Colored-noise-driven bistable systems

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We consider the escape rate in a bistable potential driven by exponentially correlated noise. Our focus is on the crossover between the small- and large-correlation time behavior. Precise numerical results obtained by using a matrix-continued-fraction technique are compared against recent theoretical predictions.

We consider the escape process in the archetype bistable potential

$$V(x)=-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4 \tag{1}$$

driven by an external Gaussian noise source with a finite correlation time τ, i.e., the Langevin equation with Gaussian noise $\xi(t)$ of vanishing mean

$$\begin{aligned}&\dot{x}=x-x^3+\xi(t)\ ,\\ &\langle \xi(t)\xi(t')\rangle=\frac{D}{\tau}\exp\left[-\frac{1}{\tau}|t-t'|\right],\end{aligned} \tag{2}$$

with x, D, and τ normalized to dimensionless quantities.[6(b)] The potential $V(x)$ has two wells located at $x=\pm 1$ which are separated by a barrier of height $\Delta V=V(0)-V(\pm 1)=0.25$ at $x=0$.

In the white-noise limit $\tau=0$ the noise becomes δ correlated and for small-noise intensity $D \ll \Delta V$, the escape rate r_k between the two attractors λ is well reproduced by the Kramers formula[1-3]

$$r_K=\frac{1}{\sqrt{2}\pi}\exp\left[-\frac{\Delta V}{D}\right]. \tag{3}$$

In the last years several groups[4-10] studied the problem of determining the corrections to the Kramers rate (3) due to small-correlation times. The difficulty of this problem lies in the fact that there are *two small parameters* D and τ, and the result crucially depends on how the limits $\tau\to 0$ and $D\to 0$ are taken. The answer reads[5]

$$r(\tau)=r_K(1-\tfrac{3}{2}\tau)\exp\left[-\frac{\Delta V}{2D}\tau^2\right], \tag{4a}$$

which for $\tau/D \ll 1$ reduces to[4]

$$r(\tau)=r_K(1-\tfrac{3}{2}\tau)\ . \tag{4b}$$

In many real systems, however, the system variables are not much slower than the environmental dynamics represented by the noise source. In these cases the small-τ approximations are of limited use.[2,4] The most difficult situation arises when the noise correlation time τ is comparable to or larger than the characteristic time scale of the system. The rate problem associated with the process in Eq. (2) has been studied numerically[6] by computing the smallest nonvanishing eigenvalue λ_0 of the equivalent two-dimensional Fokker-Planck equation

$$\frac{\partial W(x,\epsilon,t)}{\partial t}=\left[-\frac{\partial}{\partial x}(x-x^3+\epsilon)+\frac{1}{\tau}\frac{\partial}{\partial \epsilon}\epsilon+\frac{D}{\tau^2}\frac{\partial^2}{\partial \epsilon^2}\right]W(x,\epsilon,t) \tag{5}$$

by means of the matrix continued fraction technique of Ref. 11. In fact, the existence of a separatrix for any finite noise correlation time[6-8] which splits the $(x-\epsilon)$ space into two domains of attraction guarantees a clear-cut time scale separation at small-noise intensity between the hopping mechanism (with rate $r=\lambda_0/2$) and the intrawell dynamics [described by the remaining eigenvalues of Eq. (5)].[6(b)] For $0.2\lesssim\tau\lesssim 1.5$ and small D the resulting eigenvalue λ_0 is well described by the exponential law[6(b)]

$$\lambda_0(\tau)\propto\exp\left[-\alpha\frac{\tau}{D}\right], \tag{6}$$

where $\alpha\cong 0.1$ throughout the intermediate range $0.2\lesssim\tau\lesssim 1.5$.

In this paper we focus on the crossover between such an intermediate-τ regime, where Eq. (6) is valid, and the asymptotically large-τ limit. In the latter regime the authors of Refs. 5(a)–5(c), 8, and 10 all obtain for the exponential leading dependence the result

$$\lambda_\infty(\tau)\propto\exp\left[-\frac{8\Delta V}{27D}\tau\right]\quad \text{as } \tau\to\infty,\ \ D\to 0\ . \tag{7}$$

Note that this law has again the same form as in (6), but with a different value for α, i.e., $\alpha(\infty)=8\Delta V/27$. This yields a slope for $-\ln\lambda(\tau)$ versus τ of $\alpha(\infty)/D=8\Delta V/(27D)$, which is by no means amenable to the slope defined through Eq. (6), i.e., $\alpha/D\approx 0.1/D$, as shown in the discussion following by comparison with our numerical data.

