Kac-Moody symmetric spaces and universal twin buildings

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To my parents

Contents

Chapter 1

Introduction

In this thesis we develop a theory of affine Kac-Moody symmetric spaces and their buildings. There are two main parts: in the first part we construct Kac-Moody symmetric spaces and discuss their analytic and geometric properties. The main new feature of our approach is the use of holomorphic loops on $\mathbb{C}^*$. In this setting a nice complexification of the Kac-Moody groups can be constructed. Therefore it allows the definition of Kac-Moody symmetric spaces of the non-compact type.

In the second part we describe the theory of universal geometric twin buildings associated to Kac-Moody symmetric spaces. The new point of view is the use of functional analytic methods. As an application we construct a completion of twin buildings which nicely reflects the tame Fréchet structure of the Kac-Moody symmetric spaces.

1.1 The origin of the problem and the state of the art

The problem of constructing Kac-Moody symmetric spaces emerged in the 90's from the study of isoparametric submanifolds in Hilbert spaces, $P(G, H)$-actions and polar actions on Hilbert spaces as follows:

In finite dimensional differential geometry there is a remarkable link between Riemann symmetric spaces, polar representations, isoparametric submanifolds and spherical buildings: namely isotropy representations of symmetric spaces are polar; their principal orbits are isoparametric submanifolds. Conversely Dadok proved every polar representation on $\mathbb{R}^n$ to be orbit equivalent to the isotropy representation of a symmetric space. Furthermore a result of Thorbergsson shows any full irreducible isoparametric submanifold of $\mathbb{R}^n$ of rank at least three to be an orbit of some isotropy representation (see [BCO03] and references therein). The boundary of a symmetric space of non-compact type can be identified with a building. In addition the building can be embedded into the unit sphere of the representation space of the isotropy representation.

Hence, generalizing the concepts of isoparametric submanifolds and polar actions to Hilbert spaces, Chuu-Lian Terng conjectured in her foundational article [Ter95] the existence of infinite dimensional symmetric spaces completing the generalization of the finite dimensional blueprint. She also remarks (remark 3.4) that severe technical problems make the rigorous definition of those spaces difficult. The crucial point is to find an analytic framework that allows all algebraic constructions which are needed for the description of the geometric theory. A recent review of the theory of isoparametric submanifolds and polar actions on Hilbert space, which contains additional references, is given in [Hei06].

Important progress towards the construction of Kac-Moody symmetric spaces was achieved by Bogdan Popescu only in 2005/2006 in his thesis [Pop05] where he considers weak Hilbert symmetric spaces, modeled as loop spaces of H^1-Sobolev loops, equipped with a H^0 scalar product. As the differential of H^1 loops is only in H^0 this approach does not allow a convincing definition of the torus bundle extension corresponding to the c and d parts of the Kac-Moody algebra. To remedy this he investigates also the framework of smooth loops, which allows the construction of symmetric Fréchet spaces of the "compact type". However there is no convincing definition of a complexification for those groups. Hence, the definition of the dual non-compact symmetric spaces fails completely. As about half of all symmetric spaces are of non-compact type this is a serious detriment.

In this work we overcome this problem by using holomorphic loops defined on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In this ansatz we have to tackle the serious obstacle that the exponential function in general defines no longer a diffeomorphism between neighborhoods of the 0-element in $\mathfrak{g}$ and the unit element in G. Therefore one requires methods from infinite dimensional Lie theory to define Lie group- (resp. manifold-) structures on those groups and their quotients. A comprehensive review of this topic including further references is the subject of the article [Nee06], an investigation of Lie groups of holomorphic maps is contained in the article [NW07].

In the geometry of non-compact finite dimensional symmetric spaces of rank $r \geq 2$ and their quotients, Tits buildings play an important role; so the appropriate generalization of those buildings for Kac-Moody symmetric spaces is an important step in the development of a theory of Kac-Moody symmetric spaces. The algebraic theory of abstract Kac-Moody groups tells us that from a combinatorial point of view, Tits twin buildings should be the correct generalization. Nevertheless these buildings are purely algebraic objects; to make them well adapted for the geometric situation of Kac-Moody symmetric spaces, we define a class of new objects, which we call universal geometric twin buildings; they preserve the combinatorial structure of algebraic Tits twin buildings but beyond that reflect the analytic properties of Kac-Moody symmetric spaces.

Hence, the theory of Kac-Moody symmetric spaces presented in this work is situated at the melting point of 3 different areas of current research:

- The geometry of loop groups, polar actions and isoparametric submanifolds.
- The theory of infinite dimensional analysis and Lie groups.
- The theory of Kac-Moody algebras, Kac-Moody groups and their twin buildings.

Using the geometric description of extensions of loop groups we describe Kac-Moody groups as tame Fréchet Lie groups and construct Kac-Moody symmetric spaces of the compact and of the non-compact type as tame Fréchet manifolds. Then we turn to the universal twin buildings associated to affine Kac-Moody symmetric spaces.

1.2 Geometry of Kac-Moody symmetric spaces

To state our main results about the geometry of Kac-Moody symmetric spaces, let us fix some notation: we denote by $G_{\mathbb{C}}$ a complex semisimple Lie group and by G a compact real form of $G_{\mathbb{C}}$. Furthermore let σ be a diagram automorphism of order n for $\mathfrak{g}_{\mathbb{C}}$ ($n = 1$ is allowed) and $\omega := e^{\frac{2\pi i}{n}}$. Now, we define the holomorphic loop spaces

$$MG^{\sigma}_{\mathbb{C}} := \{f : \mathbb{C}^* \to G_{\mathbb{C}} | f \text{ is holomorphic and } \sigma \circ f(z) = f(\omega z)\}$$

and

$$MG^{\sigma}_{\mathbb{R}} := \{f : \mathbb{C}^* \to G_{\mathbb{C}} | f(\mathbb{S}^1) \subset G, \text{f is holomorphic and } \sigma \circ f(z) = f(\omega z)\} .$$

The complex Kac-Moody groups $\widehat{MG}_{\mathbb{C}}^{\sigma}$ are now constructed as certain $(\mathbb{C}^*)^2$-bundles over $MG_{\mathbb{C}}^{\sigma}$. To simplify notation we omit the superscript σ whenever possible.
Let ρ_* denote a suitable involution of the second kind (see [HG09]).

Theorem 1.2.1 (affine Kac-Moody symmetric spaces of the "compact" type)
Both the Kac-Moody group $\widehat{MG}_{\mathbb{R}}^{\sigma}$ equipped with its Ad-invariant metric, and the quotient space $X = \widehat{MG}_{\mathbb{R}}^{\sigma}/Fix(\rho_*)$ equipped with its $Ad(Fix(\rho_*))$-invariant metric are tame Fréchet symmetric spaces of the "compact" type with respect to their natural Ad-invariant metric. Their curvatures satisfy

$$\langle R(X,Y)X,Y\rangle \geq 0\,.$$

Theorem 1.2.2 (affine Kac-Moody symmetric spaces of the "non-compact" type)
Both quotient spaces $X = \widehat{MG}_{\mathbb{C}}^{\sigma}/\widehat{MG}_{\mathbb{R}}^{\sigma}$ and $X = H/Fix(\rho_*)$, where H is a non-compact real form of $\widehat{MG}_{\mathbb{C}}^{\sigma}$ equipped with their Ad-invariant metric, are tame Fréchet symmetric spaces of the "non-compact" type. Their curvatures satisfy

$$\langle R(X,Y)X,Y\rangle \leq 0\,.$$

Furthermore Kac-Moody symmetric spaces of the non-compact type are diffeomorphic to a vector space.

Define the notion of duality as for finite dimensional Riemann symmetric spaces.

Theorem 1.2.3 (Duality)
Affine Kac-Moody symmetric spaces of the compact type are dual to the Kac-Moody symmetric spaces of the non-compact type and vice versa.

Kac-Moody symmetric spaces have several conjugacy classes of flats. For our purposes the most important class are those of finite type. A flat is called of finite type iff it is finite dimensional. A flat is called of exponential type iff it lies in the image of the exponential map and it is called maximal iff it is not contained in another flat. Adapting a result of Bogdan Popescu (see [Pop05]) to our setting, we find:

Theorem 1.2.4
All maximal flats of finite exponential type are conjugate.

We show that all Kac-Moody symmetric spaces are Lorentz symmetric spaces.

In the finite dimensional case no complete classification of pseudo Riemann symmetric spaces is known. However there are important partial results: Marcel Berger achieved in 1957 a complete classification of pseudo Riemann symmetric spaces of "semisimple" type [Ber57]. Ines Kath and Martin Olbrich gave a classification of pseudo Riemann symmetric spaces of index 1 and 2 and described structure results that indicate that a general classification of pseudo Riemann symmetric spaces is out of reach [KO04], [KO06]. As Kac-Moody groups are the natural infinite dimensional analogue of semisimple Lie groups, it is tempting to interpret Kac-Moody symmetric spaces as an infinite dimensional analogue of the subclass of finite dimensional "semisimple" Lorentz symmetric spaces.
In contrast to this point of view, we prefer, because of their similar structure theory, to understand Kac-Moody symmetric spaces as a direct generalization of finite dimensional Riemann symmetric spaces. This point of view is further strengthened as the isotropy representations of Kac-Moody symmetric spaces induce polar actions on Hilbert spaces. Thus in the infinite dimensional setting Kac-Moody symmetric spaces take over the role played by Riemann symmetric spaces.

The material is ordered as follows:

- In chapter 2 we collect the analytic foundations: We study the tame Fréchet- and ILH-structure on loop groups and loop algebras and investigate properties of the exponential map. Furthermore we study polar actions on certain Fréchet spaces. This will be needed at several places to understand the conjugacy properties of flats (chapter 4) and to prove some embedding results about universal geometric twin buildings (see chapter 5).

- In chapter 3 we investigate the algebraic structures which we need for Kac-Moody symmetric spaces. We describe tame Fréchet realizations of the twisted and non-twisted Kac-Moody algebras and Kac-Moody groups. Following the blueprint of the finite dimensional theory we describe a classification of Kac-Moody symmetric spaces by the classification of their indecomposable orthogonal symmetric affine Kac-Moody algebras (OSAKA). After the definition of OSAKA's this is a direct application of the classification of affine Kac-Moody algebras (see [Kac90]) and their involutions (see for example [MR03] and the recent work of Ernst Heintze achieving a complete classification from the geometric point of view [HG09]; moreover a list of Satake diagrams for Kac-Moody symmetric spaces can be found in [TP06]).

- In chapter 4 we describe the construction of Kac-Moody symmetric spaces and prove some results about their geometry.

There are several important problems related to Kac-Moody symmetric spaces which are not studied here but are intended to be subject for further work:

- **The study of quotients of Kac-Moody symmetric spaces:** We conjecture the existence of a Mostow-type theorem for Kac-Moody symmetric spaces of the non-compact type, if the rank r of each irreducible factor satisfies $r \geq 4$; the theory of buildings suggests that the condition $r \geq 4$ is necessary. In the finite dimensional situation the main ingredients for the proof of Mostow rigidity are the spherical buildings which are associated to the universal covers $\widetilde{M}$ and $\widetilde{M}'$ of two homotopy equivalent locally symmetric spaces $M = \widetilde{M}/\Gamma$ and $M' = \widetilde{M}'/\Gamma'$ (suppose the rank r of each de Rham factor satisfies $\operatorname{rank}(M) \geq 2$). To prove Mostow rigidity one has to show that a homotopy equivalence of the quotients lifts to a quasi isometry of the universal covers and induces a building isomorphism. By rigidity results of Jacques Tits this building isomorphism is known to introduce a group isomorphism which in turn leads to an isometry of the quotients.

  Hence, to prove a generalization of Mostow rigidity to quotients of Kac-Moody symmetric spaces of the non-compact type along these lines, the crucial step is a thorough understanding of the twin buildings associated to Kac-Moody symmetric spaces. This is the main motivation for the study of universal geometric twin buildings in the second part of my thesis.

  We note that a generalization or adaption of the methods developed in [Cap08] and in [GM09] might lead to an algebraic proof of Mostow rigidity.

- **Kac-Moody symmetric spaces as Moduli space:** Recent results by Shimpey Kobayashi and Josef Dorfmeister show that the moduli spaces of different classes of integrable surfaces can be understood as real forms of loop groups of $\mathfrak{sl}(2, \mathbb{C})$ (see [Kob09]). We conjecture that Kac-Moody symmetric spaces can be interpreted as Moduli spaces of special classes of submanifolds in more general situations.

1.3 Universal twin buildings

For Kac-Moody symmetric spaces universal twin buildings take over the role played by spherical buildings for finite dimensional (Riemann) symmetric spaces.

The theory of finite dimensional spherical buildings associated to Lie groups, algebraic groups and symmetric spaces is quite completely understood and there is a well developed theory of algebraic twin buildings associated to algebraic Kac-Moody groups [AB08]: an exposition of different algebraic aspects of the theory of loop groups and twin buildings for groups of type A_n and $\widetilde{A}_n$ respectively, is contained in the interesting article [Kra02].

We describe some details about spherical buildings and Lie groups (see for example the book [AB08]): the building associated to a compact simple Lie group G is a chamber complex $\mathfrak{B}$, together with sets of subcomplexes $\mathcal{A}_i$, called apartments, satisfying two axioms: first, for every pair $\{c, x\} \subset \mathfrak{B}$, consisting of a chamber c and a simplex x there is an apartment $\mathcal{A}_{c,x}$ containing both c and x. Second, for every two apartments $\mathcal{A}$ and $\mathcal{A}'$ there is an isomorphism $\phi : \mathcal{A} \longrightarrow \mathcal{A}'$ fixing the intersection $\mathcal{A} \cap \mathcal{A}'$. From a combinatorial point of view, the system of apartments is the fundamental structure of a building. Each apartment can be characterized as a convex hull of two opposite chambers. The Weyl group acts transitively on the chambers of an apartment. As a simplicial complex the building associated to G can be described as the simplicial complex associated to the opposite poset (i.e. the poset with reversed inclusion relation) to the poset of parabolic subgroups of $G_{\mathbb{C}}$ ordered by inclusion.

For loop groups (resp. affine Kac-Moody groups) there appear two constructions of buildings in the literature:

- If one replaces semisimple Lie groups (or more generally reductive linear algebraic groups) by algebraic Kac-Moody groups, it turns out that algebraic twin buildings take over the role played by spherical buildings for Lie groups.

  As in the Lie group situation, we want an equivalence between Borel subgroups of the Kac-Moody group and chambers of the building. Due to the fact that Kac-Moody groups have two conjugacy classes of Borel subgroups the associated building breaks up into two pieces: hence a twin building consists of a pair $\mathfrak{B}^+ \cup \mathfrak{B}^-$ of buildings that are "twinned": the twinning can be described in different ways: from the point of view of apartments the twinning is most easily defined by the introduction of a system of twin apartments, that is subcomplexes $\mathcal{A}^+ \cup \mathcal{A}^- \subset \mathfrak{B}^+ \cup \mathfrak{B}^-$, consisting of two apartments $\mathcal{A}^+$ and $\mathcal{A}^-$, such that $\mathcal{A}^+$ is contained in $\mathfrak{B}^+$ and $\mathcal{A}^-$ is contained in $\mathfrak{B}^-$. Imposing some axioms similar to those used for spherical buildings, many features known from apartments in spherical buildings generalize to the system of twin apartments (see [AB08]).

- For groups of algebraic loops there is a theory of affine buildings (but not for twin buildings) developed from the group theoretic point of view, described in [Mit88].

To associate twin buildings to Kac-Moody symmetric spaces the main problem is again to unify algebraic and analytic aspects of the theory: twin buildings associated to affine Kac-Moody groups consist of pairs of Euclidean buildings. These are purely algebraic constructions and thus they correspond only to the subgroup of algebraic loops. Written in an algebraic notion these groups are of the form $G(\mathbb{R}[z, z^{-1}])$, that is: groups of polynomial loops in z and z^{-1}. Their affine Weyl groups act transitively on the chambers of the apartments whereas the groups $G(\mathbb{R}[z, z^{-1}])$ act transitively on the apartments.

Unfortunately a straightforward process of completion — even in only one direction z or z^{-1} — destroys the twinned structure. The classically used remedy can be found in Shrawan Kumar's book [Kum02]: a group that is completed only in one direction — (let's say in the direction of z: hence the Laurent polynomials in z and z^{-1} are replaced by holomorphic functions with finite principal part) — acts on the part of the twin building corresponding to this direction (i.e. $\mathfrak{B}^+$). For our purposes however it is not enough to restrict the theory to one half of the twin building: on the one hand, we need a completion that is symmetric in z and z^{-1} in order to be able to define the involutions of the second kind which are needed for Kac-Moody symmetric spaces. On the other hand, our purpose is a proof of Mostow rigidity and there are good rigidity results only for twin buildings themselves, but not for their affine parts. In particular, a counterexample to rigidity for affine buildings follows from the constructions in [Kra02].

Our solution is to define on the level of the building a "completion" of the two parts of the twin building that corresponds to the completed groups. For different completions of the loop group (i.e. H^1, C^∞, analytic or holomorphic loops), the associated completions of the building have to be adapted. To unify notation, we will use X as a substitution to denote some kind of completion (regularity).
The process of completing algebraic twin buildings leads to a complex $\mathfrak{B} = \mathfrak{B}^+ \cup \mathfrak{B}^-$, whose (uncountable many) positive and negative connected components are affine buildings; we call these objects "universal buildings in the X-category", where X denotes the type of completion used.

Universal buildings have a nice structure theory and allow actions of completed Kac-Moody groups:

Theorem 1.3.1 (Universal twin buildings in the X-category)
Let $\mathfrak{B} = \mathfrak{B}^+ \cup \mathfrak{B}^-$ be a universal twin building in the X-category.

- *Each pair $(\mathfrak{B}_1^+, \mathfrak{B}_2^-) \in \mathfrak{B}^+ \cup \mathfrak{B}^-$, consisting of a positive and a negative building, is an affine twin building.*
- *$\mathfrak{B}^+ \cup \mathfrak{B}^-$ has a building at infinity.*
- *The X-completion of ab algebraic Kac-Moody group acts on the universal twin building of the same type in the X-category.*

The importance of this concept for the geometry of Kac-Moody symmetric spaces results from the following theorem:

Theorem 1.3.2 (Embedding of twin buildings)
Denote by $H_{-l^2,r}$ the intersection of the sphere of radius $-l^2$ of a real affine Kac-Moody algebra $\widehat{M\mathfrak{g}}$ with the horospheres $r_d = \pm r$, where r_d is the coefficient of d in the Kac-Moody algebra. There is a 2-parameter family $\varphi_{l,r}, (l,r) \in \mathbb{R}^+ \times \mathbb{R}^+$ of $\widehat{MG}$-equivariant immersion of $\mathfrak{B}^+ \cup \mathfrak{B}^-$ into $\widehat{M\mathfrak{g}}$. It is defined by the identification of $H_{-l^2,r}$ with $\mathfrak{B}$. The two complexes $\mathfrak{B}^+$ and $\mathfrak{B}^-$ are immersed into the two sheets of $H_{-l^2,r}$.

Let $\widehat{M\mathfrak{g}}_{\mathbb{R}} = \mathcal{K} \oplus \mathcal{P}$ be the decomposition of $\widehat{M\mathfrak{g}}_{\mathbb{R}}$ into the eigenspaces of ± 1 of an involution of the second kind. There is a similar embedding result for the buildings in this situation into $\mathcal{P}$.

Via this identification both components of the universal twin building in the X-category get a structure of an X-vector space.

Specializing our results to universal geometric twin buildings in the category of holomorphic functions, we get the following result:

Theorem 1.3.3
The maximal flats of finite exponential type in a Kac-Moody symmetric space correspond to apartments of a universal geometric twin building in the category of holomorphic functions.

In particular this shows that, from a geometric point of view, universal geometric twin buildings are the correct generalization of spherical buildings for Kac-Moody groups. A careful investigation of the completion process allows to deduce a natural ultrametric pseudodistance on the universal geometric twin buildings.

Besides this application for Kac-Moody symmetric spaces, universal buildings have a nice theory in their own right: to understand buildings, often the explicit construction of the building as a flag complex is used. For example a spherical building of type A_n can be realized explicitly as the poset of flags in $\mathbb{C}^n$.
We deduce a similar description for the universal twin buildings: the universal building $\mathfrak{B}^+ \cup \mathfrak{B}^-$ of type $\widetilde{A}_n$ in the X-category can be realized as a flag complex of periodic flags in an X-vector space. This description allows explicit calculations of purely building-theoretic concepts, for example Phan involutions.

The most general setting that we study is that of H^0-loops: let $H^{(n)} := L^2(S^1, \mathbb{C}^n)$ be the space of square-summable $\mathbb{C}^n$-valued functions, and take as loop groups the groups $GL_{\text{res}}(H)$ which were used in [PS86]. We explicitly construct the universal H^0-buildings, associated to these loop groups, as complexes of periodic flags, constructed as operators. The main ingredient is the Grassmann manifold $Gr(H)$, defined in chapter 7 of [PS86]. Our building can be described as the poset of periodic flags in this Grassmann manifold. Similar constructions are possible for the other classical groups. If one uses the smallest complexes — that is: those which correspond to groups of algebraic loops — and restricts to the building associated to only one direction, one gets back the well known affine algebraic building described by [Mit88].

1.4 Universal algebraic twin buildings

An open question in the algebraic theory of twin buildings is the study of the space of all possible twinnings, that is the construction of a universal algebraic twin building

The state of the art in this direction is a paper of Jacques Tits and Marc Ronan, describing in a purely algebraic way universal twinnings for trees ([RT94] and [RT99]). But since their approach relies on a description of the twinning by a special codistance function that can only be defined in the case of trees, it does not generalize to other classes of buildings. More generally it seems difficult to describe the geometry of the universal twin buildings using purely algebraic means.

It is not clear yet if one can solve this problem by using the analytic ideas described above. Nevertheless we have promising partial results: the functional analytic approach proposed here gives a natural geometric universal twin building for affine buildings over the base fields $\mathbb{R}$ and $\mathbb{C}$. Moreover applying our ideas to the case of the algebraic universal twin tree, we can define a similar pseudo distance on the universal twin trees.

We conjecture the existence of a universal algebraic twin building, constructed analogously as complex of subspaces of certain infinite dimensional vector spaces. If this construction can be carried out, we expect applications in the study of discrete subgroups of algebraic Kac-Moody groups.

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Chapter 2

Analytic foundations

In this section we describe the analytic foundations of the theory of Kac-Moody symmetric spaces. As Kac-Moody symmetric spaces are tame Fréchet manifolds, we start by reviewing the theory of tame Fréchet structures following the presentation given by Richard Hamilton [Ham82] and describe the strongly related concept of ILH- (resp. ILB-) structures, developed by Hideki Omori [Omo97]. Then we investigate the examples necessary for the theory of Kac-Moody symmetric spaces: we construct tame Fréchet and ILH- (resp. ILB-) structures on various loop algebras and loop groups of holomorphic maps. Kac-Moody groups (resp. Kac-Moody symmetric spaces) are torus extensions of those loop groups (resp. quotients). Hence, there appear no new analytical problems.

2.1 Tame Fréchet manifolds

2.1.1 Fréchet spaces

This introductory section collects some standard results about Fréchet spaces, Fréchet manifolds and Fréchet Lie groups. Further details or omitted proofs can be found in Hamiltons article [Ham82].

Definition 2.1.1 (Fréchet space)
A Fréchet vector space is a locally convex topological vector space which is complete, Hausdorff and metrizable.

Lemma 2.1.1 (Metrizable topology)
A topology on a vector space is metrizable, iff it can be defined by a countable collection of seminorms.

Example 2.1.1 (Fréchet spaces)

1. *Every Banach space is a Fréchet space.*

2. *Let* $Hol(\mathbb{C},\mathbb{C})$ *denote the holomorphic functions* $f : \mathbb{C} \longrightarrow \mathbb{C}$. *Let* K_n *be a sequence of simply connected compact sets in* $\mathbb{C}$, *such that* $K_n \subset K_{n+1}$ *and* $\bigcup K_n = \mathbb{C}$. *Let* $\|f\|_n := \sup\limits_{z\in K_n} |f(z)|$. *Then* $Hol(\mathbb{C},\mathbb{C}; \|\ \|_n)$ *is a Fréchet space.*

3. *More generally, for every Riemann surface the sheaf of holomorphic functions carries a Fréchet structure which is defined similarly as in the trivial case* $\mathbb{C}$.

We define a Fréchet manifold in the usual way as a manifold having charts in a Fréchet space, such that the chart transition functions are smooth. While it is possible to define Fréchet manifolds in this way, there are two strong impediments to the development of analysis and geometry of those spaces:

1. There is no inverse function theorem. For counterexamples cf. [Ham82].
2. The dual space of a Fréchet space is in general not a Fréchet space.

Dealing with those problems is difficult: take as example the space of holomorphic functions: $\mathrm{Hol}(\mathbb{C},\mathbb{C})$. $\mathbb{C}$ can be interpreted as a direct limit of the sets $K_n := B_n(0)$ with respect to inclusion; the space $\mathrm{Hol}(\mathbb{C},\mathbb{C})$ should thus be interpreted as its categorical dual, that is as inverse limit of a sequence of function spaces $\mathrm{Hol}(K_n,\mathbb{C})$, where $\mathrm{Hol}(K_n,\mathbb{C}) := \bigcup_{K_n \subset U_n^m} \{\mathrm{Hol}(U_n^m,\mathbb{C})\}$. By a choice of appropriate norms on the spaces $\mathrm{Hol}(K_n,\mathbb{C})$, one can give them structures as Bergmann-(or Hardy-) spaces. See for example [HKK00] and [Dur00]. Hence $\mathrm{Hol}(\mathbb{C},\mathbb{C})$ can be interpreted as inverse limit of Hilbert spaces. The dual space of an inverse limit is a direct limit. Thus it is clear that we cannot expect the dual space of a Fréchet space to be Fréchet. It will be Fréchet iff the projective limit and the inductive limit coincide, that is if they both stabilize. This is the case exactly for Hilbert and Banach spaces.

The solution to the first problem is based on a more refined control of the projective limits. Using this structure, the inverse function theorems on the Hilbert-(resp. Banach-) spaces in the sequence piece together to give an inverse function theorem on the limit space; this is the famous Nash-Moser inverse function theorem. In the next sections we will formalise those concepts.

We have to note that there are other ways of dealing with those analytic problems. A recent example is the concept of bounded Fréchet geometry developed by Olaf Müller [Mü06].

The solution to the second problem consists in avoiding dual spaces.

2.1.2 Tame Fréchet spaces

The central problem for all further structure theory of Fréchet spaces is a better control of the set of seminorms. Our presentation follows closely the article [Ham82].
The easiest way to control norms is a grading:

Definition 2.1.2 (Grading)
Let F be a Fréchet space. A grading on F is a collection of seminorms $\{\| \,\|_n, n \in \mathbb{N}_0\}$ that define the topology and satisfy

$$\|f\|_0 \leq \|f\|_1 \leq \|f\|_2 \leq \|f\|_3 \leq \dots .$$

Lemma 2.1.2 (Constructions of graded Fréchet spaces)

1. *A closed subspace of a graded Fréchet space is a graded Fréchet space.*
2. *Direct sums of graded Fréchet spaces are graded Fréchet spaces.*

Every Fréchet space admits a grading: let $(F, \| \,\|_{n,n\in\mathbb{N}})$ be a Fréchet space. The space

$$(F, \| \,\|_{n,n\in\mathbb{N}})$$

such that $\| \,\|_n := \sum_{i=1}^{n} \| \,\|_i$ is a graded Fréchet space.

Definition 2.1.3 (Tame equivalence of gradings)
Let F be a graded Fréchet space, $r, b \in \mathbb{N}$ and $C(n), n \in \mathbb{N}$ a sequence with values in $\mathbb{R}^+$. The two gradings $\{\| \, \|_n\}$ and $\{\widetilde{\| \, \|}\}$ are called $(r, b, C(n))$-equivalent iff

$$\|f\|_n \leq C(n)\widetilde{\|f\|}_{n+r} \text{ and } \widetilde{\|f\|}_n \leq C(n)\|f\|_{n+r} \text{ for all } n \geq b\,.$$

They are called tame equivalent iff they are $(r, b, C(n))$-equivalent for some $(r, b, C(n))$.

The following example is basic:

Example 2.1.2
Let B be a Banach space with norm $\| \, \|_B$. Denote by $\Sigma(B)$ the space of all exponentially decreasing sequences $\{f_k\}$, $k \in \mathbb{N}_0$ of elements of B. On this space, we can define different gradings:

$$\|f\|_{l_1^n} := \sum_{k=0}^{\infty} e^{nk}\|f_k\|_B$$
$$\|f\|_{l_\infty^n} := \sup_{k \in \mathbb{N}_0} e^{nk}\|f_k\|_B$$

Lemma 2.1.3
On the space $\Sigma(B)$, the two gradings $\|f\|_{l_1^n}$ and $\|f\|_{l_\infty^n}$ are tame equivalent.

For the proof cf. [Ham82].

Example 2.1.3
The space of exponentially decreasing sequences of elements in $B = \mathbb{C}^2$, equipped with the euclidean norm, and the space of exponentially decreasing sequences of elements in $B = \mathbb{C}^2$ together with the supremum-norm $\|(c_1, c_1')\|_B := \sup(|c_1|, |c_1'|)$, are tame Fréchet spaces.

Notation 2.1.1
Let F and G denote graded Fréchet spaces.

Definition 2.1.4 (Tame map)
A linear map $\varphi : F \longrightarrow G$ is called $(r, b, C(n))$-tame if it satisfies the inequality

$$\|\varphi(f)\|_n \leq C(n)\|f\|_{n+r}\,.$$

φ is called tame iff it is $(r, b, C(n))$-tame for some $(r, b, C(n))$.

Definition 2.1.5 (Tame isomorphism)
A map $\varphi : F \longrightarrow G$ is called a tame isomorphism iff it is a linear isomorphism and φ and φ^{-1} are tame maps.

Definition 2.1.6 (Tame direct summand)
F is a tame direct summand of G iff there exist tame linear maps $\varphi : F \longrightarrow G$ and $\psi : G \longrightarrow F$ such that $\psi \circ \varphi : F \longrightarrow F$ is the identity.

Definition 2.1.7 (Tame space)
F is tame iff there is a Banach space B such that F is a tame direct summand of $\Sigma(B)$.

Lemma 2.1.4 (Constructions of tame spaces)

1. *A tame direct summand of a tame space is tame.*

2. *A cartesian product of two tame spaces is tame.*

There are many different examples of tame spaces. One can show that all examples in 2.1.1 are tame spaces. For proofs and further examples cf. [Ham82]. In the next section we will study some tame spaces of holomorphic functions in detail.

Definition 2.1.8 (Tame Lie algebra)
A Fréchet Lie algebra $\mathfrak{g}$ is tame iff it is a tame vector space and $ad(X)$ is a tame linear map for every $X \in \mathfrak{g}$.

We now give some definitions for nonlinear tame Fréchet objects:

Notation 2.1.2
Let F, G_1, G_2 denote tame spaces.

Definition 2.1.9
A nonlinear map $\Phi : U \subset F \longrightarrow G$ is called $(r, b, C(n))$-tame iff it satisfies the inequality

$$\|\Phi(f)\|_n \leq C(n)(1 + \|f\|_{n+r}) \, \forall n > b \, .$$

Φ is called tame iff it is $(r, b, C(n))$-tame for some $(r, b, C(n))$.

Lemma 2.1.5 (Construction of tame maps)

1. *Let $\Phi : U \subset F \longrightarrow G_1 \times G_2$ be a tame map. Define the projections $\pi_i : G_1 \times G_2 \longrightarrow G_i, i = 1, 2$. The maps*
$$\Phi_i := \pi_i \circ \Phi : U \longrightarrow G_i$$
are tame as well.

2. *Let $\Phi_i : U \subset F \longrightarrow G_i, i \in \{1, 2\}$ be $(r_i, b_i, C_i(n))$-tame maps. Then the map*
$$\Phi := (\Phi_1, \Phi_2) : U \longrightarrow G_1 \times G_2$$
is $(\max(r_1, r_2), \max(b_1, b_2), C_1(n) + C_2(n))$-tame.

Proof.

1. Projections onto a direct factor are $(0, 0, (1)_{n\in\mathbb{N}})$-tame. The composition of tame maps is tame. Thus Φ_i is tame.

2. $$\begin{aligned} \|\Phi(f)\|_n = \|\Phi(f)\|_n^1 + \|\Phi(f)\|_n^2 \leq \\ \leq C_1(n)(1 + \|f\|_{n+r_1}) + C_2(n)(1 + \|f\|_{n+r_2}) \leq \\ \leq C_1(n)(1 + \|f\|_{n+\max(r_1,r_2)}) + C_2(n)(1 + \|f\|_{n+\max(r_1,r_2)}) = \\ \leq (C_1(n) + C_2(n))(1 + \|f\|_{n+\max(r_1,r_2)}) \end{aligned}$$

   for all $n \geq \max(b_1, b_2)$.

□

Definition 2.1.10 (Tame manifold)
A tame manifold is a Fréchet manifold with charts in a tame space such that the chart transition functions are smooth tame maps.

Definition 2.1.11
Let M and N be two tame manifolds modelled on F resp. G. A map $f : M \longrightarrow N$ is tame iff for every pair of charts $\psi_i : V_i \subset N \longrightarrow V_i'$ and $\varphi_j : U_i \subset M \longrightarrow U_i'$, the map $\psi_i \circ f \circ \varphi_j^{-1}$ is tame whenever it is defined

Definition 2.1.12 (Tame submanifold of finite type)
$M \subset F$ is a n-codimensional smooth submanifold of F iff for every $m \in M$ there are open sets $U(m) \subset F$, $V(m) \subset G \times \mathbb{R}^n$ and a tame chart $\varphi_m : U(m) \longrightarrow V(m) \subset G \times \mathbb{R}^n$ such that

$$\varphi_M(M \cap U(m)) = G \cap V(m)\,.$$

Lemma 2.1.6
A tame submanifold of finite type is a tame manifold.

Proof. We have to find an atlas, i.e. we have to show that $\varphi_{M,m} : M \cap U(m) \longrightarrow G \cap V(m)$ is a tame diffeomorphism.

Let $m \in M$, $\varphi_m : U(m) \longrightarrow V(m)$ as in definition 2.1.12. Define the projection $\pi_G : G \times \mathbb{R}^n \longrightarrow G$. Lemma 2.1.5 tells us that the map $\varphi_{m,G} := \pi_G \circ \varphi_m : U \longrightarrow V_G \subset G$ is tame. Restricting the domain to $M \cap U$ gives us a tame function $\varphi_{M,G} : G \times \mathbb{R}^n \longrightarrow G$. φ_m is supposed to be a tame chart. Thus it is invertible and the inverse function is tame. Let $\varphi_m^{-1} : V(m) \longrightarrow U(m)$ be the inverse function. Restricting the domain to $V(M) \cap G$ yields a tame function $\varphi_{m,G}^{-1} : V(M) \cap G \longrightarrow U(m) \cap M$.

We have to show that it is a smooth isomorphism. As $\varphi(m)$ is a chart, the maps $\varphi(m)$ and $\varphi^{-1}(m)$ are injective and smooth. Hence, the restrictions $\varphi_{M,m}$ and $\varphi_{m,G}^{-1}$ are injective and smooth. Furthermore $\varphi_{m,M}$ is surjective onto $V(m) \cap G$. Hence also $\varphi_{m,G}^{-1}$ is surjective.

□

Lemma 2.1.7
Let the notations be as in definition 2.1.12. Let H be a tame space. A map $\Phi : H \longrightarrow M$ is tame if it is tame as a map $\Phi_F : H \to F$.

Proof. Let φ be tame as a map $\varphi_F : H \to F$. Then the cocatenation $\varphi_i \circ \Phi$ is tame for any chart φ of M. Thus Φ is tame as a map into M. □

The reason for introducing the category of tame spaces and tame maps is the Nash-Moser inverse function theorem. We cite the version of [Ham82]:

Theorem 2.1.1 (Nash-Moser inverse function theorem)
Let F and G be tame spaces and $\Phi : U \subset F \longrightarrow G$ a smooth tame map. Suppose that the equation for the derivative $D\Phi(f)h = k$ has a unique solution $h = V\Phi(f)k$ for all $f \in U$ and all k and that the family of inverses $V\Phi : U \times G \longrightarrow F$ is a smooth tame map. Then Φ is locally invertible, and each local inverse Φ^{-1} is a smooth tame map.

A description of this theorem and some of its applications is the subject of the article [Ham82]. In comparison to the classical Banach inverse function theorem, the important additional assumption is that invertibility of the differential is assumed not only in a single point but in a neighborhood U. This condition is necessary.

2.1.3 An implicit function theorem for tame maps

We aim in this section for a characterization of the inverse image of a "regular" point of a tame map into $\mathbb{R}^n$ as a tame submanifold of finite type. Following the finite dimensional blueprint we prove this theorem as a corollary to an implicit function theorem.

Morally inverse function theorems and implicit function theorems come in pairs. In the literature there appear different implicit function theorems for classes of Fréchet spaces admitting smoothing operators (i.e. tame spaces). See for example [KP02], [Ser73] and [Pop02]. An implicit function theorem for maps from topological vector spaces to Fréchet spaces, using metric estimates, is described in [Glö07].

Richard Hamilton [Ham82] p. 212 proves the following theorem:

Theorem 2.1.2 (Hamilton's implicite function theorem)
Let F, G and H be tame spaces and let Φ be a smooth tame map defined on an open subset U in $F \times G$ to H.

$$\Phi : U \subset F \times G \longrightarrow H$$

Suppose that whenever $\Phi(f, g) = 0$, the partial derivative $D_f\Phi(f, g)$ is surjective, and there is a smooth tame map $V(f, g)h$ linear in h,

$$V : (U \subset F \times G) \times H \longrightarrow F\,,$$

and a smooth tame map $Q(f, g\{h, k\})$, bilinear in h and k, such that for all (f, g) in U and all $h \in H$ we have:

$$D_f\Phi(f, g)V(f, g)h = h + Q(f, g)\{\Phi(f, g), h\}\,,$$

so that V is an approximate right inverse for $D_f\Phi$ with quadratic error Q. Then if $\Phi(f_0, g_0) = 0$ for some $(f_0, g_0 \in U)$ we can find neighborhoods of f_0 and g_0 such that for all g in the neighborhood of g_0 there exists an f in the neighborhood of f_0 with $\Phi(f, g) = 0$. Moreover the solution $f = \Psi(g)$ is defined by a smooth tame map Ψ.

For details and a proof we refer to [Ham82].

For the application we have in mind a much easier theorem is sufficient:

Theorem 2.1.3 (Implicite function theorem for tame maps)
Let F be a tame space, $V \simeq W \simeq \mathbb{R}^n$ a finite dimensional vector space and $\varphi : U \subset F \times V \longrightarrow W$ a smooth tame map, such that the partial derivative $\frac{\partial}{\partial(y)}\varphi(z)$ is invertible for all $z = (x, y) \in U$. Suppose $\varphi(z_0) = 0$. Then there exist open sets U' and U'' and a smooth tame map $\psi : U' \longrightarrow U''$ such that

$$\varphi(x, y) = 0 \Leftrightarrow y = \psi(x)\,.$$

For the proof we use the Nash-Moser inverse function theorem. Our proof follows the one described by Konrad Königsberger [Kö00] for the finite dimensional implicit function theorem.

Proof.

1. **Prepare use of the Nash-Moser inverse function theorem** Study the map $\Phi : F \times V \longrightarrow F \times W$ defined by $\Phi(x, y) := (x, \varphi(x, y))$. As Cartesian products of tame spaces are tame (lemma 2.1.4), $F \times W$ is a tame space; furthermore Φ is a tame map (lemma 2.1.5).

The differential of Φ is given by

$$D\Phi(x,y)(h',h'') = \left(h', \frac{\partial}{\partial(x)}\varphi(x,y)h' + \frac{\partial}{\partial(y)}\varphi(x,y)h''\right).$$

As $\frac{\partial}{\partial(y)}\varphi(x,y)$ is invertible, the equation $D\Phi(x,y)(h',h'') = (k',k'')$ has for every $k = (k',k'')$ a unique solution $(h',h'') = V\Phi(x,y)(k',k'')$. As Φ is smooth also $D\Phi(x,y)(h',h'')$ is. By assumption $\frac{\partial}{\partial(y)}\varphi(z)$ is invertible for all $z = (x,y) \in U$. By finite dimensionality the inverse $\frac{\partial}{\partial(y)}\varphi(z)^{-1}$ is smooth. Hence $V\Phi(x,y)(k',k'') = (k', -\frac{\partial}{\partial(y)}\varphi(z)^{-1}\frac{\partial}{\partial(x)}\varphi(x,y)k', \frac{\partial}{\partial(y)}\varphi(z)^{-1}k'')$ is a smooth family of inverses. Hence, we are now in the situation to apply the Nash-Moser inverse function theorem.

2. **Apply the Nash-Moser inverse function theorem** Application of the Nash-Moser inverse function theorem gives us open neighborhoods $U_0 \subset U$ and $\widetilde{U}_0 \subset F \times W$ such that $z_0 \in U_0$ and $\Phi(z_0) = (x_0, 0) \subset \widetilde{U}_0$ together with an inverse $\Phi^{-1} : \widetilde{U}_0 \longrightarrow U_0$ that is a smooth tame map.

   Without loss of generality suppose $\widetilde{U}_0 = \widetilde{U}'_0 \times \widetilde{U}''_0$ such that $\widetilde{U}''_0 = \widetilde{U}_0 \cap W$ and $\widetilde{U}'_0 = \widetilde{U}_0 \cap F$. Thus we can put $\Phi^{-1}(w_1, w_2) = (w_1, \psi(w_1, w_2))$ for some $\psi : F \times W \to V$. φ is tame as V is a finite dimensional vector space. Thus

$$\Phi(x,y) = 0 \Leftrightarrow y = \psi(x).$$

   This completes the proof. □

Thus the main difference to the finite dimensional implicit function theorem is that it is not enough to suppose invertibility in just one point and get the extension to an open neighborhood for free. As it is the case for the Nash-Moser inverse function theorem, we have always to take care of invertibility in an open neighborhood.

This phenomenon appears again in the definition of a regular value:

Definition 2.1.13 (regular value)
Let $\varphi : F \to \mathbb{R}^n$ be a tame map. $g \in \mathbb{R}^n$ is a regular value iff there is an open set $U \subset \mathbb{R}^n$ containing g, such that the differential has full rank for all $g \in U$.

A direct consequence is the theorem:

Theorem 2.1.4
Let F be a tame space, $W \simeq \mathbb{R}^n$ and $\varphi : F \to W$ a tame map. Let $g \in W$ be a regular value for φ. Then $\varphi^{-1}(g)$ is a tame Fréchet submanifold of finite type.

Proof. The proof follows the finite dimensional blueprint:

1. **Prepare use of theorem 2.1.3** Let $\widetilde{U}'' \subset W$ be an open set containing g such that the differential of φ has full rank on $\widetilde{U}''$. Choose now an arbitrary element $h \in \varphi^{-1}(g)$. To apply the implicit function theorem 2.1.3 we need an open set $U \subset F$ containing h and a decomposition $U = U' \times U''$ such that the partial differential $\frac{\partial}{\partial(y)}\varphi(z)$ is invertible for all $z = (x,y) \in U$. To this end, choose a vector $v_1 \in T_hV$ such that $\frac{\partial\varphi}{\partial v_1}(0) \neq 0$. By continuity of the derivative there exists an open set $U_1 \subset U$, such that $\frac{\partial\varphi}{u_1}(x) \neq 0$ for all $x \in U_1$. Now choose u_2 such that $\frac{\partial\varphi}{\partial u_2}(0) \neq 0$ and such that $\frac{\partial\varphi}{\partial u_1}(0)$ and $\frac{\partial\varphi}{\partial u_2}(0)$ are linearly independent. Those conditions are satisfied on an open set U_2. Successively we choose n vectors $\{u_1, \ldots, u_n\}$ that are linearly

independent with non-vanishing differential on an open set $U_0 \subset U \subset F$. Define now $U_0'' = \text{span}\{u_1, \dots, u_n\} \cap U_0$. Then $U_0 = U_0' \times U_0''$, where U_0' is a complementary subspace to U_0'', is the desired decomposition. Now we are in a position to apply theorem 2.1.3.

2. **Construct charts** Theorem 2.1.3 gives us an implicit function $\psi : U_0' \longrightarrow U_0''$, such that $\Phi(x, y) = g$ iff $y = \psi(x)$. This defines a chart in U_0''. This chart is a tame map. Thus $\varphi^{-1}(g)$ is a tame submanifold. □

2.1.4 Some tame Fréchet spaces

Notation 2.1.3

- *Let A_n denote the annulus $A_n := \{z \in \mathbb{C}^* | e^{-n} \leq |z| \leq e^n\}$. Let $\partial A_n^+ := \{z |\, |z| = e^n\}$ and $\partial A_n^- := \{z |\, |z| = e^{-n}\}$,*
- *Let A_n' denote the set $A_n' := \{z \in \mathbb{C} | -n \leq \Re(z) \leq n, 0 \leq \Im(z) \leq 2\pi i\}$,*
- *Let B_n denote the disc $B_n := \{z \in \mathbb{C} |\, |z| \leq e^n\}$.*

Lemma 2.1.8
Let $F := Hol(\mathbb{C}, \mathbb{C})$. The two gradings

$$\|f\|_{L_1^n} := \frac{1}{2\pi} \int_{\partial B_n} |f(z)| dz$$

and

$$\|f\|_{L_\infty^n} := \sup_{z \in B_n} \|f(z)\|$$

are tame equivalent.

Proof. cf. [Ham82]. □

Lemma 2.1.9
$F := Hol(\mathbb{C}, \mathbb{C})$ is a tame Fréchet space.

Proof. cf. [Ham82]. □

Corollary 2.1.1
The space $F := Hol(\mathbb{C}, \mathbb{C}^n)$ is a tame Fréchet space.

Our aim is now to prove that $Hol(\mathbb{C}^*, \mathbb{C})$ is a tame Fréchet space. Our strategy parallels the one developed by Hamilton for the proof of lemma 2.1.9.

Lemma 2.1.10
Let $F := Hol(\mathbb{C}^*, \mathbb{C})$. The two gradings

$$\|f\|_{L_\infty^n} := \sup_{z \in A_n} |f(z)|$$

and

$$\|f\|_{L_1^n} := \frac{1}{2\pi} \sup \left\{ \int_{\partial A_n^+} |f(z)| dz, \int_{\partial A_n^-} |f(z)| dz \right\}$$

are tame equivalent.

Proof.

1. We show: $\|f\|_{L_1^n} \leq \|f\|_{L_\infty^n}$.

$$\begin{aligned}
\|f\|_{L_1^n} &= \frac{1}{2\pi} \sup \left\{ \int_{\partial A_n^+} |f(z)| dz, \int_{\partial A_n^-} |f(z)| dz \right\} \leq \\
&\leq \frac{1}{2\pi} \sup \left\{ \int_{\partial A_n^+} \sup_{\zeta \in A_n^+} |f(\zeta)| dz, \int_{\partial A_n^-} \sup_{\zeta \in A_n^-} |f(\zeta)| dz \right\} \leq \\
&\leq \sup \left\{ \sup_{z \in \partial A_n^+} |f(z)|, \sup_{z \in \partial A_n^-} |f(z)| \right\} \leq \\
&\leq \sup_{z \in A_n} |f(z)| = \|f\|_{L_\infty^n}
\end{aligned}$$

2. We show: $\|f\|_{L_\infty^n} \leq \frac{2}{r} \|f\|_{L_1^n}$.

   To this end, we identify the space $Hol(\mathbb{C}^*, \mathbb{C})$ with the space $Hol_{2\pi i}(\mathbb{C}, \mathbb{C})$ of $2\pi i$ periodic functions. Under this identification A_n will be identified with A_n'.

$$\begin{aligned}
\|f\|_{L_\infty^n} &= \sup_{z \in A_n'} |f(z)| = \\
&= \sup_{z \in A_n'} \left| \frac{1}{2\pi i} \left\{ \int_{n+r}^{n+r+2\pi i} \frac{f(\zeta)}{z - \zeta} d\zeta - \int_{-n-r}^{-n-r+2\pi i} \frac{f(\zeta)}{z - \zeta} d\zeta \right\} \right| \leq \\
&\leq \sup_{z \in A_n'} \frac{1}{2\pi} \left\{ \int_{n+r}^{n+r+2\pi i} \left| \frac{f(\zeta)}{r} \right| d\zeta + \int_{-n-r}^{-n-r+2\pi i} \left| \frac{f(\zeta)}{r} \right| d\zeta \right\} = \\
&= \sup_{z \in A_n'} \frac{1}{2\pi r} \left\{ \int_{n+r}^{n+r+2\pi i} |f(\zeta)| \, d\zeta + \int_{-n-r}^{-n-r+2\pi i} |f(\zeta)| \, d\zeta \right\} \leq \\
&\leq \frac{2}{r} \frac{1}{2\pi} \sup \left\{ \int_{n+r}^{n+r+2\pi i} |f(\zeta)|, \int_{-n-r}^{-n-r+2\pi i} |f(\zeta)| \right\} = \\
&= \frac{2}{r} \|f\|_{L_\infty^{n+r}}
\end{aligned}$$

□

Lemma 2.1.11
$F := Hol(\mathbb{C}^*, \mathbb{C})$ is a tame Fréchet space.

