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Interaction of Periodic and Stationary Bifurcation from Multiple Eigenvalues

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Introduction. We consider nonlinear evolution equations

(0.1) $$\frac{du}{dt}+G(\lambda, u)=0$$

depending on a real parameter λ in some real Hilbert space E. For all $\lambda\in\mathbb{R}$ (0.1) has a trivial equilibrium

(0.2) $$G(\lambda, 0)=0.$$

The mapping

(0.3) $$G: \mathbb{R}\times D\to E,$$

where D is a continuously embedded dense subspace of E, has the following properties:

(0.4) G depends analytically on λ and u in a neighborhood of $(\lambda, u)=(0,0)$. In particular, if $G_u(\lambda, 0)$ = derivative with respect to u at $(\lambda, 0)$, we denote $G_u(\lambda, 0) = A(\lambda)$.

(0.5) The operator $-A(0)=-A_0: D\to E$ generates an analytic semigroup $e^{-A_0 t}$ in E.

(0.6) The spectrum of A_0 (in the complexified space E) lies in a sector $\sum_\delta$ of the complex plane with an angle δ less than π and with vertex on the negative real axis. There exists an $\varepsilon>0$ such that the spectrum of A_0 in $\sum_\delta\cap\{z\in\mathbb{C}, \operatorname{Re} z<\varepsilon\}$ consists only of finitely many eigenvalues of finite (algebraic) multiplicity.

(0.7) $i\kappa_0$ and possibly 0 are eigenvalues of $A_0=A(0)$.

Here and in this sequel we refer to our paper [10] where we introduced abstract and concrete operators which satisfy all required conditions. We mention that the class possible applications covers parabolic equations including the Navier-Stokes system.

In order to study periodic solutions of (0.1) it is convenient to introduce spaces of periodic functions defined on $\mathbb{R}$ and having values in D or E. We do not want to specify all properties of these spaces here but we simply say which abstract attributes are needed to prove our general theorem. Again we refer to [10] or [11] where such spaces were introduced in an abstract and concrete setting.

The scalar product and the norm in E are denoted by $(\ ,\ )$ and $\|\ \|$. Then

(0.8) $E_{2\pi}=L^2[(0,2\pi),E]$ with scalar product $(\ ,\ )_0=\int_0^{2\pi}(\ ,\ )dt$ and with norm $\|\ \|_0$

$$D_{2\pi}=\left\{u\colon [0,2\pi]\to D,\ u \text{ is continuous}, \frac{du}{dt}, A_0u\in E_{2\pi}, u(0)=u(2\pi)\right\}$$

with continuous embedding $D_{2\pi}\subset E_{2\pi}$.

We assume that for any $\kappa_0>0$ the evolution operator

(0.9) $J_0=\kappa_0\dfrac{d}{dt}+A_0\colon D_{2\pi}\to E_{2\pi}$ is continuous and is a Fredholm operator of index zero.

In practice this Fredholm property follows from a compact embedding of $D_{2\pi}$ into $E_{2\pi}$ together with an a priori-estimate

$$\|u\|_{D_{2\pi}}\leqq \text{const}(\|J_0u\|_{E_{2\pi}}+\|u\|_{E_{2\pi}}),\qquad u\in D_{2\pi}, \tag{0.10}$$

and from the corresponding properties of the adjoint operator J_0^*. In [10] we proved the compact embedding and the estimate (0.10) for the chosen norm in $D_{2\pi}$. Since in this case the adjoint operator is given by $J_0^*=-\kappa_0\dfrac{d}{dt}+A_0^*$ it has the same properties and condition (0.9) follows from the closed range theorem.

Finally we need that G as given by (0.3) gives rise to an operator

(0.11) $G\colon \mathbb{R}\times D_{2\pi}\to E_{2\pi}$ depending analytically on λ and u with respect to the norms in $D_{2\pi}$ and $E_{2\pi}$.

Again we mention that a proof of property (0.11) for a class of operators G can be found in [10].

In order to formulate our main Theorem we need the notion of a crossing number of $A(\lambda)=G_u(\lambda,0)$ at $\lambda_0=0$.

