Erratum: replaces pages 171–177

3rd Step

After having replaced the weighting function of the outer integration ($dF(r)$), we are allowed to return to the original sequence of integrations. We obtain

$$Q_2(F) \leq \frac{\int_0^t \widehat{G}_F(h)^k \, T^{-1} \int_h^1 R^{n-2} \, r^{-n+2} \, d\overline{F}(r) \, dh}{\int_0^t \widehat{G}_F(h)^k \, T^{-1} \int_h^1 R^{n-1} \, r^{-n+1} \, d\overline{F}(r) \, dh}.$$

For the inner denominator-integral, there is an interesting estimation:

$$\int_h^1 R^{n-1} r^{-n+1} d\overline{F}(r) = (n-1) \int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \, \sigma \, d\sigma d\overline{F}(r) \geq$$

$$\geq \; 2 \cdot \frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)} \int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \, d\sigma \, d\overline{F}(r) = 2[1 - G_{\overline{F}}(h)] = [1 - \widehat{G}_{\overline{F}}(h)].$$

The first equation results from pure integration, the second is based on the definition of $G(h)$ in Chapter 2.

Now consider the inequality. We know that

$$\frac{\int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \, \sigma \, d\sigma d\overline{F}(r)}{\int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \, d\sigma d\overline{F}(r)}$$

increases with h. So the minimal quotient will be attained for $h = 0$.

Hence the quotient can be underestimated by

$$\frac{\int_0^1 (1-\sigma^2)^{\frac{n-3}{2}} \, \sigma \, d\sigma d\overline{F}(r)}{\int_0^1 (1-\sigma^2)^{\frac{n-3}{2}} \, d\sigma d\overline{F}(r)}.$$

Here the numerator is exactly $\frac{1}{n-1}$, the value of the denominator has a geometrical meaning, namely

$$\int_0^1 (1-\sigma^2)^{\frac{n-3}{2}} \, d\sigma d\overline{F}(r) = \frac{\lambda_{n-1}(\omega_n)}{2 \cdot \lambda_{n-2}(\omega_{n-1})} \; .$$

After integration over r we have

$$\frac{\int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \, \sigma \, d\sigma d\overline{F}(r)}{\int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \, d\sigma d\overline{F}(r)} \geq 2 \cdot \frac{\lambda_{n-2}(\omega_{n-1})}{(n-1) \cdot \lambda_{n-1}(\omega_n)},$$

which justifies the inequality above.

Now consider the quotient

$$\frac{\int_h^1 R^{n-1} r^{-n+1} d\overline{F}(r)}{\int_h^1 R^{n-2} r^{-n+2} d\overline{F}(r)}.$$

The concavity of the function

$$f(x) = x^{\frac{n-2}{n-1}} \text{ for } x > 0$$

yields

$$\int_h^1 R^{n-2} r^{-n+2} d\overline{F}(r) \le \left[\int_h^1 R^{n-1} r^{-n+1} d\overline{F}(r)\right]^{\frac{n-2}{n-1}} \left[\int_h^1 d\overline{F}(r)\right]^{\frac{1}{n-1}} \le \left[\int_h^1 R^{n-1} r^{-n+1} d\overline{F}(r)\right],$$

because $\overline{F}$ is a distribution function. We conclude that

$$\frac{\int_h^1 R^{n-1} r^{-n+1} d\overline{F}(r)}{\int_h^1 R^{n-2} r^{-n+2} d\overline{F}(r)} \ge \left[\int_h^1 R^{n-1} r^{-n+1} d\overline{F}(r)\right]^{\frac{1}{n-1}} \ge \left[1 - \widehat{G}_{\overline{F}}(h)\right]^{\frac{1}{n-1}}.$$

So we may, if it is useful, exploit as an option the new estimation

$$Q_2(F) \le \frac{\int_0^t \widehat{G}_F(h)^k \, T^{-1} \int_h^1 R^{n-2} \, r^{-n+2} \, d\overline{F}(r) \, dh}{\int_0^t \widehat{G}_F(h)^k \, \left[1 - \widehat{G}_{\overline{F}}(h)\right]^{\frac{1}{n-1}} \, T^{-1} \int_h^1 R^{n-2} \, r^{-n+2} \, d\overline{F}(r) \, dh}.$$

On the other hand (for the opposite estimation-direction) , it is clear that

$$\begin{aligned}
\frac{1}{n-1} \cdot \int_h^1 R^{n-1} r^{-n+1} d\overline{F}(r) &= \int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \, \sigma \, d\sigma d\overline{F}(r) &\le& \int_h^1 \int_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \, d\sigma d\overline{F}(r) \\
&= [1 - G_{\overline{F}}(h)] \frac{\lambda_{n-1}(\omega_n)}{\lambda_{n-2}(\omega_{n-1})} &=& [1 - \widehat{G}_{\overline{F}}(h)] \frac{\lambda_{n-1}(\omega_n)}{2 \cdot \lambda_{n-2}(\omega_{n-1})}.
\end{aligned}$$

