RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 35, NUMBER 10 MAY 15, 1987

Dynamical systems: A unified colored-noise approximation

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(Received 19 January 1987)

By use of an adiabatic elimination procedure and a time scaling $\hat{t}=\tau^{-1/2}t$, where τ denotes the correlation time of colored noise $\varepsilon(t)$, one arrives at a novel colored-noise approximation which is exact both for $\tau=0$ and $\tau=\infty$. The theory is implemented for one-dimensional flows of the type $\dot{x}=f(x)+g(x)\varepsilon(t)$. The approximation has the form of a Smoluchowski dynamics, which is valid in regions of state space for which the damping $\gamma(x,\tau)\equiv\tau^{-1/2}-\tau^{1/2}[f'-(g'/g)f]$ is positive and large, and times $t\gg\tau^{1/2}/\gamma(x,\tau)$. This novel Smoluchowski dynamics combines the advantageous features of a recent decoupling theory that does not restrict the value of τ, together with those occurring in the small-correlation-time theory due to Fox. The approximative theory is applied to a nonlinear model for a dye laser driven by multiplicative noise. Excellent agreement for the stationary probability is obtained between numerical exact solution and the novel approximative theory.

Recent work on dye lasers[1,2] and the optical ring laser gyroscope[3] has emphasized the physical role of colored-noise sources. An already well-known situation in which strongly colored noise determines the physics is the phenomenon of motional narrowing in magnetic resonance.[4] Another area of active colored-noise research addresses escape problems that are currently in the limelight from both the theoretical[5] and experimental[6] points of view. In this context, a frequency-dependent friction mechanism, being coupled to colored noise via the fluctuation-dissipation relation, can considerably modify the classical barrier transmission. Except for two-state noise[7] there exists presently no exact analytical theory for nonlinear dynamical systems driven by correlated noise. Thus, one is forced to either perform laborious numerical studies or to invoke approximative schemes. The common approximative schemes developed in the recent literature[8,9] describe only corrections to the white-noise limit; i.e., they are necessarily restricted to small noise correlation times. Presently, there is only one scheme, originated by one of the authors[10] (see also Ref. 2) which does not restrict the value of the noise correlation time τ. This novel decoupling approximation, however, is, by construction, limited to probabilities with small widths. Generally, this implies a small noise intensity D. In order to improve on the present state of affairs an ideal approximation scheme should encompass the advantages of the decoupling theory,[10] but should also accurately describe the behavior for moderate-to-strong noise intensity, in a way that holds for the small correlation time theories.[8,9]

In the following we restrict the analysis to a one-dimensional dynamical flow, driven by exponentially correlated, Gaussian (central limit theorem) noise $\varepsilon(t)$:

$$\dot{x}=f(x)+\varepsilon(t) \; , \tag{1a}$$

$$\langle\varepsilon(t)\varepsilon(s)\rangle=(D/\tau)\exp[-(|t-s|/\tau)] \; . \tag{1b}$$

In one dimension, (1) describes the dynamics without loss of generality: A possible multiplicative noise structure $\varepsilon(t)\rightarrow g(x)\varepsilon(t)$ can always be transformed into additive noise, if we only transform the state variable $x\rightarrow\tilde{x}=\int^x g^{-1}(y)dy$. Equation (1) is equivalent to the two-dimensional (Markovian) flow

$$\begin{aligned}&\dot{x}=f(x)+\varepsilon \; ,\\ &\dot{\varepsilon}=-\varepsilon/\tau+(D^{1/2}/\tau)\xi(t), \;\; \langle\xi(t)\xi(s)\rangle=2\delta(t-s) \; .\end{aligned} \tag{2}$$

Because (2) does not obey detailed balance, it does not present a simplification over (1); i.e., exact solutions are beyond analytical means both for (1) and (2). Before we proceed let us emphasize the following requirements of the new approximation scheme. (i) It must be simple (Fokker-Planck form), enabling direct analytical evaluation. (ii) It must (hopefully) work for all correlation times τ, as well as for weak-to-moderate-to-strong noise intensity D.

