \left( \int_{\Omega} V_p(v_j) \, dx - M \right) \leq 2.$$

As a consequence of (2.21) this results in  $\|v_j\|_{L^p(\Omega; \mathbb{R}^N)}^p \leq c(p) (3 + M)$ , showing (6.4), when taking the supremum over all  $j \in \mathbb{N}$ . This finishes the proof.  $\square$

As further ingredients, let us recall estimate (4.27) from Corollary 4.3, namely, the Ekeland-sequence satisfies

$$\|v_j\|_{L^1(\Omega; \mathbb{R}^N)} \leq c(\Omega, \Gamma) \left[ \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) + \frac{1}{j^2} \right]. \quad (6.5)$$

Moreover, (4.29) from Corollary 4.4 reads in the quadratic setting

$$\frac{1}{2A_j j^2} \int_{\Omega} |v_j|^2 \, dx \leq c \left[ \frac{2}{j^2} + \left( \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) + \frac{1}{j^1} \right)^2 \right] \quad (6.6)$$

for a constant  $c = c(\Omega, \Gamma, \mathbb{A}) > 0$ . In light of Remark 4.7, we note that the estimates from Proposition 6.2 suffice to ensure the finiteness condition from Theorem 4.6, as it is not necessary to permit  $\mathfrak{g}$  to reach the value  $+\infty$ . In this way we obtain from (4.30) that for all  $j \in \mathbb{N}$  and all  $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$  the following Euler-Lagrange inequality

$$\begin{aligned} & \left| \int_{\Omega} \langle \nabla f_j(\mathbb{A}v_j), \mathbb{A}\varphi \rangle \, dx \right. \\ & \quad \left. + \Upsilon' \left( \int_{\Omega} V_p(v_j) \, dx - M \right) \int_{\Omega} \langle \nabla V_p(v_j), \varphi \rangle \, dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-2,1}(\Omega; \mathbb{R}^N)}, \end{aligned} \quad (6.7)$$

holds. Moreover, assuming the growth condition (4.31) from Lemma 4.8, we gain

$$v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^N) \quad (6.8)$$

for all  $j \in \mathbb{N}$ . As a consequence of Lemma 4.9 we deduce the following differentiated Euler-Lagrange inequality

$$\begin{aligned} & \left| \int_{\Omega} \langle \nabla^2 f_j(\mathbb{A}v_j) \partial_s(\mathbb{A}v_j), \mathbb{A}\varphi \rangle \, dx \right. \\ & \quad \left. + \Upsilon' \left( \int_{\Omega} V_p(v_j) \, dx - M \right) \int_{\Omega} \langle \nabla^2 V_p(v_j) \partial_s v_j, \varphi \rangle \, dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-1,1}(\Omega; \mathbb{R}^N)}, \end{aligned} \quad (6.9)$$

which holds for all  $s \in \{1, \dots, n\}$ ,  $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$  and  $j \in \mathbb{N}$ . Similar to Chapter 5 this will be the key tool to derive uniform second-order estimates in the sequel. Finally, we abbreviate  $\sigma_j := \sigma_{j,2} = \nabla f_j(\varepsilon(v_j))$ .

## 6.2 Weighted Uniform Estimates

Having the differentiated Euler-Lagrange inequality from (6.9) at our disposal, we can show weighted second-order estimates for the Ekeland-sequence  $(v_j)_{j \in \mathbb{N}}$  that are now uniform in  $j \in \mathbb{N}$ . These uniform estimates become crucial in the next section, when proving superlinear bounds for  $(\mathbb{A}v_j)_{j \in \mathbb{N}}$ .

**Theorem 6.3** (Uniform weighted second-order estimates). *Let  $f \in C^2(\mathcal{R}(\mathbb{A}))$  satisfy (1.3) and, for some  $\Lambda \in (0, \infty)$ , the upper bound (4.31). Then there exist a constant  $c > 0$  such that for every Ball  $B_{2r}(x_0) \Subset \Omega$  and every localisation function  $\varrho \in C_c^\infty(B_{2r}(x_0); [0, 1])$  with  $\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq \left(\frac{2}{r}\right)^s$  for  $s \in \{1, 2, 3\}$  we have the following estimates:*

- If  $\mathbb{A} = \nabla$ , then  $c = c(\Lambda) > 0$ , and

$$\begin{aligned} & \sum_{k=1}^n \int_{B_{2r}(x_0)} \varrho^4 \langle \nabla^2 f(\nabla v_j) \partial_k \nabla v_j, \partial_k \nabla v_j \rangle dx \\ & \leq \frac{c}{r^2} \left[ \int_{B_{2r}(x_0)} \left( |\nabla v_j| + \left( \frac{r^2}{j} + \frac{r^3}{j} \right) \frac{|v_j|}{r} \right) dx + \frac{1}{A_j j^2} \int_{B_{2r}(x_0)} (1 + |\nabla v_j|^2) dx \right]. \end{aligned} \quad (6.10)$$

- If  $\mathbb{A} = \varepsilon$ , then  $c = c(\Gamma, \Lambda) > 0$ , and

$$\begin{aligned} & \sum_{k=1}^n \int_{B_{2r}(x_0)} \varrho^4 \langle \nabla^2 f(\varepsilon(v_j)) \partial_k \varepsilon(v_j), \partial_k \varepsilon(v_j) \rangle dx \\ & \leq \frac{c}{r^2} \left[ \int_{B_{2r}(x_0)} \left( |\varepsilon(v_j)| + \frac{|v_j|}{r} \left( 1 + \frac{r^2}{j} + \frac{r^3}{j} \right) \right) dx \right. \\ & \quad + \mathcal{L}^n(B_{2r}(x_0))^{\frac{p-2}{p}} \left( \int_{B_{2r}(x_0)} \frac{|v_j|^p}{r^p} dx \right)^{\frac{2}{p}} \\ & \quad \left. + \frac{1}{A_j j^2} \int_{B_{2r}(x_0)} \left( (1 + |\varepsilon(v_j)|^2) + \frac{|v_j|^2}{r^2} \right) dx \right]. \end{aligned} \quad (6.11)$$

**Remark 6.4.** *Recalling the uniform bounds (6.2), (6.3), and (6.4) from Proposition 6.2 in conjunction with (6.5) and (6.6) we see that all terms at the right hand side of (6.11) and (6.10) are uniformly bounded in  $j \in \mathbb{N}$ . Additionally, assuming that the integrand  $f$  is  $\mu$ -elliptic in the sense of (1.5), we can conclude weighted second-order estimates using the explicit representation (4.15) for  $\nabla^2 f_j(z)$  in both cases. More precisely, utilising the inequality  $\frac{r}{j} \leq \frac{1}{r} + \frac{r^2}{j}$  then results in*

$$\begin{aligned} & \int_{B_{2r}(x_0)} \varrho^4 (1 + |\varepsilon(v_j)|^2)^{-\frac{\mu}{2}} |\nabla \varepsilon(v_j)|^2 dx + \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} \varrho^4 |\nabla \varepsilon(v_j)|^2 dx \\ & \leq \frac{c}{r^2} \left[ \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \left( 1 + \frac{r^2}{j} + \frac{r^3}{j} \right) \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx \right. \\ & \quad + \mathcal{L}^n(B_{2r}(x_0))^{\frac{p-2}{p}} \left( \int_{B_{2r}(x_0)} \frac{|v_j|^p}{r^p} dx \right)^{\frac{2}{p}} \\ & \quad \left. + \frac{1}{A_j j^2} \int_{B_{2r}(x_0)} \left( (1 + |\varepsilon(v_j)|^2) + \frac{|v_j|^2}{r^2} \right) dx \right] \end{aligned} \quad (6.12)$$

$$\begin{aligned} &\leq \frac{c}{r^2} \left[ \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) \left( 1 + \frac{1}{r} + \frac{r^2}{j} \right) + \frac{1}{j^2} \left( \frac{1}{r} + \frac{r^2}{j} \right) \right. \\ &\quad \left. + \frac{1}{r^2} \mathcal{L}^n(\Omega)^{\frac{p-2}{2}} + \frac{2}{j^2} + \frac{2}{j^2 r^2} + \frac{1}{r^2} \left( \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) + \frac{1}{j^2} \right)^2 \right], \end{aligned}$$

with  $c = c(p, \gamma, \Gamma, \lambda, \Lambda, \Omega, M) > 0$  in the case of the symmetric gradient  $\mathbb{A} = \varepsilon$ . Similarly, we infer

$$\begin{aligned} &\int_{B_{2r}(x_0)} \varrho^4 (1 + |\nabla v_j|^2)^{-\frac{p}{2}} |\nabla^2 v_j|^2 dx + \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} \varrho^4 |\nabla^2 v_j|^2 dx \\ &\leq \frac{c}{r^2} \left[ \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) \left( 1 + \frac{1}{r} + \frac{r^2}{j} \right) + \left( \frac{r}{j} + \frac{r^2}{j} \right) \frac{1}{j^2} + \frac{2}{j^2} \right], \end{aligned} \quad (6.13)$$

with  $c = c(\gamma, \Gamma, \lambda, \Lambda, \Omega) > 0$  in the full gradient case  $\mathbb{A} = \nabla$ . Finally, we notice that the seemingly unnatural scaling in  $r$  for the right-hand side vanishes in the limit  $j \rightarrow \infty$ .

*Proof of Theorem 6.3.* Although many arguments are similar to Theorem 5.5 we give a self-contained proof, since these estimates are crucial to obtain regularity. For the sake of readability we divide the proof into several steps, and we only distinguish between  $\mathbb{A} = \nabla$  and  $\mathbb{A} = \varepsilon$  at one point, as the argument simplifies a lot in the full gradient case. As usual, we take a point  $x_0 \in \Omega$ , a radius  $r > 0$  such that  $B_{2r}(x_0) \Subset \Omega$  and a localisation function  $\varrho \in C_c^\infty(\Omega; [0, 1])$  with  $\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq \left(\frac{2}{r}\right)^s$  for  $s \in \{1, 2, 3\}$ .

*Step 1: Preliminary estimate and first bounds.* Recalling the higher differentiability from (6.8) we can apply the product rule from (2.4) to obtain

$$\varrho^4 \partial_k \mathbb{A} v_j = -\partial_k v_j \otimes_{\mathbb{A}} \nabla \varrho^4 + \mathbb{A}(\varrho^4 \partial_s v_j)$$

for  $k \in \{1, \dots, n\}$ . Similarly to (5.20), summing over  $k$  gives

$$\begin{aligned} &\sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\mathbb{A} v_j) \varrho^2 \partial_k \mathbb{A} v_j, \varrho^2 \partial_k \mathbb{A} v_j \rangle dx \\ &= - \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\mathbb{A} v_j) \partial_k \mathbb{A} v_j, \partial_k v_j \otimes_{\mathbb{A}} \nabla \varrho^4 \rangle dx \\ &\quad + \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\mathbb{A} v_j) \partial_k \mathbb{A} v_j, \mathbb{A}(\varrho^4 \partial_k v_j) \rangle dx =: A + B. \end{aligned} \quad (6.14)$$

*Step 2a: Estimate for A in case  $\mathbb{A} = \nabla$ .* In the full gradient case the associated product  $\otimes_{\mathbb{A}}$  reduces to the usual tensor product  $\otimes$ . Recalling (4.36) and (4.37) allows to use a Cauchy-Schwarz type inequality which then leads to

$$\begin{aligned} A &= - \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\nabla v_j) \partial_k \nabla v_j, \partial_k v_j \otimes 4\varrho^3 \nabla \varrho \rangle dx \\ &\leq \frac{1}{2} \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\nabla v_j) \varrho^2 \partial_k \nabla v_j, \varrho^2 \partial_k \nabla v_j \rangle dx \\ &\quad + 8 \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\nabla v_j) \partial_k v_j \otimes \varrho \nabla \varrho, \partial_k v_j \otimes \varrho \nabla \varrho \rangle dx. \end{aligned}$$

The first appearing term can be absorbed into the left-hand side of (6.14), while the second can be estimated using the upper bound from (4.31) in combination with (4.17). This results in

$$\begin{aligned}
& 8 \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\nabla v_j) \partial_k v_j \otimes \varrho \nabla \varrho, \partial_k v_j \otimes \varrho \nabla \varrho \rangle dx \\
& \leq 8\Lambda \sum_{k=1}^n \int_{\Omega} \varrho^2 \frac{|\partial_k v_j \otimes \nabla \varrho|^2}{(1 + |\nabla v_j|^2)^{\frac{1}{2}}} dx + \sum_{k=1}^n \frac{c}{2A_j j^2} \int_{\Omega} \varrho^2 (1 + |\nabla v_j|^2)^{\frac{q-2}{2}} |\partial_k v_j \otimes \nabla \varrho|^2 dx \\
& \leq \frac{c(\Lambda)}{r^2} \int_{B_{2r}(x_0)} |\nabla v_j| dx + \frac{c}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\nabla v_j|^2) dx.
\end{aligned}$$

Thanks to Proposition 6.2 applied for  $\mathbb{A} = \nabla$  all appearing terms on the right hand side are under control.

*Step 2b: Estimate for A in case  $\mathbb{A} = \varepsilon$ .* In the case of the symmetric gradient the associated product  $\otimes_{\mathbb{A}}$  is given by the symmetric tensor product  $\odot$ . We recall our convention which states  $N = n$ , since otherwise the symmetric gradient is not well-defined. Similar to (5.21) we spell out A into components and smuggling in some terms to deduce

$$\begin{aligned}
\text{A} &= -\frac{1}{2} \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \left[ \partial_i \varrho^4 \partial_k v_j^{(m)} + \partial_i \varrho^4 \partial_m v_j^{(k)} \right] dx \\
&\quad + \frac{1}{2} \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \left[ \partial_i \varrho^4 \partial_m v_j^{(k)} + \partial_m \varrho^4 \partial_i v_j^{(k)} \right] dx \\
&\quad - \frac{1}{2} \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \left[ \partial_m \varrho^4 \partial_i v_j^{(k)} + \partial_m \varrho^4 \partial_k v_j^{(i)} \right] dx \\
&=: \text{I} + \text{II} + \text{III}.
\end{aligned}$$

Exploiting the symmetry of  $\sigma_j$ , i.e.  $\sigma_j(x) \in \mathbb{R}_{\text{sym}}^{n \times n}$  for all  $x \in \Omega$ , we conclude  $\text{I} = \text{III}$ . Therefore, using the Cauchy-Schwarz inequality we can follow the lines of (5.22) (of course, this time for a quadratic regularisation) to arrive at

$$\begin{aligned}
|\text{I} + \text{III}| &\leq \frac{1}{4} \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^4 \partial_k \varepsilon(v_j), \varrho^4 \partial_k \varepsilon(v_j) \rangle dx \\
&\quad + 64 \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^3 \nabla \varrho \odot (\varepsilon(v_j) e_k), \varrho^3 \nabla \varrho \odot (\varepsilon(v_j) e_k) \rangle dx.
\end{aligned}$$

While the first term can be absorbed into the left-hand side of (6.14), we can proceed analogous to (5.23). More precisely, we will use (4.43) from Lemma 4.10 for  $\mathfrak{R} = r^{-1} \varrho \mathbb{1}_{n \times n}$ ,  $w = r \nabla \varrho$  and  $e = \varepsilon(v_j) e_k$  for  $k = \{1, \dots, n\}$  in the quadratic framework, to infer

$$\begin{aligned}
& 64 \sum_{k=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho \nabla \varrho \odot (\varepsilon(v_j) e_k), \varrho \nabla \varrho \odot (\varepsilon(v_j) e_k) \rangle dx \\
& \leq \frac{c(\Lambda)}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \frac{c}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2) dx.
\end{aligned}$$

As in the previous step, all appearing terms at the very right hand side are under control by means of Proposition 6.2 applied for  $\mathbb{A} = \varepsilon$ . We continue by handling term II, which is slightly more delicate. Again utilising the symmetry of  $\sigma_j$ , we can rewrite II in components by

$$\Pi = \sum_{k,i,m=1}^n \int_{\Omega} \partial_k \sigma_j^{(im)} \partial_i \varrho^4 \partial_m v_j^{(k)} dx.$$

Since the derivative  $\partial_m v_j^{(k)}$  appearing in the integral is not estimated in terms of the symmetric gradient  $\varepsilon(v_j)$ , we integrate by parts twice, where all computations are justified because of (6.8). In this way, we obtain from (5.24) (again now in the quadratic framework) the decomposition

$$\begin{aligned} \Pi &= \sum_{k,i,m=1}^n \int_{\Omega} \left( \partial_m \sigma_j^{(im)} \partial_{ik} \varrho^4 v_j^{(k)} + \sigma_j^{(im)} \partial_{ikm} \varrho^4 v_j^{(k)} \right. \\ &\quad \left. + \partial_m \sigma_j^{(im)} \partial_i \varrho^4 \partial_k v_j^{(k)} + \sigma_j^{(im)} \partial_{im} \varrho^4 \partial_k v_j^{(k)} \right) dx \\ &=: \Pi_1 + \dots + \Pi_4, \end{aligned}$$

where now the only derivatives of  $v_j$  appearing in  $\Pi_1, \dots, \Pi_4$  are of the form  $\partial_k v_j^{(k)}$ . After summation in  $k \in \{1, \dots, n\}$  these derivatives sum up to the divergence of  $v_j$ , and can therefore be estimated pointwise by  $|\varepsilon(v_j)|$ .