Proof. Let $f := \sum_{k \in \mathbb{Z}} c_k z^k$ and set $f_0^+ := \sum_{k \in \mathbb{N}_0} c_k z^k$ and $f^- := \sum_{-k \in \mathbb{N}} c_k z^k$. Clearly $f_0^+(z)$ and $f^-(\frac{1}{z})$ are holomorphic functions on $\mathbb{C}$. Let

$$\begin{array}{rccc}
\varphi: & Hol(\mathbb{C}^*, \mathbb{C}) & \longrightarrow & \Sigma(\mathbb{C}^2) \\
& (f) & \mapsto & (\{c_k\}_{k \geq 0}, \{c_k\}_{k < 0}) \; .
\end{array}$$

We use the notation $\widetilde{c}_k := (c_k, c_{-k}) \subset \mathbb{C}^2$ and use the supremum-norm on $\mathbb{C}^2$.

1. We show: $\|f\|_{L_\infty^n} \leq \|\{\widetilde{c}_k\}\|$.

$$
\begin{aligned}
\|f\|_{L_\infty^n} &= \sup_{z\in A_n} |f(z)| \leq \\
&\leq \sup_{z\in A_n} \left\{|f_0^+(z)| + |f^-(z)|\right\} \leq \\
&\leq 2 \sup_{z\in A_n} \left\{\sup\left\{|f_0^+(z)|, |f^-(z)|\right\}\right\} = \\
&= 2 \sup \left\{ \sup_{z\in A_n} |f_0^+(z)|, \sup_{z\in A_n} |f^-(\frac{1}{z})| \right\} = \\
&= 2 \sup \left\{\|f_0^+\|_{L_\infty^n}, \|f^-\|_{L_\infty^n}\right\} \leq \\
&\leq 2 \sup \left\{\|\{c_k\}\|_{L_1^n}, \|\{c_{-k}\|_{L_1^n}\}\right\} \leq \\
&\leq 2\|\{\widetilde{c}_k\}\|_{L_\infty^n}
\end{aligned}
$$

2. We show: $\|\{c_k\}\|_{L_\infty^n} \leq \|f\|_{L_1^n}$.

$$
\begin{aligned}
\|\{\widetilde{c}_k\}\|_{L_\infty^n} &= \sup_k e^{nk} |\widetilde{c}_k| = \\
&= \sup_k e^{nk} \left|\sup\{c_k, c_{-k}\}\right| = \\
&\leq \sup \left\{ \sup_k e^{nk}|c_k|, \sup_k e^{nk}|c_{-k}| \right\} \leq \\
&\leq \sup \left\{ \sup_k e^{nk} \frac{1}{2\pi} \left| \int_n^{n+2\pi i} e^{-kz} f(z) dz \right|, \sup_k e^{nk} \frac{1}{2\pi} \left| \int_{-n}^{-n+2\pi i} e^{kz} f(z) \right| | \right\} \leq \\
&\leq \sup \left\{ \sup_k \frac{1}{2} \int_n^{n+2\pi i} |e^{nk} e^{-kn}||e^{-ki\Im(z)}||f(z)| dz \right. , \\
&\qquad \left. \sup_k \frac{1}{2\pi} \int_{-n}^{-n+2\pi i} |e^{nk} e^{-kn}||e^{ki\Im(z)}||f(z)| dz \right\} \leq \\
&\leq \sup \left\{ \sup_k \frac{1}{2} \int_n^{n+2\pi i} |f(z)| dz, \sup_k \frac{1}{2\pi} \int_{-n}^{-n+2\pi i} |f(z)| dz \right\} \leq \\
&\leq \sup \left\{ \sup_k \frac{1}{2} \int_n^{n+2\pi i} \sup_{\zeta\in\partial A_n'^+} |f(\zeta)| dz, \sup_k \frac{1}{2\pi} \int_{-n}^{-n+2\pi i} \sup_{\zeta\in\partial A_n'^+} |f(\zeta)| dz \right\} = \\
&= \sup \left\{ \sup_{\zeta\in\partial A_n'^+} |f(z)|, \sup_{\zeta\in\partial A_n'^-} |f(z)| \right\} \leq \\
&\leq \sup_{\zeta\in A_n' |f(\zeta)|} = \|f\|_{L_\infty^n}
\end{aligned}
$$

□

Lemma 2.1.12
The differential

$$\frac{d}{dz} : Hol(\mathbb{C}^* \longrightarrow \mathbb{C}),\ f \mapsto f'$$

is a tame linear map.

For the proof we need the following result:

Lemma 2.1.13
Let f be analytic on a closed disc $\overline{D}(z_0, R), R > 0$. Let $0 < R_1 < R$. Denote by $\|f\|_R$ the supremum norm of f on the circle of radius R. Then for $z \in \overline{D}(z_0, R_1)$, we have:

$$|f^{(k)}(z)| \leq \frac{k!R}{(R-R_1)^{k+1}} \|f\|_R \,.$$

Proof. This lemma is an application of Cauchy's integral formula. For details cf. [Lan99], pp. 131. □

Proof of lemma 2.1.12.

1. $\frac{d}{dz}$ is a linear operator.
2. To prove tameness, we use lemma 2.1.13. Let $z \in A_n$ and choose $R = e^{-(n+1)}(e-1)$. Thus $\overline{D}(z, R) \subset A_{n+1}$. Hence $\|f\|_R \leq \|f\|_{n+1}$.

   Using that $R_1 = 0$ and $k = 1$ we find in this way:

   $$\|f'(z)\|_n \leq \frac{R}{R^2} \|f\|_{n+1} = \frac{e^{n+1}}{e-1} \|f\|_{n+1} \,.$$

   This description is independent of z. Thus

   $$\|f'\|_n \leq \frac{e^{n+1}}{e-1} \|f\|_{n+1} \,.$$

   Thus the differential is $(1, 0, \frac{e^{n+1}}{e-1})$-tame.

□

A further class of spaces that will be important for the description of twisted Kac-Moody algebras (compare definition 2.3.2) are spaces of holomorphic functions that satisfy some functional equation. We describe first the general setting and specialize then to the two most important cases, namely symmetric or antisymmetric holomorphic functions.

Lemma 2.1.14 (Subspaces of $Hol(\mathbb{C}^*, \mathbb{C})$)
Let $k, l \in \mathbb{N}$ and $\omega = e^{\frac{2\pi i}{k}}$. The spaces $Hol^{k,l}(\mathbb{C}^*, \mathbb{C}) := \{f \in Hol(\mathbb{C}^*, \mathbb{C}), f(\omega z) = \omega^l f(z)\}$ are tame spaces.

Proof. As usual for $f \in Hol^{k,l}(\mathbb{C}^*, \mathbb{C})$, we put $\|f\|_n := \sup_{z \in A_n} |f(z)|$. As $Hol^{k,l}(\mathbb{C}^*, \mathbb{C})$ is a closed subspace of $Hol(\mathbb{C}^*, \mathbb{C})$, it is closed as a consequence of lemma 2.1.4. □

Twisted affine Kac-Moody algebras arise as fixed point algebras of diagram automorphisms of non-twisted affine Kac-Moody algebras. The list of possible diagram automorphism shows that nontrivial diagram automorphisms have order $k = 2$ or $k = 3$ — cf. [Car02]. Thus the values of k which are important for us are $k = 2$ and $k = 3$.
For $k = 2$, lemma 2.1.14 has the corollaries:

Corollary 2.1.2 (symmetric and antisymmetric loops)

- *The space $Hol^s(\mathbb{C}^*, \mathbb{C}) := \{f \in Hol(\mathbb{C}^*, \mathbb{C}), f(z) = f(-z)\}$ is a tame Fréchet space.*
- *The space $Hol^a(\mathbb{C}^*, \mathbb{C}) := \{f \in Hol(\mathbb{C}^*, \mathbb{C}), f(z) = -f(-z)\}$ is a tame Fréchet space.*

Lemmata 2.1.4 and 2.1.11 and corollary 2.1.2 include the following result:

Corollary 2.1.3
$F := Hol(\mathbb{C}^*, \mathbb{C}^n)$, $F^s := Hol^s(\mathbb{C}^*, \mathbb{C}^n)$ and $F^a := Hol^a(\mathbb{C}^*, \mathbb{C}^n)$ are tame Fréchet spaces.

Let V^n be a n-$\mathbb{C}$-dimensional complex vector space. Using the corollary 2.1.3 a tame Fréchet structure is defined on $Hol(\mathbb{C}^*, V^n)$, $Hol^s(\mathbb{C}^*, V^n)$ and $Hol^a(\mathbb{C}^*, V^n)$ only after the choice of an identification of V^n with $\mathbb{C}^n$. This structure depends a priori on this identification.

Nevertheless, the following lemma shows that the tame structure is independent of the identification $V^n \cong \mathbb{C}^n$:

Lemma 2.1.15
Let V^n be a n-dimensional complex vector space equipped with two norm $|\ |$ and $|\ |'$. Study the spaces $Hol^{k,l}(\mathbb{C}^*, V^n)$. Define gradings $\|f\|_n := \sup_{z \in A_n} |f(z)|$ and $\|f\|'_n := \sup_{z \in A_n} |f(z)|'$. Those gradings are tame equivalent.

Proof. Any two norms on a finite dimensional Lie algebra are equivalent (see any book about elementary analysis, i.e. [Kö00]). Thus there exist constants c_1 and c_2 such that $|x| \leq c_1|x|'$ and $|x|' \leq c_2|x|$. Then $\|f\|_n := \sup_{z \in A_n} |f(z)| \leq \sup_{z \in A_n} c_1|f(z)|' = c_1\|f\|'_n$ and $\|f\|'_n := \sup_{z \in A_n} |f(z)|' \leq \sup_{z \in A_n} c_2|f(z)|' = c_2\|f\|_n$. Thus they are tame equivalent. □

2.2 Inverse limit constructions

Following Omori [Omo97], we define

Definition 2.2.1 (ILE-chain)
A family of locally convex topological vector spaces $\{E, E^k; k \in \mathbb{N}\}$ is called an ILE-chain iff:

1. *E^k is continuously embedded in E^{k+1}; its image is dense.*
2. *$E = \bigcap E^k$; the topology on E is the inverse limit topology (i.e. the weakest topology such that the embedding $E \hookrightarrow E^k$ is continuous for every k).*

If the spaces E^k are Hilbert spaces, it is called an ILH-chain, if they are Banach spaces an ILB-chain.

From now on, we suppose every ILE-chain to be at least ILB.

Lemma 2.2.1
Every Fréchet F space admits an ILB-chain.

Proof. As any Fréchet space admits a grading, we can suppose $(F, \|\ \|_n, n \in \mathbb{N})$ to be a graded Fréchet space. Define B_n to be the completion of F with respect to $\|\ \|_n$. The grading assures that the embedding $B_{n+1} \hookrightarrow B_n$ is bounded and thus continuous (a linear map between Banach spaces is continuous iff it is bounded — cf. [Con90]. As F is dense in B_n, the image of B_{n+1} in B_n is dense too. Thus $F = \varprojlim B_n$. This inverse limit will be called the *canonical ILB-system associated to F.* □

The following definition is a slight variant of a concept due to Hideki Omori:

Definition 2.2.2 (ILB-regular map)
Let $\varphi : E \longrightarrow F$ be a linear map between two ILB-systems. φ is called ILB-(r,b)-regular iff for every n, φ extends to a continuous map $\varphi_n : E^n \longrightarrow F^{n+r}$ for all $n > b$. It is called ILB-regular if it is ILB-$(0,0)$-regular. It is called weak ILB-regular iff there are (r,b) such that it is ILB-(r,b)-regular

Remark 2.2.1

1. *Suppose $E \neq F$ and $\varphi : E \longrightarrow F$ is a weak ILB-regular map. After re-indexing the canonical ILB-system associated to E and F one may suppose φ to be ILB-regular. Thus without loss of generality we can suppose every weak ILB-regular map $\varphi : E \longrightarrow F$ to be ILB-regular.*
2. *Caution! This is in contrast to the case $E = F$. In this case a map is ILB-regular iff it is ILB-$(0,b)$-regular.*

Definition 2.2.3
Let $\{F, B^k; k \in \mathbb{N}\}$ and $\{\widetilde{F}, \widetilde{B}^k; k \in \mathbb{N}\}$ be two ILB-systems. They are called ILB-equivalent iff there are ILB-regular maps:

$$\varphi : \{F, B^k; k \in \mathbb{N}\} \longrightarrow \{\widetilde{F}, \widetilde{B}^k; k \in \mathbb{N}\}$$
$$\nu : \{\widetilde{F}, \widetilde{B}^k; k \in \mathbb{N}\} \longrightarrow \{F, B^k; k \in \mathbb{N}\}\,.$$

such that $\varphi \circ \nu = Id$ and $\nu \circ \varphi = Id$.

Now we investigate the relationship with tame structures: so from now on we suppose E and F denote tame Fréchet spaces.

Lemma 2.2.2
A tame map $\varphi : E \longrightarrow F$ induces an ILB-(r,b)-regular map between the canonical ILB-system associated to E and F.

Proof. Suppose φ is $(r, b, C(n))$-tame. We have to show that φ induces a family of maps $\varphi_n : E^n \longrightarrow F^{n+r}$ for all $n \geq b$. Let $e \in E^n$. Take a sequence $e_k \in E$ such that $\lim_{k\to\infty} e_k = e$. Define $\varphi_n(e) := \lim_{k\to\infty} \varphi(e_k)$. Clearly $\varphi_n(e)$ is well defined. As $\|\varphi(e_k)\|_{n+r} \leq C(n)\|e_k\|$ we get $\|\varphi_n(e)\|_{n+r} \leq C(n)\|e\| + \epsilon$ for all $\epsilon > 0$. Thus $\varphi_n(e) \subset F^{n+r}$ and φ_n is a bounded linear map and thus continuous. □

Tame equivalence of gradings translates directly into a property of the associated ILB-systems.

Definition 2.2.4 (Tame equivalence of ILB-systems)
Let F be a Fréchet spaces with two gradings $\| \|_n$ and $\widetilde{\| \|}$. Let $(F, B_n; n \in \mathbb{N})$ and $(F, \widetilde{B}_n; n \in \mathbb{N})$ be the canonical ILB-systems associated to $(F, \| \|_n)$ and $(F, \widetilde{\| \|})$.

$(F, B_n; n \in \mathbb{N})$ and $(F, \widetilde{B}_n; n \in \mathbb{N})$ are called $(r, b, C(n))$-equivalent iff there are ILB-(r,b) regular maps $\varphi : (F, B_n) \longrightarrow (F, \widetilde{B_n})$ and $\nu : (F, \widetilde{B_n}) \longrightarrow (F, B_n)$ such that

$$|\nu(b)|_n \leq C(n)|\widetilde{b}|_{n+r} \text{ and } \widetilde{|\varphi(b)|}_n \leq C(n)|b|_{n+r} \text{ for all } n \geq b\,.$$

Two inverse limit systems are called tame equivalent if they are $(r, b, C(n))$-equivalent for some set $(r, b, C(n))$.

Lemma 2.2.3
Let F be a Fréchet space. Tame equivalence of gradings on F and tame equivalence of the canonical ILB-systems associated to F is equivalent.

Proof. We have to prove two directions:

1. Suppose first $(F, B^n; n \in \mathbb{N})$ and $(F, \widetilde{B}^n; n \in \mathbb{N})$ are $(r, b, C(n))$-equivalent. The n-norms $\| \|_n$ (resp. $\widetilde{\| \|}_n$ on F are the restriction of the norm $| |_n$ on B_n (resp. $\widetilde{| |}_n$ on $\widetilde{B}^n$) to F. Thus the equivalence of the two gradings is trivial.

2. Suppose now the spaces $(F, \| \|_n)$ and $(F, \widetilde{\| \|}_n)$ are $(r, b, C(n))$-tame. Using lemma 2.2.2 we get maps $\varphi : (F, B_n) \longrightarrow (F, \widetilde{B_n})$ and $\nu : (F, \widetilde{B}^n) \longrightarrow (F, B^n)$ that are both $(r, b, C(n))$-equivalent. This completes the proof. □

Corollary 2.2.1
Let F be a Fréchet space. Tame equivalent gradings on F have ILB-equivalent ILB-systems iff the gradings are $(0, b, C(n))$-tame equivalent.

Remark 2.2.2
The concept of inverse limits sheds also new light on the Nash-Moser inverse function theorem. It gives an intuition where the additional condition – the existence of an open set U such that the differential is invertible on U – has its origins:

Let F^n and G^n be the inverse limit systems associated to F and G. Suppose ϕ is $(r, 0, C(n))$-tame. Then ϕ induces a family of maps $\phi_n : F^n \longrightarrow G^{n+r}$ for all $n \geq b$. Suppose now, we know that for $f \in F$ the derivative is invertible. Thus by the Banach inverse function theorem we get a family of open sets $U_n \subset F^n$ such that $\phi_n|_{U_n}$ is invertible. The whole family – and thus ϕ – is invertible on the intersection $\bigcap U_n$ – which as an intersection of infinitely many open sets may not be open. Thus the Nash-Moser inverse function theorem can be rephrased: ϕ is invertible if there exists an open set $U \subset \bigcap U_n$ such that ϕ_n is invertible on U.

We define some nonlinear ILB-concepts. Our reference is again the monograph [Omo97]:

Definition 2.2.5 (nonlinear ILB-mappings)
A function $f : U \cap E \longrightarrow F$ is called a $C^\infty - ILB$-mapping iff for each $n \in \mathbb{N}$, f extends to a C^∞ function $f^n : U \cap E^n \longrightarrow F^n$.

There is an inverse function theorem and an implicit function theorem for $C^{\infty,r}$-ILB normal mappings, a slight variant of ILB-regular maps. But as its assumptions are too restricted for our applications we omit the details (for a statement of the theorem and a proof see the monograph [Omo97]).

Definition 2.2.6 (ILB-manifold)
Let $\{F; F^n\}$ be an ILB-system. A Fréchet manifold M modeled on F is an ILB-manifold modeled on $\{F; F^n\}$ iff there are an open covering $\{U_i\}$ and a family of mappings φ_i satisfying the following conditions:

1. *For each i there is an open subset $V_i \subset F^n$ and φ_i is a homeomorphism of $V_i \cap F$ onto U_i.*

2. *For any U_i, U_j with $U_i \cap U_j \neq \emptyset$ there are open subsets $V_{i,j}$ and $V_{j,i}$ of F^n such that the chart transition functions $\varphi_{i,j}$ are C^∞-ILB-functions.*

Theorem 2.2.1
A manifold M modelled on the tame space F is an ILB-manifold iff it has an atlas $\mathcal{A}$ such that its chart transition functions are $(0, b, C(n))$-tame.

Proof.

1. We first show that existence of an atlas with $(0, b, C(n))$-tame chart transition function is necessary. Suppose there is no such atlas. Thus there is always a pair of charts: $\varphi_i : U_i \longrightarrow V_i$ and $\varphi_j : U_j \longrightarrow V_j$ such that the chart transition functions $\varphi_{i,j}$ are $(r \neq 0, b, C(n))$-tame maps.

   Lemma 2.2.2 tells us that it induces an ILB-$(r \neq 0, b)$-regular map of the canonical ILB-systems associated to $V_i \subset F$ and $V_j \subset F$. Thus this chart transition functions are not C^∞-ILB-functions as required by the definition.

2. Now we construct the ILB-manifold associated to M. Let $\{F; F^n\}$ be the canonical ILB-system associated to F. As the chart transition functions for M are supposed to be $(0, b, C(n))$-tame, they induce for each $n \in \mathbb{N}$ a continuous map $\varphi^n_{i,j} : V^n_j \longrightarrow V^n_i$. Hence, the chart transition functions are ILB-maps. Next we have to check that the maps $\varphi^n_{i,j}$ are smooth functions. Using that $\varphi_{i,j}$ is supposed to be a smooth function, we get smooth tame derivatives of any order. Another applications of Lemma 2.2.2 tells us that they induce maps $\varphi^{n,(k)}_{i,j}$ of the canonical associated ILB-systems. Those maps coincide with the derivatives of the ILB-maps as they coincide on a dense subspace after the embedding of $F \hookrightarrow F^n$.

□

Remark 2.2.3
An ILB-manifold M whose chart transition functions are $(0, b, C(n))$-tame has an associated series of Banach manifolds $M^n, n \in \mathbb{N}$. M^n is defined to be the space of charts $U^n_i, i \in I$ identifying points in different charts via the chart transition functions. This is a manifold. The set of spaces $U^n_i, i \in I$ defines charts, whose chart transition functions are smooth maps. It is furthermore Hausdorff as M is.
Thus one can imagine those manifolds to be surrounded by a cloud of Hilbert manifolds.

2.3 Lie algebras of holomorphic maps

Let $\mathfrak{g}$ be a complex reductive Lie algebra, that is a direct product of simple Lie algebras with an abelian factor. Simple Lie algebras are completely classified by their root systems; we give a complete list:

$$A_n, B_{n,n\geq 2}, C_{n,n\geq 3}, D_{n,n\geq 4}, E_6, E_7, E_8, F_4, G_2 \,.$$

We define a compact real form to be a reductive Lie algebra that is a direct product of the (up to conjugation) unique compact real forms of the simple factors together with a compact real form of the abelian factor.

Definition 2.3.1 (complex holomorphic non-twisted Loop algebra)
Let $\mathfrak{g}_\mathbb{C}$ be a finite-dimensional reductive complex Lie algebra.

1. *The loop algebra $A_n\mathfrak{g}_\mathbb{C}$ is the vector space*

$$A_n\mathfrak{g}_\mathbb{C} := \bigcup_{A_n \subset U\, open} \{f : U \longrightarrow \mathfrak{g}_\mathbb{C} |\ f \text{ is holomorphic}\}\,,$$

   equipped with the natural Lie bracket:

$$[f, g]_{L_n\mathfrak{g}}(z) := [f, g]_0(z) := [f(z), g(z)]_\mathfrak{g} \,.$$

2. *The loop algebra* $M\mathfrak{g}_{\mathbb{C}}$ *is the vector space*

$$M\mathfrak{g}_{\mathbb{C}} := \{f : \mathbb{C}^* \longrightarrow \mathfrak{g}_{\mathbb{C}} | \ f \text{ is holomorphic}\}$$

equipped with the natural Lie bracket:

$$[f,g]_{M\mathfrak{g}}(z) := [f,g]_0(z) := [f(z), g(z)]_{\mathfrak{g}} .$$

Lemma 2.3.1

1. $M\mathfrak{g}_{\mathbb{C}}$ *is a tame space.*
2. $A_n\mathfrak{g}_{\mathbb{C}}$ *is a Banach space.*

Proof. The first assertion is a consequence of corollary 2.1.3. The second assertion is a consequence of Montel's theorem — cf. [BG91]. □

The inclusions $S^1 = A_0 \subset \dots A_n \subset A_{n+1} \subset \dots \subset \mathbb{C}^*$ induce the reversed inclusions on the associated loop algebras:

$$M\mathfrak{g}_{_\mathbb{C}} \subset \dots \subset A_{n+1}\mathfrak{g}_{\mathbb{C}} \subset A_n\mathfrak{g}_{\mathbb{C}} \subset \dots \subset A_0\mathfrak{g}_{\mathbb{C}} = L_{\text{hol}}\mathfrak{g}_{\mathbb{C}} .$$

To describe the twisted loop algebras we recall the graph automorphism of the finite dimensional simple Lie algebras: the following list contains the simple algebras A with a nontrivial diagram automorphism σ and the type of the fixed point algebra (compare [Car02]).

A	:	A_{2k}	A_{2k+1}	D_{k+1}	D_4	E_6
Order of σ	:	2	2	2	3	2
A^1	:	B_k	C_k	B_k	G_2	F_4

Definition 2.3.2 ((twisted) loop algebra, ord$(\sigma) = 2$)
Let $\mathfrak{g}_{\mathbb{C}}$ *be a finite dimensional semisimple complex Lie algebra of type* A_k, $D_{k,k\geq 5}$ *or* E_6, σ *the diagram automorphism. Let* $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g}_{\mathbb{C}}^1 \oplus \mathfrak{g}_{\mathbb{C}}^{-1}$ *be the decomposition into the* $\pm$*-eigenspaces of* σ*. Let* $X \in \{A_n, \mathbb{C}^*\}$*. If* $X = A_n$ *holomorphic functions on* X *are understood to be holomorphic on an open set containing* X*.*
The loop algebra $(X\mathfrak{g})^{\sigma}$ *is the vector space*

$$X\mathfrak{g}^{\sigma} := \{f \in X\mathfrak{g} | f(-z) = \sigma(f(z))\} ,$$

equipped with the natural Lie bracket:

$$[f,g]_{X\mathfrak{g}^{\sigma}}(z) := [f,g]_0(z) := [f(z), g(z)]_{\mathfrak{g}} .$$

Remark 2.3.1 ((twisted) loop algebra, ord$(\sigma) = 3$)
For the algebra of type D_4 *there exists an automorphism* σ *of order* 3*. In this case we get exactly the same results as for the other types. The main difference is that we have three eigenspaces, corresponding to* $\{\omega, \omega^2, \omega^3 = 1\}$ *for* $\omega = e^{\frac{2\pi i}{3}}$*. For a function* f *in the loop algebra* $M\mathfrak{g}$*, this results in a twisting condition* $f(\omega z) = \sigma f(z)$ *(for details compare again [Car02]).*

Lemma 2.3.2 (Banach- and Fréchet structures on twisted loop algebras)

1. $A_n\mathfrak{g}^{\sigma}$ *equipped with the norm* $\| \ \|_n$ *is a Banach Lie algebra,*

2. *$M\mathfrak{g}^\sigma$ equipped with the norms $\| \|_n$ is a tame Fréchet Lie algebra.*

Proof. Closed subspaces of Banach spaces are Banach and closed subspaces of tame Fréchet spaces are tame Fréchet spaces (lemma 2.1.4). □

To unify notation suppose the identity to be an involution and define $X \in \{A_n, \mathbb{C}\}$. If $X = A_n$ holomorphic functions on X are understood to be holomorphic on an open set containing X.

Theorem 2.3.1
The system $\{M\mathfrak{g}^\sigma_\mathbb{C}; A_n\mathfrak{g}^\sigma_\mathbb{C}\}$ is an ILB-system.

Proof.

1. We have to check that $A_{n+1}\mathfrak{g}^\sigma \hookrightarrow A_n\mathfrak{g}^\sigma$ is a continuous, dense embedding. Continuity follows as $\| \|_n \leq \| \|_{n+1}$. Thus the embedding is a bounded linear map and thus continuous; the image is dense as polynomials on $\mathbb{C}^*$ are dense in $A_n\mathfrak{g}$ for all $n \in \mathbb{N}$.

2. The topology on $M\mathfrak{g}^\sigma$ is the inverse limit topology as it is the topology generated by the set of all norms $\| \|_n$.

□

The adjoint action $ad(g) : M\mathfrak{g}^\sigma \longrightarrow M\mathfrak{g}^\sigma$ is $(0, 0, 2\|g\|_n)$-tame for each $g \in M\mathfrak{g}^\sigma$. Thus it induces an adjoint action on each algebra $A_n\mathfrak{g}^\sigma$ of the associated ILB-system. Contrast this with the situation for the affine Kac-Moody algebra described in section 3.3.
Having described the holomorphic complex loop algebras which we will need, we turn now to some associated objects namely the compact real forms and spaces of differential forms. We start with real forms of compact type:

Definition 2.3.3 (compact real form of a holomorphic non-twisted loop algebra)
Let $\mathfrak{g}_\mathbb{C}$ be a finite-dimensional semisimple complex Lie algebra and $\mathfrak{g}$ its compact real form. The loop algebra $X\mathfrak{g}^\sigma_\mathbb{R}$ is the vector space

$$X\mathfrak{g}^\sigma_\mathbb{R} := \{f \in X\mathfrak{g}^\sigma_\mathbb{C} | f(S^1) \subset \mathfrak{g}_\mathbb{R}\},$$

equipped with the natural Lie bracket:

$$[f, g]_{X\mathfrak{g}}(z) := [f, g]_0(z) := [f(z), g(z)]_\mathfrak{g}.$$

As holomorphic functions on X can be expanded into Laurent series, one can represent every element of a loop algebra by a series

$$f(z) := \sum_n g_n z^n$$

with $g_n \in \mathfrak{g}$. The condition $f(S^1) \subset \mathfrak{g}_\mathbb{R}$ translates to the condition $g_n = -\bar{g}^*_{-n}$.
$M\mathfrak{g}_\mathbb{R} \subset M\mathfrak{g}_\mathbb{C}$ is a closed subspace and thus tame according to lemma 2.1.4.

Theorem 2.3.2
The system $\{M\mathfrak{g}^\sigma_\mathbb{R}, A_n\mathfrak{g}^\sigma_\mathbb{R}\}$ is an ILB-system.

Proof. The proof is as in the complex case. □

Definition 2.3.4

1. *$\Omega^1(X, \mathfrak{g}_\mathbb{C})$ is the space of $\mathfrak{g}_\mathbb{C}$-valued 1-forms on X, that is elements $\omega \in \Omega^1(X, \mathfrak{g}_\mathbb{C})$ are of the form $\omega(z) = f(z)dz$ with $f(z) \in X\mathfrak{g}_\mathbb{C}$. We define a family of norms by $\|\omega\|_n := |f(z)|_n$.*

2. *$\Omega^1(X, \mathfrak{g}_\mathbb{C})_\mathbb{R}$ is the space of $\mathfrak{g}_\mathbb{C}$-valued 1-forms on X such that $f(\mathbb{S}^1) \subset \mathfrak{g}_\mathbb{R}$.*

As $M\mathfrak{g}_\mathbb{C}$ and $M\mathfrak{g}_\mathbb{R}$ are tame Fréchet spaces, also $\Omega^1(X, \mathfrak{g}_\mathbb{C})$ and $\Omega^1(X, \mathfrak{g}_\mathbb{C})_\mathbb{R}$ are tame.

Theorem 2.3.3
The system $\{\Omega^1(M, \mathfrak{g}_\mathbb{C}), \Omega^1(A_n, \mathfrak{g}_\mathbb{C})\}$ is an ILB-system.

Proof. This is equivalent to $\{M\mathfrak{g}_\mathbb{C}, A_n\mathfrak{g}_\mathbb{C}\}$ being an ILB-system. □

Remark 2.3.2
Real forms of the algebras $X\mathfrak{g}_\mathbb{C}^\sigma$ correspond to conjugate-linear involutions of $X\mathfrak{g}_\mathbb{C}^\sigma$: assign to a real form the conjugation with respect to it. In the other direction, fixed point algebras of conjugate-linear involutions are real forms. Hence, real forms are closed subalgebras. Thus by an application of lemma 2.1.4 real forms of $M\mathfrak{g}_\mathbb{C}^\sigma$ are tame, real forms of $A_n\mathfrak{g}_\mathbb{C}^\sigma$ are Banach. The proof of theorem 2.3.1 generalizes to this setting. Thus one has ILB-systems.

2.4 Groups of holomorphic maps

Up to now we studied analytic structures on loop algebras but not on the associated loop groups. In short the main result is that all loop algebras interesting to us carry a tame structure and an ILB-structure. In this section we prove similar results for loop groups. Let G be a compact semisimple Lie group and $G_\mathbb{C}$ its complexification.

2.4.1 Foundations

We start with some definitions:

Definition 2.4.1 (Complex loop groups)

1. *The loop group A_nG is the group*

$$A_nG_\mathbb{C} := \{f : A_n \longrightarrow G_\mathbb{C} |\ f \text{ is holomorphic}\}.$$

The multiplication is defined to be $fg(z) := f(z)g(z)$ for $f, g \in A_nG$.

2. *The loop group MG is the group*

$$MG_\mathbb{C} := \{f : \mathbb{C}^* \longrightarrow G_\mathbb{C} |\ f \text{ is holomorphic}\}.$$

The multiplication is defined to be $(fg)(z) := f(z)g(z)$ for $f, g \in MG$.

Definition 2.4.2 (Real form of compact type)

1. *The real form of compact type $A_nG_\mathbb{R}$ is defined to be*

$$A_nG_\mathbb{R} := \{f \in A_nG_\mathbb{C} | f(\mathbb{S}^1) \subset G_\mathbb{R}\}.$$

2. *The real form of compact type $MG_{\mathbb{R}}$ is defined to be*

$$MG_{\mathbb{R}} := \{f \in MG_{\mathbb{C}} | f(\mathbb{S}^1) \subset G_{\mathbb{R}}\}\,.$$

There are exponential functions $A_n \exp : A_n\mathfrak{g} \longrightarrow A_nG$ and $M\exp : M\mathfrak{g} \longrightarrow MG$, defined pointwise using the group exponential function: $\exp : \mathfrak{g} \to G$:

$$(A_n \exp)(f)(z) := \exp(f(z))\,,$$
$$(M \exp)(f)(z) := \exp(f(z))\,.$$

The next important object to describe the connection between the loop algebras and the loop groups is the definition of the Adjoint action Ad:

As usual it is defined pointwise using the Adjoint action of the Lie group $G_{\mathbb{K}}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$:

$$(\mathrm{Ad}(A_nG)_{\mathbb{K}} \times A_n\mathfrak{g}_{\mathbb{K}}) \longrightarrow A_n\mathfrak{g}_{\mathbb{K}}, \qquad (f,h) \mapsto fhf^{-1}\,,$$
$$(\mathrm{Ad}(MG)_{\mathbb{K}} \times M\mathfrak{g}_{\mathbb{K}}) \longrightarrow M\mathfrak{g}_{\mathbb{K}}, \qquad (f,h) \mapsto fhf^{-1}\,,$$

where $fhf^{-1}(z) := f(z)h(z)f^{-1}(z) \simeq \mathrm{Ad}(f(z))(h(z))$.

For the Adjoint action of the compact type groups to be well-defined we have to check, that the condition $f(\mathbb{S}^1) \subset \mathfrak{g}_{\mathbb{R}}$ is preserved. This is a consequence of the adjoint action for finite dimensional compact Lie groups: for all $z \in \mathbb{S}^1$ we have $f(z) \in G_{\mathbb{R}}$ and $h(z) \in \mathfrak{g}_{\mathbb{R}}$. Thus the condition $\mathrm{Ad}(f)h(z) \in \mathfrak{g}_{\mathbb{R}}$ is preserved pointwise.

Lemma 2.4.1

The exponential function and the Adjoint action satisfy the identity: $Ad \circ \exp = e^{ad}$.

The proof consists in the application of the finite dimensional identity to show that the identity is valid pointwise.

We will now investigate the functional analytic nature of the groups A_nG and MG: to fix some notation let $X\sigma$ denote by abuse of notation an involution of $X\mathfrak{g}$ resp. of XG. Let XG_D denote a real form of non-compact type of XG, denote by $\mathrm{Fix}(X\sigma)$ the fixed point group of an involution $X\sigma$.

Ernst Heintze and Christian Groß— cf. [HG09] — show that real forms of the non-compact type of a complex simple Kac-Moody algebra are in bijection with involutions of the compact real form (which is unique up to conjugation). Let $X\mathfrak{g}_{\mathbb{R}}$ be a compact real form with involution $X\sigma$. We denote by $X\mathfrak{g}_{D,\sigma}$ the real form of non-compact type associated to $X\sigma$.

The groups A_nG are easy to understand: as A_n is compact we can follow the classical strategy to define manifold and Lie group structures. We start by defining a chart on an open set containing the identity with values in the Lie algebra via the exponential map; then we use left translation to construct an atlas of the whole group. This strategy yields the following basic results:

Theorem 2.4.1

1. *$A_nG_{\mathbb{R}}$ and $A_nG_{\mathbb{C}}$ are Banach-Lie groups.*

2. *Real forms A_nG_D of non-compact type of $A_nG_{\mathbb{C}}$ are Banach-Lie groups.*

3. *Quotients $A_nG_{\mathbb{R}}/Fix(A_n\sigma)$ and $A_nG_{D,\sigma}/Fix(A_n\sigma)$ are Banach manifolds.*

For Banach-Lie groups and Banach manifolds, there is a considerable body of work; for a classical introduction see for example [Pal68].
For the groups MG the theory is considerably more difficult. The crucial observation is the fact that the exponential map has in general not to be a local diffeomorphism.
We give an example of this strange phenomenon:

Example 2.4.1 ($MSL(2,\mathbb{C})$)
We study the Lie group $SL(2,\mathbb{C})$. As is well known, $\exp : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow SL(2,\mathbb{C})$ *is not surjective. For example, elements $g \in SL(2,\mathbb{C})$ conjugate to the element* $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ *are not in the image of* $\exp(\mathfrak{sl}(2,\mathbb{C}))$.

We want to show that there exists a sequence $f_n \in MSL(2,\mathbb{C})$ which converges to the identity map in the compact-open-(tame Fréchet) topology but is not contained in the image of $M\exp$. *To this end, we have to construct f_n in a way that it contains points that are not in the image of* $\exp(\mathfrak{sl}(2,\mathbb{C}))$. *We take*

$$f_n(z) = \begin{pmatrix} e^{\pi z/n} & -iz/n \\ 0 & e^{-\pi z/n} \end{pmatrix}.$$

Then

$$f_n(in) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

So f_n is not contained in $\Im(M\exp(\mathfrak{sl}(2,\mathbb{C})))$. On the other hand, for all $z_0 \in \mathbb{C}^*$ fixed

$$\lim_{n\to\infty} f_n(z_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Id.$$

So in the compact-open topology for every neighborhood U_k of the identity there exist $n_k \in \mathbb{N}$ such that $\forall n \geq n_k : f_n \in U_k$.

This proves that f_n is not a local diffeomorphism.

Especially, this observation contains the corollary that the groups MG are no locally exponential Lie groups in the sense of Karl-Hermann Neeb [Nee06], that is Lie groups such that exp is a local diffeomorphism.

Hence, we have to find another way to define manifold structures on MG. We will start by describing some results about the relationship between MG and $M\mathfrak{g}$. Then we will show that loop groups satisfy the weaker axioms for pairs of exponential type introduced by Hideki Omori. Finally, we will investigate the structure of $(MG; A_nG)$ as ILB-groups.
The relationship between $M\mathfrak{g}$ and MG is governed by the following three theorems:

Theorem 2.4.2 (Tangential space)
Let $\mathfrak{g}$ be the Lie algebra of G. Then

$$M\mathfrak{g} := T_e(MG).$$

As usual the tangential space T_pMG is defined as the space of path-equivalence classes of continuous paths.
Moreover, it is isomorphic to the algebra of left-invariant vector fields on MG.
On the other hand, we find:

Theorem 2.4.3 (Loop groups whose exponential map is no local diffeomorphism)
Let $G_{\mathbb{C}}$ be a complex semisimple Lie group.

$$M\exp : M\mathfrak{g} \longrightarrow MG$$

is not a local diffeomorphism.

In contrast we have for nilpotent Lie groups:

Theorem 2.4.4 (Loop groups whose exponential map is a local diffeomorphism)
Let $G_{\mathbb{C}}$ be a complex Lie group such that its universal cover is biholomorphically equivalent to $\mathbb{C}^n$. Then its exponential map $M \exp$ is a local diffeomorphism.

Corollary 2.4.1
Let G be a complex Lie group. If G is nilpotent (i.e. abelian) then $M \exp$ is a diffeomorphism.

Corollary 2.4.1 is a direct consequence of theorem 2.4.4, as abelian and nilpotent Lie groups have $\mathbb{C}^n$ as their universal cover — cf. [Kna96], corollary 1.103 and [Var84], section 3.6.

This behaviour seems contradictory as the usual intuition tells us there should be a geodesic in every direction. From a classical analysis point of view, one can describe the analytic basics of this situation as a consequence of a change of limits, arising because of the non-compactness of $\mathbb{C}^*$:

- On the one hand, theorem 2.4.2 describes locally uniform convergence on each compact subset $K \subset \mathbb{C}^*$.

- Theorem 2.4.3 shows in contrast that globally on $\mathbb{C}^*$ convergence needs no longer be uniform.

We now give the proofs that we have omitted:

Proof of theorem 2.4.2.

1. We prove: $M\mathfrak{g} \subset T_e(MG)$, that is: let $\gamma(t) : (-\epsilon, \epsilon) \to MG$ be a curve such that $\gamma(0) = e \in MG$. Then $\dot{\gamma}(0)$ is in $M\mathfrak{g}$.

   Let $z \in \mathbb{C}^*$ arbitrary but fixed. Then $\gamma_z(t)$, the evaluation of $\gamma(t)$ at $z \in \mathbb{C}^*$, is a curve in G satisfying $\gamma_z(0) = e$. Thus $\frac{d}{dt}\gamma_z(t)$ is an element in $\mathfrak{g}$. So we get a function $w : \mathbb{C}^* \longrightarrow \mathfrak{g}$; we have to prove that w is holomorphic.

   So let $\gamma(t) := \sum_n \gamma_n(t) z^n$. Then $\frac{d}{dt}\gamma(t) = \sum_n \frac{d}{dt}\gamma_n(t) z^n$, which is clearly holomorphic.

   This shows: $T_e(MG) \subset M\mathfrak{g}$.

2. The other direction is straight forward: let $\gamma \in M\mathfrak{g}$. Then $M\exp(t\gamma)$ is a curve in MG with tangential vector γ. So $M\mathfrak{g} \subset T_e(MG)$.

This completes the proof. □

Proof of theorem 2.4.3. The proof of theorem 2.4.3 relies on the fact that $M \exp$ is no local diffeomorphism for $SL(2, \mathbb{C})$. Let $\mathfrak{h} \simeq \mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{g}$ be a subalgebra of $\mathfrak{g}$. Then there is an $H = SL(2, \mathbb{C})$ subgroup in G, such that $\mathfrak{h}$ is the Lie algebra of H. See chapter VII.5 of the book [Kna96]. Study the subalgebra $M\mathfrak{h} \subset M\mathfrak{g}$ and the subgroup $MH \subset MG$. $M\mathfrak{h}$ can be identified with $T_e(MH)$. Moreover $M \exp : M\mathfrak{h} \subset MH$. But as example 2.4.1 shows, $M \exp$ is no local diffeomorphism. This completes the proof. □

The image of the exponential map consists of those loops whose image is completely contained in the image of the Lie group exponential map. We will give further comments about this situation in section 4.7, where we study the behaviour of geodesics.

Proof of theorem 2.4.4. Let $\widetilde{G}$ be the universal cover of G; the exponential map $\exp : \mathfrak{g} \longrightarrow \widetilde{G}$ is a biholomorphic map. Thus cocatenation with exp (resp. $\exp^{-1}$) induces a biholomorphic map between $M\mathfrak{g}$ and $M\widetilde{G}$. To get that $M\exp : M\mathfrak{g} \longrightarrow MG$ is a local diffeomorphism, one uses the fact that each loop in $\widetilde{MG}$ projects onto a loop in MG and, conversely, each loop in MG can be lifted to a loop in $M\widetilde{G}$, which is unique up to Deck transformation and thus locally unique. This proves that $M\exp$ is a local diffeomorphism. □

2.4.2 Manifold structures on groups of holomorphic maps

In this section, we prove that the groups $MG_{\mathbb{R}}$, $MG_{\mathbb{C}}$ and various quotients are tame Fréchet manifolds.

Theorem 2.4.5
$MG_{\mathbb{C}}$ is a tame Fréchet manifold.

The idea of the proof is to use logarithmic derivatives. The concept of logarithmic derivatives for regular Lie groups is developed in the book [KM97], chapters 38 and 40. Furthermore it is used by Karl-Hermann Neeb [Nee06] to prove the following theorem — cf. [Nee06], theorem III.1.9.):

Theorem 2.4.6 (Neeb's theorem)
Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, G be a $\mathbb{F}$-Lie group and M a finite dimensional, connected σ-compact $\mathbb{F}$-manifold. We endow the group $C^\infty_{\mathbb{F}}(M, G)$ with the compact open C^∞-topology, turning it into a topological group. This topology is compatible with a Lie group structure if $\dim_{\mathbb{F}} M = 1$ *and $\pi_1(M)$ is finitely generated.*

The main ingredients for the proof are the use of logarithmic derivatives to define charts in $\Omega^1(M, \mathfrak{g})$, the space of $\mathfrak{g}$-valued 1-forms on M and Glöckners inverse function theorem [Glö03], to take care of the monodromy if $\pi_1(M)$ is nontrivial.

If $\mathbb{F} = \mathbb{C}$ we have the equivalence $C^\infty_{\mathbb{F}}(M, G) \simeq Hol(M, G)$. Our situation is the special case $M = \mathbb{C}^*$. Hence, $\pi_1(M) = \mathbb{Z}$ is a finitely generated group. The compact open C^∞-topology coincides for holomorphic maps with the topology of compact convergence. Thus Neeb's theorem tells us that $MG_{\mathbb{F}}, \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ are locally convex topological Lie groups.

Nevertheless, we do not get tame structures. Hence, we have to prove the theorem completely new. Nevertheless, our presentation follows the proof of Karl-Hermann Neeb for the locally convex case.

We need some definitions: $\alpha \in \Omega^1(M, \mathfrak{g})$ is called integrable, iff there exists a function $f \in Hol(M, G)$ such that $\delta(f) := f^{-1}df = \alpha$. The uniqueness of solutions to linear differential equations shows that $\delta(f_1) = \delta(f_2)$ iff $f_1 = gf_2$ for some $g \in G$.

The first step of the proof is the following lemma, whose statement and proof can be found in [KM97] and [Nee06]: morally it is a straight forward application of the monodromy principle for holomorphic Pfaffian systems, as described in the article [NY02].

Lemma 2.4.2
Let M be a 1-dimensional complex manifold, $\alpha \in \Omega^1(M, \mathfrak{g})$.

1. *α is locally integrable*
2. *If M is connected, $M_0 \in M$, then there exists a homomorphism*

$$per_\alpha : \pi_1(M, m_o) \longrightarrow G$$

that vanishes iff α is integrable.

Proof. Proof of theorem 2.4.5 Using this lemma, we get an embedding:

$$\begin{array}{rcl} \varphi : MG_{\mathbb{C}} & \hookrightarrow & \Omega^1(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}}) \times G_{\mathbb{C}} \\ f & \mapsto & (\delta(f) = f^{-1}df, f(1)) \end{array}$$

Compare this embedding with the description of polar actions on Fréchet spaces in section 2.5. Let π_1 and π_2 denote the projections:

$$\begin{array}{rcl} \pi_1 : \Omega^1(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}}) \times G_{\mathbb{C}} & \mapsto & \Omega^1(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}}) \\ \pi_2 : \Omega^1(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}}) \times G_{\mathbb{C}} & \mapsto & G_{\mathbb{C}} \end{array}$$

We will describe charts for $MG_{\mathbb{C}}$ as direct sums of charts for $\pi_1 \circ \varphi(MG_{\mathbb{C}})$ and $\pi_2 \circ \varphi(MG_{\mathbb{C}})$.

- $\pi_2 \circ \varphi$ is surjective; so to describe the second factor, we can choose charts for G. Via the exponential mapping, we use charts in $\mathfrak{g}_{\mathbb{C}}$. To describe the norms, we use for $\| \|_n$ on this factor the Euclidean metric.
- The first factor is more difficult to deal with as $\pi_1 \circ \varphi$ is not surjective. While every form $\alpha \in \Omega(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}})$ is locally integrable, the monodromy may prevent global integrability. A form $\alpha \in \Omega^1(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}})$ is in the image of $\pi_1 \circ \varphi$ iff its monodromy vanishes, that is iff
$$e^{\int_{\mathbb{S}^1} \alpha} = e \in G_{\mathbb{C}}\,.$$
This is equivalent to the condition $\int_{\mathbb{S}^1} \alpha = a_{-1}(\alpha) \subset \frac{1}{2\pi i} \exp^{-1}(e)$ where $a_{-1}(\alpha)$ denotes the (-1)-Laurent coefficient of the Laurent development of $\alpha = f(z)dz$. So we can describe $\Im(\pi_1 \circ \varphi)$ as the inverse image of the monodromy map of $e \in G_{\mathbb{C}}$.

  Thus we have to show that this inverse image is a tame Fréchet manifold. To this end, we use composition with a chart $\psi : U \longrightarrow V$ for $e \in U \subset G$ with values in $\mathbb{G}_{\mathbb{C}}$. This gives us a tame map $\Omega(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathfrak{g}_{\mathbb{C}}$. This map satisfies the assumptions of theorem 2.1.4. Thus its inverse image is a tame Fréchet submanifold. This proves that $\pi_1 \circ \varphi$ is a tame Fréchet submanifold.

