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EXTREMAL EXPONENTIAL GROWTH RATES OF BILINEAR CONTROL SYSTEMS

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1. INTRODUCTION

This paper considers bilinear control systems (Σ) having the following form

$$\dot{x}(t) = \Big[A_0 + \sum_{i=1}^{m} u_i(t)A_i\Big]x(t), \quad t \geq 0 \tag{1.1}$$

$$x(0) = x_0 \in \mathbf{R}^d \tag{1.2}$$

$$u = (u_i) \in U := \{u : \mathbf{R}_+ \to \Omega, \ \text{ integrable } \} \tag{1.3}$$

where $A_i \in \mathbf{R}^{d\times d}$, $i = 0, 1, \ldots, m$, $\Omega \subset \mathbf{R}^m$ is compact and convex with $\operatorname{int}\Omega \neq \emptyset$.

We analyze the minimal and maximal exponential growth rates of individual solutions and of fundamental solutions associated with specific control functions u. Furthermore, we give a complete characterization of those initial states x_0, from which the maximal and the minimal exponential growth rate can be realized. The proofs rely on methods from geometric control theory.

2. GROWTH CONCEPTS

A solution of the linear time-varying differential equation (1.1) corresponding to the initial state x_0 and the control function u is denoted by $x(t, x_0, u)$, $t \geq 0$. The exponential growth rate of $x(\cdot, x_0, u)$ is given by

$$\lambda(x_0, u) = \limsup_{t\to\infty} \frac{1}{t} \log |x(t, x_0, u)| \tag{2.1}$$

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where $|\cdot|$ is any norm on $\mathbf{R}^d$. The minimal and maximal exponential growth rates of (Σ) are given, resp., by

$$\kappa^* := \inf_{u\in U} \inf_{x_0\neq 0} \lambda(x_0, u), \quad \kappa = \sup_{u\in U} \sup_{x_0\neq 0} \lambda(x_0, u).$$

In general, one cannot expect that κ^* can be realized from all initial states x_0; furthermore, the control function u here depends on the initial state. Different concepts are obtained by requiring uniformity with respect to the initial states:

Define

$$\hat{\kappa}^* := \inf_{u\in U} \sup_{x_0\neq 0} \lambda(x_0, u), \quad \hat{\kappa} := \sup_{u\in U} \inf_{x_0\neq 0} \lambda(x_0, u).$$

Instead of looking at individual solutions $x(\cdot, x_0, u)$ also the exponential growth rates of fundamental solutions may be considered. Let

$$N := \{A_0 + \Sigma u_i A_i : (u_i) \in \Omega\}$$

and consider the semigroup corresponding to piecewise constant control functions u

$$S_t := \{\exp(B_n t_n + \cdots + B_1 t_1) : B_i \in N,\ t_i > 0,\ \Sigma t_i = t\}.$$

The growth of $g \in S_t$ may be measured by the minimal (or maximal) absolute value of the eigenvalues or by the minimal (or maximal) absolute value that g attains on the unit sphere in $\mathbf{R}^d$.

Define the spectral radius and coradius, the norm and the conorm of g by

$$r(g) := \max\{|\lambda| : \lambda \in \sigma(g)\}, \quad cor(g) := \min\{|\lambda| : \lambda \in \sigma(g)\}$$
$$\|g\| := \max\{|g(x)| : |x| = 1\}, \quad m(g) := \min\{|g(x)| : |x| = 1\}.$$

One obtains the following concepts for extremal exponential growth rates:

$$\beta^* := \limsup_{t\to\infty} \frac{1}{t} \inf_{g\in S_t} \log cor(g), \quad \beta := \limsup_{t\to\infty} \frac{1}{t} \sup_{g\in S_t} \log r(g)$$
$$\hat{\beta}^* := \limsup_{t\to\infty} \frac{1}{t} \inf_{g\in S_t} \log r(g), \quad \hat{\beta} := \limsup_{t\to\infty} \frac{1}{t} \sup_{g\in S_t} \log cor(g).$$

$$\delta^* := \limsup_{t\to\infty} \frac{1}{t} \inf_{g\in S_t} \log m(g), \quad \delta := \limsup_{t\to\infty} \frac{1}{t} \sup_{g\in S_t} \log \|g\|$$
$$\hat{\delta}^* := \limsup_{t\to\infty} \frac{1}{t} \inf_{g\in S_t} \log \|g\|, \quad \hat{\delta} := \limsup_{t\to\infty} \frac{1}{t} \sup_{g\in S_t} \log m(g).$$

Our results, in particular, clarify the relations between these concepts.

