On the History of the Kronecker Product

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History reveals that what is today called the Kronecker product should be called the Zehfuss product.

1. INTRODUCTION

A matrix operation of wide application, defined for matrices $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B}$ of any order to be

$$\mathbf{A} \otimes \mathbf{B} = \{a_{ij}\mathbf{B}\}, \tag{1}$$

is widely referred to as the Kronecker, direct, or tensor product of $\mathbf{A}$ and $\mathbf{B}$. The intriguing history of Kronecker's name being associated with this product is outlined, and the historical development of its properties discussed, thus extending the earlier references of Searle (1966) and Henderson and Searle (1981).

2. THE ZEHFUSS DETERMINANT RESULT

Our story begins with a little-known scientist, Johann Georg Zehfuss, born April 10, 1832 in Darmstadt, who became Privatdozent at the University of Heidelberg in 1857. According to biographical notes by Poggendorff (1863, 1898), Zehfuss published some four or five papers on determinants in the decade 1858–1868, before working in other sciences, including astronomy. In particular, Zehfuss (1858) contains the determinant result

$$|\mathbf{A} \otimes \mathbf{B}| = |\mathbf{A}|^b |\mathbf{B}|^a, \tag{2}$$

for square matrices **A** and **B** of order a and b, respectively. Zehfuss wrote in terms of determinants rather than matrices, following the then customary practice of employing the term 'determinant' both for what we now call a square matrix as well as for its determinant. [The idea that a matrix could have its own identity was barely beginning, in the seminal work of Sylvester (1850) and Cayley (1855, 1858).]

3. MATRIX RESULTS

It took some time for the distinctiveness of matrices to be adopted in a manner that we would now recognize. Concerning (1), Hurwitz (1894, p. 389) uses the symbol $\times$, as a *Producttransformation* of matrices, while Stéphanos (1899a) uses the term *conjunction*. Hurwitz (ibid.) also develops the determinant result (2) and the now very familiar matrix equalities:

$$\begin{gathered} \mathbf{I}_m \otimes \mathbf{I}_n = \mathbf{I}_{mn}, \\ (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}), \\ (\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}, \end{gathered} \tag{3}$$

and

$$(\mathbf{A} \otimes \mathbf{B})' = \mathbf{A}' \otimes \mathbf{B}',$$

where, in these expressions, the necessary rank and conformability conditions for their existence are assumed satisfied, and where $\mathbf{I}_n$ is the identity matrix of order n. In addition, Stéphanos (1899a, b, 1900) formulates eigenvalues of $\mathbf{A} \otimes \mathbf{B}$ as all products of those of **A** and **B**. This provides indirect derivation of the determinant result (2), which also follows from (3), using $|\mathbf{A} \otimes \mathbf{B}| = |\mathbf{A} \otimes \mathbf{I}_b|\,|\mathbf{I}_a \otimes \mathbf{B}|$ and the block diagonal nature of $\mathbf{I}_a \otimes \mathbf{B}$ and the vec-permutation matrices of Henderson and Searle [1981, equation (3)].

4. ZEHFUSS ACKNOWLEDGED

Muir (1881) lists Zehfuss (1858), which contains the determinant result (2), and reviews it in his authoritative history (Muir, 1911, pp. 102–3), where he claims it for Zehfuss and accordingly calls $|\mathbf{A} \otimes \mathbf{B}|$ the Zehfuss determinant of **A** and **B**.

Others, notably Rutherford (1933), and Aitken (1935) and his student Ledermann (1936), following Muir's lead, have gone further and called $\mathbf{A} \otimes \mathbf{B}$ the Zehfuss matrix of **A** and **B**. Aitken's adoption of this name is of interest in light of Ledermann's later (1968) comment that Aitken "was particularly fond of stressing the claims of lesser known mathematicians of former times for discoveries erroneously attributed to their more famous contemporaries. Many of his historical references were gleaned from Sir Thomas Muir's monumental work on determinants, for which Aitken had a profound admiration." More recently, Henderson and Searle (1981) is the sole source which cites Zehfuss (1858) for (2), according to *Science Citation Index* since its inception in 1961. But, generally speaking, the name of Zehfuss has been forgotten in this connection.

