

The simplex method: a probabilistic analysis

Karl-Heinz Borgwardt

Angaben zur Veröffentlichung / Publication details:

Borgwardt, Karl-Heinz. 1987. *The simplex method: a probabilistic analysis*. Berlin: Springer.
<https://doi.org/10.1007/978-3-642-61578-8>.

Nutzungsbedingungen / Terms of use:

licgercopyright



Erratum: replaces pages 171–177

3rd Step

After having replaced the weighting function of the outer integration ($dF(r)$), we are allowed to return to the original sequence of integrations. We obtain

$$Q_2(F) \leq \frac{\int_0^t \widehat{G}_F(h)^k T^{-1} \int_h^1 R^{n-2} r^{-n+2} d\overline{F}(r) dh}{\int_0^t \widehat{G}_F(h)^k T^{-1} \int_h^1 R^{n-1} r^{-n+1} d\overline{F}(r) dh}.$$

For the inner denominator-integral, there is an interesting estimation:

$$\begin{aligned} & \int_h^1 R^{n-1} r^{-n+1} d\overline{F}(r) = (n-1) \int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \sigma d\sigma d\overline{F}(r) \geq \\ & \geq 2 \cdot \frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)} \int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} d\sigma d\overline{F}(r) = 2[1 - G_{\overline{F}}(h)] = [1 - \widehat{G}_{\overline{F}}(h)]. \end{aligned}$$

The first equation results from pure integration, the second is based on the definition of $G(h)$ in Chapter 2.

Now consider the inequality. We know that

$$\frac{\int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \sigma d\sigma d\overline{F}(r)}{\int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} d\sigma d\overline{F}(r)}$$

increases with h . So the minimal quotient will be attained for $h = 0$.

Hence the quotient can be underestimated by

$$\frac{\int_0^1 (1-\sigma^2)^{\frac{n-3}{2}} \sigma d\sigma d\overline{F}(r)}{\int_0^1 (1-\sigma^2)^{\frac{n-3}{2}} d\sigma d\overline{F}(r)}.$$

Here the numerator is exactly $\frac{1}{n-1}$, the value of the denominator has a geometrical meaning, namely

$$\int_0^1 (1-\sigma^2)^{\frac{n-3}{2}} d\sigma d\overline{F}(r) = \frac{\lambda_{n-1}(\omega_n)}{2 \cdot \lambda_{n-2}(\omega_{n-1})}.$$

After integration over r we have

$$\frac{\int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \sigma d\sigma d\overline{F}(r)}{\int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} d\sigma d\overline{F}(r)} \geq 2 \cdot \frac{\lambda_{n-2}(\omega_{n-1})}{(n-1) \cdot \lambda_{n-1}(\omega_n)},$$

which justifies the inequality above.

Now consider the quotient

$$\frac{\frac{1}{h} \int R^{n-1} r^{-n+1} d\bar{F}(r)}{\frac{1}{h} \int R^{n-2} r^{-n+2} d\bar{F}(r)}.$$

The concavity of the function

$$f(x) = x^{\frac{n-2}{n-1}} \text{ for } x > 0$$

yields

$$\frac{1}{h} \int R^{n-2} r^{-n+2} d\bar{F}(r) \leq \left[\int_h^1 R^{n-1} r^{-n+1} d\bar{F}(r) \right]^{\frac{n-2}{n-1}} \left[\int_h^1 d\bar{F}(r) \right]^{\frac{1}{n-1}} \leq \left[\int_h^1 R^{n-1} r^{-n+1} d\bar{F}(r) \right],$$

because \bar{F} is a distribution function. We conclude that

$$\frac{\frac{1}{h} \int R^{n-1} r^{-n+1} d\bar{F}(r)}{\frac{1}{h} \int R^{n-2} r^{-n+2} d\bar{F}(r)} \geq \left[\int_h^1 R^{n-1} r^{-n+1} d\bar{F}(r) \right]^{\frac{1}{n-1}} \geq [1 - \hat{G}_{\bar{F}}(h)]^{\frac{1}{n-1}}.$$

So we may, if it is useful, exploit as an option the new estimation

$$Q_2(F) \leq \frac{\int_0^t \hat{G}_F(h)^k T^{-1} \int_h^1 R^{n-2} r^{-n+2} d\bar{F}(r) dh}{\int_0^t \hat{G}_F(h)^k [1 - \hat{G}_{\bar{F}}(h)]^{\frac{1}{n-1}} T^{-1} \int_h^1 R^{n-2} r^{-n+2} d\bar{F}(r) dh}.$$

On the other hand (for the opposite estimation-direction) , it is clear that

$$\begin{aligned} \frac{1}{n-1} \cdot \frac{1}{h} \int R^{n-1} r^{-n+1} d\bar{F}(r) &= \int_h^1 \int_{\frac{h}{r}}^1 (1 - \sigma^2)^{\frac{n-3}{2}} \sigma d\sigma d\bar{F}(r) \leq \int_h^1 \int_{\frac{h}{r}}^1 (1 - \sigma^2)^{\frac{n-3}{2}} d\sigma d\bar{F}(r) \\ &= [1 - G_{\bar{F}}(h)] \frac{\lambda_{n-1}(\omega_n)}{\lambda_{n-2}(\omega_{n-1})} = [1 - \hat{G}_{\bar{F}}(h)] \frac{\lambda_{n-1}(\omega_n)}{2 \cdot \lambda_{n-2}(\omega_{n-1})}. \end{aligned}$$