A nonstationary Fokker-Planck approach has been proposed by Tsironis and Grigolini[10] to bridge the large-τ

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and small-τ behavior. In order to assess the validity of this method let us start with the exact equation of motion for the probability of x (Ref. 12),

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}(x - x^3)p_t(x) + \frac{D}{\tau}\frac{\partial^2}{\partial x^2}\int_0^t \exp\left[-\frac{1}{\tau}(t-s)\right] \times \left\langle \delta(x(t)-x)\frac{\delta(x(t))}{\delta(\xi(s))}\right\rangle ds \ . \tag{8}$$

Here the functional derivative in the integral in (8) is given in terms of the stochastic process $x(t)$ by

$$\frac{\delta x(t)}{\delta x(s)} = \Theta(t-s)\exp\left[\int_s^t [1-3x^2(t')]dt'\right] , \tag{9}$$

where $\Theta(t)$ denotes the step function

$$\Theta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \ . \end{cases} \tag{10}$$

Changing the integration variable in (8), i.e., $s \to u = (t-s)/\tau$, and expanding the argument of the exponential function in (9) up to the first order in τ (note that this expansion is not uniform in x) one finds

$$\dot{p}_t(x) \underset{\tau \to 0}{=} -\frac{\partial}{\partial x}(x - x^3)p_t(x) + D\frac{\partial^2}{\partial x^2}\int_0^{t/\tau} \exp(-u)\exp[\tau u(1-3x^2)] \times du \, p_t(x) \ . \tag{11}$$

Performing the integral in (9) without letting the upper limit go to infinity we recover the time-inhomogeneous Fokker-Planck equation (5) of Ref. 10, i.e.,

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}(x - x^3)p_t(x) + \frac{D}{\tau}\frac{\partial^2}{\partial x^2}\frac{\exp\{[1-3x^2-(1/\tau)]t\}-1}{1-3x^2-1/\tau}p_t(x) \ . \tag{12}$$

Note that our derivation of Eq. (12) is only valid under the *explicit condition that the correlation time is small.* For $\tau < \tau_0 = 1$ the time-dependent diffusion coefficient

$$D(t) = D\frac{1-\exp\{-[1-\tau(1-3x^2)]t/\tau\}}{1-3\tau x^2} \tag{13}$$

converges for $t \to \infty$ to

$$D_{\text{Fox}} = D\frac{1}{1-\tau(1-3x^2)} , \tag{14}$$

thus reproducing Fox's result.[9] For $\tau \geq \tau_0$, instead, $D(t)$ diverges to $+\infty$ in the domain I, defined by $|x| < \sqrt{(\tau-1)/3}$, thereby leading to a vanishing stationary probability in I. In fact, the *correct* stationary probability $p_{\text{st}}(x)$ computed numerically (cf. Fig. 1) does not support the existence of this divergence, i.e., $p_{\text{st}}(x)$ *does not vanish in* I *for* $\tau > \tau_0$.

The authors of Ref. 10 considered the decay of an initial population confined within one well by solving numerically Eq. (12) for small *as well as for large correlation times.* They found that the long time tail of this population decays exponentially with decay time T_{TG}, where TG represents Tsironis and Grigolini. For $\tau < \tau_0$, T_{TG} coincides, see (13) and (14), with the reciprocal of the smallest nonvanishing eigenvalue, λ_{Fox}, of the Fokker-Planck equation derived in Ref. 9, i.e.,

$$-\lambda_{\text{Fox}}\Psi(x) = -\frac{\partial}{\partial x}(x - x^3)\Psi(x) + D\frac{\partial^2}{\partial x^2}\frac{1}{1-\tau(1-3x^2)}\Psi(x) , \tag{15}$$

with the boundary conditions $\Psi(\infty) = \Psi(0) = 0$.

