Optical instabilities: new theories for colored-noise-driven laser instabilities

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Novel colored-noise theories for optical instabilities are presented and compared with conventional approaches. In contrast to the conventional approaches, valid for small noise correlation times only, these new theories allow for a description of moderate-to-strong noise color and are of relevance particularly for the dye-laser instability and noise-dithered ring-laser gyroscopes. The theories are applied to the dye-laser instability and optical bistability. For the dye laser we evaluate the stationary probability and the integral of the stationary intensity correlation function (relaxation time), which both compare favorably with exact numerical results. For optical bistability we present the stationary probability and the switching rates.

1. INTRODUCTION

The study of dynamic systems in the optical sciences is attracting rapidly growing interest. Particularly, the fields of optical bistability and optical chaos have become the main focus of interest for many researchers. Here we restrict ourselves to the influence of noise in nonlinear-optical systems. Recent applications and experiments for dye-laser systems and optical ring-laser gyroscopes strongly emphasize the role played by noise sources of finite correlation time (colored noise). For the dye-laser instability[1-5] and the optical gyroscope,[6,7] strongly correlated noise occurs in reality, and it crucially impacts the physics.

The conventional small-correlation-time colored-noise theories[8,9] cannot describe this new regime of moderate-to-strong noise color. These theories are restricted to noise color close to the white-noise limit, i.e., the typical noise correlation time τ has to be of the order of $\tau < O(10^{-1})$ in dimensionless units (ratio of the noise time scale to the system time scale). For the dye-laser problem, however, correlation times range up to $\tau \approx O(10^{-1})$, while for the ring-laser gyroscope the noise dither possesses noise correlation times τ of the order of $\tau = 0.1$ up to $\tau = 10^2$. Thus there is a definite need for new theories covering small-to-moderate-to-strong noise color.

We recently developed such novel colored-noise theories that *a priori* do not restrict the value of τ. In particular we developed two new roads to describe colored-noise dynamics; we have termed these decoupling theory[10] and unified colored-noise theory.[11] In this paper we discuss these theories and their application to two different nonlinear-optical systems in which noise color of moderate-to-strong correlation strength plays an important role.

2. THEORIES FOR COLORED-NOISE-DRIVEN DYNAMICS

Let us consider the stochastic dynamics driven by colored noise given in terms of the Langevin equation:

$$\dot{x} = h(x) + g(x)\xi(t),$$

$$\langle \xi(t)\xi(t')\rangle = \frac{D}{\tau}\exp\left(-\frac{1}{\tau}|t-t'|\right), \tag{1}$$

where $h(x)$ and $g(x)$ are arbitrary (continuous) nonlinear functions. This fundamental form matches many particular stochastic models, such as simple laser equations, locking equations, and equations for optical bistability. In Subsection 2.A we give a summary of the two novel theories.

A. Decoupling Theory

For Gaussian colored noise an exact equation of motion for the single-event probability $p_t(x)$ can be derived[12,13]:

$$\dot{p}_t = -\frac{\partial}{\partial x}h(x)p_t(x) + \frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}\int_0^t \frac{D}{\tau} \times \exp\left[-\frac{1}{\tau}(t-s)\right]\left\langle \delta[x(t)-x]\frac{\delta x(t)}{\delta\xi(s)}\right\rangle \mathrm{d}s, \tag{2}$$

where the functional derivative $\delta x(t)/\delta\xi(s)$ is given by[12-15]

$$\frac{\delta x(t)}{\delta \xi(s)} = \theta(t-s)g[x(t)]\exp\int_s^t \left\{h'[x(u)] - h[x(u)]\frac{g'[x(u)]}{g[x(u)]}\right\}\mathrm{d}u. \tag{3}$$

Equation (2) is not a closed equation for $p_t(x)$ and can only be treated approximately. In contrast to the conventional small-correlation-time approximation,[8,9] which is derived by a Taylor expansion of the functional derivative around $s = t$, the decoupling theory (often called Hänggi ansatz) proceeds in factoring the mean value under the integral of Eq. (2). This method is valid for highly concentrated distributions that are mostly related to small noise strength:

$$\left\langle \delta[x(t)-x]\frac{\delta x(t)}{\delta\xi(s)}\right\rangle \approx p_t(x)\theta(t-s)g(x) \times \left\langle \exp\int_s^t \left\{h'[x(u)] - h[x(u)]\frac{g'[x(u)]}{g[x(u)]}\right\}\mathrm{d}u\right\rangle. \tag{4}$$

When we neglect transients (thus $t \to \infty$), the mean value on the right-hand side of expression (4) is approximated by a stationary mean value (quasi-stationary approximation). Consistent with the factorization in Eq. (2), we can factor mean values of functions with different time arguments to finally obtain the decoupling result[10]

$$\dot{p}_t = -\frac{\partial}{\partial x} h(x)p_t(x) + \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \times \frac{D}{1 - \tau\left[\langle h'(x)\rangle - \left\langle \frac{g'(x)}{g(x)} h(x)\right\rangle\right]} p_t(x). \quad (5)$$

Equation (5) has the same form as the white-noise Fokker–Planck equation (FPE) but with the effective noise strength

$$D_{\text{eff}} \equiv \frac{D}{1 - \tau\left[\langle h'(x)\rangle - \left\langle \frac{g'(x)}{g(x)} h(x)\right\rangle\right]}, \quad (6)$$

which has to be determined self-consistently. Note that no further restriction is made on τ besides the small noise-strength condition. The condition for small noise strength, however, is often fulfilled in real physical systems, and the decoupling result of Eq. (5) provides an accurate approximation for the stationary behavior even for strongly correlated noise.

