Multiple eigenvalue bifurcation for Fredholm operators*)

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Introduction

In our paper on the "Principle of Reduced Stability" [10] we studied how the linearized stability of any bifurcating branch can be determined by the Jacobian of a bifurcation function about the projected branch. Whereas the individual eigenvalues may differ (i.e. the "Principle of Reduced Stability" is not true, in general) there is an invariant in the following sense: the sign of the product of all critical eigenvalues is preserved under any Lyapunov-Schmidt reduction.

In our paper [10] we studied only bifurcation from a semisimple eigenvalue, i.e. when there are no generalized eigenvectors. In this paper we admit any eigenvalue of finite geometric and algebraic multiplicity. The invariance of the sign of the product of all critical eigenvalues remains true (Theorem 3. 1). This result is not only important for the problem of stability of bifurcating branches but it can also be used for the proof of existence of bifurcating solutions.

As a matter of fact it allows to give a unifying approach to well known results of "Linearized Bifurcation Theory" where the conditions for bifurcation are only imposed on the linearizations. We link bifurcation to eigenvalue perturbation which seems to be the key for "Linearized Bifurcation Theory". In the first parts of this paper we give a kind of survey of local and global bifurcation results, in the last part we come back to the question of stability of bifurcating branches.

We consider nonlinear equations

$$G(\lambda, u) = 0 \tag{0. 1}$$

depending on a real parameter λ in some real Banach space E. To be more precise, the mapping

$$G: \mathbb{R} \times D \to E, \tag{0. 2}$$

*) Supported by the Deutsche Forschungsgemeinschaft under Ki 227/3-1

where $D \subset E$ is a continuously embedded closed subspace, satisfies the following conditions:

(0. 3) $G(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$,

(0. 4) G is continuous and has continuous partial Frechet derivatives with respect to u in a neighborhood of $(\lambda, u) = (0, 0)$. In particular we denote $G_u(\lambda, 0) = A(\lambda)$.

(0. 5) $A(0)$ is a Fredholm operator of index zero,

(0. 6) zero is an isolated eigenvalue of $A(0)$ (in the sense of [8], p. 180).

An important case is given by a closed family $A(\lambda)$: $D \to E$ and D is endowed with the graph norm.

The family $(\lambda, u) = (\lambda, 0)$, $\lambda \in \mathbb{R}$, is commonly referred to as trivial solutions of (0. 1). In Bifurcation Theory nontrivial solutions emanating from these trivial solutions are investigated.

A necessary condition for $(\lambda, u) = (0, 0)$ to be a bifurcation point is

(0. 7) $$\dim \ker (A(0)) = n > 0 .$$

Well known counterexamples show that condition (0. 7) is not sufficient for bifurcation at (0, 0).

A first general theorem due to Krasnoselskij [11] gives the following sufficient conditions for bifurcation at $(\lambda, u) = (0, 0)$:

(0. 8) $D = E$, $A(\lambda) = I + (\lambda - \lambda_0) K$, K is compact, and $\lambda_0 \neq 0$ is a characteristic value of K of odd multiplicity, i.e. $\frac{1}{\lambda_0}$ is an eigenvalue of K of odd algebraic multiplicity m.

An extension of this result is given by Sarreither [17]. He considers a family $A(\lambda)$ in its generality assuming a sufficiently high order of differentiability with respect to λ and that

(0. 9) zero is no eigenvalue of $A(\lambda)$ for λ near zero, $A(0)$ is a Fredholm operator of index zero, and the family A has an odd generalized algebraic multiplicity at zero.

Using the notation of a "Kette" (chain) the definition extends that of the classical algebraic multiplicity of an eigenvalue zero of $A(\lambda) = (\lambda - \lambda_0) I + B$ at zero.

Independently Magnus [15] defines a generalized algebraic multiplicity of a family A at zero. He needs less differentiability and for a sufficiently smooth family his definition is equivalent to the definition in [17]. A bifurcation result in case of an odd multiplicity is proved in [15], too.

Again independently, Ize [7] introduces a generalized algebraic multiplicity of a family A at zero. This definition is given by the bifurcation function obtained by a Lyapunov-Schmidt reduction. An odd algebraic multiplicity says that the determinant of the linearized bifurcation function about the trivial solutions changes its sign at $\lambda = 0$. By the homotopy invariance of Brouwer's degree this implies bifurcation. Ize's main result in this connection is the independence of this multiplicity on the choices of the projections used for the Lyapunov-Schmidt reduction. It is not explained, however, which property of the original family $A(\lambda)$ leads to an odd multiplicity.

Although we give no proof we think that all notions of multiplicities coincide. A strong hint for this suggestion is given by our Theorem 3. 1 which links the determinant of the family $A(\lambda)$ restricted to the m-dimensional invariant generalized eigenspaces $P(\lambda)E$ to the determinant of the linearized bifurcation function about the trivial solutions $(\lambda, 0)$. If E is finite dimensional we do not need this restriction and using Satz 4. 1 in [17] or Theorem 2. 6 in [15] and our formula (3. 9) we can conclude that all definitions in [7], [15], [17] coincide. (If E is infinite dimensional, however, Theorem 2. 7 in [15] does not help since D and E do not necessarily allow constant invariant decompositions for $A(\lambda)$. Therefore our remark in [9], p. 420, is not proved.)

Again independently, Laloux and Mawhin [12] define a multiplicity of a characteristic value λ_0 for a special family

$$A(\lambda) = L + (\lambda - \lambda_0) A \tag{0. 10}$$

at $\lambda = 0$. They assume that L is a Fredholm operator of index zero and that the operator A is L-compact. (For precise definitions see [12].) If we assume in addition that L is a closed operator then the family $A(\lambda)$ fits into the framework of this paper satisfying (0. 5) and (0. 6). Furthermore the notion of multiplicity in [12] should coincide with these introduced above. (See formula (3. 29) which gives a strong hint for this suggestion.) In case of an odd multiplicity of an (isolated) characteristic value λ_0 for the pair (L, A) it is proved in [12] that bifurcation takes place at $(\lambda, u) = (0, 0)$.

To summarize, all notions of generalized multiplicity are not transparent since they do not show which intrinsic property of the family $A(\lambda)$ yields an odd or an even multiplicity.

In this paper we offer a different condition for bifurcation allowing also more general families $A(\lambda)$.

We introduce the notion of a crossing number of the family $A(\lambda)$ at $\lambda = 0$. The definition of a crossing number is given by the family itself and therefore it is independent of any finite dimensional reduction. Furthermore it enlightens which property of $A(\lambda)$ actually causes local bifurcation.

Roughly spoken, any odd number of eigenvalues of $A(\lambda)$ (counted by their algebraic multiplicities) leaving (or entering) the left complex half plane through zero entails bifurcation.

Since complex eigenvalues of $A(\lambda)$ occur in pairs we may also state the result as follows: any odd number of real eigenvalues of $A(\lambda)$ leaving (or entering) the negative real axis at $\lambda = 0$ causes bifurcation at $(\lambda, u) = (0, 0)$. We mention that no "transversal" or "nondegenerate" crossing of the imaginary axis is required. Furthermore the notion of a "crossing number" is a little misleading since, for instance, the two eigenvalues $\pm\sqrt{\lambda}$ give rise to an odd crossing number at $\lambda = 0$.

