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Fritz Colonius, Christoph Kawan

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Institut für Mathematik, Universitätsstraße, D-86135 Augsburg http://www.math.uni-augsburg.de/

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Christoph Kawan
Institut für Mathematik
Universität Augsburg
86135 Augsburg

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Invariance Entropy for Outputs

Fritz Colonius · Christoph Kawan

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Abstract For continuous-time control systems with outputs, this paper analyzes invariance entropy as a measure for the information rate necessary to achieve invariance of compact subsets of the output space. For linear control systems with compact control range, relations to controllability and observability properties are studied. Furthermore, the notion of asymptotic invariance entropy is introduced and characterized for these systems.

Keywords nonlinear control · invariance entropy · outputs

Mathematics Subject Classification (2000) 94A17 · 37B40 · 93C15

1 Introduction

The purpose of this paper is to study the information needed to achieve certain invariance properties of control systems with output. In [10], Nair, Evans, Mareels and Moran introduced topological feedback entropy for a related problem in state space. Here we follow our approach in [3] for invariance properties of subsets in the state space and generalize it to systems with outputs. We count how fast—for time T tending to infinity—the number of open loop control functions grows which are needed in order to achieve desired properties of the system on $[0, T]$.

Our strategy here is to derive properties of the invariance entropy in the output space from properties of associated sets in the state space, for which a number of

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Fritz Colonius
Institut für Mathematik, Universität Augsburg, 86135 Augsburg/Germany
Tel.: +49 821 598-2246
Fax: +49 821 598-2193
E-mail: fritz.colonius@math.uni-augsburg.de

Christoph Kawan
Institut für Mathematik, Universität Augsburg, 86135 Augsburg/Germany
Tel.: +49 821 598-2244
Fax: +49 821 598-2193
E-mail: christoph.kawan@math.uni-augsburg.de

results is available. Here observability and also controllability properties will play a major role. In particular, for linear control systems the invariance entropy of subsets Q of the state space with positive Lebesgue measure has been computed in [3] as the sum over all real parts of unstable eigenvalues. In the case with outputs, one will expect that unobservable modes have to be omitted, since they should not influence the invariance entropy for the output. Theorem 3 makes this conjecture precise using a special relation between the admissible initial values and the prescribed set in the output space. Furthermore, we introduce the notion of asymptotic invariance entropy based on a definition which requires invariance only for time large enough. Here for a set K of initial values, we want to count the controls such that the corresponding outputs asymptotically approach a compact subset Q in the output space (cf. Definition 6 and Theorem 5 for precise formulations).

The analysis in this paper is essentially restricted to linear systems. The doctoral thesis Kawan [7] presents a study of invariance entropy (in the state space) for general control systems on differentiable manifolds. There also relations to data rates of symbolic controllers are established.

In Section 2 we collect some properties of observed linear control systems. Section 3 introduces invariance entropy for compact subsets of the output space of nonlinear systems. In Section 4 the linear case is discussed in relation to observability and controllability properties. Finally, Section 5 presents results for asymptotic invariance entropy, again for linear control systems.

Notation. For a set $A \subset \mathbb{R}^d$, the closure and the interior of a set A are denoted by cl$A$ and int$A$, respectively. The ε-neighborhood of A is

$$N_\varepsilon(A) := \{x \in \mathbb{R}^d, \mathrm{dist}(x, A) := \inf_{a \in A} \|x - a\| < \varepsilon\}.$$

2 Preliminaries on linear control systems

In this section, we recall some properties of linear control systems with output. In particular, we discuss their behavior under control constraints.

We consider control systems in $\mathbb{R}^d$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t),\ u \in \mathcal{U}, \tag{1}$$

with matrices $A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times m}$, and $C \in \mathbb{R}^{k\times d}$ and control range $U \subset \mathbb{R}^m$; the set $\mathcal{U}$ of admissible control functions is defined by

$$\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U \text{ for almost all } t \in \mathbb{R}\}.$$

The solution of the differential equation (1) with initial condition $x(t_0) = x_0 \in \mathbb{R}^d$ and control $u \in \mathcal{U}$ is denoted by $\varphi(t, t_0, x_0, u)$. Here one obtains the relation $\varphi(t, t_0, x_0, u) = \varphi(t - t_0, 0, x_0, u(t_0 + \cdot))$; if the initial time $t_0 = 0$, we omit it in our notation. Using the variation-of-constants formula, the outputs are given by

$$y(t) = C\varphi(t, t_0, x, u) = Ce^{A(t-t_0)}x + \int_{t_0}^{t} Ce^{A(t-s)}Bu(s)\mathrm{d}s. \tag{2}$$

The reachable set up to time $T > 0$ from an initial point $x_0 \in \mathbb{R}^d$ at time $t_0 = 0$ is

$$\mathcal{R}_{\leq T}(x_0, U) := \{\varphi(t, x_0, u) \in \mathbb{R}^d,\ 0 \leq t \leq T \text{ and } u \in \mathcal{U}\}.$$

Analogously, the reachable set $\mathcal{R}_T(x_0, U)$ at time $T > 0$ is defined. The reachability subspace (with unconstrained controls) is

$$\mathcal{R} := \mathrm{Im}[B, AB, \ldots, A^{d-1}B].$$

Using Cayley-Hamilton's Theorem, one sees that this subspace is the smallest A-invariant subspace containing $\mathrm{Im}B$ and that it coincides with the reachable set from the origin (at any time $t > 0$) with unconstrained controls,

$$\mathcal{R} = \{\varphi(t, 0, u) \in \mathbb{R}^d,\ u : [0, t] \to \mathbb{R}^m \text{ continuous}\}.$$

We call the eigenvalues of $A|_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$ the controllable eigenvalues.

The unobservable subspace $\mathcal{N}$ is

$$\mathcal{N} = \bigcap_{i=0}^{d-1} \ker CA^i.$$

Again by Cayley-Hamilton's theorem, $\mathcal{N}$ is the largest A-invariant subspace contained in $\ker C$.

Consider the induced linear control system on $\mathbb{R}^d/\mathcal{N}$ (identified with $\mathbb{R}^{\bar{d}}$) given by $(\bar{A}, \bar{B}, \bar{C}) \in \mathbb{R}^{\bar{d} \times \bar{d}} \times \mathbb{R}^{\bar{d} \times m} \times \mathbb{R}^{k \times \bar{d}}$ and denote its trajectories by $\bar{\varphi}(t, t_0, \bar{x}, u)$. The natural projection $\mathbb{R}^d \to \mathbb{R}^d/\mathcal{N} = \mathbb{R}^{\bar{d}}$ is denoted by π. For an eigenvalue $\lambda \in \mathbb{C}$ of A with real generalized eigenspace $E(\lambda)$, let $m_{\mathcal{N}}(\lambda)$ denote the dimension of $\pi(E(\lambda))$. If $m_{\mathcal{N}}(\lambda) > 0$, we call λ an observable eigenvalue and $m_{\mathcal{N}}(\lambda)$ its observable multiplicity. System $(\bar{A}, \bar{B}, \bar{C})$ is observable, i.e., its unobservable subspace is trivial, and

$$\mathrm{spec}(A) = \mathrm{spec}(\bar{A}) \cup \mathrm{spec}(A|_{\mathcal{N}}).$$

Since $\mathcal{R}$ is A-invariant, the subspace $\pi\mathcal{R} \subset \mathbb{R}^d/\mathcal{N}$ is $\bar{A}$-invariant and contains $\mathrm{Im}\bar{B} = \pi\mathrm{Im}B$. It is the smallest subspace with these properties and hence it is the controllable subspace of $(\bar{A}, \bar{B}, \bar{C})$. The system obtained by restricting the observable system to its reachable subspace $\bar{\mathcal{R}}$ is controllable and observable. The eigenvalues of the corresponding map $\bar{A}|_{\bar{\mathcal{R}}}$ are called the observable and controllable eigenvalues of A.

For observable (A, C) the observability Gramians (see, e.g., Antsaklis and Michel [1], Corollary 3.2 and Corollary 3.8 in Chapter 3), defined by

$$W(t_0, t_1) := \int_{t_0}^{t_1} e^{A^*(s-t_0)} C^* C e^{A(s-t_0)} \mathrm{d}s,\ t_1 > t_0, \tag{3}$$

are invertible. For the output $y(t) = C\varphi(t, t_0, x_0, u)$ and

$$\hat{y}(t) := y(t) - \int_{t_0}^{t} Ce^{A(t-s)} Bu(s)\mathrm{d}s$$

one has

$$x_0 = W(t_0, t_1)^{-1} \int_{t_0}^{t_1} e^{A^*(s-t_0)} C^* \hat{y}(s)\mathrm{d}s. \tag{4}$$

This shows that the initial point is uniquely determined by the control function and the output function. In particular, for $u = 0$ one has

$$x_0 = W(t_0, t_1)^{-1} \int_{t_0}^{t_1} e^{A^*(s-t_0)} C^* y(s)\mathrm{d}s. \tag{5}$$

From (3) we obtain

$$W(t_0, t_1) = \int_0^{t_1 - t_0} e^{A^* s} C^* C e^{As} \mathrm{d}s = W(0, t_1 - t_0).$$

Applying (5) on an interval $[T-1, T]$ with $T > 1$, one finds for $y(s) = C\varphi(s, x_0, 0)$, $s \geq 0$,

$$\begin{aligned}
\varphi(T-1, x_0, 0) &= W(T-1, T)^{-1} \int_{T-1}^{T} e^{A^*(s-(T-1))} C^* y(s) \mathrm{d}s \\
&= W(0,1)^{-1} \int_0^1 e^{A^* s} C^* y(s + T - 1) \mathrm{d}s.
\end{aligned}$$

This implies for $x_i \in \mathbb{R}^d$ and outputs $y_i(t) = C\varphi(t, x_i, u), i = 1, 2,$

$$\begin{aligned}
&\|\varphi(T-1, x_1, u) - \varphi(T-1, x_2, u)\| \\
&= \|\varphi(T-1, x_1, 0) - \varphi(T-1, x_2, 0)\| \\
&\leq \left\| W(0,1)^{-1} \right\| \left\| \int_0^1 \left[e^{A^* s} C^* y_1(s + T - 1) - e^{A^* s} C^* y_2(s + T - 1) \right] \mathrm{d}s \right\| \\
&\leq c_1 \max_{t \in [T-1, T]} \|y_1(t) - y_2(t)\| \qquad (6)
\end{aligned}$$

with the constant

$$c_1 = \left\| W(0,1)^{-1} \right\| \max_{s \in [0,1]} \left\| e^{A^* s} C^* \right\| > 0, \qquad (7)$$

which is independent of $T > 1$. These estimates will yield the following result.