In the following, it will be our goal to show that

$$Q_2(F) \le \left[\frac{1}{k+1}\right]^{-\frac{1}{n-1}} = (k+1)^{\frac{1}{n-1}}.$$

Since $[1 - \widehat{G}_{\overline{F}}(h)]^{\frac{1}{n-1}}$ is increasing while h decreases, there will be a value z, such that for all $h \le z$ we have

$$[1 - \widehat{G}_{\overline{F}}(h)]^{\frac{1}{n-1}} \ge (k+1)^{\frac{1}{n-1}}.$$

So, the region where $h \le z$ will be uncritical for our analysis.
Now consider the complementary region where $h > z$.
If $z > t$, then we need not care about that question at all.
So it remains to study the case $t > z$ and the region $h \in [z, t]$. Here we do not use the above mentioned option. Instead we return to the initial formulation of ${Q_2}^z(F)$, but now on the restricted interval $[z, t]$.

$$Q_2{}^z(F) \leq \frac{\int\limits_z^t \widehat{G}_F(h)^k\, T^{-1} \int\limits_h^1 R^{n-2}\, r^{-n+2}\, d\overline{F}(r)\, dh}{\int\limits_z^t \widehat{G}_F(h)^k\, T^{-1} \int\limits_h^1 R^{n-1}\, r^{-n+1}\, d\overline{F}(r)\, dh}.$$

It is clear that $\forall r: \ \overline{F}(r) \leq F(r)$ and that $\overline{F}(r) < F(r)\ \forall r < \overline{r}$.
The consequence is $\widehat{G}_F(h) \geq \widehat{G}_{\overline{F}}(h)$.
And $\forall\ h > z$ this means

$$\widehat{G}_F(h) \geq \widehat{G}_F(z) \geq \widehat{G}_{\overline{F}}(z) = 1 - \frac{1}{k+1}.$$

Since

$$\widehat{G}_{\overline{F}}(z)^k \geq (1 - \frac{1}{k+1})^k = (1 + \frac{1}{k})^{-k} > e^{-1}$$

we may get rid of the complicating factors $\widehat{G}_F(z)^k$ completely via the estimation

$$Q_2{}^z(F) \leq e \cdot \frac{\int\limits_z^t\ T^{-1} \int\limits_h^1 R^{n-2}\, r^{-n+2}\, d\overline{F}(r)\, dh}{\int\limits_z^t\ T^{-1} \int\limits_h^1 R^{n-1}\, r^{-n+1}\, d\overline{F}(r)\, dh} = e \cdot \frac{\int\limits_t^1 \int\limits_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-2}}{r^{n-2}}\, dh\, d\overline{F}(r)}{\int\limits_t^1 \int\limits_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-1}}{r^{n-1}}\, dh\, d\overline{F}(r)}.$$

For the denominator-integral we know the estimation

$$\int\limits_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-1}}{r^{n-1}}\, dh \geq \int\limits_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-2}}{r^{n-2}}\, dh \cdot \frac{1}{r \cdot 2} \sqrt{r^2 - z^2}.$$

This results from the following comparison:

$$\frac{\int\limits_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-1}}{r^{n-1}}\, dh}{\int\limits_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-2}}{r^{n-2}}\, dh} = \frac{\int\limits_z^t \frac{\sqrt{r^2-h^2}^{n-1}}{\sqrt{t^2-h^2}}\, dh}{r \cdot \int\limits_z^t \frac{\sqrt{r^2-h^2}^{n-2}}{\sqrt{t^2-h^2}}\, dh} \geq \frac{\int\limits_z^t \frac{\sqrt{r^2-h^2}^{n-1}}{\sqrt{t^2-h^2}} \cdot h\, dh}{r \cdot \int\limits_z^t \frac{\sqrt{r^2-h^2}^{n-2}}{\sqrt{t^2-h^2}} \cdot h\, dh} = \frac{\int\limits_0^{\sqrt{t^2-z^2}} \sqrt{r^2 - t^2 + u^2}^{n-1}\, dh}{r \cdot \int\limits_0^{\sqrt{t^2-z^2}} \sqrt{r^2 - t^2 + u^2}^{n-2}\, dh}$$

where $u := \sqrt{t^2 - h^2}$ and $\frac{du}{dh} = -\frac{h}{\sqrt{t^2-h^2}}$.
But

$$\begin{aligned}
&\frac{\int\limits_0^{\sqrt{t^2-z^2}} \sqrt{r^2 - t^2 + u^2}^{n-1}\, du}{r \cdot \int\limits_0^{\sqrt{t^2-z^2}} \sqrt{r^2 - t^2 + u^2}^{n-2}\, du} \geq \frac{\int\limits_0^{\sqrt{t^2-z^2}} \sqrt{r^2 - t^2 + u^2}\, du}{r \cdot \int\limits_0^{\sqrt{t^2-z^2}} du} \\
&= \frac{1}{r} \frac{\sqrt{t^2-z^2}\sqrt{r^2-z^2} + (r^2 - t^2) \ln(\sqrt{t^2-z^2} + \sqrt{r^2-z^2})}{\sqrt{t^2-z^2} \cdot 2} - \frac{1}{r} \frac{(r^2-t^2) \ln(\sqrt{r^2-z^2})}{\sqrt{t^2-z^2} \cdot 2} \\
&> \frac{1}{r \cdot 2} \sqrt{r^2 - z^2}.
\end{aligned}$$

