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Fractional diffusion without disorder in two dimensions

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We analyze how simple local constraints in two dimensions lead a defect to exhibit robust, nontransient, and tunable subdiffusion. We uncover a rich dynamical phenomenology realized in ice- and dimer-type models. On the microscopic scale, a single defect exhibits anomalously long retractions. Such retractions result in a form of dynamical caging and can be captured through an effective fractional diffusion equation. Mapping to a height field yields an effective random walk subject to an emergent (entropic) logarithmic potential, whose strength is tunable, related to the exponent of algebraic ground-state correlations. The defect's path, viewed as nonequilibrium growth process, yields a frontier of fractal dimension of $5/4$, the value for a loop-erased random walk, rather than $4/3$ for simple and self-avoiding random walks. Such frustration/constraint-induced subdiffusion is expected to be relevant to platforms, such as artificial spin ice and quantum simulators aiming to realize discrete link models and emergent gauge theories.

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I. INTRODUCTION

Diffusion is a ubiquitous and universal phenomenon: The theory of hydrodynamics posits that conserved quantities generically diffuse [1]. Deviations from this behavior can nonetheless occur, for example, as a result of disorder [2], fine-tuning of a system to the vicinity of an integrable [3,4] or critical point [5–7], or for more complex conservation laws [8,9]. Origin, nature, and robustness of the resulting subdiffusive, superdiffusive, or indeed ballistic transport are therefore natural objects of study. Such anomalous diffusion processes have found several applications in fields ranging from biology to finance [10].

Conserved quantities may be of fundamental origin—such as charge and mass of an elementary particle—or emergent properties: Simple lattice models of, say, Ising spins may yield cooperative behavior, the low-energy description of which includes an emergent conserved charge. Such models turn out to be quite commonly encountered in the study of frustrated and link models, whose low-energy description takes the form of an (emergent) $U(1)$ gauge theory. This includes familiar representatives such as the ice and hardcore dimer models [11,12].

Recent interest in, and progress toward, realizing gauge theories in quantum simulator platforms [13–20] calls for

a study of the dynamics of such emergent charges [21,22]. The basic question is how do the constraints encoded by the gauge theory manifest themselves in the hydrodynamic behavior of its conserved charges? In this work, we consider perhaps the simplest such model, a link model with ice or dimer constraints [12,23–26]. We find that the motion of defects here presents an alternative to the conventional diffusive hydrodynamics.

Concretely, we find a rich and tunable phenomenology of long-time subdiffusion, which results from the constraints imposed by the emergent gauge fields. At the microscopic level, we identify a dynamical caging effect that makes a connection to loop-erased random walks and fractional diffusion. On the level of long-wavelength descriptions, we identify a continuum theory in the form of height models and a mapping to diffusion in a logarithmic potential, a topic of recent interdisciplinary interest [27–33]. The tunable anomalous diffusion arises purely from the residual entropy of the ground-state manifold. Such dynamical phenomenology in the absence of any disorder, integrability, or fine-tuning to a critical point, in simple and paradigmatic models of geometric frustration, is a key result of our work.

Further, we find that the infinite-time subdiffusion for a single dynamical charge is modified in the presence of a finite density of such mobile charges. In this case, a length scale, corresponding to the average interparticle spacing, is introduced up to which subdiffusion persists and beyond which there is crossover to ordinary diffusion. Finally, we examine the problem through the more mathematically motivated lens of nonequilibrium frontier growth. We find that the frontier of the growing cluster sketched out by the conserved charge appears not to fall in the same universality class as a simple random walk and isoheight lines of Kardar-Parisi-Zhang. Instead, the frontier fractal dimension is consistent with $5/4$, the

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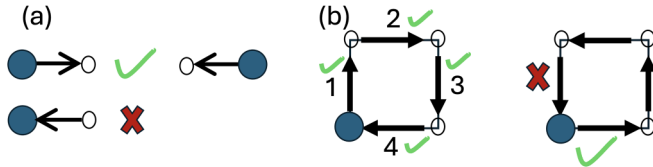


FIG. 1. (a) Constraint on defect (large blue circle) motion imposed by the Ising degrees of freedom (black arrows) on the links. A hop is allowed only if arrow points away from defect and neighboring site is empty. (b) Corresponding loop constraint: After a clockwise sequence of hops 1–4, the defect returns to its initial position, at which point the link variable flipped by its first hop blocks further clockwise hops.

value for the loop-erased random walk. Based on numerics and heuristic arguments, we conjecture that the scaling limit of the frontier of the conserved charge cluster is described by an SLE(2) conformally invariant curve, as opposed to SLE(8/3) (self-avoiding walk) for the conventional cases.