For a family of the form

$$A(\lambda) = B + (\lambda - \lambda_0) A \tag{0. 11}$$

with linear operators A and B, this local bifurcation result can be found in the book of Chow and Hale [2], Theorem 7. 4, Chapter 5. They require that λ_0 is an isolated normal eigenvalue of (B, A) which corresponds to our assumptions (0. 5), (0. 6). Our approach, however, is different since we use a different Lyapunov-Schmidt reduction. We project D onto $\ker(A(0))$ having dimension n which might be smaller than the dimension m of the invariant eigenspace $E_0 = \ker(A(0)^\alpha)$, α being the ascent of $A(0)$. This projection is no longer an eigenprojection of $A(0)$ and it is not uniquely determined. The space E is projected onto $R(A(0))$ along any n-dimensional complement. In the usual way we get an n-dimensional bifurcation equation. We show that for any choice of projections an odd crossing number is reflected on the respective bifurcation function in such a way that its Jacobian determinant at the trivial solutions changes sign at $\lambda = 0$ (Theorem 3. 1). This result does not only entail local bifurcation but it is also useful for global considerations in connection with a degree theory. Finally it is useful for the problem of stability of bifurcating solutions. We shall discuss this later.

Obviously the Krasnoselskij conditions (0. 8) are stronger than ours. By the special form of $A(\lambda)$ conditions (0. 5) and (0. 6) are fulfilled and by the linear dependence on the parameter the algebraic m-fold real eigenvalue $\frac{\lambda}{\lambda_0}$ of $A(\lambda)$ crosses zero at $\lambda = 0$.

Under the additional assumption that L is a closed operator Laloux and Mawhin's bifurcation result in [12] follows from ours, too. This is shown at the end of Section III.

Ize's local bifurcation results in [7], Chapter I, are a consequence of ours as long as they refer to Fredholm operators of index zero.

Next we mention the well known result for bifurcation from simple eigenvalues due to Crandall and Rabinowitz [3]. Under the same hypotheses for the family A they assume that $\dim\ker(A(0)) = 1$ $(n = 1)$ the algebraic multiplicity m of the eigenvalue zero being arbitrary. The additional requirement

$$\frac{d}{d\lambda} A(0)\ (\ker A(0)) \not\subset R(A(0)) \tag{0. 12}$$

means that an odd number of eigenvalues of $A(\lambda)$ leaves (or enters) the left complex half plane through zero. This interpretation is given at the end of Section III. It shows that this—at first sight—different type of bifurcation result can also be embedded into our general theorem.

The generalization of Westreich [20] on bifurcation at eigenvalues of odd multiplicities is a consequence of our present paper, too (see Section III).

The first result in the spirit of this paper is due to Weinberger [19]: any possibly degenerate crossing of an algebraically simple eigenvalue through zero entails bifurcation. In [9] we showed how the number of bifurcating branches may increase by a high degeneration of the eigenvalue crossing and that (in the analytical case) the degeneration gives an upper bound for the number of branches. Furthermore all branches can be constructed with the aid of Newton's diagram.

Finally, the result presented here embeds stationary bifurcation into the dynamic bifurcation theory for

$$\frac{du}{dt}+G(\lambda, u)=0. \tag{0. 13}$$

It is known that *any* nonzero number of eigenvalues of $A(\lambda)=G_u(\lambda, 0)$ crossing the imaginary axis at $\lambda=0$ *apart from zero* entails Hopf bifurcation for (0. 13).

Interaction of stationary and periodic bifurcation, i.e. eigenvalues of $A(\lambda)$ crossing the imaginary axis *including* zero, is not yet completely understood. We refer to results of Lauterbach [13], e.g..

As is well known by simple counterexamples an even crossing number through zero does not necessarily imply stationary bifurcation. For a double eigenvalue zero, however, we mention a remarkable result of Lauterbach [14]: For any crossing number going together with a change of stability we have at least one of three possibilities: stationary, periodic or homoclinic solutions of (0. 13) bifurcate at $(\lambda, u)=(0, 0)$.

This result is a special case of a more general fact: *any* change of stability of the trivial solutions of (0. 13) implies bifurcation in a class of functions which are bounded for all $t \in \mathbb{R}$. This—under suitable hypotheses—is a consequence of Conley's index theory.

In Section IV we give a global bifurcation result by introducing a degree for a class of proper nonlinear Fredholm operators G: $\Omega \subset D \to E$ in the sense of Smale [18]. Such a degree was introduced in [18] and refined by Elworthy and Tromba [5]. Our contribution simplifies its definition when reducing the class of operators by requiring that $G'(u)$ has only finitely many isolated eigenvalues on the negative real axis. This class contains the compact perturbations of the identity (for $D=E$) since the only possible cluster point of eigenvalues of $I+G'(u)$, $G'(u)$ being compact, is $+1$. Furthermore, by the special form of a compact perturbation of the identity, such operators are proper which means that the (bounded) inverse image of any compact set is compact.

If $D=E$ Eisenack and Fenske [4] (p. 79) introduced a class of proper nonlinear Fredholm maps having exactly the above mentioned eigenvalue property. Their degree generalizes the Leray-Schauder degree but it is more special than that of [5] and than ours.

Let us explain the simplicity of our definition. The index of a map G in $\mathbb{R}^n$ at the solution of $G(u)=p$ is given by $\operatorname{sign}\det(G'(u))$ provided it is not zero, i.e. p is locally a regular value. Obviously only eigenvalues of $G'(u)$ on the negative real axis contribute to the sign of the Jacobian determinant. Therefore this definition can be imitated in infinite dimensions if $G'(u)$ has only finitely many eigenvalues of finite multiplicity on the negative real axis: If p is a regular value of G, define the index of G at a solution of $G(u)=p$ by the sign of the product of all eigenvalues of $G'(u)$ on the negative real axis. This index gives rise to a degree for G on bounded domains in a natural way. We show that this degree has the same properties as the Brouwer or Leray-Schauder degree for their classes of maps, respectively. It is worth mentioning that the proofs are considerably simplified when using Theorem 3. 1: By the theorem of Sard-Smale it allows to reduce the problems to the one-dimensional case.

When depending on a real parameter, the index of $G(\lambda, \cdot)$ at the trivial solutions changes its sign by definition when $G_u(\lambda, 0)=A(\lambda)$ has an odd crossing number. By properness bounded solution sets of $G(\lambda, u)=0$ are compact and therefore the proof of Rabinowitz' [16] global bifurcation result holds also for our class of operators.

Our Theorem 4. 5 is a generalization of all global results of Rabinowitz [16], Ize [7], and Magnus [15] since all authors use more special families $G(\lambda, u)$. Finally, we think that our approach lays bare which properties entail local and global bifurcation for Fredholm operators: odd crossing numbers cause local bifurcation and the only form of compactness which is needed to prove global bifurcation can be expressed by properness.

In view of applications our class of operators G seems to be reasonable: If $G_u(\lambda, u)$ are suitable elliptic partial differential operators then their spectrum lies in a sector with vertex on the negative real axis. If the underlying domain is bounded the spectrum consists of isolated eigenvalues of finite multiplicity. As for bifurcation, our results allow a general dependence on the parameter λ as long as $G(\lambda, u)$ is a proper family (for results concerning properness see [1]).

Our final section is devoted to the question of stability of any local nontrivial branch of solutions of $G(\lambda, u)=0$ emanating at $(0, 0)$. This question is important in applications whenever the trivial solutions $(\lambda, 0)$ lose their stability at $\lambda=0$ and a nontrivial branch is expected to take it over thus becoming relevant for the mathematical model.

