Some Properties of Matrix Partial Orderings

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ABSTRACT

The matrix partial orderings considered are: (1) the star ordering and (2) the minus ordering or rank subtractivity, both in the set of $m \times n$ complex matrices, and (3) the Löwner ordering, in the set of $m \times m$ matrices. The problems discussed are: (1) inheriting certain properties under a given ordering, (2) preserving an ordering under some matrix multiplications, (3) relationships between an ordering among direct (or Kronecker) and Hadamard products and the corresponding orderings between the factors involved, (4) orderings between generalized inverses of a given matrix, and (5) preserving or reversing a given ordering under generalized inversions. Several generalizations of results known in the literature and a number of new results are derived.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}_{m,n}$ stand for the set of $m \times n$ complex matrices. Given $\mathbf{A} \in \mathbb{C}_{m,n}$, the symbols $\mathbf{A}^*$, $\mathscr{R}(\mathbf{A})$, $\mathrm{r}(\mathbf{A})$, and $\sigma(\mathbf{A})$ will denote the conjugate transpose, range, rank, and set of all nonzero singular values, respectively, of $\mathbf{A}$. Further, $\mathbf{A}\{1\}$ and $\mathbf{A}\{2\}$ will denote the sets of all inner and outer inverses of $\mathbf{A}$, specified as

$$\mathbf{A}\{1\} = \{\mathbf{X} \in \mathbb{C}_{n,m} : \mathbf{AXA} = \mathbf{A}\} \tag{1.1}$$

and

$$\mathbf{A}\{2\} = \{\mathbf{X} \in \mathbb{C}_{n,m} : \mathbf{XAX} = \mathbf{X}\}, \tag{1.2}$$

while $\mathbf{A}\{3\}$ and $\mathbf{A}\{4\}$ will denote the sets of all right and left symmetrizers of $\mathbf{A}$ (cf. Baksalary and Kala [6]), specified as

$$\mathbf{A}\{3\} = \{\mathbf{X} \in \mathbb{C}_{n,m} : \mathbf{AX} = (\mathbf{AX})^*\} \tag{1.3}$$

and

$$\mathbf{A}\{4\} = \{\mathbf{X} \in \mathbb{C}_{n,m} : \mathbf{XA} = (\mathbf{XA})^*\}. \tag{1.4}$$

Various intersections of the sets from among (1.1) through (1.4), denoted according to the convention $\mathbf{A}\{i_1,\dots,i_k\} = \mathbf{A}\{i_1\} \cap \cdots \cap \mathbf{A}(i_k\}$, constitute the well-known classes of generalized inverses of $\mathbf{A}$; cf. Ben-Israel and Greville [7], Rao and Mitra [29], Styan [33]. In particular, $\mathbf{A}\{1,2\}$ is the class of all reflexive generalized inverses of $\mathbf{A}$, and the unique member of $\mathbf{A}\{1,2,3,4\}$ is the Moore-Penrose inverse of $\mathbf{A}$, henceforth denoted by $\mathbf{A}^+$.

The star partial ordering $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$, the minus partial ordering or rank subtractivity $\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B}$, the space preordering $\mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B}$, and the singular-values preordering $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$ in $\mathbb{C}_{m,n}$, are defined as follows:

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B} \text{ and } \mathbf{AA}^* = \mathbf{BA}^*, \tag{1.5}$$

$$\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^-\mathbf{A} = \mathbf{A}^-\mathbf{B} \text{ and } \mathbf{AA}^= = \mathbf{BA}^= \quad \text{for some } \mathbf{A}^-, \mathbf{A}^= \in \mathbf{A}\{1\}, \tag{1.6}$$

$$\mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B} \quad \Leftrightarrow \quad \mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B}) \text{ and } \mathscr{R}(\mathbf{A}^*) \subseteq \mathscr{R}(\mathbf{B}^*), \tag{1.7}$$

$$\mathbf{A} \overset{\sigma}{\prec} \mathbf{B} \quad \Leftrightarrow \quad \sigma(\mathbf{A}) \subseteq \sigma(\mathbf{B}). \tag{1.8}$$

