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# ASYMPTOTIC COMPACTNESS OF STOCHASTIC COMPLEX GINZBURG-LANDAU EQUATION ON AN UNBOUNDED DOMAINS

DIRK BLÖMKER AND YONGQIAN HAN

**ABSTRACT.** The Ginzburg-Landau-type complex equations are simplified mathematical models for various pattern formation systems in mechanics, physics, and chemistry. In this paper, we show that the complex Ginzburg-Landau (CGL) equations in  $\mathbb{R}$  perturbed by an additive space-time white noise generates an asymptotically compact stochastic or random dynamical system in weighted  $L^2$ -spaces. This is a crucial property for the existence of a stochastic attractor for such CGL equations.

## 1. INTRODUCTION

This paper is concerned with attractors of semilinear stochastic partial differential equations (SPDEs) on unbounded domains. This is the topic of many recent papers. See for example the work of Bates, Lu, and Wang [4, 47, 48] or related for systems on unbounded lattices [5, 3], or the work by Brzezniak and Li [8, 7] or [9].

Usually, results on random attractors on unbounded domains assume some decay condition of solutions at infinity, which rules out the possibility of perturbing the equation with translation invariant noise, like space-time white noise. Here one expects the solution to be unbounded at infinity in space. In [8] estimates in  $L^2$  or  $L^4$  force a decay of solutions at infinity, while for example in [4] explicit far field estimates are used, in order to show that the solution decays in space at infinity.

For simplicity of presentation we focus in this article only on the stochastic complex Ginzburg-Landau (CGL) equation, which is an important model equation in the description of spatial pattern formation and of the onset of instabilities in non-equilibrium fluid dynamical systems (see [6, 14]). For the deterministic CGL, a large amount of work has been devoted to the study of the well-posedness of solutions, the global attractors and the related dynamical issues, see references [2, 11, 12] [16]–[27] [31]–[33] [36]–[38] [49, 50] and therein references.

Recently, stochastic complex Ginzburg-Landau equations are extensively used in physics, and frequently appear in the study of dynamical critical phenomena. For instance, for a statistical-mechanical system they may describe the time evolution of order parameter [40]. On bounded domains the well-posedness of solutions and the existence of global attractors for SCGL have been investigated in [28, 51]. The existence of invariant measures for stochastic Ginzburg-Landau equations on an unbounded domain has been studied in [42].

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Using weighted  $L^2$ -spaces (cf. for example also [10]), with a class of weights that allow for algebraic growth at infinity, we are able to show the asymptotic compactness of the stochastic dynamical system in our weighted spaces. But in order to establish this, we need to rely both on nonlinear stability given by a non-linearity with superlinear growth and the regularizing effect of the dominant linear differential operator.

Although our method of proof is quite general, for simplicity of presentation we consider only additive space-time white noise as an example for translation invariant noise. The main assumptions for the noise are verified in Lemma 3.1. We mainly need the pathwise regularity of the stochastic convolution, which is sometimes a delicate question, especially, if the covariance operator of the underlying Wiener process and the differential operator in the equation do not commute.

Also for simplicity of presentation, we focus on stochastic dynamical systems (SDS) instead of random dynamical systems (RDS). Nevertheless our main results on asymptotic compactness (AC) hold for RDS, too.

We will continue the line of research introduced in [30, 39, 41] or for stochastic systems by [13, 8]. The key of the approach, in order to obtain AC of the SDS related to SCGL equations (1.1) (1.2), are energy type estimates. It is well known that the new ingredient necessary for stochastic PDEs is the transformation of the SCGL equation (1.1) into a partial differential equation with random coefficients such that this random equation is independent of initial time  $\tau$ , but now initial data depends on  $\tau$ . Our main new feature is the use of weighted  $L^2$ -spaces in order to allow for space-time white noise.

To be more precise in the following, the stochastic complex Ginzburg-Landau (SCGL) equation is given by

$$(1.1) \quad du - \{a\partial_y^2 u + \chi_0 u - b|u|^{2\sigma} u\}dt = dW(t),$$

$$(1.2) \quad u(y, t)|_{t=\tau} = u_0(y).$$

Here  $u$  is a complex valued function of  $(y, t) \in \mathbb{R} \times \mathbb{R}_+$ , complex constants  $a = a_1 + ia_2$ , real constants  $a_1 > 0$ ,  $b > 0$ ,  $\sigma > 0$  and  $\chi_0 \in \mathbb{R}$ ,  $W(t)$  is a suitable cylindrical Wiener process, defined later on.

As phase space for some fixed  $\rho > \frac{1}{2}$  we consider the equation in the scale of weighted Hilbert spaces given by

$$L_\mu^2 = L^2(\mathbb{R}, d\mu) \quad \text{with respect to the measure } d\mu = m(y)dy, \text{ where } m(y) = (1+|y|^2)^{-\rho}.$$

The spaces  $L_\mu^p$  are defined analogously. This will allow for an algebraic growth of solutions in space, but we aim at growth as slow as possible.

We will see later (cf. Lemma 3.1) that the condition  $\rho > \frac{1}{2}$  is necessary, as only then our stochastic convolution is in  $L_\mu^2$ . Moreover, it is easy to check, that  $L_\mu^p$  is continuously embedded into  $L_\mu^2$  for  $p > 2$ , as  $\mu(\mathbb{R}) < \infty$  for  $\rho > \frac{1}{2}$ .

For simplicity we consider only space time-white noise for the random perturbation. This is given by a standard cylindrical Wiener process  $W$  on  $L^2$ . On some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  endowed with a filtration  $(\mathcal{F}_t)_{t \geq 0}$

$$W(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j,$$

where  $\{e_j\}$  is any orthonormal basis of the standard  $L^2$ -space and  $\{\beta_j\}_{j \in \mathbb{N}}$  is a sequence of mutually independent real valued two-sided Brownian motions on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let us

remark, that  $\{m^{-1/2}e_j\}$  is an orthonormal basis in  $L_\mu^2$ , and thus we can consider  $W$  as a cylindrical Wiener process in  $L_\mu^2$ , too.

In order to be self-contained, we first establish the global well-posedness of the solution for SCGL given by (1.1) (1.2). In our main result we prove AC of the SDS. To be more precise we establish AC for bounded sets. The AC for tempered sets, which is crucial for the existence of a random attractor will be proved in a following work. Finally, we comment on the existence of a global stochastic and thus random attractor  $\mathcal{A}$ , which will be independent of the weight chosen.

Analogously to [13], we are working in the framework of a general SDS on a separable Banach space. For RDS our results are similar to the results of [8], and we could establish the existence of invariant measures based on random  $\omega$ -limit sets for RDS. But for the existence of a random attractor we would need AC for tempered sets, as stated in [3].

Throughout this paper, in order to simplify the exposition, the space  $L^p(\mathbb{R}, d\mu)$  is denoted by  $L_\mu^p$  for any  $1 \leq p < \infty$ , the inner product of  $L_\mu^2$  is denoted by:

$$(f, g)_\mu = \int_{\mathbb{R}} f \bar{g} d\mu,$$

The space  $H^m(\mathbb{R}, d\mu)$  is denoted by  $H_\mu^m$  for all  $m \geq 1$ .

Different positive constants are all denoted by the same letter  $C$ . If necessary, we denote by  $C(\cdot, \cdot)$  a constant depending only on the quantities appearing in parenthesis.

The outline of this paper is as follows. In Section 2, we recall some appropriate concepts and tools from the theory of SDS. In Section 3, we prove the existence of the stochastic flow (and hence of the corresponding SDS) associated with SCGL equations (1.1) (1.2). In Section 4, we show the asymptotic compactness of the SDS corresponding to SCGL and in the final section 5, we comment on global stochastic or random attractors, which are independent of the weight chosen.

## 2. NOTATION AND PRELIMINARIES

In order to investigate the long time dynamics of SCGL (1.1) under the influence of white noise, we need some appropriate concepts and tools from the theory of *stochastic dynamical systems* (see [1, 13]). For simplicity of presentation, we restrict the results to SDS and attractors for bounded deterministic sets. Nevertheless, the results also hold for attractors of tempered random sets for a RDS.

Let  $H$  be a complete separable metric space and  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. We consider a family of mappings  $S(t, \tau; \omega) : H \longrightarrow H$ ,  $(t, \tau) \in \mathbb{R}_\Delta = \{(t, \tau) \mid -\infty < \tau \leq t < \infty\}$ , parameterized by  $\omega \in \Omega$ , satisfying for  $\mathbb{P}$ -a.e.  $\omega$  the following properties:

- (i)  $S(t, \tau; \omega)x = S(t, r; \omega) \circ S(r, \tau; \omega)x$  and  $S(\tau, \tau; \omega)x = x$  for all  $\tau \leq r \leq t$  and  $x \in H$ ,
- (ii)  $S$  is  $(\mathcal{B}(\mathbb{R}_\Delta) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H))$ -measurable.

A SDS is said to be continuous, if and only if  $S(t, \tau; \omega)$  is continuous in  $H$  for all  $(t, \tau) \in \mathbb{R}_\Delta$  and for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ .

Below we state the concept of a stochastic (global) pullback attractor of bounded deterministic sets for an SDS (see, e.g., [1, 13] and the references therein), which extends

the corresponding definition of a global universal attractor in autonomous systems (cf. [29, 43], for example).

Following [13], we define:

**Definition 2.1.** *Given  $t \in \mathbb{R}$  and  $\omega \in \Omega$ , we say that the bounded set  $K(t, \omega) \subset H$  is an attracting set if for all bounded sets  $B \subset H$ ,*

$$d(S(t, \tau; \omega)B, K(t, \omega)) \rightarrow 0 \quad \text{as } \tau \rightarrow -\infty.$$

**Definition 2.2.** *Let  $A(t, \omega)$  be a nonempty compact set for all  $t \in \mathbb{R}$ . This set is called a stochastic pullback attractor if it attracts all bounded sets from  $-\infty$ , and it is the minimal closed set with this property. Moreover, it is invariant in the sense that*

$$S(t, \tau; \omega)A(\tau, \omega) = A(t, \omega), \quad \forall \tau \leq t.$$

The first result on existence of stochastic attractors is given by the following theorem from [13].