Lemma 1 *Suppose that (A, C) is observable. Then for initial values $x_1, x_2 \in \mathbb{R}^d$ and a control $u : [0, \infty) \to \mathbb{R}^m$ with outputs $y_i(t) = C\varphi(t, x_i, u), i = 1, 2,$ the following estimate holds on every interval $[T-1, T]$, $T > 1$:*

$$\|\varphi(t, x_1, u) - \varphi(t, x_2, u)\| \leq c_0 \max_{t \in [T-1, T]} \|y_1(t) - y_2(t)\|,$$

with a constant c_0 which is independent of T.

Proof By inequality (6) we know that for all $T > 1$

$$\|\varphi(T-1, x_1, u) - \varphi(T-1, x_2, u)\| \leq c_1 \max_{t \in [T-1, T]} \|y_1(t) - y_2(t)\|.$$

Using the variation-of-constants formula, this yields for all $t \in [T-1, T]$ the desired estimate

$$\begin{aligned}
&\|\varphi(t, x_1, u) - \varphi(t, x_2, u)\| = \left\| e^{A(t-T+1)} [\varphi(T-1, x_1, u) - \varphi(T-1, x_2, u)] \right\| \\
&\leq c_0 \max_{t \in [T-1, T]} \|y_1(t) - y_2(t)\|
\end{aligned}$$

with the constant $c_0 := c_1 \max_{t \in [0,1]} \left\| e^{At} \right\|$.

3 Invariance entropy for outputs

In this section, we define controlled invariant sets in the output space and a related notion of invariance entropy. Some properties are derived.

Consider a nonlinear control system with output

$$\dot{x} = f(x, u(t)), \; y = g(x), \; u \in \mathcal{U}. \tag{8}$$

For simplicity, we assume that everything is defined in Euclidean spaces, i.e., $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^k$ are smooth (i.e., C^∞), and

$$\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m), \; u(t) \in U \text{ for almost all } t \in \mathbb{R}\}$$

with control range $U \subset \mathbb{R}^m$. The solution of the differential equation with initial condition $x(t_0) = x_0 \in \mathbb{R}^d$ and control $u \in \mathcal{U}$ is denoted by $\varphi(t, t_0, x_0, u)$. We assume that unique global solutions exist. If $t_0 = 0$, we omit this argument.

Now we analyze the invariance entropy for compact sets Q in the output space $\mathbb{R}^k$ which satisfy the following condition.

Definition 1 A nonvoid subset $Q \subset \mathbb{R}^k$ in the output space is called controlled invariant for system (8) if for all $y \in Q$ there are an initial state $x \in \mathbb{R}^d$ and a control $u \in \mathcal{U}$ with $g(x) = y$ and $g(\varphi(t, x, u)) \in Q$ for all $t \geq 0$. Then we denote

$$P(Q) := \{x \in \mathbb{R}^d, \text{ there is } u \in \mathcal{U} \text{ with } g(\varphi(t, x, u)) \in Q \text{ for all } t \geq 0\}.$$

Observe that $P(Q) \subset g^{-1}(Q)$ and that Q is controlled invariant iff $g(P(Q)) = Q$. We want to describe how the number of (open-loop) control functions which are necessary to keep the system in Q grows with time. This leads us to the following preliminary definition of an invariance entropy.

Definition 2 Let Q be a controlled invariant subset of the output space $\mathbb{R}^k$. For given $T > 0$ we call a subset $\mathcal{S}^* \subset \mathcal{U}$ a (T, Q)-*spanning* set if for all $x \in P(Q)$ there is $u \in \mathcal{S}^*$ with

$$g(\varphi(t, x, u)) \in Q \text{ for all } t \in [0, T].$$

By $r^*_{\text{inv}}(T, Q)$ we denote the minimal cardinality of a (T, Q)-spanning set. If no finite (T, Q)-spanning set exists, we set $r^*_{\text{inv}}(T, Q) := \infty$. The strict *invariance entropy* $h^*_{\text{inv}}(Q)$ is defined by

$$h^*_{\text{inv}}(Q) := \limsup_{T \to \infty} \frac{1}{T} \ln r^*_{\text{inv}}(T, Q).$$

In general, we cannot guarantee that the strict invariance entropy, or merely the numbers $r^*_{\text{inv}}(T, Q)$, are finite (compare also the discussion in [3]). Hence, we do not pursue this notion any further. Instead, we relax the condition on spanning sets of controls in the following way (additionally, a set K of admissible initial states is specified).

Definition 3 Let Q be a controlled invariant subset of the output space $\mathbb{R}^k$ and let $K \subset P(Q)$. For given $T, \varepsilon > 0$ we call a subset $\mathcal{S} \subset \mathcal{U}$ a (T, ε, K, Q)-*spanning* set if for all $x \in K$ there is $u \in \mathcal{S}$ with

$$\varphi(t, x, u) \in N_\varepsilon(P(Q)) \text{ for all } t \in [0, T].$$

By $r_{\text{inv}}(T, \varepsilon, K, Q)$ we denote the minimal cardinality of a (T, ε, K, Q)-spanning set. If no finite (T, ε, K, Q)-spanning set exists, we set $r_{\text{inv}}(T, \varepsilon, K, Q) := \infty$.

In other words: we require for a (T,ε,K,Q)-spanning set $\mathcal{S}$ that for every initial value in K there is a control in $\mathcal{S}$ such that up to time T the trajectory remains in the ε-neighborhood of $P(Q)$. Recall that by controlled invariance $g(P(Q))=Q$. Now we consider what happens for $T\to\infty$ and $\varepsilon\to 0$ and define invariance entropy for outputs.

Definition 4 Let Q be a compact controlled invariant set in the output space $\mathbb{R}^k$ and let $K\subset P(Q)$. Then the *invariance entropy* $h_{\mathrm{inv}}(K,Q)$ is defined by

$$h_{\mathrm{inv}}(\varepsilon,K,Q) := \limsup_{T\to\infty}\frac{1}{T}\ln r_{\mathrm{inv}}(T,\varepsilon,K,Q),\ \ h_{\mathrm{inv}}(K,Q) := \lim_{\varepsilon\searrow 0} h_{\mathrm{inv}}(\varepsilon,K,Q).$$

Note that $h_{\mathrm{inv}}(\varepsilon_1,K,Q)\le h_{\mathrm{inv}}(\varepsilon_2,K,Q)$ for $\varepsilon_2\le\varepsilon_1$. Hence, the limit for $\varepsilon\to 0$ exists (it may be infinite).

Remark 1 For systems with output $g=id_{\mathbb{R}^d}$, the notions of controlled invariance and (T,ε,K,Q)-spanning sets coincide with the corresponding notions in the state space introduced in [3]. We take this as a justification to use the same notation.

Next we establish a number of consequences of the definitions.

Proposition 1 *Let $\mathcal{S}\subset\mathcal{U}$ be a $(T,\varepsilon,P(Q),Q)$-spanning set for a compact, controlled invariant set $Q\subset\mathbb{R}^k$. Then for every $y\in Q$ there is an initial state $x\in P(Q)$ with $g(x)=y$ and for all such x there is a control $v\in\mathcal{S}$ with*

$$\varphi(t,x,v)\in N_\varepsilon(P(Q)) \text{ for all } t\in[0,T].$$

Proof By controlled invariance, there is for $y\in Q$ a point $x\in P(Q)$ with $g(x)=y$. Then the assertion follows, since $\mathcal{S}$ is $(T,\varepsilon,P(Q),Q)$-spanning.

The next proposition specifies assumptions guaranteeing that the invariance entropy for outputs can be related to invariance entropy in the state space. Later, we will use Proposition 2 in order to compute the invariance entropy for linear control systems.

Proposition 2 *Consider a controlled invariant subset Q of the output space. Then $P(Q)$ is controlled invariant in the state space. If Q is compact and K is a compact subset of $P(Q)$, the invariance entropies of (K,Q) (for outputs) and of $(K,P(Q))$ (for states) satisfy*

$$h_{\mathrm{inv}}(K,P(Q)) = h_{\mathrm{inv}}(K,Q).$$

Suppose, additionally, that g is uniformly continuous on a neighborhood $N_\alpha(P(Q))$, $\alpha>0$. Then for every $\varepsilon>0$ there is $\delta>0$ such that every $(T,\delta,K,P(Q))$-spanning set $\mathcal{S}$ of controls has the following property: for every $x\in K$ there is $u\in\mathcal{S}$ with

$$g(\varphi(t,x,u))\in N_\varepsilon(Q) \text{ for all } t\in[0,T]. \tag{9}$$

Proof Controlled invariance of $P(Q)$ follows, since for $x\in P(Q)$ there is $u\in\mathcal{U}$ with $g(\varphi(t,x,u))\in Q$ for all $t\ge 0$. Hence, for all $T>0$ and $t\ge 0$ one has

$$g(\varphi(t,\varphi(T,x,u),u(T+\cdot))) = g(\varphi(t+T,x,u))\in Q,$$

which implies that $\varphi(T,x,u)\in P(Q)$ for all $T\ge 0$. The equality for the invariance entropies is immediate from the definitions. Finally, let $\varepsilon>0$. By uniform continuity of g on a neighborhood $N_\alpha(P(Q)),\alpha>0$, there is $0<\delta<\alpha$ such that here $\|x_1-x_2\|<\delta$ implies $\|g(x_1)-g(x_2)\|<\varepsilon$. Thus, a $(T,\delta,K,P(Q))$-spanning set $\mathcal{S}$ of controls satisfies (9).