- On  $\Pi_1$ . To treat  $\Pi_1$  we follow the lines of (5.25) and (5.26) in our framework to infer

$$\begin{aligned} |\Pi_1| &\leq \frac{1}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho^2 \partial_{\ell} \varepsilon(v_j) \rangle dx \\ &\quad + 160 \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) (\nabla \varrho \cdot v_j) \nabla \varrho \odot e_{\ell}, (\nabla \varrho \cdot v_j) \nabla \varrho \odot e_{\ell} \rangle dx \\ &\quad + 160 \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho (\nabla^2 \varrho v_j) \odot e_{\ell}, \varrho (\nabla^2 \varrho v_j) \odot e_{\ell} \rangle dx. \end{aligned}$$

At this stage, we can continue as in (5.27), namely, we make use from (4.43) of Lemma 4.10 with  $\mathfrak{N} = r \nabla \varrho \otimes \nabla \varrho$  and  $\mathfrak{N} = r \varrho \nabla^2 \varrho$ ,  $w = r^{-1} v_j$  and  $e = e_{\ell}$  for  $\ell \in \{1, \dots, n\}$  in the quadratic framework. Observing  $|\mathfrak{N}| \leq \frac{4}{r}$  and dropping the factor  $(1 + |z|^2)^{-1/2}$  we deduce, applying Hölder's inequality with exponents  $\frac{p}{2}$  and  $\frac{p}{p-2}$  (recall  $p > 2$ ), the estimate

$$\begin{aligned} |\Pi_1| &\leq \frac{1}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho^2 \partial_{\ell} \varepsilon(v_j) \rangle dx \\ &\quad + \frac{c(\Lambda)}{r^2} \left( 1 + \frac{1}{A_j j^2} \right) \int_{\mathbb{B}_{2r}(x_0)} \frac{|v_j|^2}{r^2} dx \\ &\leq \frac{1}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_{\ell} \varepsilon(v_j), \varrho^2 \partial_{\ell} \varepsilon(v_j) \rangle dx \\ &\quad + \frac{c(\Lambda)}{r^2} \left( 1 + \frac{1}{A_j j^2} \right) \mathcal{L}^n(\mathbb{B}_{2r}(x_0))^{\frac{p-2}{p}} \left( \int_{\mathbb{B}_{2r}(x_0)} \frac{|v_j|^p}{r^p} dx \right)^{\frac{2}{p}}. \end{aligned}$$

- *On  $\Pi_3$ .* To handle  $\Pi_3$  we proceed analogous to (5.28) and using (4.43) from Lemma 4.10 for  $\mathfrak{N} = r^{-1}\mathbb{1}_{n \times n}$ ,  $w = r\varrho\nabla\varrho \operatorname{div}(v_j)$  and  $e = e_\ell$  for  $\ell \in \{1, \dots, n\}$ . In the quadratic case  $q = 2$  this results in

$$\begin{aligned} |\Pi_3| &\leq \frac{1}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_\ell \varepsilon(v_j), \varrho^2 \partial_\ell \varepsilon(v_j) \rangle dx \\ &\quad + \frac{c(\Lambda)}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \frac{c}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2) dx. \end{aligned}$$

Note, that we have used the pointwise estimate  $|\operatorname{div}(v_j)| \leq |\varepsilon(v_j)|$ .

- *On  $\Pi_2$ .* In this case we proceed similar to (5.30). More precisely, using  $|\nabla^3 \varrho| \leq \frac{8}{r^3}$  and Young's inequality, we infer

$$\begin{aligned} |\Pi_2| &\leq \frac{c(\Gamma)}{r^3} \int_{\Omega} |v_j| dx + \frac{1}{A_j j^2 r^3} \int_{\Omega} (1 + |\mathbb{A}v_j|^2) |v_j| dx \\ &\leq \frac{c(\Gamma)}{r^2} \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx \\ &\quad + \frac{c}{A_j j^2 r^2} \int_{B_{2r}(x_0)} \left( (1 + |\varepsilon(v_j)|^2) + \frac{|v_j|^2}{r^2} \right) dx. \end{aligned}$$

- *On  $\Pi_4$ .* Finally,  $\Pi_4$  can be handled similar to (5.31). Towards this aim, we will exploit the pointwise estimate  $|\operatorname{div}(v_j)| \leq |\varepsilon(v_j)|$  leading to

$$|\Pi_4| \leq \frac{c(\Gamma)}{r^2} \int_{B_{2r}(x_0)} |\varepsilon(v_j)| dx + \frac{c}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2) dx.$$

Therefore, combining all the estimates from Step 2b results in

$$\begin{aligned} A &\leq \frac{3}{4} \sum_{\ell=1}^n \int_{\Omega} \langle \nabla^2 f_j(\varepsilon(v_j)) \varrho^2 \partial_\ell \varepsilon(v_j), \varrho^2 \partial_\ell \varepsilon(v_j) \rangle dx \\ &\quad + \frac{c(\Gamma, \Lambda)}{r^2} \left[ \int_{B_{2r}(x_0)} \left( |\varepsilon(v_j)| + \frac{|v_j|}{r} \right) dx + \mathcal{L}^n(B_{2r}(x_0))^{\frac{p-2}{p}} \left( \int_{B_{2r}(x_0)} \frac{|v_j|^p}{r^p} dx \right)^{\frac{2}{p}} \right] \\ &\quad + \frac{c}{A_j j^2 r^2} \int_{B_{2r}(x_0)} \left( (1 + |\varepsilon(v_j)|^2) + \frac{|v_j|^2}{r^2} \right) dx. \end{aligned}$$

*Step 3: Estimate for B.* In order to treat term B, we aim to use to use the differentiated Euler-Lagrange inequality from (6.9). To this end, we add the term

$$\Upsilon' \left( \int_{\Omega} V_p(v_j) dx - M \right) \sum_{k=1}^n \int_{\Omega} \varrho^4 \langle \nabla^2 V_p(v_j) \partial_k v_j, \partial_k v_j \rangle dx \quad (6.15)$$

on both sides of (6.14). By construction we have  $\Upsilon' \geq 0$  and since  $V_p$  is  $C^2$  the expression (6.15) admits a non-negative sign. Recalling  $v_j \in W^{1,2}(\Omega; \mathbb{R}^N) \cap W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^N)$  we observe that  $\varphi = \varrho^4 \partial_k v_j$  belongs to  $W_0^{1,2}(\Omega; \mathbb{R}^N)$ . More precisely, we have  $v_j \in W^{2,2}(B_{2r}(x_0); \mathbb{R}^N)$

and  $\text{supp}(\varrho) = B_{2r}(x_0)$ . Hence, extending  $\varrho^4 \partial_k v_j$  from  $B_{2r}(x_0)$  to  $\Omega$  by zero shows that  $\varphi$  is admissible. Therefore, by means of the product rule  $\varrho^4 \partial_k v_j = \partial_k(\varrho^4 v_j) - (\partial_k \varrho^4) v_j$ , we deduce

$$\begin{aligned} \mathbb{B} &= \frac{1}{j} \sum_{k=1}^n \|\varrho^4 \partial_k v_j\|_{W^{-1,1}(\Omega; \mathbb{R}^N)} \\ &\leq \frac{c}{j} \left( \|\varrho^4 v_j\|_{L^1(\Omega; \mathbb{R}^N)} + \|4\varrho^3 \nabla \varrho v_j\|_{L^1(\Omega; \mathbb{R}^N)} \right) \leq \frac{c}{r^2} \left( \frac{r^2(1+r)}{j} \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx \right). \end{aligned}$$

*Step 4: Final estimate.* Gathering all the estimates gives the desired weighted inequalities in both cases: Namely, for the full gradient  $\mathbb{A} = \nabla$  we obtain

$$\begin{aligned} &\sum_{k=1}^n \int_{B_{2r}(x_0)} \varrho^4 \langle \nabla^2 f(\nabla v_j) \partial_k \nabla v_j, \partial_k \nabla v_j \rangle dx \\ &\leq \frac{c}{r^2} \left[ \int_{B_{2r}(x_0)} \left( |\nabla v_j| + \left( \frac{r^2}{j} + \frac{r^3}{j} \right) \frac{|v_j|}{r} \right) dx + \frac{1}{A_j j^2} \int_{B_{2r}(x_0)} (1 + |\nabla v_j|^2) dx \right] \end{aligned}$$

with  $c = c(\Lambda) > 0$ . If  $\mathbb{A} = \varepsilon$  is the symmetric gradient, we infer

$$\begin{aligned} &\sum_{k=1}^n \int_{B_{2r}(x_0)} \varrho^4 \langle \nabla^2 f(\varepsilon(v_j)) \partial_k \varepsilon(v_j), \partial_k \varepsilon(v_j) \rangle dx \\ &\leq \frac{c}{r^2} \left[ \int_{B_{2r}(x_0)} \left( |\varepsilon(v_j)| + \left( 1 + \frac{r^2}{j} + \frac{r^3}{j} \right) \frac{|v_j|}{r} \right) dx \right. \\ &\quad \left. + \mathcal{L}^n(B_{2r}(x_0))^{\frac{p-2}{p}} \left( \int_{B_{2r}(x_0)} \frac{|v_j|^p}{r^p} dx \right)^{\frac{2}{p}} \right. \\ &\quad \left. + \frac{1}{A_j j^2} \int_{B_{2r}(x_0)} \left( (1 + |\varepsilon(v_j)|^2) + \frac{|v_j|^2}{r^2} \right) dx \right], \end{aligned}$$

with a constant  $c = c(\Gamma, \Lambda) > 0$ . This finishes the proof.  $\square$

### 6.3 Gradient Integrability for $p > 2$ and $\mu \leq 3 - \frac{2}{p}$

Based on the weighted uniform estimates from Remark 6.4 we are now able to prove  $W^{1,1}$ -regularity for every generalized minimiser. This is the content of the main Theorem 6.1 in this section.

*Proof of Theorem 6.1.* For the sake of readability we divide the proof into several steps. To this end, we take a point  $x_0 \in \Omega$  and a radius  $r > 0$  with  $B_{2r}(x_0) \Subset \Omega$  together with a localisation function  $\varrho \in C_c^\infty(\Omega; [0, 1])$  satisfying  $\mathbf{1}_{B_r(x_0)} \leq \varrho \leq \mathbf{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq \left(\frac{2}{r}\right)^s$  for  $s \in \{1, 2, 3\}$ .

*Step 1: Derivation of a partially integrated Euler-Lagrange inequality.* Similar to the proof of Theorem 5.1 we aim to use a partially integrated version of the Euler-Lagrange inequality (6.7), thereby relaxing the regularity requirement on test functions  $\varphi$  from  $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$  to  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^N) \cap L^2(\Omega; \mathbb{R}^N)$ . Towards this aim, we notice from (6.8) that there holds

$$(\nabla f_j - \nabla f)(\mathbb{A} v_j) = \frac{1}{2A_j j^2} \mathbb{A} v_j \in W_{\text{loc}}^{1,2}(\Omega; \mathcal{R}(\mathbb{A})). \quad (6.16)$$

Therefore, all quantities are well-defined and we are in the situation to perform a partial integration in the regularisation term. More precisely, exploiting the symmetry of  $\nabla f_j$  in the case  $\mathbb{A} = \varepsilon$ , we get

$$\begin{aligned} \int_{\Omega} \langle (\nabla f_j - \nabla f)(\mathbb{A}v_j), \mathbb{A}\varphi \rangle dx &= \sum_{i,m=1}^n \int_{\Omega} (\nabla f_j - \nabla f)(\mathbb{A}v_j)^{(im)} \partial_m \varphi^{(i)} dx \\ &= - \sum_{i,m=1}^n \int_{\Omega} \partial_m ((\nabla f_j - \nabla f)(\mathbb{A}v_j)^{(im)}) \varphi^{(i)} dx. \end{aligned}$$

Consequently, we can rewrite the Euler-Lagrange-inequality

$$\begin{aligned} \left| \int_{\Omega} \langle \nabla f(\mathbb{A}v_j), \mathbb{A}\varphi \rangle dx - \sum_{i,m=1}^n \int_{\Omega} \partial_m ((\nabla f_j - \nabla f)(\mathbb{A}v_j)^{(im)}) \varphi^{(i)} dx \right. \\ \left. + \Upsilon' \left( \int_{\Omega} V_p(v_j) dx - M \right) \int_{\Omega} \langle \nabla V_p(v_j), \varphi \rangle dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-2,1}(\Omega; \mathbb{R}^N)}, \end{aligned} \quad (6.17)$$

which is a well-defined expression for  $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ . Moreover, we can approximate an arbitrary function in  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^N) \cap L^2(\Omega; \mathbb{R}^N)$  by smooth  $C_c^\infty$ -maps in the norm-topology of  $W^{1,1}(\Omega; \mathbb{R}^N) \cap L^2(\Omega; \mathbb{R}^N)$ . Since  $\Upsilon'$  and  $\nabla f(\mathbb{A}v_j)$  are bounded together with (6.16), a standard approximation argument justifies the validity of (6.17) for all  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^N) \cap L^2(\Omega; \mathbb{R}^N)$  as claimed.

*Step 2: Admissibility of the test function.* As a next step, we choose the truncated logarithmic test function

$$\varphi := \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) v_j,$$

where  $\mathbb{T}_k$  denotes the truncation at level  $k > 0$ , cf. (2.5). We tacitly suppress the  $p$ -dependence and only write  $s := s_p$  in the sequel. As a next step, we briefly argue why our choice of  $\varphi$  is admissible for (6.17) following essentially from the product rule (2.4). By construction we have  $v_j \in W^{1,2}(\Omega; \mathbb{R}^N)$  and therefore, since the cut-off as well as the truncated logarithm is bounded, we conclude  $\varphi \in L^2(\Omega; \mathbb{R}^N)$ . To proceed, we have to argue that both terms of

$$\log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \nabla v_j + v_j \otimes \nabla (\log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)))$$

are elements of  $L_{\text{loc}}^1(\Omega; \mathbb{R}^N)$ . This is clear for the first one as the truncation is bounded and  $v_j \in W^{1,2}(\Omega; \mathbb{R}^N)$ . For second, however, we consider

$$\begin{aligned} v_j \otimes \nabla (\log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))) \\ = s \log^{s-1} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{-1} \mathbf{1}_{\{1+|\mathbb{A}v_j|^2 \leq k\}} v_j \otimes \nabla |\mathbb{A}v_j|^2 \end{aligned}$$

leading to

$$\left| v_j \otimes \nabla (\log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))) \right| \quad (6.18)$$

$$\leq 2s |v_j| (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{-\frac{1}{2}} \log^{s-1} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) |\nabla \mathbb{A}v_j|. \quad (6.19)$$

Therefore, since  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^N)$ , the local integrability follows after dropping the term with negative exponent and observing that the logarithmic term remains bounded. As a next step, we compute  $\mathbb{A}\varphi$  resulting in

$$\begin{aligned} \mathbb{A}\varphi &= \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \mathbb{A}v_j \\ &\quad + \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) v_j \otimes_{\mathbb{A}} \nabla \varrho^4 + \varrho^4 v_j \otimes_{\mathbb{A}} \nabla (\log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))). \end{aligned} \quad (6.20)$$

Plugging (6.20) into (6.17) and rearranging terms yields

$$\begin{aligned} \text{I} + \text{II} &:= \int_{\Omega} \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \langle \nabla f(\mathbb{A}v_j), \mathbb{A}v_j \rangle dx \\ &\quad + \Upsilon' \left( \int_{\Omega} V_p(v_j) dx - m \right) \int_{\Omega} \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \langle \nabla V_p(v_j), v_j \rangle dx \\ &= - \int_{\Omega} \langle \nabla f(\mathbb{A}v_j), \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \otimes_{\mathbb{A}} \nabla \varrho^4 v_j \rangle dx \\ &\quad - \int_{\Omega} \langle \nabla f(\mathbb{A}v_j), \varrho^4 v_j \otimes_{\mathbb{A}} \nabla (\log^s (\mathbb{T}_k(1 + |v_j|^2))) \rangle dx \\ &\quad + \sum_{i,m=1}^n \int_{\Omega} \partial_m ((\nabla f_j - \nabla f)(\mathbb{A}v_j)^{(im)}) \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) v_j^{(i)} dx \\ &\quad + \frac{1}{j} \|\varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) v_j\|_{W^{-2,1}(\Omega; \mathbb{R}^N)} \\ &=: \text{III} + \text{IV} + \text{V} + \text{VI}. \end{aligned}$$

*Step 3: Estimating all appearing terms.* As a consequence of the linear growth condition (1.3), we use the lower bound from Lemma 2.18 to estimate I by

$$\begin{aligned} \text{I} &= \int_{\Omega} \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \langle \nabla f(\mathbb{A}v_j), \mathbb{A}v_j \rangle dx \\ &\geq \gamma \int_{\Omega} \varrho^4 |\mathbb{A}v_j| \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) dx - \Gamma \int_{\Omega} \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) dx. \end{aligned} \quad (6.21)$$

From the explicit definition (6.1) of  $\Upsilon$  it follows that  $\Upsilon' \geq 0$ . Since  $V_p$  is a convex and differentiable function of  $p$ -growth, cf. (2.21), we obtain a lower bound for  $\langle \nabla V_p(z), z \rangle$  using Lemma 2.18. Hence, we can estimate II through

$$\begin{aligned} \text{II} &\geq \Upsilon' \left( \int_{\Omega} V_p(v_j) dx - M \right) \left[ \mathfrak{a} \int_{\Omega} \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) |v_j|^p dx \right. \\ &\quad \left. - \mathfrak{b} \int_{\Omega} \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) dx \right] \\ &\geq -\Upsilon' \left( \int_{\Omega} V_p(v_j) dx - M \right) \mathfrak{b} \int_{\Omega} \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) dx, \end{aligned}$$

where we have dropped the first term due to its positive sign. Restructuring all the appearing terms leads to

$$\begin{aligned} &\gamma \int_{\Omega} \varrho^4 |\mathbb{A}v_j| \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) dx \\ &\leq \text{III} + \text{IV} + \text{V} + \text{VI} + c(\mathfrak{b}, \Gamma) \int_{\Omega} \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) dx \\ &=: \text{III} + \text{VI} + \text{V} + \text{VI} + \text{VII}. \end{aligned} \quad (6.22)$$

We now proceed to bound all appearing terms of the right-hand side of (6.22) starting with III. Using Young's inequality with exponents  $p$  and  $p'$  and the fact that the logarithm grows more slowly than any root, cf. (2.22), we obtain

$$\begin{aligned} \text{III} &\leq \Gamma \int_{\Omega} \varrho^3 |v_j| \log^s(1 + |\mathbb{A}v_j|^2) |\nabla \varrho| \, dx \\ &\leq c(\Gamma) \int_{\Omega} \varrho^p |v_j|^p \, dx + \int_{\Omega} (4\varrho^2 |\nabla \varrho|)^{\frac{p}{p-1}} \log^{\frac{sp}{p-1}}(1 + |\mathbb{A}v_j|^2) \, dx \\ &\leq c(\Gamma) \int_{B_{2r}(x_0)} |v_j|^p \, dx + \frac{1}{r^{p'}} \int_{B_{2r}(x_0)} |\mathbb{A}v_j| \, dx. \end{aligned}$$