**Theorem 2.3.** *Assume that the SDS has a compact attracting set  $K$ . Then, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , the SDS exhibits a stochastic pullback attractor.*

Similar results hold true for random attractors of RDS that are AC and exhibit a bounded (or tempered) absorbing set. Let us remark, that absorbing sets are a special type of an attracting set. But as our approach is based on energy estimates, both properties (absorbing and attracting) are established in a quite similar way.

We will discuss the following two types of AC, in contrast to [13] where the existence of a compact absorbing set defines AC.

**Definition 2.4.** *An SDS is called AC for bounded sets, if for all bounded sequences  $\{u_n\}_{n \in \mathbb{N}} \subset H$  and all families of times  $\{\tau_n\}_{n \in \mathbb{N}}$  with  $\tau_n \rightarrow -\infty$  for  $n \rightarrow \infty$ , the family  $\{S(t, \tau_n, \omega)u_n\}_{n \in \mathbb{N}}$  has a convergent subsequence.*

**Definition 2.5.** *An SDS is AC for tempered sets, if for all families of times  $\{\tau_n\}_{n \in \mathbb{N}}$  with  $\tau_n \rightarrow -\infty$  for  $n \rightarrow \infty$  and all sequences  $\{u_n\}_{n \in \mathbb{N}}$  in  $H$  such that  $\lim_{n \rightarrow \infty} e^{\epsilon \tau_n} u_n = 0$  for all  $\epsilon > 0$  one has that the family  $\{S(t, \tau_n, \omega)u_n\}_{n \in \mathbb{N}}$  exhibits a convergent subsequence.*

In our main result we verify that SCGL is AC for bounded sets. As our attracting set is not uniformly bounded in time, we would need AC for tempered sets, in order to conclude the existence of a stochastic or random attractor. This is postponed to a future work.

### 3. A PRIORI ESTIMATES, EXISTENCE AND UNIQUENESS OF SOLUTION

For the existence of global solution in time and for results on AC, in this section a priori estimates are established. We introduce the stationary Ornstein-Uhlenbeck (OU) process  $\eta(y, t, \omega)$  in  $L_\mu^2$ , in order to transform the SCGL (1.1) into a random partial differential equation. Then a priori estimates can be obtained by standard methods.

**3.1. Stochastic Convolution.** The OU-process  $\eta$  is defined as

$$\begin{aligned} \eta(t, \omega) &:= \left( \int_{-\infty}^t e^{(t-s)(a\partial_y^2 - k)} dW(s) \right)(\omega) \\ (3.1) \quad &:= \left( \int_{-\infty}^t S(t-s) dW(s) \right)(\omega), \end{aligned}$$

where  $W$  is our Wiener process defined in Section 1. The process  $\eta$  is the stationary solution of the following linear stochastic partial differential equation

$$(3.2) \quad \frac{\partial \eta}{\partial t} = \{a\partial_y^2 - k\}\eta + \dot{W},$$

with some *control* parameter  $k > 0$ . This is important, as otherwise the existence of a stationary solution is not obvious.

In the following, we also use a different representation given by a two-sided Brownian Sheet  $\mathcal{B}(t, x)$ ,  $t > 0$ ,  $x \in \mathbb{R}$ . From integration with respect to  $\mathcal{B}$  (cf. Walsh [46]) we obtain in law

$$(3.3) \quad \eta(t) = \int_{-\infty}^t \int_{\mathbb{R}} G_{t-s}(x-y) \mathcal{B}(ds, dy),$$

where

$$G_t(x) = \frac{1}{2\sqrt{\pi|a|t}} e^{-\frac{x^2}{4at} - kt}$$

is the Greens-functions corresponding to the semigroup  $S$ , such that  $[S(t)f](x) = \int_{\mathbb{R}} G_t(x-y)f(y)dy$ . Now using Gaussianity of  $\eta$  and Itô-formula (see for example [20, 46]) we obtain:

$$(3.4) \quad \begin{aligned} \mathbb{E}\|\eta(t)\|_{L_\mu^p}^p &= \int_{\mathbb{R}} \mathbb{E} \left[ \int_{-\infty}^t \int_{\mathbb{R}} G_{t-s}(x-y) \mathcal{B}(ds, dy) \right]^p m(x) dx \\ &\leq \int_{\mathbb{R}} \left[ \mathbb{E} \left( \int_{-\infty}^t \int_{\mathbb{R}} G_{t-s}(x-y) \mathcal{B}(ds, dy) \right)^2 \right]^{p/2} m(x) dx \\ &\leq \int_{\mathbb{R}} \left[ \int_{-\infty}^t \int_{\mathbb{R}} G_{t-s}(x-y)^2 dy ds \right]^{p/2} m(x) dx \\ &= \|m\|_{L^1} \|G\|_{L^2([0,\infty) \times \mathbb{R})}^{p/2} \\ &< \infty, \end{aligned}$$

as

$$\begin{aligned} \|G\|_{L^2([0,\infty) \times \mathbb{R})}^2 &= \int_0^\infty \int_{\mathbb{R}} \frac{1}{4\pi|a|t} e^{-\frac{a_1 y^2}{2|a|^2 t} - 2kt} dy dt \\ &= \int_0^\infty \frac{\sqrt{2}}{4\sqrt{\pi a_1} t} e^{-2kt} dt = C. \end{aligned}$$

Now it is easy to check, that for all  $a, b$ , and  $T$

$$\mathbb{E} \int_a^b \|\eta(t)\|_{L_\mu^p}^p dt < \infty \quad \text{and} \quad \mathbb{E} \int_{-\infty}^{T+1} e^{t-T} \|\eta(t)\|_{L_\mu^p}^p dt < \infty.$$

Moreover, from Kolmogorov continuity criterion theorem or Theorem 5.16 in [9, p. 134], we find that additionally  $\eta(\cdot, \omega) \in C_{\text{loc}}(\mathbb{R}; L_\mu^p)$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ .

We proved the following Lemma, which is the main and only regularity assumption on  $\eta$ , that we will use in the following sections.

**Lemma 3.1.** *For the OU-process  $\eta$  defined in (3.1), for all  $p \geq 1$ ,  $T \in \mathbb{R}$ , and all bounded  $I \subset \mathbb{R}$*

$$(3.5) \quad \begin{aligned} \mathbb{P}(\eta \in L^p(I, L_\mu^p) \cap C_{\text{loc}}^0(\mathbb{R}, L_\mu^p)) &= 1 \\ \mathbb{P}\left(\int_{-\infty}^{T+1} e^{t-T} \|\eta(t)\|_{L_\mu^p}^p dt < \infty\right) &= 1 \end{aligned}$$



**3.2. Energy-type estimates.** We introduce the new variable

$$(3.6) \quad \tilde{u}(t) := u(t) - \eta(t, \omega),$$

where the stationary process  $\eta$  solves the problem (3.2). Now from the standard method relying on the mild formulation, we obtain the following random partial differential equation for the new variable  $\tilde{u}$ :

$$\partial_t \tilde{u} - a \Delta \tilde{u} - \chi_0 \tilde{u} + b|\tilde{u} + \eta|^{2\sigma}(\tilde{u} + \eta) = (k + \chi_0)\eta.$$

We consider the equation path-wise and treat  $\eta$  as a known process. For convenience, we drop the tilde and rewrite the equation above. Thus, we finally get

$$(3.7) \quad \partial_t u - a \Delta u - \chi_0 u + b|u + \eta|^{2\sigma}(u + \eta) = (k + \chi_0)\eta$$

with initial data

$$(3.8) \quad u(t)|_{t=\tau} = u_0 - \eta(\tau, \omega) \in L_\mu^2,$$

where  $\eta$  is the stationary solution to (3.2).

Now we work on this random partial differential equations.

**Lemma 3.2** (Energy Estimates). *Let  $u_0 \in L_\mu^2$  and  $\sigma > 0$ . Then for all sufficiently smooth solutions  $u$  of (3.7) and (3.8), the following estimate is valid:*

$$(3.9) \quad \begin{aligned} & \|u(T)\|_{L_\mu^2}^2 + \int_T^{T+1} \left( \|\nabla u(t)\|_{L_\mu^2}^2 + \|u(t)\|_{L_{\mu_1}^{\sigma_1}}^{\sigma_1} \right) dt \\ & \leq C e^{\tau-T} \|u_0 - \eta(\tau)\|_{L_\mu^2}^2 + C \int_\tau^{T+1} e^{t-T} \{1 + \|\eta(t)\|_{L_\mu^2}^2 + \|\eta(t)\|_{L_{\mu_1}^{\sigma_1}}^{\sigma_1}\} dt, \quad \forall T \geq \tau, \end{aligned}$$

$$(3.10) \quad \|\partial_t u(t)\|_{H'(\tau, T)} \leq r_0(T, \tau, u_0, \eta), \quad \forall T \geq \tau,$$

where the constant  $r_0$  can be calculated explicitly, and the constant  $C$  is independent of  $T$  and  $\tau$ ,  $\sigma_1 = 2\sigma + 2$ , and  $H'(\tau, T)$  is the dual space of  $H(\tau, T) = L^2(\tau, T; H_\mu^1) \cap L^{2\sigma+2}(\tau, T; L_\mu^{2\sigma+2})$ .

