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LOCAL EXISTENCE AND UNIQUENESS IN THE LARGEST CRITICAL SPACE FOR A SURFACE GROWTH MODEL

DIRK BLÖMKER AND MARCO ROMITO

ABSTRACT. We show the existence and uniqueness of solutions (either local or global for small data) for an equation arising in different aspects of surface growth. Following the work of Koch and Tataru we consider spaces critical with respect to scaling and we prove our results in the largest possible critical space such that weak solutions are defined. The uniqueness of global weak solutions remains unfortunately open, unless the initial conditions are sufficiently small.

1. INTRODUCTION

The analysis of mathematical models for the study of surface growth has attracted a lot of attention in recent years, one can see for example the reviews in [1, 11] and numerous recent publications. See for example [22, 23, 21, 7, 18, 17, 9], which we comment in detail later.

In this article we consider a model arising in the growth of amorphous surfaces which is described by the following partial differential equation,

$$(1.1) \qquad \partial_t h + \Delta^2 h + \Delta |\nabla h|^2 = 0.$$

on the whole $\mathbf{R}^d$ or with periodic boundary conditions. The function $h(t, \cdot)$ models a height profile at time $t > 0$, so $d = 1$ and $d = 2$ are the physically relevant dimensions. In view of this and of Proposition 2.2 we will restrict the analysis to the case $d \leqslant 3$ throughout this paper (although most of the computation holds without restrictions on the dimension).

Equation (1.1), which is sometimes referred to as a conservative version of the Kuramoto-Sivashinsky equation, arises also in several other models for surface growth. The two–dimensional version was suggested in [22, 23, 21] as a phenomenological model for the growth of an amorphous surface ($Zr_{65}Al_{7,5}Cu_{27,5}$)

and more recently as a model in surface erosion using ion-beam sputtering [7, 18, 17]. The one-dimensional equation appeared as a model for the boundaries of terraces in the epitaxy of Silicon [9].

For simplicity of presentation we consider the rescaled version (1.1) with a–dimensional length-scales. Furthermore, we have ignored lower order terms like the Kuramoto-Sivashinsky term $-|\nabla h|^2$ or a linear instability given by $+\Delta h$. These terms can easily be incorporated in the result.

In the physical literature equation (1.1) is usually subject to space-time white noise, which we also have neglected for simplicity of presentation. Indeed, using the standard method of looking at the difference between h and the stochastic convolution, the stochastic PDE can be transformed in a random PDE. If the stochastic convolution is sufficiently regular, then for each instance of chance the path-wise solvability for the stochastic PDE is completely analogous to the results presented here and one only needs to consider additional lower order terms. This will be done with more details later in Section 5.

A crucial open problem for equation (1.1), is the fact that the uniqueness of global solutions is not known. We remark that numerical experiments do not report any problems of blow up, see Hoppe and Nash [14, 13], or the previously stated physics literature. Numerical experiments from Blömker, Gugg and Raible [4] furthermore indicate a fast convergence of spectral Galerkin methods for averaged surface roughness for the stochastic PDE.

The existence of global weak solutions in dimension $d = 1$ on bounded domains has been studied in [4] (see also the references therein), based on spectral Galerkin methods. The crucial estimates are energy-type inequalities which allow for uniform bounds on the L^2-norm. The method has been significantly extended by Blömker, Flandoli and Romito [2] in order to verify the existence of a solution that defines a Markov process. Winkler and Stein [25] used Rothe's method to verify the existence of a global weak solution, this result has been recently extended by Winkler [27] to the two–dimensional case, using energy type estimates for $\int e^h \, dx$.

The authors have showed in [5] the uniqueness of local solutions with initial values in the critical Hilbert space $H^{1/2}$ in the one dimensional case. Local uniqueness of continuous solutions in $W^{1,4}$ for the stochastic PDE in dimension $d = 1, 2$ can be found in [3]. A regularized problem with a cut-off in the nonlinearity in dimension $d = 2$ has been studied in Hoppe, Linz and Litvinov [12].

In this paper we study existence and uniqueness of solutions with initial data in a space of BMO–type, which contains all previous spaces where analogous results were proved. For periodic boundary conditions it allows for unique local solutions with arbitrary initial data in $H^{1/2}$ or the space of continuous functions C^0.

A weak solution for (1.1) with initial condition $h_0 \in L^1_{loc}(\mathbf{R}^d)$ is any distribution h on $\mathbf{R}^d$ with locally square integrable gradient $\nabla h \in L^2_{loc}([0, \infty) \times \mathbf{R}^d)$ such

that for every smooth and compactly supported test function $\phi \in C_c^\infty([0,\infty) \times \mathbf{R}^d)$,

$$(1.2) \quad \int_0^\infty \int_{\mathbf{R}^d} h(t,x) \frac{\partial \phi}{\partial t}(t,x)\,dx\,dt - \int_0^\infty \int_{\mathbf{R}^d} h(t,x)\Delta^2\phi(t,x)\,dx\,dt +$$

$$-\int_0^\infty \int_{\mathbf{R}^d} |\nabla h(t,x)|^2 \Delta\phi(t,x)\,dx\,dt = -\int_{\mathbf{R}^d} h_0(x)\phi(0,x)\,dx.$$

Note that $\nabla h \in L^2_{\text{loc}}([0,\infty) \times \mathbf{R}^d)$ implies also $h \in L^2_{\text{loc}}([0,\infty) \times \mathbf{R}^d)$ (cf. Lemma 2.1) and thus all terms in in (1.2) are well-defined, not only in the sense of distributions. Moreover, the solution is only defined up to constants.

Following the remarkable paper by Koch and Tataru [15], this article provides a local existence and uniqueness result in the largest critical space, where the above stated definition of weak solutions makes sense. As the equation is translation invariant (in space) and invariant with respect to the scaling

$$(1.3) \qquad\qquad h(t,x) \longrightarrow h(\lambda^4 t, \lambda x),$$

we consider the following scaling-aware invariant version of the L^2_{loc} space for the gradient ∇h,

$$(1.4) \qquad \|h\|_{\mathcal{X}^0} := \left(\sup_{x \in \mathbf{R}^d, R>0} \left\{ \frac{1}{R^{d+2}} \int_0^{R^4} \int_{B_R(x)} |\nabla h|^2 \, dy\, dt \right\} \right)^{\frac{1}{2}}.$$

The paper is organized as follows. In Section 2 we discuss the space defined by (1.4) and show an equivalent representation, and its relation to BMO-type spaces. Some admissible initial conditions and examples are discussed in Section 3.

Based on Banach's fixed-point iteration-scheme, Section 4 provides the existence and uniqueness results. Section 5 contains some details on the extension of such results to the stochastically forced case. We close the paper with Section 6, where we show smoothness of solutions.