II. STOCHASTIC DYNAMICAL CHARGE HOPPING MODEL

We start with a random classical configuration with Ising degrees of freedom on links of a square lattice subject to a hard-core constraint at every vertex. Such constrained models can be represented as a gauge theory where every vertex satisfies a Gauss’ law constraint. One can represent these Ising variables as arrows along the links, and the corresponding vertex constraint refers to the difference between the number of incoming and outgoing arrows. Versions of this constraint include the ice rule of two arrows pointing toward (in) and two away (out) from every vertex, and the dimer constraint of only one dimer covering every site, represented as a 3 in-1 out or 3 out-1 in for the respective sublattices of a bipartite (two inequivalent sets of sites) lattice.

We consider the constrained classical hopping problem of a dynamical charge (defect). These are created by flipping an arrow chosen at random, resulting in two monopole defects for the ice problem and two monomer defects for the dimer problem. For clarity, we focus on the square ice problem for the rest of this paper but the general theory applies equally to the dimer or any similarly constrained model (see the discussion on implications for similar models for quantitative considerations). We fix one such monopole (defines the origin) and allow the other one to hop stochastically satisfying the condition presented in Fig. 1(a). While appearing simple, such a hopping corresponds to a many-body dynamical problem due to the hard-core constraints, which can be expressed in the language of an emergent gauge theory. Although the underlying degrees of freedom are Ising-like (take on one of two values), the emergent gauge symmetry is U(1). This constraint is best understood pictorially. As shown in Fig. 1(a), Gauss’ law imposes the constraint that the dynamical charge can only hop across a link if the arrow points along the direction of the hop. Besides its theoretical simplicity, such constrained hopping is relevant for the motion of defects (matter fields) in platforms such as artificial spin ice (quantum simulators realizing lattice gauge theories) [34,35]. Due to the Ising links

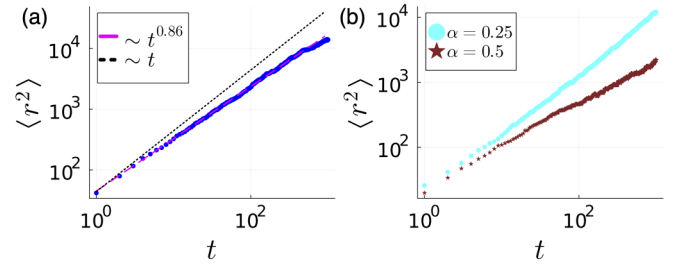


FIG. 2. (a) Subdiffusion of the defect (monopole) in square ice with $\langle r^2 \rangle \sim t^{0.86}$ (dashed pink line is the numerically obtained best fit to the data in blue), for a 400×400 lattice, where $\langle r^2 \rangle$ is in units of lattice spacing and t in units of attempted hops (Monte Carlo time). (b) Introducing a distribution of static charges (violations of the ice rule), parametrized by α , changes ground-state correlations and also the long-time subdiffusion exponent. See end matter for numerical details.

encoding a U(1) symmetry, such a hopping mimics that in the spin-1/2 U(1) quantum link model [36], where the constrained hopping term would be the three-body gauge-matter coupling.

For our classical analysis, in this work, we do not consider coherent four-spin plaquette flips. Such plaquette flips, in the absence of any dynamical charge, lead to interference-induced disorder-free localization with unique spectral signatures [21,22]. In the near future, the three-body terms are targets for quantum simulators aiming to realize U(1) gauge theories. Our analysis of such incoherent stochastic dynamics clearly separates the role of hard-core constraints from that of quantum coherence.

Dynamics of such frustrated models have been studied in a different context—purely relaxational (model A [37]) and local dynamics of stochastic plaquette flips [38]. Such stochastic dynamics always results, in the long-time limit, to standard diffusion, in which the equilibration time for the plaquette flips has a dynamical exponent $z = 2$. However, we show that the nonlocal nature of the fractionalized defects allows for a tunable dynamical exponent with asymptotic subdiffusion. The tunability can be exploited in novel synthetic platforms, which can tune lattice-level constraints.