B. Unified Colored-Noise Theory

In Subsection 2.A we presented the decoupling theory and emphasized its validity for strongly correlated noise. Here we give a summary of a theory that is also valid for strongly colored noise. The condition to be fulfilled is[11]

$$\tilde{\gamma}(z, \tau) \equiv \tau^{-1/2} - \tau^{1/2} f'(z) \gg D^{1/2} \left|\frac{\overline{f'}}{\overline{f}}\right|, \quad (7)$$

where $f(z)$ is the flow of the stochastic equations (1) transformed to additive noise $x \to z$ [see Eqs. (9) and (10)]. The overbar means that f and f' are taken to be the mean value over the typical length scale $l = \tilde{\gamma}^{-1}(z, \tau)\sqrt{D}$.[16] First we note that Eqs. (1) are stochastically equivalent to

$$\dot{x} = h(x) + g(x)\epsilon(t), \qquad \langle \Gamma(t)\Gamma(t')\rangle = 2\delta(t - t'),$$
$$\dot{\epsilon} = -\frac{1}{\tau}\epsilon + \frac{1}{\tau}\sqrt{D}\,\Gamma(t), \qquad \langle \Gamma(t)\rangle = 0,$$
$$\Gamma(t)\text{:Gaussian}, \quad (8)$$

describing a two-dimensional Markovian process. The multiplicative structure $g(x)\epsilon(t)$ can be transformed into an additive form

$$\dot{z} = f(z) + \epsilon, \qquad f(z) = \frac{h[x(z)]}{g[x(z)]}, \quad (9)$$

by changing the state variable from x to z by

$$x \to z = \int^x \frac{1}{g(x')} \mathrm{d}x'. \quad (10)$$

Elimination of ϵ and introduction of the new time scale $s = t\tau^{-1/2}$ leads to the second-order stochastic differential equation

$$\ddot{z} + \tilde{\gamma}(z, \tau)\dot{z} - f(z) = \frac{D^{1/2}}{\tau^{1/4}} \Gamma(s),$$
$$\langle \Gamma(s)\Gamma(s')\rangle = 2\delta(s - s'), \quad (11a)$$

where the z-dependent damping has the form

$$\tilde{\gamma}(z, \tau) = \tau^{-1/2} + \tau^{1/2}[-f'(z)]. \quad (11b)$$

The overdots in Eqs. (11a) indicate differentiation with respect to s. Expressed in the original variable x, the damping reads

$$\gamma(x, \tau) = \tau^{-1/2} + \tau^{1/2}\left[-h'(x) + \frac{g'(x)}{g(x)} h(x)\right]. \quad (11c)$$

If the expression in the square brackets in Eq. (11b) [and (11c)] is positive, the damping $\tilde{\gamma}$ (and γ) will be large for both small and large correlation times τ. Therefore an adiabatic elimination of $v = \dot{z}$ in Eqs. (11a) is justified for small and large correlation times, leading to the Markovian process

$$\dot{z} = \frac{f(z)}{\tilde{\gamma}(z, \tau)} + \frac{D^{1/2}}{\tau^{1/4}\tilde{\gamma}(z, \tau)} \Gamma(s). \quad (12)$$

When we transform to the state variable x and the original time scale t, we obtain the result [keeping the condition in Eq. (7) in mind]

$$\dot{x} = \frac{h(x)}{1 - \tau\left[h'(x) - \frac{g'(x)}{g(x)} h(x)\right]} + D^{1/2} \frac{g(x)}{1 - \tau\left[h'(x) - \frac{g'(x)}{g(x)} h(x)\right]} \Gamma(t),$$
$$\langle \Gamma(t)\Gamma(t')\rangle = 2\delta(t - t'), \quad (13)$$

where $\Gamma(t)$ is Gaussian white noise.