3. ANALYSIS

First, we project the system (Σ) onto the unit sphere S^{d-1}. Define

$$q(u,s) := s^T A(u)s, \qquad \text{where} \quad A(u) := A_0 + \Sigma u_i A_i$$
$$h(u,s) := [A(u) - s^T A(u)s \cdot Id]s$$

and let

$$s(t) := x(t,x_0,u)/|x(t,x_0,u)|, \quad s_0 := x_0/|x_0|\,.$$

Then, (cp. [1])

$$\dot{s}(t) = h(u(t),\, s(t)), \quad t \geq 0, \tag{3.1}$$

$$s(0) = s_0 \tag{3.2}$$

$$\lambda(x_0,u) = \limsup_{t\to\infty} \frac{1}{t}\int_0^t q(u(\tau),\, s(\tau))\, d\tau. \tag{3.3}$$

Thus, determination of κ^*, κ is reduced to an optimal control problem on the sphere with the somewhat unorthodox "average cost" functional (3.3).

In order to make use of geometric control theory, we assume that the Lie-Algebra generated by the vectorfields $h(\cdot,u)$, $u \in \Omega$, satisfies at each point $s \in S^{d-1}$

$$\dim LA\{h(\cdot,u),\ u \in \Omega\} = d-1. \tag{H}$$

The system (3.1)-(3.3) may be considered on the projective space P obtained by identifying opposite points of S^{d-1}.

Hypothesis (H) implies local accessibility (cp. e.g., [4]):

Let for $t > 0$, $s_0 \in \mathbf{P}$,

$$O^+_{\leq t}(s_0) := \{y \in \mathbf{P}: \text{ there exist } 0 < \tau \leq t \text{ and } u \in U \text{ with } s(\tau,s_0,u) = y\},$$

where $s(t,s_0,u)$, $t \geq 0$, is the trajectory of (3.1) corresponding to the initial state s_0 and the control u. Then, (H) implies $\mathrm{int} O^+_{\leq t}(s_0) \neq \emptyset$ for all $s_0 \in \mathbf{P}$, $t > 0$.

Let $O^+(s_0) := \cup_{t>0} O^+_{\leq t}(s_0)$.

Definition: A subset $C \subset \mathbf{P}$ is called an invariant control set if $\mathrm{cl}\, C = \mathrm{cl}\, O^+(s_0)$ for all $s_0 \in C$.

The following result is proved in [2].

Theorem 1. *Assume (H). Then there exists a unique invariant control set C in P. Furthermore, $\mathrm{int}\, C \neq \emptyset$, $C = \mathrm{cl}\,\mathrm{int}\, C$, and $C = \cap_{s_0 \in \mathbf{P}} \mathrm{cl}\, O^+(s_0)$.*

In [3], we have shown

Theorem 2. *Assume (H). Then*

$$\kappa = \sup\{\lambda(s_0,u) : s_0 \in C,\ (u(\cdot),\, s(\cdot,s_0,u)) \textit{ periodic }\}.$$

The following theorem presents a characterization of maximal exponential growth rates.

Theorem 3. *Assume (H). Then*

$$\kappa = \beta = \delta \quad \text{and} \quad \hat{\kappa} = \hat{\beta} = \hat{\delta}.$$

Proof: The first half of the assertion is proved on [3, Theorem 4.1]. The second half follows similarly using reduction to the periodic case and Floquet theory.

Using time reversal, one can deduce from Theorems 2 and 3, results on minimal exponential growth rates. Consider the time reversed system

$$\dot{x}(t) = -\Big[A_0 + \sum_{i=1}^{m} u_i A_i\Big] x(t), \quad x(0) = x_0, \;\; u \in U. \qquad (\Sigma^*)$$

Trajectories of (Σ^*) are denoted by $x^*(t, x_0, u)$, trajectories for the projected system by $s^*(t, s_0, u)$.