In Fig. 2 we compare $T_{\text{Fox}}(\tau) \equiv 1/\lambda_{\text{Fox}}$, which has been calculated numerically by using a shooting method, with the decay time $T_{\text{TG}}(\tau)$ of Ref. 10 and with the reciprocal

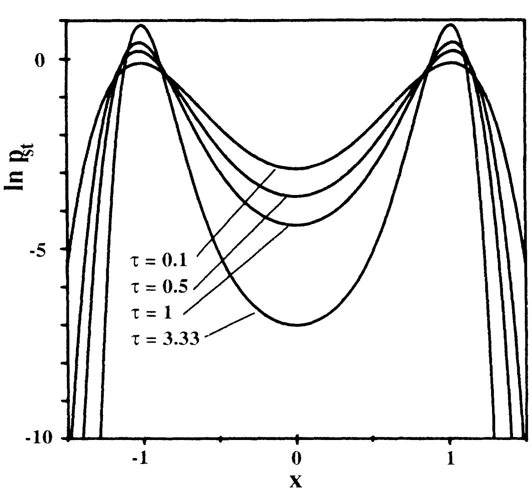

FIG. 1. The stationary probability $p_{\text{st}}(x)$, obtained by numerical integration of (5), is plotted at for several values of τ and $D=0.1$.

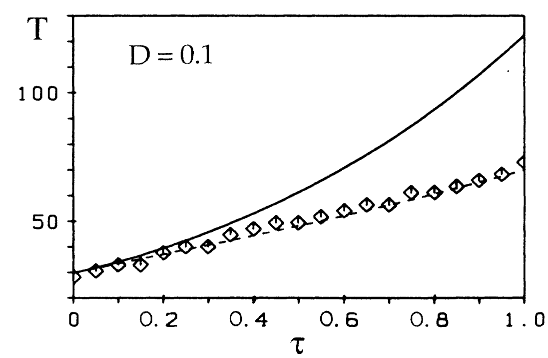

FIG. 2. The numerical values [Eq. (5)] of $T_0(\tau) = 1/\lambda_0(\tau)$ for $D=0.1$ (solid line) are compared against $T_{\text{Fox}}(\tau)$ (dashed line) and $T_{\text{TG}}(\tau)$ (diamonds) for $0 < \tau < \tau_0 = 1$.

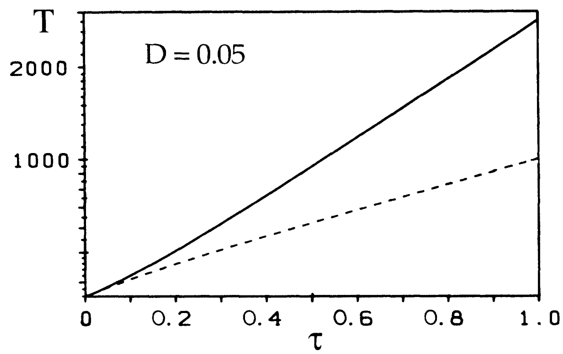

FIG. 3. The relaxation times $T_{\mathrm{Fox}}(\tau)$ (dashed line) are compared with the numerical results (solid line) for $0<\tau<\tau_0=1$ at $D=0.05$.

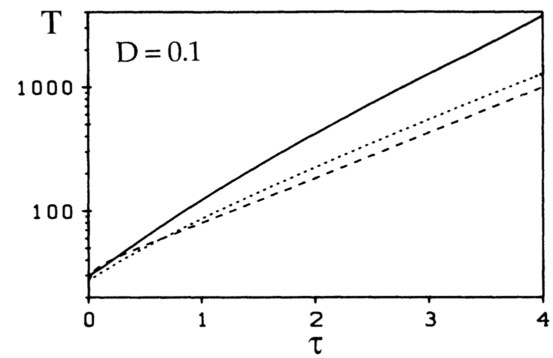

FIG. 5. The bridging formulas (16) (dashed line) and (17) (dotted line) are compared against the exact results (solid line) at small- to moderate- to large-τ values for $D=0.1$.

of the smallest eigenvalue $T_0(\tau)\equiv 1/\lambda_0(\tau)$ (with error less than 0.1%) of the two-dimensional Fokker-Planck equation in (5).[6(b)] For a detailed description of the numerical matrix continued fraction solution of (5), we refer the reader to Ref. 6(a). The data for $T_{\mathrm{TG}}(\tau)$ lie just on the curve $T_{\mathrm{Fox}}(\tau)$ in the region $0<\tau<\tau_0$, as predicted by the above argument. The agreement with the exact values $T_0(\tau)$ holds *for small-τ values only.* The discrepancies between the exact result $T_0(\tau)$ and $T_{\mathrm{TG}}(\tau)=T_{\mathrm{Fox}}(\tau)$ do not vanish with decreasing D (Fig. 3), either. In Fig. 3 the exponential behavior of $T_0(\tau)$ is clearly observable. The escape time $T_{\mathrm{Fox}}(\tau)$, or $T_{\mathrm{TG}}(\tau)$ respectively, also exhibits for $D=0.05$ an exponential behavior, but with an *incorrect slope which happens to be very close to the asymptotic value $\alpha(\infty)/D$*. This shows that the exponential behavior (7) of the rate $\lambda(\tau)$ should not be mistaken with the approximate exponential behavior for small to moderate τ values. We conclude that the validity of the nonstationary Fokker-Planck equation (FPE) (12) must be restricted, indeed, to the limit of short noise correlation times, i.e., $\tau\ll 1$.