Now since  $p > 2$  we conclude that  $s > 1$ . This in turn results in a power of a logarithm appearing on the right hand side of (6.18). Consequently, we will use Young's inequality to create a term that can be absorbed into the left-hand side of (6.22). In detail, we obtain

$$\begin{aligned} \text{IV} &\leq 2s\Gamma \int_{\Omega} \varrho^4 |v_j| (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{-\frac{1}{2}} \log^{s-1}(\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) |\nabla \mathbb{A}v_j| \, dx \\ &\leq \int_{\Omega} \varrho^4 (1 + |\mathbb{A}v_j|^2)^{-\frac{\mu}{2}} |\nabla \mathbb{A}v_j|^2 \, dx \\ &\quad + c(\Gamma) \int_{\Omega} \varrho^4 |v_j|^2 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\mu-2}{2}} \log^{2(s-1)}(\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \, dx. \end{aligned} \tag{6.23}$$

Now, the first term of the last inequality can be controlled using the weighted uniform estimates from Remark 6.4, while the restriction  $\mu \leq 3 - \frac{2}{p}$  and  $s = \frac{2p}{p+2}$  results in

$$\frac{(\mu - 2)p}{2(p - 2)} \leq \frac{1}{2} \quad \text{and} \quad \frac{2(s - 1)p}{p - 2} \leq s \leq 2.$$

Thus, by means of Young's inequality with exponents  $\frac{p}{2}$  and  $\frac{p}{p-2}$ , we can handle the second term in the last inequality of (6.23) to produce the announced absorbable term paying the price of a controllable pollution term, i.e.

$$\begin{aligned} &\int_{\Omega} \varrho^4 |v_j|^2 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\mu-2}{2}} \log^{2(s-1)}(1 + |\mathbb{A}v_j|^2) \, dx \\ &\leq c(\gamma) \int_{B_{2r}(x_0)} |v_j|^p \, dx + \frac{\gamma}{2} \int_{\Omega} \varrho^4 (1 + |\mathbb{A}v_j|^2)^{\frac{\mu-2}{2} \frac{p}{p-2}} \log^{\frac{2(s-1)p}{p-2}}(\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \, dx \\ &\leq c(\gamma) \int_{B_{2r}(x_0)} |v_j|^p \, dx + \frac{\gamma}{2} \int_{\Omega} \varrho^4 |\mathbb{A}v_j| \log^s(\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \, dx. \end{aligned}$$

For term V, coming from the integration by parts, we exploit again  $s \leq 2$ . In particular, we use Young's inequality twice to infer

$$\begin{aligned} \text{V} &= \sum_{i,m=1}^n \int_{\Omega} \partial_m((\nabla f_j - \nabla f)(\mathbb{A}v_j)^{(im)}) \varrho^4 \log^s(\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) v_j^{(i)} \, dx \\ &\leq \frac{1}{2A_j j^2} \int_{\Omega} \varrho^4 |\nabla \mathbb{A}v_j| \log^2(\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) |v_j| \, dx \\ &\leq \frac{1}{2A_j j^2} \left[ \int_{\Omega} \varrho^4 |\nabla \mathbb{A}v_j|^2 \, dx + \int_{\Omega} \varrho^4 \log^4(1 + |\mathbb{A}v_j|^2) |v_j|^2 \, dx \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{A_j j^2} \left[ \int_{\Omega} \varrho^4 |\nabla \mathbb{A} v_j|^2 dx + c(p) \int_{\Omega} \varrho^p |v_j|^p dx \right. \\
&\quad \left. + \int_{\Omega} \varrho^{\frac{2}{2-p}} \log^{\frac{4p}{p-2}} (1 + |\mathbb{A} v_j|^2) dx \right] \\
&\leq \frac{c(p)}{A_j j^2} \left[ \int_{\Omega} \varrho^4 |\nabla \mathbb{A} v_j|^2 dx + \int_{\Omega} \varrho^p |v_j|^p dx + \int_{\Omega} \varrho^{\frac{2}{2-p}} (1 + |\mathbb{A} v_j|^2) dx \right].
\end{aligned}$$

Similarly to the approach used for V, we can apply (2.16) and (2.22) to VI. Specifically, applying Young's inequality with exponents  $\frac{p}{2}$  and  $\frac{p}{p-2}$  yields

$$\begin{aligned}
\text{VI} &= \frac{1}{j} \|\varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A} v_j|)) v_j\|_{W^{-2,1}(\Omega; \mathbb{R}^N)} \\
&\leq \int_{\Omega} \varrho^4 \log^s (1 + |\mathbb{A} v_j|^2) |v_j| dx \\
&\leq c(p) \left( \int_{\Omega} \varrho^p |v_j|^p dx + \int_{\Omega} \varrho^{\frac{p}{p-1}} \log^{\frac{sp}{p-2}} (1 + |\mathbb{A} v_j|^2) dx \right) \\
&\leq c(p) \left( \int_{\Omega} \varrho^p |v_j|^p dx + \int_{\Omega} \varrho^{\frac{p}{p-1}} |\mathbb{A} v_j| dx \right).
\end{aligned}$$

Finally, VII can be handled immediately by

$$\text{VII} = \int_{\Omega} \varrho^4 \log^s (\mathbb{T}_k(1 + |\mathbb{A} v_j|^2)) dx \leq c \int_{B_{2r}(x_0)} |\mathbb{A} v_j| dx.$$

Gathering all the estimates above and noting that for  $p > 2$  we have  $\frac{p}{p-1} \leq 2$ , we infer

$$\begin{aligned}
&\int_{B_r(x_0)} |\mathbb{A} v_j| \log^s (\mathbb{T}_k(1 + |\mathbb{A} v_j|^2)) dx \\
&\leq \int_{\Omega} \varrho^4 |\mathbb{A} v_j| \log^s (\mathbb{T}_k(1 + |\mathbb{A} v_j|^2)) dx \tag{6.24} \\
&\leq c \left[ \left(1 + \frac{1}{r^2}\right) \int_{B_{2r}(x_0)} |\mathbb{A} v_j| dx + \int_{B_{2r}(x_0)} |v_j|^p dx + \frac{1}{A_j j^2} \int_{B_{2r}(x_0)} (1 + |\mathbb{A} v_j|) dx \right. \\
&\quad \left. + \int_{\Omega} \varrho^4 |\nabla \mathbb{A} v_j|^2 (1 + |\mathbb{A} v_j|^2)^{-\frac{\mu}{2}} dx + \frac{1}{A_j j^2} \int_{\Omega} \varrho^4 |\nabla \mathbb{A} v_j|^2 dx \right],
\end{aligned}$$

where  $c = c(p, \gamma, \Gamma) > 0$ . All appearing quantities at the very right-hand side are uniformly bounded in  $j \in \mathbb{N}$  by using Proposition 6.2 in combination with Remark 6.4. Especially, we observe that (6.24) implies in the full gradient case  $\mathbb{A} = \nabla$

$$\begin{aligned}
&\int_{B_r(x_0)} |\nabla v_j| \log^s (\mathbb{T}_k(1 + |\nabla v_j|^2)) dx \\
&\leq c \left[ \left(1 + \frac{1}{r^2}\right) \int_{B_r(x_0)} |\nabla v_j| dx + \int_{B_{2r}(x_0)} |v_j|^p dx \right. \\
&\quad \left. + \frac{1}{j} \left(1 + \frac{1}{r}\right) \int_{B_{2r}(x_0)} |v_j| dx + \left(1 + \frac{1}{r^2}\right) \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} (1 + |\mathbb{A} v_j|^2) dx \right], \tag{6.25}
\end{aligned}$$

whereas for the symmetric gradient case  $\mathbb{A} = \varepsilon$  we exploit (6.12) to infer

$$\begin{aligned}
& \int_{B_r(x_0)} |\varepsilon(v_j)| \log^s (\mathbb{T}_k(1 + |\varepsilon(v_j)|^2)) \, dx \\
& \leq c \left[ \left(1 + \frac{1}{r^2}\right) \int_{B_{2r}(x_0)} |\varepsilon(v_j)| \, dx + \left(\frac{1}{r^3} + \frac{1}{j} \left(1 + \frac{1}{r}\right)\right) \int_{B_{2r}(x_0)} |v_j| \, dx \right. \\
& \quad + \int_{B_{2r}(x_0)} |v_j|^p \, dx + \mathcal{L}^n(\Omega)^{\frac{p-2}{p}} \frac{1}{r^4} \left( \int_{B_{2r}(x_0)} |v_j|^p \, dx \right)^{\frac{2}{p}} \\
& \quad \left. + \left(1 + \frac{1}{r^2}\right) \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2) \, dx + \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} \frac{|v_j|^2}{r^2} \, dx \right] \tag{6.26}
\end{aligned}$$

for a constant  $c = c(p, \Omega, \gamma, \Gamma, \lambda, \Lambda, M) > 0$ . Let us note that all quantities on the right-hand side are under control due to Proposition 6.2, Corollary 4.3, and Corollary 4.4 applied in the quadratic framework. In particular, as the right-hand side is independent of the level  $k > 0$  in both cases, we use the monotone convergence theorem to conclude that

$$\lim_{k \rightarrow \infty} \int_{B_r(x_0)} |\mathbb{A}v_j| \log^s (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2)) \, dx = \int_{B_r(x_0)} |\mathbb{A}v_j| \log^s(1 + |\mathbb{A}v_j|^2) \, dx$$

is bounded, satisfying the same estimates (6.25) and (6.26) respectively.

*Step 3:  $W_{\text{loc}}^{\mathbb{A},1}$ -regularity.* We now aim to apply Reshetnyak's (lower semi-)continuity Theorem 2.14 to conclude that there holds  $\mathbb{A}^s u \equiv 0$  in  $B_r(x_0)$ . Towards this aim, we infer that (4.2), (4.7), and (4.21) imply that

$$v_j \rightarrow u \quad \text{in } W^{-2,1}(\Omega; \mathbb{R}^N) \quad \text{as } j \rightarrow \infty.$$

Using (6.2) in combination with the compact embedding  $W^{\mathbb{A},1}(\Omega) \hookrightarrow L^1(\Omega; \mathbb{R}^N)$ , cf. (2.8), we can extract a non-relabeled subsequence of  $(v_j)_{j \in \mathbb{N}}$  such that  $v_j \xrightarrow{*} u$  in  $BV^{\mathbb{A}}(\Omega)$ . Moreover, from (6.4) we deduce

$$\limsup_{j \rightarrow \infty} \|v_j\|_{L^p(\Omega; \mathbb{R}^N)}^p \leq c(3 + M)$$

and hence, in the full gradient case  $\mathbb{A} = \nabla$ , (6.25) leads to

$$\liminf_{j \rightarrow \infty} \int_{B_r(x_0)} |\nabla v_j| \log^s(1 + |\nabla v_j|^2) \, dx \leq c \left[ \left(1 + \frac{1}{r^2}\right) |Du|(\overline{B}_{2r}(x_0)) + 1 \right],$$

whereas in case of the symmetric gradient  $\mathbb{A} = \varepsilon$ , (6.26) leads to

$$\begin{aligned}
& \liminf_{j \rightarrow \infty} \int_{B_r(x_0)} |\varepsilon(v_j)| \log^s(1 + |\varepsilon(v_j)|^2) \, dx \\
& \leq c \left[ \left(1 + \frac{1}{r^2}\right) |Eu|(\overline{B}_{2r}(x_0)) + \frac{1}{r^2} \int_{B_{2r}(x_0)} \frac{|u|}{r} \, dx + \left(1 + \frac{1}{r^4}\right) \right]
\end{aligned}$$

for a constant  $c = c(p, \Omega, \gamma, \Gamma, \lambda, \Lambda, M) > 0$ . We recall from Theorem 2.10 and (4.8) that  $M$  only depends on  $\Omega, n$  and  $m$  and therefore we have  $c = c(n, p, \Omega, \gamma, \Gamma, \lambda, \Lambda, \|u\|_{L^p(U; \mathbb{R}^N)}) > 0$  in both cases. Unlike in Step 3 of the proof of Theorem 5.1, we now consider the function

$\phi(z) := \Phi_s(|z|)$  which has the same recession function as that computed in (5.49). Using  $v_j \xrightarrow{*} u$  in  $BV^{\mathbb{A}}(B_r(x_0))$  in conjunction with the lower semi-continuity part of Theorem 2.14 results in

$$\begin{aligned} \int_{B_r(x_0)} \phi(\mathcal{A}[\nabla u]) \, dx + \int_{B_r(x_0)} \phi^\infty \left( \frac{d\mathbb{A}^s u}{d|\mathbb{A}^s u|} \right) \, d|\mathbb{A}^s u| \\ = \phi(\mathbb{A}u)(B_r(x_0)) \leq \liminf_{j \rightarrow \infty} \phi(\mathbb{A}v_j)(B_r(x_0)). \end{aligned}$$

Now, the term on the right hand side is finite by Step 2, and in view of (5.49) it follows  $\mathbb{A}^s u = 0$  yielding the claimed  $W_{\text{loc}}^{\mathbb{A},1}(\Omega)$ -regularity.

*Step 4: Logarithmic integrability.* To obtain the stated estimate in case of the symmetric gradient case, we will use the scaled Korn-inequality from Lemma 3.14, together with  $s \leq 2$ , resulting in

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u| \log^{s-1}(1 + |\nabla u|^2) \, dx \\ = \int_{B_r(x_0)} \Phi_{s-1}(|\nabla u|) \, dx \\ \leq c \left[ r^n + \int_{B_r(x_0)} \Phi_{s-1} \left( \frac{|u|}{r} \right) \, dx + \int_{B_r(x_0)} \Phi_s(|\varepsilon(u)|) \, dx \right] \\ \leq c \left[ \left( 1 + \frac{1}{r^2} \right) \left( r^n + |\mathbb{E}u|(\overline{B}_{2r}(x_0)) \right) + \frac{1}{r^2} \int_{B_{2r}(x_0)} \frac{|u|}{r} \, dx + \left( 1 + \frac{1}{r^4} \right) \right]. \end{aligned}$$

This estimate holds for each ball  $B_r(x_0)$  such that  $B_{2r}(x_0) \Subset \Omega$  for a positive constant  $c = c(n, p, \Omega, \gamma, \Gamma, \lambda, \Lambda, \|u\|_{L^p(U; \mathbb{R}^n)}) > 0$ . Therefore, the proof is finished.  $\square$

**Remark 6.5** (The restriction  $p > 2$ ). *The restriction  $p > 2$  first arises, when dealing with the term  $\Pi_1$  in the proof of Theorem 6.3. In order to make use of the  $L^p$ -bounds we have to use Hölder's inequality with exponents  $\frac{p}{2}$  and  $\frac{p}{p-2}$  leading to the restriction  $p > 2$ . It is not clear to us how to omit this limitation. Motivated by the fact that setting  $p = 2$  in Theorem 6.1 would lead to  $s_p = 1$ , we believe that, in the context of regularity, we cannot expect anything more than logarithmic integrability when dealing with full gradients and using the presented method. Even worse, due to the logarithmic loss in the symmetric gradient framework, the best we could hope for is local  $W^{1,1}$ -regularity. The treatment of (sub)-quadratic constraints remain open, although of theoretical interest.*

## 6.4 Improvement for $p > 2$ and $\mu < 3 - \frac{2}{p}$

Similar to Section 5.4 we can derive higher integrability for the gradient in the strict ellipticity regime  $\mu \in (1, 3 - \frac{2}{p})$ . Unfortunately, due to technical issues, we were only able to prove a non-optimal result at the day of submission, which we will present in the upcoming section. For a concise discussion of the technicalities, including a possible solution after overcoming the missing detail, we refer to Remark 6.11. To keep the presentation clear we introduce the abbreviations

$$\mu^- := 2 - \frac{2}{p} \quad \text{and} \quad \mu^+ := 2 + \frac{4}{p^2} - \frac{2}{p}.$$

**Theorem 6.6** (Higher integrability – partial result). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded,  $p > 2$  and  $f \in C^2(\mathcal{R}(\mathbb{A}))$  be a variational integrand which satisfies (1.3) and (1.5) with  $\mu \in (1, \mu^-) \cup (\mu^+, 3 - \frac{2}{p})$ . Then any  $L^p$ -bounded local  $BV^{\mathbb{A}}$ -minimiser  $u \in BV_{\text{loc}}^{\mathbb{A}}(\Omega) \cap L_{\text{loc}}^p(\Omega; \mathbb{R}^N)$  of  $F$  is of class  $W^{1, t_p}(\Omega; \mathbb{R}^N)$  for  $t_p := \frac{(3-\mu)p-2}{p+2}$*

The strategy is similar to Section 6.4. We start by deriving a first, unfortunately non-optimal, local  $W^{1, 1+\alpha_p}$ -regularity result in Theorem 6.7. Since this statement turns out not to be optimal for  $\mu \in (1, 2 - \frac{2}{p})$  we improve it in a second step given in Theorem 6.10. This refinement is based on weighted second-order estimates at the level of the minimiser, cf. Theorem 6.9, together with the standard *Euler-Lagrange equation*.