The star ordering (1.5) is due to Drazin [11, 12]. Matrices $\mathbf{A}$ and $\mathbf{B}$ satisfying (1.5) were, however, also considered earlier by Hestenes [21, Lemma 3.4]. It was pointed out by Drazin [12] that

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^{+}\mathbf{A} = \mathbf{B}^{+}\mathbf{A} \text{ and } \mathbf{A}\mathbf{A}^{+} = \mathbf{A}\mathbf{B}^{+}, \tag{1.9}$$

as well as

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^{+}\mathbf{A} = \mathbf{A}^{+}\mathbf{B} \text{ and } \mathbf{A}\mathbf{A}^{+} = \mathbf{B}\mathbf{A}^{+}. \tag{1.10}$$

These characterizations are easily seen to be equivalent to

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}\mathbf{A}^{+}\mathbf{B} = \mathbf{A} = \mathbf{B}\mathbf{A}^{+}\mathbf{A} \quad \Leftrightarrow \quad \mathbf{B}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+} = \mathbf{A}^{+}\mathbf{A}\mathbf{B}; \tag{1.11}$$

cf. Hartwig [14].

Hartwig [15] proved that (1.6), with both $\mathbf{A}^{-}$ and $\mathbf{A}^{=}$ replaced by one and the same reflexive generalized inverse of $\mathbf{A}$, defines a partial ordering relation, and called it "plus ordering." Hartwig and Luh [16] and Hartwig and Styan [18] noted that the reflexiveness and identity of generalized inverses in the two equalities in (1.6) are immaterial, and adopted the term "minus ordering." Moreover, Hartwig [15] showed that (1.6) is equivalent to

$$\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathrm{r}(\mathbf{B} - \mathbf{A}) = \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{A}). \tag{1.12}$$

In view of Marsaglia and Styan [24, p. 288] and Cline and Funderlic [10, p. 195], (1.12) may alternatively be expressed as

$$\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{B}\mathbf{B}^{-}\mathbf{A} = \mathbf{A}\mathbf{B}^{=}\mathbf{B} = \mathbf{A}\mathbf{B}^{\equiv}\mathbf{A} = \mathbf{A} \quad \text{for some } \mathbf{B}^{-}, \mathbf{B}^{=}, \mathbf{B}^{\equiv} \in \mathbf{B}\{1\}. \tag{1.13}$$

It is clear that each of the relations $\mathbf{A} \overset{s}{\prec} \mathbf{B}$ and $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$, defined in (1.7) and (1.8), is reflexive and transitive but not antisymmetric, and therefore (cf. Marshall and Olkin [25, p. 13]) constitutes a preordering of $\mathbb{C}_{m,n}$. It is well known that

$$\mathbf{A} \overset{s}{\prec} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{B}\mathbf{B}^{-}\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{B}^{=}\mathbf{B} \quad \text{for some } \mathbf{B}^{-}, \mathbf{B}^{=} \in \mathbf{B}\{1\}. \tag{1.14}$$

Also, it may be pointed out that the space preordering $\mathbf{A} \overset{s}{\prec} \mathbf{B}$ entails the invariance of $\mathbf{AB}^{-}\mathbf{A}$ with respect to the choice of $\mathbf{B}^{-} \in \mathbf{B}\{1\}$, and that the reverse implication holds whenever both $\mathbf{A}$ and $\mathbf{B}$ are nonzero; cf. Rao and Mitra [29, pp. 21 and 43]. See also Hartwig [13] and a recent discussion on invariance properties by Carlson [8].