Assume that

(0.12) $in\kappa_0$ is an eigenvalue of $A(0)$, $n\in\mathbb{N}\cup\{0\}$, of algebraic multiplicity m

which perturbs into an m-fold family of eigenvalues of $A(\lambda)$ for λ near zero. We define

(0.13) $\sigma^<(\lambda)[\sigma^>(\lambda)]=$ sum of the algebraic multiplicities of all perturbed eigenvalues of $A(\lambda)$ near $in\kappa_0$ with negative [positive] real parts and we assume that

(0.14) $\sigma^{<}(\lambda)+\sigma^{>}(\lambda)=m$ for $\lambda \neq 0$, near zero, $\lim_{\varepsilon \searrow 0}(\sigma^{<}(-\varepsilon)-\sigma^{<}(+\varepsilon))=\chi(in\kappa_0, 0)$ exists.

Then

(0.15) $\chi(in\kappa_0, 0)$ is the crossing number of $A(\lambda)$ through $in\kappa_0$ at $\lambda_0=0$.

For $n=0$ (0.14) can be generalized in this sense that eigenvalues may perturb apart from 0 as complex conjugate pairs on the imaginary axis.

Now we are ready to give our result.

Theorem. *Let* $0, i\kappa_0$ *be eigenvalues of* $A_0=A(0)$ *and assume that*

(i) $A(\lambda)$ *has an even crossing number through* 0 *at* $\lambda_0=0$;

(ii) *the eigenvalue* $i\kappa_0$ *is (algebraically) simple and the crossing number of* $A(\lambda)$ *through* $i\kappa_0$ *is* ± 1;

(iii) *if* $in_2\kappa_0, \ldots, in_{l-1}\kappa_0$ *are all eigenvalues of* A_0 *for* $n_j \in \mathbb{N}$, $n_j \geqq 2$ *for* $j=2, \ldots, l-1$, *then* $gcd(n_2, \ldots, n_{l-1})>1$ *(gcd=greatest common divisor). Furthermore no perturbed eigenvalue of* $A(\lambda)$ *near any* $in_j\kappa_0$, $j=2, \ldots, l-1$, *stays on the imaginary axis for* λ *near* $\lambda_0=0$. *(We agree upon* $n_1=1$, $n_l=0$.)

Then $(\lambda, u)=(0,0)$ *is a bifurcation point of periodic or stationary solutions of* (0.1). *If no stationary solutions bifurcate there are periods near* $2\pi/\kappa_0$.

If 0 *is* no *eigenvalue of* A_0 *then assumption* (i) *is redundant and* $(\lambda, u)=(0,0)$ *is a bifurcation point of periodic solutions of* (0.1).

If $A(\lambda)$ *has an* odd *crossing number through* 0 *then in any case* $(\lambda, u)=(0,0)$ *is a bifurcation point of stationary solutions of* (0.1) *(see* [12], *e.g.).*

Some comments have to be made.

The Theorem is a result of „Linearized Bifurcation Theory" in the sense that the conditions for bifurcation are only imposed on the linearizations $G_u(\lambda, 0) = A(\lambda)$ along the trivial branch. In contrast to the results of [1, 2, 7, 8] we allow an eigenvalue zero. Notice that no condition is imposed on the crossing number of $A(\lambda)$ through $in_j\kappa_0$, $j=2, \ldots, l-1$.

In its essential parts our Theorem can already be found in [16]. The reason for this paper is twofold: it shows how to get rid of superfluous technical assumptions and how to simplify the proof. (In [16] the basic solution in $\mathbb{R} \times D$ is a fold and therefore the linearization along the fold has an odd crossing number through 0. The result of [16] then follows by the same arguments of degree theory under the additional assumptions (ii) and (iii) of our Theorem.)

Following the classical arguments for Hopf bifurcation a second parameter κ is introduced representing the unknown period. The main point of this paper is to eliminate this second parameter by solving one scalar equation for κ in terms of the other variables. Thus the problem is reduced completely to a one-parameter "stationary" bifurcation problem in a space of periodic functions (which contains the stationary functions as a subspace). This approach is very simple and purely analytic. On the other hand the result is certainly not the best possible of "Linearized Bifurcation Theory" for stationary or periodic solutions.