A "Principle of Reduced Stability" (see [10]) saying that the linearized stability of a branch is determined by the finite dimensional bifurcation function is not as simple as in the case when zero is a semisimple eigenvalue of $A(0)$, i.e. when there are no generalized eigenvectors. When the invariant eigenspace has a bigger dimension m than $\ker(A(0))$, the bifurcation function is n-dimensional whereas the eigenvalue zero might perturb into an m-fold family. Therefore the n eigenvalues of the linearized bifurcation function about any branch (λ, u) of solutions of (0. 1) emanating at $(0, 0)$ cannot be related to the m eigenvalues of $G_u(\lambda, u)$ near zero. There is a method, however, to connect this eigenvalue perturbation in the space E to an n-dimensional equation which is of a more complicated form than just an eigenvalue problem. The given example shows, however, that a "Principle of Reduced Stability" seems not to be valid for a similarly large class of problems for which zero is a semisimple eigenvalue (see [10]).

Nevertheless we give a criterion which holds in the case of bifurcation at simple eigenvalues in the sense of Crandall and Rabinowitz ($n=1$, m being arbitrary). No nondegeneracy condition (0. 12) is needed and it is a first general result on reduced stability for bifurcation at an eigenvalue zero for which the geometric multiplicity n and the algebraic multiplicity m may differ. For general n and m we add two conditions for the validity of the "Principle of Reduced Stability". These conditions require, however, that the lowest order term of $G_u(\lambda, u)$ in a given branch is known. Our results in [10] (in the case $n=m$) were the first to avoid this type of condition using *only* the linearized bifurcation function.

I. Lyapunov-Schmidt reduction

By assumption (0. 4) the family $A(\lambda)\colon D \to E$ is continuous with respect to λ. Therefore

$$A(\lambda) = A(0) + B(\lambda), \quad \lim_{\lambda \to 0} \|B(\lambda)\| = 0, \tag{1. 1}$$

where we use the norm for bounded linear operators from D into E.

By assumptions (0. 5), (0. 6)

$$\dim \ker(A_0) = \operatorname{codim} R(A_0) = n > 0, \tag{1. 2}$$

where $A_0 = A(0)$. Thus there exist closed complements

$$D = \ker(A_0) \oplus D_2, \; E = R(A_0) \oplus E_2 \tag{1. 3}$$

with continuous projections

$$P\colon D \to \ker(A_0) \quad \text{along } D_2, \tag{1. 4}$$

$$Q\colon E \to R(A_0) \quad \text{along } E_2. \tag{1. 5}$$

Obviously these complements, and therefore the projectors, are not uniquely determined.

Decomposing

$$G(\lambda, u) = A(\lambda) u + F(\lambda, u) \tag{1. 6}$$

the equation

$$G(\lambda, u) = 0 \tag{1. 7}$$

is equivalent to the system

$$\begin{aligned} &A_0 w + QB(\lambda)\,(v+w) + QF(\lambda, v+w) = 0, \\ &(I-Q)\,B(\lambda)\,(v+w) + (I-Q)\,F(\lambda, v+w) = 0, \\ &v = Pu, \quad w = (I-P)u, \quad u \in D. \end{aligned} \tag{1. 8}$$

Observe that

$$A_0\colon (I-P)D \to QE \tag{1.9}$$

is an isomorphism with a continuous inverse (Banach's theorem). By the implicit function theorem applied in a neighborhood of $(\lambda, v, w)=(0,0,0)$ we solve the first equation of (1. 8) and get

$$w=\Psi(\lambda, v), \quad \Psi(\lambda, 0)=0 \tag{1.10}$$

for $|\lambda|<\delta_1$, $\|v\|<\delta_2$.

The function Ψ has continuous partial Frechet derivatives with respect to v at $(\lambda, 0)$ which are given by

$$\Psi_v(\lambda, 0)=-(A_0+QB(\lambda))^{-1}QB(\lambda)\colon \ker(A_0)\to D_2. \tag{1.11}$$

For small $|\lambda|$ the operator $A_0+QB(\lambda)$ is an isomorphism from $D_2=(I-P)D$ onto $R(A_0)$. Therefore $\Psi_v(\lambda, 0)$ is a bounded linear operator from $\ker(A_0)$ into D_2.

When $w=\Psi(\lambda, v)$ is inserted into the second equation we get

$$\phi(\lambda, v)=(I-Q)\,B(\lambda)\,(v+\Psi(\lambda, v))+(I-Q)\,F(\lambda, v+\Psi(\lambda, v))=0 \tag{1.12}$$

which is usually called *bifurcation equation.* Note that

$$\phi\colon \mathbb{R}\times\ker(A_0)\to E_2, \ \dim\ker(A_0)=\dim E_2=n, \tag{1.13}$$

where the bifurcation function ϕ is defined for $|\lambda|<\delta_1$, $\|v\|<\delta_2$.

Finally, the partial Frechet derivative of ϕ with respect to v at $(\lambda, 0)$ is given by

$$\begin{aligned} &\phi_v(\lambda, 0)=(I-Q)\,B(\lambda)\,\{I_0-(A_0+QB(\lambda))^{-1}QB(\lambda)\},\\ &\phi_v(\lambda, 0)\colon \ker(A_0)\longrightarrow E_2, \end{aligned} \tag{1.14}$$

I_0 denoting the identity in $\ker(A_0)$.

II. Eigenvalue perturbation for $A(\lambda)$ near zero

By (0. 6) zero is an isolated eigenvalue of $A(0)$ considered as a closed operator in E with domain D. Let Γ be a closed curve in the resolvent set of $A(0)$ containing in its interior the unique spectral point zero. (D, E and $A(\lambda)$ are complexified in the natural way.)

For small $|\lambda|$ the curve Γ is still in the resolvent set of $A(\lambda)$ and the eigenprojector onto the generalized eigenspace corresponding to all eigenvalues in the interior of Γ is given by

$$P(\lambda)=\frac{1}{2\pi i}\int_\Gamma (z-A(\lambda))^{-1}\,dz \tag{2.1}$$

(see [8]). For $\lambda=0$, $P(0)$ projects onto the generalized eigenspace of the eigenvalue zero of A_0 which is given by

$$\bigcup_{\nu\in\mathbb{N}} \ker(A_0^\nu)=P(0)\,E=E_0. \tag{2.2}$$

By a result of Gohberg and Krein [6] we can conclude for a Fredholm operator of index zero having an isolated eigenvalue zero:

$$P(0)E = \ker(A_0^\alpha) \quad \text{for an } \alpha \in \mathbb{N}, \qquad \dim\ker(A_0^\alpha) = m < \infty. \tag{2. 3}$$

The number α is called ascent of A_0 and m is the algebraic multiplicity of the eigenvalue zero.

This number is invariant with respect to λ in the following sense

$$\dim P(\lambda)E = m \quad \text{for } |\lambda| < \delta_3. \tag{2. 4}$$

The m-fold eigenvalue zero of $A(0) = A_0$ splits into m eigenvalues of $A(\lambda)$ counted by their algebraic multiplicities. Furthermore all eigenvalue perturbations of $A(\lambda)$ near zero can be completely determined in the family of invariant m-dimensional subspaces $P(\lambda)E$ (see [8], p. 106ff.).

We define

(2. 5) $\sigma^<(\lambda)$ $[\sigma^\geqq(\lambda)]$ = sum of the algebraic multiplicities of all *nonzero* eigenvalues of $A(\lambda)$ in $P(\lambda)E$ with negative [nonnegative] real parts.

