Stochastic Nonlinear Dynamics Modulated by External Periodic Forces.

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Abstract. – The stochastic theory is developed for overdamped, nonlinear stochastic systems with periodic forcing. By use of a generalized Floquet theory we show that such systems averaged over the random phase ϕ are not strongly mixing, but exhibit ever present undamped oscillations, *e.g.* the power spectrum contains δ-function peaks at multiples of the driving frequency Ω. For the archetypal periodically driven, bistable stochastic flow, $\dot{x} = x - x^3 + A\cos(\Omega t + \phi) + \xi(t)$, we evaluate by means of matrix continued fraction techniques the stationary probability $W_{st}(x, \theta = \Omega t + \phi)$ and the ϕ averaged, complex-valued dynamical susceptibility. The stationary probability has a most interesting, rich topology in (x, θ)-phase space exhibiting several, competing modulation-induced escape paths.

The interesting behaviour of a bistable stochastic system which is driven by periodic forces has been noticed first by Benzi and coworkers [1] when they attempted to explain the periodicity of Earth's ice ages. In their model the two stable states of the potential correspond to two stable climatic configurations, a cold one (ice age) and a warm one. The nearly periodic switching between the stable states (*i.e.* occurrence of ice ages) is explained by a cooperative effect of noise and small periodic variations of the excentricity of the earth. Moreover such models are of relevance for the description of laser-assisted desorption from surfaces. In such systems there seems to occur a cooperative effect between the *internal* random behaviour of an *overdamped* stochastic process and the *external time-periodic driving* mechanism. The effect has been termed «stochastic resonance» [1, 2], due to its characteristic signature of a dramatic increase of the signal-to-noise ratio of the output. This technical important phenomenon has also been observed in numerical simulations [2] and in actual experiments [3]. The notion of «stochastic resonance» is not well defined, however, and it is actually somewhat unfortunate (see below). Nevertheless, the phenomenon is clearly distinct from the related effect of «resonance activation» which occurs in *underdamped* metastable systems [4] wherein the external periodic force resonantly interacts with the intrinsic time scale of the periodic motion around a locally stable state. Up to present time the theoretical approaches have been limited either to a phenomenological reasoning only [1-3], or have been restricted to the extreme adiabatic limit [5]. In this work we present a first theoretical treatment in terms of the underlying (time-dependent) Fokker-

Planck process. In particular, our focus is on asymptotic probability density and the asymptotically time-homogeneous autocorrelation function of the process $x(t)$, obeying

$$\dot{x} = f(x) + A\sin(\Omega t + \phi) + \xi(t), \tag{1}$$

where $\xi(t)$ denotes Gaussian white noise with correlation $\langle \xi(t)\xi(s)\rangle = 2D\delta(t-s)$ and ϕ is a random phase being uniformly distributed over one cycle $(0, 2\pi)$ with a density $\rho(\phi) = (2\pi)^{-1}$. If the random phase ϕ is held fixed the dynamics in (1) is equivalent with a *nonstationary* Fokker-Planck process $p^{\phi}(x, t)$ with a time-periodic generator, *i.e.*

$$\frac{\partial}{\partial t}p^{\phi}(x,t) = -\frac{\partial}{\partial x}(f(x)p^{\phi}(x,t)) + D\frac{\partial^2}{\partial x^2}p^{\phi}(x,t) - A\sin(\Omega t + \phi)\frac{\partial}{\partial x}p^{\phi}(x,t) \equiv$$

$$\equiv \left[\mathcal{L}_0 - A\sin(\Omega t + \phi)\frac{\partial}{\partial x}\right]p^{\phi}(x,t), \tag{2}$$

where

$$\mathcal{L}_0 = -\frac{\partial}{\partial x}f(x) + D\frac{\partial^2}{\partial x^2}.$$