**Theorem 6.7** (Higher integrability – first result). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded,  $p > 2$ , and consider a variational integrand  $f \in C^2(\mathcal{R}(\mathbb{A}))$  which satisfies (1.3) and (1.5) for  $\mu \in (1, 2 - \frac{2}{p})$ . Then every  $BV^{\mathbb{A}}$ -minimiser  $u \in BV^{\mathbb{A}}(\Omega; \mathbb{R}^N) \cap L_{\text{loc}}^p(\Omega; \mathbb{R}^N)$  of  $F$  is of class*

$$W_{\text{loc}}^{1, 1+\alpha_p}(\Omega, \mathbb{R}^N) \quad \text{for} \quad \alpha_p := \min \left\{ 1 - \frac{2}{p}, \frac{(3-\mu)p-2}{p+2} \right\}.$$

More specifically, for every subset  $U \Subset \Omega$  there exists a positive constant  $c > 0$  with  $c = c(n, p, \Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbf{a}, \mathbf{b}, \|u\|_{L^p(U; \mathbb{R}^N)}) > 0$  such that whenever  $B_{2r}(x_0) \Subset U$  we have

- for the full gradient  $\mathbb{A} = \nabla$

$$\int_{B_{2r}(x_0)} (1 + |\nabla v_j|^2)^{\frac{1+\alpha}{2}} dx \leq c \left[ \left( 1 + \frac{1}{r^2} \right) |Du|(\overline{B_{2r}(x_0)}) + 1 \right],$$

- and for the symmetric gradient  $\mathbb{A} = \varepsilon$

$$\begin{aligned} & \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{1+\alpha_p}{2}} dx \\ & \leq c \left[ \left( 1 + \frac{1}{r^2} \right) |Eu|(\overline{B_{2r}(x_0)}) + \frac{1}{r^2} \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx + \left( 1 + \frac{1}{r^4} \right) \right]. \end{aligned}$$

*Proof.* For the sake of simplicity we omit the subscript and write simply  $\alpha = \alpha_p$  throughout the proof. Since  $p > 2$  we first notice that  $p - 1 > 1$ , whereas  $1 - \frac{2}{p} < 1$ . Therefore, we conclude

$$\alpha_p = \min \left\{ p - 1, 1 - \frac{2}{p}, \frac{(3-\mu)p-2}{p+2} \right\} = \min \left\{ 1 - \frac{2}{p}, \frac{(3-\mu)p-2}{p+2} \right\}.$$

In order to derive the stated estimates, we take  $x_0 \in \Omega$  and  $r > 0$  such that  $B_{2r}(x_0) \Subset \Omega$ . Moreover, let  $\varrho \in C_c^\infty(\Omega; [0, 1])$  be a localisation function satisfying  $\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq (\frac{2}{r})^s$  for  $s \in \{1, 2, 3\}$ . For the sake of readability, we divide the proof into several steps.

*Step 1: Choice of the test function and admissibility.* As in Step 1 of the proof of Theorem 6.7, we can make use of the following Euler-Lagrange inequality

$$\begin{aligned} & \left| \int_{\Omega} \langle \nabla f(\mathbb{A}v_j), \mathbb{A}\varphi \rangle dx - \sum_{i,j=1}^n \int_{\Omega} \partial_m((\nabla f_j - \nabla f)(\mathbb{A}v_j)^{(im)}) \varphi^{(i)} dx \right. \\ & \quad \left. + \Upsilon' \left( \int_{\Omega} V_p(v_j) dx - M \right) \int_{\Omega} \langle \nabla V_p(v_j), \varphi \rangle dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-2,1}(\Omega; \mathbb{R}^N)} \end{aligned}$$

for  $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^N) \cap L^2(\Omega; \mathbb{R}^N)$ . Since  $\alpha \leq 1$  we can easily ensure that the test function  $\varphi := \varrho^4 \mathbb{T}_k((1 + |\mathbb{A}v_j|^2)^{\frac{\alpha}{2}})v_j$  is admissible, where  $\mathbb{T}_k$  denotes the truncation operator, cf. (2.5). More precisely, since the truncation as well as the localisation function is bounded, we utilise  $v_j \in W^{1,2}(\Omega; \mathbb{R}^N)$  to conclude  $\varphi \in L^2(\Omega; \mathbb{R}^N)$ . Moreover, the  $W_0^{1,1}$ -regularity can be justified in a similar manner to the proof of Theorem 5.12 after recalling  $\alpha \leq 1$ . As a next step, we compute

$$\begin{aligned} \mathbb{A}\varphi &:= \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} \mathbb{A}v_j + (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} v_j \otimes_{\mathbb{A}} \nabla \varrho^4 \\ &\quad + \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}v_j|^2 \leq k\}} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}-1} v_j \otimes_{\mathbb{A}} \nabla |\mathbb{A}v_j|^2. \end{aligned}$$

Now we plug in this expression into the partially integrated Euler-Lagrange-inequality (6.17), which yields after rearranging terms

$$\begin{aligned} \text{I} + \text{II} &:= \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} \langle \nabla f(\mathbb{A}v_j), \mathbb{A}v_j \rangle dx \\ &\quad + \Upsilon' \left( \int_{\Omega} V_p(v_j) dx - M \right) \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} \langle \nabla V_p(v_j), v_j \rangle dx \\ &\leq - \int_{\Omega} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} \langle \nabla f(\mathbb{A}v_j), v_j \otimes_{\mathbb{A}} \nabla \varrho^4 \rangle dx \\ &\quad - \int_{\Omega} \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}v_j|^2 \leq k\}} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}-1} \langle \nabla f(\mathbb{A}v_j), v_j \otimes_{\mathbb{A}} \nabla |\mathbb{A}v_j|^2 \rangle dx \\ &\quad + \sum_{i,j=1}^n \int_{\Omega} \partial_m ((\nabla f_j - \nabla f)(\mathbb{A}v_j)^{(im)}) \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} v_j^{(i)} dx \\ &\quad + \frac{1}{j} \|\varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} v_j\|_{W^{-2,1}(\Omega; \mathbb{R}^N)} \\ &=: \text{III} + \text{IV} + \text{V} + \text{VI}. \end{aligned}$$

*Step 2: Estimating all appearing terms.* The linear growth condition (1.3) in combination with Lemma 2.18 provides us with a lower bound for I, namely,

$$\text{I} \geq \gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} |\mathbb{A}v_j| dx - \Gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} dx.$$

By construction of  $\Upsilon$  we have  $\Upsilon' \geq 0$ . Moreover, by Lemma 2.18 we can find two constants  $\mathfrak{a}, \mathfrak{b} > 0$  such that  $\langle \nabla V_p(z), z \rangle \geq \mathfrak{a} |z|^p - \mathfrak{b}$ . Therefore, we obtain a lower bound for II by means of

$$\begin{aligned} \text{II} &\geq \Upsilon' \left( \int_{\Omega} V_p(v_j) dx - M \right) \left[ \mathfrak{a} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} |v_j|^p dx \right. \\ &\quad \left. - \mathfrak{b} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} dx \right] \\ &\geq -\Upsilon' \left( \int_{\Omega} V_p(v_j) dx - M \right) \mathfrak{b} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} dx. \end{aligned}$$

We continue by restructuring all the appearing terms, leading to

$$\begin{aligned}
& \gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha+1}{2}} dx \\
& \leq \text{III} + \text{IV} + \text{V} + \text{VI} + c(\mathbf{b}, \Gamma) \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} dx \quad (6.27) \\
& := \text{III} + \text{IV} + \text{V} + \text{VI} + \text{VII}.
\end{aligned}$$

We proceed by estimating all terms on the right-hand side of (6.27) gradually, starting with III. To this end, we apply Young's inequality with exponents  $p$  and  $\frac{p}{p-1}$  to obtain

$$\begin{aligned}
\text{III} & \leq \Gamma \int_{\Omega} \varrho^3 |\nabla \varrho| |v_j| (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} dx \\
& \leq c(\gamma, \Gamma) \int_{\Omega} (\varrho |\nabla \varrho|)^p |v_j|^p dx + \frac{\gamma}{8} \int_{\Omega} \varrho^{\frac{2p}{p-1}} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha p}{2(p-1)}} dx.
\end{aligned}$$

In this case, the last appearing term can be absorbed into the left-hand side of (6.27) in case

$$\frac{\alpha p}{2(p-1)} \leq \frac{\alpha+1}{2} \iff \alpha \leq p-1, \quad (6.28)$$

which is satisfied by our choice of  $\alpha$ . Towards term IV, we establish, using Young's inequality twice, that

$$\begin{aligned}
\text{IV} & = - \int_{\Omega} \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}v_j|^2 \leq k\}} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}-1} \langle \nabla f(\mathbb{A}v_j), v_j \otimes_{\mathbb{A}} \nabla |\mathbb{A}v_j|^2 \rangle dx \\
& \leq c(\Gamma, \mathbf{a}) \int_{\Omega} \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}v_j|^2 \leq k\}} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha-1}{2}} |\nabla \mathbb{A}v_j| |v_j| dx \\
& \leq c(\Gamma, \mathbf{a}) \int_{\Omega} \varrho^4 (1 + |\mathbb{A}v_j|^2)^{-\frac{\mu}{2}} |\nabla \mathbb{A}v_j|^2 dx \\
& \quad + c(\alpha, \Gamma) \int_{\Omega} \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}v_j|^2 \leq k\}} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\alpha-1+\frac{\mu}{2}} |v_j|^2 dx \\
& \leq c(\Gamma, \mathbf{a}) \int_{\Omega} \varrho^4 (1 + |\mathbb{A}v_j|^2)^{-\frac{\mu}{2}} |\nabla \mathbb{A}v_j|^2 dx + c(\gamma, \Gamma, \mathbf{a}) \int_{B_{2r}(x_0)} |v_j|^p dx \\
& \quad + \frac{\gamma}{8} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha-1+\frac{\mu}{2}}{p-2}} dx
\end{aligned}$$

We notice, that the last term can be absorbed into the left-hand side of (6.27), if

$$\left(\alpha - 1 + \frac{\mu}{2}\right) \frac{p}{p-2} \leq \frac{\alpha+1}{2} \iff \alpha \leq \frac{(3-\mu)p-2}{p+2},$$

which is again satisfied by our choice of  $\alpha$ . Moving forward, we bound the regularisation term V using again Young's inequality. In this way we get

$$\begin{aligned}
\text{V} & = \sum_{i,m=1}^n \int_{\Omega} \partial_m ((\nabla f_j - \nabla f)(\mathbb{A}v_j)^{(im)}) \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} v_j^{(i)} dx \\
& \leq \frac{1}{2A_j j^2} \int_{\Omega} \varrho^4 |\nabla \mathbb{A}v_j| (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} |v_j| dx \quad (6.29) \\
& \leq \frac{1}{2A_j j^2} \int_{\Omega} \varrho^4 |\nabla \mathbb{A}v_j|^2 dx + \frac{1}{2A_j j^2} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\alpha} |v_j|^2 dx.
\end{aligned}$$

As in the treatment of III, the first term can be controlled by means of the weighted second order estimates from Remark 6.4, whereas for the second we apply Young's inequality with exponents  $\frac{p}{2}$  and  $\frac{p}{p-2}$  to produce

$$\begin{aligned} & \frac{1}{2A_j j^2} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^\alpha |v_j|^2 \, dx \\ & \leq \frac{1}{2A_j j^2} \int_{\Omega} \varrho^p |v_j|^p \, dx + \frac{1}{2A_j j^2} \int_{\Omega} \varrho^{\frac{2p}{p-2}} (1 + |\mathbb{A}v_j|^2)^{\frac{\alpha p}{p-2}} \, dx. \end{aligned}$$

Because of  $p > 2$  we observe the equivalence

$$\frac{\alpha p}{p-2} \leq 1 \quad \iff \quad \alpha \leq 1 - \frac{2}{p},$$

and therefore, the second term in the final line of (6.29) is under control using (6.3) and (6.4). Regarding the next term, we use the estimate (2.16) for negative Sobolev-spaces, together with Young's inequality with exponents  $p$  and  $\frac{p}{p-1}$ , to obtain

$$\begin{aligned} \text{VI} &= \frac{1}{j} \|\varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} v_j\|_{W^{-2,1}(\Omega; \mathbb{R}^N)} \\ &\leq \frac{1}{j} \|\varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} v_j\|_{L^1(\Omega; \mathbb{R}^N)} \\ &\leq \int_{\Omega} \varrho^4 |v_j| (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha}{2}} \, dx \\ &\leq c(\gamma) \int_{B_{2r}(x_0)} |v_j|^p \, dx + \frac{\gamma}{8} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha p}{2(p-1)}} \, dx \end{aligned}$$

Recalling (6.28), we observe that the second term in the above inequality can also be absorbed into the very left-hand side of (6.27). Finally, in order to treat the last term VII we make use of  $\alpha \leq 1$  leading immediately to

$$\text{VII} \leq c(\Gamma, \mathbf{b}) \int_{\Omega} \varrho^4 |\mathbb{A}v_j| \, dx.$$

This closes the estimation of all terms on the right-hand side of (6.27). Therefore, after collecting all the estimates from above, we arrive at

$$\begin{aligned} & \int_{B_r(x_0)} (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha+1}{2}} \, dx \\ & \leq \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}v_j|^2))^{\frac{\alpha+1}{2}} \, dx \\ & \leq c(\gamma, \Gamma, \mathbf{a}, \mathbf{b}) \left[ \int_{\Omega} \varrho^4 (1 + |\mathbb{A}v_j|^2)^{-\frac{\mu}{2}} |\nabla \mathbb{A}v_j|^2 \, dx + \frac{1}{A_j j^2} \int_{\Omega} \varrho^4 |\nabla \mathbb{A}v_j|^2 \, dx \right. \\ & \quad \left. + \int_{B_{2r}(x_0)} |v_j|^p \, dx + \frac{1}{A_j j^2} \int_{B_{2r}(x_0)} (1 + |\mathbb{A}v_j|^2) \, dx + \int_{B_{2r}(x_0)} |\mathbb{A}v_j| \, dx \right]. \end{aligned}$$

In the full gradient case, we take into account (6.13) from Remark 6.4 to obtain

$$\begin{aligned}
& \int_{B_r(x_0)} (\mathbb{T}_k(1 + |\nabla v_j|^2))^{\frac{1+\alpha}{2}} dx \\
& \leq c \left[ \left(1 + \frac{1}{r}\right) \int_{B_{2r}(x_0)} |\nabla v_j| dx + \frac{1}{j} \left(1 + \frac{1}{r}\right) \int_{B_r(x_0)} |v_j| dx \right. \\
& \quad \left. + \int_{B_{2r}(x_0)} |v_j|^p dx + \left(1 + \frac{1}{r^2}\right) \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} (1 + |\nabla v_j|^2) dx \right]
\end{aligned} \tag{6.30}$$

for a constant  $c = c(\Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbf{a}, \mathbf{b}) > 0$ . Moreover, for the symmetric gradient we apply (6.12) from Remark 6.4 to deduce similarly

$$\begin{aligned}
& \int_{B_r(x_0)} (\mathbb{T}_k(1 + |\varepsilon(v_j)|^2))^{\frac{1+\alpha}{2}} dx \\
& \leq c \left[ \left(1 + \frac{1}{r^2}\right) \int_{B_r(x_0)} |\varepsilon(v_j)| dx + \left(\frac{1}{r^3} + \frac{1}{j} \left(1 + \frac{1}{r}\right)\right) \int_{B_{2r}(x_0)} |v_j| dx \right. \\
& \quad \left. + \int_{B_{2r}(x_0)} |v_j|^p dx + \mathcal{L}^n(\Omega)^{\frac{p-2}{p}} \frac{1}{r^4} \left(\int_{B_{2r}(x_0)} |v_j|^p dx\right)^{\frac{2}{p}} \right. \\
& \quad \left. + \left(1 + \frac{1}{r^2}\right) \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} (1 + |\varepsilon(v_j)|^2) dx + \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} \frac{|v_j|^2}{r^2} dx \right]
\end{aligned} \tag{6.31}$$

for a constant  $c = c(\Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbf{a}, \mathbf{b}) > 0$ . We note that all quantities on the right-hand side of (6.30) and (6.31) are under control independently of the level  $k > 0$  because of Proposition 6.2 in combination with Corollary 4.3 and Corollary 4.4 applied in the quadratic framework. Hence, applying the monotone convergence theorem we conclude

$$\lim_{k \rightarrow \infty} \int_{B_r(x_0)} (\mathbb{T}_k(1 + |\varepsilon(v_j)|^2))^{\frac{1+\alpha}{2}} dx = \int_{B_r(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{1+\alpha}{2}} dx \tag{6.32}$$

is bounded, satisfying the same estimates (6.30) and (6.31).

*Step 3: Higher Integrability:* Following the argument of Step 3 in the proof of Theorem 6.1, we can extract a non-re-labeled subsequence  $(v_j)_{j \in \mathbb{N}}$  such that  $v_j \xrightarrow{*} v$  in  $\text{BV}^\Lambda(\Omega)$ . Moreover, using (6.32) in combination with (6.30) and (6.31) we infer that  $(\mathbb{A}v_j)_{j \in \mathbb{N}}$  is uniformly bounded in the reflexive space  $L^{1+\alpha}(B_r(x_0); \mathbb{R}^N)$ . Hence, by the Banach-Alaoglu theorem we can extract a further subsequence such that  $\mathbb{A}v_j \rightharpoonup \mathbb{A}u$  in  $L^{1+\alpha}(B_r(x_0); \mathcal{R}(\mathbb{A}))$  (the limit can be identified by a similar argument as in Step 3 of the proof from Theorem 5.12) as well as

$$\limsup_{j \rightarrow \infty} \|v_j\|_{L^p(\Omega; \mathbb{R}^N)}^p \leq c(p) (3 + M),$$

because of (6.4). Thus, using the weak lower semi-continuity of the norm we can pass to the limit  $j \rightarrow \infty$  to obtain

$$\begin{aligned}
& \int_{B_r(x_0)} (1 + |\nabla u|^2)^{\frac{1+\alpha}{2}} dx \\
& \leq \liminf_{j \rightarrow \infty} \int_{B_r(x_0)} (1 + |\nabla v_j|^2)^{\frac{1+\alpha}{2}} dx \leq c \left[ \left(1 + \frac{1}{r^2}\right) |\text{Du}|(\overline{B}_{2r}(x_0)) + 1 \right].
\end{aligned}$$

in the full gradient case  $\mathbb{A} = \nabla$ , whereas the same reasoning yields

$$\begin{aligned} & \int_{B_r(x_0)} (1 + |\varepsilon(u)|^2)^{\frac{1+\alpha}{2}} dx \\ & \leq \liminf_{j \rightarrow \infty} \int_{B_r(x_0)} (1 + |\varepsilon(v_j)|^2)^{\frac{1+\alpha}{2}} dx \\ & \leq c \left[ \left(1 + \frac{1}{r^2}\right) |Eu|(\overline{B}_{2r}(x_0)) + \frac{1}{r^2} \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx + \left(1 + \frac{1}{r^4}\right) \right] \end{aligned}$$

in the symmetric gradient case  $\mathbb{A} = \varepsilon$  for a positive  $c > 0$ . Recalling that  $M$  depends on  $\Omega, n$  and  $m$  we obtain  $c = c(n, p, \Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbf{a}, \mathbf{b}, \|u\|_{L^p(U; \mathbb{R}^N)}) > 0$ , and in both cases also the desired estimates. This completes the proof.  $\square$

**Remark 6.8** (Consistency). *Although not sharp, the result is consistent with Theorem 5.12 from the previous section, since there holds*

$$\lim_{p \rightarrow \infty} \alpha_p = \lim_{p \rightarrow \infty} \min \left\{ 1 - \frac{2}{p}, \frac{(3 - \mu)p - 2}{p + 2} \right\} = \min\{1, 3 - \mu\}.$$

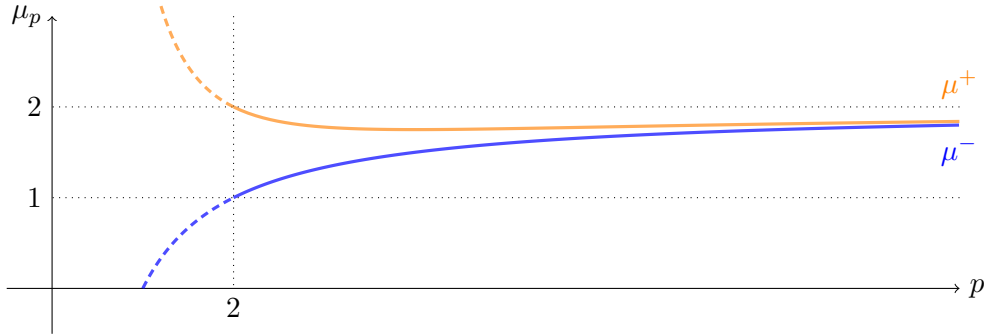


Figure 6.2: The gap in the ellipticity range

We observe that there holds  $\alpha_p = 1 - \frac{2}{p}$  if  $\mu < \mu^+$ . However, to obtain information on second derivatives, we must restrict ourselves to  $\mu < \mu^-$ , thereby creating an unnatural gap in the ellipticity range, see Figure 6.2.