As is well known from the study of adiabatic elimination of fast variables,[11,12] in the limit $\tau\rightarrow 0$ from (2), again one obtains an exact Fokker-Planck dynamics (Smoluchowski equation). Thus, we are guided to seek an approximation which for $\tau\rightarrow 0$ and $\tau\rightarrow\infty$ approaches a Smoluchowski-type limit; i.e., it becomes exact both for $\tau=0$ and $\tau=\infty$. For intermediate τ values such a scheme is then expected to give a useful approximation. A theory of this type can indeed be constructed. First, we eliminate the variable ε in (2) and introduce the new time scale $t=\tau^{1/2}\hat{t}$. This then yields (caret denotes differentiation with respect to x),

$$\ddot{x}+\gamma(x,\tau)\dot{x}-f(x)=D^{1/2}\xi(\tau^{1/2}\hat{t}) \; , \tag{3a}$$

where

$$\gamma(x,\tau)=\tau^{-1/2}+\tau^{1/2}[-f'(x)] \; . \tag{3b}$$

The derivative $-f'(x)$ is positive in regions of local stability. Most remarkable is that whenever $-f'(x)>0$ one has $\gamma(x,\tau)>0$ for all x and τ; and $\gamma(x,\tau)$ approaches infinity both for $\tau\rightarrow 0$, as well as for $\tau\rightarrow\infty$. Then the conventional adiabatic scheme[12] proceeds by setting $\ddot{x}=0$. This conventional procedure gives a multiplicative process;

i.e., a Langevin dynamics

$$\dot{x} = f(x)/\gamma(x,\tau) + [D^{1/2}\tau^{-1/4}/\gamma(x,\tau)]Q(\hat{t}) \tag{4a}$$

driven by Gaussian white noise $\langle Q(\hat{t})Q(\hat{s})\rangle = 2\delta(\hat{t}-\hat{s})$, with a corresponding (Stratonovich[13]) Fokker-Planck dynamics

$$\dot{W}_{\hat{t}}(x,\tau) = \frac{-\partial}{\partial x}(\{[f(x)/\gamma(x,\tau)] - D\tau^{-1/2}\gamma'(x,\tau)\gamma^{-3}(x,\tau)\}W_{\hat{t}}(x,\tau)) + D\tau^{-1/2}\frac{\partial^2}{\partial x^2}[\gamma^{-2}(x,\tau)W_{\hat{t}}(x,\tau)] \ . \tag{4b}$$

Note that the effective diffusion coefficient $D(x,\tau) = D\tau^{-1/2}/\gamma^2(x,\tau) > 0$ *always stays positive.* Its stationary probability is readily found to read, explicitly,[14]

$$\begin{aligned} W_{st}(x,\tau) &= N^{-1}|1-\tau f'(x)|\exp[-\tfrac{1}{2}\tau f^2(x)/D]\exp\left[D^{-1}\int^x f(y)dy\right] \\ &= Z^{-1}W_{st}^0(x,\tau=0)|1-\tau f'(x)|\exp[-\tfrac{1}{2}\tau f^2(x)/D] \ , \end{aligned} \tag{4c}$$

with N,Z denoting the corresponding normalization constants. Equations (4a)–(4c) are our main results. The usual Smoluchowski result ($\tau=0$) follows immediately. For small D we can approximate $\tau^{1/2}\gamma(x,\tau)\to 1-\tau\langle f'(x)\rangle$, and recover from (4b) the decoupling theory;[10] i.e., we only have to substitute within the ($\tau=0$) theory the diffusion $D\to D/[1-\tau\langle f'(x)\rangle]$. It should be pointed out, however, that the decoupling theory *holds* for probabilities of small width(s), *independent of the positivity condition* $\gamma(x,\tau)>0$; i.e., $1-\tau f'(x)>0$. Most remarkably is the observation that $W_{st}(x,\tau)$ in (4c) precisely equals $W_{st}(x,\tau)$ within the Fox theory;[9] i.e., the recent small-correlation-time theory which supersedes the conventional small-noise-correlation-time theories.[8] Note, however, that the Fokker-Planck equation in Fox theory substantially differs from the Smoluchowski dynamics (4b) [e.g., the diffusion coefficient in Ref. 9 takes on negative values for $1-\tau f'(x)<0$; in contrast, $D(x,\tau)$ in (4b) is strictly positive]; nevertheless, they possess identical stationary probabilities.