*Proof.* Multiplying (3.7) by  $(1 + |y|^2)^{-\rho} \bar{u}$ , integrating by parts and considering the real part in the resulting identity, we get that

$$(3.11) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L_\mu^2}^2 + a_1 \|\nabla u(t)\|_{L_\mu^2}^2 - \chi_0 \|u(t)\|_{L_\mu^2}^2 + b \|u + \eta\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} \\ & = \text{Re} \int_{\mathbb{R}} (2a\rho(1 + |y|^2)^{-1} \bar{u} y \cdot \nabla u + b|u + \eta|^{2\sigma}(u + \eta) \bar{\eta} + (k + \chi_0) \eta \bar{u}) d\mu. \end{aligned}$$

By using Hölder's inequality, we easily prove the following inequalities

$$(3.12) \quad |(k + \chi_0) \int \eta \bar{u} d\mu| \leq \frac{1}{2} \|u\|_{L_\mu^2}^2 + C \|\eta\|_{L_\mu^2}^2,$$

$$(3.13) \quad |b \int |u + \eta|^{2\sigma}(u + \eta) \bar{\eta} d\mu| \leq \frac{1}{4} b \|u + \eta\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} + C \|\eta\|_{L_\mu^{2\sigma+2}}^{2\sigma+2},$$

$$(3.14) \quad \left| \int 2a\rho(1 + |y|^2)^{-1} \bar{u} y \cdot \nabla u d\mu \right| \leq \frac{1}{2} a_1 \|\nabla u\|_{L_\mu^2}^2 + \frac{2|a\rho|^2}{a_1} \|u\|_{L_\mu^2}^2.$$

Taking  $M \gg 1$  sufficiently big, we obtain for  $|u| \geq M$  that

$$(\chi_0 + 1 + 2|a\rho|^2/a_1)|u|^2 \leq (|\chi_0| + 1 + 2|a\rho|^2/a_1)M^{-2\sigma}|u|^{2\sigma+2}.$$

Since  $\rho > 1/2$ , we thus derive

$$\begin{aligned}
 & (\chi_0 + 1 + 2|a\rho|^2/a_1)\|u\|_{L_\mu^2}^2 \\
 (3.15) \quad & \leq (|\chi_0| + 1 + 2|a\rho|^2/a_1) \left\{ \int_{|u| \geq M} |u|^2 d\mu + \int_{|u| \leq M} |u|^2 d\mu \right\} \\
 & \leq \frac{1}{4} b \|u + \eta\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} + C \|\eta\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} + C.
 \end{aligned}$$

Putting together (3.11–3.15), we have

$$\begin{aligned}
 (3.16) \quad & \frac{d}{dt} \|u(t)\|_{L_\mu^2}^2 + \|u(t)\|_{L_\mu^2}^2 + a_1 \|\nabla u(t)\|_{L_\mu^2}^2 + b \|u + \eta\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} \\
 & \leq C(1 + \|\eta\|_{L_\mu^2}^2 + \|\eta\|_{L_\mu^{2\sigma+2}}^{2\sigma+2}).
 \end{aligned}$$

Applying Gronwall inequality to (3.16), we obtain for all  $T \geq \tau$

$$\begin{aligned}
 (3.17) \quad & \|u(T)\|_{L_\mu^2}^2 + \int_\tau^T e^{t-T} \|\nabla u(t)\|_{L_\mu^2}^2 + \|u(t)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} dt \\
 & \leq C e^{\tau-T} \|u_0 - \eta(\tau)\|_{L_\mu^2}^2 + C \int_\tau^T e^{t-T} \{1 + \|\eta(t)\|_{L_\mu^2}^2 + \|\eta(t)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2}\} dt.
 \end{aligned}$$

By (3.16) and (3.17), we obtain

$$\begin{aligned}
 (3.18) \quad & \int_T^{T+1} \left( \|\nabla u(t)\|_{L_\mu^2}^2 + \|u(t)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} \right) dt \\
 & \leq C e^{\tau-T} \|u_0 - \eta(\tau)\|_{L_\mu^2}^2 + C \int_\tau^{T+1} e^{t-T} \{1 + \|\eta(t)\|_{L_\mu^2}^2 + \|\eta(t)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2}\} dt.
 \end{aligned}$$

Now (3.9) is obtained from (3.17) and (3.18) and the first part of the Lemma is proved.

For any  $T \geq \tau$  and  $v(y, t) \in L^2(T, T+1; H_\mu^1) \cap L^{2\sigma+2}(T, T+1; L_\mu^{2\sigma+2})$ , multiplying (3.7) by  $(1 + |y|^2)^{-\rho} \bar{v}$  and integrating by parts, we get that

$$\begin{aligned}
 (3.19) \quad & \int_T^{T+1} (\partial_t u(\cdot, t), v(\cdot, t))_\mu dt + a \int_T^{T+1} (\nabla u(\cdot, t), \nabla v(\cdot, t))_\mu dt \\
 & = \chi_0 \int_T^{T+1} (u + \eta, v(\cdot, t))_\mu dt - b \int_T^{T+1} (|u + \eta|^{2\sigma}(u + \eta), v(\cdot, t))_\mu dt \\
 & \quad + 2a\rho \int_T^{T+1} ((1 + |y|^2)^{-1} y \cdot \nabla u, v(\cdot, t))_\mu dt + k \int_T^{T+1} (\eta, v(\cdot, t))_\mu dt.
 \end{aligned}$$

Using Hölder inequality, we have

$$(3.20) \quad \left| a \int_T^{T+1} (\nabla u(\cdot, t), \nabla v(\cdot, t))_\mu dt \right| \leq C \|\nabla u\|_{L^2(T, T+1; L_\mu^2)} \|\nabla v\|_{L^2(T, T+1; L_\mu^2)},$$

$$(3.21) \quad \left| \chi_0 \int_T^{T+1} (u + \eta, v(\cdot, t))_\mu dt \right| \leq C \|u + \eta\|_{L^2(T, T+1; L_\mu^2)} \|v\|_{L^2(T, T+1; L_\mu^2)},$$

$$\begin{aligned}
 (3.22) \quad & \left| b \int_T^{T+1} (|u + \eta|^{2\sigma}(u + \eta), v(\cdot, t))_\mu dt \right| \\
 & \leq C \|u + \eta\|_{L^{2\sigma+2}(T, T+1; L_\mu^{2\sigma+2})}^{2\sigma+1} \|v\|_{L^{2\sigma+2}(T, T+1; L_\mu^{2\sigma+2})},
 \end{aligned}$$

$$(3.23) \quad \left| 2a\rho \int_T^{T+1} ((1+|y|^2)^{-1} y \cdot \nabla u, v(\cdot, t))_\mu dt \right| \leq C \|\nabla u\|_{L^2(T, T+1; L_\mu^2)} \|v\|_{L^2(T, T+1; L_\mu^2)},$$

$$(3.24) \quad \left| k \int_T^{T+1} (\eta, v(\cdot, t))_\mu dt \right| \leq C \|\eta\|_{L^2(T, T+1; L_\mu^2)} \|v\|_{L^2(T, T+1; L_\mu^2)}.$$

Plugging (3.17), (3.18) and (3.20)–(3.24) into (3.19) yields

$$(3.25) \quad \left| \int_T^{T+1} (\partial_t u(\cdot, t), v(\cdot, t))_\mu dt \right| \leq C \|v\|_{L^2(T, T+1; H_{\mu_1}^1)} + C \|v\|_{L^{2\sigma+2}(T, T+1; L_\mu^{2\sigma+2})}.$$

Finally, (3.10) is obtained from (3.25) and this lemma is proved.  $\square$

**3.3. Existence and Uniqueness.** The following Theorem is verified by standard methods, but as some of the estimates are used later, we state its proof for completeness.

**Theorem 3.3.** (Existence and Uniqueness) *Let  $u_0 \in L_\mu^2$  and  $\sigma > 0$ . Then equations (3.7) and (3.8) possesses a unique solution  $u(t) = u(t, \tau, \omega, u_0 - \eta(\tau, \omega))$  such that  $u(\tau, \tau, \omega, u_0 - \eta(\tau, \omega)) = u_0 - \eta(\tau, \omega)$ ,  $u(t) \in C_{loc}([\tau, \infty); L_\mu^2) \cap L_{loc}^2([\tau, \infty); H_\mu^1) \cap L_{loc}^{2\sigma+2}([\tau, \infty); L_\mu^{2\sigma+2})$  and  $\partial_t u \in H'(\tau, T)$  for any  $T \geq \tau$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  and the energy estimates (3.9) and (3.10) hold.*

*Moreover, for every two solution  $u_1(t)$  and  $u_2(t)$  of equations (3.7) and (3.8), the following estimate holds:*

$$(3.26) \quad \|u_1(t) - u_2(t)\|_{L_\mu^2}^2 \leq e^{K_0 t} \|u_1(\tau) - u_2(\tau)\|_{L_\mu^2}^2, \quad \forall t \geq \tau,$$

where  $K_0 = 2|\chi_0| + |a|^2 \rho^2 / a_1$ .

*Proof.* (I) *Existence of Solution.* For the proof we rely on the Galerkin method and a compactness argument, which extracts a convergent subsequence from the Galerkin approximations. This is used to prove path-wise the existence of solution of equations (3.7) and (3.8). Let us remark that, as we rely on subsequences, the measurability w.r.t.  $\omega$  of the solutions follow from the uniqueness of solutions.

Let  $\{e_j\}_{j=1}^\infty \subset H_\mu^2 \cap L_\mu^2$  be the orthonormal basis of  $L_\mu^2$ , fixed in the definition of  $W$ . We denote the Galerkin approximation by

$$u_N = \sum_{j=1}^N \alpha_j(t) e_j, \quad u_{0N}(\tau) := P_N(u_0 - \eta(\tau)) = \sum_{j=1}^N (u_0 - \eta(\tau), e_j) e_j \quad (\forall N \geq 1)$$

where the orthogonal projection onto the Galerkin space is denoted by

$$P_N : L_\mu^2 \longrightarrow \text{span}\{e_1, \dots, e_N\}.$$

It is obvious that  $\lim_{N \rightarrow \infty} \|u_{0N} - \{u_0 - \eta(\tau)\}\|_{L_\mu^2} = 0$  and  $P_N u_N = u_N$ . Moreover,  $\|P_N v\|_{L_\mu^2} \leq \|v\|_{L_\mu^2}$  for all  $v$ .