III. TUNABLE LONG-TIME SUBDIFFUSION

First, we present numerical data for the microscopic dynamics of the monopole hopping in the ice background. As claimed, we find that the asymptotic dynamics (long time, long distance) of the monopole is subdiffusive. As seen in Fig. 2(a) (see parameter and other details of all numerical data for all figures, in Appendix D), the mean-squared displacement of the dynamical charge has a sublinear scaling with time, $\langle r^2 \rangle \sim t^\beta$, with $\beta = 0.86$, where t is in units of number of attempted hops in the Monte Carlo and $\langle r^2 \rangle$ is in units of the lattice spacing.

Such robust subdiffusion on a simple Euclidean lattice, in the absence of disorder and away from any fine-tuned critical point is, as we will highlight in this paper, a unique feature of such many-body models with hard-core constraints.

Further, in Fig. 2(b), varying lattice-level constraints by adding static charges can continuously change the long-time

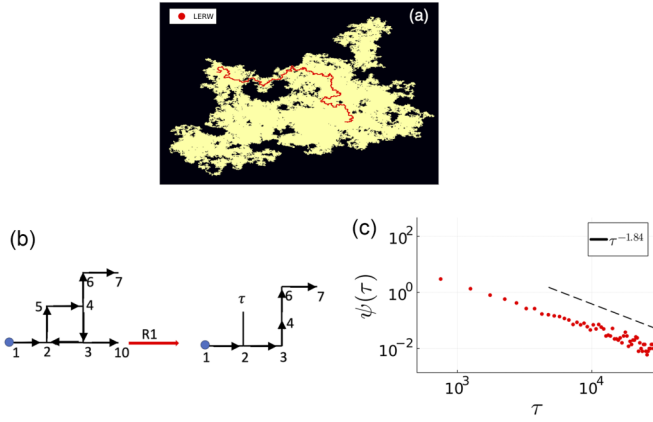


FIG. 3. (a) The trace of the defect's walk (yellow) and the corresponding LERW (red) (b) Effective description of a random walk $R1 \equiv (1, 2, 5, 4, 3, 2, 3, 4, 6, 7)$ (all these hops are allowed; see Fig. 1) as a loop-erased random walk with a waiting time τ . (c) Waiting time (erased loop-length) distribution, $\psi(\tau) \sim \tau^{-1.84}$, for the CTRW corresponding to the LERW of the dynamical charge for a 1000×1000 lattice.

subdiffusive exponent (see end matter for details on parametrization α). Our results establish such continuously varying dynamical exponents as another unique feature of such two dimensional constrained models (emergent gauge theories). In the following sections, we provide detailed theoretical explanations from varied perspectives for the numerically observed anomalous dynamics of the microscopic defect hop problem. While the numerics is for the exact microscopic dynamics, we will show that one can develop two equivalent descriptions of such anomalous diffusion through (1) an effective continuous time random walk framework and (2) a Langevin equation derived from the height-field theory of such frustrated magnets.

IV. DYNAMICAL CAGING AND FRACTIONAL DIFFUSION

First, we present an analytical, microscopic, and partly phenomenological description of the asymptotic subdiffusive dynamics of the defect using the formulation of an effective continuous-time random walk (CTRW) [10,39]. Such a lattice-level description with slight modifications is applicable to all fractionalized defects in such frustrated models.

A CTRW is governed by a random timescale, the waiting time between two jumps, τ , and a random length, the distance of the jump s . We consider the CTRW process in which these two random quantities are uncorrelated. We denote the corresponding probability density functions as $\psi(\tau)$ and $\phi(s)$. In situations where either $\langle \tau \rangle$ or $\langle s^2 \rangle$ diverges, anomalous diffusion may occur [10].

As demonstrated earlier, the Gauss' law constraint imposes a certain constraint of the random walk by not allowing the dynamical charge to circle around loops (of all length scales) along a single direction. As a result, any n -step random walk $r_w \equiv [r_0, r_1, \dots, r_n]$ can be modelled as an m -step random walk $r'_w \equiv [r_0, (m-2) \text{ pos}, r_n]$ with random waiting times $[\tau_1, \dots, \tau_{m-1}]$ (lengths of erased loops) as shown in Fig. 3(b) (see end matter for details). The reduced path r'_w is the

loop-erased random walk (LERW) of r_w , which is obtained by removing all loops from the walk as soon as each one is formed [40,41].