From (1.6), (1.10), (1.13), and (1.14) it is seen that

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A} \overset{s}{\prec} \mathbf{B}. \tag{1.15}$$

Several characterizations of the star ordering through supplementing rank subtractivity by one or more suitable extra conditions were recently given by Hartwig and Styan [18]; see also Baksalary [2] for an extension of a part of their Theorem 2. Further, from (1.13) and (1.14), it follows that

$$\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A} \overset{s}{\prec} \mathbf{B} \text{ and } \mathbf{A}\{1\} \cap \mathbf{B}\{1\} \neq \varnothing,$$

while Mitra [27, Theorem 2.1] showed that

$$\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{B}\{1\} \subseteq \mathbf{A}\{1\}.$$

Combining this result with Theorem 1 of Sambamurty [31] yields

$$\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{B}\{1,2\} \subseteq \mathbf{A}\{1\}.$$

On the other hand, from Theorem 2 of Hartwig and Styan [18], it is clear that

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A} \overset{\sigma}{\prec} \mathbf{B}. \tag{1.16}$$

The first part of (1.15) and the implication (1.16) motivated Baksalary and Hauke [4] to investigate the partial ordering defined as the conjunction of the minus partial ordering $\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B}$ and the singular-values preordering $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$.

The Löwner partial ordering $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$ in $\mathbb{C}_{m,m}$ may be defined by

$$\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{B} - \mathbf{A} = \mathbf{KK}^* \quad \text{for some } \mathbf{K}. \tag{1.17}$$

The ordering (1.17), due to Löwner [23, p. 177], is usually considered when both the matrices $\mathbf{A}$ and $\mathbf{B}$ are Hermitian or even both Hermitian nonnegative definite. This is, however, not necessary and, as in Hartwig and Styan [19], will not, in general, be assumed in the present paper.

There is no known relationship between $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$ and any of $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$, $\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B}$, $\mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B}$, and $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$ when $\mathbf{A}$ and $\mathbf{B}$ may vary over the entire $\mathbb{C}_{m,m}$. However, Hartwig and Styan [19, Theorems 2.1 and 2.2] proved that

$$\mathbf{A} = \mathbf{A}^*, \quad \mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{B}, \text{ and } \mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \tag{1.18}$$

and

$$\mathbf{A} = \mathbf{A}^2, \quad \mathbf{B} = \mathbf{B}^2, \text{ and } \mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A} \overset{*}{\leqslant} \mathbf{B}.$$

Also, it is known (cf. Baksalary and Hauke [3, p. 35]) that

$$\mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B}. \tag{1.19}$$

The implications (1.18) and (1.19) may be strengthened to the equivalence

$$\mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{A}^*, \quad \mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{B}, \quad \mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B}, \text{ and } \mathbf{A}\mathbf{B}^-\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{A} \tag{1.20}$$

for some (and hence all) $\mathbf{B}^- \in \mathbf{B}\{1\}$, which follows by applying Theorem 1 of Albert [1] to the matrices

$$\begin{pmatrix} \mathbf{A} & \mathbf{A} \\ \mathbf{A} & \mathbf{B} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{A} & \mathbf{A} \end{pmatrix};$$

see also Hartwig [14, Lemma 1]. Comparing (1.20) with

$$\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B} \text{ and } \mathbf{A}\mathbf{B}^-\mathbf{A} = \mathbf{A} \quad \text{for some (and hence all) } \mathbf{B}^- \in \mathbf{B}\{1\} \tag{1.21}$$

reveals an essential difference between the minus and Löwner partial orderings. Finally, for solutions to the problem of characterizing $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ via supplementing $\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B}$ by a suitable extra condition, when $\mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{A}$ and $\mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$, the reader is referred to Corollary 1(d) of Hartwig and Styan [18].

The purpose of this paper is to examine various properties of the star, minus, and Löwner partial orderings. Section 2 is concerned with the problem of inheriting certain characteristics under these orderings, with the problem of preserving a given ordering under some matrix multiplications, and also with the problem of establishing relationships between orderings of direct (or Kronecker) and Hadamard products and analogous orderings of the factors involved in them. Sections 3 and 4 deal with generalized inverses of matrices: the former in the context of orderings between generalized inverses of a given matrix, and the latter in the context of preserving or reversing a given ordering under generalized inversions. Several generalizations of the results known in the literature and a number of new results are derived.

2. GENERAL PROPERTIES

In the first part of this section, we collect together various results concerned with inheriting some properties of matrices under the partial orderings and preorderings considered, in the sense that if a matrix has a certain property, then all its predecessors have it as well.