We have termed this approximation the unified color-noise approximation (UCNA) because it is valid for both small and large correlation times τ and in the whole state space with positive damping $\gamma(x, \tau)$. We must emphasize that Eqs. (13) are a truly approximative Markovian description for the non-Markovian process $x(t)$. This feature is a striking advantage over the recent small-τ theory of Fox,[14,15] which supersedes the conventional small-τ theory.[8,9] Because of the Markovian character of Eqs. (13), the conditional probability obeys the very same FPE, corresponding to Eqs. (13). This fact permits the calculation of dynamic properties, such as correlation functions and mean first passage times, by using standard techniques developed for white-noise processes.[17–22]

When dealing with approximations to non-Markovian processes, the role of initial preparation is equally as important as the dynamic law.[17,23,24] The exact result in Eq. (2) implicitly assumes a correlation-free preparation, i.e., the statistics of the system (x) and the environment [noise $\epsilon(t)$] factors at initial time t_0 of preparation, yielding

$$p_{t=0}(x, \epsilon) = p_{st}(\epsilon)p_0(x). \quad (14)$$

This preparation is implicit within the theories of Fox[14,15] and the conventional small-correlation-time theories,[8,9] which all constitute approximations to the exact expression in Eq. (2). In particular, for a non-Markovian process the initial preparation $p_0(x) = p_{st}(x)$ is not invariant under time evolution; $p_t(x)$ generally deviates from $p_0(x) = p_{st}(x)$ and approaches the true stationary dynamics only asymptotically. On the other hand, the adiabatic scheme of eliminating $v = \dot{z}$ in Eqs. (11a), as performed in the UCNA, assumes an initial preparation of the form

$$p_0(z, \dot{z}) = p_{st}(\dot{z}) p_0(z), \tag{15}$$

which is correlation-free between $z(0)$ and $\dot{z}(0)$. This preparation is quite close to the so-called stationary preparation scheme,[23,24] with

$$p_0(z, \dot{z}) = W_{st}(\dot{z}|z) p_0(z),\ W_{st}(\dot{z}|z) \equiv \frac{p_{st}(z, \dot{z})}{p_{st}(z)}.$$

If z and $\dot{z}$ exactly decouple in the stationary state (which is not quite true in general), the UCNA in Eqs. (13) yields the exact stationary x probability (see footnote 15 in Ref. 11).

With the assumption in Eq. (15) one obtains, consequently,

$$0 = \langle z\dot{z}\rangle, \tag{16a}$$

i.e.,

$$\langle z\epsilon\rangle_{t=0} = -\langle zf(z)\rangle_{t=0} \neq 0, \tag{16b}$$

which compares with the exact result

$$\langle z\epsilon\rangle_{st} = -\langle zf(z)\rangle_{st}. \tag{17}$$

Thus the UCNA is close to the stationary preparation, yielding valid results on the time scale $t/\tau^{1/2} = s > \gamma^{-1}(x, \tau)$, i.e.,

$$t > \frac{\tau}{1 - \tau\left[h'(x) - \frac{g'(x)}{g(x)} h(x)\right]}. \tag{18}$$

As mentioned previously, the UCNA holds with $\gamma(x, \tau) \gg 1$ on the time scale in inequality (18) for both small and large noise correlation time, not withstanding unproven claims to the contrary.[25] This fact will be demonstrated in Section 3 when we consider colored pump fluctuations in a dye laser.

Finally, we give a generalization to a situation with vastly differing time scales for the noise. The latter situation occurs often in quantum-optical systems with internal and external noise sources present. A phenomenological approach consists of a stochastic equation with two Gaussian noise sources:

$$\dot{x} = h(x) + g(x)\epsilon + g_2(x)\sqrt{Q}\Gamma(t),$$
$$\langle \epsilon(t)\epsilon(t')\rangle = \frac{D}{\tau} \exp\left(-\frac{1}{\tau}|t - t'|\right),$$
$$\langle \Gamma(t)\Gamma(t')\rangle = 2\delta(t - t'). \tag{19a}$$

The concept of the UCNA is also applicable to this system. Until now, Eqs. (19) were solved only numerically [Monte Carlo simulation[1-4] by matrix-continued fractions (MCF's)].[26,27] A prototype approach is the approximative description in terms of an effective Markovian Langevin equation. Performing a variable transform $x \rightarrow y$, Eqs. (19a) are transformed into a stochastic process, which is additive in the noise $\epsilon(t)$,

$$\dot{y} = H(y) + \epsilon + G(y)\sqrt{Q}\Gamma(t),$$
$$y = \int^x \frac{1}{g(x')} dx',$$
$$H(y) = \frac{h[x(y)]}{g[x(y)]},$$
$$G(y) = \frac{g_2[x(y)]}{g[x(y)]}.$$

When we introduce the stochastic variable $v = H(y) + \epsilon$, the equivalent system of Langevin equations with white-noise sources reads

$$\dot{y} = v + G(y)\sqrt{Q}\Gamma(t),$$
$$\dot{v} = v\left[H'(y) - \frac{1}{\tau}\right] + \frac{1}{\tau} H(y) + \sqrt{Q}G(y)H'(y)\Gamma(t) + \frac{\sqrt{D}}{\tau}\Gamma_1(t), \tag{19b}$$

with

$$\langle \Gamma_1(t)\Gamma_1(t')\rangle = 2\delta(t - t'),$$
$$\langle \Gamma(t)\rangle = \langle \Gamma_1(t)\rangle = 0,$$
$$\langle \Gamma_1(t)\Gamma(t')\rangle = 0.$$