It is well known that under our assumptions a Lyapunov-Schmidt procedure reduces the problem to a finite dimensional one. When the critical generalized eigenspace (dependent on λ) is chosen the critical eigenvalues (depending on λ) are preserved. Therefore one-parameter bifurcation in its linearized theory is completely described by an odd crossing number of the critical eigenvalues through 0 (see [12], e.g., where more references concerning this point can be found). In this paper we provide only a first step in studying this critical eigenvalue perturbation after eliminating the second parameter κ.

There are more results comparable to ours. In [3, 15] the eigenvalues 0 and $i\kappa_0$ are (algebraically) simple and there are no eigenvalues of A_0 of the form $in\kappa_0$, $n=2,3,\ldots$ (no further resonance). In [3] the crossing of the simple eigenvalues through the imaginary axis is transversal whereas in [15] any degeneration is allowed. In both papers we find additional nonlinear conditions under which bifurcation of periodic solutions can be guaranteed. In [5] the results of [3] are generalized in this sense that the multiplicity of the eigenvalue 0 is not necessarily one. In [4] the multiplicities of the eigenvalues 0 and $i\kappa_0$ are arbitrary but they have to be semisimple (i.e. there are no generalized eigenvectors). No further resonance is allowed, too, and the additional conditions for bifurcation of stationary or periodic solutions are not related to a crossing of eigenvalues of $A(\lambda)$ through 0 or $i\kappa_0$. They refer to all terms of G up to order 3 and they correspond exactly to those given in [10].

In the papers [6, 9, 14, 17] the interaction of stationary and periodic bifurcation is studied from a different viewpoint. The parameter λ is two-dimensional and can be considered to be a bifurcation parameter and a splitting parameter which splits the degenerate bifurcation into several nondegenerate ones. The eigenvalues 0 and $i\kappa_0$ are simple and no further resonance is allowed. Additional nondegeneracy conditions are imposed on the two-parameter eigenvalue perturbation such that a whole neighborhood of (0, 0) in the parameter plane can be studied. The goal in these papers is different: exclude any additional degeneration and characterize all bifurcation diagrams by the lowest order terms of the two-dimensional bifurcation equation.

Under our conditions we do not know more about the bifurcating solution set than that it is connected. The following simple examples given in cylindrical coordinates show that stationary or periodic solutions bifurcate under the same linear conditions:

$$(0.16)\qquad \begin{aligned} \dot r &= \lambda r & \qquad \dot r &= \lambda r \\ \dot\varphi &= 1 & \qquad \dot\varphi &= 1 \\ \dot z &= \lambda^2 z + r^2 & \qquad \dot z &= \lambda^2 z + z^3. \end{aligned}$$

Proof of the Theorem. Since the period of any nontrivial solution of (0.1) is not prescribed we make the Ansatz that the periods are near $2\pi/\kappa_0$ which is the period of the linearized equation at $\lambda_0=0$. We introduce a real parameter κ, we substitute $t/(\kappa_0+\kappa)$ for t, and we obtain from (0.1)

$$(1.1)\qquad (\kappa_0+\kappa)\frac{du}{dt}+G(\lambda,u)=0, \qquad u(0)=u(2\pi).$$

We define

(1.2) $$G(\lambda, u) = A(\lambda)u + F(\lambda, u)$$

$$J(\kappa, \lambda) = (\kappa_0 + \kappa)\frac{d}{dt} + A(\lambda): D_{2\pi} \to E_{2\pi}$$

and we rewrite Eq. (1.1) as

(1.3) $$\mathscr{G}(\kappa, \lambda, u) = J(\kappa, \lambda)u + F(\lambda, u) = 0, \qquad u \in D_{2\pi}.$$

We consider (1.3) in the complexified spaces $E_{2\pi}$, $D_{2\pi}$. The solutions u to be found, however, have to be real. First we study the spectral properties of $J(\kappa, \lambda)$ which is a perturbation of $J_0 = J(0, 0)$.