THEOREM 2.1. *For $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$, the following inheritance properties hold*:

(a) $\mathbf{B}^{+} = \mathbf{B}^{*}$ *and* $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B} \Rightarrow \mathbf{A}^{+} = \mathbf{A}^{*}$,

(b) $\mathbf{B}\mathbf{B}^{*} \overset{\mathrm{L}}{\leqslant} \mathbf{I}_m$ *and* $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B} \Rightarrow \mathbf{A}\mathbf{A}^{*} \overset{\mathrm{L}}{\leqslant} \mathbf{I}_m$.

For $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,m}$, *the following inheritance properties hold*:

(c) $\mathbf{B}^2 = \mathbf{0}$ *and* $\mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B} \Rightarrow \mathbf{A}^2 = \mathbf{0}$,

(d) $\mathbf{B} = \mathbf{B}^{*} = \mathbf{B}^3$, $\mathbf{A} = \mathbf{A}^{*}$, *and* $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B} \Rightarrow \mathbf{A} = \mathbf{A}^3$,

(e) $\mathbf{B} = \mathbf{B}^2$ *and* $\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \Rightarrow \mathbf{A} = \mathbf{A}^2$,

(f) $\mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$, $\mathbf{A} = \mathbf{A}^{*}$, *and* $\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \Rightarrow \mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{A}$,

(g) $\mathbf{B} = \mathbf{B}\mathbf{B}^{*}$, $\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B}$, *and* $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B} \Rightarrow \mathbf{A} = \mathbf{A}\mathbf{A}^{*}$,

(h) $\mathbf{B}^{*}\mathbf{B}^{+} = \mathbf{B}^{+}\mathbf{B}^{*}$ *and* $\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \Rightarrow \mathbf{A}^{*}\mathbf{A}^{+} = \mathbf{A}^{+}\mathbf{A}^{*}$.

The results (a) and (b), concerning partial isometries and contractions, respectively, were given by Baksalary and Hauke [4, Theorem 1]. In view of (1.16), they strengthen the corresponding results in which the singular-values preordering $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$ is replaced by the star ordering $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$, as in Theorem 3 of Drazin [12] and Lemma 2 of Hartwig and Spindelböck [17]. On the other hand, the matrices

$$\mathbf{A}=\begin{pmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}, \qquad \mathbf{B}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

show that $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$ cannot be replaced by $\mathbf{A} \overset{\text{rs}}{\leqslant} \mathbf{B}$ or $\mathbf{A} \overset{\text{L}}{\leqslant} \mathbf{B}$.

The result (c), stating that nilpotency is inherited under the space preordering, follows by noting that if $\mathbf{B}^2 = \mathbf{0}$ and $\mathbf{A} \overset{\text{s}}{\prec} \mathbf{B}$, then (1.14) yields

$$\mathbf{A}^2 = \mathbf{A}\mathbf{B}^- \mathbf{B}\mathbf{A} = \mathbf{A}\mathbf{B}^- \mathbf{B}^2\mathbf{B}^- \mathbf{A} = \mathbf{0}.$$

In view of (1.15), this result strengthens Proposition 1(ii) of Hartwig [15], in which $\mathbf{A} \overset{\text{s}}{\prec} \mathbf{B}$ is replaced by $\mathbf{A} \overset{\text{rs}}{\leqslant} \mathbf{B}$.

Since a Hermitian matrix is tripotent if and only if it is a partial isometry, the property (d) is an immediate consequence of (a); cf. Corollary 1 in Baksalary and Hauke [4]. Again, the matrices in (2.1) show that $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$ cannot be replaced by $\mathbf{A} \overset{\text{rs}}{\leqslant} \mathbf{B}$ or $\mathbf{A} \overset{\text{L}}{\leqslant} \mathbf{B}$. Moreover, the matrices

$$\mathbf{A}=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad \mathbf{B}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

show that for non-Hermitian matrices, tripotency need not be inherited even when $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$ is strengthened to $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$. [Notice, parenthetically, that the matrices in (2.2) also show that if $\mathbf{B}$ is Hermitian, normal, or EP, then the star-predecessors of $\mathbf{B}$ need not have the same property.]