Invariance entropy is only interesting, if we can guarantee that for $T, \varepsilon > 0$ there are finite (T, ε)-spanning sets. This holds under an additional assumption.

Lemma 2 *Let $Q \subset \mathbb{R}^k$ be a compact set. Assume that for every bounded sequence (x_n) in $\mathbb{R}^d$ and every sequence of controls u_n in $\mathcal{U}$, there are $x_0 \in \mathbb{R}^d$ and $u_0 \in \mathcal{U}$ such that a subsequence of the trajectories $\varphi(t, x_n, u_n)$ converges uniformly on every compact interval $I \subset \mathbb{R}$ to $\varphi(t, x_0, u_0)$. (i) Suppose that for some $T > 0$ the set*

$$P(Q,T) := \{x \in \mathbb{R}^d, \text{ there is } u \in \mathcal{U} \text{ with } g(\varphi(t,x,u)) \in Q \text{ for all } t \in [0,T]\}$$

is nonvoid and bounded. Then $P(Q,T)$ is compact. (ii) Suppose, in addition to the assumption in (i), that Q is controlled invariant. Then $P(Q)$ is a nonvoid, compact, and controlled invariant subset of the state space and for all $\varepsilon > 0$ there is a finite $(T, \varepsilon, P(Q), Q)$-spanning set.

Proof (i) By assumption the set $P(Q,T)$ is bounded. The set $P(Q,T)$ is closed, hence compact, by the compactness assumption for the trajectories. (ii) Clearly, the set $P(Q)$ is nonvoid, compact, controlled invariant and contained in $P(Q,T)$. Let $\varepsilon > 0$. By Proposition 1, for all $y \in Q$ there is an initial state $x \in P(Q)$ with $g(x) = y$ and for all such x there is a control $u \in \mathcal{U}$ with $\varphi(t,x,u) \in P(Q)$ and $g(\varphi(t,x,u)) \in Q$ for all $t \geq 0$. By compactness of $P(Q)$ and continuous dependence on initial values it follows that there is a finite $(T, \varepsilon, P(Q), Q)$-spanning set $\mathcal{S} \subset \mathcal{U}$.

Remark 2 The compactness property in Lemma 2 is, in particular, satisfied for control-affine systems with compact and convex control range (see, e.g., Colonius and Kliemann [4]).

Next we discuss the behavior of invariance entropy under semi-conjugacy.

Theorem 1 *Consider for $i = 1, 2$ two control systems of the form (8),*

$$\dot{x}_i = f_i(x_i, u_i(t)),\ y_i = g_i(x_i),\ u_i \in \mathcal{U}_i,$$

in $\mathbb{R}^{d_i}$ with control ranges $U_i \subset \mathbb{R}^{m_i}$ and outputs $g_i : \mathbb{R}^{d_i} \to \mathbb{R}^{k_i}$. Let $\pi^s : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ and $\pi^{out} : \mathbb{R}^{k_1} \to \mathbb{R}^{k_2}$ be continuous maps and let $\pi^{in} : \mathcal{U}_1 \to \mathcal{U}_2$ be any map. Denote the corresponding trajectories by $\varphi_i(t,x,u)$ and assume that the following semi-conjugacy property holds for all $(t,x,u) \in [0,\infty) \times \mathbb{R}^{d_1} \times \mathcal{U}_1$:

$$\pi^s(\varphi_1(t,x,u)) = \varphi_2(t, \pi^s(x), \pi^{in}(u)),\ \pi^{out} \circ g_1 = g_2 \circ \pi^s. \tag{10}$$

Let $Q \subset \mathbb{R}^{k_1}$ be a compact controlled invariant set such that the restriction of g_1 to a neighborhood $N_\alpha(P(Q)) \subset \mathbb{R}^{d_1}, \alpha > 0$, is uniformly continuous. Then the following assertions hold: (i) The set $\pi^{out}(Q) \subset \mathbb{R}^{k_2}$ is compact and controlled invariant and for a compact subset $K \subset P(Q)$ the image $\pi^s(K)$ is compact and contained in $P(\pi^{out}(Q))$ with

$$h_{\text{inv}}(K,Q) \geq h_{\text{inv}}(\pi^s(K), \pi^{out}(Q)). \tag{11}$$

(ii) Equality holds in (11) if, additionally, the map $\pi^{in} : \mathcal{U}_1 \to \mathcal{U}_2$ is surjective, and the maps π^s and π^{out} are homeomorphisms with $(\pi^s)^{-1}$ uniformly continuous on a neighborhood $N_\alpha(\pi^s(P(Q))), \alpha > 0$. (iii) Semiconjugacy (10) holds, in particular, with $\pi^{in} = id$, if $\pi^s : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is a continuously differentiable function with

$$D\pi^s(x) f_1(x,u) = f_2(\pi^s(x), u) \text{ for all } (x,u) \in \mathbb{R}^{d_1} \times U_1 \text{ and } \pi^{out} \circ g_1 = g_2 \circ \pi^s. \tag{12}$$

Proof (i) Continuity of π^s and π^{out} imply that $\pi^s(K) \subset \mathbb{R}^{d_2}$ and $\pi^{out}(Q) \subset \mathbb{R}^{k_2}$ are compact sets. Semi-conjugacy property (10) implies that $\pi^{out}(Q)$ is controlled invariant. In fact: Let $y_2 \in \pi^{out}(Q)$. Then there is $y_1 \in Q$ with $\pi^{out}(y_1) = y_2$. Let $x_1 \in \mathbb{R}^{d_1}$ and $u_1 \in \mathcal{U}_1$ be such that $g_1(x_1) = y_1$ and $g_1(\varphi_1(t, x_1, u_1)) \in Q$ for all $t \geq 0$. Define $x_2 := \pi^s(x_1)$ and $u_2 := \pi^{in}(u_1) \in \mathcal{U}_2$. Then the semi-conjugacy condition implies for all $t \geq 0$

$$\begin{aligned} g_2(\varphi_2(t, x_2, u_2)) &= g_2(\varphi_2(t, \pi^s(x_1), \pi^{in}(u_1))) \\ &= g_2(\pi^s(\varphi_1(t, x_1, u_1))) \\ &= \pi^{out}(g_1(\varphi_1(t, x_1, u_1))) \in \pi^{out}(Q). \end{aligned}$$

In particular, for $t = 0$ one obtains $g_2(x_2) = \pi^{out}(g_1(x_1)) = \pi^{out}(y_1) = y_2$. The semi-conjugacy condition also implies that

$$\pi^s(K) \subset \pi^s(P(Q)) \subset P(\pi^{out}(Q)). \tag{13}$$

In fact, the first inclusion is trivial. For the second inclusion, consider $x_1 \in P(Q)$ and $u_1 \in \mathcal{U}_1$ with $g_1(\varphi_1(t, x_1, u_1)) \in Q$ for all $t \geq 0$. Then

$$g_2(\varphi_2(t, \pi^s(x_1), \pi^{in}(u_1))) = \pi^{out}(g_1(\varphi_1(t, x_1, u_1))) \in \pi^{out}(Q) \text{ for all } t \geq 0,$$

and $\pi^s(x_1) \in P(\pi^{out}(Q))$ follows. Now let $T, \varepsilon > 0$. Since π^s is uniformly continuous on a neighborhood $N_\alpha(P(Q)), \alpha > 0$, of the set $P(Q)$, there exists $\delta = \delta(\varepsilon) > 0$ with

$$\pi^s(N_\delta(P(Q))) \subset N_\varepsilon(\pi^s(P(Q))) \subset N_\varepsilon(P(\pi^{out}(Q))).$$

Let $\mathcal{S}_1 \subset \mathcal{U}_1$ be a minimal (T, δ, K, Q)-spanning set and define $\mathcal{S}_2 := \pi^{in}(\mathcal{S}_1)$. Let $x_2 = \pi^s(x_1) \in \pi^s(K)$ with $x_1 \in K$. Then there is $u_1 \in \mathcal{S}_1$ with $\varphi_1(t, x_1, u_1) \in N_\delta(P(Q))$ for all $t \in [0, T]$. Define $u_2 := \pi^{in}(u_1)$. Then one finds for all $t \in [0, T]$

$$\begin{aligned} \varphi_2(t, x_2, u_2) &= \varphi_2(t, \pi^s(x_1), \pi^{in}(u_1)) \\ &= \pi^s(\varphi_1(t, x_1, u_1)) \in \pi^s(N_\delta(P(Q))) \subset N_\varepsilon(P(\pi^{out}(Q))). \end{aligned}$$