Using time-scale arguments similar to those in Ref. 11, the variable v is eliminated adiabatically, and we get the one-dimensional stochastic process with two white-noise forces, which reads in the original variable x as

$$\dot{x} = \frac{h(x)}{1 - \tau\left[h'(x) - \frac{g'(x)}{g(x)} h(x)\right]} + \frac{g_2(x)}{1 - \tau\left[h'(x) - \frac{g'(x)}{g(x)} h(x)\right]}\sqrt{Q}\Gamma(t) + \frac{g(x)}{1 - \tau\left[h'(x) - \frac{g'(x)}{g(x)} h(x)\right]}\sqrt{D}\Gamma_1(t).$$

This equation is stochastically equivalent (i.e., corresponds to the same FPE) to the Langevin equation with one white-noise term,

$$\dot{x} = \frac{h(x)}{1 - \tau\left[h'(x) - \frac{g'(x)}{g(x)} h(x)\right]} + \frac{[Dg^2(x) + Qg_2^2(x)]^{1/2}}{1 - \tau\left[h'(x) - \frac{g'(x)}{g(x)} h(x)\right]}\Gamma(t), \tag{20}$$

being valid for small white-noise strength Q. (Further details and a discussion of the range of validity will be given in a future publication.)

3. INTENSITY STATISTICS OF A DYE LASER

Correlated pump fluctuations are of great importance to the statistical properties of the intensity of a dye laser.[28,29] When we assume a fast decay for the atomic polarization and the population inversion, the equation for the intensity then reads

$$\dot{I} = 2(a - I)I + 2\xi I, \tag{21}$$

where $\xi(t)$ denotes exponentially correlated pump noise. The fluctuations due to spontaneous emission are neglected. These fluctuations are more important to the transient dynamics of the laser.[3] The stationary behavior, however, is dominated by the pump fluctuations. In Eq. (21) we have used another normalization found in Ref. 11. The prefactor of the noise term in Ref. 11 was $\sqrt{2}$ instead of 2. To compare the results of Ref. 11 with the results [found in Eqs. (25)] of this paper, $2D$ has to be replaced by D in Eqs. (25). When the concepts of the UCNA are applied, we obtain for the damping [see Eq. (11c)]

$$\gamma(I,\tau) = \tau^{-1/2} + 2\tau^{1/2} I. \tag{22}$$

There is obviously no restriction on I (or τ); thus the UCNA is valid in the whole state space $I \geq 0$. The corresponding Langevin dynamics and FPE read

$$\dot{I} = \frac{2(a-I)I}{1+2\tau I} + \frac{2I\sqrt{D}}{1+2\tau I}\,\Gamma(t), \tag{23}$$

$$\dot{p}_t(I) = -\frac{\partial}{\partial I} D_1(I) p_t(I) + \frac{\partial^2}{\partial I^2} D_2(I) p_t(I), \tag{24a}$$

with drift coefficient $D_1(I)$ and diffusion coefficient $D_2(I)$ given by

$$D_1(I) = \frac{2(a-I)I}{1+2\tau I} + \frac{4DI}{(1+2\tau I)^3},$$

$$D_2(I) = \frac{4DI^2}{(1+2\tau I)^2}. \tag{24b}$$

The stationary probability is evaluated readily to give

$$p_{\mathrm{st}}(I) = \frac{1}{Z}(1+2\tau I)\exp\left[\frac{\tau}{2D}(2a-I)I\right] p_{\mathrm{st}}{}^{0}(I), \tag{25a}$$

where $p_{\mathrm{st}}{}^{0}(I)$ denotes the stationary white-noise probability

$$p_{\mathrm{st}}{}^{0}(I) = I^{(a/2D)-1}\exp\left(-\frac{I}{2D}\right). \tag{25b}$$

In Figs. 1(a)–1(c) the UCNA results of Eqs. (25) are compared with the exact numerical result obtained by using a MCF technique.[22,30] In the three different regimes $D < a/2$, $D = D_c = a/2$, $D > a/2$ we find remarkably good agreement for all τ values.

An important quantity characterizing the coherence properties of the light is the stationary intensity correlation function, which is defined by

$$\phi_I(t) \equiv \frac{1}{\langle I^2\rangle - \langle I\rangle^2}\langle [I(t)-\langle I\rangle][I(0)-\langle I\rangle]\rangle, \tag{26a}$$

as well as the relaxation time, which is a measure for the decay time of fluctuations,

$$T = \int_0^\infty \phi(t)\mathrm{d}t. \tag{26b}$$

In earlier work[19,20] the calculation of the correlation function and relaxation time was discussed by using the MCF technique. Here we restrict ourselves to calculating the relaxation time T with UCNA. Using the exact Markovian formula[17,19] for the statistical quantity in Eq. (26b), we obtain, in terms of the stationary distribution $P_{\mathrm{st}}(I)$ and the diffusion coefficient $D_2(I)$ [see Eqs. (24)],

$$T(\tau,D) = \frac{1}{\langle I^2\rangle - \langle I\rangle^2}\int_0^\infty \mathrm{d}x\,\frac{f^2(x)}{D_2(x)p_{\mathrm{st}}(x)}, \tag{27a}$$

with

$$f(x) = -\int_0^x \mathrm{d}x'(x'-\langle I\rangle)p_{\mathrm{st}}(x'). \tag{27b}$$