The property (e) is a particular case of Proposition 1.8(a) of Chipman and Rao [9]. In view of (1.15), it clearly implies that idempotency is inherited under the star ordering; cf. Drazin [12, Theorem 3] and Hartwig and Spindelböck [17, Lemma 2]. See also Theorem 3.1 in Hartwig and Styan [19] and a version of (e) in Hartwig and Styan [18, p. 159].

The result (f) is an immediate consequence of (1.21). In view of (1.18), the right-hand side of (f) may actually be extended to $\mathbf{0} \overset{\text{L}}{\leqslant} \mathbf{A} \overset{\text{L}}{\leqslant} \mathbf{B}$.

The result (g), concerning orthogonal projectors, was given by Baksalary and Hauke [4, Theorem 1]. In view of (1.15) and (1.16), it strengthens the corresponding result in which the partial ordering defined by $\mathbf{A} \overset{\text{rs}}{\leqslant} \mathbf{B}$ and $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$ is replaced by the star ordering $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$, as in Theorem 3 of Drazin [12] and Lemma 2 of Hartwig and Spindelböck [17]. On the other hand, taking any non-Hermitian idempotent $\mathbf{A}$ together with $\mathbf{B}$ being the identity matrix shows that the condition $\mathbf{A} \overset{\sigma}{\prec} \mathbf{B}$ on the left-hand side of (g) cannot be dropped. See also Theorem 3.2 in Hartwig and Styan [19].

Finally, the result (h) was pointed out by Drazin [11, p. 58]. Its proof is obtained by noting that, in view of (1.11),

$$\mathbf{A}^*\mathbf{A}^+ = \mathbf{A}^+\mathbf{A}\mathbf{B}^*\mathbf{B}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+\mathbf{A}\mathbf{B}^+\mathbf{B}^*\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+\mathbf{A}^*.$$

Since the property $\mathbf{A}^*\mathbf{A}^+ = \mathbf{A}^+\mathbf{A}^*$ is equivalent to $\mathbf{A}\mathbf{A}^*\mathbf{A}^+\mathbf{A} = \mathbf{A}\mathbf{A}^+\mathbf{A}^*\mathbf{A}$, it follows that it is trivially fulfilled for all partial isometries and normal matrices and also when $\mathrm{r}(\mathbf{A}) = 1$.

It is obvious that the star, minus, and Löwner partial orderings, as well as the space and singular-values preorderings, are all preserved under conjugate transposition of the matrices involved. Further, as pointed out in Baksalary and Hauke [4, p. 21], the following properties can easily be verified:

THEOREM 2.2. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$, *and let* $\overset{?}{\leqslant}$ *stand for* $\overset{*}{\leqslant}$ *or* $\overset{\mathrm{rs}}{\leqslant}$ *or* $\overset{\mathrm{s}}{\prec}$. *Then*

$$\mathbf{A} \overset{?}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \begin{cases} \mathbf{B}^*\mathbf{A} \overset{?}{\leqslant} \mathbf{B}^*\mathbf{B} \text{ and } \mathbf{A}\mathbf{B}^* \overset{?}{\leqslant} \mathbf{B}\mathbf{B}^*, \\ \mathbf{B}^+\mathbf{A} \overset{?}{\leqslant} \mathbf{B}^+\mathbf{B} \text{ and } \mathbf{A}\mathbf{B}^+ \overset{?}{\leqslant} \mathbf{B}\mathbf{B}^+. \end{cases} \tag{2.3}$$

Similar properties do not hold for the Löwner partial ordering and singular-values preordering. As an example we may take

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

In view of (1.5), it is clear that the first part of (2.3), with $\overset{?}{\leqslant}$ replaced by $\overset{*}{\leqslant}$, implies the result given originally by Drazin [11, Proposition 7.2].