In Fig. 3(c), we plot the distribution of the waiting times on the sites of the LERW for the dynamical charge (monopole) in the square-ice background. We see that it has a power-law distribution whose asymptotic behavior is given by τ^{-z} , where $z \approx 1.84$. Hence, we can write

$$\psi(\tau) = \tau_0^\beta / \tau^{1+\beta}, \quad (1)$$

where τ_0 is some constant that is related to the generalized diffusion coefficient and $\beta \approx 0.84$. Importantly since $0 < \beta < 1$, Eq. (1) implies that the mean waiting time $\langle \tau \rangle$ diverges. Note, importantly, that there is no intrinsic waiting time in this problem; the statement on diverging mean waiting time implies that the monopole dynamics is characterized by anomalously long retractions onto the sites of the LERW [see Fig. 3(a)]. This is what we refer to as dynamical caging. Such an effective caging mechanism and the anomalous dynamics is purely due to the correlations within the ice manifold of square ice, as described below [12].

Further, one can show that a CTRW with waiting time distribution given in Eq. (1) can be represented by a modified Langevin equation that can be written (in the diffusion limit) as a fractional diffusion equation [10] (also see Appendix B)

$$\frac{\partial}{\partial t} P(x, t) = K_\alpha D_t^{1-\beta} \frac{\partial^2}{\partial x^2} P(x, t), \quad (2)$$

where K_α is the generalized diffusion coefficient, D_t is the Riemann-Liouville fractional operator acting on the spatial derivative of $P(x, t)$, and x is a position on the loop-erased walk. From the above equation, we get $\langle r^2 \rangle \sim t^\beta$, which agrees well with our numerics in Fig. 2(a).

V. CONTINUUM DESCRIPTION: HEIGHT MAPPING AND DIFFUSION IN LOGARITHMIC POTENTIAL

The microscopic picture provided a semianalytical description (requiring numerics on erased-loop size/waiting time distribution) of monopole subdiffusion, we next provide a completely analytical coarse-grained description of this dynamics.

This involves a so-called height model, obtained via a mapping (familiar from frustrated magnetism) to an interface represented by a scalar height function $h(\mathbf{r})$. Crucially, this fluctuates due to residual entropy of the ice or dimer manifold of states. The resulting action for this coarse-grained field theory is

$$S = \int \frac{K}{2} |\nabla h(\mathbf{r})|^2 + \dots, \quad (3)$$

where K is the stiffness of the height field and the additional terms are irrelevant (in the RG sense) for the critical (rough) phase corresponding to the critical ground-state correlations in the ice manifold (or dimer model) [42]. The presence of a monopole, corresponds to vortex-antivortex pairs (screw dislocations) in the height model. More importantly, in two dimensions, the interaction between a vortex and antivortex is then given by an attractive potential $V(r) = K \ln(r/a)/\pi$ [42], where the coefficient of the regularized logarithmic potential

arises from a combination of the height stiffness and the Burgers vector of the defect (defect charge). This potential is purely entropic in nature.

Such an analysis makes the difference between plaquette flips and the presence of a monopole clear: While plaquette flips affect the height of only neighboring plaquettes, a monopole (fractionalized defect) has a nonlocal effect through the logarithmic potential. Moreover, the macroscopic dynamics of our defect can be expressed as a Langevin equation with a logarithmic potential. Exact solutions of the respective Fokker-Planck equations have been obtained using tools such as infinite covariant densities [43,44]. The mean-squared displacement scales as $\langle r^2 \rangle \sim t^{1.5-K/\pi}$. From exact results of the ice model ground-state correlations on the square lattice, we know that $K = 2\pi/3$ and hence $\langle r^2 \rangle \sim t^{0.83}$, in good agreement with numerics in Figs. 2(a) and 3(c) and the analysis in the previous section. A similar exponent has been obtained from a scaling ansatz and a phenomenological consideration of the different types of noise in the ice manifold [45].

Moreover, diffusion in a logarithmic potential is an extremely rich problem, since as noted initially by Bray [28], the noise term in the Langevin equation is exactly marginal in the renormalization group sense. This leads to tunable (discussed in next section) dynamical exponents for subdiffusion in the long-time limit, as we see in Fig. 2(b). Most importantly, the dynamical exponent depends on the coefficient of the logarithmic potential, which in turn depends on static correlations of the ground state. If instead of a logarithmic potential one had a power-law potential, as in three-dimensional spin ice, then in the asymptotic limit one would recover standard diffusion (in the absence of any fine-tuning).