In the following, it will be our goal to show that

$$Q_2(F) \leq \left[\frac{1}{k+1} \right]^{-\frac{1}{n-1}} = (k+1)^{\frac{1}{n-1}}.$$

Since $[1 - \hat{G}_{\bar{F}}(h)]^{\frac{1}{n-1}}$ is increasing while h decreases, there will be a value z , such that for all $h \leq z$ we have

$$[1 - \hat{G}_{\bar{F}}(h)]^{\frac{1}{n-1}} \geq (k+1)^{\frac{1}{n-1}}.$$

So, the region where $h \leq z$ will be uncritical for our analysis.

Now consider the complementary region where $h > z$.

If $z > t$, then we need not care about that question at all.

So it remains to study the case $t > z$ and the region $h \in [z, t]$. Here we do not use the above mentioned option. Instead we return to the initial formulation of $Q_2^z(F)$, but now on the restricted interval $[z, t]$.

$$Q_2^z(F) \leq \frac{\int_z^t \widehat{G}_F(h)^k T^{-1} \int_h^{R^n} r^{-n+2} d\overline{F}(r) dh}{\int_z^t \widehat{G}_F(h)^k T^{-1} \int_h^{R^{n-1}} r^{-n+1} d\overline{F}(r) dh}.$$

It is clear that $\forall r : \overline{F}(r) \leq F(r)$ and that $\overline{F}(r) < F(r) \forall r < \overline{r}$.

The consequence is $\widehat{G}_F(h) \geq \widehat{G}_{\overline{F}}(h)$.

And $\forall h > z$ this means

$$\widehat{G}_F(h) \geq \widehat{G}_F(z) \geq \widehat{G}_{\overline{F}}(z) = 1 - \frac{1}{k+1}.$$

Since

$$\widehat{G}_{\overline{F}}(z)^k \geq (1 - \frac{1}{k+1})^k = (1 + \frac{1}{k})^{-k} > e^{-1}$$

we may get rid of the complicating factors $\widehat{G}_F(z)^k$ completely via the estimation

$$Q_2^z(F) \leq e \cdot \frac{\int_z^t \frac{1}{h} T^{-1} \int_h^{R^n} r^{-n+2} d\overline{F}(r) dh}{\int_z^t \frac{1}{h} T^{-1} \int_h^{R^{n-1}} r^{-n+1} d\overline{F}(r) dh} = e \cdot \frac{\int_z^t \frac{1}{h} \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^{2n-2}}}{r^{n-2}} dh d\overline{F}(r)}{\int_z^t \frac{1}{h} \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^{2n-1}}}{r^{n-1}} dh d\overline{F}(r)}.$$

For the denominator-integral we know the estimation

$$\int_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^{2n-1}}}{r^{n-1}} dh \geq \int_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^{2n-2}}}{r^{n-2}} dh \cdot \frac{1}{r \cdot 2} \sqrt{r^2-z^2}.$$

This results from the following comparison:

$$\frac{\int_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^{2n-1}}}{r^{n-1}} dh}{\int_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^{2n-2}}}{r^{n-2}} dh} = \frac{\int_z^t \frac{\sqrt{r^2-h^{2n-1}}}{\sqrt{t^2-h^2}} dh}{r \cdot \int_z^t \frac{\sqrt{r^2-h^{2n-2}}}{\sqrt{t^2-h^2}} dh} \geq \frac{\int_z^t \frac{\sqrt{r^2-h^{2n-1}}}{\sqrt{t^2-h^2}} \cdot h dh}{r \cdot \int_z^t \frac{\sqrt{r^2-h^{2n-2}}}{\sqrt{t^2-h^2}} \cdot h dh} = \frac{\int_0^{\sqrt{t^2-z^2}} \sqrt{r^2-t^2+u^{2n-1}} du}{r \cdot \int_0^{\sqrt{t^2-z^2}} \sqrt{r^2-t^2+u^{2n-2}} du}$$

where $u := \sqrt{t^2-h^2}$ and $\frac{du}{dh} = -\frac{h}{\sqrt{t^2-h^2}}$.