5. HENSEL'S CLAIM FOR KRONECKER

In contrast, the name of Kronecker (1823–91) has long been associated with the $\otimes$ operation of (1), and with the determinant result (2). This association originated with Hensel[1] (1889, 1891) who, in presenting (2), notes that Kronecker had for some time given the result and a proof in algebra lectures, presumably during Hensel's student days (1880–1884) in Berlin. (Hensel completed his dissertation in 1884 and joined the department in 1886). According to Hasse (1950, p. 2), Kronecker's lectures were very difficult, so much so that the Russian Seliwanoff, one of the circle of young mathematicians in Berlin, recounts "Und wenn die Vorlesung aus ist, wirr rrufen alle 'wunderrvoll' und habben nicht verrstanden." ("And when the lecture

[1] Hensel can also be remembered for the spoonerism (sic) cited by Hasse (1950): "In diesem kleinen Gartenhaus Bezwing' ich selbst den harten Gauss." ("This is so dear a garden house That here I master even Gauss.")

was over, we all exclaimed 'Wonderful' but have not understood a thing.") Hasse (1950) considers it was probably Hensel who got the most from Kronecker's lectures, and indeed it was Hensel who edited publication of Kronecker's lectures and of his collected works. In particular, Kronecker's (1903) *Vorlesungen über die Theorie der Determinanten, Band I*, consists of 21 lectures that were part of a university course on *Allgemeine Arithmetik* belonging to the period 1883–1891. Although Hensel announced this work to be only a first volume, subsequent volumes did not appear, nor did *Vorlesungen über die Theorie der algebraischen Gleichungen* announced by the publisher on the last page of the published work; see the list of Hensel's editorial work in Hasse (1950, p. 11).

Bourbaki (1958, p. 151) states that the Kronecker product appears in a nonintrinsic form in Kronecker's (1903) *Vorlesungen*. The closest evidence for this appears to be equation (3b) on page 90, where scalar equations for the elements of $\mathscr{A} = \mathbf{T}'\mathbf{A}\mathbf{S}$ are presented. These could be (but are not) assembled in the form $\boldsymbol{\alpha} = (\mathbf{T} \otimes \mathbf{S})'\mathbf{a}$ where $\boldsymbol{\alpha}$ and $\mathbf{a}$ are column vectors of the elements in lexicon order of $\mathscr{A}$ and $\mathbf{A}$, respectively.

Kronecker's presentation in no way indicates, in our opinion, any perception of the product property (1) or its consequences in (3). Whereas Kronecker (1903) deals only with 2×2 matrices in Lecture 6, he discusses the general case in later chapters. For example, at the start of Lecture 20 he notes that the general case has applications to "the theory of bilinear and quadratic forms". However, these applications do not appear in volume 1 and presumably were scheduled for volume 2 which never appeared.

Nowhere else do we find any reference to Kronecker's written work that would associate his name with the product (1), not even in his collected works edited by Hensel and originally published in five volumes (1895, 1897, 1899, 1929 and 1930) in Leipzig and republished by Chelsea Publishing Company (Hensel, 1968). All that remains, as we see it, is the acknowledgment and testimony in Hensel (1889, 1891) that Kronecker presented (2) in his lectures.

6. CLAIM AND COUNTER-CLAIM

Numerous other writers of the late 1800's also developed the determinant result (2), all without reference to Zehfuss (1858). First comes Rados (1886), believing his work (using Grassmann's theory) to be original. Although Hensel (1889) attributed (2) solely to Kronecker, he does refer to Rados (1886) in Hensel (1891). But this was apparently not enough for Rados because, almost a decade later, Rados (1900) claims the result for himself, questioning Hensel's claim for Kronecker since he had found no trace of the result in a notebook of Kronecker's course, which Kronecker had himself reviewed. For Muir (1927, p. 259) this apparent lack of evidence is conclusive; he reports that Rados (1900) refers "to Hensel's claim for Kronecker and effectively disposes of it". But maybe Muir is overzealous in disposing of Kronecker, to the extent that he seems to deny the possibility that Kronecker developed the result at all.