VI. IMPLICATIONS FOR HIGHER-SPIN AND SIMILAR MODELS: TUNABILITY AND UV-IR MIXING

Higher-spin versions of these link models in two-dimensional settings have been considered in the context of the antiferromagnetic Ising model on the triangular lattice, which has a dimer representation on the honeycomb lattice [46]. In the corresponding height-field theory, the stiffness has been shown to increase on increasing spin [46], implying a change in the coefficient of the logarithmic potential and hence, according to our analysis, a different subdiffusion exponent. Alternatively, adding interaction terms can also change the ground-state correlations and thereby the subdiffusion exponent. Our results therefore identify such frustrated systems as possibly the simplest examples of tunable subdiffusion. Moreover, as we see in Fig. 2(b), changing lattice-level constraints affects asymptotic dynamics. Hence, such systems also provide a platform for probing UV-IR mixing in two dimensions.

VII. FRONTIER GROWTH CONJECTURE AND SLE

We change perspectives and view the fractionalized defect hopping as a nonequilibrium growth process. The relevant growth is that of the frontier of the cluster sketched out by the monomer, defined as the boundary of the cluster with all the holes filled in. We analyze such growth through the lens of Schramm-Loewner evolution (SLE) [47].

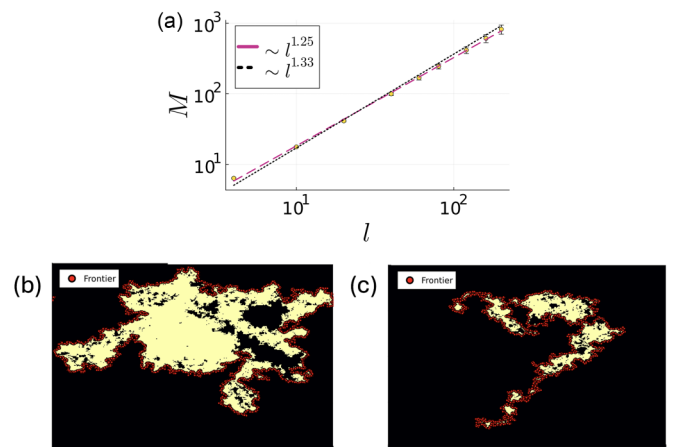


FIG. 4. (a) Fractal dimension of conserved charge motion frontier (in a 600×600 lattice) is “thinner” than that of standard Brownian motion, $d_f = 1.25 < 4/3$, just as the LERW is “thinner” than the SAW [40]. (b) Typical cluster of the conserved charge is more “space filling” than that of a simple random walk [panel (c)], as expected from duality of SLEs.

Due to their characterization of simple curves via a single parameter, κ , SLEs lend themselves naturally to describe growth in two dimensions [48]. Discrete interface growth models, such as ballistic deposition, restricted solid-on-solid, and Eden, models are all believed to fall into the two-dimensional Kardar-Parisi-Zhang (KPZ) [49] universality class with isoheight lines in these models converging (in the scaling limit) to SLE $(8/3)$. Moreover, the frontier of regular Brownian motion, as conjectured by Mandelbrot and proved later by Werner, Lawler, and Schramm using SLE methods [50], also converges to SLE $(8/3)$ —the universality class of the self-avoiding walk.

First, we obtain numerical data for the fractal dimension of the frontier in Fig. 4(a) (see end matter for details). The best linear fit to this data gives a fractal dimension $d_f \approx 1.25$ in agreement with the rigorously known growth exponent of the loop-erased random walk, $5/4$ [51,52]. Such a fractal dimension is also consistent with an SLE (2) description, since $d_f = 1 + \kappa/8$ for $\kappa \leq 8$.

Second, purely from visual inspection of Figs. 4(b) and 4(c), one can infer that the dynamical charge random walk is more “space filling” than that of a standard random walk. In SLE, this property corresponds to a higher diffusivity κ . In SLEs, there is a conjectured duality, first suggested by Duplantier, stating that frontier of the hull (cluster with all internal holes filled in) which scales as SLE (κ') is described by SLE $(16/\kappa')$ [53]. If the frontier has a lower diffusivity, then the corresponding hull is more space filling; hence, this simple observation already implies that the frontier growth is described by some $\kappa < 8/3$.

Besides, numerical observations we present a heuristic argument for the deviation from SLE $(8/3)$ behavior. Say that the dynamical charge starting at the origin, after time t is at the some point r_t . By definition, this point belongs to the frontier of the cluster at time t . Now, using the arguments presented in earlier sections, one can consider the growth of the cluster effectively as the growth of the loop-erased walk.