The algebraic invariant eigenspaces of $A(\lambda)$ corresponding to the eigenvalues $in_j\kappa_0$, $j = 1, \ldots, l$, perturb into

(1.4) $$E_{in_j\kappa_0, \lambda} \quad \text{for } \lambda \text{ near } \lambda_0 = 0.$$

Then the invariant eigenspace of $J(\kappa, \lambda)$ which perturbs from the eigenspace of $J_0 = J(0, 0)$ with eigenvalue 0 is given by

(1.5) $$E_{2\pi, \lambda} = \bigoplus_{j=1}^{l-1} E^j_{2\pi, \lambda} \oplus E_{0, \lambda},$$

$$E^j_{2\pi, \lambda} = e^{-in_jt} E_{in_j\kappa_0, \lambda} \oplus e^{in_jt} E_{-in_j\kappa_0, \lambda}, \qquad j = 1, \ldots, l-1.$$

Obviously each of these spaces $E^j_{2\pi, \lambda}$ and $E_{0, \lambda}$ is invariant for $J(\kappa, \lambda)$ and

(1.6) all eigenvalues of $J(\kappa, \lambda)$ which perturb from 0 are given by $\mu(\lambda) - in_j(\kappa_0 + \kappa)$, $\bar{\mu}(\lambda) + in_j(\kappa_0 + \kappa)$, where $\mu(\lambda)$ is an eigenvalue of $A(\lambda)$ which perturbs from $in_j\kappa_0$, $j = 1, \ldots, l$.

For a proof we refer to [10], e.g. Then

(1.7) $$\det(J(\kappa, \lambda)|_{E_{2\pi, \lambda}}) = \prod_{j=1}^{l-1} \det(J(\kappa, \lambda)|_{E^j_{2\pi, \lambda}}) \det(A(\lambda)|_{E_{0, \lambda}})$$

and by assumptions (i) to (iii)

(1.8) $$\det(J(\kappa, \lambda)|_{E^j_{2\pi, \lambda}}) > 0 \qquad \text{for } (\kappa, \lambda) \neq (0, 0),\ j = 1, \ldots, l-1$$

$$\det(A(\lambda)|_{E_{0, \lambda}}) > 0 \ (\text{or } < 0) \qquad \text{for all } \lambda \neq 0, \text{ near } 0.$$

By the simplicity of the eigenvalue $i\kappa_0$ of $A(0)$

(1.9) it is perturbed into a simple eigenvalue $\mu_1(\lambda)$ and $E^1_{2\pi, \lambda}$ has an eigenprojector $P_1(\lambda)$ depending analytically on λ of the form

$$P_1(\lambda)u = (u, \psi_1^*(\lambda))_0 \psi_1(\lambda) + (u, \bar{\psi}_1^*(\lambda))_0 \bar{\psi}_1(\lambda),$$

$$\psi_1(\lambda) = e^{-it} v(\lambda), \qquad \psi_1^*(\lambda) = e^{-it} v^*(\lambda),$$

$$v(\lambda) \in E_{i\kappa_0, \lambda}, \qquad v^*(\lambda) \in E^*_{-i\kappa_0, \lambda} \ (= \text{eigenspace of } A^*(\lambda)).$$

We introduce

$$E^0_{2\pi}=(I-P_1(0)E_{2\pi}, \quad D^0_{2\pi}=(I-P_1(0))D_{2\pi}. \tag{1.10}$$

Then

(1.11) $S(\lambda)$: $E^0_{2\pi}\to(I-P_1(\lambda))E_{2\pi}$ given by $S(\lambda)u=(I-P_1(\lambda))u$ for $\underset{0}{u}\in E^0_{2\pi}$ is an isomorphism depending analytically on λ for λ near 0 and $S(0)=I_{E^0_{2\pi}}$.