Define now for all  $N \geq 1$  the Galerkin approximation as the solution of the following problem:

$$(3.27) \quad \partial_t u_N - P_N \{a \Delta u_N + \chi_0 u_N - b|u_N + \eta|^{2\sigma}(u_N + \eta) + (k + \chi_0)\eta\} = 0,$$

$$(3.28) \quad u_N|_{t=\tau} = u_{0N}(\tau).$$

As this is basically an ODE in  $\mathbb{R}^{\mathbb{N}}$  with polynomial nonlinearity, it is straightforward to establish the existence and uniqueness of solutions.

Following the proof of Lemma 3.2, we derive in a completely analogous way

$$(3.29) \quad \begin{aligned} & \|u_N(T)\|_{L_\mu^2}^2 + \int_\tau^T \left( \|u_N(t)\|_{H_\mu^1}^2 + \|u_N(t)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} \right) dt \\ & \leq CT e^{\tau-T} \|u_0 - \eta(\tau)\|_{L_\mu^2}^2 + CT \int_\tau^T e^{t-T} (1 + \|\eta(t)\|_{L_\mu^2}^2 + \|\eta(t)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2}) dt, \quad \forall T \geq \tau, \end{aligned}$$

and

$$(3.30) \quad \|\partial_t u_N(t)\|_{H'(\tau, T)} \leq r_0(T, \tau, u_0, \eta), \quad \forall T \geq \tau,$$

where  $C$  and  $r_0$  are independent of  $N$  and given in Lemma 3.2. Define the space

$$\mathcal{W} = C([\tau, T]; L_\mu^2) \cap L^2(\tau, T; H_\mu^1) \cap L^{2\sigma+2}(\tau, T; L_\mu^{2\sigma+2}).$$

Thus there exists a subsequence of the sequence  $u_N$ , which we also denote by  $u_N$  for simplicity, such that  $u_N$  converges to  $u$  in  $\mathcal{W}$  star-weakly and  $\partial_t u_N$  converges to  $\partial_t u$  in  $H'(\tau, T)$  weakly (cf. [44]).

Let us now restrict the convergence to bounded intervals in space, in order to use compact embeddings. To be more precise, for any  $M \in \mathcal{N}$ , let  $I_M = (-M, M)$  and define for  $T \geq \tau$  the space

$$\begin{aligned} \mathcal{W}_M = \left\{ v \mid \partial_t v \in H'(\tau, T), \right. \\ \left. v \in C([\tau, T]; L^2(I_M)) \cap L^2(\tau, T; H^1(I_M)) \cap L^{2\sigma+2}(\tau, T; L^{2\sigma+2}(I_M)) \right\}. \end{aligned}$$

The energy estimates (3.29) and (3.30) imply immediately that  $u_N$  is bounded in  $\mathcal{W}_M$  uniformly with respect to  $N$ , as both estimates hold obviously with all norms restricted to  $I_M$  on the left hand side of the equations.

Due to the Aubin-Lions lemma (see [34] or [45]), we obtain that (passing to a further subsequence if necessary)  $u_N$  converges to  $u$  strongly in  $L^2(\tau, T; L^2(I_M))$ . By using a diagonalization process, we can assume that (passing to a further subsequence if necessary)  $u_N$  converges to  $u$  strongly in  $L_{loc}^2([\tau, \infty); L_{loc}^2(\mathbb{R}))$  and  $u_N(y, t)$  converges to  $u(y, t)$  for a.e.  $(y, t) \in \mathbb{R} \times [\tau, \infty)$ .

These assertions on the weak convergence imply immediately that the limit  $u$  also satisfies the energy estimates (3.9) and (3.10). Furthermore, one has enough regularity to pass to the limit in equation (3.27) and (3.28). Thus  $u$  is the solution of equations (3.7) and (3.8).

(II) *Uniqueness of Solution.* Our main task now is to prove estimate (3.26), which immediately implies the uniqueness. Let  $u_1(t)$  and  $u_2(t)$  be two solutions of equations (3.7) and (3.8) with initial conditions  $u_1(0)$  and  $u_2(0)$ , respectively, instead of  $u_0$ . Then the energy estimate (3.9) is also valid for  $u_1(t)$  and  $u_2(t)$ . Define  $v(t) = u_1(t) - u_2(t)$ , which satisfies the equation

$$(3.31) \quad \partial_t v - a\Delta v - \chi_0 v + b|u_1 + \eta|^{2\sigma}(u_1 + \eta) - b|u_2 + \eta|^{2\sigma}(u_2 + \eta) = 0,$$

$$(3.32) \quad v(\tau) = u_1(\tau) - u_2(\tau).$$

As  $v$  is sufficiently regular, we can multiply equation (3.31) by  $\bar{v}(t)$  and integrate over  $x \in \mathbb{R}$ . Using integration by parts and taking the real part, we have

$$\begin{aligned}
 (3.33) \quad & \frac{d}{dt} \|v(t)\|_{L_\mu^2}^2 + 2a_1 \|\nabla v(t)\|_{L_\mu^2}^2 - \operatorname{Re} \int_{\mathbb{R}} (4a\rho(1+|y|^2)^{-1}vy \cdot \nabla v) d\mu \\
 & + 2b \operatorname{Re} \left( |u_1 + \eta|^{2\sigma}(u_1 + \eta) - |u_2 + \eta|^{2\sigma}(u_2 + \eta), v \right)_\mu \\
 & = 2|\chi_0| \|v(t)\|_{L_\mu^2}^2.
 \end{aligned}$$

First note the fact that

$$\operatorname{Re} \left( |u_1 + \eta|^{2\sigma}(u_1 + \eta) - |u_2 + \eta|^{2\sigma}(u_2 + \eta), v \right)_\mu \geq 0.$$

Using Hölder inequality yields

$$(3.34) \quad \left| \int_{\mathbb{R}} (4a\rho(1+|y|^2)^{-1}\bar{v}y \cdot \nabla v) d\mu \right| \leq a_1 \|\nabla v(t)\|_{L_\mu^2}^2 + \frac{|a|^2\rho^2}{a_1} \|v(t)\|_{L_\mu^2}^2.$$

Inserting (3.34) into (3.33), and using Gronwall inequality, we obtain that

$$(3.35) \quad \|v(T)\|^2 + a_1 \int_\tau^T \|\nabla v(t)\|^2 dt \leq \|u_1(\tau) - u_2(\tau)\|^2 e^{K_0 t},$$

where the positive constant is  $K_0 = 2|\chi_0| + |a|^2\rho^2/a_1$ . Finally, (3.35) implies estimate (3.26) and the Theorem 3.3 is proved.  $\square$

Let us finally state a simple but yet useful fact. recall that  $u = u(t, \tau, \omega, u_0 - \eta(\tau))$  is a solution of equations (3.7) and (3.8) given by theorem 3.3. By the proof of Lemma 3.1.2 [44], we have

$$\frac{d}{dt} \|u\|^2 = 2\operatorname{Re}(\partial_t u, u)_\mu$$

in the distributional sense on  $\mathbb{R}^+$ . Thus we obtain the following lemma.

**Lemma 3.4** (Energy Equality). *Let  $u = u(t, \tau, \omega, u_0 - \eta(\tau))$  be a solution of equation (3.7) given by theorem 3.3. Then  $u$  satisfies the energy equality (3.11) in the distribution sense for  $t \geq 0$ .*

#### 4. ASYMPTOTIC COMPACTNESS

In this section, we construct the SDS corresponding to the SCGL (1.1) and (1.2), and establish AC for bounded sets.

**Definition 4.1.** *We define a map*

$$S : \mathbb{R}_\Delta \times \Omega \times L_\mu^2 \longrightarrow L_\mu^2$$

by

$$(4.1) \quad (t, \tau, \omega, u_0) \mapsto S(t, \tau; \omega)u_0 = u(t, \tau, \omega, u_0 - \eta(\tau, \omega)) + \eta(t, \omega),$$

where  $u(t, \tau, \omega, u_0 - \eta(\tau, \omega))$ , which is defined in Theorem 3.3, is the solution of the transformed equation (3.7).

By the estimate (3.26), the map  $S(t, \tau; \omega)$  is obviously continuous in  $L_\mu^2$ , for all  $t \geq \tau$  and for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . We obtain the following Lemma.

