manuscripta math. 64, 55 - 75 (1989)

manuscripta mathematica
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MAXIMUM PRINCIPLE FOR HYPERSURFACES

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We give an intrinsic proof and a generalization of the interior and boundary maximum principle for hypersurfaces in Riemannian and Lorentzian manifolds. Moreover, we show some new applications to manifolds with lower Ricci curvature bounds. E.g. we prove a local and a Lorentzian version of Cheng's maximal diameter theorem and a non-existence result for closed minimal hypersurfaces.

Contents

0. Introduction.

The maximum principle of E. Hopf is a simple and powerful analytic tool which is often used in the theory of minimal or constant mean curvature hypersurfaces. E.g. it implies that two different minimal hypersurfaces in a Riemannian manifold M cannot touch each other from one side. However, it is difficult to find a written proof for this fact (for $M = \mathbb{R}^n$ see e.g [Sc]). Using special coordinates, one can reduce the statement to the maximum principle for functions (cf. [PW], [GT]).

The present paper has a triple purpose. First we want to give a short proof of the interior and boundary maximum principle for smooth hypersurfaces with suitable mean curvature bounds, which is also valid for spacelike hypersurfaces in a Lorentzian manifold and which uses intrinsic geometry in place of coordinates (Ch.2). Second we give an extension to the case where one of the hypersurfaces satisfies the mean curvature inequality only in a generalized sense, following Calabi [C] (Ch.4). Third we give some new applications to Riemannian and Lorentzian manifolds with lower Ricci curvature bounds (Ch. 3 and 4).

1. Preliminaries

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian or Lorentzian manifold of dimension $n+1$. For any $v \in TM$ we put

$$\|v\| = |\langle v,v\rangle|^{1/2} .$$

Let $S \subset M$ be a spacelike C^2-hypersurface ("spacelike", "timelike" are void conditions in the Riemannian case) with unit normal vector field N. For any $p \in S$, there exists an open neighborhood M' of p in M and a diffeomorphism

$$e: (S \cap M') \times (-\delta,\delta) \to M', \quad e(q,t) = \exp(tN_q) .$$

We call $f = pr_2 \circ e^{-1} : M' \to (-\delta,\delta)$ the *signed distance function* of S.

Let $V = \nabla f$ be the gradient vector field of f on M'. Then $\sigma \cdot V|\{f=t\}$ is the unit normal field of the parallel hypersurface $S_t = \{f=t\}$ where

$$\sigma = \begin{cases} 1 & \text{in the Riemannian case,} \\ -1 & \text{in the Lorentzian case.} \end{cases}$$

From $\langle V,V\rangle = \sigma$ we get $D_V V = 0$, so the integral curves of V are unit speed timelike geodesics. Let $A = DV$ be the hessean tensor of f. Then $A(V) = 0$, and $A|V^\perp$ is the second fundamental tensor of the parallel hypersurfaces S_t. Moreover,

$$(1) \qquad D_V A + A \circ A + R_V = 0 ,$$

where $R_V(X) = R(X,V)V$. Since trace $A = \operatorname{div} \nabla f = \Delta f$, we get by taking the trace of (1)

$$(2) \qquad \partial_V(\Delta f) + (\Delta f)^2/n + ric(V) + \|A_o\|^2 = 0 ,$$

where $ric(V) = \operatorname{trace} R_V$ is the Ricci curvature and A_o is the trace free part of $A|V^\perp$, i.e. $A_o = A - (\operatorname{trace} A)\cdot I/n$ (cf. [E2] for details).

To fix signs, let us call $A|TS$ the *2nd fundamental tensor* (*shape operator*) of S, its eigenvalues the *principle curvatures* and

$$H = \text{trace } A|TS = \Delta f|S$$

the *mean curvature* of S . If $H \le b$ for some constant b , and if $ric(V) \ge -k$ for some positive constant k , then by (2)

$$\partial_V(\Delta f) \le k \ , \quad \Delta f|\{f=0\} \le b \ ,$$

and therefore by integration along the integral curves of V ,

(3) $$\Delta f \le b + k \cdot f$$

on $\{f \ge 0\}$.

If S is the boundary of some domain W , we choose N always to be the exterior normal vector field. Hence, $H \le 0$ means that the domain is concave "in the mean".