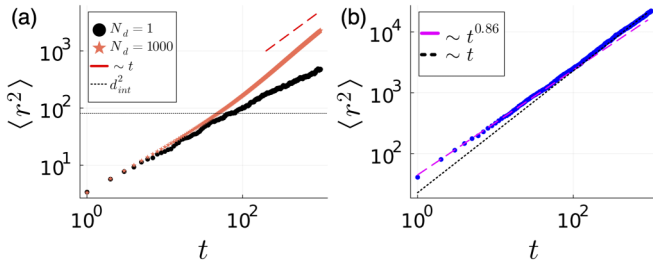


FIG. 5. Average mean-squared displacement for a finite density of defects (N_d = number of mobile defects), displaying subdiffusion upto average interparticle spacing ($\sim$ Debye screening length). (a) For an initial configuration drawn from the ensemble generated by $\alpha = 0.5$ on a 400×400 lattice [see Fig. 2(b)] and numerical details in end matter). (b) For an initial configuration drawn from the ice manifold with $N_d = 1000$ on a 1000×1000 lattice. Panel (a) visualizes the crossover to diffusion in a clearer fashion due to the greater difference in long-time exponents, as compared to that in square ice.

Unlike standard Brownian motion, the self-avoided walk or random percolation clusters, the loop-erased random walk is nonlocal, i.e., heuristically, the growth of the walk “feels” the effect of the erased loops at all length scales. Hence, the growth of the frontier in our problem is described by a new universality class, different from standard 2d KPZ and SLE (8/3).

VIII. FINITE DENSITY OF DEFECTS

We now consider the dynamics of a finite density of defects. Our coarse-grained macroscopic description would propose that as long as the defects feel the entropic logarithmic potential, the subdiffusion persists. This occurs until the mean-squared displacement is on the order of the average interparticle spacing. Beyond such a regime, an analog of Debye screening [54] would restore diffusion, as then the problem would reduce to standard Brownian motion.

As we see in Fig. 5 for both the monopole in square ice case [5(b)] and for the defect in the presence of static charges [5(a)], the average mean-squared displacement per particle displays subdiffusive behavior up to a distance (and the corresponding timescale) on the order of the average interparticle spacing. Beyond this, there is a smooth crossover to standard Brownian motion.

IX. DISCUSSION AND OUTLOOK

We have shown that a dynamical charge subject to a gauge constraint in two dimensions, arising naturally in discrete link models, can lead to anomalous dynamics and nonequilibrium growth. Such dynamics arises due to critical correlations of the ground-state manifold in such models, which induce anomalously long retractions and a form of dynamical caging. Besides rich theoretical phenomenology, such systems are being realized in various cutting-edge quantum simulation as well as artificial spin-ice systems. We highlight this more below in the end matter. Quantum extensions of the above problem have interesting consequences for nonergodicity as the Hilbert space configuration can be well approximated

by Galton-Watson trees, i.e., a tree with disorder in coordination number. Such a Hilbert space structure could be amenable to realizing nonergodicity in infinite volume. Further, three-dimensional extensions of the above work would be interesting theoretically as well as experimentally from the point of view of frustrated magnets such as pyrochlore spin ice.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The raw data that support the findings of this article is openly available at [55].

APPENDIX A: EXPERIMENTAL PLATFORMS

Remarkable recent progress has been made in realizing the PXP model in two-dimensional platforms using the Rydberg blockade mechanism. To observe such diffusion, we propose an implementation in which the Rydberg atoms are placed on the links of a honeycomb lattice. The Rydberg blockade mechanism is then implemented by choosing a blockade radius of $\sqrt{3}a/2$ (a —lattice constant), which generates dimer-type constraints, as has recently been achieved [15]. Another promising platform for the observation of such emergent phenomena is artificial spin ice—arrays of nanomagnetic islands in two dimensions [56,57]. Moreover, recent experiments have also demonstrated the ability to do real-time imaging of defect motion [58].

Finally, there is a whole host of other platforms, such as optical lattices in ultracold atoms, superconducting circuits, etc., which can realize lattice gauge theories. There has been impressive recent progress in such pursuits with a large push to realize two-dimensional lattice gauge theories in the laboratory [13–20]. Our results would be a concrete test to see if such emergent gauge theories, and thereby the ensuing rich dynamical phenomena, are indeed realized in such settings.

APPENDIX B: CONTINUOUS TIME RANDOM WALK FORMALISM

We provide some more details of the CTRW formalism that arises as an effective description for our fractionalized charge hopping problem, resulting in an effective fractional diffusion equation in the asymptotic limit.

This description emerges as follows: Due to the hopping constraint depicted in Fig. 1(b), every time the charge traverses a loop (of arbitrary size) it cannot continue in the same clockwise/anticlockwise direction. The available options are to exit the loop or to retrace along the path it took to complete the loop. Hence, a natural effective description of such

dynamics is to erase every loop, as soon as it is formed in the trajectory of the dynamical charge, and introduce a waiting time τ at the point of erasure (the point where the loop is first formed).