**Lemma 4.2.** *The map  $S$  defined in Definition 4.1 defines a SDS.*

*Proof.* All properties of a SDS with the exception of the cocycle follow from Theorem 3.3. Hence we only need to show that for any  $u_0 \in L_\mu^2$  that

$$(4.2) \quad S(t, \tau; \omega)u_0 = S(t, r; \omega)S(r, \tau; \omega)u_0, \quad \forall \tau \leq r \leq t.$$

From the definition of  $S$ , we have for all  $\tau \leq r \leq t$ ,

$$\begin{aligned} S(t, \tau; \omega)u_0 &= u(t, \tau, \omega, u_0 - \eta(\tau, \omega)) + \eta(t, \omega), \\ S(t, r; \omega)S(r, \tau; \omega)u_0 &= u(t, r, \omega, S(r, \tau; \omega)u_0 - \eta(r, \omega)) + \eta(t, \omega) \\ &= u(t, r, \omega, u(r, \tau, \omega, u_0 - \eta(\tau, \omega))) + \eta(t, \omega). \end{aligned}$$

Therefore, in order to prove (4.2), we only need to prove

$$(4.3) \quad u(t, \tau, \omega, u_0 - \eta(\tau, \omega)) = u(t, r, \omega, u(r, \tau, \omega, u_0 - \eta(\tau, \omega))), \quad \forall \tau \leq r \leq t.$$

Let us fix  $\tau$  and define two functions  $u_1$  and  $u_2$  by

$$\begin{aligned} u_1(t) &= u(t, \tau, \omega, u_0 - \eta(\tau, \omega)), \\ u_2(t) &= u(t, r, \omega, u(r, \tau, \omega, u_0 - \eta(\tau, \omega))), \quad \forall \tau \leq r \leq t. \end{aligned}$$

Because  $u(\tau, \tau, \omega, v_0) = v_0$ , we infer that

$$\begin{aligned} u_2(\tau) &= u(\tau, \tau, \omega, u(\tau, \tau, \omega, u_0 - \eta(\tau, \omega))) \\ &= u(\tau, \tau, \omega, u_0 - \eta(\tau, \omega)) = u_1(\tau). \end{aligned}$$

Since  $u(t, r, \omega, U)$  is a solution of equations (3.7) with initial data  $u|_{t=r} = U$ , we infer that  $u_1(t)$  and  $u_2(t)$  solve the problem (3.7) and (3.8). Therefore, by the uniqueness of solutions to problem (3.7) and (3.8), we have  $u_1(t) = u_2(t)$ . Finally, since  $\tau$  is arbitrary, (4.3) is proved.  $\square$

**4.1. Weak and Strong Continuity of the SDS.** Let us now prove continuous dependence in the weak and strong topology of our SDS  $S(t, \tau; \omega)$  defined in Definition 4.1. This will be used later in the proof of AC.

**Lemma 4.3** (Weak Continuity). *Fix an initial condition  $u_0 \in L_\mu^2$  and a sequence  $\{u_{0n}\}_n$  converging to  $u_0$  weakly in  $L_\mu^2$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . Then for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,*

$$(4.4) \quad \begin{cases} S(t, \tau; \omega)u_{0n} \rightharpoonup S(t, \tau; \omega)u_0 & \text{weakly in } L_\mu^2, \quad \forall t \geq \tau, \\ S(\cdot, \tau; \omega)u_{0n} \rightharpoonup S(\cdot, \tau; \omega)u_0 & \text{weakly in } H(\tau, T), \quad \forall T \geq \tau, \end{cases}$$

$$(4.5) \quad S(\cdot, \tau; \omega)u_{0n} \rightarrow S(\cdot, \tau; \omega)u_0 \text{ strongly in } L^2(\tau, T; L_{loc}^2(\mathbb{R})), \quad \forall T \geq \tau,$$

where again  $H(\tau, T) = L^2(\tau, T; H_\mu^1) \cap L^{2\sigma+2}(\tau, T; L_\mu^{2\sigma+2})$ .

*Proof.* Let us fix  $\omega$  such that  $\eta(\cdot, \omega)$  is finite in  $C_{loc}^0(\mathbb{R}; L_\mu^p)$  for any  $p \geq 2$  and fix  $T \geq \tau$ . Denote the solution  $u(t, \tau, \omega, u_{0n} - \eta(\tau, \omega))$  of equation (3.7) by  $u_n(t)$ . From the a priori estimates (3.9) and (3.10), it follows that

$$\begin{cases} \{u_n(t)\}_n & \text{is bounded in } L^\infty(\tau, T; L_\mu^2) \cap H(\tau, T), \\ \{\partial_t u_n(t)\}_n & \text{is bounded in } H'(\tau, T). \end{cases}$$

As before by a diagonalization process, we can extract a subsequence  $\{u_{n'}(t)\}_{n'}$  converging to  $\hat{u}(t)$  weakly star in  $L^\infty(\tau, T; L_\mu^2)$  and weakly in  $H(\tau, T)$ . Then  $\hat{u}(t) \in$

$L^\infty(\tau, T; L_\mu^2) \cap H(\tau, T)$ . Thanks to Aubin-Lions lemma (see [26, 33]) and a diagonalization process, we can also assume that (passing to a further subsequence if necessary)  $\{u_{n'}(t)\}_{n'}$  converges to  $\hat{u}(t)$  strongly in  $L^2(\tau, T; L_{loc}^2(\mathbb{R}))$ . Moreover,  $\hat{u}$  satisfies the energy estimates (3.9), where the initial condition is replaced by a constant, depending on a bound on the weakly convergent subsequence  $u_{0n}$ .

Passing to the limit in the weak form of the equations (3.7) and (3.8) for  $u_n(t)$ , we find that  $\hat{u}(t)$  solves the equations (3.7) and (3.8). By the uniqueness of solution, we have  $\hat{u}(t) = u(t) = u(t, \tau, \omega, u_0 - \eta(\tau, \omega))$ . Then, by a contradiction argument, we can deduce from the uniqueness of solutions that in fact the whole sequence  $\{u_n(t)\}_n$  converges to  $u(t)$  such that (4.5) holds and moreover

$$(4.6) \quad \begin{cases} u_n(t) \rightharpoonup u(t) & \text{weakly star in } L^\infty(\tau, T; L_\mu^2), \quad \forall T \geq \tau, \\ & \text{strongly in } L^2(\tau, T; L_{loc}^2(\mathbb{R})), \quad \forall T \geq \tau, \\ u_n(t) \rightharpoonup u(t) & \text{weakly in } H(\tau, T), \quad \forall T \geq \tau. \end{cases}$$

Hence, for all  $v \in H_\mu^1 \cap L_\mu^{2\sigma+2}$ ,

$$(4.7) \quad (u_n(t), v)_\mu \rightarrow (u(t), v)_\mu, \quad \text{for a.e. } t \in [\tau, \infty).$$

For all  $\tau \leq t \leq t+a \leq T$ ,

$$(4.8) \quad \begin{aligned} (u_n(t+a) - u_n(t), v)_\mu &= \int_t^{t+a} (\partial_t u_n(s), v)_\mu ds \\ &\leq \{\|v\|_{H_\mu^1} a^{1/2} + \|v\|_{L_\mu^{2\sigma+2}} a^{1/(2\sigma+2)}\} \|\partial_t u_n\|_{H'} \\ &\leq C_T \{\|v\|_{H_\mu^1} a^{1/2} + \|v\|_{L_\mu^{2\sigma+2}} a^{1/(2\sigma+2)}\}, \end{aligned}$$

where  $C_T$  is independent of  $n$ . Thus  $\{(u_n(t), v)_\mu\}_n$  is equibounded and equicontinuous on  $[\tau, T]$  for all  $T > 0$ , and by uniqueness of the Limit together with Arcela-Ascoli we derive

$$(4.9) \quad (u_n(t), v)_\mu \rightarrow (u(t), v)_\mu, \quad \forall t \in [\tau, \infty), \quad \forall v \in H_\mu^1 \cap L_\mu^{2\sigma+2}.$$

(4.4) follows from (4.6) and (4.9) by taking into account the facts that  $H_\mu^1 \cap L_\mu^{2\sigma+2}$  is dense in  $L_\mu^2$  and  $u_n$  bounded in  $L_\mu^2$ .  $\square$

The following Lemma exploits bounds in the energy estimate given by the nonlinear terms. It is crucial, in order to pass from convergence in  $L_\mu^2$  to convergence in some  $L_\mu^p$ .

**Lemma 4.4.** (Strong Convergence) *For any given  $T \geq \tau$ , suppose that*

$$(4.10) \quad u_n \rightarrow u \quad \text{strongly in } L^2(\tau, T; L_{loc}^2(\mathbb{R})),$$

*and that the sequence  $\{u_n\}_{n=1}^\infty$  is uniform bounded in  $L^{2\sigma+2}(\tau, T; L_\mu^{2\sigma+2})$ , i.e.*

$$(4.11) \quad \|u_n\|_{L^{2\sigma+2}(\tau, T; L_\mu^{2\sigma+2})} \leq C, \quad \forall n \geq 1.$$

*Then*

$$(4.12) \quad u_n \rightarrow u \quad \text{strongly in } L^p(\tau, T; L_\mu^p(\mathbb{R})), \quad \forall p \in [1, 2\sigma+2).$$

*Proof.* Recall that  $d\mu = m(y)dy$ ,  $m(y) = (1 + |y|^2)^{-\rho}$ ,  $\rho > \frac{1}{2}$  and  $\mu(\mathbb{R}) < \infty$ . Let us first prove the result for  $p = 2$ . Using (4.10), we derive for any  $r > 0$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|u_n - u\|_{L^2(\tau, T; L_\mu^2)}^2 \\ &= \lim_{n \rightarrow \infty} \int_\tau^T \int_{|y| \leq r} |u_n(s) - u(s)|^2 d\mu ds + \lim_{n \rightarrow \infty} \int_\tau^T \int_{|y| \geq r} |u_n(s) - u(s)|^2 d\mu ds \\ &=: I_1 + I_2 \end{aligned}$$

The convergence in  $L_{\text{loc}}^2$  immediately implies  $I_1 = 0$ . For  $I_2$  we obtain due to Hölder inequality

$$I_2 \leq (T - \tau)^{\sigma/(\sigma+1)} \left\{ \int_{|y| \geq r} d\mu \right\}^{\sigma/(\sigma+1)} \sup_n \|u_n - u\|_{L^{2\sigma+2}(\tau, T; L_\mu^{2\sigma+2})}^{1/(\sigma+1)}.$$

Together with (4.11) and letting  $r \rightarrow \infty$  yields

$$(4.13) \quad \lim_{n \rightarrow \infty} \|u_n - u\|_{L^2(\tau, T; L_\mu^2)}^2 = 0.$$

Let us now turn to  $p \in [1, 2)$ . Here the claim follows immediately, as

$$(4.14) \quad \|u_n - u\|_{L^p(\tau, T; L_\mu^p)}^p \leq \{(T - \tau)\mu(\mathbb{R})\}^{1-p/2} \|u_n - u\|_{L^2(\tau, T; L_\mu^2)}^{p/2} \xrightarrow{n \rightarrow \infty} 0.$$

For any  $p \in (2, 2\sigma + 2)$ , by Hölder interpolation we have

$$(4.15) \quad \lim_{n \rightarrow \infty} \|u_n - u\|_{L^p(\tau, T; L_\mu^p)}^p \leq \lim_{n \rightarrow \infty} \|u_n - u\|_{L^{2\sigma+2}(\tau, T; L_\mu^{2\sigma+2})}^{1-\frac{2\sigma+2-p}{p\sigma}} \|u_n - u\|_{L^2(\tau, T; L_\mu^2)}^{\frac{2\sigma+2-p}{p\sigma}} = 0.$$

Thus finally, the convergence in (4.12) is proved.  $\square$

**4.2. Asymptotic Compactness.** We now turn to the main result of the paper. The following Theorem verifies AC for bounded sets, as for example defined in [8] or [3] for RDS.