Thus $MG_{\mathbb{C}}$ as a product of a tame Fréchet manifold with a Lie group is a tame Fréchet manifold. This completes the proof of theorem 2.4.5. □

We now investigate different classes of quotients of loop groups:

Theorem 2.4.7
$MG_{\mathbb{R}}$ is a tame Fréchet Lie group.

Proof. The proof is similar to the proof for $MG_{\mathbb{C}}$. We have only to take care of the reality condition $f(\mathbb{S}^1) \subset G_{\mathbb{R}}$ for loops $f \in MG_{\mathbb{C}}$. Thus the embedding φ maps a loop f into $\Omega\left(\mathbb{C}^*, \mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}\right)$, which are tame Fréchet spaces. Now similar arguments apply. □

Theorem 2.4.8
The group $MG_{\mathbb{C}}/MG_{\mathbb{R}}$ is a tame Fréchet manifold.

Proof. Review the embedding

$$\psi : MG_{\mathbb{C}} \longrightarrow \Omega^1(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}}) \times G_{\mathbb{C}}\,.$$

A loop $f \cdot g$ is mapped onto $\psi(f \cdot g) = (g^{-1} f^{-1} df g + g^{-1} g, f \cdot g(1))$. This is the well-known gauge-action of the group $MG_{\mathbb{R}}$, denoted by $\mathcal{G}^*(MG_{\mathbb{R}})$. Thus there is a well defined embedding

$$\psi : MG_{\mathbb{C}}/MG_{\mathbb{R}} \longrightarrow \Omega^1(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}})/\mathcal{G}^*(MG_{\mathbb{R}}) \times G_{\mathbb{C}}/G_{\mathbb{R}}\,.$$

No we study again the projections π_1 and π_2 on the first and second factor. π_2 is surjective; $G_{\mathbb{C}}/G_{\mathbb{R}}$ is a tame manifold; so this factor is no problem.

The projection on the first factor, π_1, needs a more careful analysis: the right multiplication of $MG_{\mathbb{R}}$ on $MG_{\mathbb{R}}$ is surjective: using the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} + i\mathfrak{g}_{\mathbb{R}}$, we get for $\Omega^1(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}}) := \Omega^1_{\mathbb{R}}(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}}) + i\Omega^1_{\mathbb{R}}(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}})$.

The surjectivity of the right multiplication of $MG_{\mathbb{R}}$ on $MG_{\mathbb{R}}$ translates into the surjectivity of the MG-gauge action on the imaginary part $i\Omega^1_{\mathbb{R}}(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}}) \cap \Im(\pi_1 \circ \psi)(MG)$. Thus we can suppose to have chosen a representant $f \in f \cdot MG$, such that the imaginary part $\pi_1 \circ \psi(f)$ is 0. So all we have to check is the real part. Here we find that $\exp^{-1}(e) = 0$. Thus $a_{-1} = 0$. So we can identify the image $\pi_1 \circ \psi(MG_{\mathbb{C}}/MG_{\mathbb{R}}) \simeq \Omega^1_{\mathbb{R}}(\mathbb{C}^*, \mathfrak{g}|a_{-1} = 0)$. This is a tame Fréchet space.

Hence theorem 2.4.8 is proven. □

Having proved that $MG_{\mathbb{C}}$ and $MG_{\mathbb{C}}/MG_{\mathbb{R}}$ are tame Fréchet manifolds we have to check that the same is true for the quotients $MG_{\mathbb{R}}/\mathrm{Fix}(\rho)$ and $MG_D/\mathrm{Fix}(\rho)$.

To this end, let MG be a loop group and $M\rho$ the loop part of an involution of the second kind.

Theorem 2.4.9
Let MG_D be a non-compact real form of $MG_{\mathbb{C}}$. MG_D is a tame Fréchet manifold.

Theorem 2.4.10
The quotient spaces $MG_{\mathbb{R}}/Fix(M\rho)$ and $MG_{D,\rho}/Fix(M\rho)$ are tame manifolds.

Proof. The proof is an argument analogous to the proof that $MG_{\mathbb{C}}/MG_{\mathbb{R}}$ is a tame Fréchet space. □

The next class are twisted loop groups:

Theorem 2.4.11 (Twisted loop groups)
Let G be a compact simple Lie group of type A_n, D_n or E_6 and σ a diagram automorphism of order $m \in \{2,3\}$. Let $\omega = e^{\frac{2\pi i}{m}}$.

- *The group $A_nG^\sigma := \{f \in A^nG | \sigma \circ f(z) = f(\omega z)\}$ is a Banach-Lie group.*
- *The group $MG^\sigma := \{f \in MG | \sigma \circ f(z) = f(\omega z)\}$ is a tame manifold. Charts can be taken to be in $\Omega^1(\mathbb{C}^*, G)^\sigma$. Furthermore $M\mathfrak{g}^\sigma \simeq T_e(MG^\sigma)$.*

Proof. To generalize the proofs of the non-twisted setting to the twisted setting one has to check that the subspaces defined by diagram automorphism are preserved by the logarithmic derivative.

1. For the exponential map we calculate: we use $\sigma_{\mathfrak{g}}$ resp. σ_G to denote the realization of the diagram automorphism σ on $\mathfrak{g}$ resp. G. Any involution of a semisimple Lie group satisfies the identity: $\sigma_G \circ \exp = \exp \circ \sigma_{\mathfrak{g}}$
$$[\sigma_G \circ M\exp(f)](z) = \exp(\sigma_{\mathfrak{g}}(f(z))) = \exp(f(\omega z)) = [M\exp(f)](\omega z),$$
2. For the logarithmic derivative we calculate:
$$\delta(\sigma \circ f) = (\sigma f)^{-1} d(\sigma \circ f) = \sigma f^{-1} \sigma df = \sigma(\delta f)\,.$$
Thus we get charts in the σ-invariant subalgebra of $\Omega^1(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}})$.

□

The following definition is due to Omori [Omo97]:

Definition 2.4.3 (Exponential pair)
A pair $(G, \mathfrak{g})$ consisting of a Fréchet group and a Fréchet space $\mathfrak{g}$ is called a topological group of exponential type *if there is a continuous mapping:*

$$\exp : \mathfrak{g} \longrightarrow G$$

such that:

1. *For every $X \in \mathfrak{g}$, $\exp(sX)$ is a one-parameter subgroup of G.*
2. *For $X, Y \in \mathfrak{g}$, $X = Y$ iff $\exp(sX) = \exp(sY)$ for every $s \in \mathbb{R}$.*
3. *For a sequence $\{X_n\} \in \mathfrak{g}$, $\lim_{n\to\infty} X_n$ converges to an element $X \in \mathfrak{g}$ iff $\lim_{n\to\infty}(\exp sX_n)$ converges uniformly on each compact interval to the element $\exp(sX)$.*
4. *There is a continuous mapping $Ad : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$ with $h\exp(sX)h^{-1} = \exp sAd(h)X$ for every $h \in G$ and $X \in \mathfrak{g}$.*

Theorem 2.4.12 (Exponential type)
The pair $(MG_{\mathbb{K}}, M\mathfrak{g}_{\mathbb{K}})$ is of exponential type.

Proof. We proved that MG is a tame Fréchet Lie group, thus a topological group. To prove that $(MG_{\mathbb{K}}, M\mathfrak{g}_{\mathbb{K}})$ is of exponential type, we have to check the four conditions given in definition 2.4.3:

1. The first condition can be checked by a pointwise analysis: for $f \in M\mathfrak{g}$ and for every $z \in \mathbb{C}^*$, the curve $\exp sf(z)$ is a 1-parameter subgroup in G. This pieces for all $z \in \mathbb{C}^*$ together, to yield the condition.
2. The second condition follows analogously: let $X, Y \in M\mathfrak{g}$. $X = Y$ iff $X(z) = Y(z)$ for all $z \in \mathbb{C}^*$. The finite dimensional theory tells us that this is equivalent to the curves $\exp(sX(z)) \subset G_{\mathbb{K}}$ and $\exp(sY(z)) \subset G_{\mathbb{K}}$ to be equivalent for all $z \in \mathbb{C}^*$, but this is equivalent to $\exp(sX) = \exp(sY)$.
3. Let $\{X_n(z)\} \in M\mathfrak{g}, z \in \mathbb{C}$ be a sequence of elements such that $\lim_{n\to\infty} X_n(z) = X \in M\mathfrak{g}$. Let $T \subset \mathbb{R}$ be a compact interval, $s \in T$. As we have on MG the compact-open topology,

   $$\lim_{n\to\infty}(\exp sX_n) = \exp(sX) \Leftrightarrow \forall K \subset \mathbb{C}^* : \exp \lim_{n\to\infty}(\exp sX_n(K)) = \exp(sX)(K)\,.$$

   This assertion is correct as for every $z \in K : \exp \lim_{n\to\infty}(\exp sX_n(z)) = \exp(sX)(z)$.
4. The last assertion follows again from pointwise consideration and the validity of the assertion for finite semisimple Lie groups. □

Theorem 2.4.13 (ILB-manifold)
$MG_{\mathbb{K}}$ carries the structure of an ILB-manifold. As an ILB-manifold, it is modelled on the ILB-system $\{M\mathfrak{g}_{\mathbb{K}}, A_n\mathfrak{g}_{\mathbb{K}}\}$.

Proof. This is a consequence of theorem 2.2.1 as the atlases for loop groups are $(0, b, C(n))$-tame. □

Obviously a similar result holds for

1. non-compact real forms $\{MG_{D,\rho}; A_nG_{D,\rho}\}$,
2. the quotient spaces $MG_{\mathbb{R}}/\mathrm{Fix}(M\rho)$, $\{MG_{D,\rho}/\mathrm{Fix}(M\rho)$
3. the twisted versions of all described objects.

To prove this, one has to check in every step that the restrictions defined by the involutions are compatible.

Remark 2.4.1
The ILB-structure gives an easy interpretation of the properties of the exponential function: the exponential map $A_n \exp^{(n)} : A_{\mathfrak{g}}^{(n)} \longrightarrow A_G^{(n)}$ is defined by composition with the group exponential function $\exp : \mathfrak{g} \to G$. By the inverse function theorem for Banach spaces it is a local diffeomorphism. Thus for every $n \in \mathbb{N}$ there exist $U^{(n)}$, $V^{(n)}$ such that the map $A_n \exp^{(n)}$ is a local diffeomorphism $A_n \exp^{(n)} : U^{(n)} \longrightarrow V^{(n)}$. Of course in the limit $n \to \infty$, we get only that $M \exp$ is a diffeomorphism of $\bigcap U^{(n)}$ onto $\bigcap V^{(n)}$, which need no longer be open.

The obvious solution to this problem is the introduction of an additional assumption of the kind: there are open neighborhoods $U^\infty \subset M\mathfrak{g}$ and $V^\infty \subset MG$, such that $U^\infty \subset \left(\bigcap U^{(n)}\right) \cap M\mathfrak{g}$ and $V^\infty \subset \left(\bigcap V^{(n)}\right) \cap MG$ for all n. This assures that the intersection is open.

This corresponds to the additional condition of invertibility on an open subset in the Nash-Moser inverse function theorem.

We give some remarks about 1-parameter subgroups.

Remark 2.4.2
Let $g(t) := X\exp(tu)$ for $u \in X\mathfrak{g}_{\mathbb{K}}$ be a 1-parameter subgroup in $XG_{\mathbb{K}}$, $X \in \{A_n, \mathbb{C}^*\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then the following hold:

1. *$X\exp(tu)_{z_0}$ is a 1-parameter group in $G_{\mathbb{C}}$ for all $z_0 \in X$.*
2. *If $A_n \subset A_{n+k}$ then the embedding $A_{n+k}G \hookrightarrow A_nG$ maps 1-parameter subgroups onto 1-parameter subgroups.*

Proof. Direct calculation. □

Remark 2.4.3
As we have seen, the fact that the exponential function does not define a local diffeomorphism is responsible for several difficulties; so it is reasonable to try to use a setting in which the exponential function defines a local diffeomorphism. So let us try to take loops $f : \mathbb{S}^1 \longrightarrow G$ satisfying some regularity condition. In this case it is easy to see that the exponential map defines always a local diffeomorphism, as a neighborhood of the identity element of such a loop group is given by loops whose image lies in a small neighborhood V of the identity of the subjacent Lie group; this neighborhood can be chosen in a way such that the group exponential is a diffeomorphism from an open neighborhood U in the Lie algebra onto it. But now other problems appear:

1. *Suppose the functions to be H^1-Sobolev loops. In this setting, one can construct weak Hilbert symmetric spaces of compact and non-compact type. Nevertheless, one cannot define the double extension corresponding to the c- and d-part of the Kac-Moody algebra. As this extension is responsible for the structure theory, this setting is not useful for us.*

2. To be able to construct the extension corresponding to the derivative d, one needs loops that are C^∞. For C^∞-loops, it is possible to construct compact type symmetric spaces corresponding to the finite dimensional types I and III, but it is not possible to dualize to construct symmetric spaces of the non-compact type.

The details of both theories are developed in [Pop05]. Let us mention in this context the short summary in [Ber03] about infinite dimensional differential geometry, where the conflict between easy Hilbert space structures and good metrics is addressed.

2.5 Polar actions on tame Fréchet spaces

The isotropy representation of a (finite dimensional) Riemann symmetric space is a polar representation of the isotropy group on the tangential space. As a section one can choose any maximal flat subspace. We will see in chapter 4 that Kac-Moody symmetric spaces behave in a similar way. Nevertheless, there is a striking difference: there are orbits with finite codimension and orbits with infinite codimension. We will see, that the orbits with finite codimension correspond to gauge actions of tame loop groups on tame spaces.

Closely related is the theory of polar actions on Hilbert spaces, which is described in the article [Ter95].

The fundamental theorem due to Chuu-Lian Terng states:

Theorem 2.5.1
Define $P(G,H) := \{g \in H^1([0,1],G) | (g(0),g(1)) \in H \subset G \times G\}$ and $V = H^0([0,1],\mathfrak{g})$. Suppose the H-action on G is polar with flat sections. Let A be a torus section through e and let $\mathfrak{a}$ denote its Lie algebra. Then the gauge action of $P(G,H)$ on V is polar with section $\mathfrak{a}$.

Proof. cf. [Ter95]. □

Important special cases are the following: let $\Delta_\sigma \subset G \times G$ denote the σ-twisted diagonal subgroup of $G \times G$, that is: $(g,h) \in \Delta_\sigma$ iff $h = \sigma(g)$. We use the notation $\Delta = \Delta_{\mathrm{Id}}$ for the non-twisted subgroup.

1. The gauge action of H^1-Sobolev loop groups $P(G, G \times G) \cong H^1([0,1],G)$ on their H^0-Sobolev loop algebras $H^0([0,1],\mathfrak{g})$ is transitive.

2. The gauge action of $P(G,\Delta_\sigma)$ on $H^1([0,1],G)$ is polar with flat sections [HPTT95].

3. The gauge action of $P(G, K \times K)$ on $H^1([0,1],G)$ where K is the fixed point set of some involution of G is polar with flat sections [HPTT95].

4. The gauge action of $P(G, K_1 \times K_2)$ on $H^1([0,1],G)$ where $K_i, i \in \{1,2\}$ are the fixed point set of involutions of G is polar with flat sections [HPTT95].

5. The gauge action of Sobolev-H^1-loop groups $H^1(S^1,G)$ on their Sobolev H^0-loop algebras $H^0(S^1,\mathfrak{g})$-is polar [PT88].

In this section we describe a similar theory for the loop groups XG^σ on the tame loop algebras $X\mathfrak{g}^\sigma$. As usual let $X \in \{A_n, \mathbb{C}^*\}$. Holomorphic functions on A_n are supposed to be holomorphic in an open set containing A_n. From the embedding $XG^\sigma \hookrightarrow H^1([0,1],G)$ and $X\mathfrak{g}^\sigma \hookrightarrow H^0([0,1],\mathfrak{g})$ it is clear that the algebraic part of the theory works exactly the same in all regularity conditions. This means for example: sections for holomorphic actions correspond to sections for the Hilbert actions and the associated affine Weyl groups are the same. Hence, the crucial point is, to check that the additional regularity restrictions fit together. This breaks down to two points:

1. One has to show locally, that the additional regularity conditions are satisfied.
2. One has to show globally, that additional monodromy conditions are satisfied.

To define orthogonality on $X\mathfrak{g}$ we use the H^0-scalar product induced on $M\mathfrak{g}$ by the embedding into $H^0([0,1],\mathfrak{g})$. Hence, we can define polar actions on Banach (resp. tame) spaces like that:

Definition 2.5.1
An action of a Lie group G on a Fréchet space F is called polar iff there is a subspace S, called a section, intersecting each orbit orthogonally with respect to some scalar product.

Theorem 2.5.2
The gauge action of $XG^\sigma_\mathbb{R}$ on $X\mathfrak{g}^\sigma$ is polar; an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$ interpreted as constant loops is a section.

The proof consists of two parts:

1. We have to show that each orbit intersects the section $\mathfrak{a}$.
2. We have to show that the intersection is orthogonal.

The second part follows trivially from the embedding and Terng's result.
Thus we are left with proving the first assertion. We do this in a step-by-step way: first we study the action of C^k-loop groups on C^{k-1}-loop algebras ($k \in \{\mathbb{N},\infty\}$). Then we proceed to the holomorphic setting of theorem 2.5.2.

Lemma 2.5.1
The gauge action of L^kG^σ on $L^{k-1}\mathfrak{g}^\sigma$ is polar for $k \in \{\mathbb{N},\infty\}$.

This result is used without proof in [Pop05] in order to show that all finite dimensional flats are conjugate. We do not know if a proof can be found in the literature. For completeness we give one:

Proof of lemma 2.5.1.

1. **Orthogonality** in $L^{k-1}\mathfrak{g}^\sigma$ is (as usual) defined via the embedding into the space $H^0([0,1],\mathfrak{g})$ and the use of the H^0-scalar product. Hence, orthogonality of the intersection between sections and orbits is covered by Terng's result.
2. **Local regularity**

   Define the following spaces $P(G,H)^k := \{g \in C^k([0,1],G) | (g(0),g(1)) \in H \subset G \times G\}$.

   Furthermore we use the equivalence $P(G;e\times G) \simeq H^0([0,1],\mathfrak{g})$ defined by $h \leftrightarrow -h'h^{-1}$ — cf. [Ter95].

   Terng's polarity result — cf. [Ter95] — yields that the action of $P(G,\Delta_\sigma)$ on $P(G;e\times G)$ defined by $(g(t),h(t)) \mapsto g(t)h(t)g(0)^{-1}$ is polar with a section of constant loops $\exp t\mathfrak{a}$ where $\mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{g}$ (if $\sigma \neq 0$ we restrict to $\mathfrak{a}_\sigma$ and omit the index σ in the notation — cf. [Kac90]). Thus for every $h(t) \in P(G; e \times G)$ there exist $g(t) \in P(G, \Delta_\sigma)$ and $X \in \mathfrak{a}$, such that $g(t)h(t)g(0)^{-1} = \exp(tX)$.

Rearranging this equation we deduce for any loop $g(t) \in P(G, \Delta_\sigma)$ the explicit description $g(t) := \exp(tX)g(0)h(t)^{-1}$. Hence if $h(t) \in P^k(G; e \times G)$ then $g(t) \in P^k(G, \Delta_\sigma)$. Combining this with the orthogonality we obtain that the actions of $P^k(G, \Delta_\sigma)$ on $P^k(G; e \times G) \simeq H^{k-1}([0, 1], \mathfrak{g})$ and of $P^\infty(G, \Delta_\sigma)$ on $P^\infty(G; e \times G) \simeq H^\infty([0, 1], \mathfrak{g})$ are polar.

3. **The periodicity relation:** We want to show that L^kG^σ acts on $L^{k-1}\mathfrak{g}^\sigma$ with slice $\mathfrak{a}$ for $k \in \{\mathbb{N}, \infty\}$.

   (a) Let first $g \in L^kG^\sigma$ and $u \in L^{k-1}\mathfrak{g}^\sigma$. Then $g \cdot u = gug^{-1} - g'g^{-1}$ is in $L^{k-1}\mathfrak{g}^\sigma$. Thus L^kG^σ acts on $L^{k-1}\mathfrak{g}^\sigma$.

   (b) We have to show that any L^kG^σ-orbit intersects the section $\mathfrak{a}$. This is equivalent to: For each $u \in L^{k-1}\mathfrak{g}^\sigma$, there is $X \in \mathfrak{g}$ and $g \in P^k(G, \Delta)$ such that $\exp(tX) = g(t)h(t)g^{-1}(0)$ with $h'(t) = u(t)h(t)$ and the derivatives coincide; interpret in this last equation $u(t)$ as a quasi-periodic function on $\mathbb{R}$ (i.e. $u(t + 2\pi) = \sigma u(t)$ and $h(t)$ as a function on $\mathbb{R}$).

   Using the first part, we find a function $g(t) \in P^k(G, \Delta_\sigma)$. Hence, what remains is to check the closing condition of the derivatives: $g^{(n)} \cdot u(2\pi) = \sigma g^{(n)} \cdot u(0)$. We will prove that it is equivalent to the closing condition $g^{(n+1)}(2\pi) = \sigma g^{(n+1)}(0)$.

   We start with the case $n = 1$. For this case, we have to show

   $$\exp((t + 2\pi)X)g_0h(t + 2\pi)^{-1} = \sigma(\exp(tX)g_0h(t)^{-1}).$$

   After rearranging, this is equivalent to the identity

   $$\sigma(g_0^{-1})\exp(2\pi X)g_0 = \sigma(h(t)^{-1})h(t + 2\pi).$$

   As the left side is a constant we find:

   $$\big(\sigma(h(t)^{-1})h(t + 2\pi)\big)' = 0.$$

   Hence: $-\sigma(h(t)^{-1}h'(t)h(t)^{-1})h(t + 2\pi) + \sigma(h(t)^{-1})h'(t + 2\pi) = 0$. Rearranging this equality we get

   $$\sigma(u(t)) = -\sigma(h'(t))\sigma(h(t)^{-1}) = -h'(t + 2\pi)h(t + 2\pi)^{-1} = u(t + 2\pi)$$

   which is the desired periodicity condition.

   For $n \neq 0$ use induction. If g is k-times differentiable then u is $k - 1$-times differentiable. This proves the lemma. □

Proof of theorem 2.5.2. To prove the theorem, we have to further strengthen the used regularity conditions to holomorphic functions. The description in the proof of lemma 2.5.1 shows that the group of analytic loops $L_{an}(S^1, G)$ acts polarly with section $\mathfrak{a}$ on the algebra $L_{an}(S^1, \mathfrak{g})$ of analytic loops.

1. **The case of holomorphic loops on** $\mathbb{C}^*$ For the specialization to holomorphic maps we use the description:

   $$H_{\mathbb{C}^*}(\mathbb{C}, \mathfrak{g}_\mathbb{C})_\mathbb{R} := \{f : \mathbb{C} \longrightarrow \mathfrak{g}_\mathbb{C} | f(z + i\mathbb{Z}) = f(z), fi\mathbb{R} \subset \mathfrak{g}_\mathbb{R}\}.$$

Identifying $it \leftrightarrow t$ in this (resp. the above) description we get an embedding:

$$H_{\mathbb{C}^*}(\mathbb{C}, \mathfrak{g}_{\mathbb{C}})_{\mathbb{R}} \hookrightarrow H^{\infty}(S^1, \mathfrak{g})$$

This shows that there are no problems concerning the monodromy. So we have only to check the regularity aspect. For $g \in MG$ and $u \in M\mathfrak{g}$, $g \cdot (u) = gug^{-1} - g'g^{-1} \in M\mathfrak{u}$. On the other hand, using the description in the proof of lemma 2.5.1, we get for $u \in M\mathfrak{g}$ a transformation function $g(t) := \exp(tX)g(0)h(t)^{-1}$. A priori this function is in $L(S^1, G)$; but $\exp(tX)$ can be continued to a holomorphic function on $\mathbb{C}$, $g(0)$ is a constant and $h(t)^{-1}$ is a solution of the differential equation: $h'(t) = u(t)h(t)$; if $u(t)$ is defined on $\mathbb{C}^*$, this equation has a solution on the universal cover of $\mathbb{C}^*$, that is $\mathbb{C}$. So $g(t)$ is defined on $\mathbb{C}$, but has perhaps nontrivial monodromy; this is, of course, not possible, as the embedding tells us that $g(t) \subset L_{an}G$.

2. **The case of holomorphic loops on A_n** This case is exactly similar. $\mathbb{C}$ is replaced by A' (compare subsection 2.1.4).

Hence theorem 2.5.2 is proved. □

Thus we have proven that σ-actions and diagonal actions are polar. Those two cases corresponds to the isotropy representation of Kac-Moody symmetric spaces of types II and IV: the diagonal action corresponds to the non-twisted case, the σ-action to the twisted one.

The isotropy representations of Kac-Moody symmetric spaces of type I and III corresponds to Hermann examples. A holomorphic version of the Hermann examples can be defined in exactly the same way:

Let $XG^{\sigma}_{\mathbb{R}}$ be a simply connected loop group, ρ an involution such that $X\mathfrak{g}^{\sigma}_{\mathbb{R}} = \mathcal{K} \oplus \mathcal{P}$ is the decomposition into the ± 1-eigenspaces of the involution induced by ρ on $X\mathfrak{g}^{\sigma}_{\mathbb{R}}$. Let $XK_{\mathbb{R}} \subset XG^{\sigma}_{\mathbb{R}}$ be the subgroup fixed by ρ.

Theorem 2.5.3
The gauge action of $XK_{\mathbb{R}}$ on $\mathcal{P}$ is polar.

Proof. The proof is like the one of theorem 2.5.2. One starts with a similar result for polar actions on Hilbert action — cf. [Ter95] — and checks then step by step that the introduced higher regularity conditions fit together. □

Chapter 3

Algebraic foundations

3.1 Kac-Moody algebras

3.1.1 Algebraic approach to Kac-Moody algebras

The theory of Kac-Moody algebras was developed in the 60's independently by V. G. Kac, R. V. Moody and D.-N. Verma (unpublished) as a generalization of semisimple Lie algebras. The classical reference is the book [Kac90]; the book [Kum02] contains a short summary for the parts of the theory which are necessary for algebraic Kac-Moody groups; the recent book [Car02] is a detailed reference for the details of semisimple and affine algebras.

The main idea of Kac-Moody theory is borrowed from Cartan's classification of complex semisimple Lie algebras which proceeds by encoding the Lie algebra structure in a matrix, later called Cartan matrix. Reversing this procedure one now starts with a $n \times n$-matrix A called a generalized Cartan matrix and constructs a Lie algebra $\mathfrak{g}(A)$ realizing this matrix, i.e. a Lie algebra such that the matrix obtained by applying Cartan's procedure is A.

We start with the definition of a Cartan matrix:

Definition 3.1.1 (Cartan matrix)
A Cartan matrix $A^{n\times n}$ is a square matrix with integer coefficients such that

1. $a_{ii} = 2$ *and* $a_{i\neq j} \leq 0$,
2. $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$,
3. *There is a vector $v > 0$ (component wise) such that $Av > 0$ (component wise).*

Example 3.1.1 (2×2-Cartan matrices)
There are – up to equivalence – four different 2-dimensional Cartan matrices:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

They correspond to the algebras $A_1 \times A_1, A_2, B_2, G_2$.

Definition 3.1.2
A Cartan matrix $A^{n\times n}$ is called decomposable iff $\{1, 2, \dots, n\}$ has a decomposition in two non-empty sets N_1 and N_2 such that $a_{ij} = 0$ for $i \in N_1$ and $j \in N_2$. It is called indecomposable iff it is not decomposable.

A complete list of indecomposable Cartan matrices consists of the Cartan matrices

$$A_n, B_{n,n\geq 2}, C_{n,n\geq 3}, D_{n,n\geq 4}, E_6, E_7, E_8, F_4, G_2 .$$

They correspond to Dynkin diagrams of the same name.

Definition 3.1.3 (affine Cartan matrix)
An affine Cartan matrix $A^{n\times n}$ is a square matrix with integer coefficients, such that

1. $a_{ii} = 2$ *and* $a_{i\neq j} \leq 0$.
2. $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.
3. *There is a vector* $v > 0$ *(component wise) such that* $Av = 0$.

Example 3.1.2 (2×2-affine Cartan matrices)
There are – up to equivalence – two different 2-dimensional affine Cartan matrices:

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} .$$

They correspond to the non-twisted algebra $\tilde{A}_1$ *and the twisted algebra* $\tilde{A}'_1$.

The reason for the distinction between twisted and non-twisted affine Kac-Moody algebras will become apparent in section 3.1.2.

1. The indecomposable non-twisted affine Cartan matrices are

$$\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8, \widetilde{F}_4, \widetilde{G}_2 .$$

   They correspond to Dynkin diagrams of the same name. In fact every non-twisted affine Cartan matrix $\widetilde{X}_l$ can be transformed into the corresponding Cartan matrix X_l by the removal of the first column and the first line. This close relation is reflected in the explicit realizations; see section 3.1.2.

2. The indecomposable twisted affine Cartan matrices are

$$\widetilde{A}'_1, \widetilde{C}'_l, \widetilde{B}^t_l, \widetilde{C}^t_l, \widetilde{F}^t_4, \widetilde{G}^t_2 .$$

   They correspond to Dynkin diagrams of the same name. The Kac-Moody algebras associated to them can be constructed as fixed point algebras of certain automorphisms σ of a non-twisted Kac-Moody algebra X. This construction suggest an alternative notation describing a twisted Kac-Moody algebra by the order of σ and the type of X. This yields the following equivalences:

$\widetilde{A}'_1$	${}^2\widetilde{A}_2$
$\widetilde{C}'_l$	${}^2\widetilde{A}_{2l}, l \geq 2$
$\widetilde{B}^t_l$	${}^2\widetilde{A}_{2l-1}, l \geq 3$
$\widetilde{C}^t_l$	${}^2\widetilde{D}_{l+1}, l \geq 2$
$\widetilde{F}^t_4$	${}^2\widetilde{E}_6$
$\widetilde{G}^t_2$	${}^3\widetilde{D}_4$

There are more general types of generalized Cartan matrices

- hyperbolic Cartan matrices
- indefinite Cartan matrices
- Borchert's Cartan matrices.

For their description, cf. [Car02].
To all those classes of Cartan matrices one can associate Lie algebras, called their realizations:

Definition 3.1.4 (Realization)
Let $A^{n\times n}$ be a generalized Cartan matrix. The realization of A, denoted $\mathfrak{g}(A)$, is the algebra

$$\mathfrak{g}(A^{n\times n}) = \langle e_i, f_i, h_i, i = 1, \dots, n | R_1, \dots, R_6\rangle\,,$$

where

$$\begin{aligned} R_1: &\quad [h_i, h_j] = 0\,,\\ R_2: &\quad [e_i, f_j] = h_i\delta_{ij}\,,\\ R_3: &\quad [h_i, e_j] = a_{ji}e_j\,,\\ R_4: &\quad [h_i, f_j] = -a_{ji}f_j\,,\\ R_5: &\quad (\mathrm{ad}e_i)^{1-a_{ji}}(e_j) = 0\ (i\neq j)\,,\\ R_6: &\quad (\mathrm{ad}f_i)^{1-a_{ji}}(f_j) = 0\ (i\neq j)\,. \end{aligned}$$

The realization defines a bijection between Cartan matrices and complex simple Lie algebras on the one hand and affine Cartan matrices and complex affine Kac-Moody algebras on the other hand.

If a (generalized) Cartan matrix $A^{n+m\times n+m}$ is decomposable into the direct sum of two Cartan matrices $A^{n\times n}$ and $A^{m\times m}$ then the same decomposition holds for the realizations:

$$\mathfrak{g}(A^{n+m\times n+m}) = \mathfrak{g}(A^{n\times n}) \oplus \mathfrak{g}(A^{m\times m})\,.$$

This is a crucial fact as this decomposition has counterparts in many classes of objects associated to those algebras. In the case of finite semisimple Lie algebras this is especially appealing:

- Cartan matrix $\longrightarrow$ indecomposable Cartan matrix,
- complex semisimple Lie algebra $\longrightarrow$ direct product of complex simple Lie algebras (ideals),
- complex semisimple Lie group $\longrightarrow$ direct product of complex simple Lie groups,
- simply connected, complete Riemannian manifold $\longrightarrow$ simply connected, complete Riemannian manifold with irreducible holonomy,
- Riemann symmetric space $\longrightarrow$ direct sum of irreducible Riemann symmetric spaces.

Morally all those decompositions are equivalent. Unfortunately in the infinite dimensional situation of Kac-Moody symmetric spaces the situation is more complicated. The central problem is that the direct product construction is very ill-adapted to the geometric situation of Kac-Moody symmetric spaces as it does not preserve the Lorentz-structure of the spaces. Thus it will be necessary to review this construction in order to define a different concept of composition such that the products are still Lorentzian. In order to distinguish between those two concepts we will refer in case of ambiguity to Kac-Moody algebras as algebraic Kac-Moody algebras in contrast to geometric Kac-Moody algebras.

3.1.2 The loop algebra approach to Kac-Moody algebras

To describe the loop algebra approach to Kac-Moody algebras we follow the terminology of the article [HG09].
Let $\mathfrak{g}$ be a finite dimensional reductive Lie algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Hence, $\mathfrak{g}$ is a direct product of a semisimple Lie algebra $\mathfrak{g}_s$ with an Abelian Lie algebra $\mathfrak{g}_a$. Let furthermore $\sigma \in \mathrm{Aut}(\mathfrak{g}_s)$ denote an automorphism of finite order of $\mathfrak{g}_s$ such that the restriction of σ to any simple factor $\mathfrak{g}_i$ of $\mathfrak{g}$ is an automorphism of $\mathfrak{g}_i$ and $\sigma|_{\mathfrak{g}_a} = \mathrm{Id}$. If $\mathfrak{g}_s$ is a Lie algebra over $\mathbb{R}$ we suppose it to be of compact type.

$$L(\mathfrak{g}, \sigma) := \{f : \mathbb{R} \longrightarrow \mathfrak{g} \,|\, f(t + 2\pi) = \sigma f(t), f \text{ satisfies some regularity conditions}\}\,.$$

We use the notation $L(\mathfrak{g}, \sigma)$ to describe in a unified way constructions that can be realized with explicit constructions of loop algebras satisfying various regularity conditions. The loops may be smooth, real analytic, (after complexification) holomorphic on $\mathbb{C}^*$ or holomorphic on an annulus $A_n \subset \mathbb{C}$ or algebraic loops. If we discuss loop algebras of a fixed regularity we use other precise notations: $M\mathfrak{g}$, $L_{alg}\mathfrak{g}$, $A_n\mathfrak{g}$ ... - compare appendix A.

Definition 3.1.5 (Geometric affine Kac-Moody algebra)
The geometric affine Kac-Moody algebra associated to a pair $(\mathfrak{g}, \sigma)$ is the algebra:

$$\widehat{L}(\mathfrak{g}, \sigma) := L(\mathfrak{g}, \sigma) \oplus \mathbb{F}c \oplus \mathbb{F}d\,,$$

equipped with the lie bracket defined by:

$$\begin{aligned}&[d, f] := f'; [c, c] = [c, d] = [c, f] = [d, d] = 0\,;\\ &[f, g] := [f, g]_0 + \omega(f, g)c\,.\end{aligned}$$

Here $f \in L(\mathfrak{g}, \sigma)$ and ω is a certain antisymmetric 2-form on $M\mathfrak{g}$, satisfying the cocycle condition.

Explicit realizations using k-times differentiable, smooth, analytic, holomorphic or algebraic loops are common. For holomorphic or algebraic loops we use

$$\omega(f, g) = \mathrm{Res}(\langle f, g'\rangle)\,.$$

For integrable loops we use

$$\omega(f, g) = \frac{1}{2\pi}\int_0^{2\pi} \langle f, g'\rangle dt\,.$$

By the residuum-formula those two descriptions coincide for holomorphic functions [BG91].

Definition 3.1.6 (derived algebra)
The derived geometric affine Kac-Moody algebra associated to a pair $(\mathfrak{g}, \sigma)$ is the algebra:

$$\widetilde{L}(\mathfrak{g}, \sigma) := L(\mathfrak{g}, \sigma) \oplus \mathbb{F}c\,,$$

with the lie bracket inherited from the affine geometric Kac-Moody algebra. The name derived algebra is due to the fact that

$$\widetilde{L}(\mathfrak{g}, \sigma) \;=\; \left[\widehat{L}(\mathfrak{g}, \sigma), \widehat{L}(\mathfrak{g}, \sigma)\right]\,.$$

If $\mathfrak{g}$ is a simple Lie algebra then the associated algebraic and geometric Kac-Moody algebras coincide:

Lemma 3.1.1
Suppose $\mathfrak{g}$ is simple. The realization of $\widehat{L}(\mathfrak{g},\sigma)$ with algebraic loops $\widehat{L_{alg}\mathfrak{g}}^{\sigma}$ is a simple, algebraic affine Kac-Moody algebra.

The geometric Kac-Moody algebra of algebraic loops coincides with the loop realization — cf. [Kac90]. If one chooses realizations of a more general class of loops then $\widehat{L}(\mathfrak{g},\sigma)$ can be described as the completion of $\widehat{L_{alg}\mathfrak{g}}^{\sigma}$ with respect to some seminorms or some set of semi-norms.

Remark 3.1.1
Without proof we remark that one can define a formal completion for algebraic Kac-Moody algebras. For a simple algebra $\mathfrak{g}$ all realizations of $\widehat{L}(\mathfrak{g},\sigma)$ embed (non-surjectively) into this formal completion (for further details and proofs cf. [Kum02]).

We will now describe the splitting behaviour of geometric Kac-Moody algebras: Wwe start with the loop part.

Lemma 3.1.2
Let $\mathfrak{g}$ be semisimple and suppose $(\mathfrak{g},\sigma) := \bigoplus_i(\mathfrak{g}_{\mathfrak{i}},\sigma_i)$. Then

$$L(\mathfrak{g},\sigma) = \bigoplus_i L(\mathfrak{g}_i,\sigma_i)\,.$$

Each algebra $L(\mathfrak{g}_i,\sigma_i)$ is an ideal in $L(\mathfrak{g},\sigma)$.

Proof. Study the decomposition of any loop $f \in L(\mathfrak{g},\sigma)$ into its component loops $f_i \in L(\mathfrak{g}_i,\sigma_i)$. This yields the direct product decomposition. As the bracket is defined pointwise each $L(\mathfrak{g}_i,\sigma_i)$ is an ideal in $L(\mathfrak{g},\sigma)$. □

Nevertheless on the level of Kac-Moody algebras the behaviour is different: to get a direct sum decomposition of a Kac-Moody algebra into indecomposable ones we would need an extension $\mathbb{C}c_i \oplus \mathbb{C}d_i$ for every simple factor $\mathfrak{g}_i$ in the decomposition of the underlying Lie algebra $\mathfrak{g}$:

From the analogy with the affine algebraic Kac-Moody algebra associated to $(\mathfrak{g},\sigma)$ we would expect $\widehat{L}(\mathfrak{g},\sigma)$ to be the algebra

$$\bigoplus_{i=1}^{n}\widehat{L}(\mathfrak{g}_i,\sigma_i)\,.$$

This algebra has a n-fold extension. As we will see later the associated Ad-invariant scalar product would have index n. Nevertheless for geometric reasons – to get an Ad-invariant Lorentz scalar product (i.e. index 1) – we defined $\widehat{L}(\mathfrak{g},\sigma)$ to be an algebra with only one such extension. Thus we trivially get

Lemma 3.1.3
Suppose $\mathfrak{g}$ is a semisimple, non-simple Lie algebra. Then

$$\widehat{L}(\mathfrak{g},\sigma) \neq \bigoplus_i \widehat{L}(\mathfrak{g}_i,\sigma_i)\,.$$

Proof. The center of $\widehat{L}(\mathfrak{g},\sigma)$ is 1-dimensional. In contrast the dimension of the center of $\bigoplus_i \widehat{L}(\mathfrak{g}_i,\sigma_i)$ is equivalent to the number of simple factors of $\mathfrak{g}$. □

Thus geometric Kac-Moody algebras do not split into a direct sum of simple algebras.

Lemma 3.1.4 (Splitting of geometric Kac-Moody algebras)
Let, as above, $\mathfrak{g} := \bigoplus_i \mathfrak{g}_i$ *be the decomposition of a reductive Lie algebra into its simple factors. Let* $\widehat{L}(\mathfrak{g},\sigma)$ *be the associated Kac-Moody algebra. Then*

1. $\widetilde{L}(\mathfrak{g}_i,\sigma_i)$ *is an ideal in* $\widehat{L}(\mathfrak{g},\sigma)$.

2. *Let* $(\mathfrak{g}_i,\sigma_i)$ *be a simple factor of* $\mathfrak{g}$. $\widetilde{L}_{alg}(\mathfrak{g}_i,\sigma_i)\oplus\mathbb{F}d$ *is an indecomposable Kac-Moody subalgebra.*

3. *There is a Lie algebra homomorphism*

$$\varphi : \left(\bigoplus_{i=1}^{n}\widetilde{L}(\mathfrak{g}_i,\sigma_i)\right) \longrightarrow \widetilde{L}(\mathfrak{g},\sigma)\,,$$

defined by $\varphi(f_1,r_{c_1}),\dots,(f_n,r_{c_n}) = (f_1,\dots,f_n,r_c=\sum r_{c_i})$.

4. $\widehat{L}(\mathfrak{g},\sigma) := \left(\bigoplus \widetilde{L}(\mathfrak{g}_i,\sigma_i)\right)/Ker(\varphi)\oplus\mathbb{F}d$. *This defines an exact sequence*

$$1 \longrightarrow \mathbb{F}^{n-1} \longrightarrow \left(\bigoplus_{i=1}^{n}\widetilde{L}(\mathfrak{g}_i,\sigma_i)\right) \longrightarrow \widetilde{L}(\mathfrak{g},\sigma) \longrightarrow 1\,.$$

Proof.

1. To check 1., it is sufficient to verify the closedness of the bracket operation: let $f_i \in \widetilde{L}(\mathfrak{g}_i,\sigma_i)$, $g+\mu d \in \widehat{L}(\mathfrak{g},\sigma)$. Then $[f_i, g+\mu d] = [f_i,g]-\mu f_i'$. f_i' is in $\widetilde{L}(\mathfrak{g}_i,\sigma_i)$, the same is true for $[f_i,g]$, as it is true pointwise for elements in $\mathfrak{g}$.

2. 2. follows directly from the definition.

3. $\varphi|_{L(\mathfrak{g},\sigma)}$ is an isomorphism. So we are left with checking the behaviour of the extensions.

$$\begin{aligned}
&\varphi[((f_1,r_{c_1}),\dots,(f_n,r_{c_n}));((\bar{f}_1,\bar{r}_{c_1}),\dots,(\bar{f}_n,\bar{r}_{c_n}))] =\\
&= \varphi[f_1,\dots,f_n;\bar{f}_1,\dots,\bar{f}_n]_0 + \varphi\left(\sum_{i=1}^{n}\omega_i\left(f_i;\bar{f}_i'\right)c_i\right) =\\
&= [\varphi(f_1,\dots,f_n);\varphi(\bar{f}_1,\dots,(\bar{f}_n))]_0 + \sum_{i=1}^{n}\omega\left(\varphi(f_1,\dots f_n),\varphi(\bar{f}_1,\dots,\bar{f}_n)\right)c =\\
&= [\varphi((f_1,r_{c_1}),\dots,(f_n,r_{c_n}));\varphi((\bar{f}_1,\bar{r}_{c_1}),\dots,(\bar{f}_n,\bar{r}_{c_n}))]
\end{aligned}$$

4. is a consequence of 3. . □

Remark 3.1.2
One can construct products of geometric affine Kac-Moody algebras. If a generalized geometric affine Kac-Moody algebra is the product of n geometric affine Kac-Moody algebras, it has index n. Thus we restrict our attention to geometric affine Kac-Moody algebras.

Now we want to prove a similar splitting theorem for automorphisms of complex geometric affine Kac-Moody algebras.

The following lemma allows to restrict the study to the loop algebras:

Lemma 3.1.5

1. *Any automorphism* $\widehat{\varphi} : \widehat{L}(\mathfrak{g}_{\mathbb{R}}, \sigma) \longrightarrow \widehat{L}(\mathfrak{g}_{\mathbb{R}}, \sigma)$ *induces an automorphism of the derived algebras* $\widetilde{\varphi} : \widetilde{L}(\mathfrak{g}_{\mathbb{R}}, \sigma) \longrightarrow \widetilde{L}(\mathfrak{g}_{\mathbb{R}}, \sigma)$.

2. *Any automorphism* $\widehat{\varphi} : \widehat{L}(\mathfrak{g}_{\mathbb{R}}, \sigma) \longrightarrow \widehat{L}(\mathfrak{g}_{\mathbb{R}}, \sigma)$ *induces an automorphism of the loop algebras* $\varphi : L(\mathfrak{g}_{\mathbb{R}}, \sigma) \longrightarrow L(\mathfrak{g}_{\mathbb{R}}, \sigma)$.

Proof. cf. [HG09]. □

For loop algebras of real or complex type we find:

Lemma 3.1.6
Let $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_a$ *and* $(\mathfrak{g}_\mathfrak{s}, \sigma) := \bigoplus_i (\mathfrak{g}_\mathfrak{i}, \sigma_i)$, *where* $(\mathfrak{g}_\mathfrak{i}, \sigma_i)$ *is a simple Lie algebra of the compact type. Let* $L(\mathfrak{g}, \sigma)$ *be the associated loop algebra and* φ *an automorphism of* $L(\mathfrak{g}, \sigma)$. *Then*

1. $\varphi(L(\mathfrak{g}_a, \sigma)) = L(\mathfrak{g}_a, \sigma)$ *and* $\varphi(L(\mathfrak{g}_s, \sigma)) = L(\mathfrak{g}_s, \sigma)$.

2. $L(\mathfrak{g}_s, \sigma)$ *decomposes under the action of* φ *into* φ*-invariant ideals of two types:*

   (a) *Loop algebras of simple Lie algebras* $L(\mathfrak{g}_i, \sigma_i)$ *together with an automorphism* φ_i *(called type I-factors),*

   (b) *Loop algebras of products of simple Lie algebras* $\mathfrak{g}_\mathfrak{i} = \oplus_{i=1}^m \mathfrak{g}_\mathfrak{i}{}'$ *together with an automorphism* φ_i *of order* n, *cyclically interchanging the* m*-factors (called type II-factors). In this case* $\frac{n}{m} = k \in \mathbb{Z}$ *and* σ *induces an automorphism of order* k *on each simple factor.*

Proof. Let $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_s$. Each function $f : \mathbb{R} \longrightarrow \mathfrak{g}$ has a unique decomposition $f := (f_a, f_s)$, such that $f_s : \mathbb{R} \to \mathfrak{g}_s$ and $f_a : \mathbb{R} \to \mathfrak{g}_a$. As $f(t + 2\pi) = \sigma f(t)$ is equivalent to $f_a(t + 2\pi) = \sigma f_a(t) = f_a(t)$ and $f_s(t + 2\pi) = \sigma_s f_s(t)$ and $\sigma|_{\mathfrak{g}_a} = Id$ this induces the decomposition: $L(\mathfrak{g}, \sigma) = L(\mathfrak{g}_a, Id) \oplus L(\mathfrak{g}_s, \sigma_s)$.
Let now $\mathfrak{g}_s = \bigoplus_{i=1}^m \mathfrak{g}_i$ be a decomposition of $\mathfrak{g}_s$ such that

1. $\mathfrak{g}_i$ is invariant under $\varphi = \varphi|_{\mathfrak{g}_i}$.

2. There is no decomposition $\mathfrak{g}_i = \mathfrak{g}_i' \oplus \mathfrak{g}_i''$ such that $\varphi|_{\mathfrak{g}_i'}$ and $\varphi|_{\mathfrak{g}_i''}$ are automorphisms and $\mathfrak{g}_i'$ and $\mathfrak{g}_i''$ are invariant under the bracket operation.

Again f_s splits into m component functions $f_s = (f_1, \ldots, f_m)$ and the compatibility condition $f_s(t + 2\pi) = \sigma f_s(t)$ is equivalent to the m compatibility conditions $f_i(t + 2\pi) = \sigma f_i(t), i = 1, \ldots, m$.
There are now two cases:

1. Suppose first, $\mathfrak{g}_i$ is simple. Then φ_i is an involution of $L(\mathfrak{g}_i, \sigma_i)$. The pair

$$(L(\mathfrak{g}_i, \sigma_i), \varphi_i)$$

   is of I-type. The finite order automorphisms of simple affine geometric Kac-Moody algebras are completely classified — cf. [HG09].