2. FUNCTION SPACES

Recall first the following result, an easy consequence of Poincaré's inequality, which ensures that all integrals in (1.2) are well defined.

Lemma 2.1. *If u is a distribution on $\mathbf{R}^d$ such that $\nabla u \in L^2_{\text{loc}}([0,\infty) \times \mathbf{R}^d)$, then $u \in L^2_{\text{loc}}([0,\infty) \times \mathbf{R}^d)$ and thus $u \in L^2_{\text{loc}}([0,\infty), H^1_{\text{loc}}(\mathbf{R}^d))$.*

We consider the linear space $\mathcal{X}^0$ of functions h with $|\nabla h| \in L^2_{\text{loc}}((0,\infty) \times \mathbf{R}^d)$ and thus $h \in L^2_{\text{loc}}((0,\infty) \times \mathbf{R}^d)$ such that the quantity $\|h\|_{\mathcal{X}^0}$ is finite.

Furthermore, we define the linear space $\mathcal{X}$ of functions such that the following norm is finite.

$$(2.1) \qquad\qquad \|k\|_{\mathcal{X}} = \sup_{t>0}\left\{ t^{\frac{1}{4}} \|\nabla k(t)\|_\infty \right\}.$$

A local in time version of these spaces can be defined for any $R > 0$ by

$$\|k\|_{\mathcal{X}_R^0}^2 := \sup_{x \in \mathbf{R}^d, r \leqslant R} \left\{ \frac{1}{r^{d+2}} \int_0^{r^4} \int_{B_r(x)} |\nabla k(t,y)|^2 \, dy \, dt \right\},$$

$$\|k\|_{\mathcal{X}_R} := \sup_{t \leqslant R^4} \left(t^{\frac{1}{4}} \|\nabla k(t)\|_\infty \right).$$

for functions $k : [0, R^4] \times \mathbf{R}^d \to \mathbf{R}$. Note that we always identify functions that differ only by a constant. This is motivated by the fact that the equation is mass-conservative, if the total mass $\int h \, dx$ is finite. Furthermore, solutions are only defined up to additive constants.

In order to track the corresponding spaces for initial values, let $A = \Delta^2$. Consider the Green's function $G : [0, \infty) \times \mathbf{R}^d \to \mathbf{R}$ associated to the operator A, where $G(t, x)$ has the Fourier transform (w.r.t. x) $\widehat{G}(t, \xi) = e^{-t|\xi|^4}$. By scaling we obtain

$$G(t, x) = t^{-d/4} g(x t^{-1/4}), \quad \text{where} \quad g(x) = G(1, x).$$

The function g is in the Schwartz class since $\widehat{g}(\xi) = e^{-|\xi|^4}$.

Define the semigroup e^{-tA} by the convolution $e^{-tA} k = G(t, \cdot) \star k$. Define the space $\mathcal{B}^0$ of all functions $k : \mathbf{R}^d \to \mathbf{R}^d$ such that the *bi-caloric* extension $e^{-tA} k$ is in $\mathcal{X}^0$, endowed with the semi-norm

$$\|k\|_{\mathcal{B}^0} := \| e^{-tA} k\|_{\mathcal{X}^0},$$

and the space $\mathcal{B}$ of all functions $k : \mathbf{R}^d \to \mathbf{R}^d$ such that

$$\|k\|_{\mathcal{B}} := \| e^{-tA} k\|_{\mathcal{X}}$$

is finite, endowed with the semi-norm $\| \cdot \|_{\mathcal{B}}$. Define similarly the local versions $\mathcal{B}_R^0$ and $\mathcal{B}_R$ of these spaces.

In contrast to the case of Navier–Stokes in dimension three [15], the spaces $\mathcal{B}$ and $\mathcal{B}^0$ (as well as their local counterparts) turn out to be equivalent, as shown by the proposition stated below. This makes the analysis simpler than in the Navier–Stokes case. This proposition is the only reason, why we restrict to dimension $d \leqslant 3$, as we rely for simplicity on the equivalence of $\mathcal{B}$ and $\mathcal{B}0$.

Proposition 2.2. *Assume* $d \leqslant 3$. *Then there are constants* $c_1, c_2 > 0$ *such that*

(2.2) $$c_1 \| \cdot \|_{\mathcal{B}} \leqslant \| \cdot \|_{\mathcal{B}^0} \leqslant c_2 \| \cdot \|_{\mathcal{B}}$$

and

(2.3) $$c_1 \| \cdot \|_{\mathcal{B}_R} \leqslant \| \cdot \|_{\mathcal{B}_R^0} \leqslant c_2 \| \cdot \|_{\mathcal{B}_R},$$

for every $R > 0$.

Moreover, for every $d \geqslant 1$, *there exists* $c_3 > 0$ *such that*

$$\|k\|_{\mathcal{B}^0} \leqslant c_3 \|k\|_{BMO(\mathbf{R}^d)},$$

so $BMO(\mathbf{R}^d) \subset \mathcal{B}^0$ *and in particular* $L^\infty(\mathbf{R}^d) \subset \mathcal{B}^0$.

For a definition of the space $\mathrm{BMO}(\mathbf{R}^d)$ of functions of bounded mean oscillation and its properties, we refer to Stein [24]. Here we only use an equivalent norm on $\mathrm{BMO}(\mathbf{R}^d)$ given by the Carleson measure characterization (see (2.4)).

Proof. We start by proving (2.2). The inequality on the right holds in any dimension $d \geqslant 1$ since it is straightforward to check that there is $c > 0$ such that $\|\cdot\|_{\mathcal{X}^0} \leqslant c\|\cdot\|_{\mathcal{X}}$. For the inequality on the left, we need to show that for $k \in \mathcal{B}^0$, $x \in \mathbf{R}^d$, $t > 0$,

$$|t^{\frac{1}{4}}\nabla(e^{-tA}k)(x)| = t^{\frac{1}{4}}\left|\nabla\int_{\mathbf{R}^d} G(t, x-y)k(y)\,dy\right| \leqslant c\|k\|_{\mathcal{B}^0}.$$