**Theorem 4.5.** (Asymptotic Compactness) *Let  $\{u_j\}_j$  be bounded in  $L_\mu^2$  and  $\{\tau_j\}_j \subset \mathbb{R}$  with  $\tau_j \rightarrow -\infty$ . Then there exists  $w \in L_\mu^2$  and a subsequence  $\{j'\}$  such that  $S(t, \tau_{j'}; \omega)u_{j'} \rightarrow w$  strongly in  $L_\mu^2$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ .*

*Proof.* Let us fix  $\omega$  such that  $\eta(T, \omega)$  is sufficiently regular, as asserted by Lemma 3.1.

From a priori estimates (3.9) and (3.10) and from the fact that  $\{u_j\}_j$  is bounded in  $L_\mu^2$ , there exist constants  $J > 0$  and  $R_0 > 0$  such that

$$(4.16) \quad \|u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}))\|_{L_\mu^2} \leq R_0, \quad \forall j' \geq J,$$

where  $u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}))$  is the solution of equations (3.7) with initial data

$$(4.17) \quad u|_{t=\tau_{j'}} = u_{j'} - \eta(\tau_{j'}).$$

Thus  $\{u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}))\}_{j'}$  is weakly precompact in  $L_\mu^2$ , and we can assume that

$$(4.18) \quad u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'})) \rightharpoonup v \text{ weakly in } L_\mu^2$$

by passing to a further subsequence if necessary.

Similarly for each  $T > 0$ , we also have that there exist constants  $J = J(T) > 0$  such that  $t - T \geq \tau_{j'}$  and

$$(4.19) \quad \|u(t - T, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}))\|_{L_\mu^2} \leq R_0, \quad \forall j' \geq J.$$



Thus  $\{S(t-T, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}))\}_{j'}$  is weakly precompact in  $L_\mu^2$ , and by using a diagonal process and passing to a further subsequence if necessary we can assume that

$$(4.20) \quad u(t-T, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'})) \rightharpoonup v_T \text{ weakly in } L_\mu^2$$

for any rational number  $T > 0$ .

Then by the proof of Lemma 4.3 and Lemma 4.2 we deduce that

$$\begin{aligned} v &= w\text{-}\lim_{j' \rightarrow \infty} u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'})) \\ &= w\text{-}\lim_{j' \rightarrow \infty} u\left(t, t-T, \omega, u(t-T, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}))\right) \\ &= u\left(t, t-T, \omega, w\text{-}\lim_{j' \rightarrow \infty} u(t-T, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}))\right) = u(t, t-T, \omega, v_T), \end{aligned}$$

where  $w\text{-}\lim$  denotes the limit taken in the weak topology of  $L_\mu^2$ . Thus

$$(4.21) \quad v = u(t, t-T, \omega, v_T), \quad \forall T \in \mathbb{N}.$$

Now, from (4.18), we find

$$(4.22) \quad \|v\|_{L_\mu^2} \leq \liminf_{j' \rightarrow \infty} \|u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}))\|_{L_\mu^2},$$

and, in order to show strong convergence, we will show that

$$(4.23) \quad \limsup_{j' \rightarrow \infty} \|u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}))\|_{L_\mu^2} \leq \|v\|_{L_\mu^2}.$$

For  $T \in \mathbb{N}$  and  $t-T \geq \tau_j$  we obtain by integrating the energy equation (3.11) from  $t-T$  to  $t$  that

$$\begin{aligned} &\|u(t, \tau_j, \omega, u_j - \eta(\tau_j, \omega))\|_{L_\mu^2}^2 + 2b \int_{t-T}^t e^{\tau-t} \|U_j + \eta(\tau, \omega)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} d\tau \\ &+ \int_{t-T}^t e^{\tau-t} \{2a_1 \|\nabla U_{j'}\|_{L_\mu^2}^2 - (2\chi_0 + 1) \|U_{j'}\|_{L_\mu^2}^2\} d\tau \\ &= \|u(t-T, \tau_j, \omega, u_j - \eta(\tau_j, \omega))\|_{L_\mu^2}^2 e^{-T} + 2Re \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} (k + \chi_0) U_j \bar{\eta}(\tau, \omega) d\mu d\tau \\ &+ 2Re \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} b \left| U_j + \eta(\tau, \omega) \right|^{2\sigma} \{U_j + \eta(\tau, \omega)\} \bar{\eta}(\tau, \omega) d\mu d\tau, \end{aligned} \quad (4.24)$$

where

$$U_j = u(\tau, \tau_j, \omega, u_j - \eta(\tau_j, \omega)) = u\left(\tau, t-T, \omega, u(t-T, \tau_j, \omega, u_j - \eta(\tau_j, \omega))\right).$$

From (4.19) we find

$$(4.25) \quad \limsup_{j'} \|u(t-T, \tau_j, \omega, u_j - \eta(\tau_j, \omega))\|^2 e^{-T} \leq R_0^2 e^{-T}.$$

Also, by the proof of (4.4) and (4.5) we deduce from (4.20) that

$$(4.26) \quad U_{j'} = u\left(\cdot, t-T, \omega, u(t-T, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}, \omega))\right) \rightharpoonup u(\cdot, t-T, \omega, v_T)$$

weakly in  $L^2(t-T, t; H_\mu^1) \cap L^{2\sigma+2}(t-T, t; L_\mu^{2\sigma+2})$ ,

$$(4.27) \quad U_{j'} = u\left(\cdot, t-T, \omega, u(t-T, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}, \omega))\right) \rightarrow u(\cdot, t-T, \omega, v_T)$$

strongly in  $L^2(t-T, t; L_{loc}^2(\mathbb{R}))$  and  $U_{j'}(y, s)$  converges to  $u(s, t-T, \omega, v_T)$  for a.e.  $(y, s) \in \mathbb{R} \times (t-T, t)$ . By using Lemma 4.4, we get that  $U_{j'}$  converges to  $u(s, t-T, \omega, v_T)$  strongly in  $L^p(t-T, t; L_\mu^p)$  for any  $p \in [1, 2\sigma+2)$ . Since  $\mu(\mathbb{R}) < \infty$  and

$$\tau \mapsto e^{\tau-t} \bar{\eta}(\tau, \omega) \in L^2(t-T, t; L_\mu^2),$$

we find that

$$(4.28) \quad \begin{aligned} & \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} U_{j'} \bar{\eta}(\tau, \omega) d\mu \, d\tau \\ & \longrightarrow \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} u(\tau, t-T, \omega, v_T) \bar{\eta}(\tau, \omega) d\mu \, d\tau, \end{aligned}$$

Next, as  $\exp$  is  $L^\infty$ , the strong convergence in  $L^2(L_\mu^2)$  implies

$$(4.29) \quad \int_{t-T}^t e^{\tau-t} \|U_{j'}\|_{L_\mu^2}^2 d\tau \longrightarrow \int_{t-T}^t e^{\tau-t} \|u(\tau, t-T, \omega, v_T)\|_{L_\mu^2}^2 d\tau,$$

From strong convergence in  $L^p(L_\mu^p)$  with  $p < 2\sigma+2$  we deduce

$$(4.30) \quad \begin{aligned} & \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} \left| U_{j'} + \eta(\tau, \omega) \right|^{2\sigma} \{U_{j'} + \eta(\tau, \omega)\} \bar{\eta}(\tau, \omega) d\mu \, d\tau \\ & \longrightarrow \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} \left| u(\tau, t-T, \omega, v_T) + \eta(\tau, \omega) \right|^{2\sigma} \\ & \quad \{u(\tau, t-T, \omega, v_T) + \eta(\tau, \omega)\} \bar{\eta}(\tau, \omega) d\mu \, d\tau, \end{aligned}$$

Finally, the weak convergence bounds

$$(4.31) \quad \begin{aligned} & \int_{t-T}^t e^{\tau-t} \|u(\tau, t-T, \omega, v_T) + \eta(\tau, \omega)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} d\tau \\ & \leq \liminf_{j' \rightarrow \infty} \int_{t-T}^t e^{\tau-t} \|U_{j'} + \eta(\tau, \omega)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} d\tau, \end{aligned}$$

and

$$(4.32) \quad \int_{t-T}^t e^{\tau-t} \|\nabla u(\tau, t-T, \omega, v_T)\|_{L_\mu^2}^2 d\tau \leq \liminf_{j' \rightarrow \infty} \int_{t-T}^t e^{\tau-t} \|\nabla U_{j'}\|_{L_\mu^2}^2 d\tau.$$

Therefore

$$(4.33) \quad \begin{aligned} & \limsup_{j' \rightarrow \infty} -2a_1 \int_{t-T}^t e^{\tau-t} \|\nabla U_{j'}\|_{L_\mu^2}^2 d\tau = -2a_1 \liminf_{j' \rightarrow \infty} \int_{t-T}^t e^{\tau-t} \|\nabla U_{j'}\|_{L_\mu^2}^2 d\tau \\ & \leq -2a_1 \int_{t-T}^t e^{\tau-t} \|\nabla u(\tau, t-T, \omega, v_T)\|_{L_\mu^2}^2 d\tau. \end{aligned}$$