2. Suppose now $\mathfrak{g}_i$ is not simple and $\varphi|_{L(\mathfrak{g}_i,\sigma_i)}$ is of order n. There is a decomposition $\mathfrak{g}_i := \bigoplus_j \mathfrak{g}_i^j$ such that $\mathfrak{g}_i^j$ is a simple Lie algebra. As there is no subalgebra $L(\mathfrak{h}, \sigma_i|_{\mathfrak{h}}) \subset L(\mathfrak{g}_i, \sigma_i)$ which is both invariant under $\varphi_{\mathfrak{g}_i}$ and an ideal in $L(\mathfrak{g}, \sigma)$, we find that all $(\mathfrak{g}_i^j, \sigma_i^j)$ are of the same type and the algebras $L(\mathfrak{g}_i^j, \sigma_i^j)$ are permuted by $\varphi_{\mathfrak{g}_i}$. Thus the number of those factors, denoted m, is a divisor of $n = km$. We get
$$L(\mathfrak{g}_i, \sigma_i) := \bigoplus_{j=1}^{m} L(\mathfrak{g}_j, \sigma_j)\,.$$
On each simple factor φ induces an automorphism $\bar{\varphi}$ of order k. $\bar{\varphi}$ is again a finite order automorphism of a simple geometric affine Kac-Moody algebra. □

Using the classification result of Ernst Heintze and Christian Groß [HG09] we know that every automorphism of the loop algebra $L(\mathfrak{g}, \sigma)$ is of standard form:
$$\varphi\,(f(t)) = \varphi_t f(\lambda(t))$$
Here $\varphi(t)$ denotes a curve of automorphisms of $\mathfrak{g}$ and $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function. Not every such automorphism is extendible to the affine Kac-Moody algebra associated to $L(\mathfrak{g}, \sigma)$. We quote theorem 3.4. of [HG09]:

Theorem 3.1.1 (Heintze-Groß, 09)
Let $\widehat{\varphi} : \widehat{L}(\mathfrak{g}, \sigma) \longrightarrow \widehat{L}(\widetilde{\mathfrak{g}}, \widetilde{\sigma})$ be a linear or conjugate linear map. Then $\widehat{\varphi}$ is an isomorphism of Lie algebras iff there exists $\gamma \in \mathbb{F}$ and a linear (resp. conjugate linear) isomorphism $\varphi : L(\mathfrak{g}, \sigma)) \longrightarrow L(\widetilde{\mathfrak{g}}, \widetilde{\sigma})$ with $\lambda' = \epsilon_\varphi$ constant such that
$$\widehat{\varphi}c = \epsilon_\varphi c$$
$$\widehat{\varphi}d = \epsilon_\varphi d - \epsilon_\varphi f_\varphi + \gamma c$$
$$\widehat{\varphi}f = \varphi(u) + \mu(f)c\,.$$

We call $\widehat{\varphi}$ of first type if $\epsilon_\varphi = 1$ and of second type if $\epsilon_\varphi = -1$. We call an involution φ of a loop algebra $L(\mathfrak{g}, \sigma)$ for a reductive Lie algebra $\mathfrak{g}$ locally admissible iff its restriction to any irreducible factor is extendible to the associated simple affine Kac-Moody algebra. It is called extendible iff it can be extended to $\widehat{L}(\mathfrak{g}, \sigma)$.

Remark 3.1.3
It is important to note that we can choose on every simple factor of a loop algebra any automorphism we want, especially it is possible to use any locally admissible automorphism — i.e. the identity, automorphisms of first type and automorphism of second type — simultaneously.

This is no longer the case with geometric affine Kac-Moody algebras: study now extensions to $\widehat{L}(\mathfrak{g}, \sigma)$. Here we have an important restriction: study any factor $\widehat{L}(\mathfrak{g}_i, \sigma_i)$. The involution φ_i of $L(\mathfrak{g}, \sigma)$ defines a unique involution $\widehat{\varphi}_i$ of $\widehat{L}(\mathfrak{g}_i, \sigma_i)$ — cf. [HG09]. Thus we find:

Lemma 3.1.7
A locally admissible involution of $\varphi : L(\mathfrak{g}, \sigma) \longrightarrow L(\mathfrak{g}, \sigma)$ is admissible, that is extendible to $\widehat{L}(\mathfrak{g}, \sigma)$, iff every restriction $\varphi_i : L(\mathfrak{g}_i, \sigma_i) \longrightarrow L(\mathfrak{g}_i, \sigma_i)$ has the same extension to $\widehat{L}(\mathfrak{g}_i, \sigma_i)$.

Thus we have exactly two possibilities:

1. Every involution φ_i is of the first type or the identity. In this case φ is called of first type.

2. Every involution φ_i is of the second type. In this case φ is called of second type.

3.2 Orthogonal symmetric Kac-Moody algebras

3.2.1 The finite dimensional blueprint

The classification of finite dimensional Riemann symmetric spaces proceeds by associating a pair consisting of a real simple Lie algebra and an involution to every irreducible symmetric space — cf. [Hel01]. Those pairs are called "orthogonal symmetric Lie algebras".

We review the approach used for finite dimensional symmetric spaces — cf. [Hel01]. An orthogonal symmetric Lie algebra is defined to be a pair $(\mathfrak{l}, \rho)$, consisting of a real lie algebra $\mathfrak{l}$ and an involutive automorphism ρ of $\mathfrak{l}$ such that the fixed point algebra of ρ is a subalgebra of compact type of $\mathfrak{l}$. We distinguish three types of orthogonal symmetric Lie algebras.

Let us describe them more closely: let $\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{e}$ be the decomposition of $\mathfrak{l}$ into the $+1$ and -1-eigenspace of ρ.

1. **Euclidean type:** an effective orthogonal symmetric Lie algebra is of Euclidean type iff $\mathfrak{e}$ is Abelian.

2. **Compact type:** an orthogonal symmetric Lie algebra is of compact type iff $\mathfrak{l}$ is a compact, semisimple Lie algebra. There are two classes of irreducible orthogonal symmetric Lie algebras of compact type: the first class consists of compact real forms of simple Lie algebras together with an involution ρ, the second class consists of an algebra $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{h}$ such that $\mathfrak{h}$ is a compact real form of a simple Lie algebra together with an involution ρ, interchanging the two factors.

3. **Non-compact type:** an orthogonal symmetric Lie algebra is of non-compact type iff $\mathfrak{e}_0$ is non-compact, semisimple and $\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{e}$ is a Cartan decomposition (for the definition of a Cartan decomposition cf. [Hel01], p. 183). There are two classes of irreducible Lie algebras of non-compact type: the first class consists of complex simple Lie algebras with the involution being conjugation with respect to a compact real form, the second one consists of non compact real forms with the involution defined by the Cartan decomposition.

Every orthogonal symmetric Lie algebra can be decomposed into a direct product of three ideals $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_+ \oplus \mathfrak{l}_-$ such that $\mathfrak{l}_0$ is an orthogonal symmetric subalgebra of Euclidean type, $\mathfrak{l}_+$ is an orthogonal symmetric Lie algebra of compact type and $\mathfrak{l}_-$ is an orthogonal symmetric Lie algebra of non-compact type.

Orthogonal symmetric algebras of compact type and of non-compact type are in duality: let $\mathfrak{l}_+ = \mathfrak{u} \oplus \mathfrak{e}$ be an orthogonal symmetric Lie algebra of compact type, then $\mathfrak{l}_- = \mathfrak{u} \oplus i\mathfrak{e}$ is an orthogonal symmetric Lie algebra of non-compact type and vice versa.

Let $\mathfrak{g}_\mathbb{C}$ be a simple complex Lie algebra, $\mathfrak{g}_\mathbb{R}$ a real form of compact type of $\mathfrak{g}_\mathbb{C}$ and denote by $^-$ the conjugation with respect to $\mathfrak{g}_\mathbb{R}$. Let ρ denote an involution of $\mathfrak{g}_\mathbb{R}$. We define two involutions:

$$\begin{aligned} \rho_* : \mathfrak{g}_\mathbb{C} &\longrightarrow \mathfrak{g}_\mathbb{C}, \quad z \mapsto \overline{z} \\ \rho_0 : \mathfrak{g}_\mathbb{C} &\longrightarrow \mathfrak{g}_\mathbb{C}, \quad z \mapsto \overline{\rho(z)} \end{aligned}$$

Using this notation one can describe the compact and non-compact real forms of $\mathfrak{g}_\mathbb{C}$ as fixed point algebras of the involutions: namely we get $\mathfrak{g}_\mathbb{R} = \mathrm{Fix}(\rho_*)$ and for the non-compact dual $\mathfrak{g}_D = \mathrm{Fix}(\rho_0)$. Let $\mathfrak{g}_\mathbb{R} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition into the $+1$ and -1 eigenspaces of ρ, then $\mathfrak{g}_D = \mathfrak{k} \oplus i\mathfrak{p}$.

1. The pair $(\mathrm{Fix}(\rho_*) = \mathfrak{g}_\mathbb{R}, \rho_0)$ is an orthogonal symmetric Lie algebra of the compact type,

2. The pair $(\mathrm{Fix}(\rho_0) = \mathfrak{g}_D, \rho_*)$ is an orthogonal symmetric Lie algebra of the non-compact type.

Now we will study the infinite dimensional version of this theory which we need for Kac-Moody symmetric spaces - thus we introduce orthogonal symmetric Kac-Moody algebras. The results in this section hold for twisted and non-twisted affine Kac-Moody algebras. We use the notation of twisted affine Kac-Moody algebras together with the convention that σ may denote the identity.

3.2.2 Orthogonal symmetric affine Kac-Moody algebras

We start with some definitions:

Definition 3.2.1
A real form of a complex geometric affine Kac-Moody algebra $\widehat{L}(\mathfrak{g}_{\mathbb{C}}, \sigma)$ is the fixed point set of a conjugate linear involution.

We have described in section 3.1.2 that involutions of a geometric affine Kac-Moody algebra restrict to involutions of irreducible factors of the loop algebra. Hence, the invariant subalgebras are direct products of invariant subalgebras in those factors together with the appropriate torus extension.

Definition 3.2.2 (compact real affine Kac-Moody algebra)
A compact real form of a complex affine Kac-Moody algebra $\widehat{L}(\mathfrak{g}_{\mathbb{C}}, \sigma)$ is defined to be a subalgebra of $\widehat{L}(\mathfrak{g}_{\mathbb{C}}, \sigma)$ that is conjugate to the algebra $\widehat{L}(\mathfrak{g}_{\mathbb{R}}, \sigma)$ where $\mathfrak{g}_{\mathbb{R}}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$.

Remark 3.2.1
A semisimple Lie algebra is called of "compact type" iff it integrates to a compact semisimple Lie group. The infinite dimensional generalization of compact Lie groups are loop groups of compact Lie groups and the Kac-Moody groups, constructed as extensions of those loop groups (see section 3.4). Thus the denomination is justified by the fact that "compact" affine Kac-Moody algebras integrate to "compact" Kac-Moody groups.

To define a loop group of the compact type we define an infinite dimensional version of the Cartan-Killing form:

Definition 3.2.3 (Cartan-Killing form)
The Cartan-Killing form of a loop algebra $L(\mathfrak{g}_{\mathbb{C}}, \sigma)$ is defined by

$$B_{(\mathfrak{g}_{\mathbb{C}},\sigma)}\left(f, g\right) = \int_0^{2\pi} B\left(f(z), g(z)\right) .$$

Definition 3.2.4 (compact loop algebra)
A loop algebra of compact type is a subalgebra of $L(\mathfrak{g}_{\mathbb{C}}, \sigma)$ such that its Cartan-Killing form is negative definite.

Lemma 3.2.1
Let $\mathfrak{g}_{\mathbb{R}}$ be a compact semisimple Lie algebra. Then the loop algebra $L(\mathfrak{g}_{\mathbb{R}}, \sigma)$ is of compact type.

Proof. The Cartan-Killing form on $\mathfrak{g}_{\mathbb{R}}$ is negative definite. Hence, $B_{(\mathfrak{g}_{\mathbb{C}},\sigma)}\left(f, g\right)$ is negative definite. □

Ernst Heintze and Christian Groß prove in [HG09], theorem 7.4. for indecomposable affine Kac-Moody algebras:

Theorem 3.2.1
$\widehat{L}(\mathfrak{g}_{\mathbb{C}}, \sigma)$ has a compact real form which is unique up to conjugation.

Their proof extends directly to geometric affine Kac-Moody algebras such that $\mathfrak{g}_{\mathbb{C}}$ is semisimple (but need not be simple): we choose the (up to conjugation) unique subalgebra in every factor and have to check that the extensions fit together. This follows from the explicit construction.

To find non-compact real forms, we need the following result of Ernst Heintze and Christian Groß (Corollary 7.7. of [HG09]):

Theorem 3.2.2
Let $\mathcal{G}$ be a irreducible complex geometric affine Kac-Moody algebra, $\mathcal{U}$ a real form of compact type. The conjugacy classes of real forms of non-compact type of $\mathcal{G}$ are in bijection with the conjugacy classes of involutions on $\mathcal{U}$. The correspondence is given by $\mathcal{U} = \mathcal{K} \oplus \mathcal{P} \mapsto \mathcal{K} \oplus i\mathcal{P}$ where $\mathcal{K}$ and $\mathcal{P}$ are the ± 1-eigenspaces of the involution.

Thus to find non-compact real forms, we have to study automorphism of order 2 of a geometric affine Kac-Moody algebra of the compact type. From now on we restrict to involutions $\widehat{\varphi}$ of type 2, that is $\epsilon_{\varphi} = -1$.

Now we want to extend this result to non-irreducible geometric affine Kac-Moody algebras.

1. Suppose first that the involution $\widehat{\varphi}$ on $\widehat{L}(\mathfrak{g}, \sigma)$ is chosen in a way that every irreducible factor is of type I. In this case $\widehat{\varphi}$ restricts to an involution $\widehat{\varphi}_i$ on every irreducible factor $\widehat{L}(\mathfrak{g}_i, \sigma_i)$. Let $L(\mathfrak{g}_i, \sigma_i) = \mathcal{K}_i \oplus \mathcal{P}_i$ be the decomposition of $L(\mathfrak{g}_i, \sigma_i)$ into the eigenspaces of φ_i. The dualization is performed according to the standard pattern using $\mathcal{K} = \oplus_i \mathcal{K}_i$ and $\mathcal{P} = \oplus_i \mathcal{P}_i \oplus \mathbb{R}c \oplus \mathbb{R}d$.

2. If the decomposition of $\widehat{\varphi}$ contains simple factors of type II, we perform the same decomposition procedure: let $\widehat{L}(\mathfrak{g}_j \oplus \mathfrak{g}'_j, \sigma_j \oplus \sigma'_j)$ be irreducible with respect to $\widehat{\varphi}$, then we have the decomposition $L(\mathfrak{g}_j \oplus \mathfrak{g}'_j, \sigma_j \oplus \sigma'_j) = \mathcal{K}_j \oplus \mathcal{P}_j$. Dualization follows the standard pattern.

We have to investigate iff all real forms can be described in this way:

1. If $\mathfrak{g}$ is simple it is a result of Ernst Heintze and Christian Groß that every real form of non-compact type can be constructed in this way — cf. [HG09].

2. If $\mathfrak{g}$ is not simple we use that the restriction to the loop algebra $L(\mathfrak{g}, \sigma)$ of any real form consists of the direct product of real forms in the irreducible components of $L(\mathfrak{g}_{\mathbb{C}}, \sigma)$. Hence, according to the result of Ernst Heintze and Christian Groß, those are of the describe type, and thus the non-compact real form we started with.

Putting this together we have proved the following two theorems:

Theorem 3.2.3
Let $\widehat{L}(\mathfrak{g}, \sigma)$ be a geometric affine Kac-Moody algebra of the compact type. Let $\widehat{\varphi}$ be an involution of order 2 of the second kind of $\widehat{L}(\mathfrak{g}, \sigma)$. Let furthermore $\widehat{L}(\mathfrak{g}, \sigma) = \mathcal{K} \oplus \mathcal{P}$ be the decomposition into its ± 1-eigenspaces. Then $\mathcal{G}_D := \mathcal{K} \oplus i\mathcal{P}$ is the dual real form of the non-compact type.

Theorem 3.2.4
Every real form of a complex geometric affine Kac-Moody algebra is either of compact type or of non-compact type. A mixed type is not possible.

Note, that the combination of the identity on some factors with involutions of second type on other factors is excluded.

Theorem 3.2.5
Let $\mathfrak{g}$ be semisimple and $\widehat{L}(\mathfrak{g},\sigma)_D$ be a real form of the non-compact type. Let $\widehat{L}(\mathfrak{g},\sigma)_D = \mathcal{K}\oplus\mathcal{P}$ be a Cartan decomposition. The Cartan Killing form is negative definite on $\mathcal{K}$ and positive definite on $\mathcal{P}$

Proof. Suppose first σ is the identity. Let φ be an automorphism. Then without loss of generality $\varphi(f) = \varphi_0(f(-t))$ — cf. [HG09]. Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into the ± 1-eigenspaces of φ_0. Then $f \in \mathrm{Fix}(\varphi)$ iff its Taylor expansion satisfies

$$\sum_n a_n e^{int} = \sum \varphi_0(a_{-n})e^{int}\,.$$

Let $a_n = k_n \oplus p_n$ be the decomposition of a_n into the ± 1 eigenspaces with respect to φ_0. Hence

$$f(t) = \sum_n k_n \cos(nt) + \sum_n p_n \sin(nt)\,.$$

Then using bilinearity and the fact that $\{\cos(nt), \sin(nt)\}$ are orthonormal we can calculate $B_{\mathfrak{g}}$:

$$B_{\mathfrak{g}} = \int_0^{2\pi} \sum_n \cos^2(nt) B(k_n,k_n) - \int_0^{2\pi} \sum_n \sin^2(nt) B(p_n,p_n)\,.$$

Hence $B_{\mathfrak{g}}$ is negative definite on $\mathrm{Fix}(\varphi)$. Analogously one calculates that it is positive definite on the -1-eigenspace of φ. If $\sigma \neq Id$ then one gets the same result by embedding $L(\mathfrak{g},\sigma)$ into an algebra $L(\mathfrak{h},\mathrm{id})$ which is always possible — cf. [Kac90]. □

Theorem 3.2.6
Let $\mathfrak{g}$ be abelian. The Cartan-Killing form of $L(\mathfrak{g})$ is trivial.

Proof. Direct calculation. □

Now we can define OSAKAs:

Definition 3.2.5 (Orthogonal symmetric Kac-Moody algebra)
An orthogonal symmetric affine Kac-Moody algebra (OSAKA) is a pair $\left(\widehat{L}(\mathfrak{g},\sigma), \widehat{L}(\rho)\right)$ such that

1. *$\widehat{L}(\mathfrak{g},\sigma)$ is a real form of an affine geometric Kac-Moody algebra,*
2. *$\widehat{L}(\rho)$ is an involutive automorphism of $\widehat{L}(\mathfrak{g},\sigma)$ of the second kind,*
3. *Fix$(\widehat{L}(\rho))$ is a compact real form.*

Following again the presentation of Helgason, we define 2 types of OSAKAs:

Definition 3.2.6 (Types of OSAKAs)
Let $(\widehat{L}(\mathfrak{g},\sigma, \widehat{L}(\rho)$ be an OSAKA. Let $\widehat{L}(\mathfrak{g},\sigma) = \mathcal{K}\oplus\mathcal{P}$ be the decomposition of $\widehat{L}(\mathfrak{g},\sigma)$ into the eigenspaces of $\widehat{L}(\mathfrak{g},\sigma)$ of eigenvalue $+1$ resp. -1.

1. *If $\widehat{L}(\mathfrak{g},\sigma)$ is a compact real affine Kac-Moody algebra, it is said to be of the compact type.*

2. *If $\widehat{L}(\mathfrak{g},\sigma)$ is a non-compact real affine Kac-Moody algebra, $\widehat{L}(\mathfrak{g},\sigma) = \mathcal{U} \oplus \mathcal{P}$ is a Cartan decomposition of $\widehat{L}(\mathfrak{g},\sigma)$.*

3. *If $L(\mathfrak{g},\sigma)$ is abelian it is said to be of Euclidean type.*

Definition 3.2.7 (irreducible OSAKA)
An OSAKA $(\widehat{L}(\mathfrak{g},\sigma), \widehat{L}(\rho))$ is called irreducible iff its derived algebra has no non-trivial derived Kac-Moody subalgebra invariant under $\widehat{L}(\rho)$.

Thus we can describe the different classes of irreducible OSAKAs of compact type.

1. The first class consists of pairs consisting of compact real forms $\widehat{M\mathfrak{g}}$, where $\mathfrak{g}$ is a simple Lie algebra together with an involution of the second kind. Complete classifications are available. See for example [Hei08]. This are irreducible factors of type I. A recent paper of Tripathy and Pati gives a list of Dynkin diagrams [TP06]. They correspond to Kac-Moody symmetric spaces of type I.

2. Let $\mathfrak{g}_{\mathbb{R}}$ be a simple real Lie algebra of the compact type. The second class consists of pairs of an affine Kac-Moody algebra $M\widehat{(\mathfrak{g}_{\mathbb{R}} \times \mathfrak{g}_{\mathbb{R}})}$ together with an involution of the second kind, interchanging the factors. Those algebras correspond to Kac-Moody symmetric spaces that are compact Kac-Moody groups equipped with their Ad-invariant metrics (type II).

Dualizing OSAKAs of the compact type, we get the OSAKAs of the non-compact type.

1. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra, and $\widehat{L}(\mathfrak{g}_{\mathbb{C}},\sigma)$ the associated affine Kac-Moody algebra. This class consists of real forms of the non-compact type that are described as fixed point sets of involutions of type 2 together with a special involution, called Cartan involution. This is the unique involution on $\mathcal{G}$, such that the decomposition into its ± 1-eigenspaces $\mathcal{K}$ and $\mathcal{P}$ yields: $\mathcal{K} \oplus i\mathcal{P}$ is a real form of compact type of $\widehat{L}(\mathfrak{g}_{\mathbb{C}},\sigma)$. Those orthogonal symmetric Lie algebras correspond to Kac-Moody symmetric spaces of type III.

2. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra. The fourth class consists of negative-conjugate real forms of $\widehat{L}(\mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}, \sigma \oplus \sigma)$. The involution is given by the complex conjugation $\widehat{L}(\rho_0)$ with respect to a compact real form, i.e. $\widehat{L}(\mathfrak{g}_{\mathbb{R}},\sigma)$. Those algebras correspond to Kac-Moody symmetric spaces of type IV.

The derived algebras of the last class of OSAKAs — the ones of Euclidean type — are Heisenberg algebras — cf. [PS86]. The maximal subgroups of compact type are trivial. Hence, the involution inverts all elements.

3.3 Tame structures and ILB-structures on Kac-Moody algebras

In this section we will describe explicit realizations as central extensions of holomorphic loop algebras of the abstract affine geometric Kac-Moody algebras which we introduced in definition 3.1.5.
Let $X \in \{A_n, \mathbb{C}^*\}$. As usual, holomorphic function on A_n are understood to be holomorphic in an open set containing A_n.

Definition 3.3.1 (holomorphic affine geometric Kac-Moody algebra)
Define $\widehat{X\mathfrak{g}}$ to be the realization of $\widehat{L}(\mathfrak{g},\sigma)$ with $L(\mathfrak{g},\sigma)$ replaced by $X\mathfrak{g}^\sigma$.

Thus an element of a Kac-Moody algebra can be represented by a triple $(f(z), r_c, r_d)$, where $f(z)$ denotes a $\mathfrak{g}_\mathbb{C}$-valued holomorphic function on X and $\{r_c, r_d\} \in \mathbb{C}$.

We equip those algebras with the norms $\|(f, r_c, r_d)\|_n := \sup_{z\in A_n} |f_z| + (r_c\bar{r}_c + r_d\bar{r}_d)^{\frac{1}{2}}$. Thus we use the supremum norm on the loop algebra and complete it with an Euclidean norm on the double extension defined by c and d.

Lemma 3.3.1 (Banach- and Fréchet structures on Kac-Moody algebras)

1. *For each n, the algebras $\widehat{A_n\mathfrak{g}_\mathbb{R}}^\sigma$ and $\widehat{A_n\mathfrak{g}_\mathbb{C}}^\sigma$ equipped with the norm $\|\ \|_n$ are Banach-Lie algebras,*
2. *$\widehat{M\mathfrak{g}_\mathbb{R}}^\sigma$ and $\widehat{M\mathfrak{g}_\mathbb{C}}^\sigma$ equipped with the sequence of norms $\|\ \|_n$ are tame Fréchet-Lie algebras.*

Proof. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

1. As a consequence of lemma 2.3.1, $A_n\mathfrak{g}^\sigma$ is a Banach space. Thus $\widehat{A_n\mathfrak{g}}^\sigma$ is Banach.

   To prove that $ad(f + r_c c + r_d d)$ is continuous, we use $[f + r_c c + r_d d, g + s_c c + s_d d] = [f, g] + r_d[d, g] - s_d[d, f] = [f, g]_0 + \omega(f, g)c + r_d izg' - s_d izf'$. Continuity follows from the continuity of $\frac{d}{dz}$, which is a consequence of the Cauchy-inequality and the boundedness of multiplication on compact domains.

2. $M\mathfrak{g}^\sigma$ is a tame Fréchet space as a consequence of lemma 2.3.1. Thus $\widehat{M\mathfrak{g}}^\sigma$ is tame as a direct product of tame spaces (lemma 2.1.4). To prove tameness of the adjoint action, we need tame estimates for the norms. Those estimates follow directly from the Banach space situation:

$$
\begin{aligned}
&\|ad(f + r_c c + r_d d)(g + s_c c + s_d d)\|_n = \\
&= \|[f, g]_0 + \omega(f, g)c + r_d izg' - s_d izf'\| \leq \\
&\leq 2\|f\|_n \sup_{z\in A_n} |g(z)| + \|\omega(f, g)c\| + \|r_d izg'\| + \|s_d izf'\|_n \leq \\
&\leq 2\|f\|_n\|g(z)\|_n + 2\pi\|f\|_n\|g'\|_n + |r_d|\|z\|_n\|g'\|_n + |s_d|\|z\|_n\|f'\|_n \leq \\
&\leq 2\|f\|_n\|g\|_n + 2\pi\|f\|_n\frac{e^{n+1}}{e-1}\|g\|_{n+1} + e^n\|f\|_n\|\frac{e^{n+1}}{e-1}\|g\|_{n+1} + e^n\|s_d\|\frac{e^{n+1}}{e-1}\|f\|_{n+1} \leq \\
&\leq \left(2 + 2\pi\frac{e^{n+1}}{e-1} + 2\frac{e^{2n+1}}{e-1}\right)\|g\|_{n+1}\|f\|_{n+1} \leq \\
&\leq 6\pi e^{2n+1}\|g\|_{n+1}\|f\|_{n+1}\,.
\end{aligned}
$$

   Thus $ad(\widehat{g})$ is $(1, 0, 6\pi e^{2n+1}\|g\|_{n+1})$-tame. □

This result shows that the tame structure on the Kac-Moody algebra is preserved by the adjoint action.
For analytic details and the Cauchy-inequalities see for example [BG91].

Theorem 3.3.1
The system $\{\widehat{M\mathfrak{g}}^\sigma, \widehat{A_n\mathfrak{g}}^\sigma\}$ is an ILB-system.

Proof. According to lemma 2.3.1, the system $\{A\mathfrak{g}^\sigma, A_n\mathfrak{g}^\sigma\}$ is an ILB-system. The same is true for $\{\widehat{M\mathfrak{g}}^\sigma, \widehat{A_n\mathfrak{g}}^\sigma\}$ as it is a direct product of an ILB-system with a finite dimensional subspace. □

Nevertheless, the adjoint action on $\widehat{M\mathfrak{g}}^\sigma$ does not induce an adjoint action on each algebra $\widehat{A_n\mathfrak{g}}^\sigma$, but only an ILB-(1,0)-regular map. This is a consequence of lemma 3.3.1. This means that $\widehat{A_n\mathfrak{g}}^\sigma$ is not a Lie algebra as the bracket is not defined for all elements.

Remark 3.3.1
The algebraic theory including the proof that the algebras we describe are isomorphic to the non-twisted and twisted Kac-Moody algebras can be found in chapter 18 of [Car02].

3.4 Kac-Moody groups

3.4.1 Construction

In this section we describe Kac-Moody groups. Kac-Moody groups are torus extensions of loop groups. Our presentation follows the book [PS86]. Furthermore using a technical result of Bogdan Popescu we prove that Kac-Moody groups of holomorphic loops carry a structure as tame Fréchet manifolds.

As usual $G_\mathbb{C}$ is a complex semisimple Lie group G its compact real form. As the constructions are valid in all regularity types which we investigate we use again the regularity-independent notation $L(G_\mathbb{C}, \sigma)$ for the complex loop group and $L(G, \sigma)$ for its real form of compact type. To define groups of polynomial or analytic loops, we use the fact, that every compact Lie group is isomorphic to a subgroup of some unitary group. Hence, we can identify it with a matrix group. Similarly the complexification can be identified with a subgroup of some general linear group — cf.[PS86].
Kac-Moody groups are constructed in two steps.

1. The first step consists in the construction of an S^1-bundle in the real case (resp. a $\mathbb{C}^*$-bundle in the complex case) that corresponds via the exponential map to the central term $\mathbb{R}c$ (resp. $\mathbb{C}c$) of the Kac-Moody algebra.

2. In the second step we construct a semidirect product with S^1 (resp. $\mathbb{C}^*$). This corresponds via the exponential map to the $\mathbb{R}d$- (resp. $\mathbb{C}d$-) term

Study first the extension of $L(G, \sigma)$ with the short exact sequence:

$$1 \longrightarrow S^1 \longrightarrow X \longrightarrow L(G, \sigma) \longrightarrow 1\,.$$

There are different groups X that fit into this sequence. We need to define X in a way that its tangential Lie algebra at $e \in \widetilde{L}(G, \sigma)$ is isomorphic to $\widetilde{L}(\mathfrak{g}, \sigma)$.

As described in [PS86] this S^1-bundle is best represented by triples: take triples (g, p, z) where g is an element in the loop group, p a path connecting the identity to g and $z \in \mathbb{S}^1$ (respective $\mathbb{C}^*$) subject to the relation of equivalence: $(g_1, p_1, z_1) \sim (g_2, p_2, z_2)$ iff $g_1 = g_2$ and $z_1 = C_\omega(p_2 * p_1^{-1})z_2$. The term $z_1 = C_\omega(p_2 * p_1^{-1})z_2$ defines a twist of the bundle. Here we put: $C_\omega(p_2 * p_1^{-1}) = e^{\int_{S(p_2 * p_1^{-1})} \omega}$ where $S(p_2 * p_1^{-1})$ is a surface bounded by the closed curve $p_2 * p_1^{-1}$ and ω denotes the 2-form used to define the central extension of $L(\mathfrak{g}, \sigma)$. The law of composition is defined by

$$(g_1, p_1, z_1) \cdot (g_2, p_2, z_2) = (g_1 g_2, p_1 * g_1 \cdot p_2, z_1 z_2)\,.$$

If G is simply connected it can be shown that this object is a well defined group independent of arbitrary choices made in the construction iff ω is integral. This condition is satisfied by our definition of ω — cf. [PS86], theorem 4.4.1. If G is not simply connected, the situation is a little more complicated: let $G = H/Z$ where H is a simply connected Lie group and $Z = \pi_1(G)$. Let $(LG)_0$ denote the identity component of LG. We can describe the extension using the short exact sequence:

$$1 \longrightarrow S^1 \longrightarrow \widetilde{LH}/Z \longrightarrow (LG)_0 \longrightarrow 1$$

— cf. [PS86], section 4.6.
In case of complex loop groups, the S^1-bundle is replaced by a $\mathbb{C}^*$-bundle.

Hence, we can now give the definition of Kac-Moody groups:

Definition 3.4.1 (Kac-Moody group)

1. *The real Kac-Moody group $\widehat{MG}_{\mathbb{R}}$ is the semidirect product of $\mathbb{S}^1$ with the $\mathbb{S}^1$-bundle $\widetilde{MG}_{\mathbb{R}}$.*

2. *The complex Kac-Moody group $\widehat{MG}_{\mathbb{C}}$ is its complexification: a semidirect product of $\mathbb{C}^*$ with $\widetilde{MG}_{\mathbb{C}}$-bundle over MG.*

The action of the semidirect S^1 (resp. $\mathbb{C}^*$)-factor is in both cases given by a shift of the argument: $\mathbb{C}^* \ni re^{i\varphi} : MG \to MG : f(z) \mapsto f(z \cdot re^{i\varphi})$.

This exponential can be uniquely extended to the complete Kac-Moody algebra $\widehat{M\mathfrak{g}}$. It maps $\mathbb{C}c$ into the fiber of the $\mathbb{C}^*$-bundle and the $\mathbb{C}d$-term into the semidirect $\mathbb{C}^*$-factor.

To study more precisely the properties of the exponential function, we introduce the notion of curves in a Kac-Moody group. As a Kac-Moody group $\widehat{MG}_{\mathbb{R}}$ is locally a (topologically) direct product of the loop group $MG_{\mathbb{R}}$ with $\mathbb{R}^2$, a path: $\widehat{\gamma} : (-\epsilon, \epsilon) \to \widehat{MG}$ is locally described by three components: $\widehat{\gamma}(t) = (\gamma(t), \gamma_c(t), \gamma_d(t))$,

with $\gamma(t)$ taking values in $MG_{\mathbb{R}}$ and $\gamma_c(t)$, $\gamma_d(t)$ taking values in $\mathbb{R}$. For every $z \in \mathbb{C}^*$, γ defines a path $\gamma_z(t) : (-\epsilon, \epsilon) \to G$ by setting $\gamma_z(t) := [\gamma(t)](z)$. A path $\gamma : (-\epsilon, \epsilon) \to MG$, $t \to \gamma(t)$ is differentiable (respective smooth) iff the map $\delta : (-\epsilon, \epsilon) \times \mathbb{C}^* \to G_{\mathbb{C}}$, $(t, z) \to \delta(t, z)$ such that $\delta(t, z) = \gamma_z(t)$ is differentiable (respective smooth).

The next aim is to prove that Kac-Moody groups are tame Fréchet manifolds. To this end we use a result of Bogdan Popescu [Pop05] stating that fiber bundles whose fiber is a Banach space over tame Fréchet manifolds are tame.

We start with the definition of tame fiber bundles:

Definition 3.4.2 (tame Fréchet fiber bundle)
A fiber bundle P over M with fiber G is a tame Fréchet manifold P together with a projection map $\pi : P \longrightarrow M$ satisfying the following condition:

For each point $x \in M$ there is a chart $\varphi : U \longrightarrow V \subset F$ with values in a tame Fréchet space F such that there is a chart $\varphi : \pi^{-1}(U) \longrightarrow G \times U \subset G \times F$ such that the projection π corresponds to a projection of $U \times F$ onto U in each fiber.

The following lemma is proved in [Pop05].

Lemma 3.4.1
Let P be a fiber bundle over M whose fiber is a Banach manifold; then P is a tame Fréchet manifold.

This result contains the important corollary:

Corollary 3.4.1
$\widehat{MG}_{\mathbb{R}}$, $\widehat{MG}_{\mathbb{C}}$ and $\widehat{MG}_{\mathbb{C}}/\widehat{MG}_{\mathbb{R}}$, $\widehat{MG}_D$ and their quotients $\widehat{MG}_{\mathbb{R}}/Fix(\rho)$ and $\widehat{MG}_D/Fix(\rho)$ are tame Fréchet manifolds.

Next we prove:

Theorem 3.4.1
$\widehat{MG}_{\mathbb{C}}/\widehat{MG}_{\mathbb{R}}$ is diffeomorphic to a vector space.

Proof. By theorem 2.4.8, we know that $MG_{\mathbb{C}}/MG_{\mathbb{R}}$ is diffeomorphic to a vector space. As $MG_{\mathbb{R}}$ is a subgroup of $MG_{\mathbb{C}}$, the quotient is well defined. To prove the theorem we check the decomposition

$$\widehat{MG}_{\mathbb{C}}/\widehat{MG}_{\mathbb{R}} \simeq MG_{\mathbb{C}}/MG_{\mathbb{R}} \times (\mathbb{R}^+)^2 .$$

To this end we use the description of the elements in $\widehat{MG}_{\mathbb{C}}/\widehat{MG}$ as 4-tuples. Two 4-tuples $(g(t), p(t), r_c, r_d)$ and $(g'(t), p'(t), r'_c, r'_d)$ describe the same element of $\widehat{MG}_{\mathbb{C}}/\widehat{MG}$ iff there exists an element $(h(t), q(t), s_c, s_d) \in MG_{\mathbb{R}}$ such that $(g(t), p(t), r_c, r_d) = (g'(t) + h(t), p'(t) + q(t), r'_c + s_c, r'_d + s_d)$.

Hence, the equivalence classes for $g(t)$ are elements of $MG_{\mathbb{C}}/MG_{\mathbb{R}}$. The extension of $MG_{\mathbb{R}}$ lies in $\mathbb{S}^1$. Thus r_c and r_d are defined up to an element in S^1. So we get a description of $\widehat{MG}_{\mathbb{C}}/\widehat{MG}_{\mathbb{R}}$ as a $(\mathbb{R}^+)^2$-bundle over $MG_{\mathbb{C}}/MG_{\mathbb{R}}$.

As $MG_{\mathbb{C}}/MG_{\mathbb{R}}$ is diffeomorphic to a vector space, this bundle is trivial. Omitting p describes the diffeomorphism. □

Now we investigate the quotients $\widehat{MG}/\widehat{\mathrm{Fix}(\sigma)}$.

The group $\widehat{\mathrm{Fix}(\sigma)}$ consists of elements $(g, p, r_c, r_d) \in \widehat{MG}$ such that $\{g, p \in \mathrm{Fix}(\sigma), r_c \in \pm 1, r_d \in \pm 1\}$. As $\{r_c, r_d\} \in \pm 1$, this is a covering with four leaves. The details of the argument follow the description found in [Pop05] for the smooth C^∞-setting.
Let $H \subset MG_{\mathbb{C}}$ be a real form of non-compact type.
The description of the space $H/\mathrm{Fix}(\sigma) \subset MG_{\mathbb{C}}/\mathrm{Fix}(\sigma)$ follows similarly.

Theorem 3.4.2
The space $H/Fix(\sigma)$ is diffeomorphic to a vector space.

The proof that this space is diffeomorphic to a vector space parallels the one given for theorem 3.4.1.

3.4.2 Adjoint action and isotropy representations

The most important example is the adjoint action:

Example 3.4.1 (Adjoint action)
With $x = \{w, (g, p, z)\}$, the adjoint action of $\widehat{MG}^{\sigma}$ on $\widehat{M\mathfrak{g}}^{\sigma}$ is described by the following formulae

$$\begin{aligned} Ad(x)u &:= gw(u)g^{-1} + \langle gw(u)g^{-1}, g'g^{-1}\rangle c \\ Ad(x)c &:= c \\ Ad(x)d &:= d - g'g^{-1} + \tfrac{1}{2}\langle g'g^{-1}, g'g^{-1}\rangle c . \end{aligned}$$

Here $\omega(u)$ denotes the shift of the argument by ω.

For the proof compare [HPTT95], [PS86] and [Kac90].

Proof.

- c generates the center, thus $\mathrm{Ad}(x)c := c$.

- $\mathrm{Ad}(x)u$ follows by integrating the *Ad*-action.
- To calculate $\mathrm{Ad}(g)(d)$, we use the *Ad*-invariance of the Lie bracket. As $[d, v] = v'$ we get for all $v \in L_{alg}\mathfrak{g}^\sigma$:

$$\begin{aligned}
&gv'g^{-1} + \langle gv'g^{-1}, g'g^{-1}\rangle c = \\
=&\mathrm{Ad}(g)(v') = \mathrm{Ad}(g)[d, v] = [\mathrm{Ad}(g)(d), \mathrm{Ad}(g)(v)] = \\
=&[h + \mu c + \nu d, gvg^{-1} + \langle gvg^{-1}, g'g^{-1}\rangle c] = \\
=&[h, gvg^{-1}] + \nu[d, gvg^{-1}] = \\
=&hgvg^{-1} - gvg^{-1}h + \omega(h, (gvg^{-1})')c + \nu g'vg^{-1} + \nu gv'g^{-1} + \nu gv(g^{-1})' ,
\end{aligned}$$

with $h \in L_{alg}\mathfrak{g}^\sigma$ and $\{\mu, \nu\} \in \mathbb{R}$.

To get equality we have to choose $\nu = 1$, $h = -g'g^{-1}$. This gives us

$$\begin{aligned}
&hgvg^{-1} - gvg^{-1}h + \omega(h, (gvg^{-1})')c + \nu g'vg^{-1} + \nu gv'g^{-1} + \nu gv(g^{-1})' = \\
=& - g'vg^{-1} + gvg^{-1}g'g^{-1} + \omega(-g'g^{-1}, (gvg^{-1})')c + g'vg^{-1} + gv'g^{-1} + gv(g^{-1})' = \\
=&\omega(-g'g^{-1}, (gvg^{-1})')c + gv'g^{-1} .
\end{aligned}$$

Thus we are left with the calculation of μ. To this end we use the property that *Ad* acts by isometries. This results in $\mu = \frac{1}{2}\langle g'g^{-1}, g'g^{-1}\rangle$. □

More details can be found in [PS86, HPTT95, Pop05, Pop06].

Theorem 3.4.3 (Polarity of the Adjoint action)
Let $H_{l,r} \subset \widehat{X\mathfrak{g}}^\sigma$, $\{l, r\} \in \mathbb{R}\backslash\{0\}$ denote the intersection of the sphere with radius $-l^2$ with the horosphere $r_d = r$. The restriction of the Adjoint action to $H_{l,r}$ is polar.

Proof. The restriction of the Adjoint action to $H_{l,r}$ coincides with the gauge action on $X\mathfrak{g}$. Hence, theorem 3.4.3 is a direct consequence of theorem 2.5.2. □

Chuu-Lian Terng describes how to associate an affine Weyl group to this gauge action — cf. [Ter95]. This is exactly the affine Weyl group of the Kac-Moody group $\widehat{MG}$
This theorem gives a complete description of the Adjoint action iff $r_d \neq 0$.
Surprisingly in the remaining case $r_d = 0$ the situation is different: now the Adjoint action is reduced to the equations:

$$\begin{aligned}
\mathrm{Ad}(x)u &:= gw(u)g^{-1} + \langle gw(u)g^{-1}, g'g^{-1}\rangle c \\
\mathrm{Ad}(x)c &:= c
\end{aligned}$$

Calculate the orbit of the constant function $u \equiv 0$. u is fixed by the Adjoint action as $\mathrm{Ad}(x)u := g0g^{-1} + \langle g0g^{-1}, g'g^{-1}\rangle c = 0$. Hence, iff we can describe the restriction of the adjoint action to $X\mathfrak{g}^\sigma$ as some kind of polar action then the associated Weyl group has to be necessarily of spherical type. Furthermore the action is clearly not proper Fredholm. Hence, the Hilbert-space version is not covered by Terng's results.
We use the regularity independent notation:

We define a flat of finite type to be a flat $\mathfrak{t} \subset L(\mathfrak{g}, \sigma)$ such that $\mathfrak{t}$ is the restriction of a flat in $\widehat{L}(\mathfrak{g}, \sigma)$. Hence, all flats of finite type are conjugate in $\widehat{L}(G, \sigma)$ and as the orbits of $\widehat{L}(G, \sigma)$ and $L(G, \sigma)$ coincide on $L(\mathfrak{g}, \sigma)$ also in $L(G, \sigma)$. Hence, any flat of finite type in $L(\mathfrak{g}, \sigma)$ is isomorphic to $\mathfrak{t}_0 \subset \mathfrak{g}$ where we choose $\mathfrak{g}$ to denote the subalgebra of constant loops. Using the usual notion for regular and singular elements we find that the associated Weyl group is the spherical Weyl group of $\mathfrak{g}$.

Chapter 4

Kac-Moody symmetric spaces

4.1 Foundations

Kac-Moody symmetric spaces are tame Fréchet Lorentz manifolds. In this foundational section, we will review the differential geometry of tame Fréchet manifolds, following the presentation in [Ham82], and discuss the generalization of some results of pseudo-Riemann geometry to the tame Fréchet setting.

4.1.1 Differential geometry of tame Fréchet manifolds

Let M be a tame Fréchet manifold whose charts take values in a Fréchet space F such that its tangent space at a point T_fM is isomorphic to F_1. As the exponential map is not in general a diffeomorphism, we cannot automatically suppose $F = F_1$.

Having defined fibre bundles in section 3.4, we turn now to the special case of vector bundles, which is the most important one for differential geometry. Denote by P a vector bundle over M with fiber V. M, P and V are supposed to be tame Fréchet space and denote by TP its tangential bundle.

A vector field on M is a smooth section of TM. As an example, let $M := MG$. Then TM is a $M\mathfrak{g}$-bundle over MG. Charts can be chosen to be $U \times M\mathfrak{g}$, where U is a chart of MG. A vector field on MG is defined locally to be $(f, \phi(f))$, with $f \in U$ and $\phi : U \longrightarrow M\mathfrak{g}$ a smooth map. Following the finite dimensional theory, we define:

Definition 4.1.1 (Vertical bundle)
The vertical bundle $T_vP \subset TP$ consists of the vertical vectors, that is the vectors $v \subset TP$ such that $v \subset (d\pi)^{-1}(0, x)$ with $x \in M$.

A connection on TP consists of the assignment of a complementary tame Fréchet subbundle T_hP of TP (i.e.: such that $TP = T_vP \oplus T_hP$), that is:

Definition 4.1.2 (Connection)
A connection Γ on TP is the assignment of a complementary subspace of horizontal vectors, such that in terms of any coordinate chart of P with values in $(U \subset F) \times G$ the subspace of horizontal vectors at T_fP consists of all $(h, k) \in F_1 \times G$ such that $k = \Gamma(f)\{g, h\}$, where Γ is represented in any local chart by a smooth map $\Gamma : (U \subset F) \times G \times F_1 \longrightarrow G$, which is bilinear in g and h.

If P is the tangent bundle of M, then $G = F_1$. We call a connection symmetric if Γ is symmetric in $\{g, h\}$.

In the finite dimensional case, to define curvature one would now define a tensor field on M. As dual spaces of Fréchet spaces are (with the trivial exception of Banach spaces) not Fréchet spaces, this approach is not possible for Fréchet manifolds. In contrast we are forced to use explicit coordinate dependent descriptions in terms of component functions.

Definition 4.1.3 (curvature)
The curvature of a connection on a vector bundle P over M is the trilinear map $R : P \times TM \times TM \longrightarrow P$ such that

$$R(f)\{g,h,k\} := D\Gamma\{g,h,k\} - D\Gamma(f)\{g,k,h\} - \Gamma(f)\{\Gamma(f)\{g,h\},k\} + \Gamma f\{\Gamma(f)\{g,k\},h\}\,.$$

We now want to define metrics on TM. Again, this is only possible in a coordinate description:

Definition 4.1.4 (metrics)
Let M be a tame Fréchet manifold and TM its tangential bundle. A metric on TM is a smooth bilinear map: $g : TM \times TM \longrightarrow \mathbb{R}$. Smooth means that g can be described in any local chart as a smooth map.

Clearly, M is not complete with respect to g; so g is only a weak metric.

Following the finite dimensional convention, we define the index of g to be the maximal subspace on which g is negative definite.

We define a connection to be compatible with the metric, iff

$$\frac{d}{dt} g\left(V,W\right) = g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right)$$

for any vector fields V and W along a curve $c : I \longrightarrow M$.

Definition 4.1.5 (Levi-Civita connection)
Let (M,g) be a tame Fréchet manifold. A Levi-Civita connection is a symmetric connection which is compatible with the metric.

Lemma 4.1.1
If a Levi-Civita connection exists, it is well defined and unique.

In contrast to this result, the existence of a Levi-Civita connection seems not to be clear.

Nevertheless for the special case of Lie groups, Bogdan Popescu proves in [Pop05]:

Theorem 4.1.1 (Existence of Levi-Civita connection)
Any Fréchet Lie group G admits a unique left invariant connection such that $\nabla_X Y = \frac{1}{2}[X,Y]$ for any pair of left invariant vector fields X and Y. It is torsion free. If G admits a biinvariant (pseudo-) Riemann metric, then ∇ is the corresponding Levi-Civita connection.

4.1.2 Lorentz geometry on tame Fréchet manifolds

In the last section, we saw that one can define a connection and a metric which work locally exactly like their finite dimensional counterparts. So all local purely algebraic results will carry over to the infinite dimensional case.

Lemma 4.1.2 (Bianchi-identity and symmetry properties)
For a pseudo-Riemann tame Fréchet manifold with Levi-Civita connection, the following identities hold:

- $R(f)\{g,h,k\} + R(f)\{h,k,g\} + R(f)\{k,g,h\} = 0$ *(Bianchi-identity),*
- $\langle R(f)\{g,h,k\}, l\rangle = -\langle R(f)\{h,g,k\}, l\rangle$,
- $\langle R(f)\{g,h,k\}, l\rangle = -\langle R(f)\{g,h,l\}, k\rangle$,
- $\langle R(f)\{g,h,k\}, l\rangle = \langle R(f)\{l,k,g\}, h\rangle$.

The proof consists in algebraic manipulations, exactly like in the finite dimensional case.