By scaling and translations invariance, it is sufficient to show the statement for $t = 1$ and $x = 0$. Since

$$e^{-A} = \int_0^1 e^{-(1-s)A}\,e^{-sA}\,ds,$$

it follows by the Cauchy Schwartz inequality that

$$|\nabla(e^{-A}k)(0)| = \left|\int_0^1 (1-s)^{-\frac{d}{4}}\int_{\mathbf{R}^d} g\big(y(1-s)^{-\frac{1}{4}}\big)\nabla(e^{-sA}k)(y)\,dy\,ds\right|$$

$$\leqslant \sum_{n\in\mathbf{Z}^d}\left|\int_0^1 (1-s)^{-\frac{d}{4}}\int_{B_n} g\big(y(1-s)^{-\frac{1}{4}}\big)\nabla(e^{-sA}k)(y)\,dy\,ds\right|$$

$$\leqslant \sum_{n\in\mathbf{Z}^d}\left(\int_0^1 (1-s)^{-\frac{d}{2}}\int_{B_n} g\big(y(1-s)^{-\frac{1}{4}}\big)^2\,dy\,ds\right)^{\frac{1}{2}}$$

$$\times\left(\int_0^1\int_{B_n} |\nabla(e^{-sA}k)(y)|^2\,dy\,ds\right)^{\frac{1}{2}}$$

$$\leqslant c\|k\|_{\mathcal{B}^0}\sum_{n\in\mathbf{Z}^d}\left(\int_0^1 s^{-\frac{d}{2}}\int_{B_n} g\big(ys^{-\frac{1}{4}}\big)^2\,dy\,ds\right)^{\frac{1}{2}},$$

where B_n are the balls of centre $2d^{-1/2}n$ and radius 1 (so that their union covers $\mathbf{R}^d$). By a change of variables,

$$I_n := \int_0^1 s^{-\frac{d}{2}}\int_{B_n} g\big(ys^{-\frac{1}{4}}\big)^2\,dy\,ds = \int_0^1 s^{-\frac{d}{4}}\int_{s^{-1/4}B_n} g(z)^2\,dz\,ds.$$

First, $|I_n| \leqslant C$ for all $n \in \mathbf{Z}^d$, as $d \leqslant 3$ and $g \in L^2(\mathbf{R}^d)$. Note that $d \leqslant 3$ is necessary, as for $0 \in B_n$ we have $\int_{s^{-1/4}B_n} g(z)^2\,dz \uparrow \|g\|_{L^2}^2$ for $s \downarrow 0$.

For the convergence of the series consider for $s \in (0,1)$ and $0 \notin B_n$ (i.e. $2|n| > \sqrt{d}$) that

$$\int_{s^{-1/4}B_n} g(z)^2\,dz \leqslant \int_{\mathbf{R}^d} |g(z)|\,dz \cdot \sup\left\{|g(z)| : |z| \in s^{-1/4}B_n\right\}$$

$$\leqslant C\sup\left\{|g(z)| : |z| > 2|n|d^{-1/2} - 1\right\},$$

which can be bounded by a summable term, since g is in the Schwartz class. The inequality (2.3) for the local spaces proceeds similarly.

Let ϕ be in the Schwartz class, with $\widehat{\phi} > 0$, and set $\phi_t(x) = t^{-d}\phi(\frac{x}{t})$. By the Carleson measure characterization of BMO (see Theorem 3, Section 4.3 of Stein [24]), we have that (up to a constant)

$$(2.4) \qquad \|k\|_{\mathrm{BMO}}^2 = \sup_{x \in \mathbf{R}^d, R > 0} \left\{ \frac{1}{R^d} \int_0^R \int_{B_R(x)} \frac{1}{t} |((\nabla\phi)_t \star k)(y)|^2 \, dy \, dt \right\}.$$

Note that this is an equivalent norm to the standard definition. We could relax the conditions on ϕ, but we are going to use $\phi = g$ which satisfies the stronger condition $\widehat{g} > 0$. On the other hand the definition of $\mathcal{B}^0$ given above can be restated (up to a constant) as

$$\|k\|_{\mathcal{B}^0}^2 = \sup_{x \in \mathbf{R}^d, R > 0} \left\{ \frac{1}{R^{d+2}} \int_0^R \int_{B_R(x)} t |((\nabla\phi)_t \star k)(y)|^2 \, dy \, dt \right\},$$

with $\phi = g$, and so $\|k\|_{\mathcal{B}^0} \leqslant c_3 \|k\|_{\mathrm{BMO}(\mathbf{R}^D)}$, as $t^2 \leqslant R^2$. $\qquad\square$

3. EXAMPLES

In view of Theorem 4.3 we wish to discuss for which initial conditions it is possible to find R such that the initial condition is small in the $\mathcal{B}_R$ norm. To this aim define

$$\mathcal{Z} = \{k : \mathbf{R}^d \to \mathbf{R} : \|k\|_{\mathcal{B}_R} \to 0 \text{ as } R \downarrow 0\}.$$

We later see in Theorem 4.3 that functions in $\mathcal{Z}$ correspond to initial conditions, where it is possible to solve the equation (1.1) locally for a small time interval.

The next lemma shows that the whole $L^\infty(\mathbf{R}^d)$, although it is contained in $\mathcal{B}^0$, is not contained in $\mathcal{Z}$. We will see later that this implies that our method of proof fails to provide local uniqueness of solutions for some initial conditions in $L^\infty(\mathbf{R}^d)$, although $L^\infty(\mathbf{R}^d) \subset \mathcal{B}_0$.

Lemma 3.1. *The following statements hold,*

- *there are functions in $L^\infty(\mathbf{R}^d)$ not belonging to $\mathcal{Z}$,*
- *if $k : \mathbf{R}^d \to \mathbf{R}$ is bounded and uniformly continuous, then $k \in \mathcal{Z}$,*
- *if $k : \mathbf{R}^d \to \mathbf{R}$ has bounded gradient on $\mathbf{R}^d$, then $k \in \mathcal{Z}$.*

Proof. We prove the first statement. Since the Green's function tensorises, it is enough to find a counterexample in dimension $d = 1$. Let $k(x) = \mathbb{1}_{[-1,1]}(x)$, then

$$t^{\frac{1}{4}} |(e^{-tA} k)_x(x)| = \left| \frac{1}{t^{\frac{1}{4}}} \int_{\mathbf{R}} g'(\tfrac{x-y}{t^{1/4}}) k(y) \, dy \right| = |g(\tfrac{x+1}{t^{1/4}}) - g(\tfrac{x-1}{t^{1/4}})|,$$

hence (choosing $x = -1$ and $t = R$), since $g(x) \to 0$ as $x \to \infty$,

$$\sup_{t \leqslant R, x \in \mathbf{R}} \left\{ t^{\frac{1}{4}} |(e^{-tA} k)_x(x)| \right\} \geqslant |g(0) - g(\tfrac{-2}{R^{1/4}})| \longrightarrow g(0) > 0.$$

Assume now that k is bounded and uniformly continuous and fix $\epsilon > 0$. By uniform continuity there is $\delta > 0$ such that $|k(x) - k(y)| \leqslant \epsilon$ for all $x, y \in \mathbf{R}^d$ with $|x - y| \leqslant \delta$. Since the integral of ∇g is zero,

$$t^{\frac{1}{4}}|\nabla(e^{tA}k)(x)| = \left|\int_{\mathbf{R}^d} \nabla g(z)\big(k(x - zt^{\frac{1}{4}}) - k(x)\big)\,dz\right|$$

$$= \left|\int_{t^{\frac{1}{4}}|z|\geqslant\delta} \nabla g(z)\big(k(x - zt^{\frac{1}{4}}) - k(x)\big)\,dz\right|$$

$$+ \left|\int_{t^{\frac{1}{4}}|z|\leqslant\delta} \nabla g(z)\big(k(x - zt^{\frac{1}{4}}) - k(x)\big)\,dz\right|$$

$$\leqslant 2\|k\|_\infty \int_{t^{\frac{1}{4}}|z|\geqslant\delta} |\nabla g(z)|\,dz + \epsilon\|\nabla g\|_{L^1(\mathbf{R}^d)},$$

hence $\limsup_{R\to 0}\|k\|_{\mathcal{B}_R} \leqslant \epsilon\|\nabla g\|_{L^1}$ and as $\epsilon \downarrow 0$, the claim follows.