We decompose (1.3) as

(1.12) (a) $P_1(\lambda)\mathscr{G}(\kappa,\lambda,r(\psi_1(\lambda)+\bar{\psi}_1(\lambda))+S(\lambda)u)=0, \quad r\in\mathbb{R},$
(b) $S(\lambda)^{-1}(I-P_1(\lambda))\mathscr{G}(\kappa,\lambda,./.)=0, \quad u\in D^0_{2\pi}.$

Since $R(J_0)\cap E^0_{2\pi}$ is closed and since $\ker(J_0)$ has a closed complement D^0_c in $D^0_{2\pi}$ the map

(1.13) $(I-P_1(0))J_0$: $D^0_c\to R(J_0)\cap E^0_{2\pi}$ is an isomorphism (with continuous inverse).

When $u\in D^0_{2\pi}$ is decomposed as

$$u=v+w, \quad v\in\ker(J_0)\cap D^0_{2\pi}, \quad w\in D^0_c \tag{1.14}$$

the implicit function theorem yields the following: For any solution of equation (1.12b) in a neighborhood of $(\kappa,\lambda,r,u)=(0,0,0,0)$ the component of u in D^0_c depends on the component in $\ker(J_0)\cap D^0_{2\pi}$ and all other variables as

(1.15) $w=\psi(\kappa,\lambda,r,v)$, $\psi(\kappa,\lambda,0,0)=0$, where the function ψ depends analytically on all variables near $(0,0,0,0)$.

Thus, as indicated in the Introduction, any bifurcating solution depends only on finitely many modes in $\ker(J_0)$.

We use this function ψ in order to eliminate κ. Since any $v\in\ker(J_0)\cap D^0_{2\pi}$ is of the form

$$v=\sum(c_{j,k}e^{-in_jt}v_{j,k}+\bar{c}_{j,k}e^{in_jt}\bar{v}_{j,k}), \quad 2\leqq j\leqq l \qquad v_{j,k}\in E_{in_j\kappa_0,0}, \tag{1.16}$$

assumption (iii) implies that any m-linear form in v (depending on (κ,λ)) satisfies

$$(\mathscr{G}^{(m)}(\kappa,\lambda;v,\ldots,v),\psi_1^*(\lambda))_0\equiv 0. \tag{1.17}$$

This is a simple consequence of

$$\sum_{j=2}^{l}\alpha_j n_j\neq -1 \quad \text{for all } (\alpha_2,\ldots,\alpha_l)\in\mathbb{Z}^{l-1}. \tag{1.18}$$

This observation implies

$$P_1(\lambda)\mathscr{G}(\kappa,\lambda,S(\lambda)(v+\psi(\kappa,\lambda,0,v))\equiv 0 \quad \text{for all } v\in\ker(J_0)\cap D^0_{2\pi}. \tag{1.19}$$

The $\psi_1(\lambda)$-component of equation (1.12a) is complex conjugate to the $\bar{\psi}_1(\lambda)$-component (for real u). When decomposed into real and imaginary parts the real system being equivalent to (1.12a) can be written as (see (1.19)):

$$\begin{aligned} &\operatorname{Re}\mu_1(\lambda)r + rg_1(\kappa,\lambda,r,v)=0, \quad && g_j(\kappa,\lambda,0,0)=0,\\ &(\operatorname{Im}\mu_1(\lambda)-\kappa_0-\kappa)r + rg_2(\kappa,\lambda,r,v)=0, && j=1,2. \end{aligned} \tag{1.20}$$

After division by r the imaginary part is solved for κ in terms of the other variables:

$$\kappa=\kappa(\lambda,r,v), \qquad \kappa_r(\lambda,0,0)=0. \tag{1.21}$$

The last property follows from the fact that $g_j(\kappa,\lambda,r,0)$ is even in r (see [10], e.g.).

We insert this function into the system (1.12) and after deleting the imaginary part of (1.12a) the resulting one-parameter system has a linearization along the trivial solution given by

$$\begin{pmatrix} \operatorname{Re}\mu_1(\lambda) & 0 \\ 0 & S(\lambda)^{-1}(I-P_1(\lambda))J(\kappa(\lambda,0,0),\lambda)S(\lambda) \end{pmatrix}. \tag{1.22}$$

By (1.8) and assumption (ii) this family has an odd crossing number through 0. This observation completes the proof of our Theorem (see Theorem 3.1 in [12], e.g.).

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