In analogy to the concept of quasi-energies in quantum mechanics with periodic forcing [6], a solution of (2) is composed of expressions of the form

$$p^{\mu}_t(x) = \exp[-\mu t]\rho_{\mu}(x,t) = \exp[-\mu t]\sum_{n=-\infty}^{\infty} p^{\mu}_n(x)\exp[in\Omega t], \tag{3}$$

where $\rho_\mu(x, t)$ is time-periodic (period $T = 2\pi/\Omega$) with Fourier coefficients $\{p^{\mu}_n(x)\}$ and μ is a Floquet characteristic exponent obeying the infinite coupled set of equations

$$0 = (\mathcal{L}_0 + (\mu - in\Omega)\mathbf{1})p^{\mu}_n(x) - \frac{i}{2}A\frac{\partial}{\partial x}(p^{\mu}_{n+1}(x) - p^{\mu}_{n-1}(x)). \tag{4}$$

Alternatively, one can recast the dynamics in (1) in the form of a two-dimensional, time-homogeneous Fokker-Planck process, $W_t(x, \theta)$, in the random variables x and $\theta = \Omega t + \phi$, *i.e.*

$$\frac{\partial}{\partial t}W_t(x,\theta) = \mathcal{L}W_t(x,\theta) = \left(\mathcal{L}_0 - A\sin\theta\frac{\partial}{\partial x} - \Omega\frac{\partial}{\partial\theta}\right)W_t(x,\theta). \tag{5}$$

Because the random variable θ is observable only modulo 2π we have periodic boundary conditions, *i.e.* $W_t(x, \theta) = W_t(x, \theta + 2\pi)$. With $f(x)$ an unbounded function, which corresponds to a confining potential, we further assume for x the natural boundary condition $W_t(x \to \pm\infty, t) = 0$, and for θ the marginal steady-state density $\int W_t(x, \theta)\,dx = \rho(\theta) = (2\pi)^{-1}$. Due to the periodic boundary conditions, we expand $W_t(x, \theta)$ into a Fourier series with respect to θ, *i.e.* $W_t(x, \theta) = \sum_{n=-\infty}^{\infty} c_n(x, t)\exp[in\theta]$ where upon a substitution into (5) we find for the set $\{c_n\}$ the infinite system of linear differential equations

$$\dot{c}_n(x,t) = (\mathcal{L}_0 - in\Omega\mathbf{1})c_n(x,t) - \frac{i}{2}A\frac{\partial}{\partial x}[c_{n+1}(x,t) - c_{n-1}(x,t)]. \tag{6}$$

Hereby we made use of the orthogonality and the completeness relation of the trigonometric basis functions.

Thus we observe that with $c_n(x, t) = \exp[-\lambda_{l;k} t]\,\tilde{c}_n(x)$ *the eigenvalues* $\{\lambda_{l;k}\}$ *are identical with the Floquet exponents* $\{\mu\}$ *in* (4). *Moreover, the set of eigenvalues* $\{\lambda_{l;k}\}$ *of the two-dimensional Fokker-Planck process contains a branch of purely imaginary eigenvalues.* This can be seen from the adjoint eigenvalue equation of eq. (5) by using an ansatz for the (left-)eigenfunctions which depends only on θ. The corresponding eigenvalues are $\lambda_{0;k} = ik\Omega$, $k = 0, \pm 1, \pm 2, \ldots$. Therefore, the autocorrelation $S(\tau) = \langle x(\tau)x(0)\rangle$ undergoes undamped oscillations (see below). Moreover, $S(\tau)$ is then not strongly mixing [7], *i.e.* $\lim S(\tau \to \infty) \neq \langle x\rangle^2$. If the system is initially prepared with a uniformly distributed random phase $\rho(\phi) = (2\pi)^{-1}$ the probability $W_t(x, \theta)$ relaxes to a time-independent, stationary probability $W_{st}(x, \theta)$ ([1]). On the other hand, the long-time limit of (2) relaxes for fixed phase to a time periodic, asymptotic probability $p^{\phi}_{as}(x, t) \equiv \lim_{t_0 \to -\infty} p^{\phi}(x, t|x_0, t_0)$, where t_0 denotes the initial time. The stationary probability $W_{st}(x, \theta)$ obeys from (5)

$$\Omega\,\frac{\partial W_{st}(x, \theta)}{\partial \theta} = \left[\mathscr{L}_0 - A\sin\theta\,\frac{\partial}{\partial x}\right] W_{st}(x, \theta)\,. \tag{7}$$

Thus, noting that the time t equals, $t \equiv (\theta - \phi)/\Omega$, one obtains upon a comparison with (2) that $W_{st}(x, \theta)$ is connected with the asymptotic probability $p^{\phi}_{as}(x, t)$, *i.e.* $W_{st}(x, \theta) = (1/2\pi)\,p^{\phi}_{as}(x, t)$.