The dynamics in (4b) (including equilibrium correlations) is thus expected to represent accurately[15] the exact colored-noise behavior for *times t* obeying

$$\hat{t} \gg \gamma^{-1}(x,\tau) = [\tau^{-1/2} - \tau^{1/2}f']^{-1} \ ;$$

i.e., $t \gg \tau/(1-\tau f')$, and in *space regions x* obeying $\gamma(x,\tau) \gg D^{1/2}|f'/f|$, particularly within the whole state space if $f'<0$ for all x. The latter condition follows because the adiabatic elimination of $\dot{x}$ requires that the change of the force field over the characteristic length, $l=\gamma^{-1}(x,\tau)D^{1/2}$, is small[16] ($x$ indicates the corresponding characteristic value of $\{\gamma,f',f\}$ within the length l).

Next we test this assertion with an example widely discussed in the Laser community.[1,2,17]

A dye laser obeys near threshold a multiplicative colored-noise dynamics of the form[1,17]

$$\dot{I} = 2(a-I)I + \sqrt{2}I\varepsilon \ . \tag{5}$$

With $x=\ln I$, (5) can be transformed into additive colored noise; i.e.,

$$\dot{x} = 2(a-e^x) + \sqrt{2}\varepsilon \ . \tag{6a}$$

From (3a) we obtain with the time scale $\hat{t}=\tau^{-1/2}t$,

$$\ddot{x} + [\tau^{-1/2} + 2\tau^{1/2}e^x]\dot{x} - 2(a-e^x) = \sqrt{2}D^{1/2}\xi(\tau^{1/2}\hat{t}), \tag{6b}$$

wherein with $-f'(x)=2e^x>0$ the damping $\gamma(x,\tau)$ is positive for all x and τ. Thus, we obtain for the stationary probability $W_{st}(I,\tau)=W_{st}(x,\tau)|dx/dI|$ (see Ref. 14) the result

$$W_{st}(I,\tau) = Z^{-1}(1+2\tau I)\exp[\tau I(2a-I)/D]W_{st}^0(I,\tau=0) \ , \tag{7a}$$

with

$$W_{st}^0(I,\tau=0) = I^{[(a/D)-1]}\exp\{-I/D\} \ . \tag{7b}$$

This solution has the following behavior: For the critical noise intensity $D=D_c=a$, $W_{st}(I=0,\tau)$ approaches a finite value. For $D<D_c$, we have $W_{st}(I=0,\tau)=0$, while for $D>D_c$, $W_{st}(I,\tau)$ approaches (an integrable) infinity; i.e., $W_{st}(I\to 0,\tau)\to\infty$. For this critical D_c value [which coincides precisely with the exact value, $D_c=a$ (Ref. 17)] we depict in Fig. 1 the approximative stationary probability $W_{st}(I,\tau)$ in (7), together with the numerically exact stationary probability (dotted line) evaluated in Ref. 17. The agreement is remarkably good. In Fig. 2 we show $W_{st}(I,\tau)$ for a supercritical D value of $D=2$. The approximation and the exact result lie within line thickness. Deviations from the exact result are with $\tau=O(1)$ most pronounced for small I values, $I\le 1/4$; i.e., for

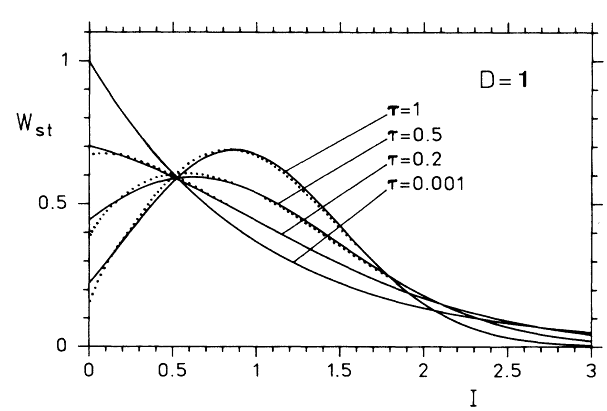

FIG. 1. Stationary probabilities for the dye-laser model (5) for the pump parameter $a=1$ and critical noise intensity $D=D_c=1$, as a function of the noise color τ. The approximation (7) (solid line) is compared with the numerical matrix-continued function solution (dotted line) of Ref. 17.