**Theorem 6.9** (Second-order estimates for minimisers). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded,  $p > 2$ , and consider a variational integrand  $f \in C^2(\mathcal{R}(\mathbb{A}))$  which satisfies (1.3) and (1.5) with  $\mu \in (1, \mu^-)$ . Then any  $L^p$ -bounded local  $BV^{\mathbb{A}}$ -minimiser  $u \in BV_{\text{loc}}^{\mathbb{A}}(\Omega) \cap L_{\text{loc}}^p(\Omega; \mathbb{R}^N)$  of  $F$  is of class  $W_{\text{loc}}^{2, s_p}(\Omega, \mathbb{R}^N)$  for  $s_p := \frac{4(p-1)}{(\mu+2)p-2}$ . Moreover, for every subset  $U \Subset \Omega$  there exists a constant  $c > 0$  such that whenever  $B_{2r}(x_0) \Subset U$  we have the weighted second-order estimates*

- for the full gradient  $\mathbb{A} = \nabla$

$$\int_{B_{2r}(x_0)} (1 + |\nabla u|^2)^{\frac{2-\mu}{2}-2} |\nabla |\nabla u|^2|^2 dx \leq \frac{c}{r^2} |Du|(\overline{B}_{2r}(x_0)) \quad (6.33)$$

with a constant  $c = c(\Omega, \gamma, \Gamma, \lambda, \Lambda) > 0$ .

- for the symmetric gradient  $\mathbb{A} = \varepsilon$

$$\begin{aligned} & \int_{B_{2r}(x_0)} (1 + |\varepsilon(u)|^2)^{\frac{2-\mu}{2}-2} |\nabla |\varepsilon(u)|^2|^2 dx \\ & \leq \frac{c}{r^2} \left[ |\mathbb{E}u|(\overline{B}_{2r}(x_0)) + \int_{B_{2r}(x_0)} \frac{|u|}{r} dx + \frac{1}{r^2} \|u\|_{L^p(B_{2r}(x_0); \mathbb{R}^N)}^2 + 1 \right] \end{aligned} \quad (6.34)$$

with a constant  $c = c(n, p, \Omega, \gamma, \Gamma, \lambda, \Lambda, \|u\|_{L^p(U; \mathbb{R}^n)}) > 0$ .

*Proof.* Let  $x_0 \in \mathbb{R}^n$  and  $r > 0$  such that  $B_{2r}(x_0) \Subset \Omega$ . As before we omit the  $p$ -dependence when dealing with  $s$  and  $\alpha$  throughout the proof. From the proof of Theorem 6.7 we recall that the sequence  $(\mathbb{A}v_j)_{j \in \mathbb{N}}$  is uniformly bounded in  $W_{\text{loc}}^{1,1+\alpha}(\Omega; \mathbb{R}^N)$ , with  $\alpha + 1 = 2 - \frac{2}{p}$ . Next, we note that under our constraints there holds  $s \in (1, 2)$ <sup>1</sup>. Thus, Young's inequality with exponents  $\frac{2}{s}$  and  $\frac{2}{2-s}$  yields

$$\begin{aligned} & \int_{B_{2r}(x_0)} |\nabla \mathbb{A}v_j|^s dx \\ & = \int_{B_{2r}(x_0)} |\nabla \mathbb{A}v_j|^s (1 + |\mathbb{A}v_j|^2)^{-\frac{s\mu}{4}} (1 + |\mathbb{A}v_j|^2)^{\frac{s\mu}{4}} dx \\ & \leq \int_{B_{2r}(x_0)} |\nabla \mathbb{A}v_j|^2 (1 + |\mathbb{A}v_j|^2)^{-\frac{s\mu}{4} \frac{2}{s}} dx + \int_{B_{2r}(x_0)} (1 + |\mathbb{A}v_j|^2)^{\frac{s\mu}{4} \frac{2}{2-s}} dx \\ & = \int_{B_{2r}(x_0)} |\nabla \mathbb{A}v_j|^2 (1 + |\mathbb{A}v_j|^2)^{-\frac{\mu}{2}} dx + \int_{B_{2r}(x_0)} (1 + |\mathbb{A}v_j|^2)^{\frac{1+\alpha}{2}} dx, \end{aligned}$$

where the last equality directly follows from the computation

$$\frac{s\mu}{4} \frac{2}{(2-s)} = \frac{1}{2} \left( \frac{4\mu(p-1)}{((\mu+2)p-1) - (4p-2)} \right) = \frac{1}{2} \left( \frac{2\mu(p-1)}{\mu p} \right) = \frac{1}{2} \left( 2 - \frac{2}{p} \right) = \frac{1+\alpha}{2}.$$

Therefore, using the uniform weighted estimates from Remark 6.4 we conclude that the sequence  $(\mathbb{A}v_j)_{j \in \mathbb{N}}$  is uniformly bounded in  $W^{1,s}(B_{2r}(x_0); \mathbb{R}^N)$ . Hence, by weak compactness, we can extract a weakly convergent non-reabeled subsequence such that

$$\mathbb{A}v_j \rightharpoonup \mathbb{A}u \quad \text{in } L^s(B_{2r}(x_0); \odot^2 \mathcal{R}(\mathbb{A})) \quad \text{as } j \rightarrow \infty.$$

Using the lower semi-continuity of the norm together with Korn's inequality from Corollary 3.6 we conclude  $u \in W_{\text{loc}}^{2,s}(\Omega; \mathbb{R}^N)$ . Moreover, using the Sobolev embedding we conclude, after passing to a further non-reabeled subsequence, that  $\mathbb{A}v_j \rightarrow \mathbb{A}u$  pointwise  $\mathcal{L}^n$ -a.e. in  $B_{2r}(x_0)$ . Similarly to the proof of Theorem 5.14 we define

$$V(z) := V_{\frac{2-\mu}{4}}(z) = (1 + |z|^2)^{\frac{2-\mu}{4}} - 1 \quad \text{for } z \in \mathcal{R}(\mathbb{A}),$$

leading to

$$|\nabla V(\mathbb{A}v_j)|^2 = \left( \frac{2-\mu}{2} \right)^2 (1 + |\mathbb{A}v_j|^2)^{\frac{2-\mu}{2}-2} |\nabla |\mathbb{A}v_j|^2|^2.$$

<sup>1</sup>We get  $s < 2$  if and only if  $4p < 2(\mu+2)p$ . Hence, cancelling out  $p$  (recall  $p > 2$ ) results in  $\mu > 0$ , which is fulfilled in our case. Similarly, we have  $s > 1$  if and only if  $4(p-1) > (\mu+2)p - 2$  which requires  $p(2-\mu) - 2 > 0$  or equivalently  $\mu < 2 - \frac{2}{p}$ .

Especially, we obtain

$$\int_{B_{2r}(x_0)} |\nabla V(\mathbb{A}v_j)|^2 dx \leq C \int_{B_{2r}(x_0)} (1 + |\mathbb{A}v_j|^2)^{-\frac{\mu}{2}} |\nabla \mathbb{A}v_j|^2 dx$$

and therefore, the sequence  $(V(\mathbb{A}v_j))_{j \in \mathbb{N}}$  for  $j \in \mathbb{N}$  is uniformly bounded in  $W^{1,2}(B_{2r}(x_0))$  using the weighted estimates from Remark 6.4. The claim follows by repeating the argument from the proof of Theorem 5.14.  $\square$

**Theorem 6.10** (Higher integrability for  $\mu \in (1, \mu^-)$  - Improvement). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded and  $f \in C^2(\mathcal{R}(\mathbb{A}))$  be a variational integrand which satisfies (1.3) and (1.5) with  $\mu \in (1, \mu^-)$ . Then for  $p > 2$  every  $L^p$ -bounded local  $BV^{\mathbb{A}}$ -minimiser  $u \in BV^{\mathbb{A}}(\Omega)_{\text{loc}}(\Omega) \cap L^p(\Omega; \mathbb{R}^N)$  of  $F$  is of class  $W_{\text{loc}}^{1,1+s_p}(\Omega; \mathbb{R}^N)$  with  $s_p := \frac{(3-\mu)p-2}{p+2}$ . More precisely, for every subset  $U \Subset \Omega$  there exists a constant  $c > 0$  such that whenever  $B_{2r}(x_0) \Subset U$  we have*

- for the full gradient  $\mathbb{A} = \nabla$

$$\int_{B_r(x_0)} (1 + |\nabla u|^2)^{\frac{1+s_p}{2}} dx \leq c \left[ \frac{1}{r^2} |Du|(\overline{B_{2r}(x_0)}) + \left(1 + \frac{1}{r^2}\right) \int_{B_{2r}(x_0)} |u|^p dx \right] \quad (6.35)$$

with  $c = c(p, \Omega, \gamma, \Gamma, \lambda, \Lambda) > 0$ .

- for the symmetric gradient  $\mathbb{A} = \varepsilon$

$$\begin{aligned} & \int_{B_r(x_0)} (1 + |\varepsilon(u)|^2)^{\frac{1+s_p}{2}} dx \\ & \leq c \left[ \frac{1}{r^2} |Eu|(\overline{B_{2r}(x_0)}) + \int_{B_{2r}(x_0)} \frac{|u|}{r} dx \right. \\ & \quad \left. + \left(1 + \frac{1}{r^2}\right) \int_{B_{2r}(x_0)} |u|^p dx + \frac{1}{r^2} \|u\|_{L^p(B_{2r}(x_0); \mathbb{R}^N)}^2 + 1 \right]. \end{aligned} \quad (6.36)$$

with  $c = c(n, p, \Omega, \gamma, \Gamma, \lambda, \Lambda, \|u\|_{L^p(U; \mathbb{R}^n)}) > 0$ .

*Proof.* Let  $p > 2$  and write  $s := s_p$  for short. Moreover, let  $x_0 \in \Omega$  and  $r > 0$  such that  $B_{2r}(x_0) \Subset \Omega$  and  $\varrho \in C_c^\infty(\Omega; [0, 1])$  be a cut-off function with  $\mathbf{1}_{B_r(x_0)} \leq \varrho \leq \mathbf{1}_{B_{2r}(x_0)}$  and  $|\nabla^s \varrho| \leq \left(\frac{2}{s}\right)^s$  for  $s \in \{1, 2, 3\}$ . Similar to the proof of Theorem 5.16 we use that  $u$  solves the Euler-Lagrange equation

$$\int_{\Omega} \langle \nabla f(\mathbb{A}u), \mathbb{A}\varphi \rangle dx = 0 \quad \text{for all } \varphi \in W_0^{1,1}(\Omega; \mathbb{R}^N) \quad (6.37)$$

which can be derived analogously.

*Step 1: Admissibility of the test function.* We aim to use  $\varphi = \varrho^4 T_k(1 + |\mathbb{A}u|^2)^{\frac{s}{2}} u$  as a test function in the Euler-Lagrange equation (6.37) and briefly argue why it is admissible. Clearly, there holds  $\varphi \in L_{\text{loc}}^1(\Omega; \mathbb{R}^N)$ , because the truncation as well as the cut-off function is bounded together with  $u \in W_{\text{loc}}^{1,1+\alpha_p}(\Omega; \mathbb{R}^N)$ , cf. Theorem 6.7. Moreover, it turns out that both terms of

$$\nabla u (T_k(1 + |\mathbb{A}u|^2))^{\frac{s}{2}} + \mathbf{1}_{\{1+|\mathbb{A}u|^2 \leq k\}} (T_k(1 + |\mathbb{A}u|^2))^{\frac{s-2}{2}} u \otimes \nabla |\mathbb{A}u|^2$$

are elements in  $L^1_{\text{loc}}(\Omega; \mathbb{R}^N)$ . While this immediately follows for the first term using the boundedness of the truncation and  $u \in W^{1,1+\alpha_p}_{\text{loc}}$  we check this for the second directly: To this end, we use Young's inequality twice to infer

$$\begin{aligned}
& \int_{B_r(x_0)} \mathbb{1}_{\{1+|\mathbb{A}u|^2 \leq k\}} (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s-2}{2}} |u| |\nabla |\mathbb{A}u|^2| \, dx \\
& \leq \int_{B_r(x_0)} (1+|\mathbb{A}u|^2)^{\frac{2-\mu}{2}-2} |\nabla |\mathbb{A}u|^2|^2 \, dx \\
& \quad + \int_{B_r(x_0)} \mathbb{1}_{\{1+|\mathbb{A}u|^2 \leq k\}} |u|^2 (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{s-1-\frac{\mu}{2}} \, dx \\
& \leq \int_{B_r(x_0)} (1+|\mathbb{A}u|^2)^{\frac{2-\mu}{2}-2} |\nabla |\mathbb{A}u|^2|^2 \, dx + c(p) \int_{B_r(x_0)} |u|^p \, dx \\
& \quad + c(p) \int_{B_r(x_0)} \mathbb{1}_{\{1+|\mathbb{A}u|^2 \leq k\}} (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\left(s-1-\frac{\mu}{2}\right)\frac{p}{p-2}} \, dx.
\end{aligned}$$

We note that all appearing integrals are finite, using Theorem 6.9,  $u \in L^p(\Omega; \mathbb{R}^N)$  and the boundedness of the truncation operator. Finally, recalling the product rule from (2.4) shows that  $\varphi \in W^{1,1}_0(\Omega; \mathbb{R}^N)$  and closes the argument for admissibility.

*Step 2: Testing the Euler-Lagrange equation.* Since  $\varphi$  is admissible by Step 1, we can plug it into the Euler-Lagrange equation. Towards this aim, we compute

$$\begin{aligned}
\mathbb{A}\varphi & := \varrho^4 (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s}{2}} \mathbb{A}u + (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s}{2}} u \otimes_{\mathbb{A}} \nabla \varrho^4 \\
& \quad + \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}u|^2 \leq k\}} (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s-2}{2}} u \otimes_{\mathbb{A}} \nabla |\mathbb{A}u|^2.
\end{aligned}$$

Substituting this expression into the Euler-Lagrange equation and rearranging terms gives

$$\begin{aligned}
\text{I} & := \int_{\Omega} \varrho^4 (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s}{2}} \langle \nabla f(\mathbb{A}u), \mathbb{A}u \rangle \, dx \\
& = - \int_{\Omega} (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s}{2}} \langle \nabla f(\mathbb{A}u), u \otimes_{\mathbb{A}} \nabla \varrho^4 \rangle \, dx \\
& \quad - \int_{\Omega} \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}u|^2 \leq k\}} (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s-2}{2}} \langle \nabla f(\mathbb{A}u), u \otimes_{\mathbb{A}} \nabla |\mathbb{A}u|^2 \rangle \, dx \\
& =: \text{II} + \text{III}.
\end{aligned}$$

*Step 3: Estimate all appearing terms.* Using the lower bound from Lemma 2.18 we obtain

$$\begin{aligned}
\text{I} & \geq \gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s}{2}} |\mathbb{A}u| \, dx - \Gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s}{2}} \, dx \\
& \geq \gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s+1}{2}} \, dx - \Gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s}{2}} \, dx,
\end{aligned}$$

which results in

$$\begin{aligned}
\gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s+1}{2}} \, dx & \leq \text{II} + \text{III} + \Gamma \int_{\Omega} \varrho^4 (\mathbb{T}_k(1+|\mathbb{A}u|^2))^{\frac{s}{2}} \, dx \\
& =: \text{II} + \text{III} + \text{IV}.
\end{aligned} \tag{6.38}$$

We now estimate all the appearing terms on the right-hand side, starting with II. Using Young's inequality with  $p$  and  $\frac{p}{p-1}$  leads to

$$\begin{aligned} \text{II} &\leq 4\Gamma \int_{\Omega} (\mathbb{T}_k(1 + |\mathbb{A}u|^2))^{\frac{s}{2}} |u| \varrho^3 |\nabla \varrho| \, dx \\ &\leq c(\gamma, \Gamma) \int_{B_{2r}(x_0)} \frac{|u|^p}{r^p} \, dx + \frac{\gamma}{4} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}u|^2))^{\frac{sp}{2(p-1)}} \, dx. \end{aligned}$$

In order to absorb the second appearing term into the left-hand side of (6.38) we note that

$$\frac{sp}{2(p-1)} \leq \frac{s+1}{2} \iff s \leq p-1.$$

Hence, using the definition of  $s$  in combination with  $p > 2$  and  $1 < \mu \leq \mu^-$  we see

$$\frac{(3-\mu)p-2}{p+2} \leq p-1 \iff (3-\mu)p < p^2+2 \iff \mu \geq 2-p,$$

which is always fulfilled in our case. In order to treat III we

$$\begin{aligned} \text{III} &\leq 2\Gamma \int_{\Omega} \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}u|^2 \leq k\}} (\mathbb{T}_k(1 + |\mathbb{A}u|^2))^{\frac{s-2}{2}} |u| |\nabla |\mathbb{A}u|^2| \, dx \\ &\leq c(\Gamma) \int_{\Omega} \varrho^4 (1 + |\mathbb{A}u|^2)^{\frac{2-\mu}{2}-2} |\nabla |\mathbb{A}u|^2|^2 \, dx \\ &\quad + 4\Gamma \int_{\Omega} \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}u|^2 \leq k\}} |u|^2 (\mathbb{T}_k(1 + |\mathbb{A}u|^2))^{s-1+\frac{\mu}{2}} \, dx \\ &=: \text{III}_a + \text{III}_b. \end{aligned}$$