This shows that $\mathcal{S}_2$ is a $(T, \varepsilon, \pi^s(K), \pi^{out}(Q))$-spanning set. Hence, for every $\delta < \delta(\varepsilon)$ one has $r_{\text{inv}}(T, \delta, K, Q) \geq r_{\text{inv}}(T, \varepsilon, \pi^s(K), \pi^{out}(Q))$ and inequality (11) follows. (ii) First we prove that

$$\pi^s(P(Q)) = P(\pi^{out}(Q)). \tag{14}$$

By (13), we only have to show that for $x_2 \in P(\pi^{out}(Q))$ there is $x_1 \in P(Q)$ with $\pi^s(x_1) = x_2$. There is $u_2 \in \mathcal{U}_2$ with $g_2(\varphi_2(t, x_2, u_2)) \in \pi^{out}(Q)$ for all $t \geq 0$. Then there are $x_1 \in \mathbb{R}^{d_1}$ and $u_1 \in \mathcal{U}_1$ with $\pi^s(x_1) = x_2$ and $\pi^{in}(u_1) = u_2$. Hence, for all $t \geq 0$

$$\begin{aligned} \pi^{out}(g_1(\varphi_1(t, x_1, u_1))) &= g_2(\pi^s(\varphi_1(t, x_1, u_1))) \\ &= g_2(\varphi_2(t, \pi^s(x_1), \pi^{in}(u_1))) \\ &= g_2(\varphi_2(t, x_2, u_2)) \in \pi^{out}(Q). \end{aligned}$$

Since π^{out} is a homeomorphism, it follows that $g_1(\varphi_1(t, x_1, u_1)) \in Q$ for all $t \geq 0$, and hence $x_1 \in P(Q)$ proving (14). Now fix $\varepsilon, T > 0$. Since $(\pi^s)^{-1}$ is uniformly continuous on a neighborhood $N_\alpha(\pi^s(P(Q))), \alpha > 0$, and equality (14) holds, there is $\delta > 0$ with

$$(\pi^s)^{-1}(N_\delta(P(\pi^{out}(Q)))) = (\pi^s)^{-1}(N_\delta(\pi^s(P(Q)))) \subset N_\varepsilon(P(Q)).$$

Let $\mathcal{S}_2 \subset \mathcal{U}_2$ be a minimal $(T, \delta, \pi^s(K), \pi^{out}(Q))$-spanning set and fix $x_1 \in K$. Then

$$x_2 := \pi^s(x_1) \in \pi^s(K) \subset \pi^s(P(Q)) = P(\pi^{out}(Q))$$

and there is $u_2 \in \mathcal{S}_2$ with

$$\varphi_2(t, x_2, u_2) \in N_\delta(P(\pi^{out}(Q))) \text{ for all } t \in [0, T].$$

Since π^{in} is surjective, we can pick $u_1 \in \mathcal{U}_1$ with $\pi^{in}(u_1) = u_2$. Define $\mathcal{S}_1 \subset \mathcal{U}_1$ as the set of these controls u_1 and note that the number of elements in $\mathcal{S}_1$ coincides with the number of elements in $\mathcal{S}_2$. Then the semi-conjugacy property implies for all $t \in [0, T]$

$$\varphi_1(t, x_1, u_1) = (\pi^s)^{-1}(\varphi_2(t, \pi^s(x_1), \pi^{in}(u_1))) \in N_\varepsilon(P(Q)).$$

Thus, $\mathcal{S}_1$ is (T, ε, K, Q)-spanning. This shows equality in (11). (iii) Finally, suppose that $D\pi^s(x)f_1(x, u) = f_2(\pi^s(x), u)$ for all $(x, u) \in \mathbb{R}^{d_1} \times U_1$. Then

$$\pi^s(\varphi_1(0, x, u)) = \pi^s(x)$$

and for almost all $t \geq 0$

$$\frac{d}{dt}\pi^s(\varphi_1(t, x, u)) = D\pi^s(\varphi_1(t, x, u))f_1(\varphi_1(t, x, u), u(t)) = f_2(\pi^s(\varphi_1(t, x, u)), u(t)).$$

Hence, $\pi^s(\varphi_1(t, x, u))$ coincides with $\varphi_2(t, \pi^s(x), u)$ for all $t \geq 0$. Together with $\pi^{out} \circ g_1 = g_2 \circ \pi^s$, this shows semi-conjugacy (10).

4 Invariance entropy for linear systems

In this section, we determine the invariance properties of compact subsets in the output space for linear control system (1).

Throughout this section, we assume that the control range U is compact and convex. We note the following consequence of Theorem 1 relating the entropy for (1) to the entropy for the induced observable system.

Lemma 3 *Consider linear control system (1) and let $Q \subset \mathbb{R}^k$ be compact. (i) If the set Q is controlled invariant for (1), then it is controlled invariant for the induced observable system with state space $\mathbb{R}^d/\mathcal{N}$,*

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}u(t),\ u \in \mathcal{U},\ \bar{y}(t) = \bar{C}z(t). \tag{15}$$

(ii) Systems (1) and (15) are semi-conjugate with the projection $\pi^s : \mathbb{R}^d \to \mathbb{R}^d/\mathcal{N}$ and identity maps π^{in} on $\mathcal{U}$ and π^{out} on $\mathbb{R}^k$. In particular, if the set $P(Q)$ for system (1) is compact, the invariance entropy of Q for system (1) is greater or equal than the invariance entropy of Q for system (15).

Proof (i) Let $u \in \mathcal{U}$. Then for all $x \in \mathbb{R}^d$ one has

$$C\varphi(t, x, u) = \bar{C}\bar{\varphi}(t, \pi^s x, u).$$

Hence, controlled invariance for the induced observable system follows. (ii) Here the assumptions of Theorem 1(i) are satisfied with π^{out} and π^{in} the identity maps and the surjective projection π^s. Thus, the inequality for the invariance entropies follows.

As another consequence of Theorem 1, we note the following observation.

Proposition 3 *Suppose that (A, C) is observable and A is totally unstable (i.e., all eigenvalues have positive real part). If there exists a compact controlled invariant set $Q \subset \mathbb{R}^k$ with nonvoid interior, then the reachable subspace $\mathcal{R}$ of the system satisfies $C\mathcal{R} = \mathbb{R}^k$.*

Proof For a system with output map C the uncontrollable quotient system has state space $\mathbb{R}^d/\mathcal{R}$ and output space $\mathbb{R}^k/C\mathcal{R}$ with induced maps $\hat{A} : \mathbb{R}^d/\mathcal{R} \to \mathbb{R}^d/\mathcal{R}$, $\hat{B} : \mathbb{R}^m \to \mathbb{R}^d/\mathcal{R}$ and output map $\hat{C} : \mathbb{R}^d/\mathcal{R} \to \mathbb{R}^k/C\mathcal{R}$. Observe that $\hat{B}$ is trivial, since $\operatorname{Im} B \subset \mathcal{R}$. This quotient system is semi-conjugate to the original system via the maps $\pi^{in} = id$ and the natural projections $\pi^s : \mathbb{R}^d \to \mathbb{R}^d/\mathcal{R}$, and $\pi^{out} : \mathbb{R}^k \to \mathbb{R}^k/C\mathcal{R}$. Hence, by Theorem 1, the image $\hat{Q}$ of Q in $\mathbb{R}^k/C\mathcal{R}$ is controlled invariant and it is compact. Since the spectrum of $\hat{A}$ is contained in the spectrum of the unstable matrix A, this can only be true if $\hat{Q}$ is trivial, i.e., $\hat{Q} = \{0\}$ which means that Q is contained in $C\mathcal{R}$. Since Q has nonvoid interior, it follows that $C\mathcal{R} = \mathbb{R}^k$.

The following lemma shows that for a compact set Q in the output space of an observable system, the set of initial values $x_0 \in \mathbb{R}^d$ which lead to outputs in Q on a finite interval, is compact. Furthermore, we can always find finite spanning sets.

Lemma 4 *Suppose that (A, C) is observable and let $Q \subset \mathbb{R}^k$ be compact and controlled invariant. (i) Then for every $T > 0$ the set*

$$P(Q, T) := \{x \in \mathbb{R}^d,\ \textit{there is } u \in \mathcal{U} \textit{ with } C\varphi(t, x, u) \in Q \textit{ for all } t \in [0, T]\}$$

is compact. (ii) The set $P(Q) \subset \mathbb{R}^d$ is compact and for all $T, \varepsilon > 0$ there are finite (T, ε, Q)-spanning sets.

Proof (i) Let $T > 0$ and pick $x \in \mathbb{R}^d$ and a control $u \in \mathcal{U}$. Then, by equation (4),

$$\begin{aligned} x &= W(0,T)^{-1} \int_0^T e^{A^* t} C^* \left[C\varphi(t,x,u) - \int_0^t C e^{A(t-s)} Bu(s) \mathrm{d}s \right] dt \\ &= W(0,T)^{-1} \left[\int_0^T e^{A^* t} C^* C\varphi(t,x,u) dt - \int_0^T \int_0^t C e^{A(t-s)} Bu(s) \mathrm{d}s\ dt \right]. \end{aligned}$$

This shows that the set $P(Q, T)$ is bounded, since here $C\varphi(t, x, u) \in Q$ for all $t \in [0, T]$ and Q and U are bounded. Furthermore, since U is compact and convex, system (1) satisfies the compactness assumption for the trajectories imposed in Lemma 2. This follows by a standard argument for existence of optimal controls (cf. Lee and Markus [8, proof of Theorem 4 in Section 4.2] or Macki and Strauss [9, Section III.1]): The set of admissible control functions is weakly compact in $L_2([0, T], \mathbb{R}^m)$ and hence contains a weakly convergent subsequence. (ii) By (i), the set $P(Q, T)$ is compact. Thus, the assertion follows by Lemma 2(ii).

From [3, Theorem 4.1] we obtain the following characterization of the invariance entropy in the state space.

Proposition 4 *Consider a linear control system with compact control range U (without output, i.e., $C = I$) in $\mathbb{R}^d$,*

$$\dot{x}(t) = Ax(t) + Bu(t),\ u \in \mathcal{U}. \tag{16}$$

Then for a compact controlled invariant subset Q of the state space $\mathbb{R}^d$

$$h_{\mathrm{inv}}(Q) \leq \sum \operatorname{Re} \lambda_i, \tag{17}$$

where the sum is taken over all positive real parts of the eigenvalues of A counted with their multiplicity. If Q has positive Lebesgue measure, then equality holds in (17).