We assume the existence of two sequences

$$\lambda_k^+ > 0, \ \lambda_k^- < 0, \ \lim_{k\to\infty} \lambda_k^\pm = 0 \tag{2. 6}$$

with the properties

$$\sigma^<(\lambda_k^\pm) + \sigma^\geqq(\lambda_k^\pm) = m \quad \text{for all } k \in \mathbb{N} \tag{2. 7}$$

and

$$\sigma^<(\lambda_k^+) - \sigma^<(\lambda_k^-) = \chi_k \in 2\mathbb{Z} + 1 \tag{2. 8}$$

for all $k \in \mathbb{N}$. In this sense, passing from λ_k^- to λ_k^+, an odd number of eigenvalues of $A(\lambda)$ have left or entered the left half plane through zero. Therefore we call conditions (2. 7), (2. 8) the property that $A(\lambda)$ has odd crossing numbers. (Condition (2. 7) means that zero is no eigenvalue of $A(\lambda_k^\pm)$.) If $\lim_{k\to\infty} \chi_k = \chi(0)$ exists we call the limit $\chi(0)$ the *crossing number of* $A(\lambda)$ *at* $\lambda = 0$. It certainly exists if $A(\lambda)$ depends analytically on λ.

The following example, however, shows that the limit does not necessarily exist although conditions (2. 6) to (2. 8) are fulfilled: Any 2×2-matrix $A(\lambda)$ having the eigenvalues $\lambda \sin\frac{1}{\lambda}$ and $\lambda\cos\frac{1}{\lambda}$ has, according to our definition, odd crossing numbers.

III. Odd crossing numbers imply bifurcation

Since $\ker(A_0) \subset P(0)E$ and $P(0)E_2$ is a complement of $R(A_0)$ in E, the spaces $P(t)\ker(A_0)$ and $P(t)E_2$ (for $t \in \mathbb{R}$) are both n-dimensional and the following decompositions hold for small $|t| \leqq \delta_4$:

$$\begin{aligned} D &= P(t)\ker(A_0) \oplus D_2, \\ E &= R(A_0) \oplus P(t)E_2. \end{aligned} \tag{3. 1}$$

The invariant eigenspaces $E_t = P(t)E$ allow the decompositions

$$\begin{aligned} E_t &= P(t)\ker(A_0) \oplus D_{2t}, & D_{2t} &= D_2 \cap E_t, \\ E_t &= R_t \oplus P(t)E_2, & R_t &= R(A_0) \cap E_t. \end{aligned} \tag{3. 2}$$

These decompositions give rise to bounded projectors

$$\begin{aligned} P_t &: D \to P(t)\ker(A_0) && \text{along } D_2, \\ Q_t &: E \to R(A_0) && \text{along } P(t)E_2. \end{aligned} \tag{3. 3}$$

E_t is an invariant space for P_t as well as for Q_t and when restricted to $E_t \subset D \subset E$ they project

$$\begin{aligned} P_t &: E_t \to P(t)\ker(A_0) && \text{along } D_{2t}, \\ Q_t &: E_t \to R_t && \text{along } P(t)E_2. \end{aligned} \tag{3. 4}$$

Furthermore, P_t and Q_t are bounded families in the space of bounded operators and $P_t u$ as well as $Q_t u$ is continuous in t for any $u \in D$ or $u \in E$, respectively.

Now we represent $A(\lambda)$: $D \to E$ as follows:

$$\begin{gathered} A(\lambda) = \begin{pmatrix} Q_t A(\lambda) & Q_t A(\lambda) \\ (I-Q_t)A(\lambda) & (I-Q_t)A(\lambda) \end{pmatrix}, \\ A(\lambda)\colon P(t)\ker(A_0) \oplus D_2 \to R(A_0) \oplus P(t)E_2. \end{gathered} \tag{3. 5}$$

For all $|t| \leqq \delta_4$ the operators

$$Q_t A(\lambda)\colon D_2 \to R(A_0) \tag{3. 6}$$

are isomorphisms provided $|\lambda|$ is sufficiently small.

Introducing

$$\begin{gathered} C(\lambda, t) = \begin{pmatrix} -(I-Q_t)A(\lambda)[Q_t A(\lambda)]^{-1} & I_{P(t)E_2} \\ [Q_t A(\lambda)]^{-1} & 0 \end{pmatrix}, \\ C(\lambda, t)\colon R(A_0) \oplus P(t)E_2 \to P(t)E_2 \oplus D_2 \end{gathered} \tag{3. 7}$$

and

$$\begin{gathered} D(\lambda, t) = \begin{pmatrix} (I-Q_t)A(\lambda)\{I_t - [Q_t A(\lambda)]^{-1} Q_t A(\lambda)\} & 0 \\ [Q_t A(\lambda)]^{-1} Q_t A(\lambda) & I_{D_2} \end{pmatrix}, \\ D(\lambda, t)\colon P(t)\ker(A_0) \oplus D_2 \to P(t)E_2 \oplus D_2,\ I_t = I_{P(t)\ker(A_0)}, \end{gathered} \tag{3. 8}$$

the following relation holds:

$$C(\lambda, t)A(\lambda) = D(\lambda, t). \tag{3. 9}$$

Since

$$C(0,t)=\begin{pmatrix} 0 & I_{P(t)E_2} \\ A_0^{-1} & 0 \end{pmatrix} \tag{3.10}$$

is an isomorphism the same holds for small $|\lambda|$ and for all $|t| \leqq \delta_4$. In the following we restrict $|\lambda| \leqq \delta_5$, $|t| \leqq \delta_4$, $\delta_5 \leqq \delta_4$, where all relations up to now hold. In view of (3. 9) we may state:

(3. 11) Zero is no eigenvalue of $A(\lambda) \Leftrightarrow (I-Q_t)\,A(\lambda)\,\{I_t-[Q_t A(\lambda)]^{-1}\,Q_t A(\lambda)\}$: $P(t)\ker(A_0) \to P(t)\,E_2$ is an isomorphism for all $|t| \leqq \delta_4$.

Now consider $A(\lambda)$ restricted to its invariant space E_λ. By the definitions of the projectors P_λ and Q_λ is follows that

(3. 12) E_λ is an invariant space for $C(\lambda,\lambda)$ and for $D(\lambda,\lambda)$.

Then by (3. 9)

$$\det\big(C(\lambda,\lambda)|_{E_\lambda}\big)\det\big(A(\lambda)|_{E_\lambda}\big)=\det\big(D(\lambda,\lambda)|_{E_\lambda}\big). \tag{3.13}$$

Since $\det\big(C(\lambda,\lambda)|_{E_\lambda}\big)$ is continuous in λ and nonzero for all $|\lambda| \leqq \delta_5$, this factor does not change sign. Let us define

$$\begin{aligned} S_0 &= \operatorname{sign}\det\big(C(\lambda,\lambda)|_{E_\lambda}\big) \\ &= \operatorname{sign}[(-1)^{(m-n)n}\det(A_0|_{D_2\cap E_0})] \neq 0 \quad \text{for } |\lambda| \leqq \delta_5, \end{aligned} \tag{3.14}$$

where the last determinant is evaluated with respect to a fixed basis in E_0.

Then

$$S_0 \operatorname{sign}\det\big(A(\lambda)|_{E_\lambda}\big)=\operatorname{sign}\det\big((I-Q_\lambda)\,A(\lambda)\,\{I_\lambda-[Q_\lambda A(\lambda)]^{-1}\,Q_\lambda A(\lambda)\}\big). \tag{3.15}$$

Assume now that zero is no eigenvalue of $A(\lambda)$, i.e. $\operatorname{sign}\det\big(A(\lambda)|_{E_\lambda}\big) \neq 0$. Then by (3. 11)

$$\operatorname{sign}\det\big((I-Q_t)\,A(\lambda)\,\{I_t-[Q_t A(\lambda)]^{-1}\,Q_t A(\lambda)\}\big) \neq 0 \tag{3.16}$$

for all t between λ and 0.