2. The maximum principle

THEOREM 1. *Let* W_+ and W_- *be disjoint open domains with spacelike connected* C^∞*-boundaries having a point in common. If the mean curvatures* H_+ *of* ∂W_+ *and* H_- *of* ∂W_- *satisfy*

$$H_- \le -a \ , \quad H_+ \le a$$

for some real number a *, then* $\partial W_- = \partial W_+$ *, and* $H_+ = -H_- = a$.

PROOF. Let $S = \partial W_-$ and let f be the signed distance function of ∂W_+ . Then $f \ge 0$ on S and $\{f=0\} \cap S$ is non-empty. It suffices to show that $S \subset \{f=0\}$. Suppose not. Then $\{f>0\} \cap S$ is a non-empty open subset of S . Let $p' \in \{f=0\} \cap S$ and choose an open coordinate ball U_r around p' in S with some radius r . Then $U_{r/2} \cap \{f>0\}$ contains a coordinate ball B . Making B larger and larger inside U_r , its boundary will finally hit the boundary of $S \cap \{f>0\}$, i.e. we still have $B \subset \{f>0\}$ but there exists a point $p \in \partial B$ with $f(p) = 0$. Now we construct a smooth function h on a neighborhood S' of p in S with

(a) $h(p) = 0$,

(b) $h > 0$ outside $B \cup \{p\}$,

(c) $\|\nabla' h\| \leq 1$,

(d) $\Delta' h \leq -\lambda < 0$

for some positive constant λ, where ∇' and Δ' denote gradient and Laplacian on S with its induced metric. This is done as follows: Let $B' \subset B$ be a coordinate ball with $p \in \partial B'$ and $\partial B' \setminus \{p\} \subset B$ and let s be the signed distance function of $\partial B'$ in S. Then

$$h = \alpha(1 - e^{-\beta \cdot s})$$

has the desired properties if α is small and β large enough.

For small $\delta > 0$ let $B(\delta) \subset S'$ be the metric ball of radius δ centered at p. The function

$$g = f + \varepsilon \cdot h : S' \to \mathbb{R}$$

is positive on $\partial B(\delta)$ and 0 at p, since $f|S \geq 0$ and

$$\partial B(\delta) \subset \{f>0\} \cup \{h>0\},$$

for sufficiently small $\varepsilon > 0$ (depending on δ). So it takes a nonpositive minimum at some point q in the interior of $B(\delta)$. In particular, $\Delta' g(q) \geq 0$. We will obtain a contradiction by showing that in fact $\Delta' g(q) < 0$ if δ and ε are small enough.

Recall that along S we have

(4) $$\Delta' f = \Delta f - H_{-} \cdot df(N) + \sigma \cdot Ddf(N,N)),$$

where N is the normal vector field and H_{-} the mean curvature of $S = \partial W_-$. In fact, if η is the mean curvature vector, i.e. the normal part of $\Sigma D_{e_i} e_i$ for any orthonormal frame $e_1, \dots, e_n$ of TS , then

$$\Delta f = \Delta' f - df(\eta) + \sigma \cdot Ddf(N,N) .$$

On the other hand, if f_- denotes the signed distance function of S , then $N = \sigma \cdot \nabla f_-$ along S and $\eta = \langle \eta, \nabla f_- \rangle \cdot N = -\Delta f_- \cdot N$ which shows (4).

Note that ∇f is a timelike unit vector whose tangent part $\nabla' f$ (with respect to S) is ε-small at q since $\nabla' g(q) = 0$ and therefore

$$\|\nabla' f(q)\| = \varepsilon \cdot \|\nabla' h(q)\| \le \varepsilon .$$

Thus its normal part $(\nabla f)^\perp = \sigma \cdot \langle \nabla f, N \rangle \cdot N = \sigma \cdot df(N) \cdot N$ has a norm which is ε^2-close to 1 . But $df(N)(p) = -1$ and $df(N)$ has no zero near p , therefore $df(N)(q)$ is ε^2-close to -1 .

Moreover, since ∇f is in the kernel of the hessean form Ddf and $\nabla f^\perp = \nabla f - \nabla' f$, we have

$$\begin{aligned} Ddf(N,N) &= df(N)^{-2} \cdot Ddf(\nabla f^\perp, \nabla f^\perp) \\ &= df(N)^{-2} \cdot Ddf(\nabla' f, \nabla' f) \end{aligned}$$

which is ε^2-close to 0 at q .