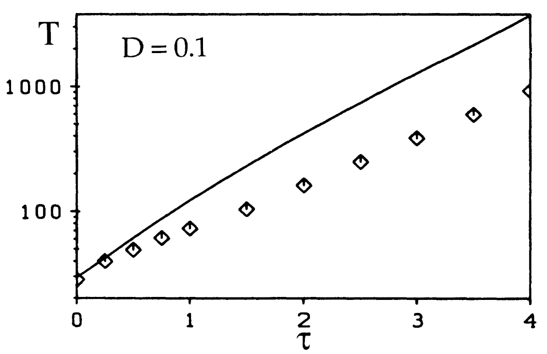

FIG. 4. The relaxation times $T_{\mathrm{TG}}(\tau)$ (diamonds) are compared with the exact results (solid line) for small- to moderate- to large-τ values at $D=0.1$.

For $\tau>\tau_0$ (cf. Fig. 4) the disagreement between $T_{\mathrm{TG}}(\tau)$ and $T_0(\tau)$ grows even further; the exact value $T_0(\tau)$ exceeds $T_{\mathrm{TG}}(\tau)$ at $\tau=4$ by a factor of 3.6. Though the absolute values of $T_{\mathrm{TG}}(\tau)$ are off by such an amount, the slope of the logarithmic plot of $T_{\mathrm{TG}}(\tau)$ and $T_0(\tau)$ seemingly converge to the same value $8\Delta V/(27D)$ of Eq. (7) when τ becomes very large. It should be noticed that for $\tau\geq\tau_0$, $T_{\mathrm{Fox}}(\tau)$ diverges. The solid line in Fig. 4 clearly shows that the regime with an exponential τ/D dependence of the rate $\lambda(\tau)$, i.e., $0.2\lesssim\tau\lesssim 1.5$, is followed by a regime with a nonexponential dependence on τ/D. On further increasing τ, within the domain of reliability of our numerical algorithm [for $0\leq\tau\leq 4$ and $D=0.1$ $\lambda_0(\tau)$ is determined with an error of less than 1%], the slope of $\ln T_0(\tau)$ seems to converge *slowly* to the asymptotic value α_∞/D.

Finally we discuss the bridging formulas[10]

$$T_{\mathrm{TG}}(\tau)=\exp\left[\frac{\Delta V}{D}\right]\times\left[\frac{\pi}{a\sqrt{2}}+\left[\frac{27D\pi\tau}{8\Delta V}\right]^{1/2}\exp\left[\frac{8\Delta V\tau}{27D}\right]\right] \tag{16}$$

and[5(b),6(b)]

$$T_{LV}(\tau)=\frac{\sqrt{2}}{\pi}(1+3\tau)^{1/2}\times\exp\left[\frac{1}{4D}\left[\frac{1+\frac{27}{16}\tau+\frac{1}{2}\tau^2}{1+\frac{27}{16}\tau}\right]\right] \tag{17}$$

proposed to interpolate between the small-τ and large-τ regime. It should be remarked that (16) does not reproduce the correct short-τ behavior (4), while the large-τ limit (7) turns out to be multiplied *ad hoc* by the (large) factor $\exp(\Delta V/D)$. In Fig. 5 the two bridging formulas are compared versus the precise numerical results for

$\lambda_0(\tau)$ obtained by solving the Fokker-Planck equation in (5). Both (16) and (17) have been derived within the steepest descent approach and thus exhibit a difference of $\sim 10\%$ from the exact result at $\tau=0$.[6(b)] While the bridging formula due to Luciani and Verga reproduces our numerical data for $\lambda_0(\tau)$ at small τ somewhat more closely, both results in (16) and (17) are off in the region of intermediate-to-large τ values by a considerable amount.

[1]For an overview of the present state of the art in rate theory see P. Hänggi, J. Stat. Phys. **42**, 105 (1986); **44**, 1003 (1986).

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