Proof By Remark 1, the invariance entropy with $C = I_n$ defined above coincides with the state space entropy from [3]. Hence, the assertions follow from [3, Theorem 5.1].

In the next lemma we impose additional conditions ensuring a property in the reachable subspace. Note that one can restrict the state space of a control system (1) to its reachable subspace $\mathcal{R}$ and obtains

$$\dot{x} = A|_{\mathcal{R}}\, x + Bu,\ y = C|_{\mathcal{R}}\, x,\ u \in \mathcal{U}. \tag{18}$$

Lemma 5 *Suppose that $A|_{\mathcal{R}}$ is hyperbolic (i.e., there are no reachable eigenvalues on the imaginary axis), and assume that $0 \in \mathrm{int}U$ and $0 \in \mathrm{int}Q$. Then the set*

$$P_{\mathcal{R}}(Q) := \{x \in \mathcal{R},\ \textit{there is a control } u \in \mathcal{U} \textit{ with } C\varphi(t, x, u) \in Q \textit{ for all } t \geq 0\}$$

has nonvoid interior in the reachable subspace $\mathcal{R}$.

Proof For $\rho > 0$, small enough, the control ranges

$$U^\rho := \{u \in \mathbb{R}^m,\ \|u\| \leq \rho\}$$

are contained in U. Denote the corresponding sets of admissible control functions by $\mathcal{U}^\rho$. Recall that control sets are maximal sets of approximate controllability. For the control range U^ρ, Colonius and Spadini [5, Theorem 4.1] shows that there exists a unique control set D^ρ in $\mathcal{R} \subset \mathbb{R}^d$ and $0 \in \mathrm{int}D^\rho$, where the interior is taken with respect to $\mathcal{R}$. Furthermore, for $\rho \to 0$, the diameter of $\mathrm{cl}D^\rho$ shrinks to zero. The initial point $x_0 = 0$ is in $P_{\mathcal{R}}(Q)$, since $C\varphi(t, 0, 0) = 0 \in \mathrm{int}Q$ for all $t \geq 0$. Since U^ρ is bounded, there is $T > 0$, such that $C\varphi(t, 0, u) \in Q$ for all $t \in [0, T]$ and all $u \in \mathcal{U}^\rho$. We may take $T > 0$ small enough such that

$$\mathcal{R}^{\rho;+}_{\leq T}(0) := \{\varphi(t, 0, u),\ 0 \leq t \leq T \text{ and } u \in \mathcal{U}^\rho\} \subset \mathrm{int}D^\rho \subset \mathcal{R}.$$

In $\mathrm{int}D^\rho$ exact reachability holds (cf. [5, Lemma 2.1] or Colonius and Kliemann [4, Lemma 3.2.13]). Hence, every point in $\mathcal{R}^{\rho;+}_{\leq T}(0)$ can be steered back to the origin, naturally without leaving D^ρ. Since $0 = C0 \in \mathrm{int}Q$ and the map C is continuous, one may take $\rho > 0$ small enough, such that all points on such a trajectory are mapped by C into Q. Extending the controls periodically to $[0, \infty)$ one sees that $\mathcal{R}^{\rho;+}_{\leq T}(0) \subset P_{\mathcal{R}}(Q)$. The small time reachable set $\mathcal{R}^{\rho;+}_{\leq T}(0)$ has nonvoid interior in $\mathcal{R}$, hence $P_{\mathcal{R}}(Q)$ has nonvoid interior in $\mathcal{R}$, as claimed.

For the following result recall that the induced observable system associated with (1) is given by the matrices $(\bar{A}, \bar{B}, \bar{C})$ and controls in $\mathcal{U}$.

Theorem 2 *Consider system (1) where the control range U is compact and convex with $0 \in \mathrm{int}U$ and let Q be a compact controlled invariant set in the output space $\mathbb{R}^k$ with $0 \in \mathrm{int}Q$. (i) Suppose that (A, C) is observable. Then the set*

$$P(Q) = \{x \in \mathbb{R}^d, \text{ there is } u \in \mathcal{U} \text{ with } C\varphi(t, x, u) \in Q \text{ for all } t \geq 0\} \tag{19}$$

is compact and the invariance entropy of Q satisfies the inequality

$$h_{\mathrm{inv}}(Q) \leq \sum_{\lambda} \mathrm{Re}\,\lambda, \tag{20}$$

where the sum on the right-hand side is taken over all eigenvalues λ of A with positive real parts. (ii) Suppose that (A, B) is reachable, that the matrix A is hyperbolic, and that the set $P(Q)$ is compact. Then

$$h_{\mathrm{inv}}(Q) \geq \sum_{\lambda} \mathrm{Re}\,\lambda, \tag{21}$$

where the sum on the right-hand side is taken over all eigenvalues λ of A with positive real parts. (iii) If (A, C) is observable and (A, B) is reachable with a hyperbolic matrix A, then equality holds in (20).

Proof (i) By observability, Lemma 4(ii) implies that the set $P(Q)$ in (19) is compact. Furthermore, this set is controlled invariant in the state space. Proposition 4 shows that the invariance entropy satisfies

$$h_{\mathrm{inv}}(P(Q)) \leq \sum_{\lambda} \mathrm{Re}\,\lambda, \tag{22}$$

where summation is over the eigenvalues λ of A with positive real parts. Finally, the equality

$$h_{\mathrm{inv}}(P(Q)) = h_{\mathrm{inv}}(Q), \tag{23}$$

follows from Proposition 2. (ii) By Lemma 3, the invariance entropy $h_{\mathrm{inv}}(Q)$ of Q for system (1) is greater than or equal to the invariance entropy of Q for the induced observable system. It is easily seen that this system is also reachable. Hence, we may assume without loss of generality that (A, C) is observable and (A, B) is reachable. By Lemma 5 and reachability, the set $P(Q)$ has nonvoid interior. Thus, Proposition 4 shows that equality holds in (22). (iii) This is immediate from (i) and (ii) noting that observability implies compactness of $P(Q)$.

The following simple example illustrates some of the results above.

Example 1 Consider the two-dimensional system ($d = 2$, $m = k = 1$)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t),$$

and controls satisfying $u(t) \in U = [-1, 1]$. The solutions are

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^t x_1^0 \\ e^{-t} x_2^0 \end{pmatrix} + \int_0^t u(s) \begin{pmatrix} e^{t-s} \\ e^{s-t} \end{pmatrix} \mathrm{d}s.$$

There is a unique control set D with nonvoid interior,

$$D = (-1, 1) \times [-1, 1]$$

and $h_{\mathrm{inv}}(\mathrm{cl}D) = 1$. The system is reachable, since the matrix

$$(B, AB) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

has rank $d = 2$. (i) The system with output matrix $C = (1, 1)$, i.e.,

$$y = (1, 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + x_2,$$

is observable, since the matrix

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

has rank $d = 2$. Clearly, $Q := C(\mathrm{cl}D) = \{x_1 + x_2,\ x_1, x_2 \in [-1, 1]\} = [-2, 2]$ and $C^{-1}(Q) = \{(x_1, x_2),\ |x_1 + x_2| \leq 2\}$ is unbounded, since $\ker C$ is nontrivial. By Theorem 2, it follows that

$$h_{\mathrm{inv}}([-2, 2]) = h_{\mathrm{inv}}([-1, 1] \times [-1, 1]) = 1.$$

By observability, the set $P(Q)$ is bounded. This follows from Lemma 4, since already the set of points x with $C\varphi(t, x, u) \in Q$ on any interval $[0, T]$ is bounded. Observe that for $x_0 \in \ker C$, the trajectory $\varphi(t, x_0, 0)$ $t > 0$, immediately leaves $\ker C$. (ii) The system with output matrix $C = (0, 1)$, i.e.,

$$y = (0, 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2,$$

is not observable. Here for $Q := C(\mathrm{cl}D) = [-1, 1]$ one obtains

$$\{x \in \mathbb{R}^2,\ Cx \in Q\} = \mathbb{R} \times [-1, 1],$$

and

$$P(Q) = [-1, 1] \times [-1, 1].$$

(iii) Similarly, the system with output matrix $C = (1, 0)$, i.e.,

$$y = (1, 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1,$$

is not observable. Again for $Q = [-1, 1]$, one obtains

$$\{x \in \mathbb{R}^2,\ Cx \in Q\} = [-1, 1] \times \mathbb{R},$$

and the set $P(Q)$ is unbounded and given by

$$P(Q) = [-1, 1] \times \mathbb{R}.$$

Observe that this system has the observable eigenvalue $\lambda = 1$ with positive real part.

Next we discuss invariance entropy for systems where the set $P(Q)$ need not be bounded. Example 1 shows that for linear systems which are not observable, the set $P(Q)$ may be noncompact. In this situation, in order to obtain that nonobservable eigenvalues do not contribute to the invariance entropy, we consider special (noncompact) sets $K \subset P(Q)$ of initial values. The following theorem estimates the entropy of nonobservable systems in such a case.

Theorem 3 *Consider system (1), where the control range U is compact and convex with $0 \in U$ and let Q be a compact controlled invariant neighborhood of the origin in the output space $\mathbb{R}^k$. Let Q_0 and U_0 be compact neighborhoods of the origin in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, with*

$$Q_0 + Q_0 \subset Q \text{ and } U_0 + U_0 \subset U.$$

Let $\mathcal{U}_0 := \{u_0 \in \mathcal{U},\ u_0(t) \in U_0 \text{ for all } t \geq 0\}$ and define

$$K := \{x \in \mathbb{R}^d,\ \text{there is } u_0 \in \mathcal{U}_0 \text{ with } C\varphi(t,x,u_0) \in Q_0 \text{ for all } t \geq 0\}.$$

Then the invariance entropy satisfies

$$h_{\text{inv}}(K,Q) \leq \sum_{\lambda} \operatorname{Re}\lambda,$$

where the sum on the right-hand side is taken over all observable eigenvalues λ with positive real parts; i.e., the eigenvalues of $\bar{A}$ with positive real parts.