Fix bases in $\ker(A_0)$ and E_2 and define continuously varying bases in $P(t)\ker(A_0)$ and $P(t)E_2$ by the isomorphisms $P(t)$. Then the determinant in (3. 16) with respect to these bases is continuous in t and therefore does not change sign. By (3. 15) we therefore proved the relation

$$S_0 \operatorname{sign}\det\big(A(\lambda)|_{E_\lambda}\big)=\operatorname{sign}\det\big((I-Q_0)\,A(\lambda)\,\{I_0-[Q_0 A(\lambda)]^{-1}\,Q_0 A(\lambda)\}\big). \tag{3.17}$$

The spaces E_2 and $P(0)E_2$ are both complements of $R(A_0)$ in E. There exists a continuous and linear homotopy $H\colon [0,1]\times E \to E$ with the properties that $H(t)E_2$ is a complement of $R(A_0)$ in E for all $t \in [0,1]$ and $H(0)E_2=P(0)E_2$, $H(1)E_2=E_2$. Take simply

$$H(t)u=P(0)u+t\big(I-P(0)\big)u, \quad u \in E, \quad t \in [0,1]. \tag{3.18}$$

The decompositions

$$E = R(A_0) \oplus H(t) E_2 \tag{3. 19}$$

define a continuous family of projectors

$$\tilde{Q}_t : E \to R(A_0) \quad \text{along } H(t) E_2 . \tag{3. 20}$$

Observe that $\tilde{Q}_0 = Q_0$ and $\tilde{Q}_1 = Q$, where Q is defined by (1. 5). Repeating the above argument when replacing P_t by P_0 and Q_t by $\tilde{Q}_t$ we get

(3. 21) zero is no eigenvalue of $A(\lambda) \Leftrightarrow (I - \tilde{Q}_t) A(\lambda) \{I_0 - [\tilde{Q}_t A(\lambda)]^{-1} \tilde{Q}_t A(\lambda)\}$: $\ker(A_0) \to H(t) E_2$ is an isomorphism for all $t \in [0, 1]$.

The bases in $H(t) E_2$ are given by the isomorphisms $H(t)$ and vary continuously. Then with respect to those bases

$$\begin{aligned} &\operatorname{sign} \det\left((I - Q_0) A(\lambda) \{I_0 - [Q_0 A(\lambda)]^{-1} Q_0 A(\lambda)\}\right) \\ &= \operatorname{sign} \det\left((I - Q) A(\lambda) \{I_0 - [Q A(\lambda)]^{-1} Q A(\lambda)\}\right) \\ &= \operatorname{sign} \det \phi_v(\lambda, 0) \end{aligned} \tag{3. 22}$$

where we used (1. 14). By (3. 17) we therefore proved

Theorem 3. 1. *Fix any bases in* $\ker(A_0)$ *and* E_2. *Then*:

(i) *Zero is an eigenvalue of* $A(\lambda)$ *if and only if* $\det \phi_v(\lambda, 0) = 0$.

(ii) *Any change of sign of* $\det(A(\lambda)|_{E_\lambda})$ *implies a change of sign of* $\det \phi_v(\lambda, 0)$ *and vice versa.*

If the orientation of $\ker(A_0)$ *and of* E_2 *is suitably chosen we may state for all* $|\lambda| \leqq \delta_5$:

$$\operatorname{sign} \det(A(\lambda)|_{E_\lambda}) = \operatorname{sign} \det \phi_v(\lambda, 0). \tag{3. 23}$$

In other words: Odd crossing numbers of $A(\lambda) = G_u(\lambda, 0)$ imply odd crossing numbers of any linearized bifurcation function $\phi_v(\lambda, 0)$ and vice versa.

Applying Theorem 3. 1 to the sequences $(\lambda_k^{\pm})$ for some $k \geqq k_0$ we get the local bifurcation result using finite dimensional degree theory for the bifurcation equation $\phi(\lambda, v) = 0$.

Under the additional requirement that $\frac{d}{d\lambda} B(0)$ exists formula (1. 14) gives

$$\phi_{v\lambda}(0, 0) = (I - Q) \frac{d}{d\lambda} B(0) I_0 . \tag{3. 24}$$

If $n = 1$ and $\frac{d}{d\lambda} B(0) \left(\ker(A_0)\right) \not\subset R(A_0)$ then

$$\phi_{v\lambda}(0, 0) \neq 0 \tag{3. 25}$$

which, by Theorem 3. 1, implies a change of sign at $\lambda = 0$ of $\det(A(\lambda)|_{E_\lambda})$. This means that $A(\lambda)$ has an odd crossing number at $\lambda = 0$ and therefore the Crandall-Rabinowitz theorem is embedded into our result.

If n is odd and instead of (0. 12),

$$\frac{d}{d\lambda} A(0)v \notin R(A(0)) \quad \text{for all } v \in \ker(A(0)) \setminus \{0\} \tag{3. 26}$$

then the first nonvanishing derivative of $\det(\phi_v(\lambda, 0))$ is given by

$$\frac{d^n}{d\lambda^n} \det(\phi_v(\lambda, 0)) \neq 0 \tag{3. 27}$$

which, by Theorem 3. 1 again, implies an odd crossing number of $A(\lambda)$ at $\lambda = 0$. The corresponding bifurcation result is due to Westreich [20].

By the hypotheses on the pair (L, A) imposed in [12] and under the assumption that L is a closed operator the family $A(\lambda) = L + (\lambda - \lambda_0) A$ is an isomorphism from D onto E for each $0 < |\lambda| < \delta_6$. From the identity

$$A(\lambda) = (L + (\lambda_1 - \lambda_0) A)\,(I_D + (\lambda - \lambda_1)\,(L + (\lambda_1 - \lambda_0) A)^{-1} A) \tag{3. 28}$$

for some fixed $0 < |\lambda_1| < \delta_6$ we deduce in a similar way as above for small $|\lambda|$:

$$\operatorname{sign} \det(A(\lambda)|_{E_\lambda}) = s_0 \operatorname{sign} \lambda^{\beta(\lambda_0)}, \quad s_0 \in \{-1, 1\}. \tag{3. 29}$$

Here $\beta(\lambda_0)$ denotes the multiplicity of the characteristic value λ_0 for the pair (L, A) as defined in [12]. Obviously $A(\lambda)$ has an odd crossing number at $\lambda = 0$ if $\beta(\lambda_0)$ is odd. This fact embeds the bifurcation result of Laloux and Mawhin into this paper, too.

IV. A degree for a class of proper Fredholm operators and a global bifurcation result

In this section $D \subset E$ are both separable Banach spaces. We define

Definition 4. 1. A linear operator $A\colon D \to E$ is called admissible if it satisfies

(4. 1) A is a Fredholm operator of index zero.

(4. 2) There exist $c > 0$, $\varepsilon > 0$ such that the spectrum $\sigma(A)$ of A in the strip $S_A = (-\infty, c) \times (-i\varepsilon, i\varepsilon)$ consists of finitely many eigenvalues of finite (algebraic) multiplicity.

Definition 4. 2. Let $\Omega \subset D$ be a bounded domain. A map $G\colon \Omega \to E$ is called admissible if

$$G \in C^2(\Omega, E). \tag{4. 3}$$

(4. 4) Its Frechet derivative $DG(u) = G'(u)$ is admissible in the sense of Definition 4. 1 for all $u \in \Omega$.

(4. 5) G is proper, i.e. the inverse image in $\bar{\Omega}$ of any compact set in E is compact in D.