Definition 4.1.6 (Sectional curvature)
If $|g \wedge h|^2 = \langle g,g\rangle\langle h,h\rangle - \langle g,h\rangle^2 \neq 0$, *we define the sectional curvature, to be*

$$K_f(g,h) = \frac{\langle R(f)\{g,h,g\}, h\rangle}{|g \wedge h|^2} .$$

A nice structure result for finite dimensional Lorentz geometry is the theorem of Kulkarni. It states that for pseudo-Riemann manifolds that are not Riemann, bounded sectional curvature implies constant curvature. This theorem generalizes to tame Fréchet manifolds:

Theorem 4.1.2 (generalized Kulkarni-type)
Let M be a Lorentz tame Fréchet manifold with Levi-Civita connection. Then the following conditions are equivalent:

- $K_f(g,h)$ *is constant.*
- $a \leq K_f(g,h)$ *or* $K_f(g,h) \leq b$ *for some* $a, b \in \mathbb{R}$.
- $a \leq K_f(g,h) \leq b$ *on all definite planes for some* $a \leq b \in \mathbb{R}$.
- $a \leq K_f(g,h) \leq b$ *on all indefinite planes for some* $a \leq b \in \mathbb{R}$.

For the finite dimensional proof of this theorem — and more generally finite dimensional Lorentz geometry — cf. [O'N83]. The proof of our theorem is again a straight forward generalization based on local algebraic manipulations, so we do not detail it.

The second important ingredient is the Lorentz structure. The most important fact for us is that finite dimensional (pseudo-) Riemann symmetric spaces are not classified; there is recent work by Ines Kath and Martin Olbrich — cf. [KO04], [KO06] — which gives a good description of the structure and classifies pseudo-Riemann symmetric spaces of index 1 and 2. The surprising difficulty of the classification of pseudo-Riemann symmetric spaces in comparison to the Riemann case has its roots in two facts:

1. There is no splitting theorem of a pseudo-Riemann symmetric space in the direct sum of simple factors.
2. There are pseudo-Riemann symmetric spaces that have no semisimple groups of isometries.

In contrast to the difficulty of a complete classification, the subclass of pseudo-Riemann symmetric spaces corresponding to semisimple Lie groups is well understood. A classification is achieved in the paper [Ber57].

4.1.3 Tame Fréchet symmetric spaces

We use the definition:

Definition 4.1.7 (tame Fréchet symmetric space)
A tame Fréchet manifold M with a weak metric having a Levi-Civita connection is called a symmetric space, iff for all $p \in M$ there is an involutive isometry ρ_p, such that p is an isolated fixed point of ρ_p.

It is not clear if the isometry group of a tame Fréchet manifold is in general a tame Fréchet Lie group.

Lemma 4.1.3 (geodesic symmetry)
Let M be a tame Fréchet pseudo-Riemann symmetric space. For each $p \in M$ there exists a normal neighborhood N_p of p such that s_p coincides with the geodesic symmetry on all geodesics through p in N_p.

As the exponential map for Fréchet manifolds is in general not a diffeomorphism, it is not true that there is an open set N_p for each p, such that there is a geodesic connecting p and q for all $q \in N_p$. Thus the notion of geodesic symmetries at a point is only defined for $N_p \cap \exp_p T_pM$.

Proof of lemma 4.1.3: Let $\gamma(t)$ be a geodesic through p and $\mu(t) := \rho_p(\gamma(t))$ its image under the isometry ρ_p.

- As ρ_p is an isometry we find that $\mu(t)$ is a geodesic.
- $(d\rho)(\dot\gamma(t)|_{t=0}) = \dot\mu(t)|_{t=0}$. If $\dot\gamma(t)|_{t=0} = -\dot\mu(t)|_{t=0}$ for all geodesics $\gamma(t)$, we are done. So suppose by contradiction, there exists a $\gamma(t)$ such that $\dot\gamma(t)|_{t=0} \neq -\dot\mu(t)|_{t=0}$. As ρ_p is an isometry, we have $|\dot\gamma(t)|_{t=0} = |\dot\mu(t)|_{t=0}$. Hence, there are two cases:
  - There is a geodesic such that $\dot{\gamma(0)} = \dot{\mu(0)}$. Then $\gamma(t) = \mu(t)$ so p is no isolated fixed point.
  - $\dim \operatorname{span}\{\dot\gamma(0), \dot\mu(0)\} = 2$. Then take the geodesic $\nu(t) := \exp(\dot\gamma(0) + \dot\mu(0))$. ρ_p is the identity on $\nu(t)$, so again p is no isolated fixed point. This is again a contradiction, so the lemma is proved. □

Corollary 4.1.1
For each p, the involution ρ_p induces $-Id$ on the tangent space T_pM.

The corollary is a straight forward application of lemma 4.1.3.

Lemma 4.1.4
A tame Fréchet symmetric space with a Levi-Civita connection is locally symmetric, that is it has parallel curvature.

This lemma is again based on local algebraic manipulations. The finite dimensional proof — cf. [Hel01] — generalizes.

Definition 4.1.8 (Kac-Moody symmetric space)
An (affine) Kac-Moody symmetric space M is a tame Fréchet Lorentz symmetric space such that its isometry group $I(M)$ contains a transitive subgroup isomorphic to an affine geometric Kac-Moody group H and the intersection of the isotropy group of a point with H is a loop group of compact type.

Lemma 4.1.5
Let M be a Kac-Moody symmetric space with isometry group $I(M)$; let H be the Kac-Moody subgroup of $I(M)$. For $x \in M$, let $K_x = Fix(x) \cap (\widehat{H})_0$.

1. *For $x, y \in M$ $K_x \simeq K_y$.*

2. *M is the isomorphic to the quotient $M \simeq (\widehat{H})_0/K_x$.*

Proof.

1. Transitivity of H assures the existence of an element $h_{xy} \in H$ such that $h_{xy}(x) = y$. Then $K_y = hK_xh^{-1}$.

2. The second assertion follows as H acts transitive on M. □

Thus we describe now how to construct the OSAKA associated to M. Let $\widehat{H}$ be the affine Kac-Moody group acting transitively on M and $\widehat{\mathfrak{h}}$ its tangential geometric affine Kac-Moody algebra. Let ρ_* be the involution induced on $\widehat{\mathfrak{h}}$.

Theorem 4.1.3
The pair $(\widehat{\mathfrak{h}}, \rho_*)$ is an OSAKA.

Proof.

1. $\mathfrak{h}$ is a geometric affine Kac-Moody algebra.

2. ρ_* is an involution of the second kind; hence we can suppose $\rho_*(c) = -c$ and $\rho_*(d) = -d$. The fixed point algebra $\text{Fix}(\rho_*) \subset \widehat{\mathfrak{h}}$ is a loop algebra. Let $\mathfrak{h} = \bigoplus_i \mathfrak{h}_i \oplus \mathbb{R}c \oplus \mathbb{R}d$ be the decomposition of $\mathfrak{h}$ into ideals invariant invariant under ρ_*.

   (a) If $\mathfrak{h}_i$ is abelian then the Cartan-Killing form of the subjacent Lie algebra vanishes – hence also the averaged Cartan Killing form. Hence, the maximal subgroup of compact type is trivial.

   (b) If $\mathfrak{h}_i$ is of compact type then the fixed point algebra of any involution of the second kind is of compact type as it is a subalgebra of compact type — cf. [HG09].

   (c) If $\mathfrak{h}_i$ is of non-compact type then there is the Cartan involution whose fixed algebra is the maximal compact subalgebra $\mathcal{K}$. Hence, if ρ_* is the Cartan involution we are done. Suppose ρ_* it is not the Cartan involution. Then there are two possibilities: suppose first there is an element $x \in \mathcal{K} \backslash \text{Fix}(\rho_*)$. Then the resulting space is not Lorentzian as the Cartan Killing form is no longer positive definite on the loop part of the complement of $\text{Fix}(\rho)$. Thus $\mathcal{K} \subset \text{Fix}(\rho_*)$. But then $\text{Fix}(\rho_*)$ is not of compact type. □

Definition 4.1.9 (indecomposable Kac-Moody symmetric space)
A Kac-Moody symmetric space is called indecomposable iff its OSAKA is irreducible.

In the next sections we investigate the three types of Kac-Moody symmetric spaces.

4.2 Kac-Moody symmetric spaces

From now on, let X denote a Kac-Moody symmetric space. A transitive subgroup H of its isometry group $I(M)$ — called the Kac-Moody isometry group — is a real form of a Kac-Moody group $\widehat{MG}$. Following the finite dimensional theory, we define the scalar product on a Kac-Moody symmetric space via the scalar product on its Kac-Moody isometry group.

Suppose H is a real form of a Kac-Moody group $\widehat{MG}$. We first need an Ad-invariant scalar product on $\widehat{MG}_{\mathbb{C}}$. This induces a scalar product on each real form.

The scalar product on $\widehat{XG}$, $X \in \{A_n, \mathbb{C}^*\}$, can be defined in the following way:

Let $\langle , \rangle$ be an Ad-invariant scalar product on $\mathfrak{g}$.

$$\begin{aligned} \langle u, v\rangle &= \frac{1}{2\pi}\int \langle u(t), v(t)\rangle dt \\ \langle c, d\rangle &= -1 \\ \langle c, c\rangle &= \langle d, d\rangle = \langle f, c\rangle = \langle f, d\rangle = 0 \end{aligned}$$

This is a Lorentz scalar product. As in the finite dimensional case, we call a vector v

- space-like iff $|v| > 0$,
- light-like iff $|v| = 0$,
- time-like iff $|v| < 0$.

The Lie algebra $M\mathfrak{g}$ lies completely in the space-like part, while for example the direction $c + d$ is a time-like vector, and c and d are light-like vectors.

If $\mathfrak{g}$ is simple, this scalar product is essentially (that is: up to the choice of d and a global scaling constant) unique: the definition $\langle d, d\rangle = 0$ is arbitrary, but d is only defined up to an element $f \in \widetilde{L\mathfrak{g}}$. So, suppose $\langle \widetilde{d}, \widetilde{d}\rangle = -k^2 < 0$, choose f such that $\langle f, f\rangle = k^2$ and put $d := \widetilde{d} + f$. Then $\langle d, d\rangle = 0$. If in contrast $\langle \widetilde{d}, \widetilde{d}\rangle = k^2 > 0$, choose $d = \widetilde{d} + \frac{k^2}{2}c$; again $\langle d, d\rangle = 0$.

Ad-invariance can be checked elementary. No we want to study the complexification: We identify $\widehat{M\mathfrak{g}_{\mathbb{C}}} = \widehat{M\mathfrak{g}_{\mathbb{R}}} \oplus i\widehat{M\mathfrak{g}_{\mathbb{R}}}$ and put $\langle ix, iy\rangle := \langle x, y\rangle$ and $\langle x, iy\rangle = 0$. We have to check that this scalar product is invariant under $\widehat{MG}$. The crucial point here is to check Ad-invariance under the action of $\exp id$. We have $Ad(\exp ird)(v(t)) := v(te^{-r})$, $v(t) \in M\mathfrak{g}$, which changes the radius. Ad-invariance follows like that:

We use the equivalence $\langle u, v\rangle = 2\pi\mathrm{Res}(uv')$. Thus

$$\begin{aligned} Ad(e^{ird})\langle v, u\rangle &= \langle v(te^{-r}), u(te^{-r}) =\rangle \\ &= 2\pi Res(v(te^{-r})u(te^{-r}) = \\ &= 2\pi Res(\textstyle\sum_n(\sum_k v_k u_{n-k} r(n-k))r^n t^n) \\ &= \textstyle\sum_k v_k u_{-k-1} = \\ &= 2\pi Res(uv) = \langle u, v\rangle\,. \end{aligned}$$

Now theorem 4.1.1 guarantees the existence of an associated Levi-Civita Connection ∇.

4.3 Kac-Moody symmetric spaces of compact type

4.3.1 Foundations

In this chapter we want to equip Kac-Moody groups of compact type with an Ad-invariant metric, in order to obtain symmetric spaces. Of course these spaces are — as they are

infinite dimensional — not compact. So we want to call a Kac-Moody symmetric space "of compact type"' if it is associated to a real form of compact type of a Kac-Moody group or to a quotient of such a real form or, equivalently, if $\langle R(f)\{g,h,g\},h\}\rangle \geq 0$.
A consequence of theorem 4.1.1 is:

Theorem 4.3.1 (Levi-Civita connection)
$(\widehat{MG}, g)$ admits a unique Levi-Civita connection ∇. For g,h left invariant vector fields, $\nabla_g h = \frac{1}{2}[g,h]$.

Elementary calculation shows that $\langle R(f)\{g,h,g\},h\}\rangle \geq 0$.
So, the resulting symmetric spaces have "compact type behaviour".

4.3.2 Kac-Moody symmetric spaces of type II

In this section we describe the Kac-Moody analogue of irreducible "type II" symmetric spaces, that is a symmetric space structure on Kac-Moody groups of compact type. Let $G_{\mathbb{C}}$ be a simple complex Lie group. Let $f : \mathbb{C} \longrightarrow G_{\mathbb{C}} \in MG_{\mathbb{R}}$. We use f^* for the element: $f^*(z) := \overline{f\left(\frac{1}{\bar{z}}\right)}$ and define the symmetry by the mapping

$$\widehat{M\rho} : M\widehat{(G \times G)} \longrightarrow M\widehat{(G \times G)}$$
$$\{(f_1(z), p_1)\,(f_2(z), p_2)\,, r_c, r_d\} \mapsto \{f_2^*(z), p_2^*(z), f_1^*(z), p_1^*(z), -r_c, -r_d\}\ .$$

As $f^{**} = Id$, $\rho(r_c) = -r_c$, $\rho(r_d) = -r_d$, $M\rho$ is an involution of the second kind and the pair $\left(M\widehat{(\mathfrak{g} \times \mathfrak{g})}, d\widehat{M\rho}\right)$ is an irreducible OSAKA.

The fixed points of $\widehat{M\rho}$ consist of elements that have a description of the form:

$$((f_1(z), p_1(z,t))\,, (f_2(z), p_2(z,t))\,, r_c, r_d)\ ,$$

such that $f_1(z) = f_2^*(z), p_1(z) = p_2^*(z)$ and $r_c = r_d \equiv 0$. So we see that the fixed point group is isomorphic to the loop group MG.

The Lorentz structure on $M\widehat{(G \times G)}$ is $M(G \times G)$ invariant, iff the corresponding structure on G is biinvariant. Thus by example 3.4.1 the resulting Lorentz structure on $\widehat{MG}$ is biinvariant.

4.3.3 Kac-Moody symmetric spaces of type I

The aim of this section is to prove the following theorem:

Theorem 4.3.2 (Kac-Moody symmetric space of type I)
The space $\widehat{MG}/\widehat{MG}^{\rho}$ carries an $\widehat{MG}$-invariant Lorentz metric such that it is a symmetric space.

We denote by π the projection

$$\pi : \widehat{MG} \longrightarrow \widehat{MG}/\widehat{MG}^{\rho}, f \mapsto f \cdot \widehat{MG}^{\rho}\ .$$

For the proof we need a lemma:

Lemma 4.3.1
Let $\widehat{M\mathfrak{g}}^{\rho}_{-1}$ denote the eigenspace of ρ to the eigenvalue -1. There is an equivalence

$$T_{\pi(e)}\widehat{MG}/\widehat{MG}^{\rho} \simeq \widehat{M\mathfrak{g}}^{\rho}_{-1}\,.$$

Proof.

- The inclusion $T_{\pi(e)}\widehat{MG}/\widehat{MG}^{\rho} \subset \widehat{M\mathfrak{g}}^{\rho}_{-1}$ follows as for every curve $\gamma(t)$ in $\widehat{MG}/\widehat{MG}^{\rho}$ there is a curve $\gamma'(t)$ in $\widehat{MG}$, such that $\pi(\gamma'(t) = \gamma(t)$. So let $\gamma'(0) = e$ and $\gamma(0) = \pi(e)$; then $d\pi(\dot{\gamma}'(0)) = \gamma(0)$.
- The inclusion $T_{\pi(e)}\widehat{MG}/\widehat{MG}^{\rho} \supset \widehat{M\mathfrak{g}}^{\rho}_{-1}$ is clear: let $X \in \widehat{M\mathfrak{g}}^{\rho}_{-1}$, $\gamma(t) := \pi \circ exp(tX)$ is a curve in $\widehat{MG}/\widehat{MG}^{\rho}$; thus $\widehat{MG}/\dot{\gamma(t)} := d\pi()T_{\pi(e)}\widehat{MG}^{\rho}$. □

Proof of theorem 4.3.2: To define a metric on $\widehat{MG}/\widehat{MG}^{\rho}$, we use the projection

$$\pi : \widehat{MG} \longrightarrow \widehat{MG}/\widehat{MG}^{\rho}.$$

Lemma 4.3.1 shows that π induces the identification of $T_{\pi(e)}\widehat{MG}/\widehat{MG}^{\rho} \simeq \widehat{M\mathfrak{g}}^{\rho}_{-1}$. We can thus define a metric on $T_{\pi(e)}\widehat{MG}/\widehat{MG}$ by

$$\langle x, y\rangle_{T_{\pi(e)}\widehat{MG}/\widehat{MG}^{\rho}} := \langle \pi^{-1}(x), \pi^{-1}(y)\rangle_{\widehat{M\mathfrak{g}}^{\rho}_{-1}}.$$

This scalar product is $Ad(\widehat{MG}^{\rho})$-invariant, thus well defined. By left translation it induces an $\widehat{MG}$-invariant metric on $\widehat{MG}/\widehat{MG}^{\rho}$.

Via π, the connection on $\widehat{MG}$ induces a connection on $\widehat{MG}/\widehat{MG}^{\rho}$. □

Again, this metric and connection induce metric and connections on the quotient spaces $\widehat{MG}_{\mathbb{R}}/\mathrm{Fix}(\rho)$.

4.4 Symmetric spaces of non-compact type

One can construct dual symmetric spaces in the canoncial way, known from finite dimensional symmetric spaces.

The complexification of the Ad-invariant Lorentz metric on $\widehat{M\mathfrak{g}}_{\mathbb{R}}$ is an Ad-invariant metric on $\widehat{M\mathfrak{g}}_{\mathbb{C}}$ and induces thus a left invariant metric on $\widehat{MG}_{\mathbb{C}}$. This induces a unique left invariant Lorentz metric on the quotient space $\widehat{MG}_{\mathbb{C}}/\widehat{MG}_{\mathbb{R}}$. The proof follows the pattern of theorem 4.3.2.

Elementary calculation shows that $\langle R(f)\{g, h, g\}, h\}\rangle \geq 0$.

Let H be a real form of non-compact type of $\widehat{MG}_{\mathbb{C}}$. The Ad-invariant scalar product on $\widehat{MG}_{\mathbb{C}}$ restricts to an Ad-invariant scalar product on H. Let $\mathrm{Fix}(\rho)$ be the maximal compact subgroup. The projection

$$\pi : H \longrightarrow H/\mathrm{Fix}(\rho)$$

induces a unique left invariant Lorentz metric on $H/\mathrm{Fix}(\rho)$. With respect to this metric, the space $X = H/\mathrm{Fix}(\rho)$ is a symmetric space.

Theorems 3.4.1 and 3.4.2 include the following important corollary.

Corollary 4.4.1
Kac-Moody symmetric spaces of non-compact type are diffeomorphic to a vector space.

This last fact suggest the existence of a nice boundary at infinity of the Kac-Moody symmetric spaces; the existence of rigidity of quotient spaces and a Mostow-type behaviour seem quite probable.

4.5 Kac-Moody symmetric spaces of the Euclidean type

A Kac-Moody symmetric space of Euclidean type is the double extension of a loop algebra $\widehat{L}(\mathfrak{g}_{\mathbb{R}})$ where $\mathfrak{g}$ denotes a finite dimensional real abelian Lie algebra. We omit the σ in the notation as σ is the identity for Euclidean factors. This real abelian Lie algebra is a real form of a complex abelian Lie algebra $\widehat{L}(\mathfrak{g}_{\mathbb{C}})$. $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \oplus i\mathfrak{g}_{\mathbb{R}}$. Hence, we have two possible real forms: $\widehat{L}(\mathfrak{g}_{\mathbb{C}})$ and $\widehat{L}(i\mathfrak{g}_{\mathbb{C}})$. While the loop algebras are isomorphic, the extensions are different.

The most important feature of Kac-Moody symmetric spaces of the Euclidean type is that its exponential map behaves well: it describes a tame diffeomorphism $M\exp : \widehat{U} \longrightarrow \widehat{V}$, where $\widehat{U} \subset \widehat{M\mathfrak{g}}$ and $\widehat{V} \subset \widehat{MG}$.
This is a direct consequence of corollary 2.4.1.
In connection to this result, we also want to mention the following theorem of Galanis [Gal96], that describes in some sense the inverse situation:

Theorem 4.5.1 (Galanis)
Let G be a commutative Fréchet Lie group modelled on a Fréchet space F. Assume the group is a strong exponential Lie group (i.e. a group such that exp *is a local diffeomorphism). Then it is a projective limit Banach Lie group.*

As the loop group part of an Euclidean Kac-Moody group is exponential we get immediately that an Euclidean Kac-Moody symmetric space carries is strong exponential. Furthermore it carries a projective limit Banach structure modelled via the exponential maps on the tangential Kac-Moody algebra.

Theorem 4.5.2
Let G be a simple compact Lie group, $T \subset G$ a maximal torus. The space $\widehat{MT}$ is a Kac-Moody symmetric space of Euclidean type.

As is the case for finite dimensional symmetric spaces of the Euclidean type the isometry group is much bigger, namely a semidirect product of an Euclidean Kac-Moody group with the isotropy group of a point.
The group $\widetilde{MG}$ is a Heisenberg group — cf. [PS86], chapter 9.5. .

4.6 The structure of finite dimensional flats

Let M be a finite dimensional symmetric space and $p \in M$. It is well-known that all maximal flats containing p are conjugate. The proof consists of several parts. For irreducible symmetric spaces of type II, that is compact Lie groups, this assertion is equivalent to the fact that all maximal tori are conjugate. For symmetric spaces of type I one has to study the decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and prove that all maximal flats in $\mathfrak{p}$ are conjugate. For the dual symmetric spaces of non-compact type one can deduce the assertion via the isotropy representation: the isotropy representation is isomorphic for a symmetric space of compact type and its non-compact dual. Thus all maximal abelian subalgebras are conjugate and thus via the exponential map also all maximal flats. For details cf. [Hel01].

To understand the infinite dimensional situation we start with the investigation of flats in a Kac-Moody group of compact type $\widehat{MG}$ and its Kac-Moody algebra $\widehat{M\mathfrak{g}}$. Then we study flats in Kac-Moody symmetric spaces of type I. Third we turn to the Kac-Moody symmetric spaces of non-compact type.

Flats in Kac-Moody symmetric spaces of type II

In a Kac-Moody algebra $\widehat{L}(\mathfrak{g},\sigma)$ there are two types of abelian subalgebras: infinite dimensional ones and finite dimensional ones.

1. The first class — called subalgebras of infinite type — consists of subalgebras that are contained in $\widetilde{L}(\mathfrak{g},\sigma)$. They are infinite dimensional. For the affine Kac-Moody groups of smooth loops this is proved by Bogdan Popescu — cf. [Pop06]; his argument applies as well to the situation of $\widehat{A_n\mathfrak{g}_{\mathbb{R}}}$ and to $\widehat{M\mathfrak{g}_{\mathbb{R}}}$.

2. The second class — called subalgebras of finite type — consists of subalgebras that are not contained in $\widetilde{L}(\mathfrak{g},\sigma)$. Subalgebras of this type always contain an element of the form $f+c+d$, where $f\in L(\mathfrak{g},\sigma)$. Furthermore they contain $c\mathbb{R}$. Thus a subalgebra of finite type $\widehat{\mathfrak{a}}$ can be described as $\widehat{\mathfrak{a}}=\mathfrak{a}\oplus\mathbb{R}c\oplus\mathbb{R}(d+f)$, where $\mathfrak{a}$ is an abelian subalgebra of $M\mathfrak{g}$. Popescu proves in the setting of affine Kac-Moody algebras of C^∞-loops that $\dim(\widehat{\mathfrak{a}})=\operatorname{rank}(\mathfrak{g})+2$.

   The crucial observation in his proof is that any element $v\in\mathfrak{a}$ satisfies:

   $$[v,d+f]=0\Leftrightarrow v'=[v,f]\Leftrightarrow v'=[v,f]_0,\omega(v,f)=0\,.$$

   Thus one has to calculate the space of solutions of this Lax equation under the condition $\omega(v,f)=0$. The proof applies as well to the setting of holomorphic loops.

Remark 4.6.1
Those two classes of flats reappear in different guises: a prominent example is the study of co-adjoint actions of loop resp. affine Kac-Moody groups. The orbits of the co-adjoint action of loop groups $L(G,\sigma)$ have infinite codimension, while the orbits of affine Kac-Moody groups $\widehat{L}(G,\sigma)$ have finite codimension provided one studies an orbit such that $r_d\neq 0$. Identifying the dual loop algebra $L^*(\mathfrak{g},\sigma)$ with the $\{r_d=0\}$-section of $\widehat{L}^*(\mathfrak{g},\sigma)$, the loop group action coincides with the action of the affine group. For further details and the definitions we omitted see the recent reference [KW09], remark 1.20.

From our point of view, the important type of flats are the finite dimensional ones as they correspond to apartments of the universal twin building (see 5.4):
By definition an abelian subalgebra $\widehat{\mathfrak{a}}\subset\widehat{M\mathfrak{g}}$ of finite type contains an element $\widetilde{u}+d$ with $\widetilde{u}\subset\widetilde{M\mathfrak{g}}$.
Thus every finite dimensional flat intersects the sphere of radius $-l^2, l\in\mathbb{R}\backslash\{0\}$. We will now prove the converse, namely that any element in a sphere of radius $-l^2$ in an affine Kac-Moody algebra is contained in some finite dimensional flat.

To this end we identify the two sheets of the intersection of the sphere of radius l with the planes $r_d=\pm r\neq 0$ with two copies of the tame Fréchet space $M\mathfrak{g}$, called $H_{l,r}$. Under this identification the adjoint action restricted to $H_{l,r}$ coincides with the gauge action. Hence, we can use the analytic results of section 2.5.

As consequence of theorem 2.5.2 we get:

Lemma 4.6.1
Let $H_{l,r}:=\{u\in\widehat{M\mathfrak{g}}|\|u\|=l,r_d=r\neq 0\}$. For every $u\in H_{l,r}$ there is an abelian subalgebra $\widehat{\mathfrak{a}}$ containing u. This abelian subalgebra is $\widehat{\mathfrak{a}}=\{\mathfrak{a}+\mathbb{R}c+\mathbb{R}(d+u)\}$ where $\mathfrak{a}=\widehat{\mathfrak{a}}\cap M\mathfrak{g}$ is a finite dimensional abelian subalgebra whose dimension is the rank $\mathfrak{g}$.

Note that this lemma is stated for Kac-Moody algebras of C^∞-loops in [Hei06], in [Pop05] and in [Pop06].

Proof. For the proof one has to check that an element $v \in M\mathfrak{g}$ commutes with an element of the form $\widetilde{u}+d$, iff $[v,u]=v'$. This differential equation has the solution $v(t)=\mathrm{Ad}\phi(t)v_0$ for a solution $\phi(t)$ of the differential equation $\phi'(t)=u(t)\phi(t)$ — cf. [Gue97].
To get closed loops we need the condition $v(2\pi)=\mathrm{Ad}\phi(2\pi)v_0=v_0$. Furthermore for two solutions v and v' we need that $[v,v']=0$; hence we get the condition $[v(0),v'(0)]=0$ – thus for every element $\widetilde{u}+d$ there is an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g} = \mathrm{Lie}(G)$, such that $v(0) \in \mathfrak{a}$ for every element $v(t)$ in the flat containing $\widetilde{u}+d$. □

Lemma 4.6.2
All finite dimensional abelian subalgebras in $\widehat{M\mathfrak{g}}$ are conjugate by elements in $\widehat{MG}$.

Proof. Bogdan Popescu proves a similar result for the real Kac-Moody algebras $\widehat{L^\infty\mathfrak{g}}$ of compact type, constructed with C^∞ loops — cf. [Pop06]). Embedding

$$\widehat{M\mathfrak{g}} \hookrightarrow \widehat{L^\infty\mathfrak{g}}\,,$$

we find that all finite abelian subalgebras in $\widehat{M\mathfrak{g}}$ are conjugate by elements in $\widehat{L^\infty G}$; we have to check that the conjugating element can be chosen to be in $\widehat{MG}$. This is done — following the blueprint of Popescu's proof — in two steps:

1. First, we have to check that a flat $\widehat{\mathfrak{a}}$ is conjugate to a flag $\widehat{\mathfrak{b}}$, such that $\widehat{\mathfrak{b}} = \mathfrak{b} \oplus \mathbb{R}c \oplus \mathbb{R}(d+x)$ with x constant by an element in $\widehat{MG}$. This is a direct consequence of lemma 2.5.2, stating that the gauge action of MG on $M\mathfrak{g}$ is polar.
2. Second, for a given x we have to check that all flats of the form of $\widehat{\mathfrak{b}}$ are conjugate. This relies on the solution of a differential equation being holomorphic iff the equation is holomorphic. □

Definition 4.6.1 (Flat of exponential type)
A flat in $\widehat{\mathfrak{a}} \subset \widehat{MG}$ such that $e \in \widehat{\mathfrak{a}}$ is called of exponential type iff it is in the image of the exponential map.

Theorem 4.6.1
All finite dimensional flats $\widehat{\mathfrak{a}} \subset \widehat{MG}$ of exponential type are conjugate.

Proof. This is a direct consequence of lemma 4.6.2. □

Equivalent theorems hold for the Kac-Moody algebras $\widehat{A_nG}$ and their associated Kac-Moody groups $\widehat{A_nG}$:

Lemma 4.6.3
All finite dimensional abelian subalgebras in $\widehat{A_n\mathfrak{g}}$ are conjugate by elements in $\widehat{A_nG}$.

Theorem 4.6.2
All finite dimensional flats $\widehat{\mathfrak{a}} \subset \widehat{A_nG}$ are conjugate.

As the exponential map $\widehat{A_n \exp} : \widehat{A_n\mathfrak{g}} \longrightarrow \widehat{A_nG}$ is a local diffeomorphism, every flat containing e is of exponential type. Thus we can omit this condition.
Study now the ILB-system: $\lim \widehat{A_nG} = \widehat{MG}$. As $\widehat{MG} \subset \widehat{A_nG}$ for each $n \in \mathbb{N}$, any flat in $\widehat{MG}$ is contained in a flat in $\widehat{A_nG}$.
This observation contains the result:

Theorem 4.6.3
Any two finite flats in $\widehat{MG}$ are conjugate by elements in $\widehat{A_nG}$ for every $n \in \mathbb{N}$.

Flats in Kac-Moody symmetric spaces of type I

Let M be an affine Kac-Moody symmetric space of compact type and let $\widehat{L}(\mathfrak{g},\sigma) = \mathcal{K} \oplus \mathcal{P}$ the decomposition of its associated Kac-Moody algebra into the ± 1-eigenspaces of the involution defining M.

Lemma 4.6.4
All subalgebras of finite type contained in $\mathcal{P}$ are conjugate.

Bogdan Popescu proves a similar theorem for the affine Kac-Moody algebra of C^∞-loops. His proof generalizes to the holomorphic situation — cf. [Pop06]).

As we did for Kac-Moody symmetric spaces of type II we call a flat in M of exponential type iff it is in the image of the exponential map.

Theorem 4.6.4
All flats of finite exponential type are conjugate by elements in the isotropy group of $M_{\mathbb{C}^*}$.

Proof. Via the exponential map we use lemma 4.6.4. □

Analogously to the case of type I we study the inverse limit structure: let M_n be the Kac-Moody symmetric space corresponding to $\widehat{A_n G}$. We have then a series of spaces M_n such that $\lim M_n = M$ and get the result:

Theorem 4.6.5
All flats of finite type in M_n are conjugate by elements in the isotropy group of M_n.

As $M \subset M_n$ for all n, we find

Theorem 4.6.6
All flats of finite type in M are conjugate by elements in the isotropy group of M_n for all n.

Flats in Kac-Moody symmetric spaces of non-compact type

Let M be an affine Kac-Moody symmetric space of non-compact type; we use the canonical identification of $T_eM \simeq \mathcal{P}$. Then there is an exponential map $\widehat{M \exp} : \mathcal{P} \longrightarrow M$. Let M_c denote the affine Kac-Moody symmetric space of compact type that is dual to M. Its tangential space is isomorphic to $\mathcal{P}$. The isotropy representation is the same – thus on the Lie algebra level, abelian subspaces in T_eM correspond bijectively to abelian subspaces in T_eM_c. Especially, the adjoint action is transitive on the space of all flats.

Via the exponential map, this shows that flats of exponential type (defined as in the case of Kac-Moody symmetric spaces of compact type) are conjugate. Investigation of the inverse limit construction yields that all flats are conjugate by elements in the inverse limit groups.

4.7 Some remarks concerning the geometry

We will concentrate on three points which show that the geometry of Kac-Moody symmetric spaces differs from the classical well-known finite dimensional Riemannian geometry:

- Geodesics
- Curvature

- Hermitian symmetric spaces

The most important fact concerning the structure of geodesics is that geodesic completeness fails even locally. There is no chance to get a Hopf-Rinow-type theorem. This is a typical feature of Lorentz geometry.

There are hard theorems showing the impossibility of Lorentz manifolds to be geodesically complete. For details cf. [BE81].

The fact that finite dimensional Lorentz manifolds are not globally geodesically complete leads directly to the fact that Kac-Moody symmetric spaces are not locally geodesically complete (compare the construction for $SL(2,\mathbb{C})$.

We have some remarks about the behaviour of geodesics:

Theorem 4.7.1 (1-parameter subgroups)
A curve in $\widehat{MG}$ through e is a geodesic iff it is a 1-parameter subgroup.

Proof. The proof is as in the finite dimensional case. □

A very important point is the close connection to geodesics in the subjacent Lie groups:

Theorem 4.7.2
$\gamma(t) \subset MG$ is a geodesic iff $\gamma_z(t) \subset G$ is a geodesic for every $z \in \mathbb{C}^*$.

The d-extension twists those geodesics by a function $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$. This gives rise to the following theorem for $\widehat{MG}$:

Theorem 4.7.3 (Geodesic through e)
Let $\pi : \widehat{MG} \longrightarrow MG$ be the bundle projection. Let $\dot{\gamma} := (f, r_c, r_d) \in M\mathfrak{g}$ and $\gamma(t)$ the associated geodesic. Then $(\pi\gamma)_{\varphi(z,r_d)}(t)$ is a geodesic in G.

As Kac-Moody symmetric spaces — by their construction — have flats of dimension at least 3 their sectional curvature has to be unbounded by Kulkarni's theorem 4.1.2. This is also in contrast to the finite dimensional theory.

The last point to mention is the question of existence of hermitian Kac-Moody symmetric spaces. One can show that — as in finite dimensions — the existence of a complex structure leads to a direct $\mathbb{S}^1$-factor in the isometry group. Unfortunately — as with finite dimensional Lorentz symmetric spaces — it is not possible to construct Hermitian Kac-Moody symmetric spaces:

Theorem 4.7.4
There are no hermitian Kac-Moody symmetric spaces.

Proof. For the existence of a complex structure we need an S^1-factor in the isometry group. This is in contradiction to the existence of the time-like vector $v = c + d$. □

The question of the existence of para-hermitian Kac-Moody symmetric spaces is open and deserves further study.

Chapter 5

Universal twin building complexes

5.1 Introduction

Felix Klein's important insight was the existence of a fundamental equivalence between groups and spaces: geometric properties of spaces are reflected by algebraic properties of groups acting on them and vice versa. For a wide class of groups G that appear in geometry, namely groups with a BN-pair, one can construct certain simplicial complexes called buildings that permit actions of G with particularly nice properties. The class of groups with a BN-pair for example contains all semisimple lie groups, loop groups and affine algebraic Kac-Moody groups.

In this chapter we develop the theory of buildings and twin buildings for those three types of groups from a geometric point of view. The loop group approach to affine Kac-Moody groups shows that semisimple Lie groups and Kac-Moody groups are closely related. The same is true for their buildings.

Therefore the plan for this chapter is the following:

In the first section we review the finite dimensional theory of Lie groups, symmetric spaces and their buildings. This is the guideline for the theory we want to construct in the infinite dimensional setting of Kac-Moody groups and Kac-Moody symmetric spaces. We collect some well known foundational material about buildings. Standard sources are the books [Bro89], [Gar97], [Ron89] and the recent reference book [AB08].

We focus on aspects on whose generalization we are interested:

- the algebraic approach defining the building as a simplicial complex,
- the geometric approach introducing metrics on the building and describing the building as a metric space (here the definition of a building via the Weyl distance is of particular interest for our approach),
- the connection to Lie groups via parabolic subgroups and the adjoint representation,
- the connection to symmetric spaces via the isotropy representation,
- the connection to symmetric spaces of the non-compact type via the boundary.

Having described the finite dimensional blueprint we start in section 5.3 the development of the same program for loop groups and affine Kac-Moody groups. Several important differences appear: as the set of Borel subgroups of an algebraic Kac-Moody group consists of two conjugacy classes, buildings have to be replaced by pairs of isomorphic buildings, called twin buildings. To get a building adapted to completed Kac-Moody groups one has to "thicken" the twin building to get universal geometric twin buildings. Completed Kac-Moody groups are Lorentzian manifolds. The two connected components of those universal geometric twin buildings correspond roughly to the two sheets of the intersection of any sphere with certain subspaces, i.e. opposite horospheres in the two sheets of any sphere of negative radius. This correspondence can be realized by immersions of the universal twin building into a sphere of radius $l, l \in \mathbb{R}$ of the affine (Kac-Moody)-Lie algebra. More generally, while G-equivariant immersions of the spherical building into the finite simple Lie algebra are characterized by 1-parameter, we now get a two-parameter family of immersions. For the spherical building at infinity, we find a particularly nice embedding in the hyperplane, separating (and closing) the images of the two parameter families of the two components of the twin building.

A valuable reference describing the algebraic theory in the special case of Kac-Moody algebras of type $\widetilde{A}_n$ is the article [Kra02].

For the classical groups, the spherical building can be described as a complex of flags. A similar description is possible for the affine universal twin building associated to non-twisted affine Kac-Moody algebras of classical type. This approach relies heavily on the analytic theory of loop groups, infinite dimensional Grassmanians and periodic flags developed in Pressley Segals Book [PS86], chapters 6–8. This theory is the content of chapter 6.

The close connection between the root systems of Lie groups and their associated loop- (resp. affine Kac-Moody-) groups is reflected in numerous appearances of objects of the finite dimensional theory in the study of their infinite dimensional analogues.

5.2 Spherical buildings for Lie groups

5.2.1 Combinatorial theory of buildings

As the combinatorial structure of a building is governed by its Coxeter group, we will start this section with some remarks about Coxeter groups and Coxeter complexes:

Definition 5.2.1 (Coxeter group)
A Coxeter group is a group W admitting a presentation of the following type:

$$W := \langle s_1, \dots, s_n, n \in \mathbb{N} | s_i^2 = 1, (s_i s_k)^{m_{ik}} = 1, i, k = 1, \dots, n \text{ and } m_{ik} \in \mathbb{N} \cup \infty \rangle .$$

$K := \{1, \dots, n\}$ is called the indexing set. If $K' \subset K$, we use the notation $W_{K'} \subset W \cong W_K$ for the sub-Coxeter group generated by K'.
The matrix $M = (m_{ik})_{i,k \in K}$ is called the Coxeter matrix.

1. The easiest example of a Coxeter group is $\mathbb{Z}_2$. It admits the presentation:

   $$\mathbb{Z}_2 := \langle s_1 | s_1^2 = 1 \rangle .$$

   It is the only Coxeter group with one generator.

2. Coxeter groups with 2 generators are the dihedral groups D_m. They admit presentations of the form:

   $$D_m := \langle s, t | s^2 = t^2 = 1, (st)^m = 1 \rangle .$$

3. An example which is slightly more complicated is the symmetric group in n letters, denoted S_n. It is the Weyl group of type A_n. As generators one can use the elements $s_{(i,i+1)}, i = 1, \dots, n-1$, which interchanges the i-th and $(i+1)$-th element and fixes all the others.

Thus S_n has the presentation

$$S_n := \left\langle s_{(1,2)}, \ldots, s_{(n-1,n)} \mid \begin{array}{c} s^2_{(i,i+1)} = 1, (s_{(i,i+1)} s_{(i+1,i+2)})^3 = 1 \\ (s_{(i,i+1)} s_{(j,j+1)})^2 = 1 \text{ iff } j \neq i \pm 1 \end{array} \right\rangle .$$

4. More generally Weyl groups of semisimple Lie groups are Coxeter groups.

5. Affine Weyl groups are Coxeter groups.

Definition 5.2.2 (Coxeter system)
A Coxeter system is a pair (W, S) consisting of a Coxeter group W and a set of generators S such that $ord(s) = 2$ for all $s \in S$.

Any group element $w \in W$ is a word in the generators $s_i \in S$. We define the length $l(w)$ of an element $w \in W$ to be the length of the shortest word representing w. The specific word — and thus the length $l(w)$ — depends on the specified set S of generators. Nevertheless many global properties are preserved by a change of generators — cf. [Dav08].

To each Coxeter System (W, S) one can associate a simplicial complex C of dimension $|S| - 1$, called the Coxeter complex of type (W, S).

We start with (W, S): let $\mathcal{S}$ denote the power set of S and define a partial order relation on $\mathcal{S}$ by $S' < S'' \in \mathcal{S}$ iff $(S')^c \subset (S'')^c$ as subsets of S. Here $(S')^c$ denotes the complement of S' in S. Now construct a simplicial complex $\Sigma(S)$ associated to $\mathcal{S}$ by identifying a set $S' \subsetneq \mathcal{S}$ with a simplex $\sigma(S')$ of dimension $|(S')^c| - 1$ and defining the boundary relations of $\Sigma(S)$ via the partial order on $\mathcal{S}$: $\sigma(S')$ is in the boundary of $\sigma(S'')$ iff $S' < S''$. In this simplicial complex $(\emptyset)^c = S$ corresponds to a simplex of maximal dimension and the $|S|$ sets consisting of the single elements $s_i \in S$ correspond to faces.

The simplicial complex $\Sigma(W, S)$ consists of all W-translates of elements in $\Sigma(S)$. Its elements $\sigma(w, S')$ correspond to pairs consisting of an element $w \in W$ and an element $S' \subset \mathcal{S}$ subject to the equivalent relation $\sigma(w_1, S'_1) \cong \sigma(w_1, S'_1)$ iff $S'_1 = S'_2$ and $w_1 \cdot \langle S'_1 \rangle = w_2 \cdot \langle S'_2 \rangle$. It carries a natural W-action that is transitive on simplices of maximal dimension (called chambers). For a simplex $\sigma(w, \{s_i, i \in S'\})$, the stabilizer subgroup is $W_{S'}$ as elements in $W(S')$ stabilize $\{s_i, i \in S'\}$.

One can show that, as a simplicial complex, the Coxeter complex is independent of the choice of S.

The action of elements $s_i \in S$ on $\Sigma(W, S)$ can be interpreted geometrically as reflection at the faces $\sigma(e, s_i)$ of $\sigma(e, \emptyset)$, where e denotes the identity element of W. For further details we refer to [Dav08].

We are now ready for the abstract definition of a building — cf. [Bro89]:

Definition 5.2.3 (Building)
A building $\mathfrak{B}$ is a thick chamber complex Σ together with a set $\mathcal{A}$ of thin chamber complexes $A \in \mathcal{A}$, called apartments, satisfying the following axioms:

1. *For every pair of simplices $x, y \in \Sigma$ there is an apartment $A_{x,y} \subset \mathcal{A}$ containing both of them.*

2. *Let A and A' be apartments, x a simplex and C a chamber such that $\{x, C\} \subset A \cap A'$. Then there is a chamber complex isomorphism $\varphi : A \longrightarrow A'$ fixing x and C pointwise.*

A chamber complex is a simplicial complex satisfying two taming properties: first it is required that every simplex is contained in the boundary of a simplex of maximal dimension and second that for every pair of simplices x and y there exists a sequence of

simplices of maximal dimension $S := \{z_1, \ldots, z_n\}$ such that $x \in z_1$, $y \in z_n$ and $z_i \cap z_{i+1}$ contains a codimension 1 simplex. Simplices of maximal dimension are called chambers; simplices of codimension 1 are called walls. A sequence S is called a gallery connecting x and y.
A chamber complex is called thin if every wall is a face of exactly 2 chambers. It will be called thick if every wall is a face to at least 3 chambers.
A chamber complex map $\varphi : A \longrightarrow A'$ is a map of simplicial complexes, mapping k-simplices onto k-simplices and respecting the face relation. It is a chamber complex isomorphism iff it is bijective.
We give some remarks that clarify the mathematical meaning of the definition:

1. Axiom 2 can be replaced by a seemingly much more powerful version: let A and A' be apartments. Then there is a chamber complex isomorphism, $\varphi : A \longrightarrow A'$ fixing $A \cap A'$ pointwise. Especially all apartments are isomorphic as chamber complexes. Suppose they are isomorphic to an abstract chamber complex A_0. Then one can describe every apartment via a chart map $\varphi_A : A_0 \longrightarrow \Sigma$, such that $A = \varphi(A_0)$. As the axioms for buildings impose important restrictions on the way in which the apartments are pieced together, this reduces many properties of buildings to properties of the much easier complex A_0 (an example for this strategy is the definition of the distance function on $\mathfrak{B}$ – see section 5.2.2).

2. An important object is the group of chamber complex isomorphisms of A_0. It has a subgroup that acts simply transitively on the chambers; this group – called the Weyl group – is automatically a Coxeter group; it is generated by the reflections at the faces of one – arbitrarily chosen – chamber. The Weyl group action shows that every apartment is a Coxeter complex. This fact dramatically reduces the possibilities to construct buildings and is the main source for all classification results.

3. Axiom 1 is some kind of completeness axiom. Compare this with the differential geometric situation: in a complete Riemann manifold, by the Hopf-Rinow theorem every pair of points can be connected by a geodesic. In a sphere all geodesics are isometric as metric spaces – the same is true in a spherical building.

4. There is a striking analogy between the definition of buildings and the characterisation of symmetric spaces as k-flat homogeneous spaces [HPTT94].

A variant we will need later is the cone building:

Remember the definition of the cone complex: let Σ be a simplicial complex and $E_\Sigma = \{e_1, \ldots, e_l\}$ its vertex set. Let $S := \{(e_{i_1}, \ldots, e_{i_n}), (i_1, \ldots, i_n) \subset P(E_\Sigma)\}$ be the set of simplices of Σ. Usually S is much smaller than $P(E_\Sigma)$. Let e_0 be an additional vertex. The Cone complex $\mathfrak{C}_\Sigma$ over Σ is defined to be the simplicial complex with simplices $S_{\mathfrak{C}} := S \cup e_0 \cup S_{e_0}$, where $S_{e_0} := \{(e_0, e_{i_1}, \ldots, e_{i_n}) | (e_{i_1}, \ldots, e_{i_n}) \subset S\}$ with the evident boundary relations.

Definition 5.2.4 (Cone building)
The cone building $\mathfrak{C}_{\mathfrak{B}}$ over a building $\mathfrak{B}$ is the cone complex over $\mathfrak{B}$.

Apartments in the cone building $\mathfrak{C}_{\mathfrak{B}}$ are exactly the cone complexes of the apartments in $\mathfrak{B}$.

Example 5.2.1 (Trees)
Let $n \geq 3$ and define Σ to be the n-Cayley tree T_n, namely a 1-dimensional simplicial

complex without cycles, such that at each vertex there meet exactly n *edges. We identify each edge with the open interval* $(0,1) \subset \mathbb{R}$. *As* $n > 2$ *edges meet at each vertex, this complex is thick. Define* A_0 *to be a* 2*-Cayley tree, that is a path graph;* A_0 *can be identified with* $\mathbb{R}$*, the vertices corresponding to integer points and the edges to open intervals* $(n, n+1), n \in \mathbb{Z}$. A_0 *is thin.*

Now we want to define an apartment structure on T_n*: to this end label the edges by the numbers* $\{1, \dots, n\}$*, such that at each vertex there is exactly one edge labeled with each number. If we start with a vertex* $a \in T_n$ *and take a sequence* $S = \langle s_1, \dots, s_k \rangle$ *of* k *numbers* $\{s_1, \dots, s_k\} \in \{1, \dots, n\}^k$*, then we get a unique vertex* $S * a$ *by following, starting from* a*, the edges labeled with* $s_1, s_2, \dots, s_k$*. If consecutive numbers* s_i *and* s_{i+1} *are identical, an edge will be run twice – once in every direction – which means that we are back at our starting point. Thus we can delete those two numbers. If two consecutive numbers are different, every edge will be run through only once; so from now on we suppose that consecutive numbers are chosen differently.*

As a tree is connected for every pair of vertices a *and* b *we can find a series of edges connecting the two. As there are no nontrivial cycles allowed, there will be a unique shortest connection, characterized by the condition that edges are only run across once. By writing down the labels associated to the edges connecting* a *and* b*, we get a set* $S[a, b]$*; by construction this set satisfies* $S[a, b] * a = b$.