The dynamics is then described as follows: The charge moves along the loop-erased path (formed by erasing every loop chronologically, i.e., as soon as a loop is formed in the trajectory) but with a waiting time on every site of this path (time spent at this site before jumping to the next site) characterized by the length of the erased loop. Such a dynamics represents that of a CTRW. As mentioned in the main text, the CTRW is characterized by two quantities: the mean waiting time $\langle \tau \rangle$ and the step-size variance $\langle s^2 \rangle$. These quantities are expressed as

$$\begin{aligned}\langle \tau \rangle &= \int_0^\infty \tau \psi(\tau) d\tau, \\ \langle s^2 \rangle &= \int_{-\infty}^\infty s^2 \phi(s) ds,\end{aligned}\quad (\text{B1})$$

where $\psi(\tau)$ and $\phi(s)$ are the waiting time and step-size distributions. In our constrained hopping problem, the step size is fixed to the lattice spacing. The crucial step in our CTRW modeling is representing the erased-loop sizes as effective waiting times. There is no in-built waiting time in our problem. However, due to the constraints mentioned in the main text and above, the erased loop distribution acts as an effective waiting time distribution, giving a waiting time to every site of the loop-erased walk and thereby allowing us to use CTRW methods to model the asymptotic subdiffusion.

As we saw in the main text, the waiting time (erased-loop size) distribution is obtained numerically, and is a heavy-tailed distribution, i.e., $\psi(\tau) \sim \tau^{-1.84}$. Such heavy-tailed distributions of the form $\sim \tau^{-(1+\beta)}$, for $0 < \beta < 1$, result in diverging mean waiting times. Such a diverging mean waiting time, as mentioned in the main text for our hopping problem, represents a dynamics characterized by anomalously long retractions to points on the loop-erased path—dynamical caging.

Starting from such a heavy-tailed distribution for the waiting times, one can show that the mean-squared displacement is given by $\langle r^2 \rangle \sim t^\beta$. Further, one can obtain the continuum stochastic equation describing the asymptotic dynamics in the form of a fractional diffusion equation of the form

$$\begin{aligned}\frac{\partial}{\partial t} P(x, t) &= K_\alpha D_t^{1-\beta} \frac{\partial^2}{\partial x^2} P(x, t), \\ D_t^{1-\beta} &= \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t \frac{P(x, t')}{(t-t')^{1-\beta}} dt',\end{aligned}\quad (\text{B2})$$

where D is known as the Riemann-Liouville fractional operator. The above equation reduces to the standard diffusion equation in the limit $\beta \rightarrow 1$. Thus, while the asymptotic dynamics of a simple random walk results in a continuum diffusion equation, the asymptotic dynamics of such fractionalized defects in frustrated models result in effective fractional diffusion equations.

APPENDIX C: HEIGHT MODELS AND LOGARITHMIC POTENTIAL

Besides the CTRW, one can also use the height model to obtain a coarse-grained and fully analytical description of the asymptotic subdiffusion. Further, the height description elucidates the role of the ground-state correlations in the dynamics.

The height model describes the residual entropy of a class of frustrated models in terms of a rough surface (see Ref. [42] for a very clear exposition). The height field lives on the dual lattice, which in our case is defined by the centers of the square plaquettes. The height difference between adjacent plaquettes is determined by the direction of the arrow on the link shared by them for the ice model, or the presence/absence of a dimer on the shared link for the dimer model. The prescription for the height differences is chosen such that the net height difference for any contractible closed loop on the dual lattice vanishes.

Such a microscopic map identifying ice/dimer with a unique height configuration encodes the hard-core constraint of the ice/dimer model. This microscopic height field is then amenable to coarse graining into $h(r)$, to describe the long-wavelength physics. This results in Eq. (3), a simple Gaussian field theory. Other terms are irrelevant (in an RG sense) in the rough phases of such height models that capture the residual entropy of dimer and ice models in the regime we are interested in.