COROLLARY 2.1. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$. *Then*

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A}^*\mathbf{A} \overset{*}{\leqslant} \mathbf{B}^*\mathbf{B} \text{ and } \mathbf{A}\mathbf{A}^* \overset{*}{\leqslant} \mathbf{B}\mathbf{B}^*. \tag{2.4}$$

In view of (1.8), another consequence of Theorem 2.2 is that

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A}^{+}\mathbf{A} \overset{*}{\leqslant} \mathbf{B}^{+}\mathbf{B} \text{ and } \mathbf{A}\mathbf{A}^{+} \overset{*}{\leqslant} \mathbf{B}\mathbf{B}^{+}. \tag{2.5}$$

This was established by Drazin [11, Corollary 7.4] by combining (2.4) with the properties

$$\mathbf{A}^{*}\mathbf{A} \overset{*}{\leqslant} \mathbf{B}^{*}\mathbf{B} \quad \Rightarrow \quad \mathbf{A}^{+}\mathbf{A} \overset{*}{\leqslant} \mathbf{B}^{+}\mathbf{B}$$

and

$$\mathbf{A}\mathbf{A}^{*} \overset{*}{\leqslant} \mathbf{B}\mathbf{B}^{*} \quad \Rightarrow \quad \mathbf{A}\mathbf{A}^{+} \overset{*}{\leqslant} \mathbf{B}\mathbf{B}^{+},$$

given in his Proposition 7.3. Since the relations of the star ordering, minus ordering, Löwner ordering, and space preordering are actually all identical in the set of orthogonal projectors (cf. Theorem 5.8 in Hartwig and Styan [19]), the result (2.5) may be strengthened to the form revealed in Corollary 2.2 below. On the other hand, (2.4) cannot be modified to the statement $\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B} \Rightarrow \mathbf{A}^{*}\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B}^{*}\mathbf{B}$ or $\mathbf{A}\mathbf{A}^{*} \overset{\mathrm{rs}}{\leqslant} \mathbf{B}\mathbf{B}^{*}$ or to the statement $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \Rightarrow \mathbf{A}^{*}\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}^{*}\mathbf{B}$ or $\mathbf{A}\mathbf{A}^{*} \overset{\mathrm{L}}{\leqslant} \mathbf{B}\mathbf{B}^{*}$, a counterexample being the matrices in (2.1).

Corollary 2.2. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$. *Then*

$$\mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^{+}\mathbf{A} \overset{*}{\leqslant} \mathbf{B}^{+}\mathbf{B} \textit{ and } \mathbf{A}\mathbf{A}^{+} \overset{*}{\leqslant} \mathbf{B}\mathbf{B}^{+}.$$

It is obvious that the space preordering is preserved under multiplication of the matrices involved by any (possibly different) nonzero scalars. The singular-values preordering and the Löwner ordering are much more sensitive to such manipulations, although the use of different nonzero scalars is still possible. The star ordering and minus ordering, however, are extremely sensitive, as shown in the theorem below, which follows directly from (1.11) and (1.13).

Theorem 2.3. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$, *and let* $a, b \in \mathbb{C}$. *If* $\mathbf{A} \neq \mathbf{0}$ *and* $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ *or* $\mathbf{A} \overset{\mathrm{rs}}{\leqslant} \mathbf{B}$, *then neither* $a\mathbf{A} \overset{*}{\leqslant} b\mathbf{B}$ *nor* $a\mathbf{A} \overset{\mathrm{rs}}{\leqslant} b\mathbf{B}$ *can hold except for the trivial cases where* $a = 0$ *or* $a = b$.

The last part of this section is concerned with two special products of matrices. First, it is shown (in Theorem 2.4) that if $\mathbf{A},\mathbf{B} \in \mathbb{C}_{m,n}$ and $\mathbf{C},\mathbf{D} \in \mathbb{C}_{p,q}$ are star-ordered, minus-ordered, or space-preordered, then so are the corresponding direct products, also referred to in the literature as Kronecker products or sometimes (cf. [20]) as Zehfuss products. It is also shown that the reverse implications for the star and minus orderings require certain minor modifications.