Finally, let $k : \mathbf{R}^d \to \mathbf{R}$ be such that $\|\nabla k\|_{L^\infty} < \infty$, then

$$t^{\frac{1}{4}}|\nabla(e^{tA}k)(x)| = t^{-\frac{d-1}{4}}\left|\int_{\mathbf{R}^d} g\big(\tfrac{x-y}{t^{1/4}}\big)\nabla k(y)\,dy\right| \leqslant t^{\frac{1}{4}}\|g\|_{L^1}\|\nabla k\|_{L^\infty},$$

and hence $\|k\|_{\mathcal{B}_R} = R^{\frac{1}{4}}\|g\|_{L^1}\|\nabla k\|_{L^\infty}$. $\qquad\square$

Example 3.2. For $d = 1$ consider $k(x) = \log|x|$. This function has paramount importance since it is a stationary solution for problem (1.1) (see [5]). It is an interesting fact that k is neither a weak nor a mild solution (for instance due to Theorem 6.1 which ensures smoothness of solutions according to Definition 4.1). Here we will show that $k \in \mathcal{B}$ but $k \notin \mathcal{Z}$.

Indeed, consider first,

$$t^{\frac{1}{4}}\left|(e^{-tA}k)_x(x)\right| = \left|\int_{\mathbf{R}} g'(z)\log|x - t^{\frac{1}{4}}z|\,dz\right|$$

$$= \left|\int_{\mathbf{R}} g'(z)(\log(t^{\frac{1}{4}}) + \log|t^{-\frac{1}{4}}x - z|)\,dz\right|$$

$$= \left|\int_{\mathbf{R}} g'(z)\log|t^{-\frac{1}{4}}x - z|\,dz\right|,$$

where we used that the integral over g' is zero. Now substitute $\tilde{x} = t^{-\frac{1}{4}}x$ to obtain

$$\sup_{x\in\mathbf{R}}\left\{t^{\frac{1}{4}}|(\partial_x e^{-tA}k)(x)|\right\} = \sup_{x\in\mathbf{R}}\left\{\left|\int_{\mathbf{R}} g'(z)\log|x - z|\,dz\right|\right\} = \|\partial_x e^{-A}k\|_\infty.$$

So it is easy to see that $\|k\|_\mathcal{B}$ is finite, but $\|k\|_{\mathcal{B}_R}$ is independent of R and does not converge to 0.

Example 3.3. Consider $d = 1$ and, for $\alpha > 0$, $k_\alpha(x) = |x|^\alpha$. Then $k_\alpha \notin \mathcal{B}$, but $\|k_\alpha\|_{\mathcal{B}_R} \to 0$. So $\mathcal{Z}$ may contain certain unbounded functions, which are not in $\mathcal{B}$.

As in the previous example,

$$t^{\frac{1}{4}}|(\partial_x e^{-tA} k_\alpha)(x)| = \left|\int_{\mathbf{R}} g'(z)|x - t^{\frac{1}{4}}z|^\alpha \, dz\right| = t^{\frac{\alpha}{4}}\left|\int_{\mathbf{R}} g'(z)|t^{-\frac{1}{4}}x - z|^\alpha \, dz\right|$$

Thus $\|t^{\frac{1}{4}}\partial_x(e^{-tA} k_\alpha)\|_\infty = t^{\frac{\alpha}{4}}\|\partial_x(e^{-A} k_\alpha)\|_\infty$, hence $\|k_\alpha\|_{\mathcal{B}_R} \to 0$ for $R \to 0$ but $\|k_\alpha\|_{\mathcal{B}} = \infty$.

Next lemma, together with the main Theorem 4.3, shows that problem (1.1) has locally a unique solution for any $\dot{H}^{d/2}(\mathbf{R}^d)$ initial conditions. This recovers and extends a result proved in dimension $d = 1$ in [5].

Lemma 3.4. *The homogeneous space $\dot{H}^{d/2}(\mathbf{R}^d)$ is contained in $\mathcal{Z}$, where*

$$\dot{H}^{d/2}(\mathbf{R}^d) = \left\{k : \mathbf{R}^d \to \mathbf{R} : \|k\|_{\dot{H}^{d/2}} := \int_{\mathbf{R}^d} |\xi|^d |\widehat{k}(\xi)|^2 \, d\xi < \infty\right\}$$

and $\widehat{k}$ denotes Fourier transform of k.

Proof. If $t > 0$ and $x \in \mathbf{R}^d$, by using the properties of Fourier transform and convolution,

$$\begin{aligned}
t^{\frac{1}{4}}|\nabla(e^{tA} k)(x)| &= \frac{1}{t^{\frac{d}{4}}}\left|\int_{\mathbf{R}^d} (\nabla g)\big((x-y)t^{-\frac{1}{4}}\big) k(y) \, dy\right| \\
&= \frac{1}{t^{\frac{d}{4}}}\left|\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} (\nabla g)\big((x-y)t^{-\frac{1}{4}}\big)\widehat{k}(\xi) \, e^{i\xi \cdot y} \, dy \, d\xi\right| \\
&= \left|\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \nabla g(y) \, e^{-i\xi z t^{\frac{1}{4}}} \, \widehat{k}(\xi) \, e^{i\xi \cdot x} \, dy \, d\xi\right| \\
&\leqslant t^{\frac{1}{4}}\int_{\mathbf{R}^d} |\xi| \, |\widehat{k}(\xi)| \, e^{-|\xi|^4 t} \, d\xi.
\end{aligned}$$