Finally, by (3) we have

$$\Delta f \le a + k \cdot f$$

on $\{f \ge 0\}$ if $-k$ is a lower bound of $ric(\nabla f)$ near p . Since $f(q) \ge 0$ and $(f+\varepsilon h)(q) \le 0$, we get

$$0 \le f(q) \le \varepsilon \cdot |h(q)| \le \varepsilon \cdot \delta$$

by property (c) since $dist(p,q) \le \delta$. Therefore,

$$\Delta f(q) \le a + k \cdot \varepsilon \cdot \delta .$$

Plugging in all these estimates, we get from (4)

$$\begin{aligned}\Delta' g(q) &= (\Delta f - H_-\cdot df(N) - \sigma\cdot Ddf(N,N)) + \varepsilon\cdot\Delta' h)(q) \\ &\le a + k\cdot\varepsilon\cdot\delta - a + k'\cdot\varepsilon^2 - \varepsilon\cdot\lambda \\ &= \varepsilon\cdot(k\cdot\delta + k'\cdot\varepsilon - \lambda)\end{aligned}$$

where k' is another constant. If δ and ε are sufficiently small, this is negative and the proof is finished.

We may easily extend this argument to get a maximum principle in a manifold with boundary:

THEOREM 1a. *Let* M *be a manifold with boundary* ∂M *and* W_+ and W_- *disjoint open domains with spacelike connected* C^2*-boundaries intersecting* ∂M *transversally. Suppose that there exists a point in* $\partial W_+ \cap \partial W_- \cap \partial M$ *where* ∂W_+ *and* ∂W_- *have common tangent hyperplanes. If the mean curvatures* H_+ *of* ∂W_+ *and* H_- *of* ∂W_- *satisfy*

$$H_- \le -a \ , \quad H_+ \le a$$

for some real number a *, then* $\partial W_- = \partial W_+$ *, and* $H_+ = -H_- = a$.

PROOF. We continue the proof of Theorem 1. Note that now $S = \partial W_-$ is a manifold with boundary $\partial S = S \cap \partial M$. Let $S^o = S \setminus \partial S$. If $\{f=0\} \cap S^o \neq \phi$, we find a coordinate ball B with closure in S^o such that $f > 0$ on B but $f(p) = 0$ for some $p \in \partial B$, and we can proceed as before. Hence we may assume that $f > 0$ on S^o . By assumption, there exists $p \in \partial S$ with $f(p) = 0$ and $\nabla f(p) = 0$. Let B be a coordinate ball in S^o which touches ∂S at

p . Now we may proceed as before. However, $B(\delta)$ now is the metric half ball in S with radius δ and center p . By the choice of h , the function $g = f + \epsilon h$ is positive on $\partial B(\delta) \cap S^{o}$ and on $(B(\delta) \cap \partial S) \setminus \{p\}$, and $\nabla g(p)$ is a nonzero vector pointing into the interior, since $\nabla f(p) = 0$. Thus g takes a minimum on $B(\delta) \setminus \partial S$ and we get the same contradiction as before.

REMARK. To illustrate the last theorem, we recall the argument of Alexandrov [A]: Let $W \subset \mathbb{R}^n$ a relatively compact open subset with connected boundary of constant mean curvature $H = a$ in euclidean n-space. Fix any unit vector $v \in \mathbb{R}^n$. The corresponding hight function $h_v(x) = \langle x,v \rangle$ takes a minimum t_o on ∂W . Let M_t be the half space $\{h_v \geq t\}$ and α_t be the reflection at the hyperplane $\{h_v = t\}$. For $t \geq t_o$ put $W_{t+} = M_t \setminus \text{Clos}(W)$, $W_{t-} = M_t \cap \alpha_t(W)$. Then $W_{t-} \cap W_{t+} \cap \partial M_t = \phi$ for all t , and for t close to t_o we have $W_{t-} \cap W_{t+} = \phi$ while $W_{t-} \subset W_{t+}$ for large t . Let u be the supremum of all t such that $W_{t-} \cap W_{t+} = \phi$. Then ∂W_{u-} touches ∂W_{u+} from the interior. If this contact takes place inside M_u , we get $\partial W_{u-} = \partial W_{u+}$ from Theorem 1. If the contact point lies on ∂M_u , then ∂M_u intersects ∂W_{u+} and ∂W_{u-} orthogonally at this point and we may apply Theorem 1a to get the same conclusion. Hence ∂W is invariant under α_u . The isometries of $\mathbb{R}^n$ which leave the compact subset ∂W invariant form a compact subgroup G which has a common fixed point, say 0 . We have shown that any linear hyperplane reflection occurs as the linear part of such an isometry, so $G = O(n)$ and ∂W is a sphere centered at 0 . (The same argument also works if ∂W is a compact constant mean curvature hypersurface in hyperbolic space or in an open half sphere.)