THEOREM 2.4. *Let* $\mathbf{A},\mathbf{B} \in \mathbb{C}_{m,n}$ *and* $\mathbf{C},\mathbf{D} \in \mathbb{C}_{p,q}$ *be all nonzero. Then*:

(a) $\mathbf{A}\otimes\mathbf{C} \overset{*}{\leqslant} \mathbf{B}\otimes\mathbf{D} \Leftrightarrow \mathbf{A} \overset{*}{\leqslant} s\mathbf{B}$ *and* $s\mathbf{C} \overset{*}{\leqslant} \mathbf{D}$ *for some* $s \neq 0$,

(b) $\mathbf{A}\otimes\mathbf{C} \overset{\mathrm{rs}}{\leqslant} \mathbf{B}\otimes\mathbf{D} \Leftrightarrow \mathbf{A} \overset{\mathrm{rs}}{\leqslant} s\mathbf{B}$ *and* $s\mathbf{C} \overset{\mathrm{rs}}{\leqslant} \mathbf{D}$ *for some* $s \neq 0$,

(c) $\mathbf{A}\otimes\mathbf{C} \overset{\mathrm{s}}{\prec} \mathbf{B}\otimes\mathbf{D} \Leftrightarrow \mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B}$ *and* $\mathbf{C} \overset{\mathrm{s}}{\prec} \mathbf{D}$.

Proof. On account of (1.5), the ordering $\mathbf{A}\otimes\mathbf{C} \overset{*}{\leqslant} \mathbf{B}\otimes\mathbf{D}$ is equivalent to

$$\mathbf{A}^*\mathbf{A}\otimes\mathbf{C}^*\mathbf{C} = \mathbf{A}^*\mathbf{B}\otimes\mathbf{C}^*\mathbf{D} \quad \text{and} \quad \mathbf{A}\mathbf{A}^*\otimes\mathbf{C}\mathbf{C}^* = \mathbf{B}\mathbf{A}^*\otimes\mathbf{D}\mathbf{C}^*. \tag{2.6}$$

Lemma 1 in [5] asserts that, for any nonzero $\mathbf{K}_1,\mathbf{K}_2 \in \mathbb{C}_{m,n}$ and $\mathbf{L}_1,\mathbf{L}_2 \in \mathbb{C}_{p,q}$, the equality $\mathbf{K}_1\otimes\mathbf{L}_1 = \mathbf{K}_2\otimes\mathbf{L}_2$ holds if and only if $\mathbf{K}_1 = s\mathbf{K}_2$ and $s\mathbf{L}_1 = \mathbf{L}_2$ for some $s \neq 0$. Consequently, (2.6) is equivalent to

$$\mathbf{A}^*\mathbf{A} = s_1\mathbf{A}^*\mathbf{B}, \qquad s_1\mathbf{C}^*\mathbf{C} = \mathbf{C}^*\mathbf{D}, \qquad \mathbf{A}\mathbf{A}^* = s_2\mathbf{B}\mathbf{A}^*, \qquad s_2\mathbf{C}\mathbf{C}^* = \mathbf{D}\mathbf{C}^*. \tag{2.7}$$

Observing that s_2 in (2.7) must be identical with s_1 concludes the proof of (a). The statements (b) and (c) follow similarly in view of (1.13), (1.14), and the fact that $\mathbf{B}^-\otimes\mathbf{D}^- \in (\mathbf{B}\otimes\mathbf{D})\{1\}$ for any $\mathbf{B}^- \in \mathbf{B}\{1\}$ and $\mathbf{D}^- \in \mathbf{D}\{1\}$. ■

Combining Theorems 2.3 and 2.4 leads to the following:

COROLLARY 2.3. *Let* $\mathbf{A},\mathbf{B} \in \mathbb{C}_{m,n}$, *and let* $\overset{?}{\leqslant}$ *stand for either* $\overset{*}{\leqslant}$ *or* $\overset{\mathrm{rs}}{\leqslant}$. *Then*

$$\mathbf{A}\otimes\mathbf{A} \overset{?}{\leqslant} \mathbf{B}\otimes\mathbf{B} \quad \Leftrightarrow \quad \mathbf{A} \overset{?}{\leqslant} \mathbf{B} \text{ or } \mathbf{A} \overset{?}{\leqslant} -\mathbf{B}.$$