Given $a > 0$, split the integral in the last line of formula above in two pieces ⓛ and ⓗ, corresponding to the domains of integration $\{|\xi| \leqslant a\}$ and $\{|\xi| > a\}$ respectively. We estimate ⓛ using the Cauchy–Schwarz inequality,

$$ⓛ \leqslant t^{\frac{1}{4}}\int_{|\xi|\leqslant a} |\xi|^{1-\frac{d}{2}}\big(|\xi|^{\frac{d}{2}}|\widehat{k}(\xi)|\big) \, d\xi \leqslant t^{\frac{1}{4}}\|k\|_{\dot{H}^{d/2}}\left(\int_{|\xi|\leqslant a} |\xi|^{2-d} \, d\xi\right)^{\frac{1}{2}} \leqslant c a t^{\frac{1}{4}}\|k\|_{\dot{H}^{d/2}},$$

while by a change of variables and Cauchy–Schwarz' inequality again,

$$ⓗ \leqslant t^{\frac{1}{4}}\left(\int_{|\xi|\geqslant a} |\xi|^d |\widehat{k}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbf{R}^d} |\xi|^{2-d} \, e^{-2t|\xi|^4} \, d\xi\right)^{\frac{1}{2}} \leqslant c\left(\int_{|\xi|\geqslant a} |\xi|^d |\widehat{k}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}.$$

In conclusion

$$\|k\|_{\mathcal{B}_R} \leqslant c a R \|k\|_{\dot{H}^{d/2}} + c\left(\int_{|\xi|\geqslant a} |\xi|^d |\widehat{k}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}},$$

so we see that $\limsup_{R\to 0} \|k\|_{\mathcal{B}_R}$ is bounded by a quantity which converges to 0 as $a \uparrow \infty$. $\qquad\square$

4. THE FIXED POINT ARGUMENT

Define the map

$$\mathscr{V}(h, k)(t) = \int_0^t \Delta\big(e^{-(t-s)A} \nabla h(s) \cdot \nabla k(s)\big) \, ds$$

and set

(4.1)
$$\mathscr{F}(h)(t) = e^{-tA} h_0 - \mathscr{V}(h, h)(t).$$

We will use the following concept of a mild solution, which is given as a solution of the variation of constants formula in (4.2).

Definition 4.1. We say that $h \in \mathcal{X}$ solves (1.1) with initial condition $h_0 \in \mathcal{B}$, if for all $t > 0$

(4.2)
$$h(t) = e^{-tA} h_0 - \mathscr{V}(h, h)(t) \,.$$

We call $h \in \mathcal{X}_R$ a local solution, if (4.2) holds only for $t \in [0, R^4]$.

The following Lemma is crucial for the proof of uniqueness and existence. It verifies that the nonlinear part is locally Lipschitz.

Lemma 4.2. *The map $\mathscr{V}$ is bi-linear continuous from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{X}$ and from $\mathcal{X}_R \times \mathcal{X}_R$ to $\mathcal{X}_R$, for all $R > 0$.*

Proof. The bilinearity is obvious. For the boundedness let $x \in \mathbf{R}^d$ and $t > 0$, then

$$
\begin{aligned}
|\nabla \mathscr{V}(h, k)(t, x)| &= \left| \int_0^t \int_{\mathbf{R}^d} \nabla \Delta G(t - s, x - y) \nabla h(s, y) \nabla k(s, y) \, dy \, ds \right| \\
&= \left| \int_0^t \frac{1}{(t - s)^{\frac{d+3}{4}}} \int_{\mathbf{R}^d} (\nabla \Delta g)\left(\frac{x - y}{(t - s)^{1/4}}\right) \nabla h(s, y) \nabla k(s, y) \, dy \, ds \right| \\
&\leqslant \|h\|_{\mathcal{X}} \|k\|_{\mathcal{X}} \int_0^t \frac{1}{(t - s)^{\frac{d+3}{4}} \sqrt{s}} \int_{\mathbf{R}^d} \left| (\nabla \Delta g)\left(\frac{x - y}{(t - s)^{1/4}}\right) \right| dy \, ds \\
&\leqslant \|h\|_{\mathcal{X}} \|k\|_{\mathcal{X}} \|g\|_{W^{3,1}(\mathbf{R}^d)} \int_0^t \frac{1}{(t - s)^{3/4} \sqrt{s}} \, ds \\
&\leqslant t^{-1/4} B(\tfrac{1}{2}, \tfrac{1}{4}) \|h\|_{\mathcal{X}} \|k\|_{\mathcal{X}} \|g\|_{W^{3,1}(\mathbf{R}^d)} \\
&= c_4 t^{-1/4} \|h\|_{\mathcal{X}} \|k\|_{\mathcal{X}},
\end{aligned}
$$

where B is the Beta function. The corresponding inequality for the local space $\mathcal{X}_R$ proceeds similarly. $\qquad \square$

Using the previous Lemma, we can now state and prove our main result. The first part states global existence of unique solutions, while the second part is about local existence of solutions.

Theorem 4.3. *There is $\delta > 0$ such that if $\|h_0\|_{\mathcal{B}^0} \leqslant \delta$, then there exists a unique (global) solution in $\mathcal{X}$ of (1.1) with initial condition h_0.*

Moreover, if $\|h_0\|_{\mathcal{B}^0_R} \leqslant \delta$, then there is a unique local solution in $\mathcal{X}_R$ of (1.1) on $[0, R^4]$ with initial condition h_0.

Finally, if h_0 is periodic and small in $\mathcal{B}^0_R$ for some $R > 0$ (or it is small in $\mathcal{B}^0$), then the solution is also periodic.

In particular, $\|h_0\|_{\mathcal{B}^0_R} \leqslant \delta$ is true for a suitable value of R for all $h_0 \in \mathcal{Z}$.

Proof. We prove the first statement by a fixed point iteration argument. Let c_4 be the constant defined in the proof of Lemma 4.2 and choose $\delta > 0$, $K > 0$ such that

$$1 - \frac{4c_4\delta}{c_1} > 0, \qquad \frac{1}{2c_4}\left(1 - \sqrt{1 - \frac{4c_4\delta}{c_1}}\right) \leqslant K < \frac{1}{2c_4}.$$

Define

$$(4.3) \qquad H_0 = 0, \qquad H_{n+1} = \mathscr{F}(H_n) = e^{-tA}h_0 - \mathscr{V}(H_n, H_n),$$

then $\|H_1\|_{\mathcal{X}} \leqslant \frac{\delta}{c_1}$ and it is easy to check by induction (and by the choice of δ and K) that $\|H_n\|_{\mathcal{X}} \leqslant K$ for all n. Then

$$\|H_{n+1} - H_n\|_{\mathcal{X}} = \|\mathscr{V}(H_n, H_n) - \mathscr{V}(H_{n-1}, H_{n-1})\|_{\mathcal{X}} \leqslant 2c_4 K \|H_n - H_{n-1}\|_{\mathcal{X}}$$

and so $(H_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{X}$ to a fixed point of $\mathscr{F}$.

The same proof works for local spaces, since both constants c_1 and c_4 do not depend on R. Finally, if h_0 is periodic, the statement follows by translation invariance and uniqueness. $\square$

Remark 4.4 (Forward self–similar solutions). The theorem above allows to show the existence of self–similar solutions, namely solutions invariant for the scaling (1.3). Indeed, assume to have $h_0 \in \mathcal{B}^0$ (or in a local space) such that $h_0(\lambda x) = h_0(x)$ for all $\lambda > 0$, then it is easy to verify that H_1 is invariant for the scaling (1.3) and that $\mathscr{V}(h, h)$ is also invariant if so is h. In conclusion the whole sequence $(H_n)_{n\in\mathbb{N}}$ defined in (4.3) is invariant, as well as its limit.