Whereas  $\text{III}_a$  is under control because of Theorem 6.9, we use Young's inequality with exponents  $\frac{2}{p}$  and  $\frac{2}{2-p}$  to deduce

$$\text{III}_b \leq c(\gamma, \Gamma) \int_{B_{2r}(x_0)} |u|^p \, dx + \frac{\gamma}{4} \int_{\Omega} \varrho^4 \mathbb{1}_{\{1+|\mathbb{A}u|^2 \leq k\}} (\mathbb{T}_k(1 + |\mathbb{A}u|^2))^{(s-1+\frac{\mu}{2})\frac{2}{p-2}} \, dx.$$

To absorb the term on the very right into the left-hand side of (6.38) we observe

$$s = \frac{(3-\mu)p-2}{p+2} \iff \left(s-1+\frac{\mu}{2}\right) \frac{2}{2-p} = \frac{s+1}{2}$$

for  $p > 2$ . Finally, we bound IV by applying Young's inequality with exponents  $\frac{s+1}{s}$  and  $s+1$ . Hence, we get

$$\text{IV} \leq \frac{\gamma}{4} \int_{\Omega} \varrho^4 (\mathbb{T}_k(1 + |\mathbb{A}u|^2))^{\frac{s+1}{2}} \, dx + c(\Omega, \gamma, \Gamma).$$

Now the first term can be again absorbed into the left-hand side of (6.38), whereas the second is constant. Since  $u \in L^p(\Omega; \mathbb{R}^N)$  with  $p > 2$ , we may apply Theorem 6.9. After absorbing the relevant terms, the right-hand side of (6.38) is bounded independently of the level  $k$ , which yields

$$\begin{aligned}
& \int_{B_r(x_0)} (\mathbb{T}_k(1 + |\mathbb{A}u|^2))^{\frac{s+1}{2}} dx \\
& \leq c \left[ \left(1 + \frac{1}{r^2}\right) \int_{B_{2r}(x_0)} |u|^p dx + \int_{B_{2r}(x_0)} (1 + |\mathbb{A}u|^2)^{\frac{s-2}{2}-2} |\nabla |\mathbb{A}u|^2|^2 dx + 1 \right] \quad (6.39)
\end{aligned}$$

with a constant  $c = c(\Omega, \gamma, \Gamma) > 0$ . In conclusion, we can pass to the limit  $k \rightarrow \infty$  using the monotone convergence theorem, i.e.

$$\lim_{k \rightarrow \infty} \int_{B_r(x_0)} (\mathbb{T}_k(1 + |\mathbb{A}u|^2))^{\frac{s+1}{2}} dx = \int_{B_r(x_0)} (1 + |\mathbb{A}u|^2)^{\frac{s+1}{2}} dx. \quad (6.40)$$

Taking into consideration (6.33) and (6.34) we obtain from (6.39) and (6.40) the claimed estimates (6.35) and (6.36) respectively. This finishes the proof.  $\square$

**Remark 6.11** (Closing the gap). *The gap that appears in the range of ellipticity is a remnant of the quadratic regularisation. It turns out that the latter is not sufficient for proving a sharp version in Theorem 6.7. It seems that the correct approach is to work with a  $p$ -dependent subquadratic  $L^q$ -regularisation with  $q = \frac{2p}{p+2}$  instead of the quadratic one. More precisely, with the above choice we can improve  $\alpha_p$  to*

$$\alpha_p := \min \left\{ 1 - \frac{2}{p+2}, \frac{(3-\mu)p-2}{p+2} \right\},$$

which then closes the gap, since

$$\alpha_p = 1 - \frac{2}{p+2} \quad \text{if } \mu \leq \mu^-.$$

We can now proceed similarly to Theorem 6.9 and Theorem 6.10 to obtain a sharp result. The main technical reason for our restriction to quadratic regularisation is the absence of both uniform and non-uniform estimates in the subquadratic framework.

## 7 | Gradient Integrability for Planar $BV^{\mathbb{A}}$ -Minimisers

In this section, we present the results of [EL25] extending a result of [Gme20] to more general  $\mathbb{C}$ -elliptic operators in dimension two. More precisely, we prove gradient integrability for relaxed minimisers  $u \in BV^{\mathbb{A}}(\Omega)$  without further constraints of the variational principle

$$\text{to minimise } F[v; \Omega] := \int_{\Omega} f(\mathbb{A}v) \, dx \quad \text{over } v \in \mathcal{D}_{u_0} := u_0 + W_0^{\mathbb{A},1}(\Omega),$$

where  $\mathbb{A} = \mathcal{A}[\cdot \otimes \nabla]$  is a first-order  $\mathbb{C}$ -elliptic differential operator between  $V = \mathbb{R}^2$  and  $W = \mathbb{R}^{2 \times 2}$  in two dimensions induced by a projection  $\mathcal{A}: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ . Specifically, we will prove the following theorem:

**Theorem 7.1** (Gradient integrability). *Let  $\Omega \subset \mathbb{R}^2$  be open and bounded and consider a variational integrand  $f \in C^2(\mathcal{R}(\mathbb{A}))$  which satisfies (1.3) and (1.5) with  $\mu \in (1, 2)$ . Then any local  $BV^{\mathbb{A}}$ -minimiser  $u \in BV_{\text{loc}}^{\mathbb{A}}(\Omega)$  of  $F$  is of class  $W^{\mathbb{A},1}(\Omega) \cap W_{\text{loc}}^{1,q}(\Omega; \mathbb{R}^2)$  for all  $q \in [1, \infty)$ . More specifically, for every subset  $U \Subset \Omega$  there exists a positive constant  $c = c(\Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbb{A}, \mu) > 0$  such that whenever  $B_{2r}(x_0) \Subset U$  we have*

$$\begin{aligned} & \|\nabla u\|_{\exp L^{\frac{3-\mu}{2-\mu}}(B_r(x_0); \mathbb{R}^{2 \times 2})} & (7.1) \\ & \leq c \left( \left(1 + \frac{1}{r}\right) \left[ \left(1 + \frac{1}{r^2}\right) |\mathbb{A}u|(\overline{B}_{2r}(x_0)) + \int_{B_{2r}(x_0)} \frac{|u|}{r} \, dx \right]^{\frac{1}{2-\mu}} + \int_{B_r(x_0)} \frac{|u|}{r} \, dx \right). \end{aligned}$$

*Structure of the chapter:* In Section 7.1 we implement a quadratic viscosity regularisation without a further penalisation, building on Chapter 4. This is sufficient as we impose no further assumption on the minimiser. As in the previous chapters, we derive uniform weighted second-order estimates Section 7.2 exploiting the specific structure of  $\mathbb{C}$ -elliptic operators in the aforementioned situation. Section 7.3 is devoted to the proof of Theorem 7.1, which is based on direct estimates using  $V$ -functions and Trudinger's embedding. We also briefly discuss how a comparable result can be derived for the deviatoric gradient in all dimensions, as this operator has similar structural properties.

### 7.1 Ekeland-type Viscosity Approximation

At the beginning, we notice that Theorem 7.1 contains a regularity result for  $BV^{\mathbb{A}}$ -minimisers without any further constraint. Hence, the results from Chapter 4 simplify a lot since we do not have to keep track of any additional information. Especially, we do not have to use any additional penalisation, which results in  $\Upsilon \equiv 0$ . Moreover, in this case it is

sufficient to use a quadratic regularisation, Therefore, we consider  $f_j: \mathcal{R}(\mathbb{A}) \rightarrow \mathbb{R}$  given by  $f_j := f_{j,2}$ , i.e.

$$f_j(z) := f(z) + \frac{1}{2A_j j^2} (1 + |z|^2) \quad \text{with} \quad A_j := A_{j,2} = 1 + \int_{\Omega} (1 + |\mathbb{A}\tilde{u}_j|^2) dx.$$

This allows to define the extended and regularised functionals  $F_j$  on the negative Sobolev space  $W^{-2,1}(\Omega; \mathbb{R}^2)$  through

$$F_j[w; \Omega] := \begin{cases} \int_{\Omega} f_j(\mathbb{A}w) dx & \text{if } w \in \mathcal{D}_j \\ +\infty & \text{if } w \in W^{-2,1}(\Omega; \mathbb{R}^2) \setminus \mathcal{D}_j. \end{cases}$$

In particular, we can apply the results from Chapter 4 giving rise to a Ekeland-type approximation sequence  $(v_j)_{j \in \mathbb{N}}$ . This sequence satisfies the following a priori estimates, as an immediate consequence of Proposition 4.2:

$$\int_{\Omega} |\mathbb{A}v_j| dx \leq \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right), \quad (7.2)$$

$$\frac{1}{2A_j j^2} \int_{\Omega} (1 + |\mathbb{A}v_j|^2) dx \leq \frac{2}{j^2} \quad (7.3)$$

for all  $j \in \mathbb{N}$ . Obviously, (4.25) trivialises as there is no penalisation term present. Moreover, from Corollary 4.3 we obtain

$$\|v_j\|_{L^1(\Omega; \mathbb{R}^2)} \leq c(\Omega, \Gamma) \left[ \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) + \frac{1}{j^2} \right]. \quad (7.4)$$

Also the Euler-Lagrange inequality from Theorem 4.6 simplifies and reads

$$\left| \int_{\Omega} \langle \nabla f_j(\mathbb{A}v_j), \mathbb{A}\varphi \rangle dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-2,1}(\Omega; \mathbb{R}^2)}, \quad (7.5)$$

for all  $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^2)$ . Assuming the growth condition (4.31) from Lemma 4.8 we obtain  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^2)$  for all  $j \in \mathbb{N}$ . Furthermore, we recall that this higher regularity allows to differentiate the Euler-Lagrange inequality resulting in

$$\left| \int_{\Omega} \langle \partial_s \sigma_j, \mathbb{A}\varphi \rangle dx \right| = \left| \int_{\Omega} \langle \partial_s \sigma_j, \nabla \varphi \rangle dx \right| \leq \frac{1}{j} \|\varphi\|_{W^{-1,1}(\Omega; \mathbb{R}^2)}, \quad (7.6)$$

for arbitrary  $s \in \{1, \dots, n\}$  and  $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^2)$ , cf. Lemma 4.9. From the same Lemma it follows  $\sigma_j \in W_{\text{loc}}^{1,2}(\Omega; \mathcal{R}(\mathbb{A}))$ , where we have set  $\sigma_j := \nabla f_j(\mathbb{A}v_j)$ . The differentiated Euler-Lagrange inequality is again the starting point for the derivation of uniform weighted second-order estimates. Since  $\mathbb{A}$  is a  $\mathbb{C}$ -elliptic operator induced by a projection  $\mathcal{A}: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$  we have  $\mathbb{A} = \mathcal{A}[\cdot \otimes \nabla]$  or  $\mathbb{A}u = \mathcal{A}[\nabla u]$ . We use both notations interchangeably.

## 7.2 Weighted Uniform Estimates

The major ingredient for proving weighted second-order estimates is the decomposition (2.10) from Theorem 2.7. In particular, this characterisation allows to overcome the challenging interplay between the structure of the operator and Ornstein's Non-Inequality.

**Theorem 7.2** (Uniform second-order estimates). *Let  $f \in C^2(\mathcal{R}(\mathbb{A}))$  satisfy (1.3) and, for some  $\Lambda \in (0, \infty)$  the bound (4.31). Then there exists a constant  $c = c(\Lambda, \mathbb{A}) > 0$  such that for every ball  $B_{2r}(x_0) \Subset \Omega$  and every localisation function  $\varrho \in C_c^\infty(\Omega; [0, 1])$  with  $\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_{2r}(x_0)}$  and  $|\nabla \varrho| \leq \frac{2}{r}$  there holds*

$$\begin{aligned} & \sum_{k=1}^2 \int_{B_{2r}(x_0)} \varrho^4 \langle \nabla^2 f_j(\mathbb{A}v_j) \partial_k \mathbb{A}v_j, \partial_k \mathbb{A}v_j \rangle dx \\ & \leq \frac{c}{r^2} \left[ \int_{B_{2r}(x_0)} |\mathbb{A}v_j| dx \right. \\ & \quad \left. + \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} (1 + |\mathbb{A}v_j|^2) dx + \left(1 + \frac{r^2}{j} + \frac{r^3}{j}\right) \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx \right] \end{aligned} \quad (7.7)$$

for all  $j \in \mathbb{N}$ .

**Remark 7.3.** *Recalling the estimates (7.2), (7.3), and (7.4) for the Ekeland-type approximation sequence  $(v_j)_{j \in \mathbb{N}}$ , we observe that all terms on the right-hand side of (7.7) are uniformly bounded in  $j \in \mathbb{N}$ . Therefore, assuming in addition that the integrand  $f$  is  $\mu$ -elliptic in the sense of (1.5), we obtain the estimate*

$$\begin{aligned} & \int_{B_r(x_0)} (1 + |\mathbb{A}v_j|^2)^{-\frac{\mu}{2}} |\nabla \mathbb{A}v_j|^2 dx + \frac{1}{2A_j j^2} \int_{B_r(x_0)} |\nabla \mathbb{A}v_j|^2 dx \\ & \leq \frac{c(\lambda, \Lambda, \mathbb{A})}{r^2} \left[ \int_{B_{2r}(x_0)} |\mathbb{A}v_j| dx \right. \\ & \quad \left. + \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} (1 + |\mathbb{A}v_j|^2) dx + \left(1 + \frac{r^2}{j} + \frac{r^3}{j}\right) \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx \right] \quad (7.8) \\ & \leq \frac{c}{r^2} \left(1 + \frac{r^2}{j} + \frac{r^3}{j}\right) \left[ \frac{1}{\gamma} \left( \inf_{\mathcal{D}_{u_0}} F[-; \Omega] + \frac{2}{j^2} \right) + \frac{1}{j^2} \right] + \frac{c}{r^2 j^2}, \end{aligned}$$

where  $c = c(\Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbb{A}) > 0$ .

*Proof of Theorem 7.2.* Let  $x_0 \in \Omega$  and  $r > 0$  such that  $B_{2r}(x_0) \Subset \Omega$  and consider a localisation function  $\varrho \in C_c^\infty(\Omega; [0, 1])$  with  $\mathbb{1}_{B_r(x_0)} \leq \varrho \leq \mathbb{1}_{B_{2r}(x_0)}$  and  $|\nabla \varrho| \leq \frac{2}{r}$ . For the sake of clarity we divide the proof into several steps.

*Step 1:* First, we recall that  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^2)$  and therefore we can apply the product rule to obtain

$$\begin{aligned} \mathbb{A}(\varrho^4 \partial_k v_j) &= \partial_k v_j \otimes_{\mathbb{A}} \nabla \varrho^4 + \varrho^4 \mathbb{A}(\partial_k v_j) \\ &= \mathcal{A}[\partial_k v_j \otimes \nabla \varrho^4] + \varrho^4 \mathcal{A}[\partial_k \nabla v_j] = \mathcal{A}[\nabla(\varrho^4 \partial_k v_j)]. \end{aligned}$$

This allows to rewrite the left-hand side of (7.7) to infer

$$\begin{aligned} & \int_{\Omega} \varrho^4 \langle \nabla^2 f_j(\mathbb{A}v_j) \partial_k \mathbb{A}v_j, \partial_k \mathbb{A}v_j \rangle dx \\ &= - \int_{\Omega} \langle \nabla^2 f_j(\mathbb{A}v_j), \partial_k v_j \otimes_{\mathbb{A}} \nabla \varrho^4 \rangle dx \\ & \quad + \int_{\Omega} \langle \nabla^2 f_j(\mathbb{A}v_j) \partial_k \mathbb{A}v_j, \mathbb{A}(\varrho^4 \partial_k v_j) \rangle dx \\ &= - \int_{\Omega} \langle \partial_k \sigma_j, \partial_k v_j \otimes_{\mathbb{A}} \nabla \varrho^4 \rangle dx + \int_{\Omega} \langle \partial_k \sigma_j, \mathbb{A}(\varrho^4 \partial_k v_j) \rangle dx =: \text{A} + \text{B}. \end{aligned} \quad (7.9)$$

We proceed by estimating both terms gradually.