Now passing to the lim sup as  $j$  goes to infinity in (4.24) and using (4.25), (4.28) – (4.31) and (4.33), we get that

$$\begin{aligned}
& \limsup_{j' \rightarrow \infty} \|u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}, \omega))\|_{L_\mu^2}^2 \\
& \leq R_0^2 e^{-T} - 2b \int_{t-T}^t e^{\tau-t} \|u(\tau, t-T, \omega, v_T) + \eta(\tau, \omega)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} d\tau \\
& \quad - \int_{t-T}^t e^{\tau-t} \{2a_1 \|\nabla u(\tau, t-T, \omega, v_T)\|_{L_\mu^2}^2 - (2\chi_0 + 1) \|u\|_{L_\mu^2}^2\} d\tau \\
& \quad + 2Re \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} b \left| u(\tau, t-T, \omega, v_T) + \eta(\tau, \omega) \right|^{2\sigma} \{u + \eta\} \bar{\eta} d\mu d\tau \\
& \quad + 2Re \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} (k + \chi_0) u(\tau, t-T, \omega, v_T) \bar{\eta}(\tau, \omega) d\mu d\tau.
\end{aligned} \tag{4.34}$$

On the other hand, we obtain from the energy equality (3.11) applied to  $v = u(t, t-T, \omega, v_T)$  that

$$\begin{aligned}
\|v\|_{L_\mu^2}^2 &= \|u(t, t-T, \omega, v_T)\|_{L_\mu^2}^2 \\
&= e^{-T} \|v_T\|^2 - 2b \int_{t-T}^t e^{\tau-t} \|u(\tau, t-T, \omega, v_T) + \eta(\tau, \omega)\|_{L_\mu^{2\sigma+2}}^{2\sigma+2} d\tau \\
& \quad - \int_{t-T}^t e^{\tau-t} \{2a_1 \|\nabla u(\tau, t-T, \omega, v_T)\|^2 - (2\chi_0 + 1) \|u\|_{L_\mu^2}^2\} d\tau \\
& \quad + 2Re \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} b \left| u(\tau, t-T, \omega, v_T) + \eta(\tau, \omega) \right|^{2\sigma} \{u + \eta\} \bar{\eta} d\mu d\tau \\
& \quad + 2Re \int_{t-T}^t e^{\tau-t} \int_{\mathbb{R}} (k + \chi_0) u(\tau, t-T, \omega, v_T) \bar{\eta}(\tau, \omega) d\mu d\tau.
\end{aligned} \tag{4.35}$$

From (4.34) and (4.35) we deduce that

$$\begin{aligned}
\limsup_{j' \rightarrow \infty} \|u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}, \omega))\|_{L_\mu^2}^2 &\leq \|v\|_{L_\mu^2}^2 + (R_0^2 - \|v_T\|_{L_\mu^2}^2) e^{-T} \\
&\leq \|v\|_{L_\mu^2}^2 + R_0^2 e^{-T}, \quad \forall T \in \mathbb{N}.
\end{aligned} \tag{4.36}$$

Letting  $T \rightarrow \infty$  in (4.36), we have

$$\limsup_{j' \rightarrow \infty} \|u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}, \omega))\|_{L_\mu^2}^2 \leq \|v\|_{L_\mu^2}^2, \tag{4.37}$$

as claimed. Since  $L_\mu^2$  is a Hilbert space, the bound (4.37) on the lim sup together with the weak convergence from (4.18) imply

$$u(t, \tau_{j'}, \omega, u_{j'} - \eta(\tau_{j'}, \omega)) \xrightarrow{j' \rightarrow \infty} v \text{ strongly in } L_\mu^2. \tag{4.38}$$

□

## 5. COMMENTS ON THE ATTRACTOR

Let us again by Lemma 3.1 fix  $\omega$  such that  $\eta$  is sufficiently regular. According to Lemma 3.2, it is easy to show that for all  $t > \tau$

$$\|S(t, \tau, \omega)u_0\|_{L_\mu^2} \leq Ce^{\tau-t}\|u_0 - \eta\|_{L_\mu^2} + r(t, \omega)$$

where the random constant

$$r(t, \omega) = C \int_{-\infty}^{t+1} e^{s-t} \left( 1 + \|\eta(s, \omega)\|_{L_\mu^2}^2 + \|\eta(s, \omega)\|_{L_{\mu}^{\sigma_1}}^{\sigma_1} \right) dt$$

is  $\mathbb{P}$ -almost sure finite due to Lemma 3.1.

Thus the SDS  $S(t, \tau; \omega)$  possesses an attracting set

$$K(t, \omega) = B_{r(t, \omega)} \subset L_\mu^2.$$

It is obvious that by enlarging the radius, one obtains a bounded absorbing set  $\mathcal{B}(t, \omega)$  in  $L_\mu^2$ . Moreover in the setting of RDS, it is easy to show that the set  $K$  and thus  $\mathcal{B}$  is a tempered set, which means that  $K(t, \omega)$  grows at most subexponential for  $t \rightarrow -\infty$ .

Let us remark that for more regular forcing, where  $\eta$  is uniformly bounded in time, using Theorem 4.5, it would be easy to establish the existence of a random pull-back attractor for bounded sets for the RDS, as the absorbing set is then uniformly bounded in time.

Let us also remark, that using Theorem 4.5 and following the proof of Proposition 3.1–Theorem 3.4 in [8] for RDS, one could show that there is a compact  $\Omega$ -Limit set of the RDS, which supports an invariant measure. This would recover results of [42].

It remains an open problem, whether one can use the concept of  $\omega$ -limit sets for SDS, too. Moreover, as the absorbing set is tempered, we would need AC for tempered sets, in order to show that the  $\omega$ -limit set of the absorbing set is a random attractor. In that case it will also attract tempered sets.

Nevertheless, if we suppose that our stochastic forcing is more regular in time, such that  $\eta(t)$  is uniformly bounded in  $t$ , then it is easy to see, that in this case there is a deterministic bounded absorbing set. Using our results on AC for bounded sets, it is easy to show (cf. e.g. [3]) that the RDS has a bounded random attractor for bounded deterministic sets.

Let us conclude the paper by showing that the possible attractor is actually independent of the weight chosen. This is not obvious, as illustrated by the following example. For the deterministic PDE  $\partial_t u = \partial_x^2 u$  the stationary solutions in  $L_\mu^2$  are  $\{0\}$  for  $\rho < \frac{1}{2}$ ,  $\text{span}\{1\}$  for  $\rho \in (\frac{1}{2}, 1)$ , and  $\text{span}\{1, x\}$  for  $\rho > 1$ .

For any  $\rho > \frac{1}{2}$  denote the space  $L_\mu^2$  now by  $L_\rho^2$ , in order to allow for different weights.

**Theorem 5.1.** *Suppose that for all  $\rho > \frac{1}{2}$ , the SDS  $S(t, \tau; \omega)$  associated to SCGL (1.1) and (1.2) possesses a tempered pull-back global attractor  $\mathcal{A}_\rho(t, \omega)$  for tempered sets, i.e., for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , and for all  $t \in \mathbb{R}$ , the set  $\mathcal{A}_\rho(t, \omega)$  is a nonempty compact subset in  $L_\rho^2$  which attracts all tempered sets from  $-\infty$ , it is invariant in the sense that*

$$S(t, \tau; \omega)\mathcal{A}_\rho(\tau, \omega) = \mathcal{A}_\rho(t, \omega), \quad \forall \tau \leq t,$$

*and is the minimal closed set with this property.*

*Then  $\mathcal{A}_\rho$  is independent of  $\rho > \frac{1}{2}$ .*

Let us remark, that for more regular noise, similar results hold true for the stochastic or random attractor of bounded sets.

*Proof.* Let us first show more regularity for  $\mathcal{A}_\rho$ . For  $t_0$  and  $\omega$  fixed, take  $u_0 \in \mathcal{A}_\rho(t_0, \omega) \subset L_\rho^2$ , and define  $u(t) = S(t, t_0, \omega)u_0$ . From energy estimates  $u \in L^{2\sigma+2}(t_0, t, L_\rho^{2\sigma+2})$ , and thus there is a  $t_1(\omega) \in (t_0, t)$  such that  $u(t_1) \in L_\rho^{2\sigma+2}$ .

Now we can use the following fact that for  $0 < \theta < \sigma(2\rho - 1)/(2\sigma + 2)$  Hölder inequality yields

$$\begin{aligned} \|u\|_{L_{\rho-\theta}^2}^2 &= \int_{\mathbb{R}} (1 + |x|^2)^{-(\rho-\theta)} |u(x)|^2 dx \\ &\leq \left( \int_{\mathbb{R}} (1 + |x|^2)^{-(\rho-\theta-(\frac{\rho}{\sigma+1}))\frac{\sigma+1}{\sigma}} dx \right)^{\sigma/(\sigma+1)} \cdot \left( \int_{\mathbb{R}} m_\rho(x) |u(x)|^{2\sigma+2} dx \right)^{1/(\sigma+1)} \\ &= C \|u\|_{L_\rho^{2\sigma+2}}^2. \end{aligned}$$

Thus  $u(t_1) \in L_{\rho-\theta}^2$  and again from energy estimates it is straightforward to obtain  $u \in C^0([t_1, t], L_{\rho-\theta}^2)$  and thus  $u(t) \in L_{\rho-\theta}^2$ .

Finally, as  $u_0$  was arbitrary, this yields

$$\mathcal{A}_\rho(t, \omega) = S(t, t_0, \omega)\mathcal{A}_\rho(t_0, \omega) \subset L_{\rho-\theta}^2.$$

Repeating the previous argument, this is actually true for all  $\theta \in (0, \rho)$ .