Given a (forward) self–similar solution h, one can write $h(t, x) = \psi(x/t^{1/4})$, where $\psi(x) = h(1, x)$ solves the equation

$$\Delta^2\psi + \Delta|\nabla\psi|^2 - \frac{1}{4}x \cdot \nabla\psi = 0.$$

The simplest case corresponds to $d = 1$, where the only admissible initial conditions are all functions h_0 constants on $(-\infty, 0)$ and on $(0, \infty)$ (possibly with different values on the two half–lines), with $\|h_0\|_{\mathcal{B}_R} = |h_0(1) - h_0(-1)| \, \|g\|_{L^\infty}$.

Backward self–similar solutions might provide examples of solutions with blow–up. Due to the scaling of the problem, the quantity blowing up is related to the derivative of the solution. We do not know if backward self–similar solutions exist (notice that backward self–similar solutions do not exist for the Navier–Stokes equations, see [19]).

5. The stochastic problem

In this section we give a short outline of the proof of local existence for the stochastic PDE, without many details on probability theory. For details we refer to [8, 6, 16]. Consider

$$(5.1) \qquad \partial_t h + \Delta^2 h + \Delta |\nabla h|^2 = \partial_t W \,,$$

where $\partial_t W$ is the generalized derivative of a Hilbert-space value Wiener process. Define the corresponding Ornstein-Uhlenbeck process for $t > 0$ as the following Itô-integral

$$(5.2) \qquad Z(t) = \int_0^t e^{-(t-s)A} dW \,.$$

Note that Z solves $\partial_t Z + \Delta^2 Z = \partial_t W$ with $Z(0) = 0$. The mild solution of (5.1) is analogous to Definition 4.1 given by a solution of

$$h(t) = e^{-tA} h_0 - \mathscr{V}(h, h)(t) + Z(t)$$

Now the main problem in the stochastic setting is to determine the regularity of Z. Once we know this, we can solve the equation using Banach's fixed point argument, as in Theorem 4.3. Moreover $v = h - Z$ solves the following random PDE

$$(5.3) \qquad \partial_t v + \Delta^2 v + \Delta |\nabla v|^2 = -\Delta |\nabla Z|^2 - 2\Delta |\nabla v \cdot \nabla Z|^2, \qquad v(0) = h_0,$$

which only contains lower order terms that do not change the proofs, once Z is sufficiently regular.

In the case of bounded intervals (i. e. $d = 1$) with periodic boundary conditions and space–time white noise the stochastic convolution Z and its derivative $\partial_x Z$ are continuous in both space and time, which can be verified using the methods in [8]. See for example [20]. This implies that almost surely $\|Z\|_{\mathcal{X}_R} \to 0$ for $R \to 0$, and we can solve the stochastic PDE (5.2) (or the random PDE (5.3)) uniquely in $\mathcal{X}_R$, for some small (random) $R > 0$ if the initial condition h_0 is such that the PDE (1.1) has a unique local solution.

An interesting question appears in the case of periodic boundary conditions and $d = 2$, as for space-time white noise the convolution Z just fails to be differentiable in space. Nevertheless, Z will be differentiable, if we consider slightly more regular noise.

For stochastic PDEs on unbounded domains one can use the formulation of Walsh [26], although one has to consider that for space–time white noise the stochastic convolution $Z(t, x)$ is unbounded for $|x| \to \infty$.

6. SMOOTHNESS OF SOLUTIONS

Following the same methods of [10], we show that solutions in $\mathcal{X}$ (or $\mathcal{X}_R$) are smooth. Define for $m \geqslant 1$,

$$\|k\|_{\mathcal{X},m} := \sup_{t>0} \left\{ t^{\frac{m+1}{4}} \sum_{|\alpha|=m+1} \|D^\alpha k\|_\infty \right\}$$

and denote by $\|\cdot\|_{\mathcal{X}_R,m}$ the corresponding local version, where for $\alpha = (\alpha_1, \ldots, \alpha_d)$ we used $D^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

Let $\mathcal{X}^m$ be the space

$$\mathcal{X}^m = \{k : \mathbf{R}^d \to \mathbf{R} : \|k\|_{\mathcal{X}^m} := \max_{0 \leqslant j \leqslant m} \|k\|_{\mathcal{X},j} < \infty\},$$

and denote by $\mathcal{X}_R^m$ the corresponding local version. For simplicity of notations we understand that $\|\cdot\|_{\mathcal{X},0} = \|\cdot\|_{\mathcal{X}}$ and for $R = \infty$ that $\mathcal{X}_\infty^m = \mathcal{X}^m$. The main theorem of this section is the following result on smoothness in space. Smoothness in time then follows from the PDE by a standard bootstrapping argument.

Theorem 6.1. *Let h be a solution of* (1.1) *in $\mathcal{X}_R$, with $0 < R \leqslant \infty$. Then $h(t) \in C_b^\infty(\mathbf{R}^d)$ for all $t \in (0, R)$.*

Proof. If the initial condition is small enough in $\mathcal{B}_R^0$, the statement follows from Proposition 6.2 below. In the general case we notice that if $h \in \mathcal{X}_R$, then $\nabla h(t)$ is bounded for all $t \in (0, R)$, therefore $h(t) \in \mathcal{Z}$, by Lemma 3.1. The conclusion then follows again from Proposition 6.2. $\qquad\square$

In order to complete the proof of the above theorem, we need the following proposition, which gives also a better estimate of the solution near $t = 0$ if the initial condition is small enough.

Proposition 6.2. *There exists $\delta > 0$ such that if $\|h_0\|_{\mathcal{B}^0} < \delta$, then the solution to* (1.1) *granted by Theorem 4.3 is in $\mathcal{X}^m$ for all $m \geqslant 1$.*

If $R > 0$ and $\|h_0\|_{\mathcal{B}_R^0} < \delta$, then the solution to (1.1) *granted by Theorem 4.3 is in $\mathcal{X}_R^m$ for all $m \geqslant 1$.*

We start by giving a slight generalization of (2.2) and (2.3).