The integrals in Eqs. (27a) and (27b) can be evaluated numerically. In Fig. 2 the results are compared with those obtained by using MCF's,[22] and we find remarkably good agreement. For larger values of τ, there are small deviations

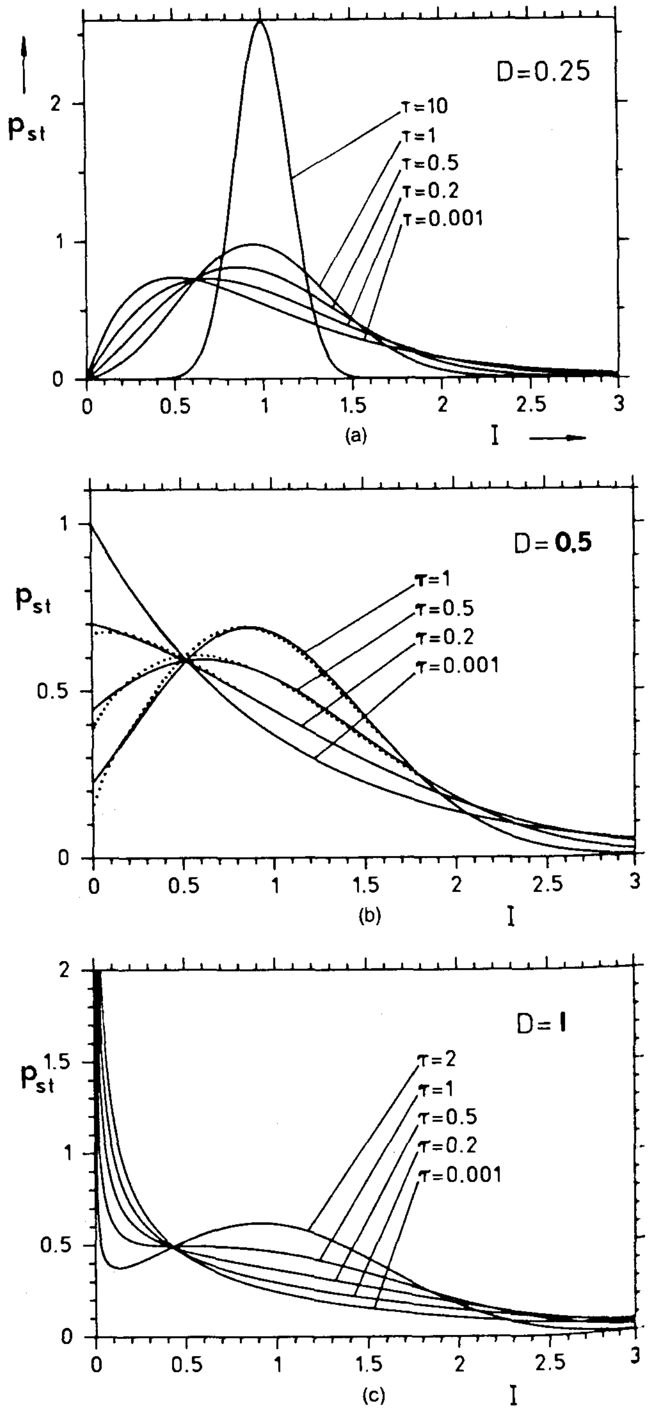

Fig. 1. The stationary distribution $p_{\mathrm{st}}(I)$ of Eq. (25a) is plotted (solid line) for $a = 1$ and (a) $D = 0.25$, (b) $D = D_c = 0.5$, and (c) $D = 1$ for various values of the correlation time τ of the noise. The MCF results, indicated by the dotted lines in (b) coincide within line thickness for $D = 0.25$ and $D = 1$.

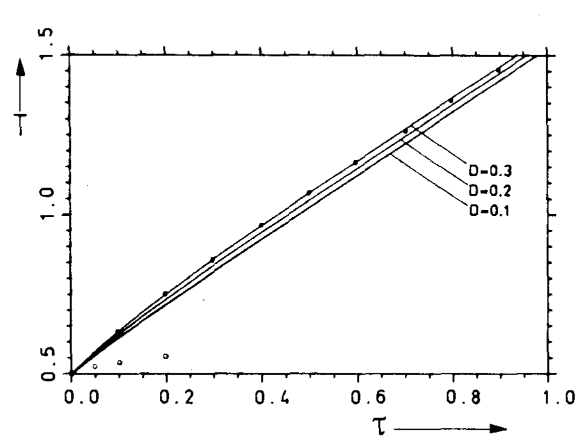

Fig. 2. The relaxation time is shown as a function of the correlation time τ. The UCNA result [Eq. (27a)] for $D = 0.3$ is indicated by full circles that are close to the exact MCF result (solid line). The same holds true for $D = 0.2$ and $D = 0.1$. The result using the theory of Fox is indicated by open circles.

from the exact values, which occur because of the invalidity of the correlation function for small times, as given by the restriction (18). The slope of the exact correlation function at $t = 0$ vanishes[21,31] as a consequence of the non-Markovian character of Eq. (21). This property cannot be described within the effective Markovian UCNA.