We refer to [1] and [18] for equivalent definitions of properness. In particular we mention that nonlinear Fredholm operators G (in the sense that $G'(u)$ are Fredholm operators for all $u \in \Omega$) are locally proper (see [18]).

For admissible maps G we define a degree in a natural way imitating the finite dimensional case.

We assume in the following that

$$p \notin G(\partial\Omega). \tag{4. 6}$$

By properness $G(\partial\Omega)$ is closed and p has a positive distance to $G(\partial\Omega)$.

Definition 4. 3. *Step* 1. Suppose that p is a regular value of G, i.e. $G'(u)$ is surjective for any $u \in G^{-1}(p)$. By properness $G^{-1}(p) \subset \Omega$ is compact, and by the Fredholm property $G'(u)$ are isomorphisms on $G^{-1}(p)$, which implies that $G^{-1}(p)$ is a finite set. Then define

$$\deg(G, \Omega, p) = \sum_{u \in G^{-1}(p)} i(G, u) \tag{4. 7}$$

where

$$i(G, u) = \operatorname{sign} \prod_{\mu \in \sigma(G'(u)) \cap S_{G'(u)}} \mu. \tag{4. 8}$$

In this product the eigenvalues μ are counted by algebraic multiplicity,

$$\sum_{u \in \emptyset} = 0 \text{ and } \prod_{\mu \in \emptyset} = 1.$$

Step 2. If p is not a regular value we find a sequence $p_n \to p$ in E such that $p_n \notin G(\partial\Omega)$ and all p_n are regular values for G. This is the result of Sard-Smale [18]. Then define

$$\deg(G, \Omega, p) = \lim_{n \to \infty} \deg(G, \Omega, p_n). \tag{4. 9}$$

Obviously it has to be proved that the definition (4. 9) makes sense. If p_0, p_1 are regular values in the same open component of $E \setminus G(\partial\Omega)$, by Smale [18] there exists a C^1-path p: $[0, 1] \to E$ satisfying $p(0) = p_0$, $p(1) = p_1$, $p(t) \notin G(\partial\Omega)$ for all $t \in [0, 1]$, which is transversal to G. Therefore $G^{-1}(p([0, 1]))$ is a compact one-dimensional manifold in Ω with boundary $G^{-1}(p_0) \cup G^{-1}(p_1)$. (Observe that by the properness of G the set of regular values of G is open.) The proof that

$$\deg(G, \Omega, p_0) = \deg(G, \Omega, p_1) \tag{4. 10}$$

is exactly the same as that of the following Theorem 4. 4, choosing $H(t, u) = G(u) - p(t)$, $\bar{p} = 0$. Therefore we postpone it and we assume for the moment the validity of (4. 10) and therefore of (4. 9).

Next we show that this degree for the class of admissible maps has the same properties as the Brouwer or Leray-Schauder degree for their classes of maps, respectively.

Theorem 4. 4. *The degree of Definition* 4. 3 *is invariant under admissible homotopies.*

To be more presice, assume

$$H\colon [0, 1] \times \bar{\Omega} \to E, \; H \in C^2([0, 1] \times \Omega, E), \tag{4. 11}$$

H is proper, $H(t, \cdot)$ is admissible for any $t \in [0, 1]$, $p \notin H(t, \partial\Omega)$ for all $t \in [0, 1]$.

Then, if $H(0, u) = G_0(u)$, $H(1, u) = G_1(u)$,

$$\deg(G_0, \Omega, p) = \deg(G_1, \Omega, p). \tag{4. 12}$$

Proof. Since $H(t, \cdot)$ is admissible for any $t \in [0, 1]$, $D_{(t,u)} H(t, u)$ is a Fredholm operator from $\mathbb{R} \times D$ into E of index 1. By the Sard-Smale-Theorem there exists a $\bar{p}$ in any neighborhood of p such that $\bar{p}$ is a regular value for G_0, G_1 and H. Therefore $H^{-1}(\bar{p})$ is a one-dimensional compact manifold with boundary

$$(0, G_0^{-1}(\bar{p})) \cup (1, G_1^{-1}(\bar{p}))$$

Pick one curve C in $H^{-1}(\bar{p})$ starting in $(0, G_0^{-1}(\bar{p}))$. It ends at a different point in $(0, G_0^{-1}(\bar{p}))$ or at a point in $(1, G_1^{-1}(\bar{p}))$.

As long as $H_u(t, u)$ is injective for $(t, u) \in C$ the index $i(H(t, \cdot), u)$ is constant. By definition (4. 8) this (nonzero) index is locally constant. The global constancy follows from the fact that C is connected.

If $H_u(t, u)$ is not injective for some $(t, u) \in C$ then $\dim \ker(H_u(t, u)) = 1$. Define

$$T = \{(t, u) \in C, \dim \ker(H_u(t, u)) = 1\} \tag{4. 13}$$

which is a compact set. In any point of T the local Theorem 3. 1 applies: Choose a parameterization of the curve C near $(t_0, u_0) \in T$ like $\{(t(\lambda), u(\lambda)), \lambda \in (-1, 1)\}$, $t(0) = t_0$, $u(0) = u_0$, and define $G(\lambda, u) = H(t(\lambda), u(\lambda) + u) - \bar{p}$. Obviously $G(\lambda, 0) = 0$ and $G_u(\lambda, 0) = H_u(t(\lambda), u(\lambda))$.

There are finitely many points $(t_k, u_k) \in T$ such that T is covered by open sets O_k in $(0, 1) \times \Omega$ where Theorem 3. 1 is valid. For the sake of simplicity the complementary spaces of $R(H_u(t_k, u_k))$ in E are chosen to be $E_{2k} = \operatorname{span}\{H_t(t_k, u_k)\}$.

Let $(t_1, u_1) \in O_1 \cap T$ be the first point on C when starting at a boundary point in $(0, G_0^{-1}(\bar{p}))$. Choose an orientation of $\ker(H_u(t_1, u_1))$ and of $\operatorname{span}\{H_t(t_1, u_1)\}$ such that

$$\operatorname{sign} \phi_{1v}(t, v) = \operatorname{sign} \det(H_u(t, u)|_{E_{(t,u)}}) \tag{4. 14}$$

for all $(t, u) \in O_1 \cap C$ (see Theorem 3. 1). Here $v = P_1 u$, where P_1 is a projector onto $\ker(H_u(t_1, u_1))$, and ϕ_1 is the scalar bifurcation function obtained by the method of Lyapunov-Schmidt (see Section II). Finally, $E_{(t,u)}$ denotes the invariant finite-dimensional eigenspace of $H_u(t, u)$ for $(t, u) \in O_1 \cap C$.

Now there are two possibilities: $O_1 \cap O_2 = \emptyset$ or $O_1 \cap O_2 \neq \emptyset$. In the first case we distinguish when at $\partial O_1 \cap C$ the tangent vectors of C point into equal or opposite t-directions. In the first case the index $i(H(t, \cdot), u)$ for $(t, u) \in \partial O_1 \cap C$ is not changed whereas in the second case it has opposite signs. For a proof use formula (4. 14) and observe that the projected plane solution curve of the scalar equation

$$\phi_1(t, v) - (I - Q_1)\bar{p} = 0$$

is oriented like $O_1 \cap C$.