But

$$\begin{aligned} \frac{\int_0^{\sqrt{t^2-z^2}} \sqrt{r^2-t^2+u^{2n-1}} du}{r \cdot \int_0^{\sqrt{t^2-z^2}} \sqrt{r^2-t^2+u^{2n-2}} du} &\geq \frac{\int_0^{\sqrt{t^2-z^2}} \sqrt{r^2-t^2+u^2} du}{r \cdot \int_0^{\sqrt{t^2-z^2}} du} \\ &= \frac{1}{r} \frac{\sqrt{t^2-z^2} \sqrt{r^2-z^2} + (r^2-t^2) \ln(\sqrt{t^2-z^2} + \sqrt{r^2-z^2})}{\sqrt{t^2-z^2} \cdot 2} - \frac{1}{r} \frac{(r^2-t^2) \ln(\sqrt{r^2-z^2})}{\sqrt{t^2-z^2} \cdot 2} \\ &> \frac{1}{r \cdot 2} \sqrt{r^2-z^2}. \end{aligned}$$

This leads to

$$Q_2^z(F) \leq e \cdot 2 \cdot \frac{\int_z^t \left[\int_z^t \frac{1}{h} \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^{2n-2}}}{r^{n-2}} dh \right] d\overline{F}(r)}{\int_z^t \left[\int_z^t \frac{1}{h} \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^{2n-2}}}{r^{n-2}} dh \right] \frac{\sqrt{r^2-z^2}}{r} d\overline{F}(r)}.$$

The integrals in brackets can be seen as weights for different values of r . The bound on the right side will become larger, if more weight is given to smaller values of r , since then the "expected value" of $\frac{\sqrt{r^2-z^2}}{r}$ becomes smaller.

We achieve such a transformation of weights in favour of smaller r -values, if we replace the \int_z^t -integrals by integrals like \int_z^τ , where $t > \tau > z$. This follows from the following comparison, where $t \leq r_1 < r_2$ leads to

$$\begin{aligned} \left[\int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_1^2-h^2}^{n-2}}{r_1^{n-2}} dh \right] &\geq \left[\int_\tau^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_1^2-h^2}^{n-2}}{r_1^{n-2}} dh \right] + \left[\int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_1^2-h^2}^{n-2}}{r_1^{n-2}} dh \right] \\ \left[\int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_2^2-h^2}^{n-2}}{r_2^{n-2}} dh \right] &\leq \left[\int_\tau^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_2^2-h^2}^{n-2}}{r_2^{n-2}} dh \right] + \left[\int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_2^2-h^2}^{n-2}}{r_2^{n-2}} dh \right] \end{aligned}$$

because $\frac{\sqrt{r_1^2-h^2}}{\sqrt{r_2^2-h^2}}$ decreases with increasing $h \rightarrow r_1^-$. And in that way the smallest quotients are dropped when we reduce to \int_z^τ . Therefore

$$\begin{aligned} Q_2^z(F) &\leq e \cdot 2 \cdot \frac{\int_t^1 \left[\lim_{\tau \rightarrow z_+} \int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-2}}{r^{n-2}} dh \right] d\bar{F}(r)}{\int_t^1 \left[\lim_{\tau \rightarrow z_+} \int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-1}}{r^{n-1}} dh \right] \frac{\sqrt{r^2-z^2}}{r} d\bar{F}(r)} \\ &= e \cdot 2 \cdot \frac{\int_t^1 \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-z^2}^{n-2}}{r^{n-2}} d\bar{F}(r)}{\int_t^1 \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-z^2}^{n-2}}{r^{n-2}} \frac{\sqrt{r^2-z^2}}{r} d\bar{F}(r)} = e \cdot 2 \cdot \frac{\int_t^1 \frac{\sqrt{r^2-z^2}^{n-2}}{r^{n-2}} d\bar{F}(r)}{\int_t^1 \frac{\sqrt{r^2-z^2}^{n-1}}{r^{n-1}} d\bar{F}(r)}. \end{aligned}$$

But we had just before proven that

$$\frac{\int_z^1 \frac{\sqrt{r^2-h^2}^{n-2}}{r^{n-2}} d\bar{F}(r)}{\int_z^1 \frac{\sqrt{r^2-z^2}^{n-1}}{r^{n-1}} d\bar{F}(r)} \leq \left[1 - \hat{G}_{\bar{F}}(z) \right]^{-\frac{1}{n-1}} = \left[\frac{1}{k+1} \right]^{-\frac{1}{n-1}} = [k+1]^{\frac{1}{n-1}}$$

according to our choice of z .

Consequently

$$Q_2^z(F) \leq 2 \cdot e \cdot [k+1]^{\frac{1}{n-1}}.$$

Let us summarize. We know that

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} \leq C(n) \cdot Q_1(F) \cdot Q_2(F)$$

and that

$$\begin{aligned} Q_1(F) &\leq \frac{n \cdot \lambda_{n-1}(\Omega_{n-1})}{2 \cdot \lambda_{n-2}(\Omega_{n-2})} \\ Q_2(F) &\leq 2 \cdot e \cdot [k+1]^{\frac{1}{n-1}}. \\ C(n) &= \frac{n^2 \cdot (n-1) \cdot \lambda_n(\Omega_n)}{\lambda_{n-1}(\Omega_{n-1})}. \end{aligned}$$

Multiplication yields

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} \leq [k+1]^{\frac{1}{n-1}} \cdot \frac{e \cdot n^3 \cdot (n-1) \cdot \lambda_n(\Omega_n)}{\lambda_{n-1}(\Omega_{n-1})} = [k+1]^{\frac{1}{n-1}} \cdot \pi \cdot n^3 \cdot 2 \cdot e \leq [m-n+1]^{\frac{1}{n-1}} \cdot \pi \cdot n^3 \cdot 2 \cdot e.$$