3. Application to spaces of positive Ricci curvature

In this chapter, we examine a Riemannian or Lorentzian manifold $M = M^{n+1}$ with

(R) $\qquad\qquad ric(v) \geq n$

for any timelike unit vector $v \in TM$. The standard spaces with this condition are

$$Q_\sigma = \{x \in \mathbb{R}^{n+2} \; ; \; \langle x,x\rangle = \sigma\}$$

where we use the scalar product

$$\langle x,y\rangle = \sum_{j=1}^{n} x_j y_j + \sigma(x_{n+1}y_{n+1} + x_{n+2}y_{n+2})$$

on $\mathbb{R}^{n+2}$. For $\sigma = +1$ (riemannian case), this is the unit (n+1)-sphere. For $\sigma = -1$ we get the Lorentzian analogue called *anti-de Sitter space time* (cf. [HE]) which is diffeomorphic to $S^1 \times \mathbb{R}^n$. Its curvature tensor satisfies $R_v = I$ on $v^\perp$ for any timelike unit vector v, and $v^\perp$ is a tangent space of a totally geodesic spacelike hypersurface which is isometric to hyperbolic n-space of curvature -1.

A timelike geodesic $\gamma : I \to M$ is called *locally extremal* if there exists a neighborhood M' of $\gamma(I)$ such that γ is the shortest (in the Riemannian case) or the longest (in the Lorentzian case) among all timelike curves in M' connecting any two points of γ. Due to (R), no geodesic of length $> \pi$ can be locally extremal since there are conjugate points at distance $\leq \pi$

(cf. [E], Prop. 2.4). If M is Lorentzian and globally hyperbolic (cf [HE]), then there exists a longest timelike geodesic between any two points which are timelike seperated, and so we get the

<u>OBSERVATION.</u> *If* M *is Lorentzian and globally hyperbolic and satisfies (R), then no timelike curve in* M *can have length* $> \pi$.

Next we discuss the equality case.

<u>THEOREM 2.</u> *Let* M *be any Riemannian or Lorentzian manifold satisfying* (R) . *If there exists a timelike unit speed geodesic* γ *of length* π *which is locally extremal, then a neighborhood of* $\gamma((0,\pi))$ *is isometric to an open subset of* Q_σ .

<u>PROOF.</u> Let $p_- = \gamma(0)$, $p_+ = \gamma(\pi)$, and put

$$v_- = ((\pi/2)+s)\cdot\gamma'(0) \ , \quad v_+ = -((\pi/2)-s)\cdot\gamma'(\pi)$$

for any $s \in (-\pi/2, \pi/2)$. Since $p := \gamma((\pi/2)+s)$ is not conjugate to p_- nor to p_+ , there exist neighborhoods $V_\pm$ of $v_\pm$ in $T_{p_\pm}M$ where $e_\pm := \exp_{p_\pm}$ are diffeomorphisms. Let

$$B_\pm = \{v \in V_\pm \ ; \ \sigma\cdot\|v\| < \sigma\cdot((\pi/2)\mp s)\} \ .$$

Since γ is locally extremal, there exists an open neighborhood U of p such that

$$W_\pm = U \cap e_\pm(B_\pm)$$

are disjoint open subsets of M with connected boundaries having the point p in common. The comparison theorem for the mean curvature of distance spheres (cf. [E2], 4.2) shows

$$H_\pm \le \pm\tan(s)$$

where $H_\pm$ are the mean curvatures of $\partial W_\pm$. Thus by Theorem 1, W_- and W_+ have common boundary with $H_+ = -H_- = \sigma \cdot \tan(s)$. So the geodesics from p_- and p_+ to any point $q \in \partial W_+ = \partial W_-$ join up to an unbroken geodesic of length π. Hence there is an open neighborhood V of $\gamma'(0)$ in the unit tangent space $S_{p_-}M$ such that $\pi \cdot V$ lies in the domain of e_- and is mapped to p_+. Moreover, if we consider Equation (2) for $S = \partial W_-$ with $s = 0$, we have $\Delta f = -\tan(f)$ and therefore $\mathrm{ric}(V) = n$ and $\|A_o\| = 0$. Hence by (1), $R_V = I$ on $V^\perp$. This finishes the proof.