The time-homogeneous, real-valued autocorrelation $S(\tau) = S(-\tau)$ is obtained from

$$S(\tau) = \langle x(\tau)\,x(0)\rangle = \int \mathrm{d}x \int \mathrm{d}\theta\, x \exp[\mathscr{L}[x;\theta]\,\tau]\,\{x\,W_{st}(x, \theta)\}\,, \tag{8a}$$

with $\mathscr{L}$ given in (5). Alternatively, this very same correlation $S(\tau)$ can be evaluated from the nonstationary dynamics in (2), which becomes time-homogeneous after an *averaging procedure over the uniformly distributed random phase* ϕ, *i.e.* with $t_2 - t_1 \equiv \tau$

$$S(\tau) = \frac{1}{2\pi}\int \mathrm{d}\phi \int \mathrm{d}x\,\mathrm{d}y\, xy\, R^{\phi}(x, t_2|\,y, t_1)\,p^{\phi}_{as}(y, t_1)\,, \tag{8b}$$

where R^{ϕ} denotes the time inhomogeneous propagator (conditional probability) of (2). Note that the dependence of the quantities R^{ϕ}, p^{ϕ}_{as} on ϕ enters solely via the combination $z = t_1 + \Omega^{-1}\phi$, which plays the role of a random time. With $\tilde{\rho}(z) = \rho(\phi)|\mathrm{d}\phi/\mathrm{d}z| = \Omega/2\pi = T^{-1}$, one finds with the substitution $\phi \to z$ for (8b)

$$S(\tau) = \frac{1}{T}\int_0^T \mathrm{d}z \int \mathrm{d}x\,\mathrm{d}y\, xy\, R^{z}(x, \tau + z|y, z)\,p^{z}_{as}(y, z)\,, \tag{8c}$$

which clearly exhibits the (continuous) time-translation symmetry ([2]). The dynamical

([1]) In other words, initial probabilities which include a weighting corresponding to relaxation frequencies $\lambda_{0;k}$ are excluded; then all such transient probabilities relax uniquely to the stationary probability $W_{st}(x, \theta)$ which corresponds to the eigenvalue $\lambda_{0;0} = 0$. In contrast, the relaxation of an initial δ-function in (x, θ)-phase space, *i.e.* the conditional probability of $R_{\tau}(x, \theta|x', \theta')$ of (5) involves all eigenvalues, and thus is not strongly mixing [7].

([2]) In contrast, with a nonuniform distribution $\rho(\phi)$, the constant weight $\tilde{\rho}(z) = 1/T$ in (8c) becomes time dependent (*i.e.* a function of t_1), thereby breaking the continuous time translation symmetry in (8c).

susceptibility

$$\chi(\omega) = \int_0^\infty S(\tau)\exp[-i\omega\tau]\,d\tau = \mathrm{Re}\,\chi(\omega) + i\,\mathrm{Im}\,\chi(\omega)$$

can readily be expressed in terms of the spectral properties of (5) (°). In particular, the asymptotic result $S_{as}(\tau) \equiv S(\tau \to \infty)$ exhibits ever present undamped oscillations

$$S_{as}(\tau) = \langle x \rangle^2 + \sum_{n=1}^{\infty} \alpha_n \cos(n\Omega\tau), \tag{9}$$

with the real-valued weights α_n given in terms of the right eigenfunctions $\{\psi_{0n}\}$ of the operator $\mathscr{L}$ in (5) corresponding to eigenvalues $\lambda_{0;n}$ and the left-eigenfunctions $\{\varphi_{0n}\}$, respectively [8].