The fluctuations of this height field have a logarithmic divergence, i.e., $\langle |h(0) - h(r)|^2 \rangle \approx \ln(r/a)/(\pi K)$ (a is the lattice spacing), from which one can calculate the Ising correlators as well. The height description also incorporates the description of defects/excitations. A single spin flip defect creates two monopoles, which when separated by a distance r are described as two screw dislocations of opposite vorticity in the height description. This highlights the effectively non-local nature of the defect. While a plaquette-flip excitation only changes the height locally, the monopole at the origin leads to a long-range disturbance following a $1/r$ Gauss' law. As a result, two monopoles when separated by a distance r , have an entropic attractive potential $V(r) = V_0 \ln(r/a)$. The coefficient of this potential is determined by the Burgers vector of the screw dislocation (“magnetic charge” of the defect in the field-theory language) and the stiffness of the height field. The latter quantity is directly related to the defect-defect correlations (monopole/monomer or others) in the rough phase. From known exact results in square ice, one finds that $V_0 = 2/3$, which as shown in the main text gives us $\langle r^2 \rangle \sim t^{0.83}$. Since the coefficient of the log-potential can be tuned continuously (e.g., by adding an interaction between dimers), the subdiffusion exponent is also straightforwardly continuously variable.

APPENDIX D: NUMERICAL DETAILS FOR FIGURES

Figure 2(a): We start with a random state from the ice manifold for a 400×400 square lattice. We then choose a link at random and flip it. This creates a “charge” at a 3 out-1 in vertex and another at a 3 in-1 out vertex. We fix the latter and let the former hop according to the rules laid out in the main

text. The mean-squared displacement in Fig. 2(a) is obtained by averaging over 500 different initial states, where for each state we perform stochastic hops of the monopole. Each time step is defined as 100 attempted hops in the Monte Carlo procedure, and $\langle r^2 \rangle$ is in units of the lattice spacing. We then plot the best fit for the data shown in the main text, which gives us an exponent of 0.86. The nominal statistical error from the fit of this quantity is tiny (in the third decimal digit), so that the actual error, which we estimate to be around 0.03, is dominated by systematic issues, such as possible corrections to the asymptotic behavior.

Figure 2(b) calculates $\langle r^2 \rangle$ in the same way, but instead of starting from a random configuration in the ice manifold, the initial state contains a density of defects violating the ice rule parametrized by α . On each link, we assign a probability $\sin^2(\alpha + \pi/4)$ of the link variable being the same as it would be if the arrows were arranged in one of the two clockwise/anticlockwise checkerboard configurations, and a probability $\cos^2(\alpha + \pi/4)$ of the link variable being the same as it would be if the arrows were arranged in the other possible one. Each value of α gives us an ensemble of configurations, and we average over 500 different initial configurations for each α .

Figure 3(c) analyzes the resulting statistical properties of the trajectory of the mobile defect. The loop-erased path and the corresponding distribution of erased loops are obtained. The length distribution of the latter yields the effective waiting time, τ , distribution, $\psi(\tau)$. Data are shown for a 1000×1000 square lattice, keeping track of the trajectory for 10 000 time units, i.e., 10^6 attempted hops. For the plot, we bin the data in bins of length 500 (in units of τ) and plot the

average of all data corresponding to all the τ s in each bin. We fit the range of data covered by the dashed curve to focus on the tail.

Figure 4(a): Again starting from a random-ice state, we record the cluster sketched out by the monopole after 65 000 time units (65×10^5 attempted hops) in a 600×600 lattice. The cluster is defined as the connected network of all bonds of the lattice across which the monopole has hopped at least once. We identify the frontier of the cluster by recording the outermost boundary of the cluster, i.e., the perimeter with all the holes filled in [see Fig. 4(b)]. We calculate the local fractal dimension of the frontier using the box-counting method, where we choose a point on the frontier at random as the center of the box and then calculate the number of frontier points, M , that lie within a square of length l . We choose $l = [4, 10, 20, 40, 60, 80, 120, 160, 200]$ and obtain the fractal dimension using the relation $M \sim l^{d_f}$. Both the stochastic and systematic errors we estimate to be around 1%–2% in the last digit.

Figure 5: In panel (a), we use the same parametrization as Fig. 2(b) with a random state from the ensemble generated by $\alpha = 0.5$ on a 400×400 lattice, and measure $\langle r^2 \rangle/N_d$ after averaging over 500 initial states (N_d = number of mobile defects). We choose N_d random vertices that satisfy the ice rule, flip a bond (chosen randomly) attached to the site, and then perform the same procedure as in Fig. 2, for the constrained hopping. Here, time is in units of $10N_d$ attempted hops. In panel (b), we do the same as panel (a) for $N_d = 1000$, but now starting from a random ice configuration on a 1000×1000 lattice as opposed to configurations in the ensemble defined by $\alpha = 0.5$ in panel (a).

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