In a completely analogous way we may prove:

THEOREM 2a *Let* M *be any Riemannian or Lorentzian manifold satisfying* (R) *and* $S \subset M$ *a hypersurface with* $H = 0$. *If there exists a timelike unit speed geodesic* γ *of length* $\pi/2$ *from* $p \in M$ *to* $p_o \in S$ *which has extremal length among all timelike curves from* p *to* S *close to* γ, *then there exists a neighborhood* S' *of* p_o *in* S *which is totally geodesic and a neighborhood* M' *of* $\gamma((0,\pi/2))$ *being isometric to an open subset of* Q_σ *with* $S' \subset \partial M'$.

COROLLARY 1. (Cheng [Ch, Sh]) *Let* M *be a complete connected Riemannian manifold with diameter* π *which satisfies (R). Then* M *is isometric to the unit sphere* Q_+.

PROOF. Let $p_-, p_+ \in M$ be points of distance π. The subset

$$V = \{v \in S_{p_-}M \; ; \; \exp_{p_-}(\pi \cdot v) = p_+\} .$$

of $S_{p-}M$ is closed and non-empty by assumption and open by Theorem 2, thus $V = S_{p-}M$. In particular, the geodesics with initial vector $v \in V$ are minimal from p_- to p_+ , and hence $\exp_{p-}$ is a diffeomorphism on the open disk of radius π . By Theorem 2 again, M is locally isometric and hence isometric to Q_+ .

COROLLARY 2. *Let M be a time oriented Lorentzian manifold which satisfies (R). Let $p_-, p_+ \in M$ be connected by a future directed timelike geodesic γ of length π which has maximal length in its homotopy class. Suppose that the domain of $\exp_{p-}$ contains all timelike future directed vectors of length $\leq \pi$. Then all future directed timelike geodesics of length π starting from p_- join at p_+ and cover a domain which is isometric to*

$$Q_-' = \{x \in Q_- \ ; \ -1 < x_{n+2} < 1 \ , \ \ x_{n+1} > 0\} \ .$$

PROOF. Let $S^+_{p-}M$ be the set of future directed timelike unit vectors and consider the closed nonempty subset

$$V = \{v \in S^+_{p-}M \ ; \ \ \exp_{p-}(\pi \cdot v) = p_+\} \ .$$

By Theorem 2, V is also open and hence $V = S^+_{p-}M$. All geodesics with initial vector $v \in V$ join p_- to p_+ with maximal length in the homotopy class of $\gamma|[0,\pi]$. So they have no conjugate points and cannot meat each other within the parameter interval $(0,\pi)$. Hence $\exp_{p-}$ is a diffeomorphism on the open set $B = \{r \cdot v \ ; \ v \in S^+_{p-}M \ , \ 0 < r < \pi\}$, and $\exp_{p-}(B)$ is locally and therefore globally isometric to Q_-' , by Theorem 2 again.

Last we give an example showing that similar arguments are applicable also in the case where

(R_0) $\qquad\qquad$ $ric(v) \geq 0$

for all timelike $v \in TM$. Two spacelike hypersurfaces S, S' in M are called *strongly parallel* if they bound a domain W diffeomorphic to $S \times [0,L]$ with metric $g = g_S + \sigma \cdot dt^2$ where g_S is the induced metric on S. In particular, S and S' are totally geodesic. The following theorem is well known in the Riemannian case (cf. [I],[K],[GR]).

THEOREM 3. *Let* M *satisfy* (R_0) *and let* S, S' *be disjoint spacelike* C^2*-hypersurfaces with zero mean curvature. If there exists a timelike geodesic* $\gamma : [0,L] \to M$ *which realizes the distance between these two hypersurfaces, then* S *and* S' *are strongly parallel near* γ.