Take now an arbitrary series S_∞ *of elements* $\beta_i \in \{1, \dots, n\}, i \in \mathbb{N}$*, such that two consecutive elements are different. To define an apartment* $A[a, b]$ *containing* a *and* b *we use a series* $S_\infty[a, b] = \{\dots, \beta_i, \beta_{i-1}, \dots, \beta_1, s_1, \dots, s_k, \beta_1, \beta_2, \dots\}$ *where* $s_i \subset S$ *and* $\beta_i \subset S_\infty$*. If* $s_k = \beta_1$ *or* $s_1 = \beta_1$*, skip* β_1 *at this side to make sure that two consecutive elements are different.*

A chart map $\varphi_A : \mathbb{Z} \longrightarrow A[a, b]$ *is a map defined by mapping* 0 *onto* a *and* $i \in \mathbb{Z}$ *onto* $S_\infty(i) * a$*, where* $S_\infty(i)$ *consists of the* 0*-th to* i*-th element of* $S_\infty[a, b]$*. This map maps* $|S[a, b]| \in \mathbb{Z}$ *onto* $\varphi_A(|S[a, b]|) = S(|S[a, b]|) * a = b$.

The apartment system now consists of the apartments defined for all pairs of points a, b *modulo an equivalence relation* $\sim$*, where* $A[a, b] \sim A[a', b']$ *iff* $A[a, b] \subset A[a', b']$ *and* $A[a', b'] \subset A[a, b]$*. It is elementary to verify the axioms.*

It is important to note that this is NOT a maximal apartment system. Moreover for different choices of S_∞*, the resulting apartment system will look differently. This is a countable apartment system, that is dense in the maximal apartment system; its relationship with the maximal apartment system is similar to the one between* $\mathbb{Q}$ *and* $\mathbb{R}$*. One could define a complete apartment system via a completion: define a half ray to be an isometric embedding of* $\mathbb{R}^+$ *into* T_n*; define two half rays* R *and* R' *to be equivalent, if there is a half ray* R'' *which is contained in both of them. Then the set of equivalence classes of half rays defines a boundary for* T_n *which is similar to the boundary of a Riemann symmetric spaces of non-compact type. The complete apartment system is isomorphic to the set of all pairs of points in the boundary.*

5.2.2 Metric theory of buildings

We have introduced buildings as special simplicial complexes. This approach hides the metric structure of buildings. To describe a metric theory of buildings, we will introduce two different but closely related distance functions:

1. One can define an $\mathbb{R}$-valued distance function by the definition of metric structures on the apartments. For further details cf. [AB08], chapter 12.

2. One can define a W-valued distance function that takes into account the combinatorial structure of the building (called Weyl distance).

We concentrate on the second distance function, the Weyl distance:

Lemma 5.2.1
Let $\mathcal{C}_{\mathfrak{B}}$ denote the set of chambers of $\mathfrak{B}$ and $C, C', D \in \mathcal{C}_{\mathfrak{B}}$.

There is a function $\delta_W : \mathcal{C}_{\mathfrak{B}} \times \mathcal{C}_{\mathfrak{B}} \longrightarrow W$ with the following properties:

1. $\delta_W(C, D) = e$ *iff* $C = D$,
2. $\delta_W(D, C) = \delta_W(C, D)^{-1}$,
3. *if* $\delta_W(C', C) = s$ *and* $\delta_W(C, D) = w$, *then* $\delta_W(C', D) \in \{sw, w\}$. *If furthermore* $l(sw) = l(w) + 1$, *then* $\delta_W(C', D) = sw$,
4. *if* $\delta_W(C, D) = w$, *then for any* $s \in S$ *there is a chamber* C' *such that* $\delta_W(C, C') = s$ *and* $\delta_W(C', D) = sw$. *If* $l(sw) = l(w) - 1$ *then there is a unique chamber* C' *with this property.*

For a proof cf. [AB08], section 4.8.
Geometrically this function works like that: for two chambers a and b in $\mathcal{C}_{\mathfrak{B}}$ we choose an apartment $A_{a,b}$ containing them both. By transitivity of the Weyl group action on $A_{a,b}$ there is an element $w \in W$ such that $w \cdot a = b$. One can check that this element is independent of the apartment $A_{a,b}$ chosen. Some combinatorial arguments are needed to show that this is a metric on the whole of the building.

Definition 5.2.5 (Weyl distance)
This distance function $\delta_W : \mathcal{C}_{\mathfrak{B}} \times \mathcal{C}_{\mathfrak{B}} \longrightarrow W$ is called the Weyl distance.

It is important to note that it is possible to define a building via its metric. We quote a second definition for a building from the monograph [AB08], chapter 5.

Definition 5.2.6 (W-metric building)
A building of type (W, S) is a pair $(\mathcal{C}, \delta)$ consisting of a nonempty set $\mathcal{C}$ whose elements are called chambers together with a map $\delta : \mathcal{C} \times \mathcal{C} \longrightarrow W$, called the Weyl distance function, such that for all $C, D \in \mathcal{C}$ the following conditions hold:

1. $\delta(C, D) = 1$ *iff* $C = D$.
2. *If* $\delta(C, D) = w$ *and* $C' \in \mathcal{C}$ *satisfies* $\delta(C', C) = s \in S$ *then* $\delta(C', D) = sw$ *or* w. *If in addition* $l(sw) = l(w) + 1$ *then* $\delta(C', D) = sw$.
3. *If* $\delta(C, D) = w$ *then for any* $s \in S$ *there is a chamber* $C' \subset \mathcal{C}$, *such that* $\delta(C', C) = s$ *and* $\delta(C', D) = sw$.

This definition coincides with the classical definition of a building as a simplicial complex (for a proof cf. [AB08]).

It is a very convenient description to describe the connection between buildings and Lie groups. Compare remark 5.2.1.

We will use a variant of this definition to define a universal geometric twin building.

5.2.3 Buildings and Lie groups

Let G be a simple compact Lie group, $G_{\mathbb{C}}$ its complexification, hence G (resp. $G_{\mathbb{C}}$) is of type X_l, where $X_l \in \{A_n, B_{n,n\geq 2}, C_{n,n\geq 3}, D_{n,n\geq 4}, E_6, E_7, E_8, F_4, G_2\}$.

In this subsection, we describe how to construct the spherical building associated to G (resp. $G_{\mathbb{C}}$) in terms of properties of the Lie group. There are mainly two different ways to attack this problem:

1. One can describe a building of type X_l using the orbits of the adjoint action of the compact Lie group G of type X_l on its Lie algebra $\mathfrak{g}$. The set of chambers in this setting is naturally identified with the points of a principal orbit.
2. One can understand a building as the simplicial complex associated to the set of parabolic subgroups of a complex simple Lie group. (For the classical groups, this is equivalent to a flag complex construction – see chapter 6).

Buildings and the Adjoint representation of a compact simple Lie group

Let G be a real simple Lie group of the compact type and $\mathfrak{g}$ its Lie algebra. We study the action:

$$\varphi : G \times G \longrightarrow G, \quad (g,h) \mapsto ghg^{-1}$$

We want first to restrict the domain of definition to a fundamental domain for this action. As G is covered by conjugate maximal tori, the map

$$\varphi : G \times T \longrightarrow G, \quad (g,t) \mapsto gtg^{-1}$$

is surjective for every fixed torus $T \subset G$.
As T is abelian, conjugation with elements in T is trivial; hence we get a well defined action

$$\varphi : G/T \times T \longrightarrow G, \quad (gT,t) \mapsto gtg^{-1}\,.$$

As the Weyl group $W := N(T)/T \subset G/T$ consists of automorphisms of T, we may further restrict T to a fundamental domain of the action of the Weyl group, denoted $\overline{\Delta}_G$. Hence, the map

$$\varphi : G/T \times \overline{\Delta}_G \longrightarrow G, \quad (gT,t) \mapsto gtg^{-1}$$

is again surjective.

In the theory of compact Lie groups it is shown that for regular elements, i.e. all those in the interior Δ_G of $\overline{\Delta}_G \subset T$, this map is injective — cf. [BD98]. For all elements in the boundary there exist nontrivial stabilizer subgroups.

We can now lift this construction on the Lie algebra level.
We get the Adjoint action

$$\mathrm{Ad}(G):\ \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (g,X) \mapsto gXg^{-1}\,.$$

As in the group case this action may be reduced first to a maximal torus $\mathfrak{t} \subset \mathfrak{g}$ and then further to a fixed fundamental domain $\overline{\Delta}_{\mathfrak{g}} \subset \mathfrak{t}$ for the Weyl group action on $\mathfrak{t}$:

$$\mathrm{Ad}(G/T) : \overline{\Delta}_{\mathfrak{g}} \longrightarrow \mathfrak{g}, \quad (gT,X) \mapsto gXg^{-1}\,.$$

$\overline{\Delta}_{\mathfrak{g}}$ is a closed cone with rank(G)-many faces. The Weyl group is generated by reflections s_i at theses faces. For an element $X \subset \overline{\Delta}_{\mathfrak{g}}$ in the intersection of the faces $\Delta_{i_1}, \ldots, \Delta_{i_k}, i \in I$ the stabilizer is $W_I = \langle S_I \rangle$.

So if X is in the interior $\Delta_{\mathfrak{g}}$, the map will be injective, if it is on the boundary there are nontrivial stabilizer subgroups generated by T and the Weyl group elements that fix X.

As the adjoint action preserves the Cartan-Killing form $B(X, Y)$, it is an isometry with respect to the induced metric $\langle X, Y\rangle = -B(X, Y)$. Hence, it preserves every sphere $\mathbb{S}^n_R \subset \mathfrak{g}$ of fixed radius R. Hence, we can define an Ad-invariant subspace $\mathfrak{g}_R := \mathfrak{g} \cap \mathbb{S}^m_R \subset \mathfrak{g}$. A fundamental domain for the Adjoint action on $\mathfrak{g}_R$ is the set $\overline{\Delta}_{(\mathfrak{g},R)} := \overline{\Delta}_{\mathfrak{g}} \cap \mathbb{S}^n_R$.
Thus we get a surjective map:

$$\mathrm{Ad}(G/T) : \overline{\Delta}_{(\mathfrak{g},R)} \mapsto \mathfrak{g}_R, \qquad (g, X) \mapsto gXg^{-1}.$$

Now construct a simplicial complex like that: for each element $X \in \overline{\Delta}_{(\mathfrak{g},R)}$ there is a subgroup $W_X \subset W$ stabilizing X. If X is in the interior, then $W_X = \{e\}$. If it is in the boundary, W_X it is the group $W_I \subset W$ generated by the reflections s_i that fix X. Thus we can replace $\overline{\Delta}_{(\mathfrak{g},R)}$ by the complex $\Sigma(S)$. It has exactly $n := \operatorname{rank} \mathfrak{g}$ faces $f_1, \ldots, f_n$. Let $S = \{s_i\}, i = 1, \ldots, n$ where s_i denotes the reflection at f_i. (W, S) is a Coxeter system. The cells in $\overline{\Delta}_{(\mathfrak{g},R)}$ correspond bijectively to subsets $S' \subset S$.

The W-translates $\Sigma(S)$ tessalate $\mathfrak{t}_R := \mathfrak{t} \cap \mathbb{S}^n_R$; thus the tessalation of $\mathfrak{t}_R$ corresponds to the thin Coxeter complex $\Sigma(W, S)$. The G/T-translates tessalate the whole sphere $\mathfrak{g}_R$.

This simplicial complex is a building:

$$\mathfrak{B}_G := (G/T \times \overline{\Delta})/\sim,$$

where $\sim$ denotes the equivalence relation defined by the stabilizer subgroups.

The construction we have performed defines in a natural way an embedding of the building in every sphere $\mathbb{S}_R$, hence also in the unit sphere. It can be understood also as the orbit under the adjoint action of the outer wall of the standard alcove — cf. also [Mit88].

We define the filled building $\mathfrak{B}_f$ to be the orbit of the standard alcove under the adjoint action. It can be interpreted as the union of the embeddings of the building in the spheres of radius r for $r \in [0; R]$ – but with a new cell structure. It is easy to see that $\mathfrak{B}_{\mathfrak{f}} = \mathfrak{C}$, where $\mathfrak{C}$ is the cone building. To describe the embedding of $\mathfrak{C}$ explicitly, map $\{e_0\} \mapsto \{0\}$. So $\mathfrak{B}_{\mathfrak{f}}$ is simply connected.

This shows that $h \in G$ acts on the building $\mathfrak{B}_G$ (and on $\mathfrak{B}_f$ via left multiplication:

$$G: \ \mathfrak{B}_G \longrightarrow \mathfrak{B}_G, \qquad h \cdot (gT, X) = (hgT, X)\,.$$

Let $i : \mathfrak{B}_G \longrightarrow \mathfrak{g}_R$ denote the described embedding. Then the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{B}_G & \xrightarrow{G} & \mathfrak{B}_G \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathfrak{g}_R & \xrightarrow{\mathrm{Ad}(G)} & \mathfrak{g}_R \end{array}$$

In this model we see all the chambers of the building, but we do not see all apartments. Let $\mathcal{A}_{\mathfrak{g}}$ denote the set of apartments corresponding to maximal abelian subalgebras $\mathfrak{t}$ in $\mathfrak{g}$. Our discussion shows the following:

Lemma 5.2.2 (Properties of $\mathcal{A}_{\mathfrak{g}}$)

- *For every chamber $c \in \mathfrak{B}$, there is exactly one apartment $A_c \in \mathcal{A}_{\mathfrak{g}}$ containing c.*

- *The action of G is transitive on $\mathcal{A}_{\mathfrak{g}}$ and the isotropy subgroup of any $A \in \mathcal{A}_{\mathfrak{g}}$ is isomorphic to W.*

- *The action of G is transitive on the set of chambers.*

We know from the definition of the building that we need for every pair of chambers $\{x, y\} \in \mathfrak{B}$ an apartment $A_{x,y}$, such that $\{x, y\} \in A_{x,y}$. This condition is clearly not satisfied by the apartments in $\mathcal{A}_{\mathfrak{g}}$. The missing apartments appear only in the study of the complexified Lie group.

Buildings and complex Lie groups: the structure of parabolic subgroups

As explained in the last section, maximal abelian subgroups in a compact Lie group correspond to apartments in the building. Hence, the structure of abelian subgroups is reflected in the geometry of the building. Let G be a group and let $K(G)$ denote the commutator subgroup of G. Then G is abelian iff $K(G) = 0$.
A generalization of abelian groups are solvable groups. A group is called solvable iff $K^n(G) = \{1\}$ for some $n \in \mathbb{N}$.

Definition 5.2.7 (Borel subgroup)
Let $G_{\mathbb{C}}$ be a complex simple Lie group. A Borel subgroup B is a maximal solvable subgroup.

Example 5.2.2
Let $G_{\mathbb{C}} = SL(n, \mathbb{C}) \simeq \{A \in Mat^{n\times n}(\mathbb{C}) | det(A) = 1\}$. The standard Borel subgroup B_0 is the group of upper triangular matrices in $SL(n, \mathbb{C})$. Any other Borel subgroup is conjugate to B_0 in $SL(n, \mathbb{C})$.

A slight generalization of Borel subgroups are

Definition 5.2.8 (Parabolic subgroup)
A subgroup $P \subset G_{\mathbb{C}}$ is called parabolic iff it contains a Borel subgroup.

Example 5.2.3
Let $G_{\mathbb{C}} = SL(n, \mathbb{C})$. A standard parabolic subgroup P_0^i is an upper block-triangular matrix, that is an upper triangular matrix having blocks on its diagonal. There are different types of parabolic subgroups. Any other parabolic subgroup of the same type is conjugate to P_0^i in $SL(n, \mathbb{C})$.

Similar explicit descriptions exist for all classical Lie groups — cf. [Bro89].
Borel subgroups and Weyl groups are the fundamental structural objects of simple Lie groups. The BN-pair structure formalises the way those groups are assembled to yield a simple Lie group:

Definition 5.2.9 (BN-pair)
Let $G_{\mathbb{C}}$ be a complex simple Lie group. A set (B, N, W, S) is a BN-pair for G iff:

1. *$G = \langle B, N\rangle$. Moreover $T = B \cap N \lhd N$ and $W = N/T$.*
2. *$s^2 = 1\ \forall s \in \mathbb{S}$ and $W = \langle S\rangle$ and (W, S) is a Coxeter system.*
3. *Let $C(w) := BwB$. Then $C(s)C(w) \subseteq C(w) \cup C(sw)\ \forall s \in S$ and $w \in W$.*
4. *$\forall s \in S : sBs \not\subseteq B$.*

Theorem 5.2.1
Every complex simple Lie group has a BN-pair structure.

Proof. cf. [Bum04], section 30. □

Groups with a BN-pair have a Bruhat decomposition:

Theorem 5.2.2 (Bruhat decomposition)
Let G be a group with a BN-pair structure and Weyl group W. Then

$$G = \coprod_{w \in W} C(w)\,.$$

Proof. cf. [Bum04], section 30. □

Sometimes the refined Bruhat decomposition is useful:
Let S' and S'' be subsets of S and let $W' = \langle S' \rangle$ and $W'' = \langle S'' \rangle$ and $P' := BW'B$ (resp. $P'' = BW''B$).

Theorem 5.2.3 (Refined Bruhat decomposition)
Let G be a group with a BN-pair structure and Weyl group W.

$$G = \coprod_{w \in W' \backslash W / W''} P'wP''$$

Proof. cf. the paper [Mit88] or the book [Bum04], section 30. □

The connection between the building as we introduced it for compact simple Lie groups and the complexified groups is provided by the Iwasawa decomposition: it states that a complex semisimple Lie group $G_{\mathbb{C}}$ may be decomposed as $G_{\mathbb{C}} = G \times A \times N$, where G is a compact real form (unique up to conjugation), A is a diagonal matrix with positive entries and N is nilpotent. Then $B := T \times A \times N$ is a maximal solvable subgroup, hence a Borel subgroup. As $G/T := G_{\mathbb{C}}/B$ we find:

$$\mathfrak{B}_G := (G_{\mathbb{C}}/B \times \overline{\Delta})/\sim$$

$G_{\mathbb{C}}$ acts from the left on the cosets $G_{\mathbb{C}}/B$ and thus on the building $\mathfrak{B}_G = (G_{\mathbb{C}}/B \times \overline{\Delta})/\sim$.
The embedding $\mathfrak{B}_G \hookrightarrow \mathfrak{g}$ is $G_{\mathbb{R}}$-equivariant but not $G_{\mathbb{C}}$-equivariant.
This approach shows another possible way to understand buildings:

Theorem 5.2.4
Let $G_{\mathbb{C}}$ be a simple Lie group of type X_l and $\mathfrak{B}$ the building of the same type.

1. *The chambers of $\mathfrak{B}$ correspond to the Borel subgroups of $G_{\mathbb{C}}$, simplices of $\mathfrak{B}$ correspond to parabolic subgroups. The correspondence can be realized by associating to every simplex $c \in \mathfrak{B}$ its stabilizer subgroup P_c in $G_{\mathbb{C}}$ with respect to the left action, described above.*

2. *A simplex c is in the boundary of a cell d iff $P_d \subset P_c$.*

From this description one can recover directly the W-metric description of a building:

Remark 5.2.1
Starting with a complex simple Lie group $G_{\mathbb{C}}$ with a Borel subgroup $B \subset G_{\mathbb{C}}$ one can define the building associated to $G_{\mathbb{C}}$ using the metric approach $\mathcal{C} := G_{\mathbb{C}}/B$, and $\delta : \mathcal{C} \times \mathcal{C} \longrightarrow W$ is defined by $\delta(gB, fB) = w$ iff gf^{-1} is in the $C(w)$ class of the Bruhat decomposition of $G_{\mathbb{C}}$.

If we slightly change the notation and define $G_{\mathbb{C}}$ to be a parabolic subgroup as well then the resulting simplicial complex is the cone building $\mathfrak{C}$.

Remark 5.2.2
It is an important observation that in a simple complex Lie group all Borel subgroups are conjugate. In building theoretical language this translates to the fact that the building is a connected simplicial complex.

5.2.4 Buildings and symmetric spaces

Let M be an irreducible symmetric space of type I, II, III or IV and G the identity component of its isometry group and $K \subset G$ the isotropy subgroup of a point $x \in M$ — cf. [Hel01].

Definition 5.2.10
The isotropy representation of K on T_xM is the representation of K on T_xM induced by the action of K on M.

We call a representation of a Lie group K on a vector space V polar iff there is a subspace $\Sigma \subset V$, called a section, which meets every K-orbit orthogonally.

Theorem 5.2.5 (Isotropy representation and polar sections)

1. *The isotropy representation of a symmetric space is polar.*
2. *Dadok's Theorem: every polar representation on $\mathbb{R}^n$ is orbit equivalent to the isotropy representation of a symmetric space.*

Proof. cf. [BCO03], chapter 3. □

The isotropy representation of a symmetric space of compact type and its dual symmetric space of non-compact type are isomorphic.

Via the isotropy representation one can associate a building to a symmetric space in the same way as we did it for a Lie group of compact type via the Adjoint action. This complex is a spherical building.

We omit the details — cf. [Hel01], [Mit88], [Ebe96] — but describe the two main cases.

1. A (simply connected) simple symmetric space of "type II" (resp. "type IV") is isomorphic to G (resp. $G_{\mathbb{C}}/G$) (here G is a compact simple Lie group); the identity component of the isometry group is isomorphic to $G \times G$ (resp. $G_{\mathbb{C}}$); their Lie algebras are isomorphic to $\mathfrak{g} \times \mathfrak{g}$ resp. to $\mathfrak{g}_{\mathbb{C}}$; the isotropy group of the identity element is in both cases isomorphic to G. Thus the tangential space at the identity element is isomorphic to $\mathfrak{g}$ (resp. $i\mathfrak{g}$). The isotropy representation of the symmetric space is just the Adjoint representation of the Lie group G on the Lie algebra $\mathfrak{g}$. Hence, the embedding of the building into $\mathfrak{g}$ defined via the isotropy representation is just the embedding we investigated in the last section. Let $G_{\mathbb{C}}$ be of type X_l, then the spherical building associated to G is of type X_l as well.

2. A (simply connected) simple symmetric space of "type I" is isomorphic to a quotient space G/K, where G is supposed to be a connected Lie group of compact type and $\mathrm{Fix}(\sigma)_0 \subset K \subset \mathrm{Fix}(\sigma)$ is the fixed point group of some involution σ of G. The Lie algebra $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$, where $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{p} \simeq T_{eK}(G/K)$. Here $\mathfrak{k}$ (resp. $\mathfrak{p}$) corresponds to the $+1$-eigenspace (resp. -1-eigenspace) of the induced involution $d\sigma : \mathfrak{g} \to \mathfrak{g}$.

   The real form of $\mathfrak{g}_{\mathbb{C}}$ which is dual to $(\mathfrak{g}, \sigma)$ is the Lie algebra $\mathfrak{g}_D = \mathfrak{k} + i\mathfrak{p}$. Via the exponential map it corresponds to a non compact real form $G_D \subset G_{\mathbb{C}}$; the maximal compact subgroup of G_D is isomorphic to K. The dual symmetric space M_D is isomorphic to the quotient G_D/K. Those spaces are called "type III".

The isotropy group of the identity element is in both cases isomorphic to K. Furthermore, the tangential algebras are isomorphic to $\mathfrak{p}$; thus the isotropy representation is in both cases the conjugation action $K : \mathfrak{p} \longrightarrow \mathfrak{p}$. The type of the building depends on K. As K may be not simple there are several technical problems.

Via the exponential map this construction also defines an embedding of the building into the symmetric space:

1. In the case of symmetric spaces of compact type, one can identify the building with the tangent cut locus — cf. [Mit88].

2. In the case of non-compact symmetric spaces M, the exponential map defines an isomorphism
$$\exp : T_pM \longrightarrow M_D\,.$$
Via this isomorphism we get for any embedding of the building in T_pM a realization of the building in the symmetric space. Thus the building also immerses into the image of any sphere around p of the dual symmetric space. The condition $[X, Y] = 0$ in $T_pM \simeq \mathfrak{p}$ is equivalent to thevanishing of the sectional curvature $K(X, Y)$, which means that X and Y span a flat subspace. Thus abelian subalgebras in $\mathfrak{p}$ correspond to flats in M_D.

   As with all other immersions, based on the study of a restricted symmetry group (i.e. G in place of $G_{\mathbb{C}}$) we do see every cell of the building, but we do not see every apartment. As we saw in the adjoint representation of a compact Lie group on its Lie algebra only apartments corresponding to maximal abelian subalgebras, we now only see the apartments corresponding to flats passing through the point p.

Suppose M_D to be a symmetric space of the non-compact type. The trivial fact that the apartments one sees in this description correspond bijectively to flats passing through p is the key to finding all other apartments: if p runs through all points in M, we get for every point the isotropy group K_p. Every apartment of the associated building appears for some points. More precisely, we have:

Theorem 5.2.6
Let M be a symmetric space and $\mathfrak{B}$ its building. For every apartment A in $\mathfrak{B}$ there is a flat submanifold M in X such that A appears in the unit sphere of T_mX for all points $m \in M$.

Example 5.2.4
Study the space $X = Sl(n, \mathbb{C})/SU(n, \mathbb{C})$. As is well known, this space can be identified with the space of hermitian scalar products on $V^n = \mathbb{C}^n$. The associated building is the flag complex of flags in V^n. Apartments correspond to frames $\{U\} = \{U_1, \dots, U_n\}$ such that $U_i \simeq \mathbb{C}$ — cf. [Bro89]. Let $x \in X$ be a point. The apartments corresponding to flats containing x correspond to frames that are orthogonal with respect to the corresponding scalar product g_x.

To get a realization of the building that includes all chambers at the same time one has to pass to the boundary. Let M be a symmetric space of non-compact type. Then M admits a compactification by the set of equivalence classes of geodesic rays modulo an equivalent relation defined by the condition: two rays γ_1 and γ_2 are identified iff the distance between γ_1 and γ_2 is finite. This boundary is topologically a sphere, sometimes called the sphere at infinity.

It is well known (cf. for example [BH89], [Ji06] or [Ebe96]) that the building associated to a symmetric space of non-compact type can be canonically identified with its sphere at infinity. The collection of all flats now forms a collection of apartments, satisfying all the building axioms.

5.3 Buildings for Loop groups and Kac-Moody groups

5.3.1 Some remarks and notations

We saw that one can associate to a finite dimensional Lie group in a natural way a spherical building that reflects the symmetry properties of the group. There is a unique way to associate a building to a Lie group. This is different for loop groups and Kac-Moody groups $L(G,\sigma)$ and $\widehat{L}(G,\sigma)$. We use again this notation to describe in a unified way constructions that we need for different types of regularity conditions. Here we have two different possibilities to construct buildings, one using the spherical Weyl group W of G and one using the affine extension W_{aff} of W.

The fact that there exist two types of buildings leads to the necessity for a number of notations. We list for later reference the most important notations, without giving a detailed definition:

Let $G_{\mathbb{C}}$ denote a finite-dimensional complex simple Lie group, G its compact real form.

1. $L(\mathfrak{g},\sigma)$, $\widehat{L}(\mathfrak{g},\sigma)$, $L(G,\sigma)$ and $\widehat{L}(G,\sigma)$ as always,
2. W is the spherical Weyl group of G, generated by a set $S=\langle s_i, i\in I\rangle$,
3. W_{aff} is the affine Weyl group associated to W,
4. a Borel subgroup B is always a Borel subgroup in $G_{\mathbb{C}}$,
5. a parabolic subgroup P is a subgroup of G, containing a Borel subgroup,
6. an Iwahori subgroup B_A is a subgroup of $L(G,\sigma)$, that is the loop group analogue of a Borel subgroup,
7. a parahori subgroup P_A of $L(G,\sigma)$ is a subgroup that contains an Iwahori subgroup (analogue of a parabolic subgroup),
8. an affine Borel subgroup $\widehat{B}_A$ of $\widehat{L}(G,\sigma)$ is a maximal solvable subgroup of a Kac-Moody group,
9. an affine parabolic subgroup $\widehat{P}_A$ is a subgroup of $\widehat{L}(G,\sigma)$ that contains an affine Borel subgroup.

5.3.2 Algebraic theory: twin BN-pairs and twin buildings

In this section we present well-known results about the algebraic twin building associated to algebraic Kac-Moody algebras. Similar to finite dimensional Lie groups, also Kac-Moody groups possess a BN-pair structure and a Bruhat decomposition. Nevertheless there are some striking differences:

1. The associated Weyl group is infinite.

2. There are two conjugacy classes of Borel subgroups.
3. The associated building is contractible and CAT(0). Thus it has a boundary which is itself a building.

The material of this section is standard. For further details we refer to the monographs [AB08], [Gar97] and [Kum02].

Definition 5.3.1 (Iwahori group)
An Iwahori group of a group $LG_{\mathbb{C}}$ is a maximal countably solvable subgroup.

Definition 5.3.2 (normal Iwahori groups for $LG_{\mathbb{C}}$)
Let $B = B^+$ denote the Borel subgroup of upper triangular matrices in $G_{\mathbb{C}}$ and B^- its opposite Borel subgroup. The positive (resp. negative) standard Iwahori subgroups are defined to be:

$$\begin{aligned} B_A^+ &= \{f : \mathbb{C} \longrightarrow G_{\mathbb{C}} \mid f(1) \subset B^-\} \\ B_A^- &= \{f : \widehat{\mathbb{C}}\backslash\{0\} \longrightarrow G_{\mathbb{C}} \mid f(1) \subset B^+\} \end{aligned}$$

Definition 5.3.3
A parahori group of $LG_{\mathbb{C}}$ is a subgroup $P_A \subset LG_{\mathbb{C}}$ that contains an Iwahori subgroup.

Any Iwahori group is a parahori group.

Definition 5.3.4 (affine Borel group)
An affine Borel subgroup is a maximal solvable subgroup of a Kac-Moody group $\widehat{L_{alg}G}_{\mathbb{C}}$.

Definition 5.3.5 (affine parabolic group)
An affine parabolic group is a subgroup of a Kac-Moody group $\widehat{L_{alg}G}^{\sigma}$ that contains an affine Borel subgroup.

An affine Borel group is an affine parabolic group.

Lemma 5.3.1 (Affine Borel groups and Iwahori groups)
Affine Borel subgroups of $\widehat{L_{alg}G_{\mathbb{C}}}$ are extensions of the Iwahori subgroups of $L_{alg}G_{\mathbb{C}}$

$$\widehat{B_A} = \widetilde{B_A} \ltimes \mathbb{C}^*\,,$$

where $\widetilde{B_A}$ denotes the central extension of B_A inherited by the central extension of $L_{alg}G_{\mathbb{C}^*}$.

Proof. cf. [Kum02]. □

Thus to simplify notation we use B_A to design both the parahori group $B_A \subset L_{alg}G$ and the affine parabolic subgroup $\widehat{B}_A \subset \widehat{L_{alg}G}$.

Lemma 5.3.2
B_A^+ and B_A^- (and thus $\widehat{B_A^+}$ and $\widehat{B_A^-}$) are not conjugate in $L_{alg}G$ (resp. $\widehat{L_{alg}G}$). Every Borel subgroup is conjugate either to B_A^+ or B_A^-.

Proof. cf. [Kum02]. □

Definition 5.3.6
We call an affine parabolic (resp. parahori) group positive (resp. negative) iff it is conjugate to the standard positive (resp. negative) affine parabolic (resp. parahori) group.

This double structure makes it natural to introduce twin BN-pairs — cf. [Tit92] and [MGH07].

Definition 5.3.7 (Twin B_AN-pair)
Let $\widehat{L_{alg}G_{\mathbb{C}}}$ be a Kac-Moody group. Data $(\widehat{B}_A^+, \widehat{B}_A^-, N, W, S)$ is a twin BN-pair for $\widehat{L_{alg}G_{\mathbb{C}}}$ iff:

1. $(\widehat{B}_A^+, N, W, S)$ *is a BN-pair for $\widehat{L_{alg}G_{\mathbb{C}}}$ (called B_A^+N),*
2. $(\widehat{B}_A^-, N, W, S)$ *is a BN-pair for $\widehat{L_{alg}G_{\mathbb{C}}}$ (called B_A^-N),*
3. *B^+N and B^-N are compatible, i.e.:*
   (a) *If $l(ws) < l(w)$ then $B^\epsilon wB^{-\epsilon}sB^{-\epsilon} = B^\epsilon wsB^{-\epsilon}$ for $\epsilon = \pm$, $w \in W, s \in S$,*
   (b) *$B^+s \cap B^- = \emptyset\, \forall s \in S$.*

The existence of the twin BN-pairs yields Bruhat decompositions similar to the finite dimensional case:

Theorem 5.3.1 (Bruhat decomposition)
Let $\widehat{L_{alg}G}$ be an affine algebraic Kac-Moody group with affine Weyl group W_{aff}. Let furthermore $\widehat{B}_A^\pm$ denote a positive (resp. negative) Borel group. There are decompositions

$$\widehat{L_{alg}G} = \coprod_{w \in W_{aff}} \widehat{B}^+ w \widehat{B}^+ = \coprod_{w \in W_{aff}} \widehat{B}^- w \widehat{B}^- .$$

Theorem 5.3.2 (Bruhat twin decomposition)
Let $\widehat{L_{alg}G}$ be an affine algebraic Kac-Moody group with affine Weyl group W_{aff}. Let furthermore $\widehat{B}_A^\pm$ denote a positive and its opposite negative Borel group. There are decompositions

$$\widehat{L_{alg}G} = \coprod_{w \in W_{aff}} B^\epsilon w B^{-\epsilon} \;\epsilon \in \{\pm\} .$$

Note, that the Bruhat decompositions and the Bruhat-Twin decompositions are defined on the whole group $\widehat{L_{alg}G}$. This translates into the fact that any two chambers in $\mathfrak{B}^+$ resp. $\mathfrak{B}^-$ have a well-defined Weyl distance and a well-defined Weyl codistance (compare definition 5.3.11). Similar results hold for $L_{alg}G$.

Definition 5.3.8 (BN-flip)
An involution φ of a Kac-Moody group is called a BN-flip iff

1. $\varphi^2 = 1$,
2. $\varphi(B_A^+) = B_A^-$,
3. *φ centralizes the Weyl group.*

A BN-flip interchanges the two BN-pairs. The existence of a BN-flip is a sign of symmetry of the group structure, which can be lost for generalized Tits systems.

The same is true for affine parabolic subgroups. The set of affine parabolic subgroups breaks up into two conjugacy classes. The first one consists of affine parabolic subgroups containing a conjugate of $\widehat{B}_A^+$, the second one of those containing a conjugate of $\widehat{B}_A^-$. The two sets of parabolic subgroups admit a partial order relation exactly as in the finite dimensional case. So one can associate Tits buildings to them. As their Weyl groups are infinite, they are contractible — cf. [Mit88]. The apartments are infinite Coxeter complexes.

A further investigation of those two buildings shows that a major obstacle is the absence of opposite chambers (recall, that two chambers are called opposite if their Weyl distance is maximal). Opposite chambers appear at some prominent places. Probably the most important one is the following theorem:

Theorem 5.3.3
In a spherical building, apartments are exactly the convex hulls of a pair of opposite chambers.

Proof. cf. [AB08]. □

This theorem implies the important corollary:

Corollary 5.3.1
The apartmentsystem in a spherical building is unique.

This is in striking contrast to the situation of non-spherical buildings: even in the most basic case of the tree T_m, we get an uncountable family of different apartment systems (compare example 5.2.1).

So there is a need for a version of the oppositeness-relation also for affine buildings, which should then lead to a generalization of theorem 5.3.3. It is clear that an opposite chamber cannot be in the same building, as it would have a Weyl distance of maximal length. Thus the solution lies in a twinning of the two buildings associated to the two BN-pairs. The resulting object, called a twin building, behaves in many respects like a spherical building.

It can be described most conveniently using a codistance function.

Definition 5.3.9 (Twin building)
A twin building consists of a pair of buildings $\mathfrak{B}^+$ and $\mathfrak{B}^-$ together with a codistance function $\delta^*_W : \mathfrak{B}^+ \times \mathfrak{B}^- \longrightarrow W$ which is subject to the following conditions: Let $X \in \mathfrak{B}^+$ and $Y, Z \in \mathfrak{B}^-$,

1. $\delta^*_W(X,Y) = \delta^*_W(Y,X)^{-1}$,
2. $\delta^*_W(X,Y) = w, \delta_W(Y,Z) = s \in S$ *and* $l(ws) = l(w) - 1$, *then* $\delta^*_W(X,Z) = ws$,
3. $\delta^*_W(X,Y) = w, s \in S$. *Then* $\exists Z \in \mathfrak{B}^-$ *such that* $\delta^*_W(Y,Z) = s$ *and* $\delta^*_W(X,Z) = ws$.

Definition 5.3.10
X and Y are called opposite iff $\delta^*_W(X,Y) = 1$.

For a pair of affine buildings the twinning is not uniquely determined. There are uncountable many non-isomorphic twinnings. In the case of rank 1-buildings – trees – , a universal twin building has been constructed by Mark Ronan and Jacques Tits — cf. [RT94], [RT99]. For more general classes of buildings this is an open problem.

we restate the definition, using the W-metric approach.

Definition 5.3.11 (W-metric twin building)
A twin building of type (W,S) is a quintupel $(\mathcal{C}^+, \mathcal{C}^-, \delta^+, \delta^-, \delta^*)$ such that

1. $(\mathcal{C}^+, \delta^+)$ *is a building of type* (W,S),
2. $(\mathcal{C}^-, \delta^-)$ *is a building of type* (W,S),

3. *δ^* is a codistance.*

Lemma 5.3.3
Let $\widehat{L_{alg}G}$ be an algebraic affine Kac-Moody group, $\widehat{B}_A^{\pm}$ two opposite affine Borel groups. Define $\mathcal{C}^+ = \widehat{L_{alg}G}/\widehat{B}_A^+$ and $\mathcal{C}^- = \widehat{L_{alg}G}/\widehat{B}_A^-$. Put furthermore

1. *$\delta^+(f\widehat{B}_A^+, g\widehat{B}_A^+) = w$ iff $f^{-1}g$ is in the $\widehat{B}_A^+ w\widehat{B}_A^+$-class of the positive Bruhat decomposition.*
2. *$\delta^-(f\widehat{B}_A^-, g\widehat{B}_A^-) = w$ iff $f^{-1}g$ is in the $\widehat{B}_A^- w\widehat{B}_A^-$-class of the negative Bruhat decomposition.*
3. *$\delta(f\widehat{B}_A^+, g\widehat{B}_A^-) = w$ iff $f^{-1}g$ is in the $\widehat{B}_A^+ w\widehat{B}_A^-$-class of the Bruhat twin decomposition.*

The quintupel $(\mathcal{C}^+, \mathcal{C}^-\delta^+, \delta^-, \delta^*)$ is a twin building.

Proof. Verification of the axioms of definition 5.3.11. □

For the readers' convenience we cite two possible ways to characterize twin buildings: the first characterization due to Peter Abramenko and Hendric Van Maldeghem [AVM01] characterizes twin buildings using the concept of 1-twinnings developed by Bernhard Mühlherr:

Definition 5.3.12 (1-twinning)
Let $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) denote the chambers of $\mathfrak{B}^+$ (resp. $\mathfrak{B}^-$). A nonempty symmetric relation $\mathcal{O} \subset (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+)$ is called a 1*-twinning if the following axiom holds: for a pair of chambers $(c_+, c_-) \in \mathcal{O}$ and two panels $P_+ \subset \mathcal{C}_+$ and $P_- \subset \mathcal{C}_-$ of the same type such that $c_\epsilon \in P_\epsilon, \epsilon \in \{+,-\}$ and $x_\epsilon \in P_\epsilon$ there exists a unique $y_{-\epsilon} \in P_{-\epsilon}$ such that $(x_\epsilon, y_{-\epsilon}) \notin \mathcal{O}$.*

Theorem 5.3.4 (Criterion for a twin building)
A 1*-twinning induces a twin building iff for $\epsilon \in \{+,-\}$ there exists a chamber $c_{-\epsilon} \in \mathcal{C}_{-\epsilon}$ such that for any chamber x_ϵ with $(c_{-\epsilon}, x_\epsilon) \subset \mathcal{O}$ there is an apartment Σ_ϵ of $\mathfrak{B}_\epsilon$ satisfying $\{x_\epsilon\} = \{y_\epsilon \in \mathcal{C}_\epsilon | y_\epsilon \in \Sigma_\epsilon$ and $(c_{-\epsilon}, y_\epsilon) \in \mathcal{O}\}$.*

The second characterization due to Peter Abramenko and Mark Ronan characterizes twin buildings via twin apartments — cf. [AR98]. In complete analogy to the finite dimensional situation, twin apartments are characterized as the coconvex hull of opposite chambers. We quote the definition from [AR98] (adapting the notation):

Definition 5.3.13 (Twin apartment)
For every pair of opposite chambers c_+ and c_-, $\{d_\epsilon \in \mathcal{C}^\epsilon | \delta^*(c_{-\epsilon}, d_\epsilon) = \delta_\epsilon(c_\epsilon, d_\epsilon)\}$ is the set of chambers of an apartment in $\mathfrak{B}^\epsilon$ ($\epsilon \in \{+,-\}$). This apartment is called a twin apartment in $\mathfrak{B}$.

We define a complete twin apartment to be a twin apartment together with the spherical apartments of the building at infinity.

Focusing on the geometric description of loop and Kac-Moody groups, a description of the affine building due to Mitchell is very convenient: he shows in [Mit88] that the affine algebraic building of type $\widetilde{X}_l$ over the base field $\mathbb{C}$ can be described in complete analogy to the finite dimensional case as

$$\mathfrak{B} = (L_{alg}G_{\mathbb{C}}/B_A \times \Delta)/\sim\ .$$

In this description Δ is a fundamental alcove in a fixed abelain subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and $\sim$ is the equivalence relation defined by $(f_1, Y_1) \sim (f_2, Y_2)$ iff $Y_1 = Y_2 \simeq Y$ and $f_1 \simeq f_2(\text{mod}\,(\text{Fix } \exp tY))$. Using the Iwasawa decomposition of $L_{alg}G_{\mathbb{C}}$ we get a second description:

$$\mathfrak{B} = (L_{alg}G_{\mathbb{R}}/T \times \Delta)/\sim\ .$$

Here T denotes a maximal torus of the finite dimensional compact Lie group G. From the point of view of loop groups the appearance of a torus T of the finite dimensional Lie group is counterintuitive. But this becomes natural from the point of view of Kac-Moody groups:

Using the bijection of affine parabolic groups in $\widehat{L_{\text{alg}}G_{\mathbb{C}}}$ with parahori groups in $L_{\text{alg}}G$, we can deduce a new description for the building:

$$\mathfrak{B} = (\widehat{L_{alg}G_{\mathbb{C}}}/\widehat{B_A} \times \Delta)/\sim\ .$$

The Iwasawa decomposition yields again a version using the maximal compact subgroup:

$$\mathfrak{B} = (\widehat{L_{alg}G}_{\mathbb{R}}/\widehat{T} \times \Delta)/\sim\ .$$

$\widehat{T}$ is a maximal torus in $\widehat{L_{alg}G}$. Thus we get a structure mirroring the finite dimensional one. Nevertheless there is an important difference: the chambers in this building correspond only to the Iwahori subgroups (resp. affine parabolic subgroups) of one of the two equivalence classes. Furthermore there is no opposite-relation on the set of all chambers.

To resolve those problems it is useful to double this construction by introducing two affine buildings. The first one is denoted $\mathfrak{B}^+$. Its cells correspond to the positive affine parabolic subgroups. The second one is denoted $\mathfrak{B}^-$. Its cells correspond to the negative affine parabolic subgroups.

Let $\mathfrak{B} = \mathfrak{B}^* \cup \mathfrak{B}^-$. In order to define a codistance on $\mathfrak{B}$ that makes it into a twin building, we need the Bruhat twin decomposition. Define the codistance between two chambers $(f, \Delta_1) \in \mathfrak{B}^+$ and $(g, \Delta_2) \in \mathfrak{B}^-$ to be the element $w \in W_{\text{aff}}$, corresponding to the class of $f^{-1}g$ of the Bruhat twin decomposition. The verification of the axioms of definition 5.3.11 is straight forward.

For a particularly nice description of case $\widetilde{A}_n$ see the paper of Linus Kramer [Kra02]. His arguments generalize to the other classical types.

5.4 Universal geometric twin buildings

In this section we construct universal geometric twin buildings associated to simple geometric affine Kac-Moody algebras $\widehat{L}(\mathfrak{g}, \sigma)$ and their Kac-Moody groups $\widehat{L}(G, \sigma)$.

There are two major obstacles:

1. Algebraic twin buildings correspond only to subsets of algebraic loops.

2. The completions of Kac-Moody groups do not act properly on the twin buildings.

To resolve those problems, we define universal BN-pairs and their associated universal geometric twin buildings. Those twin buildings are chamber complexes such that $\mathfrak{B}^+$ and $\mathfrak{B}^-$ each consist of an infinite number of connected components, each of which is an affine building, such that each pair consisting of a building in $\mathfrak{B}^+$ and a building $\mathfrak{B}^-$ is a twin building in the classical algebraic sense.

We start by quoting some results about Iwahori and parahori groups in loop groups $L(G,\sigma)$:

Theorem 5.4.1
Let $L(G,\sigma)$ be a twisted loop group.

1. *The completions of Iwahori (resp. parahori) groups in $L_{alg}G^\sigma$ are Iwahori (resp. parahori) groups of $L(G,\sigma)$.*
2. *The Weyl group of $L(G,\sigma)$ is the affine Weyl group of $L_{alg}G^\sigma$.*

Proof. See Pressley-Segal [PS86]. □

Using this result we get the same assertion for the affine parabolic subgroups in the affine Kac-Moody groups $\widehat{L}(G,\sigma)$. $L_{alg}G^\sigma$ is just one subgroup of $L(G,\sigma)$ isomorphic to $L_{alg}G^\sigma$, which is distinguished by the representation, we choose. All those subgroups are conjugate.

Definition 5.4.1 (Universal geometric BN-pair for $\widehat{L}(G,\sigma)$)
Let $\widehat{L}(G,\sigma)$ be an affine Kac-Moody group. Data $(\widehat{B}_A^+,\widehat{B}_A^-,N,W,S)$ is a twin BN-pair for $\widehat{L}(G,\sigma)$ iff there are subgroups $\widehat{L}(G,\sigma)^+$ and $\widehat{L}(G,\sigma)^-$ of $\widehat{L}(G,\sigma)$ such that $\widehat{L}(G,\sigma) = \langle \widehat{L}(G,\sigma)^-,\widehat{L}(G,\sigma)^+\rangle$ subject to the following axioms:

1. *$(\widehat{B}_A^+,N,W,S)$ is a BN-pair for $\widehat{L}(G,\sigma)^+$ (called B^+N),*
2. *$(\widehat{B}_A^-,N,W,S)$ is a BN-pair for $\widehat{L}(G,\sigma)^-$ (called B^-N),*
3. *$(\widehat{B}_A^+\cap\widehat{L}(G,\sigma)^-,\widehat{B}_A^-\cap\widehat{L}(G,\sigma)^+,N,W,S)$ is a twin BN-pair for $\widehat{L}(G,\sigma)^+\cap\widehat{L}(G,\sigma)^-$*

Remark 5.4.1
For an algebraic Kac-Moody group a generalized BN-pair coincides with a BN-pair. We can choose $\widehat{L}(G,\sigma)^+ = \widehat{L}(G,\sigma)^- = \widehat{L}(G,\sigma)$.

We use the equivalent definition for the loop groups $L(G,\sigma)$.

Lemma 5.4.1
The groups $L(G,\sigma)^+$ and $L(G,\sigma)^-$ have Bruhat decompositions. The group $L(G,\sigma)$ has a Bruhat twin decomposition but no Bruhat decomposition.