On one hand  $\mathcal{A}_\rho$  is an invariant set in  $L_{\rho-\theta}^2$ , which has to be attracted by  $\mathcal{A}_{\rho-\theta}$ . Thus  $\mathcal{A}_\rho \subset \mathcal{A}_{\rho-\theta}$ .

On the other hand  $\mathcal{A}_{\rho-\theta} \subset L_{\rho-\theta}^2 \subset L_\rho^2$  is also invariant, and thus  $\mathcal{A}_{\rho-\theta} \subset \mathcal{A}_\rho$ .  $\square$

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#### REFERENCES

- [1] L. Arnold. *Random Dynamical Systems*. Springer, New York, 1998.
- [2] M. V. Bartuccelli, P. Constantin, C. Doering, J. Gibbon, M. Gisselalt, On the possibility of soft and hard turbulence in the complex Ginzburg-Landau equation, *Physica D* 44 (1990), 421–444.
- [3] P.W. Bates, H. Lisei, K. Lu, Attractors for stochastic lattice dynamical systems. *Stoch. Dyn.* 6(1) (2006), 1–21.
- [4] P.W. Bates, K. Lu, B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains. *J. Differ. Equations* 246(2) (2009), 845–869.
- [5] P.W. Bates, K. Lu, B. Wang, Attractors for lattice dynamical systems. *Int. J. Bifurcation Chaos Appl. Sci. Eng.* 11(1) (2001), 143–153.
- [6] H. R. Brand, R. J. Deissler, Interaction of localized solutions for subcritical bifurcations, *Phys. Rev. Lett.* 63 (1989), 2801–2804.
- [7] Z. Brzeźniak and Y. Li, Asymptotic behaviour of solutions to the 2D stochastic Navier-Stokes equations in unbounded domains – new developments. Alberverio, Sergio (ed.) et al., *Recent developments in stochastic analysis and related topics. Proceedings of the first Sino-German conference on stochastic analysis (a satellite conference of ICM 2002)*, Beijing, China, 29 August – 3 September 2002. River Edge, NJ: World Scientific. 78–111 (2004).
- [8] Z. Brzeźniak and Y. Li, Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations on some unbounded domains. *Trans. Amer. Math. Soc.* 358(12), 2006, 5587–5629.
- [9] Z. Brzeźniak, Y. Li, and Zhou J. Conceptual Analysis and Random Attractor for Dissipative Random Dynamical Systems. *Acta Mathematica Scientia* 28(2) (2008), 253–268.
- [10] Z. Brzeźniak, S. Peszat, Space-time continuous solutions to SPDE’s driven by a homogeneous Wiener process. *Stud. Math.* 137(3) (1999), 261–299.

- [11] C. Bu, On the Cauchy problem for the 1+2 complex Ginzburg-Landau equation. *J. Austral. Math. Soc. Ser. B* 36(1994), 313–324.
- [12] P. Collet, J. P. Eckmann, Extensive properties of the complex Ginzburg-Landau equation. *Comm. Math. Phys.* 200(3), 1999, 699–722.
- [13] H. Crauel, A. Debussche and F. Flandoli. Random attractors. *J. Dynam. Differential Equations*, 9:307–341, 1995.
- [14] M. C. Cross, P. C. Hohenberg, Pattern formation outside of equilibrium, *Rev. Mod. Phys.* 65 (1993), 851–1089.
- [15] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimension*. University Press, Cambridge, 1992.
- [16] C. Doering, J. Gibbon, C. C. Levermore, Weak and strong solutions of the complex Ginzburg-Landau equation, *Physica D* 71 (1994), 285–318.
- [17] J. Duan, P. Holmes, On the Cauchy problem of a generalized Ginzburg-Landau equation, *Nonlinear Anal. TMA* 22 (1994), 1033–1040.
- [18] J. Duan, P. Holmes, S. Titi, Global existence theory for a generalized Ginzburg-Landau equation, *Nonlinearity* 5(1992), 1303–1314.
- [19] J. Duan, P. Holmes, S. Titi, Regularity approximation and asymptotic dynamics for a generalized Ginzburg-Landau equation, *Nonlinearity* 6 (1993), 915–933
- [20] Chunrong Feng and Huaizhong Zhao, A generalized Ito’s formula in two-dimensions and stochastic Lebesgue-Stieltjes integrals. *Electr. J. Probability* 12 (2007), 1568–1599.
- [21] H. Gao, Exponential attractors for a generalized Ginzburg-Landau equation. *Appl. Math. Mech.* 16(1995), 877–882.
- [22] H. Gao, J. Duan, On the initial value problem for the generalized 2D Ginzburg-Landau equation, *J. Math. Anal. Appl.* 216 (1997), 536–548.
- [23] H. Gao, B. Guo, Numbers of determining nodes for a generalized Ginzburg-Landau equation. *Prog. Natur. Sci.* 5(1995), 636–638.
- [24] H. Gao, B. Guo, Finite dimensional inertial forms for 1D generalized Ginzburg-Landau equation. *Sci. China Ser. A* 25(1995), 1233–1247.
- [25] J. M. Ghidaglia, B. Heron, Dimension of the attractor associated to the Ginzburg-Landau equation, *Physica D* 28 (1987), 282–304.
- [26] J. Ginibre, G. Velo, The Cauchy problem in local spaces for the complex Ginzburg-Landau equation. I. Compactness method, *Physica D* 95(1996), 191–228; II. Contraction method, *Commun. Math. Phys.* 187(1997), 45–79.
- [27] Boling Guo and B. Wang, Finite dimensional behaviour for the derivative Ginzburg-Landau equation in two spatial dimensions, *Physica D* 89 (1995), 83–99.
- [28] Boling Guo and G. Wang, The attractor of the stochastic generalized Ginzburg-Landau equation. *Sci. in China (Ser. A)* 51(5), 2008, 955–964.
- [29] J. K. Hale, Asymptotic behavior of dissipative systems, *Mathematical Surveys and Monographs*, 25. Amer. Math. Soc., Providence, R. I. , 1998
- [30] O. Ladyzhenskaya. A dynamical system generated by the Navier–Stokes equations. *Journal of Soviet Mathematics*, 3:458–479, 1975.
- [31] C. D. Levermore, M. Oliver, The complex Ginzburg-landau equation as a model problem, in *Dynamical Systems and Probabilistic Methods for Nonlinear Waves*, *Lectures in Applied Mathematics*, Vol. 31, 1996, 141–189.
- [32] D. Li, Boling Guo, Existence of periodic solutions for 3-D complex Ginzburg-Landau equation, *J. Partial Diff. Eq.* 17(2004), no. 1, 12–28.
- [33] Y. Li, Boling Guo, Global existence of solutions to the 2D Ginzburg-Landau equation, *J. Math. Anal. Appl.* 249 (2000), 412–432.
- [34] J. L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications. Vol.1. *Travaux et Recherches Mathématiques*, 17. Dunod, Paris, 1968.
- [35] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems. *Progress in Non-linear Differential Equations and Their Applications*, Vol. 16. Basel; Boston; Berlin: Birkhäuser, 1995.
- [36] A. Mielke, The complex Ginzburg-Landau equation on large and unbounded domain: sharper bounds and attractors. *Nonlinearity* 10 (1997), no.1, 199–222.

- [37] A. Mielke, Bounds for the solutions of the complex Ginzburg-Landau equation in terms of the dispersion parameters. *Phys. D* 117 (1998), no. 1-4, 106–116.
- [38] A. Mielke, The Ginzburg-Landau equation in its role as a modulation equation. *Handbook of dynamical systems*, Vol.2, 759–834. North-Holland, Amsterdam, 2002.
- [39] I. Moise, R. Rosa and X. Wang, Attractors for non-compact semigroups via energy equations. *Nonlinearity* 11 (1998), 1369–1393.
- [40] E. Pereira, Relaxation properties of weakly coupled stochastic Ginzburg-Landau models under intense noise. *Phys. Rev. E* 65(5), 2002.
- [41] R. Rosa, The global attractor for the 2D Navier-Stokes flow on some unbounded domains. *Nonlinear Anal.* 32(1), 1998, 71–85.
- [42] J. Rougemont, Space-time invariant measures, entropy and dimension for stochastic Ginzburg-Landau equations. Preprint.
- [43] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. Springer-Verlag, Berlin-Heidelberg-New York, second edition, 1997.
- [44] R. Temam. *Navier-Stokes Equation*. Amsterdam: North-Holland, 3rd edition, 1984.
- [45] M. I. Vishik, V. V. Chepyzhov, Averaging of trajectory attractors of evolution equations with rapidly oscillating terms, *Sbornik: Math.* 192 (2001), no. 1, 11–47.
- [46] J. B. Walsh, An Introduction to Stochastic Partial Differential Equations, *Lecture Notes in Math.* 1180 (1986), 265–439.
- [47] B. Wang, Random attractors for the stochastic Benjamin-Bona-Mahony equation on unbounded domains. *J. Differ. Equations* 246(6) (2009), 2506–2537.
- [48] B. Wang, Random Attractors for the Stochastic FitzHugh-Nagumo System on Unbounded Domains. Preprint, 2008.
- [49] B. X. Wang, The limit behavior of solutions for the Cauchy problem of the complex Ginzburg-Landau equation, *Commun. Pure Appl. Math.* 55 (2002), 481–508.
- [50] B. X. Wang, B. Guo, L. Zhao, The global well-posedness and spatial decay of solutions for the derivative complex Ginzburg-Landau equation in  $H^1$ . *Nonlinear Analysis* 57 (2004), 1059–1076.
- [51] D. Yang, The asymptotic behavior of the stochastic Ginzburg-Landau equation with multiplicative noise. *J. Math. Phys.* 45(11), 2004, 4064–4076.

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