Proof Recall from Section 2 that $\mathcal{N}$ is the unobservable subspace. Consider the linear semiflow $\bar{\Phi}(t,x) = e^{\bar{A}t}x$, $\bar{\Phi} : [0,\infty) \times \mathbb{R}^d/\mathcal{N} \to \mathbb{R}^d/\mathcal{N}$. With respect to any norm $\|\cdot\|$ on $\mathbb{R}^d/\mathcal{N}$, this semiflow satisfies the following uniform continuity condition: for all $t_0 > 0$, $t \in [0,t_0]$, and $x_1, x_2 \in \mathbb{R}^d/\mathcal{N}$ one has

$$\|e^{\bar{A}t}x_1 - e^{\bar{A}t}x_2\| = \|e^{\bar{A}t}(x_1 - x_2)\| \leq \|e^{\bar{A}t}\|\|x_1 - x_2\| \leq \left(\max_{t\in[0,t_0]} \|e^{\bar{A}t}\|\right)\|x_1 - x_2\|.$$

Hence (cf. [3, Lemma 2.1]), the topological entropy $h_{\text{top}}(\bar{\Phi})$ equals the topological entropy of the time-one-map $\bar{\Phi}_1(x) = e^{\bar{A}}x$. Recall from Bowen [2] (cf. also Katok and Hasselblatt [6] or Robinson [11]) that the topological entropy of a linear map Ψ on $\mathbb{R}^d$ can be defined in the following way: For a compact set $K \subset \mathbb{R}^d$, numbers $n \in \mathbb{N}$ and $\varepsilon > 0$ an (n,ε,K,Ψ)-spanning set is a subset $R \subset K$ such that for all $x \in K$ there is $y \in R$ with $\left\|\Psi^i(x) - \Psi^i(y)\right\| < \varepsilon$ for all $i \in \{0,1,\ldots,n-1\}$. Similarly, a subset $S \subset K$ is called (n,ε,K,Ψ)-separated if for all $x,y \in S, x \neq y$, there is $i \in \{0,1,\ldots,n-1\}$ with $\left\|\Psi^i(x) - \Psi^i(y)\right\| \geq \varepsilon$. Denote the minimal cardinality of an (n,ε,K,Ψ)-spanning set by $r(n,\varepsilon,K,\Psi)$ and the maximal cardinality of an (n,ε,K,Ψ)-separated set by $s(n,\varepsilon,K,\Psi)$. Then

$$h_{\text{top}}(K,\Psi) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r(n,\varepsilon,K,\Psi) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s(n,\varepsilon,K,\Psi),$$

and

$$h_{\text{top}}(\Psi) := \sup_K h_{\text{top}}(K,\Psi),$$

where the supremum is taken over all compact $K \subset \mathbb{R}^d$. By Bowen [2] the topological entropy of the linear map $\bar{\Phi}_1$ on $\mathbb{R}^d/\mathcal{N}$ is given by

$$h_{\text{top}}(\bar{\Phi}_1) = \sum_{i:\ |\nu_i|>1} \ln|\nu_i|,$$

where $\nu_1, \ldots, \nu_{\bar{d}}$ are the eigenvalues of $e^{\bar{A}}$. Here $\mathbb{R}^d/\mathcal{N}$ is endowed with the metric

$$\mathrm{d}(x+\mathcal{N}, y+\mathcal{N}) = \inf_{z\in\mathcal{N}} \|x-y-z\|.$$

Since $|\nu_i| = \left|e^{\lambda_i}\right| = e^{\operatorname{Re}\lambda_i}$, where $\lambda_1, \ldots, \lambda_{\bar{d}}$ are the eigenvalues of $\bar{A}$, we obtain

$$h_{\text{top}}(\bar{\Phi}) = h_{\text{top}}(\bar{\Phi}_1) = \sum_{i:\ |e^{\lambda_i}|>1} \operatorname{Re}\lambda_i = \sum_{i:\ \operatorname{Re}\lambda_i>0} \operatorname{Re}\lambda_i.$$

Denote the natural projection of $\mathbb{R}^d$ to $\mathbb{R}^d/\mathcal{N}$ by π and define $\bar{K} := \pi(K)$. Since for all $t \geq 0,\ x \in \mathbb{R}^d$, and $u \in \mathcal{U}$ one has $C\varphi(t,x,u) = \bar{C}\bar{\varphi}(t,\pi x,u)$, it follows that

$$\bar{K} = \{\pi x \in \mathbb{R}^d/\mathcal{N},\ \text{there is } u_0 \in \mathcal{U}_0 \text{ with } \bar{C}\bar{\varphi}(t,\pi x,u_0) \in Q_0 \text{ for all } t \geq 0\}.$$

By observability and Lemma 4 the set $\bar{K} \subset \mathbb{R}^d/\mathcal{N}$ is compact. We also observe that $\mathcal{N} \subset K$, since for $x \in \mathcal{N}$ one has $C\varphi(t,x,0) = 0 \in Q_0$ for all $t \geq 0$. For $x_1, x_2 \in K$ there are controls $u_1, u_2 \in \mathcal{U}_0$ such that for $t \geq 0$

$$C\varphi(t,x_1+x_2,u_1+u_2) = C\varphi(t,x_1,u_1) + C\varphi(t,x_2,u_2) \in Q_0 + Q_0 \subset Q,$$
$$u_1(t) + u_2(t) \in U_0 + U_0 \subset U,\ \text{hence } u_1 + u_2 \in \mathcal{U}.$$

This shows that

$$K + K \subset P(Q).$$

Now fix $T, \varepsilon > 0$ and let $E \subset \bar{K} = \pi(K)$ be a maximal $(T,\varepsilon,\bar{K},\bar{\Phi})$-separated set with respect to the semiflow $\bar{\Phi}$ on $\mathbb{R}^d/\mathcal{N}$, say $E = \{\pi x_1, \ldots, \pi x_n\}$ with $x_1, \ldots, x_n \in K$, and $n := s(T,\varepsilon,\bar{K},\bar{\Phi})$. Then E is also $(T,\varepsilon,\bar{K},\bar{\Phi})$-spanning for $\bar{\Phi}$, which means that for every $x \in K$ there is $x_j,\ j \in \{1,\ldots,n\}$, with

$$\begin{aligned}\max_{t\in[0,T]} \mathrm{d}(e^{\bar{A}t}\pi x, e^{\bar{A}t}\pi x_j) &= \max_{t\in[0,T]} \operatorname{dist}(e^{At}x - e^{At}x_j, \mathcal{N}) \\ &= \max_{t\in[0,T]} \inf_{z\in\mathcal{N}} \left\|e^{At}x - e^{At}x_j - z\right\| < \varepsilon.\end{aligned}$$

The set K is controlled invariant with respect to controls in $\mathcal{U}_0$. Hence, we can assign to each $x_j \in K, j = 1, \ldots, n$, a control function $u_j \in \mathcal{U}_0$ such that $\varphi([0,\infty), x_j, u_j) \subset K$. Let $\mathcal{S} := \{u_1, \ldots, u_n\} \subset \mathcal{U}_0$. Using $\mathcal{N} \subset K$ and linearity, we obtain that for all $x \in K$ there is j such that

$$\begin{aligned}&\max_{t\in[0,T]} \operatorname{dist}(\varphi(t,x,u_j) - \varphi(t,x_j,u_j), K) \leq \max_{t\in[0,T]} \operatorname{dist}(\varphi(t,x,u_j) - \varphi(t,x_j,u_j), \mathcal{N}) \\ &= \max_{t\in[0,T]} \operatorname{dist}(e^{At}x - e^{At}x_j, \mathcal{N}) < \varepsilon.\end{aligned}$$

Since $\varphi(t,x_j,u_j) \in K$ for all $t \in [0,T]$ and $K + K \subset P(Q)$, this implies that for all $x \in K$ there is $u_j \in \mathcal{S}$ such that

$$\max_{t\in[0,T]} \operatorname{dist}(\varphi(t,x,u_j), P(Q)) \leq \max_{t\in[0,T]} \operatorname{dist}(\varphi(t,x,u_j), K+K) < \varepsilon.$$

This shows that $\mathcal{S}$ is (T,ε,K,Q)-spanning and hence

$$r_{\text{inv}}(T,\varepsilon,K,Q) \le s(T,\varepsilon,\pi(K),\bar{\Phi}) \text{ for all } T,\varepsilon > 0,$$

and consequently

$$h_{\text{inv}}(K,Q) \le h_{\text{top}}(\pi(K),\bar{\Phi}) \le h_{\text{top}}(\hat{\Phi}) = \sum_{i:\ \operatorname{Re}\lambda_i > 0} \operatorname{Re}\lambda_i .$$

5 Asymptotic Invariance Entropy

In the following, we propose a modified version of invariance entropy. We weaken the assumption that spanning sets of controls keep the system near the set $P(Q)$ for all times. Instead we only require this for all times large enough which may appear reasonable, since entropy is an asymptotic property. This will allow us to deal with unbounded states leading to outputs in Q, without the additional assumptions imposed in Theorem 3 on the set of admissible initial values K. More precisely, we introduce the following notions which we formulate for general system (8).

