REDUCTION OF CODIMENSION OF SURFACES

Let M be a 2-dimensional smooth manifold, Q an n-dimensional space form of constant sectional curvature and $x: M \to Q$ an immersion. We say that we can reduce the codimension to $k < n - 2$ if there is a $(k + 2)$-dimensional totally geodesic submanifold $Q' \subset Q$ such that $x(M) \subset Q'$. We always equip M with the induced metric. Let N be the normal bundle with its induced connection D and H the mean curvature vector of the immersion. Our first theorem generalizes a result of [1] and [6]:

THEOREM 1. *Let $x: M^2 \to Q^n$ be an analytic immersion such that its mean curvature vector H lies in a parallel subbundle E of N. Then either we can reduce the codimension to* $\dim(L) + 1$, *or $x(M)$ is minimal in an* $(n\text{-}\dim(L))$-*dimensional totally umbilic submanifold.*

THEOREM 2. *Let M be homeomorphic to a 2-sphere and $x: M \to Q^n$ a smooth immersion such that H lies in a parallel subbundle E of N. Then either the codimension can be reduced to* $\dim(L)$ *or $x(M)$ is minimal in an* $(n\text{-}\dim(L))$-*dimensional totally umbilic submanifold of Q^n.*

Proof of Theorem 1. Let F be the orthogonal complement of E in N; this is also a parallel subbundle. Let $\alpha: TM \otimes TM \to N$ be the second fundamental form of x and α_E, α_F its components in E and F. For any $\xi \in N_p$ define $A_\xi: T_pM \to T_pM$ by

$$\langle A_\xi(v), w\rangle = -\langle \alpha(v, w), \xi\rangle.$$

The Ricci equation gives for $\xi, \eta \in N_p$

$$\text{(R)} \qquad \langle R^N(v, w)\xi, \eta\rangle = \langle [A_\xi, A_\eta]v, w\rangle.$$

By analyticity we have the following two cases:

(a) F is a flat subbundle, i.e. the curvature tensor of the normal connection D vanishes on F.

(b) The normal curvature tensor on F does not vanish on an open dense subset of M.

CASE (a). This is proved by the following more general lemma:

LEMMA 1. *Let M be any manifold of dimension m and $x: M \to Q^n$ an immersion. Suppose that H lies in a parallel subbundle E of N whose orthogonal complement F is flat with respect to the normal connection. Then we may reduce the codimension to* $\dim(E) + m - 1$.

Proof of Lemma 1. By (R), the endomorphisms A_ξ and A_η commute for any two vectors $\xi, \eta \in F_p$ since F is flat. Thus the linear maps A_ξ are simultaneously diagonalizable for all $\xi \in F_p$. Hence there is an orthonormal basis $e_1, \ldots, e_n$ of T_pM such that

$$(*) \qquad \alpha_F(e_j, e_k) = 0 \quad \text{for } j \neq k.$$

Let N_1 denote the first normal space, i.e. the linear span of the values of α. We claim that $E + N_1$ is a parallel subbundle of N. In fact, if $\xi \in F \cap (N_1)^\perp$, then

$$\begin{aligned} A_{ijk} &:= \langle D_{e_i}\xi, \alpha(e_j, e_k)\rangle \\ &= \langle D_{e_i}\xi, \alpha_F(e_j, e_k)\rangle = -\langle \xi, D_{e_i}\alpha(e_j, e_k)\rangle \end{aligned}$$

is symmetric in all three indices, by Codazzi's equation, and by $(*)$ it vanishes if at least two indices are different. But since H lies in E, we have trace $\alpha_F = 0$ and so

$$A_{kkk} = -\sum_{j \neq k} \langle D_{e_k}v, \alpha_F(e_j, e_j)\rangle = 0.$$

Thus $D_{e_i}v \in F \cap (N_1)^\perp$, which shows that $F \cap (N_1)^\perp$ and therefore also $E + N_1$ are parallel. Since N_1 is spanned by $\alpha(e_j, e_j)$ for $j = 1, \ldots, n$ and $\Sigma_{j=1}^n \alpha(e_j, e_j) \in E$, we have $\dim(E + N_1) \leqslant \dim(E) + n - 1$ and the result follows by Erbacher's theorem (cf. [3]).