Lemma. *Assume (H). Then, the maximal exponential growth rate of (Σ^*) satisfies*

$$\sup_{u\in U}\sup_{x_0\neq 0}\limsup_{t\to\infty}\frac{1}{t}\log|x^*(t,x_0.u)| = -\inf\{\lambda(s_0,u), \quad s_0\in C^*, \qquad (3.4)$$
$$(u(\cdot),\ s(\cdot,s_0,u)) \text{ periodic with } s(t,s_0,u)\in int\, C^* \text{ for all } t\geq 0\}$$

where C^* is the unique invariant control set of (Σ^*) projected onto **P**.

Proof: Let $(u(\cdot),\ s(\cdot, s_0, u))$ be τ-periodic. Extend $s(t)$ and $u(t)$ τ-periodically for $t < 0$ and define $\sigma(t) := s(-t)$, $t > 0$. Then,

$$\dot{\sigma}(t) = -\dot{s}(-t) = [-A(u(-t)) - \sigma^T(t)(-A(u(-t)))\sigma(t)Id]\sigma(t)$$

and

$$\lambda(s(0),\ u) = \frac{1}{\tau}\int_0^\tau s(t)^T A(u(t))s(t)\,dt = \frac{1}{\tau}\int_{-\tau}^{0} s(t)^T A(u(t))s(t)\,dt$$
$$= \frac{1}{\tau}\int_0^\tau s(-t)^T A(u(-t))s(-t)\,dt = \frac{1}{\tau}\int_0^\tau \sigma(t)^T A(u(-t)\sigma(t)\,dt$$
$$= -\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma(t)^T(-A(u(-t)))\sigma(t)\,dt.$$

The last expression is minus the exponential growth rate of the trajectory of (Σ^*) corresponding to the initial value $s(0)$ and the (periodic) control $u(-t)$.

Hence, using Theorem 2, we find that the left hand side of (3.4) equals

$$\sup\Big\{\limsup_{t\to\infty}\frac{1}{t}\int_0^t (s^*)^T A(u)s^*\,d\tau,\ (u(\cdot),\ s^*(\cdot)) \text{ periodic},$$
$$s^*(t) = s^*(t,s_0,u)\subset int\, C^*,\ t\geq 0\Big\}$$
$$= -\inf\Big\{\lambda(s_0,u),\ (u(\cdot),\ s(\cdot),s_0,u)) \text{ periodic},\ s(\cdot,s_0,u)\subset int\, C^*\Big\}.$$

This lemma and use of Floquet theory yields the following characterization of minimal exponential growth rates.

Theorem 5. *Assume (H). Then,*

$$\kappa^* = \beta^* = \delta^* \quad \text{and} \quad \hat{\kappa}^* = \hat{\beta}^* = \hat{\delta}^*$$

Furthermore,

$$\kappa^* = \inf\{\lambda(s_0, u) : (u(\cdot),\ s(\cdot, s_0, u)) \text{ periodic with } s(\cdot, s_0, u) \subset int\, C^*\}$$

where C^* is the unique invariant control set in **P** *of the time reversed system* (Σ^*).

Remark. In general, the minimal exponential growth rate κ^* can only be realized from $S_0 \in \operatorname{int} C^*$

REFERENCES

[1] L. Arnold and W. Kliemann, Qualitative theory of stochastic systems. *Probabilistic Analysis and Related Topics, Vol. 3 (A.T. Bharucha-Reid, ed.), Academic Press, New York (1983), 1-79.*

[2] L. Arnold, W. Kliemann and E. Oeljeklaus, Lyapunov exponents of linear stochastic systems. Lyapunov Exponents, *Proceedings, Bremen 1984, (L. Arnold, V. Wihstutz, eds.), Springer-Verlag, Berlin (1985), 85-125.*

[3] F. Colonius and W. Kliemann, Infinite time optimal control and periodicity, submitted to: Appl. Math. Optim.

[4] A. Isidori, Nonlinear Control Systems: An Introduction, *Springer-Verlag, Berlin, 1985.*