It seems reasonable to accept, without doubt, Hensel's testimony that Kronecker presented (2) in lectures. But was Kronecker indebted to Zehfuss? Kronecker, it will be recalled, completed his dissertation in 1845 and until 1853 was a successful business man, although he maintained a lively scientific correspondence with his former master, Kummer. Further, as Bell (1937, p. 478) notes, "from 1861 to 1883 Kronecker [as a member of the Berlin Academy] conducted regular courses at the university, principally on his personal researches, after the necessary introductions. In 1883 Kummer . . . retired, and Kronecker succeeded his old master as ordinary professor" until his death in 1891. Kronecker's interest in determinants was long-standing, according to Frobenius (1893, p. 722), and even in his early scientific days he had dealt with determinants. He may have read the short note by Zehfuss (1858) and noticed the result, but it would seem that he derived the result independently, since at least his method of proof, noted by Hensel (1891) as an elegant reformulation of the determinants using multiplicative rules, seems to differ from that of Zehfuss.

7. THE NEGLECT OF ZEHFUSS AND ASSOCIATION OF KRONECKER

The claim made by Rados (1900) did not appear to affect the growing association of Kronecker with the determinant result (2). This association which, as we have seen, originated with Hensel (1889, 1891), was referred to by Netto (1893, 1898) and was included in popular text-books of the day. Evidence of this is Muir's lament (1927, p. 259) in reporting Rados (1900) that "the text-book of Scott and Mathews (1904, p. 72), which appeared four years after the publication of Rados' paper, gave new life to the old error. This was probably due to the teaching of Pascal, whose second edition (1923) still propagates the error" of the first edition (1897). In addition, Loewy's (1903) extensions of Hurwitz (1894) follow Netto's (1898) reference to Kronecker and later, in *Pascal's Repertorium*, Loewy (1910) refers without discussion to (2) as Kronecker's theorem, which is surprising since he cites relevant works of both Hensel and Rados. Following on from this, MacDuffee (1933) cites Loewy (1910), Hensel (1891) and Netto (1893), but nowhere mentions Kronecker in this regard. However, this is less surprising as MacDuffee uses the term direct product and quotes group theory references as the motivation.

Other writers of the 1890's, such as Igel (1892), Mertens (1893), Escherich (1892), Hurwitz (1894), Sterneck (1895) and Stéphanos (1899a, b, 1900), also developed (2), and Metzler (1899), Moore (1900) and Petr (1906) effected certain generalizations thereto—all of them, without reference to Zehfuss which, as Muir (1923, p. 49) points out in reporting Igel (1892) and Escherich (1892), was a neglect that "not a few subsequent writers were equally guilty of".

An early example of making the easy step from associating the Kronecker name with (2) to attaching it to the matrix operation (1), which is involved in (2), is Murnaghan (1938, p. 68). Thus "Kronecker product" for (1) is now widely used: e.g., Vartak (1955), Cornish (1957), Shah (1959), Searle (1966), Neudecker (1968, 1969) and Graybill (1969) in the context of statistics; Barnett (1979) in engineering, and Jacobson (1953, 1964), Halmos (1958) and Bellman (1970) in algebra texts. Thus today, the name Kronecker product is widely associated with the $\otimes$ operation of (1), with little thought as to its historical validity. It seems to us that Muir's (1911) claim of priority for Zehfuss demands that if any name be attached to the $\otimes$ operation it should be Zehfuss and not Kronecker—unless history can reveal Kronecker's priority.

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References

[1] A. C. Aitken, The normal form of compound and induced matrices, *Proceedings of the London Mathematical Society* **38** series 2 (1935), 354–376.

[2] S. Barnett, *Matrix Methods for Engineers and Scientists*, McGraw-Hill, U.K. (1979).