Now we can compare our results with the theory of Fox.[15,16] Within the theory of Fox, the drift and diffusion coefficients for the dye-laser model [Eq. (21)] read[15,16]

$$D_1^F = 2(a - I)I - 4D \frac{I}{1 + 2\tau I},$$

$$D_2^F = 4D \frac{I^2}{1 + 2\tau I} > 0. \tag{28}$$

Indeed the stationary distribution obtained from Fox's theory coincides with that of the UCNA [Eqs. (25)]. Interpreting Fox's theory as a Markovian Fokker–Planck approach of a non-Markovian process (which is possible in this case of a strictly positive diffusion coefficient), we can calculate the correlation functions and the relaxation time in Eqs. (27). The result for the relaxation time is indicated by open circles in Fig. 2. As expected for any small-correlation-time theory (such as the Fox theory) we find agreement at small τ. For larger τ values, however, the theory of Fox breaks down. The difference between the exact result and that using Fox's theory is due to the (nonstationary) correlation-free preparation $\langle x\epsilon\rangle_{t=0} = 0$, as already mentioned in Section 3. Contrary to statements of Faetti and Grigolini,[25] we find good agreement between the UNCA and the exact results for large τ as well as small τ.

4. OPTICAL BISTABILITY WITH COLORED NOISE

Optical bistability arises in many different situations (absorptive optical bistability, dispersive optical bistability, optical bistability in lasers with saturable absorber) governed by different equations (for an overview, see Ref. 32). Here the bistable character is modeled by a bistable flow of the Ginzburg–Landau type. The noise $\xi(t)$ models external colored noise on the control parameter that is the injected laser signal $y = \langle y\rangle + \xi(t)$[33]; internal noise is neglected. The Langevin equation then reads[10,34,35]

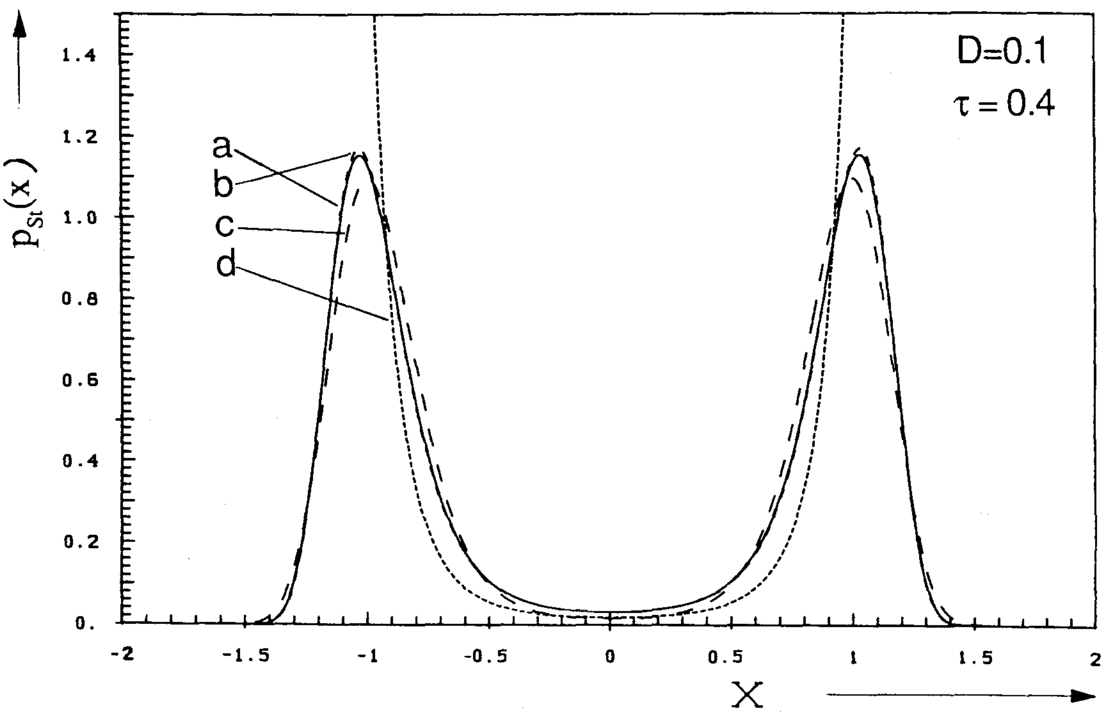

Fig. 3. The numerical exact distribution, a, of Eqs. (29) is compared with the UCNA result, b, in Eq. (34), the decoupling result, c, in Eq. (33) and the conventional small-τ theory, d, in Eq. (35) for $D = 0.1$ and $\tau = 0.4$ ($a = b = 1$).