Proof. This result is a restatement of the decomposition results in chapter 8 of [PS86]. □

Theorem 5.4.2 (Bruhat decomposition)
Let $\widehat{L}(G,\sigma)$ be an affine Kac-Moody group with affine Weyl group W_{aff}. Let furthermore $\widehat{B}_A^\pm$ denote a positive (resp. negative) Borel group. There are decompositions

$$\widehat{L}(G,\sigma)^+ = \coprod_{w\in W_{aff}} \widehat{B}_A^+ w \widehat{B}_A^+$$

and

$$\widehat{L}(G,\sigma)^- = \coprod_{w\in W_{aff}} \widehat{B}_A^- w \widehat{B}_A^- .$$

Proof. This is a consequence of lemma 5.4.1. □

Theorem 5.4.3 (Bruhat twin decomposition)
Let $\widehat{L}(G,\sigma)$ be an affine algebraic Kac-Moody group with affine Weyl group W_{aff}. Let furthermore $\widehat{B}_A^{\pm}$ denote a positive and its opposite negative Borel group. There are two decompositions

$$\widehat{L}(G,\sigma) = \coprod_{w \in W_{aff}} \widehat{B}_A^{\pm} w \widehat{B}_A^{\mp} .$$

Remark 5.4.2
Note that the Bruhat twin decomposition is defined on the whole group $\widehat{L}(G,\sigma)$. This translates into the fact that any two chambers in $\mathfrak{B}^+$ resp. $\mathfrak{B}^-$ have a well-defined Weyl codistance. In contrast Bruhat decomposition are only defined for subgroups. This translates into the fact that there are positive (resp. negative) chambers without a well-defined Weyl distance.

Example 5.4.1
Kumar studies Kac-Moody groups and algebras that are completed "in one direction" (In the setting of affine Kac-Moody groups of holomorphic loops this corresponds to holomorphic functions with finite principal part). There is an associated twin BN-pair; the positive Borel subgroups are completed affine Borel subgroups while the negative ones are the algebraic affine Borel subgroups. Thus for a universal geometric twin BN-pair we have to use: $\widehat{L}(G,\sigma)^+ = \widehat{L}(G,\sigma)$ and $\widehat{L}(G,\sigma)^- = \widehat{L_{alg}G}^{\sigma}$ — cf. [Kum02].

Definition 5.4.2
An involution $\varphi : \widehat{L}(G,\sigma) \longrightarrow \widehat{L}(G,\sigma)$ is called a universal BN-flip iff

1. $\varphi^2 = 1$,
2. $\varphi(B^+) = B^-$,
3. *φ centralizes W.*

Definition 5.4.3
A universal geometric twin BN-pair is called symmetric iff it has a BN-flip.

Example 5.4.2
For a twin BN-pair to be symmetric we need that $\widehat{B}_A^+$ and $\widehat{B}_A^-$ are isomorphic groups. Especially this means that the completion has to be symmetric in both directions. Thus the example 5.4.1 is not a symmetric twin BN-pair.

Example 5.4.3
An algebraic affine twin BN-pair is symmetric. As involution we can use a BN-flip.

Example 5.4.4
The universal geometric BN-pair associated to any group $\widehat{L}(G,\sigma)$ is symmetric.

If we try to associate to a universal geometric twin BN-pair a twin building $\mathfrak{B}_0 = \mathfrak{B}_0^+ \cup \mathfrak{B}_0^-$, as we did in the previous cases by identifying affine parabolics with cells, we get the problem that the action of $\widehat{L}(G,\sigma)^+$ on $\mathfrak{B}_0^-$ and the action of $\widehat{L}(G,\sigma)^-$ on $\mathfrak{B}_0^+$ are not well defined. The maximal group acting on $\mathfrak{B}_0$ is $\widehat{L}(G,\sigma)^0 = \widehat{L}(G,\sigma)^+ \cap \widehat{L}(G,\sigma)^-$.

Lemma 5.4.2
The intersection $\widehat{L}(G,\sigma)^0$ of $\widehat{L}(G,\sigma)^+$ with $\widehat{L}(G,\sigma)^-$ is isomorphic to the group of algebraic loops

$$\widehat{L}(G,\sigma)^0 \simeq \widehat{L_{alg}G}^{\sigma}$$

Proof. By the amalgam description, $\widehat{L_{alg}G}^{\sigma}$ is the maximal subgroup of $\widehat{L}(G,\sigma)$ having both Bruhat decompositions. □

We now define a universal geometric twin building using the W-metric approach:

Definition 5.4.4 (Universal geometric twin building)
Let $\widehat{L}(G,\sigma)$ be an affine Kac-Moody group with a universal geometric BN-pair. Define $\mathcal{C}^+ := \widehat{L}(G,\sigma)/\widehat{B}_A^+$ and $\mathcal{C}^- := \widehat{L}(G,\sigma)/\widehat{B}_A^-$.

1. *The distance $\delta^\epsilon : \mathcal{C}^\epsilon \times \mathcal{C}^\epsilon \longrightarrow W$, $\epsilon \in \{+,-\}$ is defined as usual via the Bruhat decompositions: $\delta^\epsilon(gB^\epsilon, fB^\epsilon) = w$ iff $g^{-1}f$ is in the w-class of the Bruhat decomposition of $\widehat{L}(G,\sigma)^\epsilon$. Otherwise it is ∞.*

2. *The codistance $\delta^* : \mathcal{C}^+ \times \mathcal{C}^- \cup \mathcal{C}^- \times \mathcal{C}^+ \longrightarrow W$ is defined as usual by $\delta^*(gB^-, fB^+) = w$ (resp. $\delta^*(gB^+, fB^-) = w$ iff $g^{-1}f$ is in the w-class of the corresponding Bruhat twin decomposition of $\widehat{L}(G,\sigma)$.*

The elements of $\mathcal{C}^\pm$ are called the positive (resp. negative) chambers of the universal geometric twin building. The building is denoted $\mathfrak{B} = \mathfrak{B}^+ \cup \mathfrak{B}^-$. One can define a simplicial complex realization in the usual way using the Weyl metric. We define connected components in $\mathfrak{B}^\pm$ in the following way: two elements $\{c_1, c_2\} \in \mathfrak{B}^\pm$ are in the same connected component iff $\delta^\pm(c_1, c_2) \in W_{\text{aff}}$. This is an equivalence relation.

Remark 5.4.3
Let $L(G,\sigma)$ be an algebraic affine Kac-Moody group. Then the universal geometric twin building coincides with the algebraic twin building.

Lemma 5.4.3 (Properties of a universal geometric twin building)

1. *$\widehat{L}(G,\sigma)$ acts by left multiplication on $\mathfrak{B}$ isometrically.*

2. *The connected components of $\mathfrak{B}^\epsilon$ are buildings of type (W,S).*

3. *Each pair consisting of one connected component in $\mathfrak{B}^+$ and one in $\mathfrak{B}^-$ is an algebraic twin building of type (W,S).*

4. *The connected components of $\mathfrak{B}^\epsilon$ are indexed by elements in $\widehat{L}(G,\sigma)/\widehat{L}(G,\sigma)^\epsilon$.*

5. *$\widehat{L}(G,\sigma)^\epsilon$ acts on the identity component $\mathfrak{B}_0^\epsilon$ by isometries.*

6. *The Borel subgroups are exactly the stabilizers of the chambers.*

7. *Let $f\widehat{B}^\epsilon$ and $g\widehat{B}^\epsilon$ be two chambers in the same connected component of $\mathfrak{B}^\epsilon$, let $h, h' \in \widehat{L}(G,\sigma)^{-\epsilon}$. The left translates $hf\widehat{B}^{-\epsilon}$ and $h'g\widehat{B}^{-\epsilon}$ are in the same connected component iff $f^{-1}h^{-1}h'g \in \widehat{L}(G,\sigma)^\epsilon$.*

Proof.

1. The first assertion follows from the definition of $\mathcal{C}^\pm$ as coset spaces of $\widehat{L}(G,\sigma)$.

2. Each connected component fulfills the axioms of definition 5.2.6.

3. Each pair consisting of a connected component in $\mathfrak{B}^+$ and one in $\mathfrak{B}^-$ fulfills the axioms of definition 5.3.11. As the Bruhat decomposition is defined on $\widehat{L}(G,\sigma)$, the codistance is defined between arbitrary chambers in $\mathfrak{B}_A^\epsilon$ resp. $\mathfrak{B}^{-\epsilon}$.

4. $\widehat{L}(G,\sigma)$ has a decomposition into subsets of the form $\widehat{L}(G,\sigma)^\epsilon$. Those subsets are indexed with elements in $\widehat{L}(G,\sigma)/\widehat{L}(G,\sigma)^\epsilon$. The class corresponding to the neutral element is $\widehat{L}(G,\sigma)^\epsilon \subset \widehat{L}(G,\sigma)$. Thus it corresponds to a connected component and a building of type (W,S). The result follows via translation by elements in $\widehat{L}(G,\sigma)/\widehat{L}(G,\sigma)^\epsilon$: a connected component of $\mathfrak{B}^\epsilon$ containing $f\widehat{B}_A^\epsilon$ consists of all elements $f\widehat{L}(G,\sigma)^\epsilon\widehat{B}_A^\epsilon$ as $\delta(fh\mathfrak{B}_A^\epsilon, fh'\mathfrak{B}_A^\epsilon) = w((fh)^{-1}fh) = w(h^{-1}f^{-1}fh') = w(h^{-1}h') \in W$ as $h,h' \in \widehat{L}(G,\sigma)$.

5. Follows from the last statement.

6. The action of $\widehat{L}(G,\sigma)^\epsilon \subset \widehat{L}(G,\sigma)^\epsilon$ is clear. Connected components of $\mathfrak{B}^\epsilon$ correspond to classes in $\widehat{L}(G,\sigma)/\widehat{L}(G,\sigma)^\epsilon$. Those are fixed by the action of $\widehat{L}(G,\sigma)^\epsilon$.

7. The chamber corresponding to fB_A^ϵ is stabilized by the Borel subgroup $B_f^\epsilon := fB_A^\epsilon f^{-1}$. The converse follows as each Borel subgroup is conjugate to a standard one.

8. $f^{-1}h^{-1}h'g \in \widehat{L}(G,\sigma)^\epsilon$ is equivalent to $\delta^\epsilon(hf\widehat{B}_A^{-\epsilon}, h'g\widehat{B}_A^{-\epsilon}) \in W_{\text{aff}}$. □

Definition 5.4.5
A universal geometric twin building $\mathfrak{B}$ is symmetric iff there is a simplicial complex involution $\varphi_{\mathfrak{B}} : \mathfrak{B} \longrightarrow \mathfrak{B}$ such that $\varphi_{\mathfrak{B}}(\mathfrak{B}^\epsilon) = \mathfrak{B}^{-\epsilon}$

Lemma 5.4.4
The universal geometric twin building which is associated to a symmetric universal geometric BN-pair is symmetric.

Proof. The BN-pair involution induces a building involution. □

5.5 Geometric twin buildings and Kac-Moody algebras

In this section we describe an explicit realization of the universal geometric twin building. Define the two simplicial complexes:

$$\mathfrak{B}^+ = (L(G_\mathbb{C},\sigma)/B \times \Delta)/\sim,$$
$$\mathfrak{B}^- = (L(G_\mathbb{C},\sigma)/B \times \Delta)/\sim\ .$$

In this description Δ denotes the fundamental alcove in a fixed abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and $\sim$ is the equivalence relation defined by $(f_1,Y_1) \sim (f_2,Y_2)$ iff $Y_1 = Y_2 \simeq Y$ and $f_1 \simeq f_2 (\mathrm{mod}\,(\mathrm{Fix} \exp tY))$. Using the Iwasawa decomposition of $L(G_\mathbb{C},\sigma)$ we get a second description:

$$\mathfrak{B}^+ = (L(G_\mathbb{R},\sigma)/T \times \Delta)/\sim,$$
$$\mathfrak{B}^- = (L(G_\mathbb{R},\sigma)/T \times \Delta)/\sim\ .$$

Furthermore, we set

$$\mathfrak{B} = \mathfrak{B}^+ \cup \mathfrak{B}^-\ .$$

Definition 5.5.1 (Apartment)
By abuse of notation, let $W_{aff} \subset L(G_\mathbb{C},\sigma)/B = L(G_\mathbb{R},\sigma)/T$ be a realization of the affine Weyl group of $G_\mathbb{C}$, $W_{aff}^f := fW_{aff}f^{-1}$. An apartment $\mathcal{A}_f^\pm \in \mathfrak{B}^+$ is the simplicial complex

$$\mathcal{A}_f^\pm := (W_{aff}^f \times \Delta)/\sim\ .$$

Proof.

- To check that the embedding $W_{\text{aff}} \subset G_{\mathbb{C}}/B = G_{\mathbb{R}}/T$ is well defined, let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian subalgebra. Let $H := \{g \in G | g\mathfrak{t}g^{-1} = \mathfrak{t}\}$. H is a group. Let $X \in \mathfrak{t}$ be a regular element, $K := \text{Fix}(X) \simeq T$. Then $W = H/T \subset G/T$.
- $\mathcal{A}_f^{\pm}$ is a thin Coxeter complex of type W. Thus $\mathcal{A}_f^{\pm}$ is an apartment. □

Lemma 5.5.1
Two elements $(f, X), (g, Y) \in \mathfrak{B}^{\epsilon}$ are contained in the same connected component if and only if $f^{-1}g \in \widehat{L_{alg}G}$.

Proof. This is a restatement of lemma 5.4.3.

The "if" part is easy: just take left translation, using the action of f^{-1} – then both elements are translated into $\mathfrak{B}^0$, which is connected; the "only if" part follows as the identity component is $\mathfrak{B}^0$. Thus all connected components are isomorphic to $\mathfrak{B}^0$. □

Corollary 5.5.1
$\widehat{L_{alg}G}$ fixes every connected component in $\mathfrak{B}$.

This lemma and its corollary show that the subgroup of algebraic loops has an accentuated role.
It is a nice observation of Ernst Heintze that the algebraic loop group can be characterized via the exponential map:

Lemma 5.5.2 (Characterization of $L_{alg}G$)

$$L_{alg}G := \{e^{tX_1}e^{tY_1} \dots e^{tX_n}e^{tY_n}g \,|\, g \in G, X_i, Y_i \in \mathfrak{g}, e^{tX_i}e^{tY_i} = e\}$$

Before describing the proof, let us note as a corollary an application: we give a characterization of elements $(f, X), (g, Y \in \mathfrak{B}^{\epsilon})$ that are in the same connected component.

Corollary 5.5.2
Two simplices $\bar{f}B^{\epsilon}$ and $\bar{g}B^{\epsilon} \in \mathfrak{B}^{\epsilon}$ are contained in the same connected component Δ_1^{ϵ} of $\mathfrak{B}^{\epsilon}$ iff there are representatives $f, g \in \widehat{L}(G, \sigma)$, $\{X_1, \dots, X_n, Y_1, \dots, Y_n \in \mathfrak{h} | e^{tX_i}e^{tY_i} = e\}$ where $\mathfrak{g} \simeq \mathfrak{h} \subset \widehat{L}(\mathfrak{g}, \sigma)$ and a constant c such that $fB^{\epsilon} = \bar{f}B^{\epsilon}$ $gB^{\epsilon} = \bar{g}B^{\epsilon}$ and $f(t) = e^{tX_1}e^{tY_1} \dots e^{tX_n}e^{tY_n}c \cdot g(t)$.

We now give the proof of Lemma 5.5.2:

Proof of lemma 5.5.2. Define:

$$L'_{alg}G := \{e^{tX_1}e^{tY_1} \dots e^{tX_n}e^{tY_n}g \,|\, g \in G, X_i, Y_i \in \mathfrak{g}, e^{tX_i}e^{tY_i} = e\}$$

We have to show: $L'_{alg}G = L_{alg}G$.

- We show: $L'_{alg}G \subset L_{alg}G$. First remark that $L'_{alg}G$ is a group of periodic mappings $c : \mathbb{R} \to G$ with period 1. As $ge^{tX} = e^{tAd(g)X}g$ the product of two elements is again in $L'_{alg}G$. Checking the group axioms is then elementary. Thus $L'_{alg}G$ is a subgroup of LG. From theorem 4.7. in [Mit88] it follows that $c(t) = \exp tX \exp tY$ is in $L_{alg}G$ iff $\exp tX \exp tY = e$. As each element in $L'_{alg}G$ is generated by elements in $L_{alg}G$, we get: $L'_{alg}G \subset L_{alg}G$.
- We show: $L_{alg}G \subset L'_{alg}G$. To prove this direction, we study the action of $L'_{alg}G$ on the building. We show:

1. $L'_{alg}G$ acts transitively on the set of chambers.
2. The isotropy group of a chamber is the same for $L_{alg}G$ and $L'_{alg}G$.

Those two assertions contain the theorem, as for $g \in L_{alg}G$ we find the existence of a $g' \in L'_{alg}G$ such that $g\Delta_0 = g'\Delta_0$ for some fixed chamber Δ_0. Thus $g'^{-1}g\Delta_0 = \Delta_0$. Thus the product $g'^{-1}g$ is in the isotropy group of Δ_0 with respect to the $L_{alg}G$-action, called $L_{alg}G_{\Delta_0}$.

Now the second assertion tells us: $g'^{-1}g \in L'_{alg}G_{\Delta_0} = L_{alg}G_{\Delta_0}$ Set $g'' := g'^{-1}g \in L'_{alg}G$. Then $g = g'g'' \in L'_{alg}G$. Thus $L_{alg}G \subset L_{alg}G'$ and the lemma is proved. Thus we are left with checking assertions 1. and 2.:

- We prove: the isotropy group of a chamber is the same for $L_{alg}G$ and $L'_{alg}G$. Let $\mathfrak{B}_{alg}$ be the affine building, associated to $L_{alg}G$, $X \in \mathfrak{B}$ a cell of type I, P_I its stabilizer in $LG_{\mathbb{C}}$. We know from [Mit88]:

$$\begin{aligned} L_{alg}G \cap P_I &= \{\overline{h} \in L_{alg}G | h(t)\exp(tX)h^{-1}(1) = \exp(tX)\} = \\ &= \{\overline{h} \in L_{alg}G | h(t) = \exp(tX)h(1)\exp(-tX)\} \subset \\ &\subset L'_{alg}G \end{aligned}$$

  The last inclusion is true, as $\exp(tX)h(1)\exp(-tX) = \exp(tX)\exp(tY)h(1)$ with $y = -\mathrm{Ad}h(1)X$ and $[h(1), \exp(tX)] = 0$.

- We prove: $L'_{alg}G$ acts transitively on the set of chambers. To this end, we remark that the action of $L_{alg}G \cap P_i \simeq SU(2)$ is transitive on the chambers having the panel corresponding to i in its boundary. Transitivity on the building follows now as every pair of chambers can be connected by a gallery, which we can follow by repeated application of the transitivity on the chambers surrounding a panel.

  As those groups are in $L'_{alg}G$, the action of $L'_{alg}G$ is transitive on $\mathfrak{B}$ - the result follows now. □

We focus our attention now on the twin-structure:

To turn $\mathfrak{B}$ into a twin building we have to define either a twin apartment structure or a codistance function on $\mathfrak{B}^+ \times \mathfrak{B}^- \cup \mathfrak{B}^- \times \mathfrak{B}^+$.

Using the Bruhat twin decomposition we define the codistance δ^* between chambers $(f, \Delta) \in \mathfrak{B}^+$ and $(g, \Delta) \in \mathfrak{B}^-$ by

$$\delta^*\left((f, \Delta), (g, \Delta)\right) = w \text{ iff } f^{-1}g \text{ is in the class of } w\,.$$

A universal geometric twin building carries a pseudo distance;

For $\widehat{f} \in \widehat{L}(G, \sigma)$ let $f = \sum a_n z^n$ be the (matrix valued) associated Fourier series of the loop part. f is in L^rG iff $\sum |(a_n)|n^r < \infty$, f is holomorphic on A_n iff $\sum |a_k|e^{kn} < \infty$, f is holomorphic on $\mathbb{C}^*$ iff $\sum |a_k|e^{kn} < \infty$ for all $n \in \mathbb{N}$. Let $\Delta_0, \Delta_1 \in \mathfrak{B}^\epsilon$ be two affine buildings and $x \in \Delta_0, y \in \Delta_1$. Define

$$\nu(x, y) = \max_n\{\text{There is } f \in A_nG \text{ such that } f(x) = y\}$$

and $d(x, y) = e^{-\nu(x,y)}$. Then we put $d(\Delta_0, \Delta_1) = d(x, y)$

This is equivalent to

$$\nu(x, y) = \max_n\{\text{There is a function } f \text{ such that } f(x) = y \text{ satisfying } \sum |a_k|e^{kn} < \infty\}.$$

Lemma 5.5.3 (pseudo distance)
d is an ultrametric pseudo distance on the space of buildings in $\mathfrak{B}^\epsilon$.

Proof of lemma 5.5.3.

1. We prove, that d is an ultrametric pseudo distance on the space of chambers. To this end let $x, y, z \in \mathfrak{B}^\epsilon$ be chambers. We have to check:
   - symmetry: $f \in A^nG \Leftrightarrow f^{-1} \in A^nG$. Thus $\nu(x,y) = \nu(y,x)$ and $d(x,y) = d(y,x)$.
   - strong Δ-inequality: Let $d(x,y) = e^{-\nu(x,y)}$, $d(y,z) = e^{-\nu(y,z)}$. Thus there is a function $f_{xy} \in A^{\nu(x,y)}G$ such that $f(x) = y$ and a function $f_{yz} \in A^{\nu(y,z)}G$ such that $f(y) = z$. Without loss of generality suppose: $\nu(x,y) \leq \nu(y,z)$. Thus $A^{\nu(x,y)}G \supset A^{\nu(y,z)}G$. So $f_{xz} = f_{xy}f_{yz} \in A^{\nu(x,y)}G$. Thus $d(x,z) = e^{-\nu(x,z)} \leq e^{-\nu(x,y)} = d(x,y)$.
2. We have to check, that the distance on the space of buildings is well defined. To this end, let $x, x' \in \Delta_0$ and $y \in \Delta_1$. There is a quasi-algebraic subgroup $G(\Delta_0)$ acting transitively on Δ_0. Let $h \in G(\Delta_0)$ such that $x' = h(x)$. Clearly $d(x,x') = 0$. The result follows now from the triangle inequality. □

We want now to embed the universal twin building in the compact real form $\widehat{L}(\mathfrak{g}_\mathbb{R}, \sigma)$ of a Kac-Moody Lie algebra. It will appear as a tessalation of a space $H_{l,r}$ defined as the intersection of the sphere of radius $l, l \in \mathbb{R}$ with a horosphere $r_d = \pm r$. The two sheets of this sphere will correspond to $\mathfrak{B}^+$ resp. $\mathfrak{B}^-$.

For the regularity of $\widehat{L}(G,\sigma)$ we require the following condition: the gauge action of $L(G,\sigma)$ on $L(\mathfrak{g},\sigma)$ is polar. This condition is fulfilled for $\widehat{A_nG}^\sigma$ and for $\widehat{MG}^\sigma$ as a consequence of lemma 4.6.1 and theorem 4.6.1. For Kac-Moody groups of H^1-loops acting on the Kac-Moody algebra of H^0-loops it is a consequence of Terng's work [Ter95].

Following the blueprint we used for finite dimensional Lie groups, we start with the conjugation action:

$$\widehat{\varphi} : \widehat{L}(G,\sigma) \times \widehat{L}(G,\sigma) \longrightarrow \widehat{L}(G,\sigma), \qquad (g,h) \mapsto ghg^{-1}$$

By differentiation we get the adjoint action on the Lie algebra:

$$\widehat{\varphi} : \widehat{L}(G,\sigma) \times \widehat{L}(\mathfrak{g},\sigma) \longrightarrow \widehat{L}(\mathfrak{g},\sigma), \qquad (g,\widehat{u}) \mapsto g\widehat{u}g^{-1}$$

As in the finite dimensional case we would like to find a fundamental domain for this action. In contrast to the finite dimensional theory it is not possible to cover $\widehat{L}(\mathfrak{g},\sigma)$ with maximal conjugate flats.

But for the restriction to $H_{l,r}$ (which is invariant under the adjoint action), this is different. Our assumptions at the regularity yield the conclusion that $H_{l,r}$ is covered with finite dimensional conjugate abelian subalgebras. So the situation is exactly as in the finite dimensional case; and the algebra works out exactly the same:

We find for a finite dimensional flat $\widehat{\mathfrak{a}}$

$$\begin{array}{rccl} \widehat{\varphi} : & \widehat{L}(G,\sigma) \times H_{l,r} & \longrightarrow & H_{l,r}, \quad (g,\widehat{u}) \mapsto g\widehat{u}g^{-1} \\ \widehat{\varphi} : & \widehat{L}(G,\sigma) \times \widehat{\mathfrak{a}} \cap H_{l,r} & \longrightarrow & H_{l,r}, \quad (g,\widehat{u}) \mapsto g\widehat{u}g^{-1}\,. \end{array}$$

Taking $\widehat{\mathfrak{a}}$ the standard flat (i.e. for non-twisted groups: $\mathfrak{a}$ consists of constant loops), we find that $\mathfrak{a}_H := \widehat{\mathfrak{a}} \cap H$ consists of tripels $\widehat{X} = (X, r_c, r_d)$ where r_c is defined by the condition $|\widehat{X}| = l$.

The exponential image of $\widehat{\mathfrak{a}}$ is the Cartan subalgebra $\widehat{T} \simeq T \oplus \mathbb{S}^1 \oplus \mathbb{S}^1$. As $\widehat{\mathfrak{a}}$ is stabilized by $\widehat{T}$, we get a well defined surjective action

$$\widehat{\varphi}: \quad \widehat{L}(G,\sigma)/\widehat{T} \times \mathfrak{a}_H \quad \longrightarrow \quad H, \quad (g, u_H) \mapsto g u_H g^{-1}\,.$$

The surjectivity of this map follows from the polarity of the adjoint action (see theorem 2.5.2).

Using the equivalence $\widehat{L}(G,\sigma)/\widehat{T} \simeq L(G,\sigma)/T$ we get:

$$\varphi: \quad L(G,\sigma)/T \times \mathfrak{a}_H \quad \longrightarrow \quad H, \quad (g, u_H) \mapsto g u_H g^{-1}\,.$$

Now the inner automorphisms of $\widehat{\mathfrak{a}}$ are the elements of the affine Weyl group $W_A := N(T)/T$, so we may further restrict $\widehat{\mathfrak{a}}_H$ to a fundamental domain of the action of W_A, denoted Δ. Then the map

$$\varphi: \quad L(G,\sigma)/T \times \Delta \quad \longrightarrow \quad H, \quad (gT, \widehat{u}_H) \mapsto g \widehat{u}_H g^{-1}$$

is again surjective.

We can now construct a chamber complex by identifying Δ with a simplex $\mathfrak{B}$ with boundary and taking its $\widehat{L}(G,\sigma)$-translates.

We have the theorem:

Theorem 5.5.1
The connected components of this simplicial complex are affine buildings of the type of $\widehat{L}(\mathfrak{g},\sigma)$.

This construction proves the following theorems:

Theorem 5.5.2 (Embedding of the universal twin building)
For each algebra $L(\mathfrak{g},\sigma)$ there is a 2-parameter family of embeddings for the universal twin building, parametrised by r and the norm l. This embedding is equivariant in the sense that:

$$\begin{array}{ccc} \mathfrak{B}_G & \xrightarrow{\widehat{L}(G,\sigma)} & \mathfrak{B}_G \\ \downarrow i & & \downarrow i \\ \widehat{L}(\mathfrak{g}_R,\sigma) & \xrightarrow{Ad(\widehat{L}(G,\sigma))} & \widehat{L}(\mathfrak{g}_R,\sigma) \end{array}$$

We call this 2-parameter family the thickened universal twin building.

Theorem 5.5.3
Every flats of finite type corresponds to a complete twin apartment of the thickened universal twin building.

We concentrate now on the situation of Kac-Moody symmetric spaces, that is $\widehat{L}(\mathfrak{g},\sigma) \in \{\widehat{M\mathfrak{g}}^{\sigma}, \widehat{A_n\mathfrak{g}}^{\sigma}\}$.

This embedding of the generalized building into $H_{l,r}$ shows:

Theorem 5.5.4

1. *The universal building associated to $\widehat{MG}$ carries a natural tame Fréchet structure.*

2. *The universal building associated to $\widehat{A_nG}$ carries a natural Banach space structure.*

3. *The universal building associated to $\widehat{L^\infty G}$ carries a natural tame Fréchet structure.*

Using the description of $\widehat{M\mathfrak{g}}$ as inverse limit of the algebras $\widehat{A_n\mathfrak{g}}$, we find this structure reflected in a inverse limit system $\{\mathfrak{B}_{\widehat{MG}}, \varprojlim \mathfrak{B}_{\widehat{A_nG}}\}$. Thus the generalized twin building for $\widehat{MG}$ is surrounded by a cloud of buildings of weaker regularity.

5.6 Topology and geometry of $\mathfrak{B}$

Theorem 5.5.4 shows that the simplicial realization of $\mathfrak{B}$ carries a natural structure as a tame Fréchet space. There are two other ways to define a topology (resp. geometry) on this space:

1. A structure on the set of chambers in $\mathfrak{B}$.
2. A structure on the set of buildings in $\mathfrak{B}$.

As chambers in $\mathfrak{B}$ correspond bijectively to elements in MG/T, the space of chambers inherits the tame Fréchet topology of MG/T. Study the gauge action of MG on $M\mathfrak{g}$. By theorem 2.5.2 it is a polar action. Let $X \in \mathfrak{t}$ be an element in the Lie algebra of T, such that $\{MG{\cdot}X\}$ is a principal orbit. As the stabilizer of X is T, we have $\{MG{\cdot}X\} \simeq MG/T$. So the space of chambers can be identified with an isoparametric submanifold. Hence, the structure on the space of chambers is well understood.

While the space of chambers and the simplicial realization allow a Fréchet structure, the situation is completely different for $\mathfrak{B}$ itself. The simple fact that the chambers belonging to a single building are dense in the space of all chambers shows that no refinement of a topology on the space of chambers will give a topology on the space of buildings.

As we choose to define buildings and twinnings in terms of Kac-Moody group data, we will also defined the geometry and topology on this building in terms of regularity conditions on these functions — 5.5.3. Nevertheless, a purely algebraic approach is possible.

5.7 The spherical building at infinity

An affine building $\mathfrak{B}_{\text{aff}}$ has a spherical building $\mathfrak{B}^\infty$ at infinity. Let W_{aff} be the Weyl group of $\mathfrak{B}_{\text{aff}}$, W the Weyl group of $\mathfrak{B}^\infty$. Then W_{aff} can be described as a semidirect product of W with $\mathbb{Z}$. Every affine apartment in $\mathfrak{B}_{\text{aff}}$ is bounded by a spherical apartment in $\mathfrak{B}^\infty$. The chambers of the spherical building $\mathfrak{B}^\infty$ correspond to sectors in the interior building. For details see the book [AB08] or the article [Ron03]. The same construction is possible for an affine twin building. Now both affine buildings have a common spherical building. We quote the following theorem of Marc Ronan [Ron03].

Theorem 5.7.1
Let E_+ denote the subsystem of $(\mathfrak{B}_+)^\infty$ comprising all sectors S^∞, where S is an interior sector of $\mathfrak{B}^+_{\mathit{aff}}$ and similarly for E_-. Then E_+ and E_- are spherical subbuildings of $(\mathfrak{B}_+)^\infty$ and $(\mathfrak{B}_-)^\infty$, and the twinning of $\mathfrak{B}^+_{\mathit{aff}}$ and $\mathfrak{B}^-_{\mathit{aff}}$ defines a canonical isomorphism between E_+ and E_-. This building corresponds to the fraction field of the ring of Laurent polynomials.

Of course one can ask oneself:
What happens with the other apartments in the spherical building $\mathfrak{B}^\infty$?
To answer this question, we start with the description of the spherical building $\mathfrak{B}^\infty$.

Definition 5.7.1
Let $G_{\mathbb{C}}$ be a complex simple Lie group, B a spherical Borel subgroup

$$\mathfrak{B}^{\infty}G := (L_{frac}(G,\sigma)/L_{frac}(B,\sigma) \times \Delta_1)\,/\sim,$$

where $\sim$ describes as usual equivalence in the boundary simplices of Δ under the conjugation action. The subscript frac indicates, that the group is defined over the fraction field of the ring of functions in question.

Note that Δ_1 is not Δ used for the description of the affine buildings.

Lemma 5.7.1
$\mathfrak{B}^{\infty}G$ is the spherical building for $\mathfrak{B}^{\pm}_{aff}G$.

Proof. The proof consists of several steps:

- Let W be the Weyl group of G. Then $\mathfrak{B}^{\infty}G$ is a spherical building of type W. This is a classical result – see [AB08].

- By the work of Marc Ronan, we know that the sectors define a spherical building of type W. Let $\mathcal{A}$ be an apartment in $\mathfrak{B}^{+}$, S_i the sectors. Thus to prove the lemma we have to identify the sectors S_i with chambers in $\mathfrak{B}^{\infty}$. So let S_i be a sector in an apartment, $[\gamma_i] \in S$ a geodesic half-ray passing through the chambers C_k. Let B_k be the affine Borel subgroup corresponding to C_k. The spherical Borel subgroup corresponding to the cell S_{∞} is defined to be

$$B_{\infty} := \lim_{k\to\infty} B_k := \{g \in G | \exists j > 0 : g \in B_k \forall k \geq j\}\,.$$

  This is a spherical subgroup. As two sectors in an affine building converge to the same sector at infinity iff they share a common subsector this definition is independent of the sector chosen. □

Theorem 5.7.2 (Embedding of the spherical building at infinity)
The cone over $\mathfrak{B}^{\infty}$ embeds equivariantly and surjectively into the loop algebra $L(\mathfrak{g},\sigma)$.

We can interpret this loop algebra as the algebra $\{\widehat{u} \in \widehat{L}(\mathfrak{g},\sigma) | r_d = 0\}$.

Proof. The proof follows directly the pattern of the proof of theorem 5.5.2. □

Continuity of the Adjoint action yields the result:

Theorem 5.7.3
The embedding of the cone building is the closure of the embedding on the thickened universal twin building. The Adjoint action of $\widehat{L}(G,\sigma)$ on $\widehat{L}(\mathfrak{g},\sigma)$ preserves the identification of sectors in the universal twin building with chambers in the building $\mathfrak{B}^{\infty}$.

5.8 The Hilbert space setting of H^1-loops

In many papers describing the geometry of Kac-Moody groups — cf. e.g. [HPTT95], [Ter89], [Ter95] —, the setting of H^1-loops with values in a compact simple Lie group G, acting on the space of H^0-loops in $\mathfrak{g}$, is used. Our results carry over to this setting:

One can describe $\mathfrak{B}^{\pm} = (LG \times \Delta)/\sim$ and $\mathfrak{B}^{\infty}$ in exactly the same way. One change has to be made by the definition of the pseudodistance. As we defined it, the distance between two buildings depends on the convergence radius of the functions transforming one building into the other. For H^1-functions this definition is useless: Tthe space of buildings such that the distance is infinite is just to big. So it seems meaningful to introduce another distance function:

Definition 5.8.1 (H^1-distance)
Let $\Delta_1, \Delta_2 \in \mathfrak{B}^\epsilon$. Let $fB \in \Delta_1$, $gB \in \Delta_2$ and let $fg^{-1} = \sum a_n e^{int}$ be the Fourier series expansion. $\nu_r(\Delta_1, \Delta_2) = max_r\{\sum n^r a_n e^{int} < \infty\}$ and $d_r(\Delta_1, \Delta_2) = e^{-\nu(\Delta_1, \Delta_2)}$.

Lemma 5.8.1
The H^1-distance is an ultrametric pseudodistance.

Proof. The proof follows the pattern of the proof for lemma 5.5.3. □

At the moment of this writing, it is unclear if there is a suitable distance function which is meaningful in the whole range of regularity conditions.

5.9 Universal algebraic twin building

In the theory of algebraic buildings one is trying to construct a universal algebraic twin building. This should be a twin building that realizes all possible twinnings. In two papers — cf. [RT94] and [RT99] — Marc Ronan and Jaques Tits succeeded to construct for a tree T_n a universal twin tree T_n^{univ} that realizes all possible twinnings.

There are two possibilities:

1. One can construct a non symmetric universal twin building consisting of an algebraic building and a universal algebraic twin building (non-symmetric universal algebraic twin building).

2. One can aim for a symmetric description consisting of two universal twin buildings (symmetric universal algebraic twin building).

We propose the following conjectures:

Conjecture 5.9.1
Let $\widehat{L}(G, \sigma)$ denote the formal completion of an algebraic affine Kac-Moody group in one direction. The associated geometric universal twin building is a non-symmetric universal algebraic twin building.

Conjecture 5.9.2
Let $\widehat{L}(G, \sigma)$ denote the formal completion of an algebraic affine Kac-Moody group. The associated geometric universal twin building is a symmetric universal algebraic twin building.

Chapter 6

Flag complexes for universal twin buildings

6.1 The finite dimensional blueprint: Flag complexes and buildings

For the classical groups there exists a very explicit description of buildings in terms of flag complexes. The buildings are constructed as a complex of flags in a certain vector space. In the case of groups of type A_n one uses arbitrary subspaces, in case of types B_n, C_n and D_n one restricts to flags constructed of subspaces that satisfy certain restrictions with respect to the invariant form.

Type A_n

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $G = SL(n;\mathbb{K})$ act on $V^n \simeq \mathbb{K}^n$ and let $\mathcal{F} = \{F = (V_1 \subset V_2 \subset \cdots \subset V_i)\}$ be the set of all flags in V_n. A flag is called maximal if $\dim_{\mathbb{K}} V_i/V_{i+1} = 1$ for $i \in \{1, \ldots, n-1\}$.

Let $\mathcal{K} = (k_1, \ldots, k_j) \subset (1, \ldots, n)$. A flag $F \subset \mathcal{F}$ is called of type $\mathcal{K}$ iff $\dim(V_i) = k_i, i = 1, \ldots, j$. We denote by $\mathcal{F}_{\mathcal{K}}$ the set of flags of type $\mathcal{K}$ and by $\mathbb{P}(n)$ the power set of $(1, \ldots n)$. Hence

$$\mathcal{F} = \bigcup_{\mathcal{K} \in \mathbb{P}(n)} \mathcal{F}_{\mathcal{K}} \,.$$

The $SL(n)$-action on V defines a natural action on $\mathcal{F}$:

$$SL(n) : \mathcal{F} \longrightarrow \mathcal{F} \qquad g \cdot (V_1 \subset \cdots \subset V_i) \mapsto (g \cdot V_1 \subset \cdots \subset g \cdot V_i)$$

Let $F = (V_1 \subset V_2 \subset \cdots \subset V_j)$ be a flag. The stabilizer subgroup $P_F \subset SL(n, \mathbb{C})$ of F consists of all $g \in SL(n, \mathbb{C})$ such that $g \cdot V_i = V_i$ for all i.

It is elementary to check the following well-known facts:

1. The Borel subgroups of $SL(n, \mathbb{K})$ are exactly the stabilizer subgroups of maximal flags. The standard Borel subgroup B_0 stabilizes the standard flag

$$F = (V_i = \text{span}(e_1, \ldots, e_i) \text{ for } i = 1, \ldots, n) \,.$$

2. The parabolic subgroups are stabilizers of partial flags.

3. The $SL(n, \mathbb{K})$-action preserves the $\mathcal{K}$-type of flags.

4. The $SL(n, \mathbb{K})$-action is transitive on $\mathcal{F}_{\mathcal{K}}$.

Hence, there is a $G_{\mathbb{K}}$-equivariant bijection between parabolic subgroups of $SL(n, \mathbb{K})$ (remember that Borel subgroups are parabolic) and flags in $V^n \simeq \mathbb{K}^n$.

In section 5.2.3 we identified the building $\mathfrak{B}$ associated to $Sl(n, \mathbb{K})$ with the complex of parabolic subgroups of $SL(n, \mathbb{K})$.

Suppose $\mathbb{K} = \mathbb{C}$. Then the associated building can be identified with the complex of flags in $V = \mathbb{C}^n$: We define a partial ordering on the set of flags: Let $f \leq f'$ iff $f \subset f'$. Identify a flag $f = \{V_i\}_{i \in I}$ with a simplex Δ_f of dimension $|I| - 1$. Δ_f is in the boundary of $\Delta_{f'}$ iff $f \leq f'$. The resulting simplicial complex is as a simplicial complex isomorphic to the building.

In this description apartments can be characterized by frames: We define a frame A_f for V^n to be a set of n 1-$\mathbb{K}$-dimensional subspaces of V^n, denoted U_i, such that $V = \text{span}\langle U_1, \ldots, U_n\rangle$. A subspace $V_i \subset V^n$ is called subjacent to the frame A_f iff there are elements $\{U_{i_1}, \ldots, U_{i_l}\} \in A_f$ such that $V_i = \text{span}\langle U_{i_1}, \ldots, U_{i_l}\rangle$.

The apartment A corresponding to the frame A_f consists of the simplices associated to the flags subjacent to A_f.

The Weyl group of $SL(n, \mathbb{K})$ is the symmetric group in n letters. It is realized as the permutation group of the elements of any frame. This action induces an action on the subspaces subjacent to the frame and hence on the flags constructed of those subspaces and thus on the apartment.

If $\mathbb{K} = \mathbb{C}$ then the associated building corresponds (for example via the isotropy representation) to the compact symmetric space A_n of type II resp. to its non-compact dual of type IV hence to the spaces $M = SU(n)$ and $M_D = SL(n, \mathbb{C})/SL(n, \mathbb{R})$.

If $\mathbb{K} = \mathbb{R}$ then the associated building corresponds to the symmetric space $A\,I_n$ of type I resp. type III hence to the spaces $M = SU(n)/SO(n)$ and $M_D = SL(n, \mathbb{R})/SO(n)$.

The symplectic groups: type C_n

We start with the symplectic groups, thus groups of type C_n:

Let $G_{\mathbb{C}} := Sp(n)$; the group $Sp(n)$ acts on $V^{2n} := \mathbb{C}^{2n}$. Let $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a basis of V^{2n}; a symplectic form can be defined by $\langle e_i, f_j\rangle = \delta_{ij} = -\langle f_j, e_i\rangle$ and $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$. We can define $Sp(n)$ to be the group of linear transformations preserving the symplectic form. To describe the building for $Sp(n)$ we construct a flag complex of isotropic subspaces:

Definition 6.1.1
A subspace $V \subset \mathbb{C}^n$ is called isotropic if $V \subset V^{\perp}$. A chain of isotropic subspaces is called an isotropic flag.

An isotropic flag F in V^{2n} has a length $l(V) \leq n$. Let $\{V_1, \ldots, V_k\}$ be an isotropic flag. There is a unique completion $\{V_1, \ldots, V_k, V_{k-1}^{\perp}, \ldots, V_1^{\perp}, V^{2n}\}$.

Definition 6.1.2
$\Delta_{Sp(n)}$ is defined to be the poset of completed isotropic flags with a partial order defined by inclusion.

Theorem 6.1.1
$\Delta_{Sp(n)}$ is a thick spherical building of type C_n.

Proof. See [AB08], section 6.6 or [Gar97], chapter 10. □

The orthogonal groups: type B_n and D_n

Similar to the symplectic groups we start again with an m-dimensional vector space $V^m(\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and define a non-degenerate quadratic form Q on V^m in the following way: For a standard basis $e_1, \dots, e_n$ of V^m use

$$Q(e_i, e_j) = \delta_{i+j,m+1} .$$

If $\mathbb{K} = \mathbb{C}$ then this quadratic form is equivalent to the standard quadratic form, if $\mathbb{K} = \mathbb{R}$ then the form has Witt index $n = \lfloor \frac{m}{2} \rfloor$ on V^m. The orthogonal group $O_Q(m, \mathbb{K})$ is the group of linear automorphisms of V^m that preserve Q. We study again the complex of isotropic subspaces. If $m = 2n + 1$ then the group $O_Q(m, \mathbb{K})$ is of type B_n; the associated building is constructed completely equivalent to the case of C^n as the complex of isotropic subspaces with respect to Q.

In contrast if $m = 2n$, hence $O_Q(m, \mathbb{K})$ of type D_n then the complex of isotropic subspaces is not thick. The spherical building of type D_n is the oriflamme complex associated to Q. For details see [AB08], section 6.7. and [Tit74].

6.2 Grassmannians and periodic flag varieties

In this section, we describe the infinite dimensional analogue of flags, that we will use for the description of universal geometric twin buildings.

To this end, review the finite dimensional construction in a formal way: define the Grassmannian $\mathrm{Gr}(k, n; \mathbb{K})$ to be the space of all k-dimensional subspaces in $V^n(\mathbb{K})$. Then every maximal flag is in a natural way an element of $\mathrm{Gr}(n, \mathbb{K}) = \prod_k \mathrm{Gr}(k, n; \mathbb{K})$ and any partial flag is an element of $\mathrm{Gr}(\mathcal{K}, n; \mathbb{K}) = \prod_{k \in \mathcal{K}} \mathrm{Gr}(k, n; \mathbb{K})$. Because of the inclusion relation in the definition of flags not every element in $\mathrm{Gr}(\mathcal{K}, n; \mathbb{K})$ corresponds to a flag.

Hence, to define an infinite dimensional analogue we start with the definition of two infinite dimensional Grassmannians denoted $Gr^+(H)$ and $Gr^-(H)$ that replace the finite dimensional Grassmannian. The dimension of subspaces is replaced by the notion of a virtual dimension. The two components of the universal geometric twin building correspond to flags in those two infinite dimensional Grassmannians.

These Grassmannians were introduced by Mikio Sato [SS83], to describe integrable systems; their theory is treated in detail in chapters 6, 7 and 8 of the monograph [PS86]. We describe the foundations and refer to [PS86] for further details.

Let $H^n = L^2(S^1, \mathbb{C}^n)$ denote the separable Hilbert space of square summable functions on S^1 with values in $\mathbb{C}^n$. Let $H = H^{++} \oplus H^0 \oplus H^{--}$ be a polarization (for example the one induced by the action of $-i\frac{d}{d\theta}$). Set $H^+ = H^{++} \oplus H^0$ and $H^- = H^0 \oplus H^{--}$.

While a Hilbert space — as does any finite dimensional vector space — has no distinguished direction, it is the combination of the polarization and a "grading operator", which we will define later, that breaks the symmetry and makes the twin structures appear that we found in Kac-Moody theory.
Following [PS86], definition 7.1, the positive Grassmannian is defined as follows:

Definition 6.2.1 (Positive Grassmanian)
The positive Grassmannian $Gr^+(H)$ is the set of all closed subspaces W of H such that

1. *the orthogonal projection $pr_+ : W \longrightarrow H^+$ is a Fredholm operator,*

2. *the orthogonal projection* $pr_{--} : W \longrightarrow H^{--}$ *is a Hilbert-Schmidt operator.*

One can assign to each element of $Gr^+(H)$ an integer called the virtual dimension:

Definition 6.2.2 (virtual dimension)
Let $W \in Gr^+(n)$. *The virtual dimension of* W *is defined by* $\nu(W) = dim(ker\, pr_+) - dim(coker\, pr_+)$.

For many applications $Gr^+(H)$ is just too big. Hence, it proves necessary to define different versions of restricted Grassmannians. This is done by imposing stronger conditions on the projection operators pr_+ and pr_{--}. It should be chosen in a way that loop groups (resp. Kac-Moody groups) satisfying usual regularity conditions act nicely on the Grassmannians. Fundamentally, this type of conditions describes how "far" away a subspace can be maximally from H^+ (resp. H^-).
Some important examples are:

Definition 6.2.3 (positive algebraic Grassmanian)
The positive algebraic Grassmannian $Gr_0^+(H)$ *consists of subspaces* $W \subset Gr^+(H)$ *such that* $z^kH^+ \subset W \subset z^{-k}H^+$.

Using the explicit description $H = L^2(\mathbb{S}^1, \mathbb{C}^n)$, $Gr_0^+(H)$ consists exactly of the elements $W \in Gr(H)$ such that the images of $pr_{--} : W \longrightarrow H^{--}$ and $pr_+ : W^\perp \longrightarrow H^+$ are polynomials. Stated in an operator theoretic way it consists of those operators, such that there is a $k > 0$ such that for $|i - j| > kn$ the coefficient $a_{ij} = 0$. In the language of n-Laurent operators this is equivalent to W being defined by an algebraic loop of degree less than k — cf. [PS86] and [GGK03].
For the definition of similar subspaces – the rational Grassmannian $Gr_1(H)$, $Gr_\omega(H)$ and the smooth Grassmannian $Gr_\infty(H)$ — cf. [PS86].
For the applications we have in mind for Kac-Moody symmetric spaces and polar actions we need two new regularity conditions: to describe the building in the situation of polar actions on Hilbert spaces as considered by Chuu-Lian Terng, we use positive Sobolev Grassmannians:

Definition 6.2.4 (positive H^1-Grassmanian)
The H^1*-positive Sobolev Grassmannian* $Gr_{H_1}^+(H)$ *consists of the graphs of operators* $T : H_s \longrightarrow H_s^\perp$ *whose entries* T_{pq} *satisfy:* $|T_{pq}|(p - q)^2 < \infty$.

As Kac-Moody symmetric spaces are tame Fréchet we need a tame Grassmannian:

Definition 6.2.5 (positive tame Fréchet Grassmanian)
The tame Fréchet Grassmannian $Gr_t^+(H)$ *consists of the graphs of operators* $T : H_s \longrightarrow H_s^\perp$ *whose entries* T_{pq} *are exponentially decreasing:* $|T_{pq}|e^{(p-q)n} < \infty\ \forall n$.