*Step 2: Estimate for A.* First, since  $\mathbb{A}$  is induced by a projection  $\mathcal{A}: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ , we have  $\partial_k v_j \otimes_{\mathbb{A}} \nabla \varrho^4 = \mathcal{A}[\partial_k v_j \otimes \nabla \varrho^4]$ . This allows to rewrite

$$A = - \int_{\Omega} \langle \partial_k \sigma_j, \mathcal{A}[\partial_k v_j \otimes \nabla \varrho^4] \rangle dx = - \int_{\Omega} \langle \partial_k \sigma_j, \partial_k v_j \otimes \nabla \varrho^4 \rangle dx,$$

cf. Section 2.5. Since  $\partial_k v_j \otimes \nabla \varrho^4$  is in general not a full gradient, the treatment of  $\mathbb{A}$  is more subtle. Here, the representation formula (2.10) comes into play. As a consequence, we obtain

$$\begin{aligned} \partial_k v_j \otimes \nabla \varrho^4 &= (\nabla v_j) e_k \otimes \nabla \varrho^4 \\ &= \mathfrak{L}(\mathcal{A}[\nabla v_j]) e_k \otimes \nabla \varrho^4 + \mathfrak{h}(\nabla v_j) \mathfrak{G} e_k \otimes \nabla \varrho^4. \\ &= \mathfrak{L}(\mathcal{A}[\nabla v_j]) e_k \otimes \nabla \varrho^4 + \mathfrak{h}(\nabla v_j) \nabla(\mathfrak{G} e_k \varrho^4), \end{aligned}$$

with  $\mathfrak{L} \in \text{Hom}(\mathbb{R}^{2 \times 2}; \mathbb{R}^{2 \times 2})$ ,  $\mathfrak{G} \in \text{GL}(2)$  and  $\mathfrak{h} \in \text{Hom}(\mathbb{R}^{2 \times 2}; \mathbb{R})$ . This allows to split A into two parts, namely

$$\begin{aligned} |A| &\leq \left| \int_{B_{2r}(x_0)} \langle \partial_k \sigma_j, \mathfrak{L}(\mathcal{A}[\nabla v_j]) e_k \otimes \nabla \varrho^4 \rangle dx \right| + \left| \int_{B_{2r}(x_0)} \langle \partial_k \sigma_j, \mathfrak{h}(\nabla v_j) \nabla(\mathfrak{G} e_k \varrho^4) \rangle dx \right| \\ &=: \text{I} + \text{II}. \end{aligned}$$

*Ad I:* Recalling (4.36) and (4.37) we can apply the Cauchy-Schwarz inequality which leads to

$$\begin{aligned} &\sum_{k=1}^2 \int_{B_{2r}(x_0)} \langle \nabla^2 f_j(\mathbb{A} v_j) \partial_k \mathbb{A} v_j, \mathfrak{L}(\mathcal{A}[\nabla v_j]) e_k \otimes \nabla \varrho^4 \rangle dx \\ &\leq \frac{1}{5} \sum_{k=1}^2 \int_{B_{2r}(x_0)} \langle \nabla^2 f_j(\mathbb{A} v_j) \varrho^2 \partial_k \mathbb{A} v_j, \varrho^2 \partial_k \mathbb{A} v_j \rangle dx \\ &\quad + 20 \int_{B_{2r}(x_0)} \langle \nabla^2 f_j(\mathbb{A} v_j) \mathfrak{L}(\mathcal{A}[\nabla v_j]) e_k \otimes \varrho \nabla \varrho, \mathfrak{L}(\mathcal{A}[\nabla v_j]) e_k \otimes \varrho \nabla \varrho \rangle dx \end{aligned}$$

Now the first term can be absorbed in the left-hand side of (7.9), while the second can be estimated using Lemma 4.10. Namely, setting  $\mathfrak{R} = r^{-1} \varrho \mathbb{1}_{2 \times 2}$ ,  $w = r \nabla \varrho$  and  $e = \mathfrak{L}(\mathcal{A}[\nabla v_j])$ , we infer

$$\begin{aligned} &20 \sum_{k=1}^2 \int_{B_{2r}(x_0)} \langle \nabla^2 f_j(\mathbb{A} v_j) \mathfrak{L}(\mathcal{A}[\nabla v_j]) e_k \otimes \varrho \nabla \varrho, \mathfrak{L}(\mathcal{A}[\nabla v_j]) e_k \otimes \varrho \nabla \varrho \rangle dx \\ &\leq \frac{c(\Lambda, \mathbb{A})}{r^2} \int_{B_{2r}(x_0)} |\mathbb{A} v_j| dx + \frac{c(\mathbb{A})}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\mathbb{A} v_j|^2) dx. \end{aligned}$$

Both terms are under control because of the estimates (7.2) and (7.3).

*Ad II:* The second term II deserves a more thorough analysis. First, exploiting the linearity of  $\mathfrak{h} \in \text{Hom}(\mathbb{R}^{2 \times 2}; \mathbb{R})$  we can write  $\mathfrak{h}(\cdot) = \langle Q, \cdot \rangle$  for a suitable matrix  $Q \in \mathbb{R}^{2 \times 2}$ . Moreover,

since  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^2)$ , we can integrate by parts to obtain

$$\begin{aligned}
& \int_{B_{2r}(x_0)} \langle \partial_k \sigma_j, \mathfrak{h}(\nabla v_j) \nabla(\mathfrak{E} e_k \varrho^4) \rangle dx \\
&= - \int_{B_{2r}(x_0)} \langle \sigma_j, \langle Q, \partial_k \nabla v_j \rangle \nabla(\mathfrak{E} e_k \varrho^4) \rangle dx - \int_{B_{2r}(x_0)} \langle \sigma_j, \mathfrak{h}(\nabla v_j) \nabla(\mathfrak{E} e_k \partial_k \varrho^4) \rangle dx \quad (7.10) \\
&=: \text{II}_1 + \text{II}_2.
\end{aligned}$$

In order to handle the second integral  $\text{II}_2$ , we use the product rule to deduce

$$\begin{aligned}
\mathfrak{h}(\nabla v_j) \nabla(\mathfrak{E} e_k \partial_k \varrho^4) &= \nabla(\mathfrak{h}(\nabla v_j) \mathfrak{E} e_k \partial_k \varrho^4) - \mathfrak{E} e_k \partial_k \varrho^4 \otimes \nabla(\mathfrak{h}(\nabla v_j)) \\
&= \nabla(\mathfrak{h}(\nabla v_j) \mathfrak{E} e_k \partial_k \varrho^4) - \mathfrak{E} e_k \partial_k \varrho^4 \otimes \begin{pmatrix} \langle Q, \partial_1 \nabla v_j \rangle \\ \langle Q, \partial_2 \nabla v_j \rangle \end{pmatrix}.
\end{aligned}$$

Next, in view of Lemma 2.8 there exist two linear maps  $\mathfrak{J}_1, \mathfrak{J}_2: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$  such that

$$\mathfrak{h}(\nabla v_j) \nabla(\mathfrak{E} e_k \partial_k \varrho^4) = \nabla(\mathfrak{h}(\nabla v_j) \mathfrak{E} e_k \partial_k \varrho^4) - \partial_k \varrho^4 \sum_{\ell=1}^2 \partial_\ell (\mathfrak{J}_\ell(\mathbb{A} v_j)). \quad (7.11)$$

Since the first term of (7.11) contains a full gradient, we use the differentiated Euler-Lagrange inequality (7.6). More precisely, using the linearity of  $\mathfrak{h} \in \text{Hom}(\mathbb{R}^{2 \times 2}; \mathbb{R})$ , the boundedness of  $\varrho$  and setting  $Q = (Q^{(im)})_{i,m=1}^{2,2} \in \mathbb{R}^{2 \times 2}$ , we deduce after spelling out  $\mathfrak{h}(\nabla v_j) = \langle Q, \nabla v_j \rangle$  into components

$$\begin{aligned}
& \left| \int_{B_{2r}(x_0)} \langle \sigma_j, \nabla(\mathfrak{h}(\nabla v_j) \mathfrak{E} e_k \partial_k \varrho^4) \rangle dx \right| \\
& \leq \frac{1}{j} \|\mathfrak{h}(\nabla v_j) \mathfrak{E} e_k \partial_k \varrho^4\|_{W^{-1,1}(B_{2r}(x_0); \mathbb{R}^2)} \\
& \leq \frac{1}{j} \sum_{i,m=1}^2 \|\partial_m (Q^{(im)} v_j^{(i)}) \mathfrak{E} e_k \partial_k \varrho^4\|_{W^{-1,1}(B_{2r}(x_0); \mathbb{R}^2)} \\
& \leq \frac{1}{j} \sum_{i,m=1}^2 \left( \|\partial_m (Q^{(im)} v_j^{(i)}) \mathfrak{E} e_k \partial_k \varrho^4\|_{W^{-1,1}(B_{2r}(x_0); \mathbb{R}^2)} \right. \\
& \quad \left. + \|\mathfrak{E} e_k Q^{(im)} v_j^{(i)} \partial_m \varrho^4\|_{W^{-1,1}(\Omega; \mathbb{R}^2)} \right) \\
& \leq \frac{c(\mathbb{A})}{j} \left( 1 + \frac{1}{r} \right) \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx.
\end{aligned}$$

For the second term coming from (7.11), we integrate by parts resulting in

$$\begin{aligned}
& \left| \sum_{\ell=1}^2 \int_{B_{2r}(x_0)} \partial_k \varrho^4 \langle \sigma_j, \partial_\ell (\mathfrak{J}_\ell(\mathbb{A} v_j)) \rangle dx \right| \\
& \leq \left| \sum_{\ell=1}^2 \int_{B_{2r}(x_0)} \partial_k \varrho^4 \langle \partial_\ell \sigma_j, \mathfrak{J}_\ell(\mathbb{A} v_j) \rangle dx \right| + c(\mathbb{A}) \int_{B_{2r}(x_0)} |\sigma_j| |\mathbb{A} v_j| |\nabla^2 \varrho^4| dx \\
& := \text{II}_{2,a} + \text{II}_{2,b}.
\end{aligned}$$

The second term  $\Pi_{2,b}$  can be estimated directly using (4.16), namely,

$$\Pi_{2,b} \leq \frac{c(\mathbb{A})}{r^2} \int_{B_{2r}(x_0)} |\mathbb{A}v_j| \, dx + \frac{c(\mathbb{A})}{A_j j^2 r^2} \int_{B_{2r}(x_0)} |\mathbb{A}v_j|^2 \, dx.$$

The latter two terms are under control in view of the a priori bounds (7.2) and (7.3). For the first integral  $\Pi_{2,a}$ , we again apply the Cauchy-Schwarz inequality for the bilinear form associated to  $\nabla^2 f_j$ . This leads to

$$\begin{aligned} |\Pi_{2,a}| &= \sum_{\ell=1}^2 \int_{B_{2r}(x_0)} \partial_k \varrho^4 \langle \partial_\ell \sigma_j, \mathfrak{J}_\ell(\mathbb{A}v_j) \rangle \, dx \\ &= \sum_{\ell=1}^2 \int_{B_{2r}(x_0)} \langle \nabla^2 f_j(\mathbb{A}v_j) \varrho^2 \partial_\ell \mathbb{A}v_j, 4\varrho \partial_k \varrho \mathfrak{J}_\ell(\mathbb{A}v_j) \rangle \, dx \\ &\leq \frac{1}{5} \sum_{\ell=1}^2 \int_{B_{2r}(x_0)} \langle \nabla^2 f_j(\mathbb{A}v_j) \varrho^2 \partial_\ell \mathbb{A}v_j, \varrho^2 \partial_\ell \mathbb{A}v_j \rangle \, dx \\ &\quad + 20 \sum_{\ell=1}^2 \int_{B_{2r}(x_0)} \langle \nabla^2 f_j(\mathbb{A}v_j) \varrho \partial_k \varrho \mathfrak{J}_\ell(\mathbb{A}v_j), \varrho \partial_k \varrho \mathfrak{J}_\ell(\mathbb{A}v_j) \rangle \, dx \end{aligned}$$

Now the first term on the right-hand side can be absorbed in the left-hand side of (7.9). For the second we recall (4.17) leading to

$$\begin{aligned} &20 \sum_{\ell=1}^2 \int_{B_{2r}(x_0)} \langle \nabla^2 f_j(\mathbb{A}v_j) \varrho \partial_k \varrho \mathfrak{J}_\ell(\mathbb{A}v_j), \varrho \partial_k \varrho \mathfrak{J}_\ell(\mathbb{A}v_j) \rangle \, dx \\ &\leq \frac{c(\Lambda, \mathbb{A})}{r^2} \int_{B_{2r}(x_0)} \frac{|\mathbb{A}v_j|^2}{(1 + |\mathbb{A}v_j|^2)^{\frac{1}{2}}} \, dx + \frac{c(\mathbb{A})}{A_j j^2 r^2} \int_{B_{2r}(x_0)} |\mathbb{A}v_j|^2 \, dx \\ &\leq \frac{c(\Lambda, \mathbb{A})}{r^2} \int_{B_{2r}(x_0)} |\mathbb{A}v_j| \, dx + \frac{c(\mathbb{A})}{A_j j^2 r^2} \int_{B_{2r}(x_0)} (1 + |\mathbb{A}v_j|^2) \, dx. \end{aligned}$$

Both terms on the right-hand side are under control by (7.2) and (7.3). It remains to treat  $\Pi_1$  from (7.10). Towards this aim, we can again use Lemma 2.8 to find linear maps  $\mathfrak{R}_\ell, \mathfrak{Q}_\ell: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$  for  $\ell \in \{1, 2\}$  such that

$$\Pi_1 = \sum_{\ell=1}^n \int_{B_{2r}(x_0)} \langle \sigma_j, \partial_\ell(\mathfrak{R}_\ell(\mathbb{A}v_j)) \rangle \varrho^3 \partial_1 \varrho \, dx + \sum_{\ell=1}^2 \int_{B_{2r}(x_0)} \langle \sigma_j, \partial_\ell(\mathfrak{Q}_\ell(\mathbb{A}v_j)) \rangle \varrho^3 \partial_2 \varrho \, dx.$$

We observe that the appearing terms are of the same type as in (7.11) and therefore we can proceed as above after integrating by parts.

*Step 3: Estimate for B.* The second term B can be treated immediately using the differentiated Euler-Lagrange inequality (7.6). Clearly, there holds  $\varrho^4 \partial_k v_j \in W_0^{1,2}(\Omega; \mathbb{R}^2)$  resulting in

$$\begin{aligned} |\mathbf{B}| &\leq \frac{1}{j} \|\varrho^4 \partial_k v_j\|_{W^{-1,1}(\Omega; \mathbb{R}^2)} \\ &\leq \frac{1}{j} \|\partial_k(\varrho^4 v_j) - v_j \partial_k \varrho^4\|_{W^{-1,1}(\Omega; \mathbb{R}^2)} \leq \frac{c}{r^2} \left( \frac{r^2(1+r)}{j} \int_{B_{2r}(x_0)} \frac{|v_j|}{r} \, dx \right). \end{aligned} \tag{7.12}$$

*Step 4: Final estimate.* Summarising all the above estimates results, and taking into account the  $L^1$ -estimate from (7.4) with  $\frac{r}{j} \leq 1 + \frac{r^2}{j}$ , we observe

$$\begin{aligned} & \sum_{k=1}^2 \int_{B_{2r}(x_0)} \varrho^4 \langle \nabla^2 f_j(\mathbb{A}v_j) \partial_k \mathbb{A}v_j, \partial_k \mathbb{A}v_j \rangle dx \\ & \leq \frac{c}{r^2} \left[ \int_{B_{2r}(x_0)} |\mathbb{A}v_j| dx \right. \\ & \quad \left. + \frac{1}{2A_j j^2} \int_{B_{2r}(x_0)} (1 + |\mathbb{A}v_j|^2) dx + \left(1 + \frac{r^2}{j} + \frac{r^3}{j}\right) \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx \right], \end{aligned}$$

for a constant  $c = c(\Lambda, \mathbb{A}) > 0$ . This gives the desired estimate and finishes the proof.  $\square$

**Remark 7.4** (Deviatoric gradient). *The previous proof relies heavily on the structure of first-order  $\mathbb{C}$ -elliptic differential operators in two dimensions. The arguments presented in (7.9) and (7.12) are applicable to any first-order differential operator  $\mathbb{A}$ , regardless of the dimension. However, in order to estimate the remaining part  $\mathbb{A}$  we make use of the  $\mathbb{C}$ -ellipticity in two dimensions in form of (2.10). Indeed, whenever such a decomposition persists we can follow the above arguments, which is especially the case when dealing with the deviatoric or trace-free gradient in all dimensions, cf. Remark 2.9. For the latter, we recall from Example 2.2 that the projection  $\mathcal{A} = \text{dev} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  is given by*

$$\text{dev}(P) := P - \frac{\text{tr}(P)}{n} \mathbb{1}_{n \times n},$$

leading to

$$\mathfrak{L} = \text{id}_{\mathbb{R}^{n \times n}}, \quad \mathfrak{h}(P) = \frac{\langle P, \mathbb{1}_{n \times n} \rangle}{n} \quad \text{and} \quad \mathfrak{G} = \mathbb{1}_{n \times n}$$

for  $P \in \mathbb{R}^{n \times n}$ . We note that  $\text{dev} \nabla$  is a  $\mathbb{C}$ -elliptic differential operator in all dimensions whose complementary part, i.e. the orthogonal complement of  $\mathfrak{sl}(n)$ , is always one-dimensional. It is precisely this structure, that was used in the proof for  $\mathbb{C}$ -elliptic differential operators in 2d which possess a one-dimensional so called almost complementary part, cf. [GLN23, Proposition 4.1]. This situation changes drastically if one focuses on the trace-free symmetric gradient. Note, that the trace-free symmetric gradient is not  $\mathbb{C}$ -elliptic in 2d and already in 3d the complementary part of the trace-free symmetric gradient is no longer one-dimensional, cf. [BDG20, Example 2.2].

### 7.3 Gradient Integrability

In this section we prove our main Theorem 7.1 and comment on the corresponding result for the deviatoric gradient.

*Proof of Theorem 7.1.* For the sake of readability we split the proof into three parts. As usual we consider  $x_0 \in \Omega$  and  $r > 0$  such that  $B_{2r}(x_0) \Subset \Omega$ .