$$\dot{x} = ax - bx^3 + \xi(t), \qquad a > 0, \qquad b > 0,$$
$$\langle \xi(t)\xi(t')\rangle = \frac{D}{\tau}\exp\left(-\frac{1}{\tau}|t - t'|\right), \tag{29}$$

with the deterministic stable points $x_{1,2} = \pm(a/b)^{1/2}$ and an intervening potential barrier located at $x = 0$. For a numerical treatment (MCF; Ref. 18) the equivalent two-dimensional FPE has the form[10]

$$\dot{p}_t(x,\epsilon) = -\frac{\partial}{\partial x}(ax - bx^3 + \epsilon)p_t + \frac{1}{\tau}\frac{\partial}{\partial \epsilon}\epsilon p_t + \frac{D}{\tau^2}\frac{\partial^2}{\partial \epsilon^2}p_t. \tag{30}$$

The Fokker–Planck approximation of the decoupling theory [Eq. (5)] reads

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}(ax - bx^3)p_t + \frac{\partial^2}{\partial x^2}\frac{D}{1 - \tau(a - 3b\langle x^2\rangle)}p_t, \tag{31}$$

whereas the UCNA provides the FPE:

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}\left\{\frac{ax - bx^3}{1 - \tau(a - 3bx^2)} - \frac{6Db\tau x}{[1 - \tau(a - 3bx^2)]^3}\right\}p_t(x) + \frac{\partial^2}{\partial x^2}\frac{D}{[1 - \tau(a - 3bx^2)]^2}p_t(x), \tag{32}$$

with the restriction

$$\tau^{-1/2} - \tau^{1/2}(a - 3bx^2) \gg D^{1/2}\left|\frac{a - 3b\overline{x^2}}{a\bar{x} - b\overline{x^3}}\right|.$$

A. Stationary Probability

The stationary probability is obtained numerically (exact) by solving Eq. (30) in terms of MCF.[20] This exact distribution (Fig. 3a) is compared with the stationary probability of the decoupling theory (Fig. 3c),

$$p_{st}(x) = \frac{1}{Z_1}\exp\left\{\frac{1}{D}[1 - \tau(a - 3b\langle x^2\rangle)]\left(\frac{a}{2}x^2 - \frac{b}{4}x^4\right)\right\}, \tag{33}$$

and with the UCNA (Fig. 3b),

$$p_{st}(x) = \frac{1}{Z_2}|1 - \tau(a - 3bx^2)| \times \exp\left\{\frac{1}{D}\left(\frac{a}{2}x^2 - \frac{b}{4}x^4\right) - \frac{\tau}{2D}(ax - bx^3)^2\right\}, \tag{34}$$

where $a = b = 1$, $\tau = 0.4$, and $D = 0.1$ (Z_1 and Z_2 are normalization constants). The stationary solution, obtained with the conventional small-τ approximation[8,9]

$$p_{st}(x) = \frac{1}{Z_3}|1 + \tau(a - 3bx^2)|^{[(1-2a\tau)/(18bD\tau^2) - 1]}\exp\left(\frac{x^2}{6\tau D}\right) \tag{35}$$

is plotted in Fig. 3d. We find nearly perfect agreement between the UCNA and the exact solution. The decoupling theory shows some small deviations. The conventional small-τ approximation provides a rather poor result in this case because the small-τ approximation is valid only in the region of positive diffusion coefficients, i.e.,

$$|x| < \left(\frac{1 + a\tau}{3b\tau}\right)^{1/2}. \tag{36}$$

The small $-\tau$ approximation loses its bistable character if the sites of the maxima,