In the following we present explicit results for the archetypal bistable flow $f(x) = x - x^3$. The corresponding potential, $U(x) = -(1/2)x^2 + (1/4)x^4$, provides attractors at $x_{1;2} = \pm 1$, with a barrier height $\Delta U = 0.25$ at $x = 0$, and $\langle x \rangle = 0$ due to the symmetry. For our numerical investigation we use the matrix-continued fraction technique [8], *i.e.* we expand the set $\{c_n(x, t)\}$ in (6) into the complete and orthogonal set of Hermite functions (see, *e.g.*, 10.1.4 in ref. [9]). The stationary probability is then obtained in terms of the time-independent solutions in (6). Figures 1a), *b*) show the altitude charts of the two-dimensional stationary probability density $W_{st}(x, \theta)$ for $D = 0.1$, *i.e.* $\Delta U/D = 2.5$ and $\Omega = 1$, for two values of the modulation strength $A = 0.1$ and $A = 1$. For $A = 0.1$ (fig. 1*a*)) the probability is very high near the attractors $x_{1;2} = \pm 1$ over the total period $\theta = (0, 2\pi)$. It decreases rapidly for $|x| < 1$, and exhibits two craters close to the unstable position $x = 0$. In order to switch between the locally stable state $x_{1;2}$ the system is forced to surmount high barriers near $x = 0$, *i.e.* the most probable path describes the «running» along the contour lines at $|x| \approx 1$.

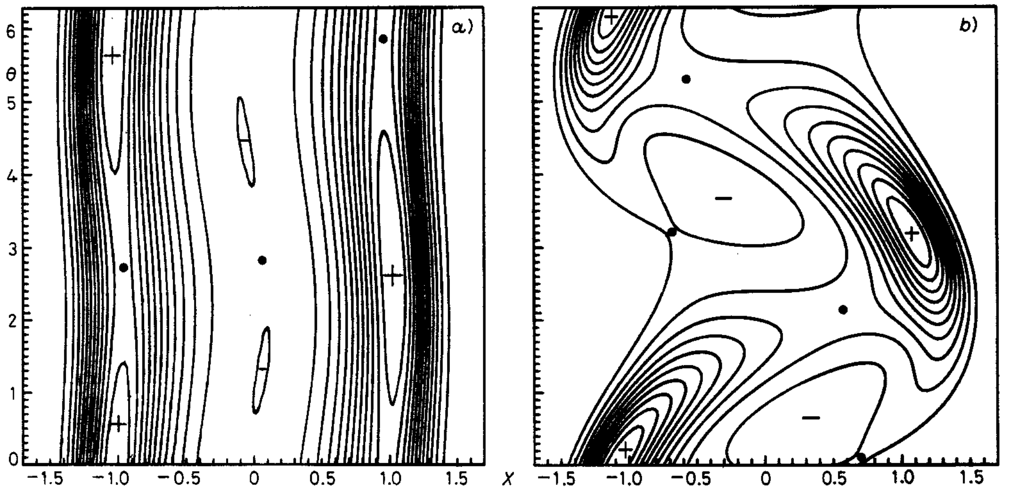

Fig. 1. – Altitude charts of the stationary distribution $W_{st}(x, \theta)$ for $D = 0.1$ and $\Omega = 1$ for $A = 0.1$ (*a*)) and $A = 1$ (*b*)). The full dots indicate saddle points and the + (−) signs denote regions of high (low) probability. The contour lines are equidistant with respect to the corresponding probability density.

(°) See section 3.2 and eq. (5.2.14) in ref. [7].

For $A = 0.5$, the situation is qualitatively the same as for $A = 0.1$; the flow lines along θ are, however, more distorted towards the x-direction, while the craters have grown considerably. For $A = 0.75$, the situation has changed drastically. The saddle points (depicted by full dots in fig. 1) along the θ-flow and those along the escape path, $x = \mp 1 \rightarrow x = \pm 1$, are now of comparable probabilistic weight. This fact implies a strong modulation-induced enhancement of the escape process. Moreover, for $A = 1$ (fig. 1*b*)), the situation is almost contrary to that for $A = 0.1$. Now the most probable path describes the escape along $x = -1 \rightarrow x = +1$, and *vice versa*. Thus, it is quite improbable to stay within one attractor during one cycle $\theta = (0, 2\pi)$.