Lemma 6.3. *Let $0 < R \leqslant \infty$ and $k \in \mathcal{B}_R^0$, then for every $m \geqslant 0$,*

(6.1) $$\sup_{t \leqslant R} \left\{ t^{\frac{m+1}{4}} \sum_{|\alpha|=m+1} \left\|D^\alpha(e^{-tA} k)\right\|_\infty \right\} \leqslant c m^d (m+1)^{\frac{m+1}{4}} \|\nabla g\|_{L^1(\mathbf{R}^d)}^m \|k\|_{\mathcal{B}_R^0}.$$

Proof. Since for $|\alpha| = m + 1$,

$$D^\alpha(e^{-tA} k) = \prod_{i=1}^d (\partial_{x_i}^{\alpha_i} e^{-\frac{\alpha_i}{m+1} tA}) k,$$

it is sufficient to show that the operator $\partial_{x_i} e^{-tA}$ maps $L^\infty(\mathbf{R}^d)$ into itself with operator norm $\|\partial_{x_i} e^{-tA}\|_{L^\infty \to L^\infty} \leqslant t^{-1/4}\|\nabla g\|_{L^1(\mathbf{R}^d)}$. This is immediate since by a change of variables,

$$t^{\frac{1}{4}}|\partial_{x_i}(e^{-tA}k)(x)| = t^{\frac{d}{4}}\left|\int_{\mathbf{R}^d}(\partial_{x_i}g)\left(\tfrac{x-y}{t^{1/4}}\right)k(y)\,dy\right| \leqslant \|\nabla g\|_{L^1(\mathbf{R}^d)}\|k\|_\infty.$$

Finally, $\#(\{\alpha : |\alpha| = m + 1\}) = \binom{m+d}{d-1} \leqslant cm^d.$ $\qquad\square$

Lemma 6.4. *There is $c_5 > 0$ such that for $m \geqslant 1$, $0 < R \leqslant \infty$ and $h, k \in \mathcal{X}_R^m$,*

(6.2)
$$\begin{aligned}
\|\mathcal{V}(h,k)\|_{\mathcal{X}_{R,m}} &\leqslant c_5 m^d (m+1)^{\frac{m+3}{2}}\|\nabla g\|_{L^1(\mathbf{R}^d)}^m\|g\|_{W^{3,1}(\mathbf{R}^d)}\|h\|_{\mathcal{X}_R}\|k\|_{\mathcal{X}_R} \\
&\quad + c_5\|h\|_{\mathcal{X}_R}\|k\|_{\mathcal{X}_{R,m}} + c_5\|h\|_{\mathcal{X}_{R,m}}\|k\|_{\mathcal{X}_R} \\
&\quad + c_5 m^d \sum_{j=1}^{m-1}\binom{m}{j}\|h\|_{\mathcal{X}_{R,j}}\|k\|_{\mathcal{X}_{R,m-j}}.
\end{aligned}$$

Proof. Fix $m \geqslant 1$, $0 < R \leqslant \infty$, $t \leqslant R$ and $h, k \in \mathcal{X}_R^m$. Consider a value $\epsilon \in (0,1)$ which will be specified later, and let $|\alpha| = m + 1$. Since $|\alpha| \geqslant 1$, there is $i \leqslant d$ such that $\alpha_i \geqslant 1$. So assume without loss of generality that $a_1 \geqslant 1$ and let $\alpha' = \alpha - (1, 0, \ldots, 0)$.

$$\begin{aligned}
D^\alpha\mathcal{V}(h,k)(t) &= \int_0^t D^\alpha\Delta\big(e^{-(t-s)A}(\nabla h(s)\nabla k(s))\big)\,ds \\
&= \int_0^{t(1-\epsilon)} D^\alpha\Delta\big(e^{-(t-s)A}(\nabla h(s)\nabla k(s))\big)\,ds \\
&\quad + \int_{t(1-\epsilon)}^t D^\alpha\Delta\big(e^{-(t-s)A}(\nabla h(s)\nabla k(s))\big)\,ds \\
&= \text{①} + \text{②}.
\end{aligned}$$

For the term ① we use the factorization introduced in the previous lemma and we proceed as in the proof of Lemma [4.2],

$$\begin{aligned}
|\text{①}| &\leqslant (m+1)^{\frac{m+3}{4}}\|\nabla g\|_{L^1}^m\|\partial_{x_1}\Delta g\|_{L^1}\|h\|_{\mathcal{X}_R}\|k\|_{\mathcal{X}_R}\int_0^{t(1-\epsilon)}s^{-\frac{1}{2}}(t-s)^{-\frac{m+3}{4}}\,ds \\
&\leqslant 2\sqrt{1-\epsilon}\left(\tfrac{m+1}{\epsilon}\right)^{\frac{m+3}{4}}t^{-\frac{m+1}{4}}\|\nabla g\|_{L^1}^m\|\partial_{x_1}\Delta g\|_{L^1}\|h\|_{\mathcal{X}_R}\|k\|_{\mathcal{X}_R}.
\end{aligned}$$

For the second term we use Leibniz formula,

$$\text{②} = \sum_{\beta \leqslant \alpha'}\binom{\alpha'}{\beta}\int_{t(1-\epsilon)}^t \partial_{x_1}\Delta e^{-(t-s)A}(D^\beta\nabla h)(D^{\alpha'-\beta}\nabla k)\,ds$$

and, as in the proof of Lemma 4.2,

$$
② \leqslant \sum_{\beta \leqslant \alpha'} \binom{\alpha'}{\beta} \|\partial_{x_1}\Delta g\|_{L^1} \|h\|_{\mathcal{X}_{R,|\beta|}} \|k\|_{\mathcal{X}_{R,m-|\beta|}} \int_{t(1-\epsilon)}^t \frac{1}{s^{\frac{m+2}{4}}(t-s)^{\frac{3}{4}}}\, ds
$$

$$
\leqslant \frac{4\epsilon^{\frac{1}{4}}}{(1-\epsilon)^{\frac{m+2}{4}}} t^{-\frac{m+1}{4}} \|\partial_{x_1}\Delta g\|_{L^1} \sum_{\beta \leqslant \alpha'} \binom{\alpha'}{\beta} \|h\|_{\mathcal{X}_{R,|\beta|}} \|k\|_{\mathcal{X}_{R,m-|\beta|}}
$$

If we set $\epsilon = \frac{1}{(m+d)^{4d}}$ the term $4\epsilon^{1/4}(1-\epsilon)^{-(m+2)/4}(m+d)^d$ is uniformly bounded in m (we recall that the number of multi–indices α such that $|\alpha| = m + 1$ is bounded by $(m + d)^d$) and so by summing up over α the estimates for ① and ② together show the lemma. $\qquad\square$

As in the proof of Theorem 4.3, define $H_0 = 0$ and

$$
H_{n+1}(t) = e^{-tA} h_0 - \mathscr{V}(H_n, H_n)(t).
$$

Lemma 6.5. *There is $\delta' > 0$ such that if $0 < R \leqslant \infty$ and $\|h_0\|_{\mathcal{B}^0_R} < \delta$, then for every $m \geqslant 0$ there is $K_m > 0$ such that*

$$
\|H_n\|_{\mathcal{X}_{R,m}} \leqslant K_m.
$$

Let us remark that with an explicit estimate of the constants K_m, and in particular their growth in terms of m, one could show that solutions are analytic in space. For simplicity of presentation, we will not focus on this.