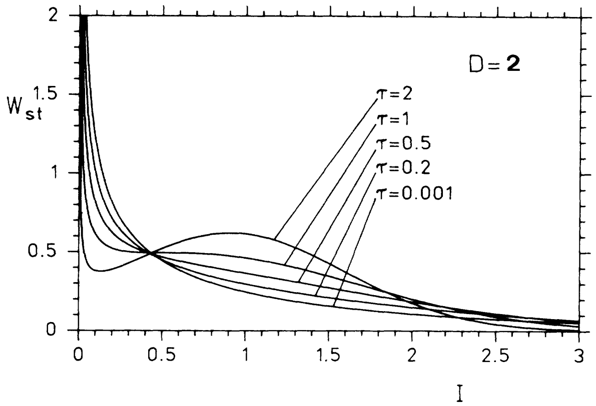

FIG. 2. The stationary probability at a supercritical noise intensity $D=2$ and pump-parameter value $a=1$ for various noise correlation times τ. The approximation in (7) and the numerical exact solution coincide within line thickness. Note the development of a local maximum at finite intensity I with increasing noise color.

$\gamma(I,\tau=1)=1+2I \sim O(1)$.

In conclusion, we have developed a unified colored-noise theory which gives surprisingly accurate approximate results for the noise driven nonlinear dynamics. The theory holds without restriction on the noise intensity D and strength of noise color τ, only if $\gamma(x,\tau)$ attains a sufficiently large (positive) value. If the probabilities are narrow, the novel approximation gives results equivalent to the decoupling theory in Ref. 10. Most surprisingly is its connection with Fox theory.[9] Originally devised as a small-correlation-time approximation, we now note that it also actually holds for moderate-to-strong noise color, subject to the restriction $\gamma(x,\tau) \gg D^{1/2}|f'/f|$. Particularly, if $f'(x)<0$, for all x, the (additive) colored-noise approximation in (4b) holds in the whole state space. Unfortunately, this novel approximation does, with $f'(x)>0$, not cover the case of multistability at moderate-to-strong noise color τ, and cannot describe exponentially small (or large) statistical quantities.[15] Presently, we attempt to extend the adiabatic elimination principle inherent in (3) to higher-dimensional systems.

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[13]We use standard transformation rules, and thus stick to the Stratonovich interpretation.

[14]For multiplicative noise, $\dot{x}=f(x)+g(x)\xi(t)$, i.e., $\gamma(x,\tau)=\tau^{-1/2}-\tau^{1/2}[f'-(g'/g)f]$, $W_{st}(x,\tau)$ reads, explicitly,

$$W_{st}(x,\tau)=Z^{-1}|\{[1-\tau(f'-fg'/g)]/g\}|\times\exp\left[\int^x dyf[1-\tau(f'-fg'/g)]/(Dg^2)\right].$$

[15]In this context, we touch upon an important problem, emphasized originally in P. Hänggi, F. Marchesoni, and P. Grigolini, Z. Phys. B **36**, 333 (1984); F. Marchesoni, Phys. Lett. **101A**, 11 (1984). In the usual Kramers case (see Refs. 11 and 12), where $(x,\dot{x})$ obeys detailed balance with a fluctuation-dissipation theorem for the noise, and where the stationary probability separates, i.e., $W_{st}(x,\dot{x})=W_{st}(x)W_{st}(\dot{x})$, the adiabatic elimination scheme (to all finite orders) always yields the exact stationary x probability. This is not true for (4c). In our case (x,ε) or $(x,\dot{x})$ do not obey detailed balance, and the (unfortunately, unknown) stationary probability for (x,ε) or $(x,\dot{x})$ does not separate. An adiabatic scheme which would yield the exact stationary probability could only be implemented in the presence of an analytical exact expression for $W_{st}(x,\dot{x})$. Corrections to the conventional Smoluchowski scheme in (4b) undoubtedly will change drift and diffusion, as well as introduce non-Fokker-Planck terms. Thus the exponential leading part of $W_{st}(x,\tau)$ will also be modified. This fact must be taken into account when describing, e.g., exponentially sensitive quantities, such as escape rates in bistable situations as studied in Ref. 10 and P. Hänggi, F. Marchesoni, and P. Grigolini, Z. Phys. B **36**, 333 (1984).

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