The correlation function and the associated dynamical susceptibility can be evaluated by use of coupled vectorial recursion relations (see, *e.g.*, in ref. [8]). Figure 2 depicts the (ω-

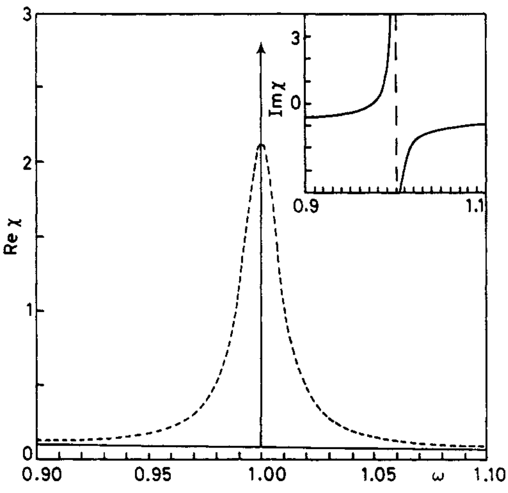

Fig. 2. – Real part and imaginary part (inset) of the dynamical susceptibility $\chi(\varphi)$ are plotted as a function of frequency ω for $D = 0.1$ and $A = 0.5$. The real part exhibits δ-spikes (solid line), which become broadened in the presence of an additional diffusion in θ (dotted line for $Q = 0.01$).

symmetric) real part and the (ω-antisymmetric) imaginary part (see inset) of the dynamical susceptibility, respectively. The real part, $\mathrm{Re}\,\chi(\omega)$, exhibits the theoretically derived (see eq. (9)) δ-function peak at the driving frequency $\omega = \Omega$, while $\mathrm{Im}\,\chi(\omega)$ exhibits a characteristic pole structure at $\omega = \Omega$. This behaviour follows from (9) upon performing the one-sided Fourier transform, *i.e.*

$$\chi_{\mathrm{as}}(\omega) = \frac{1}{2}\pi \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \alpha_k \, \delta(\omega + k\Omega) - i \sum_{k=1}^{\infty} \frac{\alpha_k \, \omega}{\omega^2 - k^2 \Omega^2} \, . \tag{10}$$

The singular behaviour can be regularized if we allow for an additional diffusion for the angle variable θ, *i.e.* if we add a diffusion $Q(\partial^2/\partial\theta^2)$ in (5). This *noisy modulation for the driving frequency Ω implies now strong mixing* for the dynamics in (5), *i.e.* all nonvanishing eigenvalues $\lambda_{l;k}$ assume a positive real part. For $Q = 0.01$ we show the effect of this regularization (finite peak height) by the dotted line in fig. 2. Moreover, the inversion-symmetric cubic flow in (14) implies for the Fokker-Planck operator in (5) the symmetry, $\mathcal{L}(x, \theta) = \mathcal{L}(-x, \theta + \pi)$, with the corresponding symmetry for left and right eigenfunctions. This in turn implies the vanishing of even numbered weights α_k, *i.e.* $\alpha_{2k} = 0$ with

$k = 1, 2, \ldots$. Thus, only odd multiples of Ω are selected as the observable resonance frequencies $\omega_{res} = (2n - 1)\Omega$.

In conclusion, we have studied the long-time properties of periodically driven, overdamped stochastic systems. For the correlation function $S(\tau)$, we find ever present oscillations in spite of a phase (time) averaging procedure, see (9). Thus, a characterization of «stochastic resonance» via its «signal-to-noise enhancement» [1-3], which is bounded (regularized) by the ever present experimental minimal bandwidth, seems somewhat unfortunate. A more physical characterization could be given in terms of the weight $\alpha_1(D, \Omega)$, or by the enhancement of the rate of escape in bistable flows, which at weak noise is governed by the ratio between the smallest, nonzero Floquet exponent and the smallest, nonvanishing eigenvalue of the force-free ($A = 0$) bistable stochastic system.

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