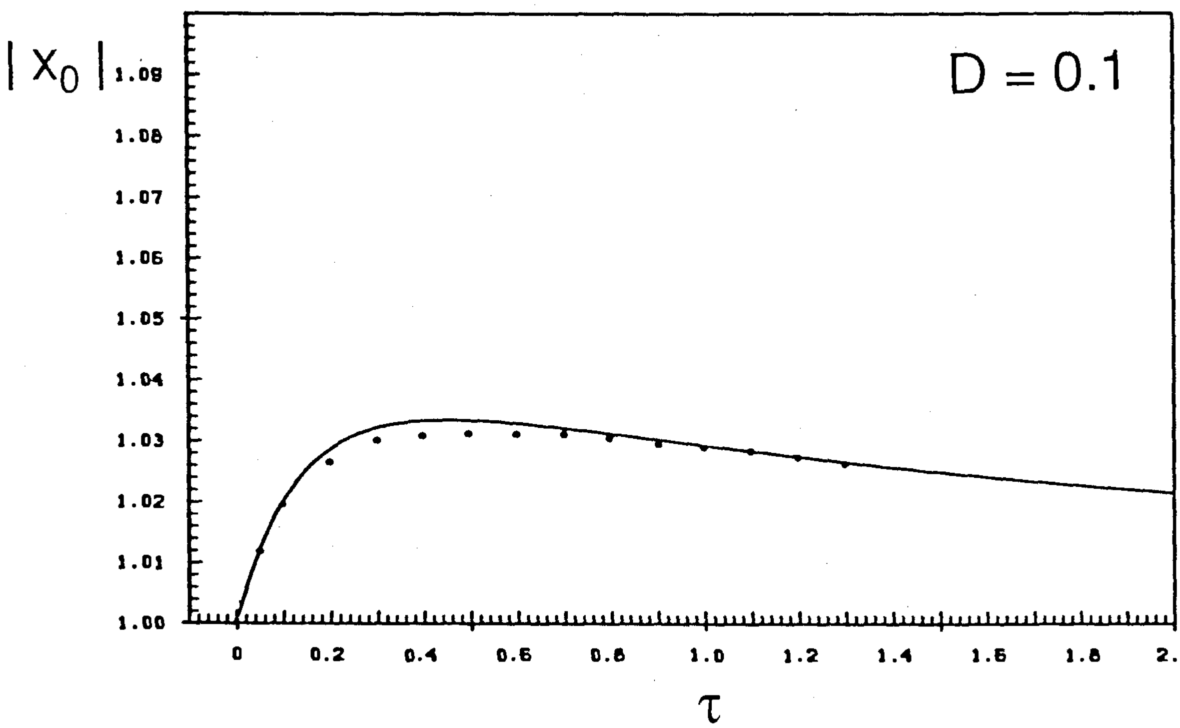

Fig. 4. The location of the shift of the maximum of the stationary probability $|x_0|$ is plotted versus the correlation time τ for $D = 0.1$. The solid line represents the positive root of Eq. (39) (UCNA), whereas the exact values are indicated by full circles.

$$x = \pm\left(\frac{a}{b} + \frac{6D\tau}{}\right)^{1/2}, \tag{37}$$

are outside the region given in expression (36). Thus there is a restriction on τ of the form

$$\tau < \frac{a}{18Db}\left[\left(1 + \frac{18Db}{a^2}\right)^{1/2} - 1\right]. \tag{38}$$

When the parameters are $a = b = 1$ and $D = 0.1$ (Fig. 3), we find that $\tau < 0.374$ (< 0.4). Therefore the small-τ approximation has already lost its bistable character at $\tau = 0.4$.

B. Shift of the Maxima

As seen in Fig. 3 the maxima of the stationary probability are shifted from the white-noise values $\pm(b/a)^{1/2}$ toward values with a larger amount of x. For larger correlation times τ, however, this shift becomes smaller and finally vanishes for infinite τ because the noise itself vanishes in the limit $\tau \to \infty$. Within the UCNA, the equation for the site x_0 of the maxima is evaluated as

$$x_0^2 = \frac{a}{b} + 6D\tau\left[\frac{1}{1 - \tau(a - 3bx_0^2)}\right]^2. \tag{39}$$

The positive root x_0 is plotted in Fig. 4 as a function of τ for $D = 0.1$. The dots indicate the exact values obtained by the MCF method. The agreement between the UCNA [Eq. (32)] and the exact result is indeed very satisfactory.

C. Transition Rates

An important quantity in a bistable system is the transition rate between the stable positions. For white noise ($\tau = 0$) this problem was first solved by Kramers, leading to the famous Kramers rate,[36]

$$r_K = \frac{a}{\sqrt{2}\pi}\exp\left(-\frac{a^2}{4bD}\right). \tag{40}$$

The conventional small-τ approximation modifies this result to yield only a τ-dependent prefactor.[35] MCF calculations,[20] however, show exponential decay of the form

$$r \propto \exp\left(-\alpha\frac{\tau}{D}\right). \tag{41}$$

Recently this exponential decay was demonstrated for a periodic double-well potential.[37]

The decoupling theory was the first theory to predict this exponential τ dependence. This dependence is easily seen by substituting D by the effective noise strength D_{eff}:

$$D_{\text{eff}} = \frac{D}{1 - \tau(a - 3b\langle x^2\rangle)}. \tag{42}$$

The corrected Kramers rate then reads[10]

$$r(\tau) = r_K\exp\left[\frac{\tau}{D}\frac{a^2}{4b}(a - 3b\langle x^2\rangle)\right]. \tag{43}$$

This correction becomes the leading contribution for small D (which must be in the first place for the rate description to be valid) and not too small τ. Recently Marchesoni[38] developed an approach that bridges between the small-τ answer (prefactor correction) and the decoupling result (exponential correction). His approach is based on ideas for the derivation of the rate from the Kramers equation.[36]

5. CONCLUSIONS

We have presented Fokker–Planck approximations to colored-noise-driven instabilities that work for weak-to-moderate-to-strong noise color. In particular, we model the stationary dynamics of the colored-noise-driven dye-laser instabilities in Eq. (21). The stationary probability and the relaxation time (integrated intensity correlation function) are described by the UCNA [see Eq. (24a)] with excellent accuracy.

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