[3] E. T. Bell, *Men of Mathematics*, Simon and Schuster, New York (1937).
[4] R. Bellman, *Introduction to Matrix Analysis*, Second edition, McGraw-Hill, New York (1970).
[5] N. Bourbaki, *Algèbre Multilinéaire*, Livre II, Chapitre 3, Hermann, Paris (1958).
[6] A. Cayley, Sept différents mémoires d'analyse. No. 3. Remarque sur la notation des fonctions algébriques, *Journal für die reine und angewandte Mathematik* **50** (1855), 282–285.
[7] A. Cayley, A memoir on the theory of matrices, *Philosophical Transactions of the Royal Society of London* **148** (1858), 17–37.
[8] E. A. Cornish, An application of the Kronecker product of matrices in multiple regression, *Biometrics* **13** (1957), 19–27.
[9] G. von Escherich, Über einige Determinanten, *Monatschefte für Mathematik und Physik* **3** (1892), 68–80.
[10] F. G. Frobenius, Gedächtnisrede auf Leopold Kronecker (1893), *Ferdinand Georg Frobenius. Gesammelte Abhandlungen*, ed. J. S. Serre, Springer-Verlag, Berlin (1968), Band III, pp. 705–724.
[11] F. A. Graybill, *Introduction to Matrices with Applications in Statistics*, Wadsworth, Belmont, California (1969).
[12] P. R. Halmos, *Finite-Dimensional Vector Spaces*, Second edition, D. Van Nostrand Company, Princeton, New Jersey (1958).
[13] H. Hasse, Kurt Hensel zum Gedächtnis, *Journal für die reine und angewandte Mathematik* **187** (1950), 1–13.
[14] H. V. Henderson and S. R. Searle, The vec-permutation matrix, the vec operator and Kronecker products: A review, *Linear and Multilinear Algebra* **9** (1981), 271–288.
[15] K. Hensel, Über Gattungen, welche durch Composition aus zwei anderen Gattungen entstehen, *Journal für die reine und angewandte Mathematik* **105** (1889), 329–344.
[16] K. Hensel, Über die Darstellung der Determinante eines Systems, welches aus zwei anderen componirt ist, *Acta Mathematica* **14** (1891), 317–319.
[17] K. Hensel, *Leopold Kronecker's Werke*, Chelsea Publishing Company, New York (1968).
[18] A. Hurwitz, Zur Invariantentheorie, *Mathematische Annalen* **45** (1894), 381–404.
[19] B. Igel, Zur Theorie der Determinanten, *Monatschefte für Mathematik und Physik* **3** (1892), 55–68.
[20] N. Jacobson, *Lectures in Abstract Algebra, Volumes II and III*, D. Van Nostrand, New York (1953, 1964).
[21] L. Kronecker, *Vorlesungen über die Theorie der Determinanten, Erster Band*, Bearbeitet und fortgeführt von K. Hensel, B. G. Teubner, Leipzig (1903).
[22] W. Ledermann, On singular pencils of Zehfuss, compound and Schläflian matrices, *Proceedings of the Royal Society of Edinburgh* **56** (1936), 50–89.
[23] W. Ledermann, A. C. Aitken's work in pure mathematics, *Proceedings of the Edinburgh Mathematics Society* **16** Series 2 (1968), 162–165.
[24] A. Loewy, Zur Gruppentheorie mit Anwendungen auf die Theorie der linearen homogenen Differentialgleichungen, *Transactions of the American Mathematical Society* **5** (1903), 61–80.
[25] A. Loewy, Kombinatorik, Determinanten und Matrices, *E. Pascal, Repertorium der Höheren Mathematik, Erster Band, erste Hälfte*, Zweite Auflage (Hrsg. P. Epstein) (1910).
[26] C. C. MacDuffee, *The Theory of Matrices*, Julius Springer, Berlin (1933).
[27] F. Mertens, Über ganze Functionen von m Systemen von je n Unbestimmten, *Monatschefte für Mathematik und Physik* **4** (1893), 193–228 and 297–329.
[28] W. H. Metzler, On a determinant each of whose elements is a product of k factors, *American Mathematical Monthly* **7** (1899), 151–153.