This leads to

$$Q_2{}^z(F) \leq e \cdot 2 \cdot \frac{\int\limits_t^1 \left[\int\limits_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-2}}{r^{n-2}}\, dh \right]\, d\overline{F}(r)}{\int\limits_t^1 \left[\int\limits_z^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-2}}{r^{n-2}}\, dh \right] \frac{\sqrt{r^2-z^2}}{r}\, d\overline{F}(r)}.$$

The integrals in brackets can be seen as weights for different values of r. The bound on the right side will become larger, if more weight is given to smaller values of r, since then the "expected value" of $\frac{\sqrt{r^2-z^2}}{r}$ becomes smaller.
We achieve such a transformation of weights in favour of smaller r-values, if we replace the $\int_z^t$- integrals by integrals like $\int_z^\tau$, where $t > \tau > z$. This follows from the following comparison, where $t \le r_1 < r_2$ leads to

$$\frac{\left[\int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_1^2-h^2}^{n-2}}{r_1^{n-2}} dh\right]}{\left[\int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_2^2-h^2}^{n-2}}{r_2^{n-2}} dh\right]} \ge \frac{\left[\int_\tau^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_1^2-h^2}^{n-2}}{r_1^{n-2}} dh\right] + \left[\int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_1^2-h^2}^{n-2}}{r_1^{n-2}} dh\right]}{\left[\int_\tau^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_2^2-h^2}^{n-2}}{r_2^{n-2}} dh\right] + \left[\int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r_2^2-h^2}^{n-2}}{r_2^{n-2}} dh\right]}$$

because $\frac{\sqrt{r_1^2-h^2}}{\sqrt{r_2^2-h^2}}$ decreases with increasing $h \to r_1^-$ And in that way the smallest quotients are dropped when we reduce to $\int_z^\tau$. Therefore

$$Q_2{}^z(F) \le \quad e \cdot 2 \cdot \frac{\int_t^1 \left[\lim_{\tau \to z_+} \int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-2}}{r^{n-2}} dh\right] d\overline{F}(r)}{\int_t^1 \left[\lim_{\tau \to z_+} \int_z^\tau \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{n-1}}{r^{n-1}} dh\right] \frac{\sqrt{r^2-z^2}}{r} d\overline{F}(r)}$$

$$= \quad e \cdot 2 \cdot \frac{\int_t^1 \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-z^2}^{n-2}}{r^{n-2}} d\overline{F}(r)}{\int_t^1 \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-z^2}^{n-2}}{r^{n-2}} \frac{\sqrt{r^2-z^2}}{r} d\overline{F}(r)} = \quad e \cdot 2 \cdot \frac{\int_t^1 \frac{\sqrt{r^2-z^2}^{n-2}}{r^{n-2}} d\overline{F}(r)}{\int_t^1 \frac{\sqrt{r^2-z^2}^{n-1}}{r^{n-1}} d\overline{F}(r)}.$$

But we had just before proven that

$$\frac{\int_z^1 \frac{\sqrt{r^2-h^2}^{n-2}}{r^{n-2}} d\overline{F}(r)}{\int_z^1 \frac{\sqrt{r^2-z^2}^{n-1}}{r^{n-1}} d\overline{F}(r)} \le \left[1 - \widehat{G}_{\overline{F}}(z)\right]^{-\frac{1}{n-1}} = \left[\frac{1}{k+1}\right]^{-\frac{1}{n-1}} = [k+1]^{\frac{1}{n-1}}$$

according to our choice of z.
Consequently

$$Q_2{}^z(F) \le 2 \cdot e \cdot [k+1]^{\frac{1}{n-1}} .$$

Let us summarize. We know that

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} \le C(n) \cdot Q_1(F) \cdot Q_2(F)$$

and that

$$Q_1(F) \le \frac{n \cdot \lambda_{n-1}(\Omega_{n-1})}{2 \cdot \lambda_{n-2}(\Omega_{n-2})}$$

$$Q_2(F) \le 2 \cdot e \cdot [k+1]^{\frac{1}{n-1}} .$$

$$C(n) = \frac{n^2 \cdot (n-1) \cdot \lambda_n(\Omega_n)}{\lambda_{n-1}(\Omega_{n-1})}.$$

Multiplication yields

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} \le [k+1]^{\frac{1}{n-1}} \cdot \frac{e \cdot n^3 \cdot (n-1) \cdot \lambda_n(\Omega_n)}{\lambda_{n-1}(\Omega_{n-1})} = [k+1]^{\frac{1}{n-1}} \cdot \pi \cdot n^3 \cdot 2 \cdot e \le [m-n+1]^{\frac{1}{n-1}} \cdot \pi \cdot n^3 \cdot 2 \cdot e.$$