PROOF. Let f, f' be the signed distance functions of S, S' for the normals $\gamma'(0)$, $-\gamma'(L)$. Since γ realizes distance, f and f' are defined in a neighborhood M_0 of $\gamma([0,L])$. For $t \in (0,L)$ put

$$W_- = \{f < t\}\ , \quad W_+ = \{f' > L-t\}\ .$$

∂W_+ and ∂W_- are smooth on M_0 and touch each other at $\gamma(t)$ from outside. By (2), they have both mean curvature ≤ 0. So by Theorem 1 they must agree on M_0, and from (2) and (1) we get $A = 0$, $R_V = 0$ between S and S' where $V = \nabla f$. This shows the statement.

4. Generalized mean curvature inequalities

As in the case of convexity, a mean curvature inequality can be generalized to non-smooth boundaries. To avoid technical complications, we assume from now on that M is a *Riemannian* manifold. However, most of the discussion can be transferred to the Lorentzian case (cf. [E3],[G]). Let $W \subset M$ be an arbitrary open domain with topological boundary ∂W . Let $b \in \mathbb{R}$.

DEFINITION. ∂W has *generalized mean curvature* $\leq b$ if for any $p \in \partial W$ there are open domains $W_{p,j}$, $j = 1,2,\ldots$, called *support domains,* whose boundaries are C^2-hypersurfaces near p with shape operator $A_{p,j}$ and mean curvature $H_{p,j}$ at the point p , with the following properties:

(a) $W_{p,1} \subset W_{p,2} \subset \ldots \subset W$,

(b) $p \in \partial W_{p,j}$,

(c) there is a locally uniform upper bound for $A_{p,1}$,

(d) $H_{p,j} \leq b + \varepsilon_j$ for some sequence $\varepsilon_j \to 0$.

Note that (a) yields $A_{p,1} \geq A_{p,2} \geq \ldots$ so that (c) also gives an upper bound for $A_{p,j}$. If ∂W is itself a C^2-hypersurface, then clearly $A_{p,j} \geq A_p$ from (a) , and ∂W has mean curvature $H \leq b$ by (d).

Let $f : M \setminus W \to \mathbb{R}_+$ be the exterior distance of ∂W , i.e.

$$f(q) = d(q,\partial W) = d(q,\mathrm{Clos}(W)) .$$

On the other hand, for any $p \in \partial W$ let $f_{p,j}$ be the signed distance functions of $\partial W_{p,j}$, defined on an open neighborhood

$M_{p,j}$ of p. Then

$$f_{p,j}(q) = d(q, \mathrm{Clos}(W_{p,j}))$$

on $M_{p,j} \setminus W_{p,j}$, and since $W_{p,j} \subset W$, we have $f_{p,j} \geq f$ on $M_{p,j} \setminus W_{p,j}$. On the other hand, there is a neighborhood M' of $\mathrm{Clos}(W)$ such that for any $q \in M'\setminus W$ there is a closest point p on ∂W. So $f(q) = d(p,q) \geq f_{p,j}(q)$, and hence $f_{p,j}$ is a smooth upper support function of f at q, which means $f = f_{p,j}$ at q and $f \leq f_{p,j}$ near p. If we assume $\mathrm{Ric} \geq -k$ on $M'\setminus W$, by (3) we get for this upper support function

(3') $$\Delta f_{p,j}(q) \leq b + \varepsilon_j + k \cdot f(q) .$$

<u>EXAMPLE 1.</u> Let M be complete, connected and $C \subset M$ a closed subset. For fixed $r > 0$ let

$$W = B_r(C) = \{x \in M \; ; \; d(x,C) < r\} .$$

For any $p \in \partial W$ there exists a shortest geodesic γ with $\gamma(0) = p$ and $\gamma(r) = q \in C$. Let $r_j \to r$, $0 < r_j < r$, and put

$$W_{p,j} = B_{r_j}(\gamma(r_j)) .$$

Then $\partial W_{p,j}$ is smooth near p since $p = \gamma(0)$ is outside the cut locus of $\gamma(r_j)$, and (a),(b),(c) are satisfied; note that for small r_1, an upper curvature bound near p yields (c). Moreover, if $\mathrm{Ric} \geq -k$ on W for some $k \in \mathbb{R}$, then it is well known (e.g. cf [E2]) that

$$H_{p,j} \geq ct_k(r_j)$$

where $ct_k = c_k/s_k$ and (s_k,c_k) is the solution of

$$s_k' = c_k \; , \quad c_k' = k \cdot s_k \; , \quad s_k(0) = 0 \; , \quad c_k(0) = 1 \; .$$

Thus ∂W has generalized mean curvature $\leq ct_k(r)$.