[29] E. H. Moore, A fundamental remark concerning determinantal notations with the evaluation of an important determinant of special form, *Annals of Mathematics* **2** (1900), 177–188.

[30] T. Muir, A list of writings on determinants, *Quarterly Journal of Mathematics* **18** (1881), 110–149.

[31] T. Muir, *The Theory of Determinants in the Historical Order of Development, Volumes 1–4*, Dover, New York (1890, 1911, 1920, 1923; reprinted 1960).

[32] T. Muir, The theory of orthogonants and latent roots from 1881 to 1918, *Proceedings of the Royal Society of Edinburgh* **47** (1927), 252–282.

[33] F. D. Murnaghan, *The Theory of Group Representations*, Johns Hopkins Press, Baltimore (1938).

[34] E. Netto, Zwei Determinantensätze, *Acta Mathematica* **17** (1893), 199–204.

[35] E. Netto, Kombinatorik, *Encyklopädie der Mathematischen Wissenschaften, mit Einschluss ihrer Anwendungen, Erster Band, erster Teil*, B. G. Teubner, Leipzig, (1898), pp. 29–46.

[36] H. Neudecker, The Kronecker matrix product and some of its applications in econometrics, *Statistica Neerlandica* **22** (1968), 69–81.

[37] H. Neudecker, Some theorems on matrix differentiation with special reference to Kronecker matrix products, *Journal of the American Statistical Association* **64** (1969), 953–963.

[38] E. Pascal, *I Determinanti: teoria ed applicazioni: con tutte le più recente ricerch*, Hoepli, Milano (1897).

[39] K. Petr, Einige Bemerkungen über die Determinanten, *Mathematische und Naturwissenschaftliche Berichte aus Ungarn* **25** (1906), 95–105.

[40] J. C. Poggendorff, *Biographisch-Literarisches Handwörterbuch zur Geschichte der Exacten Wissenschaften*, Zweiter Band, Dritter Band, J. A. Barth, Leipzig (1863, 1898).

[41] G. Rados, Zur Theorie der Determinanten, Math. és természelt. *Ertesitö* **4** (1886), 268–271. German translation in *Mathematische und Naturwissenschaftliche Berichte aus Ungarn* **8** (1891), 60–64.

[42] G. Rados, Notes sur les substitutions orthogonales, *Mathematische und Naturwissenschaftliche Berichte aus Ungarn* **18** (1900), 231–235 and 426.

[43] D. E. Rutherford, On the condition that two Zehfuss matrices be equal, *Bulletin of the American Mathematical Society* **39** (1933), 801–808.

[44] R. F. Scott, *The Theory of Determinants and Their Applications*, 2nd Ed., revised by G. B. Mathews, Cambridge University Press (1904).

[45] S. R. Searle, *Matrix Algebra for the Biological Sciences*, John Wiley and Sons, New York (1966).

[46] B. V. Shah, On a generalisation of the Kronecker product designs, *Annals of Mathematical Statistics* **30** (1959), 48–54.

[47] C. Stéphanos, Sur un mode de composition des déterminants et des formes bilinéaires, *Giornale di Matematiche di Battaglini* **36** (1899a), 376–379.

[48] C. Stéphanos, Sur une extension du calcul des substitutions linéaires, *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences (Paris)* **128** (1899b), 593–596.

[49] C. Stéphanos, Sur une extension du calcul des substitutions linéaires, *Journal de Mathématiques pures et appliqúes*, 5 Série, **6** (1900), 73–128.

[50] R. D. von Sterneck, Beweis eines Satzes über Determinanten, *Monatschefte für Mathematik und Physik* **6** (1895), 205–207.

[51] J. J. Sylvester, Additions to the articles, "On a new class of theorems", and "On Pascal's theorem", *Philosophical Magazine* **37** (1850), 363–370.

[52] N. N. Vartak, On an application of Kronecker product of matrices to statistical designs, *Annals of Mathematical Statistics* **26** (1955), 420–438.

[53] G. Zehfuss, Über eine gewisse Determinante, *Zeitschrift für Mathematik und Physik* **3** (1858), 298–301.