Definition 5 Let Q be a controlled invariant subset of the output space $\mathbb{R}^k$. For a set $K \subset \mathbb{R}^d$ and $\varepsilon > 0$, fix times $T > T_0 \ge 0$. We call a subset $\mathcal{S} \subset \mathcal{U}$ a (T,T_0,ε,K,Q)-*spanning* set if for all $x \in K$ there is $u \in \mathcal{S}$ with

$$\varphi(t,x,u) \in N_\varepsilon(P(Q)) \text{ for all } t \in [T_0,T].$$

By $r_{\text{as}}(T,T_0,\varepsilon,K,Q)$ we denote the minimal cardinality of a (T,T_0,ε,K,Q)-spanning set. If no finite (T,T_0,ε,K,Q)-spanning set exists, we set $r_{\text{as}}(T,T_0,\varepsilon,K,Q) := \infty$.

In other words: we require for a (T,T_0,ε,K,Q)-spanning set $\mathcal{S}$ that for every initial value in $K \subset \mathbb{R}^d$, there is a control in $\mathcal{S}$ such that for time t between T_0 and T the trajectory remains in the ε-neighborhood of $P(Q)$. Now we consider what happens for $T \to \infty$, then $T_0 \to \infty$, and, finally, $\varepsilon \to 0$ and obtain the following variant of invariance entropy.

Definition 6 Let Q be a compact controlled invariant set in the output space $\mathbb{R}^k$ and let $K \subset \mathbb{R}^d$. Then the *asymptotic invariance entropy* $h_{\text{as}}(K,Q)$ is defined by

$$\begin{aligned} h_{\text{as}}(\varepsilon,K,Q) &:= \lim_{T_0\to\infty} \limsup_{T\to\infty} \frac{1}{T} \ln r_{\text{as}}(T,T_0,\varepsilon,K,Q), \\ h_{\text{as}}(K,Q) &:= \lim_{\varepsilon\searrow 0} h_{\text{as}}(\varepsilon,K,Q). \end{aligned}$$

The expression $h_{\text{as}}(\varepsilon,K,Q)$ is well defined, since the limit for $T_0 \to \infty$ exists: For $T_0 > T_1$ every (T,T_1,ε,K,Q)-spanning set is also (T,T_0,ε,K,Q)-spanning, hence $r_{\text{as}}(T,T_0,\varepsilon,K,Q) \le r_{\text{as}}(T,T_1,\varepsilon,K,Q)$ and, by monotonicity of the logarithm, the limit for $T_0 \to \infty$ equals the infimum. It is also immediate from the definition that the asymptotic invariance entropy is not greater than the invariance entropy. Note that for systems without output, i.e., $g = \text{id}$, and a controlled invariant set $Q \subset \mathbb{R}^d$, one has $P(Q) = Q$ and one obtains a notion of asymptotic invariance entropy in the state space. Finally, note that for $K_1 \subset K_2$ one has $h_{\text{as}}(K_1,Q) \le h_{\text{as}}(K_2,Q)$.

For linear control systems we obtain the following estimate from above, without observability assumption. Instead we require an asymptotic reachability condition for the unobservable subspace.

Theorem 4 *Consider system (1), where the control range U is compact and convex with $0 \in U$, let Q be a compact controlled invariant set with $0 \in Q$ in the output space $\mathbb{R}^k$ and fix a compact subset $K \subset \mathbb{R}^d$. Assume that for all $\varepsilon > 0$ there is a time $T_0(\varepsilon) \geq 0$ such that for all $x \in K$ there is a control $u \in \mathcal{U}$ with*

$$\mathrm{dist}(\varphi(t,x,u), \mathcal{N}) < \varepsilon \text{ for all } t \geq T_0(\varepsilon).$$

Then the asymptotic invariance entropy satisfies

$$h_{\mathrm{as}}(K,Q) \leq \sum_{\lambda} \mathrm{Re}\,\lambda,$$

where the sum on the right-hand side is taken over all observable eigenvalues λ with positive real parts; i.e., the eigenvalues of $\bar{A}$ with positive real parts.

Proof The proof proceeds along the lines of Theorem 3. Now K is a compact subset of $\mathbb{R}^d$ and for the induced semiflow $\bar{\Phi}$, the topological entropy of $\bar{K} = \pi(K) \subset \mathbb{R}^d/\mathcal{N}$ is bounded above by the sum of the positive real parts of the observable eigenvalues. Fix $\varepsilon > 0$, $T > 0$ and let $E \subset \bar{K} = \pi(K)$ be a maximal $(T, \varepsilon, \bar{K}, \bar{\Phi})$-separated set with respect to the semiflow $\bar{\Phi} = e^{\bar{A}\cdot}$ on $\mathbb{R}^d/\mathcal{N}$, say $E = \{\pi x_1, \ldots, \pi x_n\}$ with $x_j \in K$, and let $n := s(T, \varepsilon, \bar{K}, \bar{\Phi})$. Then E is also $(T, \varepsilon, \bar{K}, \bar{\Phi})$-spanning which means that for all $x \in K$ there is $x_j,\ j \in \{1, \ldots, n\}$, with

$$\begin{aligned}\max_{t \in [0,T]} \mathrm{d}(e^{\bar{A}t}\pi x, e^{\bar{A}t}\pi x_j) &= \max_{t \in [0,T]} \mathrm{dist}(e^{At}x - e^{At}x_j, \mathcal{N}) \\ &= \max_{t \in [0,T]} \inf_{z \in \mathcal{N}} \left\| e^{At}x - e^{At}x_j - z \right\| < \varepsilon.\end{aligned}$$

By assumption, we can assign to each $x_j \in K$ a control function $u_j \in \mathcal{U}$ such that

$$\mathrm{dist}(\varphi(t, x_j, u_j), \mathcal{N}) < \varepsilon \text{ for all } t \geq T_0(\varepsilon).$$

Let $\mathcal{S} := \{u_1, \ldots, u_n\} \subset \mathcal{U}$. Note that $\mathcal{N} + P(Q) = P(Q)$, since for $x_1 \in \mathcal{N}$ and $x_2 \in P(Q)$ there is a control $u \in \mathcal{U}$ with

$$C\varphi(t, x_1 + x_2, u) = C\varphi(t, x_1, 0) + C\varphi(t, x_2, u) = C\varphi(t, x_2, u) \in P(Q).$$

Using $\mathcal{N} \subset P(Q)$ and linearity, we obtain that for all $x \in P(Q)$ there is j such that

$$\begin{aligned}&\max_{t \in [0,T]} \mathrm{dist}(\varphi(t,x,u_j) - \varphi(t,x_j,u_j), P(Q)) \leq \max_{t \in [0,T]} \mathrm{dist}(\varphi(t,x,u_j) - \varphi(t,x_j,u_j), \mathcal{N}) \\ &= \max_{t \in [0,T]} \mathrm{dist}(e^{At}x - e^{At}x_j, \mathcal{N}) < \varepsilon.\end{aligned}$$

Then it follows that for all $t \in [T_0(\varepsilon), T]$

$$\mathrm{dist}(\varphi(t,x,u_j), P(Q)) < 2\varepsilon.$$

In fact, using $\mathcal{N} + P(Q) = P(Q)$, one finds for all $t \in [T_0(\varepsilon), T]$

$$\begin{aligned}&\mathrm{dist}(\varphi(t,x,u_j), P(Q)) \\ &= \inf\{\left\|\varphi(t,x,u_j) - \varphi(t,x_j,u_j) + \varphi(t,x_j,u_j) - p\right\|,\ p \in P(Q)\} \\ &= \inf\{\left\|\varphi(t,x,u_j) - \varphi(t,x_j,u_j) + \varphi(t,x_j,u_j) - p - n\right\|,\ p \in P(Q) \text{ and } n \in \mathcal{N}\} \\ &\leq \inf\{\left\|\varphi(t,x,u_j) - \varphi(t,x_j,u_j) - p\right\| + \left\|\varphi(t,x_j,u_j) - n\right\|,\ p \in P(Q) \text{ and } n \in \mathcal{N}\} \\ &= \inf\{\left\|\varphi(t,x,u_j) - \varphi(t,x_j,u_j) - p\right\|,\ p \in P(Q)\} + \inf\{\left\|\varphi(t,x_j,u_j) - n\right\|,\ n \in \mathcal{N}\} \\ &< \varepsilon + \varepsilon.\end{aligned}$$

This shows that $\mathcal{S}$ is $(T, T_0(\varepsilon), 2\varepsilon, K, Q)$-spanning and hence

$$r_{\mathrm{as}}(T, T_0, 2\varepsilon, K, Q) \leq s(T, \varepsilon, \pi(K), \bar{\Phi}) \text{ for all } T, \varepsilon > 0,$$

and consequently

$$\begin{aligned} h_{\mathrm{as}}(2\varepsilon, K, Q) &= \lim_{T_0 \to \infty} \limsup_{T \to \infty} \frac{1}{T} \ln r_{\mathrm{as}}(T, T_0, 2\varepsilon, K, Q) \\ &\leq h_{\mathrm{top}}(\pi(K), \bar{\Phi}) \leq h_{\mathrm{top}}(\bar{\Phi}) = \sum_{i:\ \mathrm{Re}\,\lambda_i > 0} \mathrm{Re}\,\lambda_i. \end{aligned}$$

In order to combine this result with controllability properties, we show the following lemma which is similar to [3, Theorem 4.1] (here we restrict the analysis to linear control systems, consider asymptotic invariance entropy instead of invariance entropy, and do not require that $K \subset P$.)