Now let $O_1 \cap O_2 \neq \emptyset$. Then the orientations of $\ker(H_u(t_k, u_k))$ and of $\operatorname{span}\{H_t(t_k, u_k)\}$, $k=1,2$, have to be compatible in the following sense: Pick $(t, u) \in O_1 \cap O_2$ and consider the projections $P_k u$ onto $\ker(H_u(t_k, u_k))$, $k=1,2$. If $(t, u) \notin C$ then $\phi_k(t, P_k u) - (I - Q_k)\bar{p} \neq 0$. If

$$\operatorname{sign}(\phi_1(t, P_1 u) - (I - Q_1)\bar{p}) = \operatorname{sign}(\phi_2(t, P_2 u) - (I - Q_2)\bar{p})$$

and if $(t, P_k u)$ are on the same (right or left) side of the projected plane curves, respectively, then we orient $\mathbb{R} \times \ker(H_u(t_2, u_2))$ in the same way as $\mathbb{R} \times \ker(H_u(t_1, u_1))$. The modifications of the orientation in the three remaining cases is obvious. Thus we guarantee that at points $(t, u) \in O_1 \cup O_2 \cap C$ where the tangent vector of C points into the same nonzero t-direction we preserve the sign of $\phi_{kv}(t, P_k u)$, $k=1,2$.

Continuing this argument, we see in view of Theorem 3.1 that the index $i(H(t,\cdot), u)$ along C behaves like the index of a plane isolated solution curve of $\phi(t, v) = q$ in $[0, 1] \times \mathbb{R}$ joining two points in $\{0\} \times \mathbb{R} \cup \{1\} \times \mathbb{R}$. By considering regions where $\phi(t, v) - q$ is positive and negative it is obvious that the index $\phi_v(t, v)$ is different if both boundary points are in $\{0\} \times \mathbb{R}$ or $\{1\} \times \mathbb{R}$ and that the index is equal for boundary points in $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$.

Since this is true for all (finitely many) curves $C \subset H^{-1}(\bar{p})$ we showed that

$$\deg(G_0, \Omega, \bar{p}) = \deg(G_1, \Omega, \bar{p}). \tag{4.15}$$

Using (4.10) and (4.9), Theorem 4.4 is proved.

Copying the proofs for the Leray-Schauder degree, a homotopy-invariance of this degree for proper maps defined on non-cylindrical domains of $\mathbb{R} \times D$ follows.

Together with the (trivial) additivity-property of our degree these are the tools of the proof of Rabinowitz' [16] global bifurcation result. Therefore we may state:

Theorem 4.5. *Let $G: \mathbb{R} \times D \to E$ be a C^2-map satisfying the following conditions:*

(i) *G is proper on any bounded and closed domain in $\mathbb{R} \times D$.*

(ii) *$G(\lambda, \cdot)$ is admissible for any $\lambda \in \mathbb{R}$.*

Assume that $G(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$, that at some $\lambda_0 \in \mathbb{R}$ the operator $A(\lambda) = G_u(\lambda, 0)$ has an eigenvalue zero, and that $A(\lambda)$ for $0 < |\lambda - \lambda_0| < \delta_7$ has no eigenvalue zero.

If $A(\lambda)$ has an odd crossing number $\chi(\lambda_0)$ at $\lambda = \lambda_0$ (i.e. $A(\tilde{\lambda})$ has an odd crossing number $\chi(0)$ at $\tilde{\lambda} = 0$ for $\tilde{\lambda} = \lambda - \lambda_0$) then $(\lambda_0, 0)$ is an (isolated) bifurcation point for $G(\lambda, u) = 0$.

Call

$$NS = \operatorname{cl}\{(\lambda, u) \in \mathbb{R} \times D,\ G(\lambda, u) = 0,\ u \neq 0\} \tag{4.16}$$

the closure of the nontrivial solution set. Then the component $NS_{(\lambda_0, 0)}$ of NS connected to the bifurcation point $(\lambda_0, 0)$ is either unbounded in $\mathbb{R} \times D$ or $NS_{(\lambda_0, 0)}$ meets a different bifurcation point $(\lambda_1, 0)$.

We finally remark that under our hypotheses the set of possible bifurcation points $(\lambda_1, 0)$ met by $NS_{(\lambda_0, 0)}$ is not necessarily isolated.

V. Stability of bifurcating branches

Let us assume that a continuously parameterized branch $(\lambda(s), u(s))$ bifurcates (after a normalization) from $(\lambda(0), u(0)) = (0, 0)$. When studying the linearized stability of that branch (considered as stationary solutions of (0. 13)) we are led to investigate how the eigenvalue zero of $G_u(0, 0) = A_0$ perturbs for

$$T(s) = G_u(\lambda(s), u(s)). \tag{5. 1}$$

We assume only the conditions (0. 1) to (0. 6) and that $G_u(\lambda, u)$: $D \to E$ exists and is continuous in a full neighborhood of $(0, 0) \in \mathbb{R} \times D$. Then

$$T(s) = A_0 + T_1(s), \quad \lim_{s \to 0} \|T_1(s)\| = 0. \tag{5. 2}$$

As before we decompose ($I_0 =$ identity in $\ker(A_0)$)

$$T(s) - \mu I = \begin{pmatrix} Q(T(s) - \mu I_0) & Q(T(s) - \mu I_{D_2}) \\ (I-Q)(T(s) - \mu I_0) & (I-Q)(T(s) - \mu I_{D_2}) \end{pmatrix}, \tag{5. 3}$$

$$T(s) - \mu I\colon \ker(A_0) \oplus D_2 \to R(A_0) \oplus E_2.$$

Multiplying from the left by the isomorphism (for small $|s|$ and $|\mu|$)

$$\begin{pmatrix} -(I-Q)(T(s) - \mu I_{D_2})[Q(T(s) - \mu I_{D_2})]^{-1} & I_{E_2} \\ [Q(T(s) - \mu I_{D_2})]^{-1} & 0 \end{pmatrix} \tag{5. 4}$$

we end up with

(5. 5)

$$\begin{pmatrix} (I-Q)(T(s) - \mu I_0) - (I-Q)(T(s) - \mu I_{D_2})[Q(T(s) - \mu I_{D_2})]^{-1} Q(T(s) - \mu I_0) & 0 \\ [Q(T(s) - \mu I_{D_2})]^{-1} Q(T(s) - \mu I_0) & I_{D_2} \end{pmatrix}$$

such that the critical eigenvalues of $T(s)$ near zero satisfy the equation

(5. 6)

$$\det((I-Q)(T(s) - \mu I_0) - (I-Q)(T(s) - \mu I_{D_2})[Q(T(s) - \mu I_{D_2})]^{-1} Q(T(s) - \mu I_0)) = 0.$$

The operator in (5. 6) maps $\ker(A_0)$ into E_2 which are both given fixed bases. (The inverse operator appearing in (5. 6) maps $R(A_0)$ onto D_2.)

The projection $Pu(s) = v(s)$ on $\ker(A_0)$ of that branch is a solution of an n-dimensional bifurcation equation

$$\phi(\lambda(s), v(s)) = 0 \tag{5. 7}$$

(see (1. 12)).

The main question in connection with the linearized stability of that branch is the following: Is it possible to determine the sign of the real parts of the critical eigenvalues satisfying (5. 6) when $\phi_v(\lambda(s), v(s))$: $\ker(A_0) \to E_2$ is known (or is known in its lowest term)?

Introducing

$$R(s) = \phi_v(\lambda(s), v(s)) \tag{5. 8}$$

a derivation similar to that in Section I yields

$$R(s) = (I-Q)\,T(s) - (I-Q)\,T(s)\,[QT(s)]^{-1}\,QT(s) \tag{5. 9}$$

where $T(s)$ is given by (5. 1) and $[QT(s)]^{-1}$: $R(A_0) \to D_2$. Obviously $R(0)=0$.