*Step 1: Preliminary estimates.* Let  $(v_j)_{j \in \mathbb{N}}$  be the Ekeland-type approximation sequence from above and consider the auxiliary function

$$V_\mu(\xi) := (1 + |\xi|^2)^{\frac{2-\mu}{4}} \quad \text{for} \quad \xi \in \mathcal{R}(\mathbb{A}).$$

Next, we recall that  $v_j \in W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^2)$  for every  $j \in \mathbb{N}$ , and therefore we can estimate the derivatives for  $k \in \{1, 2\}$  by

$$\begin{aligned} |\partial_k V_\mu(\mathbb{A}v_j)|^2 &\leq \left(\frac{2-\mu}{2}\right)^2 (1 + |\mathbb{A}v_j|^2)^{\frac{-2-\mu}{2}} |\mathbb{A}v_j|^2 |\partial_k \mathbb{A}v_j|^2 \\ &\leq c(\mu) \frac{|\partial_k \mathbb{A}v_j|^2}{(1 + |\mathbb{A}v_j|^2)^{\frac{\mu}{2}}}. \end{aligned} \quad (7.13)$$

Furthermore, we observe that (4.2), (4.7), and (4.21) imply  $v_j \rightarrow u$  in  $W^{-2,1}(\Omega; \mathbb{R}^2)$ . Hence, using (7.2) in conjunction with the compact embedding  $W^{\mathbb{A},1}(\Omega) \hookrightarrow L^1(\Omega; \mathbb{R}^2)$ , cf. Section 2.2.3 we can extract a non-relabeled subsequence of  $(v_j)_{j \in \mathbb{N}}$  such that  $v_j \xrightarrow{*} u$  in  $BV^{\mathbb{A}}(\Omega)$  as  $j \rightarrow \infty$ . Using (7.13) in conjunction with (7.8) for  $\mu \in (1, 2)$ , we obtain

$$\begin{aligned} \|V_\mu(\mathbb{A}v_j)\|_{W^{1,2}(B_r(x_0))}^2 &= \|V_\mu(\mathbb{A}v_j)\|_{L^2(B_r(x_0))}^2 + \|\nabla V_\mu(\mathbb{A}v_j)\|_{L^2(B_r(x_0); \mathbb{R}^2)}^2 \\ &\leq \int_{B_r(x_0)} (1 + |\mathbb{A}v_j|^2)^{\frac{1}{2}} dx + c(\mu) \int_{B_{2r}(x_0)} \frac{|\nabla \mathbb{A}v_j|^2}{(1 + |\mathbb{A}v_j|^2)^{\frac{\mu}{2}}} dx \\ &\leq c \left(1 + \frac{1}{r^2}\right) \int_{B_{2r}(x_0)} |\mathbb{A}v_j| dx \\ &\quad + \frac{c}{r^2} \left[ \frac{2}{j^2} + \left(1 + \frac{r^2}{j} + \frac{r^3}{j}\right) \int_{B_{2r}(x_0)} \frac{|v_j|}{r} dx \right] \end{aligned} \quad (7.14)$$

with a constant  $c = c(\Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbb{A}, \mu) > 0$ . Therefore, the sequence  $(V_\mu(\mathbb{A}v_j))_{j \in \mathbb{N}}$  is uniformly bounded in  $W^{1,2}(B_r(x_0))$ . Now, the Trudinger theorem in two dimensions, cf. [AF03, Theorem 8.27], provides us with the embedding  $W^{1,2}(B_r(x_0)) \hookrightarrow \exp L^2(B_r(x_0))$  and hence

$$\|V_\mu(\mathbb{A}v_j)\|_{\exp L^2(B_r(x_0))}^2 \leq c(r) \|V_\mu(\mathbb{A}v_j)\|_{W^{1,2}(B_r(x_0))}^2 \quad (7.15)$$

Next, we utilise  $(V_\mu(\xi))^2 \geq |\xi|^{2-\mu}$  to conclude the estimate

$$\|\mathbb{A}v_j\|_{\exp L^{2-\mu}(B_r(x_0); \mathcal{A}(\mathbb{A}))} \leq \|V_\mu(\mathbb{A}v_j)\|_{\exp L^2(B_r(x_0))}^{\frac{2}{2-\mu}}, \quad (7.16)$$

which can be seen as follows: Setting  $\kappa_j := \|V_\mu(\mathbb{A}v_j)\|_{\exp L^2(B_r(x_0))}^{\frac{2}{2-\mu}}$  and using the definition of the Luxemburg norm, leads to the chain of inequalities

$$1 \geq \int_{B_r(x_0)} \exp\left(\frac{(V_\mu(\mathbb{A}v_j))^2}{\|V_\mu(\mathbb{A}v_j)\|_{\exp L^2(B_r(x_0))}^2}\right) dx \geq \int_{B_r(x_0)} \exp\left(\frac{|\mathbb{A}v_j|^{2-\mu}}{\kappa_j^{\frac{2}{2-\mu}}}\right) dx,$$

implying  $\|\mathbb{A}v_j\|_{\exp L^{2-\mu}(B_r(x_0))} \leq \kappa_j$ . Therefore, estimate (7.16) is proved. Hence, recalling  $v_j \xrightarrow{*} u$  in  $BV^{\mathbb{A}}$  together with (7.15) and (7.16) allows to pass to the limit in (7.14) resulting in

$$\begin{aligned} \liminf_{j \rightarrow \infty} \|\mathbb{A}v_j\|_{\exp L^{2-\mu}(B_r(x_0); \mathcal{A}(\mathbb{A}))} &\leq c(\Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbb{A}, \mu) \left[ \left(1 + \frac{1}{r^2}\right) |\mathbb{A}u|(\bar{B}_{2r}(x_0)) + \int_{B_{2r}(x_0)} \frac{|u|}{r} dx \right]^{\frac{1}{2-\mu}} =: \mathbf{m} < \infty. \end{aligned} \quad (7.17)$$

Since  $\mathbb{A}$  is a linear differential operator and  $\mathbf{m} > 0$ , we can replace  $v_j$  by  $\tilde{v}_j := \frac{v_j}{\mathbf{m}}$  and consequently, we have  $\|\mathbb{A}\tilde{v}_j\|_{\exp L^{2-\mu}(\mathbb{B}_r(x_0); \mathcal{R}(\mathbb{A}))} \leq 1$ . In conclusion, applying [BS88, Lemma 8.8] yields

$$\begin{aligned} \int_{\mathbb{B}_r(x_0)} \exp\left(\frac{|\mathbb{A}v_j|^{2-\mu}}{\mathbf{m}^{2-\mu}}\right) dx &= \int_{\mathbb{B}_r(x_0)} \exp(|\mathbb{A}\tilde{v}_j|^{2-\mu}) dx \\ &\leq \|\mathbb{A}\tilde{v}_j\|_{\exp L^{2-\mu}(\mathbb{B}_r(x_0); \mathcal{R}(\mathbb{A}))}. \end{aligned} \quad (7.18)$$

*Step 2: Exponential integrability of  $\mathbb{A}u$ .* We now aim to use the Reshetnyak lower semi-continuity theorem to conclude that  $\mathbb{A}^s u \equiv 0$  in  $\mathbb{B}_r(x_0)$ . To this end, we introduce the function

$$\Phi: \mathcal{R}(\mathbb{A}) \rightarrow \mathbb{R}, \quad \Phi(z) = \exp\left(\frac{|z|^{2-\mu}}{\mathbf{m}^{2-\mu}}\right),$$

and observe that the corresponding recession function is given by

$$\Phi^\infty(z) := \lim_{t \rightarrow \infty} \frac{\Phi(tz)}{t} = \lim_{t \rightarrow \infty} \frac{1}{t} \exp\left(\frac{(t|z|)^{2-\mu}}{\mathbf{m}^{2-\mu}}\right) = \begin{cases} 0 & \text{if } |z| = 0 \\ +\infty & \text{if } |z| > 0. \end{cases}$$

Applying the Reshetnyak Theorem 2.14 in combination with (7.18) and (7.17), then results in

$$\begin{aligned} \int_{\mathbb{B}_r(x_0)} \Phi(\mathbb{A}u) dx + \int_{\mathbb{B}_r(x_0)} \Phi^\infty\left(\frac{d\mathbb{A}^s u}{d|\mathbb{A}^s u|}\right) d|\mathbb{A}^s u| &= \Phi(\mathbb{A}u)(\mathbb{B}_r(x_0)) \\ &\leq \liminf_{j \rightarrow \infty} \Phi(\mathbb{A}v_j)(\mathbb{B}_r(x_0)) \leq 1. \end{aligned}$$

Hence, by the definition of the recession function we must have  $\mathbb{A}^s u \equiv 0$  on  $\mathbb{B}_r(x_0)$ . Since the ball  $\mathbb{B}_r(x_0)$  was arbitrarily chosen it follows  $\mathbb{A}^s u \equiv 0$  in  $\Omega$  and therefore  $u \in W^{\mathbb{A},1}(\Omega)$ . Furthermore, there holds

$$\int_{\mathbb{B}_r(x_0)} \Phi(\mathbb{A}u) dx = \int_{\mathbb{B}_r(x_0)} \exp\left(\frac{|\mathbb{A}u|^{2-\mu}}{\mathbf{m}^{2-\mu}}\right) dx \leq 1.$$

Recalling once more the definition of the Luxemburg norm, we conclude the exponential integrability of  $\mathbb{A}u$ , that is  $\mathbb{A}u \in \exp L^{2-\mu}(\mathbb{B}_r(x_0))$ , together with the estimate

$$\begin{aligned} \|\mathbb{A}u\|_{\exp L^{2-\mu}(\mathbb{B}_r(x_0); \mathcal{R}(\mathbb{A}))} &\leq \mathbf{m} \\ &= c \left[ \left(1 + \frac{1}{r^2}\right) |\mathbb{A}u|(\overline{\mathbb{B}_{2r}(x_0)}) + \int_{\mathbb{B}_{2r}(x_0)} \frac{|u|}{r} dx \right]^{\frac{1}{2-\mu}} \end{aligned}$$

for a constant  $c = c(\Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbb{A}, \mu) > 0$ .

*Step 3: Exponential integrability of  $\nabla u$ .* In order to derive the precise inequality stated as in (7.1) we use that  $\mathcal{K} := \ker(\mathbb{A})$  is finite dimensional as  $\mathbb{A}$  is a  $\mathbb{C}$ -elliptic differential operator. Therefore, all norms are equivalent on  $\mathcal{K}$ . In particular, there holds

$$\|\mathbf{p}\|_{\exp L^{\frac{2-\mu}{3-\mu}}(\mathbb{B}_r(x_0); \mathbb{R}^2)} + r \|\nabla \mathbf{p}\|_{\exp L^{\frac{2-\mu}{3-\mu}}(\mathbb{B}_r(x_0); \mathbb{R}^{2 \times 2})} \leq c(\mu) \int_{\mathbb{B}_r(x_0)} |\mathbf{p}| dx$$

for all  $\mathbf{p} \in \mathcal{K}$ . In view of (2.11), there exists a bounded linear projection operator  $\mathbb{P}_B: L^1(B_r(x_0); \mathbb{R}^2) \rightarrow \mathcal{K}$  with  $v \mapsto \mathbb{P}_B v$  such that

$$\int_{B_r(x_0)} |\mathbb{P}_B v| \, dx \leq c \int_{B_r(x_0)} |v| \, dx$$

for all  $v \in L^1(B_r(x_0); \mathbb{R}^2)$ . Towards the exact statement of (7.1), we apply the Korn-type inequality (3.19) for  $\mathbb{C}$ -elliptic operators from Corollary 3.12 with  $\beta = 2 - \mu > 0$ . Finally, noting  $\frac{\beta}{\beta+1} = \frac{2-\mu}{3-\mu}$  leads to

$$\begin{aligned} & \|\nabla u\|_{\exp L^{\frac{2-\mu}{3-\mu}}(B_r(x_0); \mathbb{R}^{2 \times 2})} \\ & \leq \|\nabla(u - \mathbb{P}_B u)\|_{\exp L^{\frac{2-\mu}{3-\mu}}(B_r(x_0); \mathbb{R}^{2 \times 2})} + \|\nabla \mathbb{P}_B u\|_{\exp L^{\frac{2-\mu}{3-\mu}}(B_r(x_0); \mathbb{R}^{2 \times 2})} \\ & \leq c \left( \left(1 + \frac{1}{r}\right) \|\mathbb{A}u\|_{\exp L^{2-\mu}(B_{2r}(x_0); \mathcal{A}(\mathbb{A}))} + \int_{B_r(x_0)} \frac{|u|}{r} \, dx \right) \\ & \leq c \left( \left(1 + \frac{1}{r}\right) \left[ \left(1 + \frac{1}{r^2}\right) |\mathbb{A}u|(\overline{B}_{2r}(x_0)) + \int_{B_{2r}(x_0)} \frac{|u|}{r} \, dx \right]^{\frac{1}{2-\mu}} + \int_{B_r(x_0)} \frac{|u|}{r} \, dx \right) \end{aligned}$$

with  $c = c(\Omega, \gamma, \Gamma, \lambda, \Lambda, \mathbb{A}, \mu) > 0$ . Clearly, exponential integrability for the gradient implies  $\nabla u \in L^q(\Omega; \mathbb{R}^{2 \times 2})$  for all  $q \in (1, \infty)$ . This finishes the proof.  $\square$

**Remark 7.5** (Deviatoric gradient - Gradient integrability). *As described in Remark 7.4 the deviatoric gradient always admits a one-dimensional complementary part, which allows to prove uniform weighted second-order estimates for the Ekeland sequence in all dimensions. While the two-dimensional case is included in Theorem 7.1 we can adapt the proof of [Gme20, Theorem 1.1] for  $n \geq 3$  to obtain a similar result for  $\mathbb{A} = \text{dev } \nabla$ . Specifically, one can show whenever  $f \in C^2(\mathfrak{sl}(n))$  satisfies (1.3) and (1.5) for  $\mu \in (1, 1 + \frac{2}{n})$ , every generalised minimiser is of class  $W^{\text{dev } \nabla, 1}(\Omega; \mathbb{R}^n) \cap W_{\text{loc}}^{1,q}(\Omega; \mathbb{R}^n)$  for some  $q \geq 1$ .*

## 8 | Conclusion and Outlook

### 8.1 Conclusion

In this thesis, we presented a regularity theory for relaxed minimisers of  $\mu$ -elliptic variational integrals with linear growth integrands depending on  $\mathbb{C}$ -elliptic operators  $\mathbb{A}$ , subject to an additional  $L^p$ -constraint. The class of  $\mathbb{C}$ -elliptic operators includes the full, symmetric, and deviatoric gradient. The natural compactness space for such problems is given by  $BV^{\mathbb{A}}$ , and existence can be demonstrated using standard techniques based on the direct method in the Calculus of Variations. However, this comes at the cost of potential non-uniqueness, as the relaxed functional is generally no longer strictly convex. To overcome this issue, we followed the strategy of BECK and SCHMIDT [BS13] based on a vanishing viscosity approach using the Ekeland variational principle. In order to treat the constraints we were putting this construction into an abstract framework in Chapter 4 allowing for general constraints.

In Chapter 5 we completed the picture for the symmetric gradient leading to a parallel Sobolev regularity theory to that known from the full gradient or BV-case. Especially, we have shown in Theorem 5.1 that every bounded BD-minimiser of a 3-elliptic linear growth variational problem is locally  $W^{1,1}$ -regular. While the boundedness can be justified directly for the full gradient it has to be assumed via an  $L^\infty$ -constraint in case of the symmetric gradient. In the strict ellipticity regime for  $\mu < 3$  we improved the result, achieving local  $W^{1,4-\mu}$ -regularity, cf. Theorem 5.11.

Since verifying the  $L^\infty$ -constraint for the symmetric gradient can be difficult in applications, we assumed an intermediate  $L^p$ -constraint for  $p > 2$ . This was the content of Chapter 6. Since many arguments were similar for the full and the symmetric gradient, we were treating both cases together. More precisely, we have shown in Theorem 6.1 that every  $L^p$ -bounded  $BV^{\mathbb{A}}$ -minimiser of a  $\mu$ -elliptic linear growth integral with  $\mu \in (1, 3 - \frac{2}{p})$  is locally  $W^{1,1}$ -regular interpolating the results of GMEINER [Gme20] and Chapter 5. Again, these results can be improved in the strict ellipticity regime. Unfortunately, due to technical issues, we were only able to prove an incomplete result.

Finally, since the explicit structure of the symmetric gradient was essential for the preceding results, we restricted our attention in Chapter 7 to two-dimensional  $\mathbb{C}$ -elliptic operators  $\mathbb{A}$  induced by orthogonal projections  $\mathcal{A}: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ . This class of operators can be structurally characterised by the presence of a one-dimensional almost complementary part. In this way we established in Theorem 7.1 that every  $BV^{\mathbb{A}}$ -minimiser of a  $\mu$ -elliptic variational problem with linear growth integrand enjoys  $W^{1,q}$ -regularity for all  $q \geq 1$  and  $\mu \in (1, 2)$ .

## 8.2 Outlook

As already mentioned above, we were able to complete the picture to obtain similar regularity results for the symmetric gradient as known for the full gradient. However, we had to restrict ourselves to a very specific situation when dealing with general  $\mathbb{C}$ -elliptic operators. At the present stage, it seems extremely challenging to apply the presented methods in the general framework, since the explicit structure of the operator has become a significant factor when dealing with second-order estimates. Nevertheless, from a research point of view it would be interesting to discuss the following questions in the future:

- \* From a technical perspective it remains to close the gap in the ellipticity range as outlined in Remark 6.11. It is not clear to the author whether this can be approached with a subquadratic regularisation or if a different technique is required.
- \* Even for the full gradient it is unclear whether BV-minimisers of autonomous  $\mu$ -elliptic variational integrands of linear growth are Sobolev regular if  $\mu > 3$ . It is commonly believed that  $\mu = 3$  displays a borderline case. It would be of potential interest to answer this regularity question in a positive or negative way.
- \* Because of the Sobolev-type embedding  $BV^{\mathbb{A}}(\Omega) \hookrightarrow L^{1^*}(\Omega)$  with  $1^* = \frac{n}{n-1}$  it would be interesting to handle also (sub)quadratic  $L^p$ -constraints for  $p \in (1^*, 2]$ . This is of particular interest, given the observation that  $BV^{\mathbb{A}}$ -minimisers invariably fall within a higher local  $L^p$ -space than the conventional embedding space  $L^{1^*}(\Omega)$ . This can be obtained by a lower-order Gehring-type self-improvement. More precisely, we fix a  $BV^{\mathbb{A}}$ -minimiser and recall from [Woz23, Theorem 3.3] that there exists a minimising sequence  $(u_j)_{j \in \mathbb{N}}$  converging to  $u$  in  $L^1(\Omega)$ . Now, one can adapt the argument of [BS13, Section 4] to apply the Ekeland variational principle for the Dirichlet class  $\mathcal{D}_{u_0} = u_0 + W^{\mathbb{A},1}(\Omega)$ . This allows to construct a new minimising sequence  $(v_j)_{j \in \mathbb{N}}$  satisfying an Euler-Lagrange inequality. Testing the latter with  $\varrho^2 v_j$ , where  $\varrho$  is a localisation function  $\varrho \in C_c^\infty(\Omega; [0, 1])$ , one can pass to the limit results in a uniform reverse Hölder inequality for  $u$ . This, in combination with the standard Gehring Lemma, cf. [GM79, Lemma 5.1], establishes the desired higher integrability for the minimiser  $u$ .
- \* Finally, it is evident that unresolved issues remain concerning general  $\mathbb{C}$ -elliptic operators in all dimensions. Analogous to the approach taken for the deviatoric gradient, the first step would be to extend the result of WOZNIAK [Woz23] to the optimal range  $\mu \in (1, 1 + \frac{2}{n})$ .

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