Compare this definition with the notion of exponential weights in [GW84].
In the language of n-Laurent operators this condition is equivalent to the loop being in MG, i.e. the restriction to holomorphic loops defined on the whole of $\mathbb{C}^*$ — cf. [GGK03].
The next tool we will use is a grading operator:

Definition 6.2.6 (Grading operator)
An operator $G : H^n \longrightarrow H^n$ *is called a (positive) grading operator iff*

1. *there is a basis* $E := \{e_i\}$ *of* H^n *such that* G *is a permutation of those basis vectors,*

2. $G(H^+) = H^{++}$.

3. *Let* $E^{++} \subset E$ *be a basis of* H^{++}*. Then for every* $e_1 \in E^{++}$ *there is* $e_0 \in E^0$ *such that* $G^k(e_0) = e_1$ *(in this case* k *is called the grade of* e_1*).*

Example 6.2.1
In the explicit realization $H = L^2(S, \mathbb{C}^n)$ *we can take the grading operator to be multiplication by* $z = e^{it}$*.*

Definition 6.2.7 (reduced Grassmanian)
The reduced positive Grassmannian $Gr^{n,+}(H^n)$ *consists of subspaces* $W \subset Gr^+(H^n)$ *such that* $G(W) \subset W$ *(or explicitly* $zW \subset W$*).*

The definition of the other types of reduced Grassmannians, especially reduced algebraic Grassmannians, reduced H^1- and reduced tame Grassmannians is self explaining.

The following theorem — cf. theorem 8.3.2, [PS86] — shows this to be the correct notion to work well with the action of loop groups.

Theorem 6.2.1
The group $L_{\frac{1}{2}}U_n$ *acts transitively on* $Gr^{n,+}$ *and the isotropy group of* H^+ *is the group* U_n *of constant loops.*

This theorem yields the equivalences $\Omega_{\frac{1}{2}}U_n = Gr^n(H)$ and $\Omega_{alg}U_n = Gr^n_0$. Similar statements hold for $Gr^{n,+}_1(H)$, $Gr^{n,+}_\omega(H)$, $Gr^{n,+}_\infty(H)$ and $Gr^{n,+}_t(H)$; for $Gr^{n,+}_1(H)$, $Gr^{n,+}_\omega(H)$, $Gr^{n,+}_\infty(H)$ a proof can be found in [PS86]; this proof adapts to the case of $Gr^{n,+}_t(H)$ straight forwardly.
The next step is the definition of the flag varieties: following again [PS86], definition 8.7.5, we define (full) periodic flags:

Definition 6.2.8 (full positive periodic flag manifold)
The full positive periodic flag manifold $Fl^{n,+}$ *consists of all sequences* $W_k, k \in \mathbb{Z}$ *of subspaces in* H^n *such that*

1. $W_k \subset Gr^{n,+}H$*,*

2. $W_{k+1} \subset W_k \forall k$ *and* $dim(W_k/W_{k+1}) = 1$*,*

3. $zW_k = W_{k+n}$*.*

As elements of a flag satisfy $zW_k = W_{k+n} \subset W_k$, all elements of a flag are taken from $Gr^{n,+}$.

Let $\{e_1, \ldots, e_n\}$ be a basis of $V^n \simeq \mathbb{C}^n$ and $V_i := \text{span}\langle e_{i+1}, \ldots, e_n\rangle$. We define the positive normal flag to be the flag $\{W_{k'}\}_{k' \in \mathbb{Z}}$ such that $W_{k'} = W_{kn+l} := z^k W_l$ for $k' = kn+l, k \in \mathbb{Z}, l \in \{0, \ldots, n-1\}$ and $W_l := \{f : \mathbb{C} \longrightarrow V^n | f \text{ is holomorphic and } f(0) \subset V_i\}$.

To define the manifolds of partial periodic flags the virtual dimension is used. It behaves well with respect to inclusions:

as the index is additive under composition of Fredholm operators, i.e. for two Fredholm operators A and B we have $\text{ind}(AB) = \text{ind}(A) + \text{ind}(B)$ — cf. [Con90], chapter 11—, one calculates for the virtual dimension: let $W \subset W' \in Gr^+(n)$ and $\dim(W/W') = l$. Then $\nu(W') = \nu(W) + l$. For $W \in \text{Gr}^{n,+}$ an important special case is $\nu(W_{k+n}) = \nu(zW_k) = \nu(W_k) + n$.

We now aim for the definition of partial flags. For this purpose let $\mathcal{K} \subset \mathbb{Z}$ such that with $k \in \mathcal{K}$ also $k + nl \in \mathcal{K}, \forall l \in \mathbb{Z}$. Furthermore set $m_\mathcal{K} := \#\{\mathcal{K} \cap \{1, \ldots, n\}\}$. Denote those $m_\mathcal{K}$-numbers $k_1, \ldots, k_{m_\mathcal{K}}$.

Definition 6.2.9 (partial positive periodic flag manifold)
The positive periodic flag manifold $Fl^{n,+}_{\mathcal{K}}$ *consists of all flags* $\{W_k\}, k \in \mathbb{Z}$ *in* H^n *such that*

1. $W_k \subset Gr^+H$,
2. $W_{k+1} \subset W_k \forall k$,
3. $zW_k = W_{k+n}$.
4. *For every flag* $\{W_k\}_{k\in\mathbb{Z}}$ *the map* $\nu : (\{W_k\}_{k\in\mathbb{Z}}) \longrightarrow \mathbb{Z}$ *mapping every subspace* W_k *to its virtual dimension is a surjection onto* $\mathcal{K}$.

This definition contains the one for full flags by using $\mathcal{K} = \mathbb{Z}$. In contrast if $m_{\mathcal{K}} = 1$, we call a flag trivial. Trivial flags are in bijection with elements of $Gr^{n,+}(H)$ under the identification $Gr^{n,+}(H) \ni W_0 \leftrightarrow \{z^k W_0\}_{k\in\mathbb{Z}} \in Fl^{n,+}_{\nu(W_0)+n\mathbb{Z}}$.
We normalize our notation by requiring that W_0 is the element with the lowest nonnegative virtual dimension. In the case of full flags this is 0, in the case of partial flags it may be between 0 and $n-1$.
The space of all partial flags carries a natural poset structure via inclusion of subspaces.

Definition 6.2.10
Let $\{W_k\}$ *be a flag. Then the set of flags* $\{W'_l\}$ *such that* $\{W'_l\} \subset \{W_k\}$ *is denoted by* $\{\leq W_k\}$ *and called the boundary of* $\{W_k\}$.

Lemma 6.2.1
The flag complex of all periodic flags is an n-dimensional simplicial complex, such that every simplex is contained in a simplex of maximal dimension.

Nevertheless it is not a chamber complex as the definition of a chamber complex includes connectedness. But we will check that every connected component is a chamber complex (compare 5.2.1).

Proof. The first assertion follows as for any (partial) periodic flag $\{W_k\}_{k\subset\mathcal{K}}$ of type $\mathcal{K}$ for every $\mathcal{K}' \subset \mathcal{K}$, there is a partial flag $\{W'_k\}_{k\in\mathcal{K}'}$ of type $\mathcal{K}'$ such that $\{W'_k\}_{k\in\mathcal{K}'} \leq \{W_k\}_{k\in\mathcal{K}}$. The second assertion follows as every partial periodic flag may be completed to a maximal periodic flag. Hence, every flag is contained in the boundary of a chamber. □

The description of the structure of Fl^n shows a direct connection with results of [Mit88]. The crucial point is that all manifolds of periodic flags fiber over the Grassmannian with the fibers being finite (partial) flag manifolds.

For the maximal flag manifold this fiber structure can be described by the exact sequence:

$$1 \longrightarrow Fl(\mathbb{C}^n) \longrightarrow Fl^{(n)} \longrightarrow Gr^{(n)} \longrightarrow 1\,.$$

To see this, note that the periodicity condition determines a whole maximal flag as soon as the spaces $\{W_0, W_1, \ldots, W_{n-1}\}$ are specified. After the identification of $W_0/W_n \simeq \mathbb{C}^n$ the spaces $\{W_1, \ldots, W_{n-1}\}$ determine a maximal flag in $\mathbb{C}^n$.

To construct a similar fibration for manifolds of partial flags the fiber has to be replaced by manifolds of partial flags. In doing so we have to construct the fibers in a way that the resulting flags get the correct virtual dimensions:

so let $\{W\}_{k\in\mathbb{Z}}$ be a flag in $Fl^{n,+}_{\mathcal{K}}$. Take the projection: $\{W_k\}_{k\in\mathbb{Z}} \longrightarrow W_0$. Again $W_0/zW_0 \simeq \mathbb{C}^n$. The intersections $U_l := W_l \cap W_0/zW_0, l \in \{1, m_{\mathcal{K}}\}$ determine the flag completely. They describe a finite dimensional partial flag $U_{m_{\mathcal{K}}-1} \subset \cdots \subset U_1$ in $zW_0/W_0 \simeq \mathbb{C}^n$. The dimensions are $\dim(U_l) = \text{virt}\dim(W_l) - \text{virt}\dim(zW_0)$.

This shows that $Fl^{n,+}_{\mathcal{K}}$ fibers over $Gr^{n,+}$, the fiber being $Fl_{\kappa_1,\ldots,\kappa_{m_{\mathcal{K}}}}(\mathbb{C}^n)$ where $\kappa_i := k_i + \text{virt}\dim(W_0)$.

$$1 \longrightarrow Fl_{\kappa_1,\ldots,\kappa_{m_{\mathcal{K}}}}(\mathbb{C}^n) \longrightarrow Fl^{(n)}_{\mathcal{K}} \longrightarrow Gr^{(n)} \longrightarrow 1\,.$$

Up to now we focused our attention to the positive part. A completely symmetric theory can be developed for H^-:

Definition 6.2.11 (Negative Grassmanian)
The negative Grassmannian $Gr^-(H)$ is the set of all closed subspaces W of H such that

1. *the orthogonal projection $pr_- : W \longrightarrow H^-$ is a Fredholm operator,*
2. *the orthogonal projection $pr_{++} : W \longrightarrow H^{++}$ is a Hilbert-Schmidt operator.*

The subspaces $Gr^-_0(H)$, $Gr^-_1(H)$, $Gr^-_\omega(H)$, $Gr^-_\infty(H)$ and $Gr^-_t(H)$ and $Gr^{n,-}_0(H)$, $Gr^{n,-}_1(H)$, $Gr^{n,-}_\omega(H)$, $Gr^{n,-}_\infty(H)$ and $Gr^{n,-}_t(H)$ are defined as in the positive case.

We define full negative flags by:

Definition 6.2.12 (full negative periodic flag manifold)
The full negative periodic flag manifold $Fl^{-,n}$ consists of all sequences $V_k, k \in \mathbb{Z}$ of subspaces in H^n such that

1. *$W_k \subset Gr^-(H)$,*
2. *$W_{k+1} \subset W_k \forall k$, $dim(W_k/W_{k+1}) = 1$,*
3. *$\frac{1}{z}W_k = W_{k+n}$.*

The negative normal flag and partial flags are defined similarly as in the case of positive flags.

6.3 The special linear groups: type $\widetilde{A}_n$

6.3.1 The affine building

As in the previous section let $H^n = L^2(S^1, \mathbb{C}^n)$. To describe the combinatorial structure of buildings in the language of the simplicial complex of partial flags, the easiest way is to define apartments:

Definition 6.3.1 (frame)
A frame is a sequence of subspaces $\{U_k\}_{k\in\mathbb{Z}} \subset H^n$ such that $U_{k+n} = zU_k$ and $H^n = \bigcup U_k$.

We now need a description of regularity conditions for frames:

Lemma 6.3.1
Let $\{U_k\}$ be a periodic frame and let $Gr^{n,+}_x$ be a Grassmannian of some prescribed regularity. The following are equivalent:

1. *The space $W_0 := \bigoplus_{i=0}^{\infty}\{U_i\}$ is in $Gr^{n,+}_x$.*
2. *All spaces $W_k := \bigoplus_{i=k}^{\infty}\{U_i\}$ are in $Gr^{n,+}_x$.*
3. *Let $\pi : \{U_k\} \longrightarrow \{U_k\}$ be an admissible permutation. All subspaces $W_{\pi,k} := \bigoplus_{i=k}^{\infty}\{U_{\pi(i)}\}$ are in $Gr^{n,+}_x$.*

Proof. The implications $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are trivial. The implication $(i) \Rightarrow (ii)$ follows as $W_k = U_k \oplus W_{k+1}$. Hence $W_k \in Gr_x^{n,+}$ iff $W_{k+1} \in Gr_x^{n,+}$. The implication $(ii) \Rightarrow (iii)$ follows, as $W_{\pi,k} \cap W_k$ has finite codimension in $W_{\pi,k}$ and in W_k. Hence $W_k \in Gr_x^{n,+}$ (resp. $W_{\pi,k} \in Gr_x^{n,+}$) is equivalent to $W_{\pi,k} \in Gr_x^{n,+}$. □

Definition 6.3.2
A frame satisfying one — and hence all — equivalent conditions of lemma 6.3.1 is called a frame of regularity X.

Let $\{W_k\}_{k\in\mathbb{Z}}$ denote a flag. A frame, such that $W_k := W_{k+1} \oplus U_k$, will be called normal with respect to $\{W_k\}$. The choice of a normal frame is not unique. We call a frame $\{U_k\}_{k\in\mathbb{Z}}$ orthogonal if $U_k \perp W_{k+1}$.

Definition 6.3.3 (admissible permutation)
We call a permutation $\pi : \{U_k\}_{k\in\mathbb{Z}} \longrightarrow \{U_k\}_{k\in\mathbb{Z}}$ admissible if $\pi(U_{k+n}) = \pi(U_k) + n$.

Definition 6.3.4 (affine Weyl group associated to $\{U_k\}_{k\in\mathbb{Z}}$)
The affine Weyl group W_{aff} is defined to be the group of admissible permutations of $\{U_k\}_{k\in\mathbb{Z}}$.

W_{aff} is independent of the choice of the periodic frame $\{U_k\}_{k\in\mathbb{Z}}$. In the algebraic setting, using $Gr_0(W)$, this definition coincides with the classical definition for the affine Weyl group of the Kac-Moody algebra of type $\widetilde{A}_n$ — cf. [Kac90]. Hence, it is well-known, that $W_{\text{aff}} \simeq \mathbb{Z}^{n-1} \rtimes \text{Sym}(n)$.

W_{aff} is generated by a set of transformations $S := \langle s_{i,i+1} | i = 1, \dots, n\rangle$ such that $s_{i,i+1}(U_{ln+i}) = U_{ln+i+1}$, $s_{i,i+1}(U_{ln+i+1}) = U_{ln+i}$ and $s_{i,i+1}(U_{ln+k}) = U_{ln+k}, k \neq i, i+1$.

Much insight can be gained by a description of the action of the two factors in the decomposition $W_{\text{aff}} \simeq \mathbb{Z}^{n-1} \rtimes \text{Sym}(n)$:

An element $\pi \in \text{Sym}(n)$ acts on $\{U_{ln+k}\}$ via permutation of k:

$$\pi(\{U_{ln+k}\}) = \{U_{ln+\pi(k)}\}\,.$$

An element $A = (a_1, \dots a_{n-1}) \in \mathbb{Z}^{n-1}$ acts on $\{U_{ln+1}, U_{ln+2}, \dots, U_{ln+n-1}\}$ via

$$A(\{U_{ln+1}, U_{ln+2}, \dots, U_{ln+n-1}\}) = \{U_{(l-a_1)n+1}, U_{(l+a_1-a_2)n+2}, \dots, U_{(l+a_{n-1})n+n-1}\}.$$

Admissible permutations induce maps on the flag manifolds: let $w : \{U_l\} \longrightarrow w(U_l)$. Then the flag $\{W_k\} := \bigcup U_l, l \geq k$ is mapped onto the flag $w(\{W_k\}) = \bigcup w(U_l), l \geq k$. For each element W_k in a flag $\{W_k\}$ one can check: $\nu(W_k) = \nu(w(W_k))$. Hence we get:

Lemma 6.3.2
The action of the Weyl group on the restricted Grassmannian preserves the virtual dimension. Furthermore on the space of all flags $\mathcal{F}$, it preserves the $\mathcal{K}$-type of flags.

In other words: the W_{aff}-orbit of a flag $\{W_k\} \in Fl_{\mathcal{K}}^{n,+}$ is completely contained in $Fl_{\mathcal{K}}^{n,+}$.

Proof. The second part of this lemma is a direct consequence of the first part. For the first part, it is sufficient to show that it is satisfied for all generators $s_{i,i+1}$; to verify this, one has to distinguish between two cases: let $s_{i,i+1}$ interchange U_{i+nl} and $U_{i+1+nl}, l \in \mathbb{Z}$. If with $U_{i+nl} \subset W_k$ also $U_{i+1+nl} \subset W_k$, then W_k is stable under $s_{i,i+1}$, so also the virtual dimension is stable; if there is $U_{i+1+nl} \subset W_k$ such that $U_{i+nl} \not\subset W_k$, then $\dim(W_k/W_k \cap s_{i,i+1}(W_k)) = 1$. So virt $\dim(W_k) = 1 +$ virt $\dim(W_k \cap s_{i,i+1}(W_k))$. Symmetrically, $\dim(s_{i,i+1}(W_k)/(W_k \cap s_{i,i+1}(W_k))) = 1$ leads to virt $\dim(s_{i,i+1}(W_k)) = 1 +$ virt $\dim(W_k \cap s_{i,i+1}(W_k))$, which shows the equivalence. □

Definition 6.3.5 (flag apartment)
Let $\{W_k\}$ be a full flag, $\{\leq W_k\}$ the boundary of $\{W_k\}$ and $\{U_k\}_{k\in\mathbb{Z}}$ a frame. The flag apartment $\mathcal{A}(\{W_k\}, U_k)$ consists of all flags that are admissible permutations of flags in $\{\leq W_k\}$ with respect to $\{\{U_k\}_{k\in\mathbb{Z}}\}$.

A flag apartment is an abstract simplicial complex. It can be identified with a simplicial chamber complex, whose chambers correspond to full flags. As W_{aff} is a Coxeter group that acts transitively on the chambers of the flag apartment, it is in fact a Coxeter complex.

The flag apartments containing a given flag $\{W_k\}$ are in bijection with frames $\{U_k\}$ associated to this flag.

For a pair of two flags of the same $\mathcal{K}$-type $\{W_k\}$ and $\{W'_{k'}\}$, there exists a flag apartment containing both of them iff there is a frame $\{U_k\}$ for $\{W_k\}$, such that for some $w \in W_{\text{aff}}$ $w(\{U_k\})$ is a frame for $\{W'_{k'}\}$.

Unfortunately – in sharp contrast to the finite dimensional situation – it is not true, that every pair of full flags is contained in a joint flag apartment. Nevertheless, we have the following: in contrast to the finite dimensional and even the infinite dimensional algebraic situation – it is not true, that every pair of full flags is contained in a joint apartment. In contrast we have:

Lemma 6.3.3
For a pair of two flags $\{W_k\}$ and $\{W'_{k'}\}$, there exists an apartment containing both of them, iff they are compatible in the sense that for all elements $W_j \in \{W_k\}$ there are elements $W'_{j'}, W'_{l'} \in \{W'_{k'}\}$ such that $W'_{j'} \subset W_j \subset W'_{l'}$ and vice versa. Compatibility defines an equivalence relation on the space of flags.

Corollary 6.3.1
If the flags $\{W_k\}$ and $\{W'_{k'}\}$ are in $Gr_x^{n,+}$ then all flags in the apartment containing them are in $Gr_x^{n,+}$.

Proof. This is a consequence of lemma 6.3.1, condition (iii). □

Corollary 6.3.2
A flag $\{W_k\}$ is contained in a apartment containing the standard normal flag iff $W_k \subset Gr_0^{n,+}$.

Proof of lemma 6.3.3. The complex of all flags is a chamber complex. Hence, without loss of generality we can assume that $\{W_k\}$ and $\{W'_k\}$ are two maximal compatible flags. For each $k \in \mathbb{Z}$, we define the set $\pi(k) := \{m | \exists v \in (W_m \backslash W_{m+1}) \cap (W'_k \backslash W'_{k+1})\}$. We have to show that $|\pi(k)| = 1$ for all k. So for $i \in \{0, \dots, n-1\}$ we choose vectors $v_i \subset \pi(i)$ and put $U_i = \text{span}\langle v_i \rangle$. Furthermore, for $i' = ln + i$ set $U_{i'} = U_{ln+i} = z^l U_i$.

The proof now consists of several steps:

- $\{U_k\}$ is a periodic frame. As the flags $\{W_k\}$ and $\{W'_k\}$ are periodic, the condition $v \in (W_m \backslash W_{m+1}) \cap (W'_k \backslash W'_{k+1})$ is equivalent to the condition $z^l v \in (W_{m+ln} \backslash W_{m+1+ln}) \cap (W'_{k+ln} \backslash W'_{k+1+ln})$.

- $W_k = W_{k+1} \oplus U_k$, $W_m = W_{m+1} \oplus U_k$ for all k and $m \subset \pi(k)$. So the apartment associated to $\{U_k\}$ contains $\{W_k\}$ and $\{W'_k\}$.

- $\pi(k+n) = \pi(k) + n$ follows from the periodicity of $\{W_k\}$ and $\{W'_k\}$.

- So we are left with showing that π is a permutation, that is $|\pi(k)| = 1\ \forall k$. The compatibility condition gives

$$z^{l+1}\{W_k\} = \{W_{k+(l+1)n}\} \subset z\{W'_k\} = \{W'_{k+n}\} \subset \{W'_{k+1}\} \subset \{W'_k\} \subset z^{-l}\{W_k\}\,.$$

  So $W'_k \backslash W'_{k+1} \subset W_{k-ln}$. This shows that there are numbers m such that the set $(W_m \cap W'_k \backslash W'_{k+1})$ is nonempty. On the other hand $W_{k+(l+1)n} \subset W'_{k+1}$ shows that the set of those m is bounded from above. So there is for every k a maximal m such that $(W_m \cap (W'_k \backslash W'_{k+1})$ is nonempty. But then $(W_{m+1} \cap (W'_k \backslash W'_{k+1}))$ is empty. So $(W_m \backslash W'_{m+1}) \cap (W'_k \backslash W'_{k+1})$ is nonempty. So $\pi(k)$ is nonempty for all k. Symmetrically also $\pi^{-1}(m)$ is nonempty for all m. We use now the periodicity condition: $\pi(n+k) = \pi(k) + n$. As each set $\pi(k)$ is nonempty, for every $l \in \{0, \ldots, n-1\}$ there is k, such that $\pi(k) = l (\mathrm{mod}\ n)$.

  This means that $\pi(k + n\mathbb{Z}) = l + n\mathbb{Z}$ as for each l, $l + n\mathbb{Z}$ is in the image. The pigeon hole principle asserts that $|\pi(k)| = 1(\mathrm{mod\ n})$. Then the periodicity shows that $|\pi(k)| = 1$. Hence, π is a permutation and thus an element of W_{aff}. □

Next we investigate isomorphisms between apartments:

Lemma 6.3.4 (apartments are isomorphic)
For every pair of apartments A and A', there is an isomorphism $\varphi : A \longrightarrow A'$. If the intersection $A \cap A'$ is nonempty, then one can choose φ in a way that it fixes $\{W_k\}$ and $\{W'_{k'}\}$ pointwise.

Proof. Isomorphisms between apartments correspond to isomorphisms of their frames. So let A and A' denote two apartments with frames $\{U_k\}$ and $\{U'_l\}$. Then every bijective map $\varphi : \{U_k\} \longrightarrow \{U'_l\}$, that preserves the periodicity condition, induces an isomorphism of the apartments. If there is a full flag $\{W_k\} \subset A \cap A'$, then without loss of generality we can assume that $W_k := \bigoplus_{l \geq k} U_l$ and $W_k := \bigoplus_{l' \geq k} U'_l$. Then every map $\varphi_m : U_k \longrightarrow U'_{k+m}$ for all $m \in \mathbb{N}$ induces an isomorphism $\varphi : A \longrightarrow A'$, stabilizing $\{W_k\}$. If $\{W_k\}$ is a partial flag, then the preservation of the $\mathcal{K}$-type restricts the possibilities for m. φ_m preserves the cells in the boundary of $\{W_k\}$ iff $m \in n\mathbb{Z}$.

If $A \cap A'$ is nonempty and $\{W'_k\} \subset A \cap A'$, then this means just, that $W'_k =:= \bigoplus_{l' \geq k} U'_{\pi(l)} = \bigoplus_{l' \geq k} U_{\pi(l)}$. Then ϕ_0 trivially fixes the intersection. □

Theorem 6.3.1 (Tits building)
Each equivalence class of compatible flags is an affine Tits building of type A_n.

Corollary 6.3.3
The simplicial complex of positive (partial) flags in $Gr_0^{n,+}$ is an algebraic affine Tits building, called $\mathfrak{B}_0^+$.

The proof of theorem 6.3.1 is mainly contained in the previous lemmas:

Proof. We have to show that those equivalence classes satisfy the axioms for a building: they clearly are chamber complexes and the apartments are thin chamber complexes. The Weyl group, acting on each apartment is $\widetilde{A}_n$ The first axiom is contained in lemma 6.3.3 the second axiom is contained lemma 6.3.4.

Hence, one finds an uncountable family of buildings which are all isomorphic to the well known building associated to the algebraic Kac-Moody group.

□

As in the case of the flag manifolds, the same constructions can be performed for the negative flag manifolds. In this way one gets a second set of buildings, denoted $\mathcal{B}^-$. In section 6.3.3 we show that $\mathfrak{B} = \mathfrak{B}^+ \cup \mathfrak{B}^-$ is a universal twin building.

This complex consists of all flags whose subspaces are in $Gr^{n,\pm}$. Those Grassmannians correspond to the loop group (resp. Kac-Moody groups) of $\frac{1}{2}$-differentiable loops — cf. [PS86]. To get restrictions associated to smaller loop groups (i.e. loop and Kac-Moody groups of holomorphic loops) one needs the same constructions but applied to flags whose subspaces are in the described sub-Grassmannians. Hence, we get a family of universal geometric twin buildings corresponding to the different regularity conditions.

For the algebraic case, this construction coincides with the well-known lattice description, as described in [Gar97], [AN02], [Kra02].

6.3.2 The spherical building at infinity

In this section, we will extend the description of the twin building via flag manifolds to incorporate the spherical building at infinity. In contrast to the flags used for the universal geometric twin building where we used flags whose subspaces are elements of the infinite dimensional Sato Grassmannians, flags for the spherical buildings are not elements of the positive resp. negative Sato Grassmannians.

Definition 6.3.6
Let $\mathcal{V}$ be the set of subspaces $V \subset H^n$ such that $zW \subset W$ and $W \subset zW$, containing a basis of $\mathcal{V}$ of some regularity condition. The inclusion relation makes $\mathcal{V}$ into a poset. Define $F(\mathcal{V})$ to be the associated flag complex.

Theorem 6.3.2
$F(\mathcal{V})$ is a spherical building of type A_n. It contains the buildings at infinity of all affine buildings in $\mathfrak{B}^+$ and $\mathfrak{B}^-$. More precisely, for every affine apartment A in a building in $\mathfrak{B}^\pm$ there is a spherical apartment in $F(\mathcal{V})$ as the closure. The restriction to the respective regularity conditions H_0, H_1, H_ω or H_t yields subbuildings that correspond exactly to all buildings in $\mathfrak{B}^\pm$ of the respective regularity.

To emphasize, that it is the spherical building at infinity we will use for $F(\mathcal{V})$ the notation $\mathfrak{B}^\infty$.

Proof. The chambers of the spherical building at infinity correspond to sectors of the affine building — cf. [AB08]. Hence, to prove this theorem we have to review sectors in the language of periodic frames: let $\{U_{kn+i}\}, k \in \mathbb{Z}, i \in \{1, \dots, n\}$ be a frame defining an apartment. Without loss of generality let $\{W_k\} := \{\bigcup\{U_{jn+i}\}, i \in \{1, \dots, n\}, j \geq k\}$ denote a vertex. The fundamental sector centered at $\{W_k\}$ consists of all cells spanned by vertices $\{W'_l\}$ such that $\{W'_l\}\{U_{(l_1+l)n+1}, U_{(l_2+l)n+2}, \dots, U_{(l_n+l)n+n-1}\}$, such that $l_1 \leq l_2 \leq \dots \leq l_n$. A sector associated to $\pi \in W_{\text{spherical}}$ consists of all elements such that $l_{\pi(1)} \leq l_{\pi(2)} \leq \dots \leq l_{\pi(n)}$.

Sectors such that $l_1 \leq l_2 \leq \dots \leq l_n$ correspond to chambers (hence $n-1$-dimensional cells, sectors such that k inequalities are replaces by equalities correspond to k-codimensional cells.

Let S denote a given sector. A subsector $T = (t_1, \dots, t_{n-1}), t_i \geq 0$ is defined by the condition that $l_{\pi(1)} + t_1 \leq l_{\pi(2)} + (t_1 + t_2) \leq \dots \leq l_{\pi(n)} + \sum t_i$.

As usual, we define two sectors to be equivalent if they contain a common subsector.

To get the cell in $\mathfrak{B}^\infty$ which corresponds to a given sector, we can use a series of subsectors $T(n) = (t_1(n), \dots, t_{n-1}(n)), t_i(n) \geq 0, \lim_{n\to\infty} t_i(n) = \infty$. For example take $t_i(n) = nt_i(1)$ for $t_i(1) > 0$ iff $l_{\pi(i)} > l_{\pi(i+1)}$ and $t_i(1) = 0$ iff $l_{\pi(i)} = l_{\pi(i+1)}$.

Then every sequence of periodic flags $\{W_k(n)\}$, such that the flag $\{W_k(n)\} \subset T(n)$ converges to the "flag" $\{V_l\} \in F(\mathcal{V})$ such that $\{V_1\}$ is generated by $U_{l(\pi(1))}$, $\{W_i\}$ by $U_{l(\pi(1))}, \ldots, U_{l(\pi(i))}$.

By definition, the limit is equivariant with respect to the action of $W_{spherical}$. Hence, we get apartments of type $W_{spherical}$ Moreover, any affine apartment in a building in $\mathfrak{B}^{\pm}$ defines a spherical apartment in $\mathfrak{B}^{\infty}$. This shows that $\mathfrak{B}^{\infty}$ is the joint spherical building for all buildings in $\mathfrak{B}^{+}$ and $\mathfrak{B}^{-}$.

□

6.3.3 A universal geometric twin building

We start by defining a twinning between $\mathcal{B}^{+}$ and $\mathcal{B}^{-}$. Focusing on the use of theorem 5.3.4, we do this by the definition of a nonempty symmetric relation $\mathcal{O}$.

We define $\mathcal{O}$ via opposite flags:

Definition 6.3.7 (Opposite flags)
Let $\{W_k^+\}$ be a positive chamber and $\{W_k^-\}$ be a negative chamber. $\{W_k^+\}$ is opposite to $\{W_k^-\}$ iff dim $W_k^+ \cap W_{n-k-1}^- = 1$.

Using this definition, we define the symmetric relation $\mathcal{O}$ via the condition:

$$(\{W_k^+\}, \{V_k^-\}) \in \mathcal{O} \text{ iff } \{W_k^+\} \text{ is opposite to } \{V_k^-\}\,.$$

This oppositeness relation induces a uniquely defined codistance function d^* on $\mathcal{B}^+ \times \mathcal{B}^- \cup \mathcal{B}^- \times \mathcal{B}^+$.

Theorem 6.3.3
The triple $(\mathcal{B}^+, \mathcal{B}^-, d^*)$ is a twin building.

To show, that this induces a twin building, we verify the criterion of Ronan and van Maldeghem, (compare theorem 5.3.4); it consists of verifying lemma 6.3.5 and 6.3.6:

Lemma 6.3.5
$\mathcal{O}$ defines a 1-twinning.

Proof. Let $(\{W_k^+\}, \{V_k^-\}), k \in \mathbb{Z}$ be a pair of opposite chambers, and let $(\{W_k'^+\}, \{V_k'^-\})$ be walls of type s_i. Hence, $(\{W_k'^+\}, \{V_k'^-\}), k \in \mathcal{K}_{\hat{i}}$ are of type $m_{\mathcal{K}} = \{1, 2, \ldots, \hat{i}, \ldots, n\}$. Hence, at the i-th position, the frame $\mathcal{U}$ such that $(\{W_k'^+\}$ and $\{V_k'^-\})$ are subjacent to $\mathcal{U}$, is defined only up to a subspace V^2 of dimension 2. The possible frames containing this wall correspond to the space of 1-dimensional subspaces in V^2 hence to $\mathbb{C}P_2$. Let (e_1, e_2) denote a base of V_2 such that span$(e_1) = V_2 \cap W_i^+$. Then all chambers $\{V_k^-\}$ are opposite but the one defined by $V_2 \cap \{V_k^-\} = \text{span}(e_1)$. □

Lemma 6.3.6
For $\epsilon \in \{+, -\}$, there exists a chamber $c_{-\epsilon} \in \mathcal{C}_{-\epsilon}$ such that for any chamber x_ϵ with $(c_{-\epsilon}, x_\epsilon) \subset \mathcal{O}$ there is an apartment Σ_ϵ of $\mathfrak{B}_\epsilon$ which satisfies $\{x_\epsilon\} = \{y_\epsilon \in \mathcal{C}_\epsilon | y_\epsilon \in \Sigma_\epsilon$ and $(c_{-\epsilon}, y_\epsilon) \in \mathcal{O}\}$.

This condition follows directly from the description of apartments via frames: a frame gives rise to two apartments, one in $\mathfrak{B}^+$ and one in $\mathfrak{B}^-$. This trivially satisfies the required condition.

So we have proved the existence of a twinning between every pair of a positive and a negative building.

Furthermore, the symmetry of the positive and the negative building yields the result:

Theorem 6.3.4
This universal geometric twin building constructed in theorem 6.3.3 is symmetric.

This is not the only possible twinning:
let $e_i z^k, i \in \{1, \ldots, n\}, k \in \mathbb{Z}$ and $f_i z^k, i \in \{1, \ldots, n\}, k \in \mathbb{Z}$ be two bases of H^n. Define $\varphi : H^n \longrightarrow H^n$ to be the map induced by $\varphi(e_i z^k) = f_{n-i} z^{-k}$.

Remark 6.3.1
Applying a suitable Hilbert space isometry before the construction of $\mathfrak{B}^-$, we can assume $e_i = f_i$. Then $\varphi(e_i z^k) = e_{n-i} z^{-k}$.

Lemma 6.3.7
φ induces involutions on Grassmannians, flag varieties etc., that are by abuse of notation also denoted φ:

1. $\varphi : Gr_k^+(H) \longrightarrow Gr_k^-(H)$,
2. $\varphi : Gr_k^{+,n}(H) \longrightarrow Gr_k^{-,n}(H)$,
3. $\varphi : Fl_{k,\mathcal{K}}^{n,+} \longrightarrow Fl_{k,\mathcal{K}}^{n,-}$,
4. $\varphi : \{U_k^+\} \longrightarrow \{U_k^-\}$,

where $k \in \{0, 1, H^1, \omega, \infty, t\}$.

For the proof, we have to check that φ preserves the property of being Hilbert-Schmidt or Fredholm. The first fact follows as φ maps orthonormal bases on orthonormal bases; so the summation condition for Hilbert-Schmidt operators is preserved under φ; the second assertion follows as kernels and cokernels of pr_+ are mapped on kernels and cokernels for pr_-.

To show that φ preserves the regularity conditions, we remark that φ is the composition of the linear continuation of the maps $e_k \longrightarrow f_k$ and $z \longrightarrow \frac{1}{z}$. Both preserve the regularity. This decomposition also shows that φ is an involution.

The third statement of the lemma implies that φ is an involution between $\mathcal{B}^+$ and $\mathcal{B}^-$. As it preserves the regularity, different flags in the same component of $\mathcal{B}^+$ will be mapped in the same component of $\mathcal{B}^-$. Together with the bijectivity, the surjectivity and the preservation of incidences this shows that φ maps buildings onto buildings.

Theorem 6.3.5
φ induces a twinning between pairs of positive and negative buildings.

Proof. Remark 6.3.1 reduces this theorem to theorem 6.3.3. □

For completeness, we state the new oppositeness relation:

Definition 6.3.8 (φ-opposite flags)
Let $\{W_k^+\}$ be a positive chamber and $\{W_k^-\}$ be a negative chamber. $\{W_k^+\}$ is φ-opposite to $\{W_k^-\}$ iff dim $W_k^+ \cap \varphi(W_{-k+1}^-) = 1$.

Theorem 6.3.6
Every canonical twinning of $\mathfrak{B}^+$ and $\mathfrak{B}^-$ induces a twinning of $\mathfrak{B}^\infty$.

For a spherical building the possible twinnings are characterized by a theorem of Jacques Tits [Tit92]:

Theorem 6.3.7
Let $(\mathfrak{B}, d)$ be a spherical building and d_0 the longest element in its Weyl group. Then a quintupel $(\mathfrak{B}^+, \mathfrak{B}^-, d^+, d^-, d^*)$ such that $\mathfrak{B}^+ \simeq \mathfrak{B}^- \simeq \mathfrak{B}$, $d^+ = d, d^- = d^0 d d^0$ and $d^* = d^0 d$ on $\mathfrak{B}^+ \times \mathfrak{B}^-$ and $d^* = d d^0$ on $\mathfrak{B}^- \times \mathfrak{B}^+$ is a twin building; we call this a canonical twinning; all twin buildings are isomorphic to a canonical one.

Hence, we have a complete characterization of all possible twinnings of $\mathfrak{B}^\infty$. Theorems 6.3.6 and 6.3.7 contain the trivial corollary:

Corollary 6.3.4
All twinnings of $\mathfrak{B}^+$ and $\mathfrak{B}^-$ induce a twinning isomorphic to the canonical twinning of $\mathfrak{B}^\infty$.

6.4 The symplectic groups $\widetilde{C}_n$

To construct a building for $\widehat{L}(Sp(n), Id)$ we need to define a symplectic Grassmannian: let $\{z^k e_i, z^l f_j, \{i,j\} \in \{1, \ldots, n\}, \{k,l\} \in \mathbb{Z}\}$ be a basis of $H^{2n} := L^2(S^1, V^{2n})$. The symplectic form on H^{2n} is defined by the condition

$$\langle z^k e_i, z^l f_j \rangle = \delta_{i,j}\delta_{k,-l} = -\langle z^l f_j, z^k e_i \rangle .$$

Definition 6.4.1
Let H^{2n} as above, $J : H^{2n} \longrightarrow H^{2n}$ defined by $J(e_i) = f_i, J(f_i) = -e_i$.

- *The component of virtual dimension 0 of the symplectic Grassmannian $Gr_{symp}(2n)$ is defined by:*
$$Gr^{\pm}_{symp,0}(2n) := \{W \in Gr^{\pm} | (JW)^{\perp} \subset zW\}$$
- *The component of virtual dimension kn of the symplectic Grassmannian $Gr_{\perp}(2n)$ is defined by:*
$$Gr^{\pm}_{symp,kn}(2n) := \{z^k W | W \in Gr^{\pm}_{symp,0}(2n)\}$$
- *The components of dimension $kn < l < (k+1)n$ consist of the direct sum of an element $W_{k+1} \in Gr^{\pm}_{symp,(k+1)n}(2n)$ and an isotropic subspace of the space $W_{k+1}/\frac{1}{z}W_{k+1}$.*

Definition 6.4.2 (positive isotropic periodic flag manifold)
The full positive periodic isotropic flag manifold $Fl^{+,n}_{\perp}$ consists of all sequences $W_k, k \in \mathbb{Z}$ of subspaces in H^n such that

1. *$W_k \subset Gr^+_{\perp} H$,*
2. *$W_{k+1} \subset W_k \forall k$ and $dim(W_k/W_{k+1}) = 1$,*
3. *$zW_k = W_{k+n}$.*

The definition of partial periodic submanifolds and the negative counterparts follows the pattern described for the case A_n.

Theorem 6.4.1
The simplicial complex, which is the opposite complex to the poset of isotropic periodic flags, is a universal twin building of type $\widetilde{C}_n$.

To describe apartments we use frames:

Definition 6.4.3
A frame consists of elements $\{e_{ik}, f_{jl}, \{i,j\} \in \{1,\ldots,n\}, \{k,l\} \in \mathbb{Z}\}$ *such that*

- $\{e_{i,k}, f_{j,l}, \{i,j\} \in \{1,\ldots,n\}, \{k,l\} \in \mathbb{Z}\}$ *span* H^{2n}.
- $ze_{ik} = e_{i,k+1}$ *and* $zf_{jl} = f_{j,l+1}$.

To complete the proof we have to verify the details. This is completely analogous to the case $\widetilde{A}^n$. This consists of the following steps:

1. Check that the Weyl group is of type $\widetilde{C}_n$.
2. The apartments are thin Coxeter complexes of type $\widetilde{C}_n$.
3. The complex is thick.
4. Any connected component is a building (the main part is to check that every pair of chambers is contained in a common apartment.
5. Check that the whole complex is a universal twin building.

The constructions for $\widetilde{B}_n$ and $\widetilde{D}_n$ are similar. The problem is that the complex of flags is not thick. Hence, it is necessary to construct more refined types of nested subspaces to represent the simplices of the building. The simplicial complexes constructed in this way are called oriflamme complexes. For $\widetilde{D}_n$ we need the double oriflamme complex, for $\widetilde{B}_n$ the simple oriflamme complex. The analytic part of the theory proceeds exactly as in the case of $\widetilde{A}_n$ and $\widetilde{C}_n$. For the algebraic details of the oriflamme construction — cf. [Gar97], chapter 20. For groups of algebraic loops, a similar construction is provided in [Gar97], [AN02].

Appendix A

Notation

Ad	Adjoint action	55
$A^{n\times n}$	Cartan matrix	39
A_n	$A_n := \{z \in \mathbb{C}^* \| e^{-n} \leq \|z\| \leq e^n\}$	16
$\widehat{A_n\mathfrak{g}}$	Kac-Moody algebra of holomorphic loops on A_n	23
$A_nG_{\mathbb{C}}$	group of holomorphic loops on $A_n \in G_{\mathbb{C}}$	26
$A_nG_{\mathbb{R}}$	compact type real form of $A_nG_{\mathbb{C}}$	26
$\widehat{A_n\mathfrak{g}}^{\sigma}$	Kac-Moody algebra of holomorphic loops on A_n, twisted with σ	24
B	Borel group	83
B_A	Iwahori group	83
$\widehat{B}_A^+$	positive affine parabolic subgroup	85
$\widehat{B}_A$	affine Borel group	83
$\widehat{B}_A^-$	negative affine parabolic subgroup	85
$\mathfrak{B}^{\infty}$	spherical building at infinity	111
$\mathfrak{B}^+$	positive part of a twin building	86
$\mathfrak{B}^-$	negative part of a twin building	86
$\mathfrak{B}$	building	73
$\mathcal{C}_{\mathfrak{B}}$	set of chambers of $\mathfrak{B}$	76
$\mathcal{C}^+$	set of chambers of $\mathfrak{B}^+$	87
$\mathcal{C}^-$	set of chambers of $\mathfrak{B}^-$	87
δ	Weyl distance (also denoted δ_W)	76
δ^*	codistance	86
δ^+	Weyl distance on $\mathfrak{B}^+$	86
δ^-	Weyl distance on $\mathfrak{B}^-$	86
$Gr^+(H)$	positive infinite Grassmannian	103
$Gr_0^+(H)$	positive algebraic Grassmannian	104
$Gr^{n,+}(H^n)$	reduced Grassmannian	105
$Gr_t^+(H)$	positive tame Fréchet Grassmannian	104
$L(\mathfrak{g}, \sigma)$	abstract loop algebra, loops in $\mathfrak{g}$, twist σ	42
$\widehat{L}(\mathfrak{g}, \sigma)$	abstract affine geometric Kac-Moody algebra	42
$\widetilde{L}(\mathfrak{g}, \sigma)$	abstract derived affine Kac-Moody algebra	42
$\widehat{M\mathfrak{g}}$	Kac-Moody algebra of holomorphic loops on $\mathbb{C}^*$	23
$\widehat{MG}$	Kac-Moody group of holomorphic loops	54
$\widehat{M\mathfrak{g}}^{\sigma}$	Kac-Moody algebra of holomorphic loops on $\mathbb{C}^*$, twisted with σ	24
$\nu(W)$	virtual dimension	104

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Appendix B

Curriculum vitae of Walter Freyn

Personal information

- **born:** 10.05.1980
- **Nationality:** German

Education

- **Gymnasium bei St. Anna**, Augsburg, 1990–1999, Abitur: grade 1,0: 791/840 points
- **Studies of Physics** in Augsburg
  - Vordiplom in Physik, 8/2002, grade 1,08
- **Internship with Tofwerk AG Thun/EMPA**, 7/2004–9/2004
  - Optimization of ion optics in TOF-MS
  - Investigation of grid technics in TOF-MS
- **Studies of Mathematics** in Augsburg and Bordeaux: 10/2000–12/2005
  - Vordiplom in Mathematik, 7/2002, grade 1,00
  - Master mathématiques (Bordeaux), 6/2004, mention „bien“, 2^{nd} en classement,
  - Diplom in Mathematik, 12/2005, grade 1,07

Scholarships

- **Jahrespreis der Societas Annensis** 7/1999
- **Bayerisches Begabtenstipendium** 9/2000–9/2005
- **e-fellows.net**, Scholarship 6/2001–08/2010
- **Studienstiftung des Deutschen Volkes**, Scholarship 04/2002–09/2006
- **Studienstiftung des Deutschen Volkes**, Scholarship for the thesis since 4/2007–08/2010, interrupted several times to replace absent colleagues

Talks

- **Géométrie hyperbolique et théorie de Teichmüller**, Opening Lecture of "Rencontres du 3. cycle", 01.06.2005, Université Bordeaux 1

- **Une démonstration du théorème de Teichmüller par le biais de la géométrie hyperbolique**, 26.10.2005, Université Bordeaux 1

- **Une démonstration du théorème de Teichmüller**, 28.10.2005, Séminaire de Cryptographie, Université Rennes 1

- **An elementary proof of Teichmüller's theorem**, 14.12.2005, Institut Bernoulli, EPFL Lausanne

- **Foundations of the theory of Kac-Moody symmetric spaces**, 09.01.2007, Oberseminar für Differentialgeometrie, LMU München

- **Kac-Moody symmetrische Räume**, 32. süddeutsches Kolloquium über Differentialgeometrie, 04.05.2007, Stuttgart

- **Infinite dimensional symmetric spaces associated to Kac-Moody algebras** 14.05.2007, Einstein Institut of the Hebrew University, Jerusalem

- **Foundations of Kac-Moody symmetric spaces**, 16.05.2007, Tel Aviv University

- **Foundations of Kac-Moody symmetric spaces**, 17.05.2007, Annual meeting of Israel mathematical Society, Be'er Sheva, Israel

- **Kac-Moody symmetric spaces**, 28.06.2007 Kolloquium des ZMP, Hamburg

- **Foundations of a general theory of Kac-Moody symmetric spaces**, 07.01.2008, Workshop in geometry, University of Pune, India

- **Kac-Moody symmetric spaces**, 04.07.2008, YMMB, Universität Bonn

- **Twin buildings for Kac-Moody symmetric spaces**, 30.09.2008, Buildings 2008, Universität Münster

- **Affine buildings for polar actions and Kac-Moody symmetric spaces**, 16.12.2008, TU Darmstadt

- **Kac-Moody geometry: Universal twin buildings, polar actions and infinite dimensional symmetric spaces**, 27.04.2009, Universität Paderborn

- **Kac-Moody symmetric spaces**, 13.05.2009, University of Leicester

Publication

- **A general theory of affine Kac-Moody symmetric spaces**, appeared in: Konferenzbericht des 32. süddeutschen Kolloquiums über Differentialgeometrie.

Teaching

- **Klausurenkurs für Staatsexamenskandidaten:**, Exercise course, summer term 2001 - summer term 2003, (Chair for number theory, Prof. Dr. Reinhard Schertz)
- **Seminar: Complex dynamics**, summer term 2006 (Chair for Differential geometry, Prof. Dr. Ernst Heintze)
- **Seminar: Riemann surfaces**, winter term 2006/2007 (Chair for Differential geometry, Prof. Dr. Ernst Heintze)
- **Seminar: Mostow rigidity theorem**, summer term 2007 (Chair for Differential geometry, Prof. Dr. Ernst Heintze)
- **Seminar: Morse theory**, winter term 2007/2008 (Chair for Differential geometry, Prof. Dr. Ernst Heintze)
- **Seminar: Hyperbolic Geometry**, summer term 2008 (Chair for Differential geometry, Prof. Dr. Ernst Heintze)
- **Teaching Assistant: Differential geometry**, winter term 2007/2008 (Chair for Differential geometry, Prof. Dr. Ernst Heintze)
- **Seminar: Differential topology**, summer term 2009 (Chair for Differential geometry, Prof. Dr. Ernst Heintze)