Case (b). Let $p \in M$ be a point where the normal curvature tensor R^N does not vanish. So there exist $\sigma, \tau \in F_p$ such that $\langle R^N(v, w)\sigma, \tau\rangle \neq 0$. On the other hand, $R^N(v, w)\xi \in E_p$ for any $\xi \in E_p$ since E is parallel in N. Hence by (R),

$$[A_\sigma, A_\tau] \neq 0, \qquad [A_\sigma, A_\xi] = [A_\tau, A_\xi] = 0.$$

It follows that $A_\xi = \lambda(\xi)\cdot\mathrm{Id}$ for all $\xi \in E_p$, i.e. ξ is an umbilic normal vector.

Next we claim that DH has values in $\mathbb{R}\cdot H$ ('$DH \subset \mathbb{R}\cdot H$' for short). In fact, if ξ is a section of E with $\xi \perp H$, then ξ is normal to N_1, the span of the values of α. Moreover, since each $\eta \in E_p$ is umbilic,

$$\langle \eta, \alpha(e_1, e_1)\rangle = \langle \eta, \alpha(e_2, e_2)\rangle = \langle \eta, H\rangle, \qquad \langle \eta, \alpha(e_1, e_2)\rangle = 0$$

for an orthonormal base e_1, e_2 of T_pM. We apply this to $\eta = D_{e_i}\xi$ and using Codazzi's equations, we get

$$\begin{aligned} \langle D_{e_i}\xi, H\rangle &= \langle D_{e_i}\xi, \alpha(e_j, e_j)\rangle = -\langle \xi, D_{e_i}\alpha(e_j, e_j)\rangle \\ &= -\langle \xi, D_{e_j}\alpha(e_j, e_i)\rangle = \langle D_{e_j}\xi, \alpha(e_j, e_i)\rangle = 0 \end{aligned}$$

for $i \neq j$, similar to Case (a). Thus $E \cap (H)^\perp$ and hence $\mathbb{R}\cdot H$ are parallel subbundles of N.

Now by the following lemma, H itself is parallel in N. Thus α has values in the parallel subbundle $F + \mathbb{R}\cdot H$ of N, by umbilicity of E. If $H = 0$, we may reduce the codimension to $\dim(F)$. If $H \neq 0$, then $x(M)$ has codimension $\dim(F) + 1$ in a totally geodesic submanifold $Q' \subset Q$, and since H is parallel, it follows from a theorem of Yau [7] that $x(M)$ is minimal in an umbilic hypersurface of Q'. This finishes the proof of Theorem 1.

LEMMA 2. (Chen [1]). *Let M be any manifold and x: $M \to Q^n$ an immersion with umbilic mean curvature vector H and parallel mean curvature direction (i.e. $DH \subset \mathbb{R}\cdot H$). Then the mean curvature vector is parallel, i.e. $DH = 0$.*

Proof of Lemma 2. Choose an orthonormal tangent frame $e_1, \ldots, e_m$ (where $m = \dim(M)$) which is parallel at the point $p \in M$ which we consider. Put $\alpha_{ij} = \alpha(e_i, e_j)$. By umbilicity, we have

$$\langle \alpha_{ij}, H\rangle = \langle H, H\rangle \cdot \delta_{ij},$$

and in particular, $\langle H, H\rangle = \langle \alpha_{kk}, H\rangle$ for any k. Moreover,

$$D_{e_j} H = \lambda_j \cdot H$$

for some function λ_j. Thus $e_j\langle H, H\rangle = 2\lambda_j\langle H \cdot H\rangle$. On the other hand,

$$e_j\langle H, H\rangle = e_j\langle \alpha_{kk}, H\rangle = \langle D_{e_j}\alpha_{kk}, H\rangle + \lambda_j\langle H, H\rangle.$$

Thus at the point p we get for $k \neq j$, using Codazzi equations

$$\begin{aligned}\tfrac{1}{2} e_j\langle H, H\rangle &= \langle D_{e_j}\alpha_{kk}, H\rangle = \langle D_{e_k}\alpha_{jk}, H\rangle \\ &= e_k\langle \alpha_{jk}, H\rangle - \lambda_k\langle \alpha_{jk}, H\rangle = 0\end{aligned}$$

which finishes the proof.

REMARK. The umbilicity of H need not be assumed but can be concluded from $DH \subset \mathbb{R}\cdot H$.