EXAMPLE 2. Let M be complete, connected and non-compact. Then there exists a *ray* $\gamma : \mathbb{R}_+ \to M$ (i.e. γ is shortest on any finite segment). For any $r,t \in \mathbb{R}$ with $r+t > 0$ let

$$B_{t,r} = B_r(\gamma(r+t)) .$$

By triangle inequality, we have $B_{t,r'} \subset B_{t,r}$ if $r' < r$. Then

$$B_t = \bigcup_r B_{t,r}$$

is called a *horo-ball* of γ , for any $t \in \mathbb{R}$. In other words,

$$B_t = \{b_\gamma \geq t\}$$

where b_γ is the Busemann function of γ , i.e.

$$b_\gamma(p) = \lim_{r \to \infty} (r - d(p,\gamma(r))) .$$

The exterior distance f_t of B_t on $M \setminus B_t$ satisfies

$$f_t = t - b_\gamma ,$$

in particular $f_{t+u} = f_t + u$ on $M \setminus B_{t+u}$ for $u > 0$.

Put $W = B_t$. For any $p \in \partial W$ let γ_p be the asymptotic ray for γ , i.e. there exists a sequence $s_j \to \infty$ and shortest geodesics $\gamma_{p,j}$ from p to $\gamma(s_j)$ which converge to γ_p . For any sequence $r_j \to \infty$ put

$$W_{p,j} = B_{r_j}(\gamma_p(r_j)) .$$

Then $W_{p,j} \subset W$ by triangle inequality (e.g. cf. [E1]) and (a),(b),(c) are satisfied as above. If moreover $\mathrm{Ric} \geq -k$ on W for some $k \geq 0$, then

$$H_{p,j} \leq ct_k(r_j) \to k^{1/2} ,$$

and so ∂W has generalized mean curvature $\leq k^{1/2}$.

THEOREM 1b. *Let* W_+ and W_- *be disjoint open domains with connected boundaries having a point in common. Suppose that* $S := \partial W_-$ *is a* C^∞*-hypersurface with mean curvature* $H_- \le -a$ *and that* ∂W_+ *has generalized mean curvature* $\le a$. *Then* $\partial W_- = \partial W_+$ *and* $H_- = -a$.

PROOF. We modify the proof of Theorem 1, keeping the notation, where now f is the exterior distance of ∂W_+ . Note however that the functions f and $g = f + \epsilon h$ on S' are only continuous. Therefore, we pass to the support functions: Let p' be a point of ∂W_+ which is closest to the point $q \in S'$ where g attains its minimum, and let $f_j = f_{p',j}$ be the corresponding upper support functions of f at q . Then also $g_j := f_j + \epsilon h$ takes a minimum at q . Using (c) and a lower curvature bound near p , we get an upper bound for the Hessean $Ddf_j(q)$ for all j , independent of q . On the other hand, since q is a minimum of g_j , we have

$$(*) \qquad D'd(f_j|S)(q) \ge -\epsilon \cdot D'dh(q)$$

where D' denotes the induced connection on T^*S . Since

$$Ddf_j|TS = D'd(f_j|S) - \langle \alpha, N\rangle\langle N, \nabla f_j\rangle$$

where α is the second fundamental form and N the unit normal field of S , (*) gives a uniform lower bound for $Ddf_j(q)|T_qS$. Replacing f and g with f_j and g_j , we get as in the proof of Theorem 1

$$\Delta' g_j(q) \le \epsilon \cdot (k\delta + k'\epsilon - \lambda) + \epsilon_j \ .$$

If we choose δ and ϵ small enough, the first term is negative. Then we may choose j large enough to make the right hand side negative which is a contradiction since q was a minimum of g_j .

REMARK. The proof remains correct if the assumption for W_+ is slightly weaker, namely $W_+ = \bigcup_j W_j$ for open domains $W_1 \subset W_2 \subset W_3 \subset \ldots$ where W_j has generalized mean curvature $\le a+\varepsilon_j$ for some $\varepsilon_j \to 0$. This was essentially shown by Galloway [G, 2.4] in the Lorentzian case. Further, a similar theorem is true if both ∂W_+ and ∂W_- have only generalized mean curvature $\le \pm a$, cf. [E3]. However, the proof is different since we can no longer restrict everything to ∂W_-.