For $s=0$ (5. 6) reduces to

$$\det\left(-\mu(I-Q)\,I_0 - \mu^2(I-Q)\,(A_0-\mu QI_{D_2})^{-1}\,QI_0\right)=0 \tag{5. 10}$$

which, according to formula (3. 9) for $A(\lambda) \equiv A_0 - \mu I$, $P_t \equiv P_0$, $Q_t \equiv Q_0$, and $E_t \equiv E_0$ for all small $|\mu|$, can be computed as follows:

$$\begin{aligned}(-1)^{(m-n)n} \det[Q_0(A_0-\mu I)]^{-1}|_{R(A_0)\cap E_0} \det\left((A_0-\mu I)|_{E_0}\right)\\ = \det\left((I-Q_0)\,\{-\mu I_0 - \mu^2\,[Q_0(A_0-\mu I_{D_2})]^{-1}\,Q_0 I_0\}\right).\end{aligned} \tag{5. 11}$$

Now, using the same arguments as in Section III, the right hand side of (5. 11) differs from (5. 10) by a positive factor, and therefore

$$\begin{aligned}\det\left(-\mu(I-Q)\,I_0 - \mu^2(I-Q)\,(A_0-\mu QI_{D_2})^{-1}\,QI_0\right)\\ = a_m\mu^m\left(1+0(\mu)\right), \quad a_m \neq 0\end{aligned} \tag{5. 12}$$

where, as before, m denotes the algebraic multiplicity of the eigenvalue zero of A_0.

For $\mu=0$ (5. 6) reduces to

$$\det R(s)=0 \tag{5. 13}$$

which follows from (5. 9).

Let us assume that G as well as the given branch depend analytically on λ, u, and s, respectively. Then (5. 6) is an analytic equation for the real variable s and the complex variable μ. The Newton diagram for (5. 6) starts on the μ-axis in m and ends on the s-axis in l if $\det R(s) = r_l s^l + O(s^{l+1})$, $r_l \neq 0$.

Introducing

$$r(\mu) = \mu(I-Q)\,I_0 + \mu^2(I-Q)\,(A_0-\mu QI_{D_2})^{-1}\,QI_0 \tag{5. 14}$$

equation (5. 6) is of the form

$$\det\left(R(s) - r(\mu) + O(\mu s)\right)=0. \tag{5. 15}$$

Observe that in the case $m=n$ and $Q=I-P$, $r(\mu)$ reduces to μI_0.

It would be desirable that the solutions of

$$\det\big(R(s)-r(\mu)\big)=0 \tag{5. 16}$$

would give the correct lowest order terms of the solutions of (5. 15) which are the critical eigenvalues of $T(s)$. Consider, however, the following counterexample:

$G\colon \mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$ is given by

$$G(\lambda, u)=\begin{pmatrix}\lambda & 1\\ 0 & -\lambda^2\end{pmatrix}u-\begin{pmatrix}u_1^2\\ 0\end{pmatrix}. \tag{5 17}$$

Here $n=1$ and $m=2$.

Consider the branch $\big(\lambda(s),\ u(s)\big)=\big(s,\ (s, 0)\big)$. Then

$$T(s)=\begin{pmatrix}-s & 1\\ 0 & -s^2\end{pmatrix} \tag{5. 18}$$

and

$$\mu_1(s)=-s,\quad \mu_2(s)=-s^2 \tag{5. 19}$$

are the critical eigenvalues.

For the Lyapunov-Schmidt reduction choose the projectors

$$P=Q=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \tag{5. 20}$$

Then

$$R(s)=-s^3,\quad r(\mu)=\mu^2 \tag{5. 21}$$

and (5. 16) is solved by

$$\mu_{1,2}(s)=\pm\sqrt{-s^3} \tag{5. 22}$$

which are different from the eigenvalues (5. 19) in lowest order terms. Obviously the Newton diagram for (5. 16) is perturbed by an $O(\mu s)$-term of (5. 15).

Nevertheless a simple sufficient condition for the validity of a "Principle of Reduced Stability" in the sense that equation (5. 16) gives the correct lowest order terms of the critical eigenvalues is given by the following

Theorem 5. 1. *Assume that* $n=1$, *i.e.* $\dim\ker(A_0)=1$, *and that* $R(s)=\phi_v\big(\lambda(s), v(s)\big)$ *is given by*

$$R(s)=sR_1+O(s^2),\ R_1\neq 0. \tag{5. 23}$$

Then the lowest order terms of all m critical eigenvalues of (5. 1) *are precisely*

$$\left(-\frac{R_1}{a_m}\right)^{\frac{1}{m}} s^{\frac{1}{m}} \tag{5. 24}$$

where a_m *is given by* (5. 12). *In particular*

$$\operatorname{sign} a_m=(-1)^{(m-n)n+m}\operatorname{sign}\det(A_0|_{D_2\cap E_0}). \tag{5. 25}$$

Proof. The Newton diagram for (5. 6) starts on the μ-axis in m and ends on the s-axis in 1, thus being the same for

$$R(s)-r(\mu)=sR_1+a_m\mu^m(1+O(\mu))+O(s^2)=0. \tag{5. 26}$$

Formula (5. 25) follows from (5. 11).

We emphasize that Theorem 5. 1 applies for bifurcation at simple eigenvalues in the sense of Crandall and Rabinowitz [3]. We do not need the nondegeneracy condition (0. 12). If it is fulfilled, however, the unique nontrivial branch $(\lambda(s), u(s))$ is parameterized by s which is the real coordinate in the one-dimensional kernel of A_0:

$$v=sv_0,\ v_0\in\ker(A_0),\quad s\in\mathbb{R}. \tag{5. 27}$$

Then (5. 23) is satisfied if for $\lambda(s)$

$$\lambda'(0)\neq 0 \tag{5. 28}$$

holds. Condition (5. 28) is fulfilled if the bifurcation equation (1. 12) contains a quadratic term in v which is quadratic in s, too.

Theorem 5. 1 shows that a branch satisfying (5. 28) is linearly unstable if $m\geqq 3$ or if $m=2$ and $\operatorname{sign} R_1=-\operatorname{sign} a_2$. If $m=1$ then $a_1=-1$ and we regain the Principle of Exchange of Stability (see [9]).

For general n and $R(s)$ we give the following

Theorem 5. 2. *Assume one of the following conditions*:

$$O\big((I-Q)\,T(s)|_{\ker(A_0)}\big)=O\big(QT(s)|_{\ker(A_0)}\big)=k \tag{5. 29}$$

or

$$O\big((I-Q)\,T(s)|_{\ker(A_0)}\big)=O\big((I-Q)\,T(s)|_{A_0^{-1}\,Q(\ker(A_0))}\big)=k. \tag{5. 30}$$

Then

$$R(s)=s^kR_k+O(s^{k+1}), \tag{5. 31}$$

where s^kR_k is also the lowest order term of $(I-Q)\,T(s)|_{\ker(A_0)}$.

If

$$\det R_k\neq 0, \tag{5. 32}$$

then equation

$$\det\big(s^kR_k-r(\mu)\big)=0 \tag{5. 33}$$

gives the lowest order terms of the Puiseux series of the eigenvalue perturbations of $T(s)=G_u(\lambda(s), u(s))$ near zero.

Proof. Both conditions imply that (5. 6) is of the form

$$\det\left(s^k R_k - r(\mu) + O(s^{k+1}) + O(\mu s^k)\right) = 0. \tag{5. 34}$$

Therefore (5. 33) and (5. 34) have the same Newton diagrams.

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Eingegangen 25. Januar 1984, in revidierter Form 17. September 1984