Lemma 6 *Consider system (1) and let $K, P \subset \mathbb{R}^d$ be nonvoid compact sets with P being controlled invariant. Then, if the Lebesgue measure $\lambda^d(K)$ of K is positive, the following estimate holds:*

$$h_{\mathrm{as}}(K, P) \geq \mathrm{tr}A. \tag{24}$$

Proof Fix $\varepsilon > 0$ and $T \geq T_0 > 0$, and let $\mathcal{S} = \{u_1, \ldots, u_n\}$ be a minimal $(T, T_0, \varepsilon, K, P)$-spanning set. Define the following sets

$$K_j := \left\{x \in K \mid \varphi([T_0, T], x, u_j) \subset N_\varepsilon(P)\right\}, \quad j = 1, \ldots, n.$$

Then the sets $K_j, j = 1, \ldots, n$, cover K and, by openness of $N_\varepsilon(P)$ and continuous dependence on initial conditions, the subsets K_j are open in K. Since $\varphi(t, K_j, u_j) \subset N_\varepsilon(P)$ for all $t \in [T_0, T]$ and $j = 1, \ldots, n$ we obtain, in particular,

$$\lambda^d(\varphi(T, K_j, u_j)) \leq \lambda^d(N_\varepsilon(P)) \text{ for all } j = 1, \ldots, n.$$

Moreover, by the transformation theorem and Liouville's formula we get for all $j \in \{1, \ldots, n\}$

$$\lambda^d(\varphi(T, K_j, u_j)) = \det(e^{AT}) \cdot \lambda^d(K_j) = \exp(T \cdot \mathrm{tr}A) \cdot \lambda^d(K_j).$$

Then it follows that

$$\lambda^d(K) \leq \sum_{j=1}^n \lambda^d(K_j) \leq n \cdot \max_{j=1,\ldots,n} \lambda^d(K_j) \leq n \cdot \frac{\lambda^d(N_\varepsilon(P))}{\exp(T \cdot \mathrm{tr}A)}.$$

Consequently, we obtain the estimate

$$r_{\mathrm{as}}(T, \varepsilon, K, P) = n \geq \frac{\lambda^d(K)}{\lambda^d(N_\varepsilon(P))} \exp(T \cdot \mathrm{tr}A).$$

Taking the logarithm on both sides, dividing by T and letting T tend to infinity yields the inequality

$$h_{\mathrm{as}}(\varepsilon, K, P) \geq \limsup_{T \to \infty} \frac{1}{T} \left[\ln \lambda^d(K) - \ln \lambda^d(N_\varepsilon(P)) + T \cdot \mathrm{tr}A\right] = \mathrm{tr}A.$$

Letting ε tend to zero we obtain (24).

The next lemma describes the behavior of the asymptotic invariance entropy under semiconjugacy. For brevity, we only state and prove the analogue of Theorem 1(i) in the case relevant here.

Lemma 7 *Consider for $i = 1, 2$ two control systems of the form (1),*

$$\dot{x}_i = A_i x_i + B_i u(t),\ y_i = C_i x_i,\ u \in \mathcal{U},$$

in $\mathbb{R}^{d_i}$ with control range $U \subset \mathbb{R}^m$ and outputs $C_i : \mathbb{R}^{d_i} \to \mathbb{R}^{k_i}$. Let $\pi^s : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ and $\pi^{out} : \mathbb{R}^{k_1} \to \mathbb{R}^{k_2}$ be linear. Denote the corresponding trajectories by $\varphi_i(t, x, u)$ and assume that the following semi-conjugacy property holds for all $(t, x, u) \in [0, \infty) \times \mathbb{R}^{d_1} \times \mathcal{U}$:

$$\pi^s(\varphi_1(t, x, u)) = \varphi_2(t, \pi^s(x), u) \text{ and } \pi^{out} \circ C_1 = C_2 \circ \pi^s. \tag{25}$$

Let $Q \subset \mathbb{R}^{k_1}$ be a compact controlled invariant set for the first system. Then the set $\pi^{out}(Q) \subset \mathbb{R}^{k_2}$ is compact and controlled invariant. For a compact subset $K \subset \mathbb{R}^{k_1}$ the image $\pi^s(K) \subset \mathbb{R}^{d_2}$ is compact and the asymptotic invariance entropies for the two systems satisfy

$$h^1_{\mathrm{as}}(K, Q) \geq h^2_{\mathrm{as}}(\pi^s(K), \pi^{out}(Q)). \tag{26}$$

Proof The proof of Theorem 1(i) applies literally to show that $\pi^s(K)$ is compact and that $\pi^{out}(Q)$ is compact and controlled invariant. In the same way, it follows that

$$\pi^s(P_1(Q)) \subset P_2(\pi^{out}(Q)),$$

where $P_1(Q)$ and $P_2(\pi^{out}(Q))$ denote the sets of initial values for the first and the second system leading to outputs in Q and in $\pi^{out}(Q)$, respectively. Now let $T \geq T_0 > 0$ and $\varepsilon > 0$. By linearity, one finds $\delta > 0$ such that

$$\pi^s(N_\delta(P_1(Q))) \subset N_\varepsilon(\pi^s(P_1(Q)) \subset N_\varepsilon(P_2(Q)).$$

Let $\mathcal{S}_1 \subset \mathcal{U}$ be a minimal (T, T_0, δ, K, Q)-spanning set for the first system. We claim that it is $(T, T_0, \varepsilon, \pi^s(K), Q)$-spanning set for the second system. In fact, for $x_2 \in \pi^s(K)$ there is $x_1 \in K$ with $x_2 = \pi^s(x_1)$. Then there is $u \in \mathcal{S}$ with $\varphi_1(t, x_1, u) \in N_\delta(P_1(Q))$ for all $t \in [T_0, T]$. One finds for all $t \in [T_0, T]$

$$\varphi_2(t, x_2, u) = \varphi_2(t, \pi^s(x_1), u) = \pi^s(\varphi_1(t, x_1, u)) \in \pi^s(N_\delta(P_1(Q))) \subset N_\varepsilon(P_2(Q)).$$

This shows that $\mathcal{S}$ is a $(T, T_0, \varepsilon, \pi^s(K), Q)$-spanning set. Hence, for every $\delta < \delta(\varepsilon)$, one finds for the minimal cardinalities of spanning sets of the first and the second system that $r^1_{\mathrm{as}}(T, T_0, \delta, K, Q) \geq r^2_{\mathrm{as}}(T, T_0, \varepsilon, \pi^s(K), Q))$, and inequality (26) follows.

Combining Theorem 4 with controllability properties, we obtain the following characterization of the asymptotic invariance entropy.

Theorem 5 *Suppose, in addition to the assumptions of Theorem 4 that $0 \in \mathrm{int} U$ and that K is a compact subset of the reachability subspace $\mathcal{R}$ which contains the origin in its interior. Assume, furthermore, that there are no reachable and observable eigenvalues on the imaginary axis. Then the asymptotic invariance entropy satisfies*

$$h_{\mathrm{as}}(K, Q) = \sum_\lambda \mathrm{Re}\,\lambda, \tag{27}$$

where the sum on the right-hand side is taken over all observable and reachable eigenvalues λ with positive real parts.

Proof (i) For $x \in \mathcal{R}$ and $u \in \mathcal{U}$ one has $C\varphi(t,x,u) \in \mathcal{R}$ for all $t \geq 0$. This shows that

$$P(Q) \cap \mathcal{R} = P_{\mathcal{R}}(Q) := \{x \in \mathcal{R}, \text{ there is } u \in \mathcal{U} \text{ with } C\varphi(t,x,u) \in Q \text{ for all } t \geq 0\}.$$

Since $K \subset \mathcal{R}$ it follows for $t > 0,\ x \in K,\ u \in \mathcal{U}$ that

$$\varphi(t,x,u) \in N_\varepsilon(P(Q)) \text{ implies } \varphi(t,x,u) \in N_\varepsilon(P(Q)) \cap \mathcal{R} = N_\varepsilon(P_{\mathcal{R}}(Q)).$$

It follows that a (T,ε,K,Q)-spanning set of the system with state space restricted to $\mathcal{R}$ is also (T,ε,K,Q)-spanning for the original system. Thus $h_{\mathrm{as}}(K,Q)$ is less than or equal to the corresponding invariance entropy for the system restricted to $\mathcal{R}$. Then Theorem 4 implies the upper bound for $h_{\mathrm{as}}(K,Q)$.
(ii) For the lower bound, consider the projection π^s to the induced observable system and take for π^{out} the identity. By Lemma 7

$$h^1_{\mathrm{as}}(K,Q) \geq h^2_{\mathrm{as}}(\pi^s(K),Q).$$

Next restrict the observable system to its reachability subspace $\bar{\mathcal{R}}$. Since $0 \in \mathrm{int}K$, it follows that $0 \in \mathrm{int}\pi^s(K)$ and hence 0 is also in the interior of $\pi^s(K) \cap \bar{\mathcal{R}}$; in particular, this set has nonvoid interior. Clearly, the invariance entropies satisfy

$$h^2_{\mathrm{as}}(\pi^s(K),Q) \geq h^3_{\mathrm{as}}(\pi^s(K) \cap \bar{\mathcal{R}},Q),$$

where the right-hand side denotes the asymptotic invariance entropy for the system restricted to $\bar{\mathcal{R}}$. It is immediate from the definitions that the asymptotic invariance entropy satisfies

$$h^3_{\mathrm{as}}(\pi^s(K) \cap \bar{\mathcal{R}},Q) = h^4_{\mathrm{as}}(\pi^s(K) \cap \bar{\mathcal{R}},P(Q)),$$

where on the right-hand side we consider the asymptotic invariance entropy in the state space. Since this is a reachable system, the corresponding set $P(Q)$ is compact by Lemma 4(ii). Now we can apply Lemma 6 to estimate $h^4_{\mathrm{as}}(\pi^s(K) \cap \bar{\mathcal{R}},P(Q))$ from below: If the linear map $\bar{A}|_{\bar{\mathcal{R}}}$ induced by A for the observable system restricted to $\bar{\mathcal{R}}$ is totally unstable (i.e., all eigenvalues have positive real parts), the assertion immediately follows. Otherwise, one has to project this system to its unstable part along the center-stable subspace. Since, again, the asymptotic invariance entropy does not increase, the assertion also follows in the general case (see Step 3 in the proof of [3, Theorem 5.1] for these arguments.).

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