Theorem 1b has several applications for open manifolds of nonnegative Ricci curvature. The next theorem generalizes the fact that complements of horo-balls are totally convex in manifolds of nonnegative sectional curvature [CG2]:

THEOREM 4. *Let* M *be a complete non-compact manifold with* $\mathrm{Ric} \ge 0$ *and* $W \subset M$ *a horo-ball. Then* $M \setminus W$ *has the following convexity property: Any compact minimal hypersurface* S *with boundary* $\partial S \subset M \setminus W$ *is contained in* $M \setminus W$.

PROOF. Otherwise, the Busemann function b of the corresponding ray takes a maximum t on $S^o = S \cap W$, say at $p \in S^o$. Thus S^o bounds an open domain $W_- \subset W$ where $b \le t$. Put $W_+ = B_t = \{b > t\} \subset W$. Then W_+ and W_- satisfy the assumptions of Theorem 1b with $a = 0$ and M replaced with W. Thus $S^o = \partial W_+$ which is impossible since ∂W_+ is closed in M and S^o is not.

REMARK. The preceding theorem was proved in [AR, §2] in the case that S is absolutely minimizing. A corresponding statement for Lorentzian manifolds was proved by Galloway [G, Lemma 2.4]; our proof is an adaptation of his ideas to the Riemannian case. Our last theorem was proved by Anderson [An] for minimal hypersurfaces which are area minimizing in their homology class.

THEOREM 5. *Let* M *be a complete connected non-compact manifold with* $\mathrm{Ric} \geq 0$ *which contains a compact connected two-sided minimal hypersurface* S *(without boundary). Then* S *is totally geodesic and bounds a domain* W *isometric to* $S \times (0,\infty)$ *, and* $M \backslash W$ *is compact unless* M *is isometric to* $S \times \mathbb{R}$ *.*

PROOF. Since M is non-compact, it contains a ray γ . Let b be the corresponding Busemann function. There is a point $p \in S$ where $b|S$ takes a maximum, say t . Thus $b|S \leq t$. Put $W = \{b > t\}$. Choose open p-neighborhoods $S' \subset S$ and $M' \subset M$ such that $M' \backslash S'$ has two connected components M'_+, M'_- . At least one of these, say M'_+, contains a connected component W_+ of $W \cap M'$. Put $W_- = M'_-$. Now Theorem 1b shows $\partial W_+ = \partial W_- = S'$. It follows that $S \cap \partial W$ is open (and closed) in S . Hence S is a connected component of ∂W , and in particular, S is embedded. Let N be a unit normal field on S which points into W . Put

$$e : S \times (0,\infty) \to M , \quad e(p,t) = \exp(tN_p) .$$

Since for any $p \in S \subset \partial W$ the asymptotic ray γ_p realizes the distance to ∂W (cf. Example 2), it must agree to the normal geodesic $t \to e(p,t)$. So e is an embedding and its range is contained in W . Since S is compact, the range of e is open and closed in W and therefore, e is a diffeomorphism onto W. By an argument of [EH], e is an isometry. Namely, if $f = pr_2 \circ e^{-1}$ and $\phi = \Delta f \circ \gamma_p$ for fixed $p \in S$, we get from (2) the inequality $\phi' + \phi^2/n \leq 0$. Put $\psi(t) = \int_0^t \phi(s)ds$. Then e^{ψ} is a concave positive function on $[0,\infty)$ with initial derivative 0, hence $e^{\psi} \equiv 1$ and $\phi \equiv 0$. Now from (2), $A = Ddf = 0$, hence $e : S \times (0,\infty) \to W$ is an isometry.

Suppose that $M\backslash W$ is not compact. Then it contains at least one end. Since also W has an end, we can construct a line β (i.e. a complete shortest geodesic) connecting these ends. Then $\beta_+ = \beta \cap W$ is a ray in W, hence $\beta_+ = e(\{q\} \times (0,\infty))$ for some $q \in S$. Now using the Cheeger-Gromoll splitting theorem ([CG1], [EH]), we see that $M = S \times \mathbb{R}$. ■

Acknowledgements. We like to thank G. Galloway for several hints.

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(Received December 16, 1988;
in revised form January 17, 1989)