Proof of Theorem 2. We use the same notation as above. The metric induces a conformal, hence complex, structure on M. Let $z = x + iy$ be a holomorphic chart and put $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. After complex-linear extension we have in general (cf. [5, 4.1])

$$D_{\bar{z}}(\alpha(\partial_z, \partial_z)) = \tfrac{1}{4}\lambda^2 D_z H,$$

where D is the normal connection and $D_{\bar{z}} = \frac{1}{2}(D_x + iD_y)$. Since F is a parallel subbundle of N normal to H, we get

$$(**) \qquad D_{\bar{z}}(\alpha_F(\partial_z, \partial_z)) = 0.$$

This implies that $\langle \alpha_F(\partial_z, \partial_z), \alpha_F(\partial_z, \partial_z)\rangle$ is a holomorphic function, where the metric $\langle\,,\rangle$ has been complex linearly extended. Hence the quartic form

$$\Lambda = \langle \alpha_F(\partial_z, \partial_z), \alpha_F(\partial_z, \partial_z)\rangle\, dz^4$$

(which is the (4, 0) part of the quartic form $\langle \alpha_F, \alpha_F \rangle$) is holomorphic. If M is homeomorphic to the 2-sphere, such a global holomorphic differential must be zero (cf. [4] for a simple proof), so $\alpha_F(\partial_z, \partial_z)$ is an isotropic vector (i.e. zero scalar product). In geometric terms this means that the projection to F of the ellipse of curvature $\{\alpha(v, v);\ v \in T_pM,\ \|v\| = 1\}$ is a circle centered at 0. By a lemma of Chern (cf. [2], [4]), (**) implies that either $\alpha_F(\partial_z, \partial_z) \equiv 0$ (Case (a)), or the zeros of $\alpha_F(\partial_z, \partial_z)$ are isolated (Case (b)). Since H is normal to F, we have $\alpha_F(\partial_x, \partial_x) = -\alpha_F(\partial_y, \partial_y)$, and so $\alpha_F(\partial_z, \partial_z) = 0$ if and only if $\alpha_F = 0$. Hence in Case (a), α takes values in the parallel subbundle E of N, and so the codimension can be reduced to $\dim(E)$, by Erbacher's theorem.

Now assume that the zeros of $\alpha(\partial_z, \partial_z)$ are isolated (Case (b)). In general, if e_1, e_2 is an orthonormal tangent frame, e.g. $e_1 = \partial_x/\lambda$, $e_2 = \partial_y/\lambda$, and if we put

$$\varphi = \alpha_{11} - H = H - \alpha_{22}, \qquad \psi = \alpha_{12},$$

where $\alpha_{ij} = \alpha(e_i, e_j)$, then for any normal vectors ξ, η we have

$$\langle [A_\xi, A_\eta] e_1, e_2 \rangle = \langle \varphi \wedge \psi, \xi \wedge \eta \rangle.$$

In our case, since $\alpha(\partial_z, \partial_z)$ is isotropic, the F-components φ_F, ψ_F of φ, ψ are linearly independent. So it follows from the Ricci equation (R) that F has vanishing normal curvature tensor only at isolated points, and we may apply Case (b) of the proof of Theorem 1. This finishes the proof of Theorem 2.

REMARK. One may try to extend these theorems to more general target spaces Q, e.g. $Q = \mathbb{C}P^m$. However, if we wish E to behave nicely with respect to the complex structure, then only the minimal case ($H = 0$) is possible:

THEOREM 3. *Let (Q, J) be a Kähler manifold and $x\colon M^2 \to Q$ an immersion whose mean curvature vector H lies in a parallel J-invariant subbundle E of the normal bundle N. Then $H = 0$.*

Proof. Let ξ be a section of E and X a tangent vector field on M. Let ∇ denote the connection in x^*TQ and $D = \nabla^\perp$ the induced connection on N. Then

$$\nabla_X J\xi = J\nabla_X \xi = JD_X\xi + JA_\xi X,$$

and $JD_X\xi$ is a section of E while $JA_\xi X$ lies in $E^\perp$ which is the orthogonal complement of E in x^*TQ (containing TM and hence JTM). On the other hand,

$$\nabla_X J\xi = D_X J\xi + A_{J\xi} X$$

which is again a decomposition with respect to E and $E^\perp$. Therefore we get in particular

$$JA_\xi X = A_{J\xi} X,$$

and $JA_\xi X$ is tangent. Consequently, at any point either the tangent space is J-invariant which implies $H = 0$, or $A_\xi X = 0$ for all ξ and X and, in particular, $A_H = 0$ which also implies $H = 0$.

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