Proof. If $\|h_0\|_{\mathcal{B}^0_R}$ is small enough, the proof of Theorem 4.3 shows that there is K_0 such that $\|H_n\|_{\mathcal{X}_R} \leqslant K_0$. By possibly taking $\|h_0\|_{\mathcal{B}^0_R}$ smaller, we can assume that $\lambda = 2c_5K_0 < 1$, where c_5 is given in Lemma 6.4. We prove the statement by induction: the case $m = 0$ has been already proved. Set $a_m = c_5 m^d (m + 1)^{(m+3)/2} \|\nabla g\|_{L^1}^m \|g\|_{W^{3,1}}$ (this is the coefficient appearing in the first line of formula (6.2)) and $b_m = c m^d (m + 1)^{(m+1)/4} \|\nabla g\|_{L^1}^m$ (this appears in formula (6.1)), then by Lemmas 6.3 and 6.4,

$$
\|H_{n+1}\|_{\mathcal{X}_{R,m}} \leqslant \|H_1\|_{\mathcal{X}_{R,m}} + \|\mathscr{V}(H_n, H_n)\|_{\mathcal{X}_{R,m}}
$$

$$
\leqslant b_m K_0 + a_m K_0^2 + 2c_5 K_0 \|H_n\|_{\mathcal{X}_{R,m}}
$$

$$
+ c_5 m^d \sum_{j=1}^{m-1} \binom{m}{j} \|H_n\|_{\mathcal{X}_{R,j}} \|H_n\|_{\mathcal{X}_{R,m-j}}
$$

$$
\leqslant \left(b_m K_0 + a_m K_0^2 + c_5 m^d \sum_{j=1}^{m-1} \binom{m}{j} K_j K_{m-j} \right) + \lambda \|H_n\|_{\mathcal{X}_{R,m}},
$$

so that by recurrence and again Lemma 6.3,

$$\|H_{n+1}\|_{\mathcal{X}_R,m} \leqslant$$

$$\leqslant \Big(b_m K_0 + a_m K_0^2 + c_5 m^d \sum_{j=1}^{m-1} \tbinom{m}{j} K_j K_{m-j} \Big)(1 + \cdots + \lambda^{n-1}) + \lambda^n \|H_1\|_{\mathcal{X}_R,m}$$

$$\leqslant \frac{1}{1-\lambda} \Big(b_m K_0 + a_m K_0^2 + c_5 m^d \sum_{j=1}^{m-1} \tbinom{m}{j} K_j K_{m-j} \Big) + b_m K_0,$$

and the last line in the formula above provides K_m. $\qquad\square$

Proof of Proposition 6.2. Theorem 4.3 ensures that if $\|h_0\|_{\mathcal{B}_R^0}$ is small enough, then there is $\lambda = 2c_4 K_0 < 1$ (where the number K_0 is given by previous lemma) such that $\|H_{n+1} - H_n\|_{\mathcal{X}_R} \leqslant c\lambda^n$. We prove by induction that there are numbers $C_m > 0$ and $\mu \in (0,1)$ such that

$$\|H_{n+1} - H_n\|_{\mathcal{X}_R,m} \leqslant C_m \mu^n, \qquad m \geqslant 0,$$

if $\|h_0\|_{\mathcal{B}_R^0}$ is small enough. Let $\lambda = 2c_5 K_0$ (where c_5 has been introduced in Lemma 6.4), assume $\lambda < 1$ and let $\lambda < \mu < 1$. We have already verified that the inductive claim is true for $m = 0$. Assume the claim is true for $0, \ldots, m-1$, then by Lemma 6.4 and the inductive assumption,

$$\|H_{n+1} - H_n\|_{\mathcal{X}_R,m}$$
$$\leqslant \|\mathcal{V}(H_n, H_n - H_{n-1})\|_{\mathcal{X}_R,m} + \|\mathcal{V}(H_n - H_{n-1}, H_{n-1})\|_{\mathcal{X}_R,m}$$
$$\leqslant \big[a_m(\|H_n\|_{\mathcal{X}_R} + \|H_{n-1}\|_{\mathcal{X}_R}) + c_5(\|H_n\|_{\mathcal{X}_R,m} + \|H_{n-1}\|_{\mathcal{X}_R,m}) \big] \|H_n - H_{n-1}\|_{\mathcal{X}_R}$$
$$\quad + c_5(\|H_n\|_{\mathcal{X}_R} + \|H_{n-1}\|_{\mathcal{X}_R})\|H_n - H_{n-1}\|_{\mathcal{X}_R,m}$$
$$\quad + c_5 m^d \sum_{j=1}^{m-1} \tbinom{m}{j} (\|H_n\|_{\mathcal{X}_R,j} + \|H_{n-1}\|_{\mathcal{X}_R,j})\|H_n - H_{n-1}\|_{\mathcal{X}_R,m-j}$$
$$\leqslant \lambda \|H_n - H_{n-1}\|_{\mathcal{X}_R,m} + \widetilde{K}_m \mu^{n-1},$$

where we have set $a_m = c_5 m^d (m+1)^{(m+3)/2} \|\nabla g\|_{L^1}^m \|g\|_{W^{3,1}}$ (the coefficient in the first line of (6.2)), $\widetilde{K}_m = 2C_0(a_m K_0 + c_5 K_m) + 2c_5 m^d \sum_{j=1}^{m-1} \tbinom{m}{j} K_j C_{m-j}$, and the constants K_j are given by the previous lemma. By recurrence (notice that $\mu > \lambda$), it is easy to see that for every n,

$$\|H_{n+1} - H_n\|_{\mathcal{X}_R,m} \leqslant \lambda^{n+1} \|H_1 - H_0\|_{\mathcal{X}_R,m} + \widetilde{K}_m \big(\lambda^{n-1} + \lambda^{n-2}\mu + \cdots + \mu^{n-1} \big)$$
$$\leqslant \big(\lambda K_m + \tfrac{1}{\mu-\lambda} \widetilde{K}_m \big) \mu^n,$$

which concludes the induction. In conclusion, the sequence $(H_n)_{n \in \mathbb{N}}$ converges in all spaces $\mathcal{X}_R^m$. $\qquad\square$

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INSTITUT FÜR MATHEMATIK, UNIVERSITÄT AUGSBURG, D-86135 AUGSBURG, GERMANY
E-mail address: dirk.bloemker@math.uni-augsburg.de
URL: http://www.math.uni-augsburg.de/ana/bloemker.html

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI FIRENZE, VIALE MORGAGNI 67/A, I-50134 FIRENZE, ITALIA
E-mail address: romito@math.unifi.it
URL: http://www.math.unifi.it/users/romito