Since $\sigma(\mathbf{K}\otimes\mathbf{L})$ consists of all possible products of the nonzero singular values of $\mathbf{K}$ with those of $\mathbf{L}$, it is clear that, for any $\mathbf{A},\mathbf{B} \in \mathbb{C}_{m,n}$ and

$\mathbf{C},\mathbf{D}\in\mathbb{C}_{p,q}$,

$$\mathbf{A}\overset{\sigma}{\prec}\mathbf{B} \text{ and } \mathbf{C}\overset{\sigma}{\prec}\mathbf{D} \quad\Rightarrow\quad \mathbf{A}\otimes\mathbf{C}\overset{\sigma}{\prec}\mathbf{B}\otimes\mathbf{D},$$

but not the other way around.

Considering the Löwner ordering between Hadamard products of matrices (cf. Styan [32]), Johnson [22, p. 590] established a result which is generalized here to the following form:

THEOREM 2.5. *Let* $\mathbf{A},\mathbf{B}\in\mathbb{C}_{m,n}$, *let* $\mathbf{C},\mathbf{D}\in\mathbb{C}_{n,n}$, *and let* $\mathbf{A}$ *with* $\mathbf{C}$, $\mathbf{A}$ *with* $\mathbf{D}$, *or* $\mathbf{B}$ *with* $\mathbf{C}$ *be Hermitian nonnegative definite. Then*

$$\mathbf{A}\overset{L}{\leqslant}\mathbf{B} \text{ and } \mathbf{C}\overset{L}{\leqslant}\mathbf{D} \quad\Rightarrow\quad \mathbf{A}\otimes\mathbf{C}\overset{L}{\leqslant}\mathbf{B}\otimes\mathbf{D}.$$

Proof. If $\mathbf{0}\overset{L}{\leqslant}\mathbf{A}$ and $\mathbf{0}\overset{L}{\leqslant}\mathbf{D}$, then $\mathbf{A}\overset{L}{\leqslant}\mathbf{B}$ and $\mathbf{C}\overset{L}{\leqslant}\mathbf{D}$ imply that

$$\mathbf{0}\overset{L}{\leqslant}\mathbf{A}\otimes(\mathbf{D}-\mathbf{C})+(\mathbf{B}-\mathbf{A})\otimes\mathbf{D}=\mathbf{B}\otimes\mathbf{D}-\mathbf{A}\otimes\mathbf{C}, \tag{2.8}$$

as desired. Similarly, if $\mathbf{0}\overset{L}{\leqslant}\mathbf{B}$ and $\mathbf{0}\overset{L}{\leqslant}\mathbf{C}$, then

$$\mathbf{0}\overset{L}{\leqslant}\mathbf{B}\otimes(\mathbf{D}-\mathbf{C})+(\mathbf{B}-\mathbf{A})\otimes\mathbf{C}=\mathbf{B}\otimes\mathbf{D}-\mathbf{A}\otimes\mathbf{C}.$$

Since $\mathbf{0}\overset{L}{\leqslant}\mathbf{C}$ and $\mathbf{C}\overset{L}{\leqslant}\mathbf{D}$ entail $\mathbf{0}\overset{L}{\leqslant}\mathbf{D}$, the case where $\mathbf{0}\overset{L}{\leqslant}\mathbf{A}$ and $\mathbf{0}\overset{L}{\leqslant}\mathbf{C}$ is covered by (2.8). ∎

The assumption in Theorem 2.5, which actually means that at least three of the matrices involved are Hermitian nonnegative definite, is essential. The quadruplets

$$\mathbf{A}_1=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad \mathbf{B}_1=\begin{pmatrix}1&2\\2&5\end{pmatrix},\quad \mathbf{C}_1=\begin{pmatrix}-1&0\\0&0\end{pmatrix},\quad \mathbf{D}_1=\begin{pmatrix}0&-1\\-1&1\end{pmatrix}, \tag{2.9}$$

$$\mathbf{A}_2=\begin{pmatrix}-1&0\\0&0\end{pmatrix},\quad \mathbf{B}_2=\begin{pmatrix}1&-1\\-1&1\end{pmatrix},\quad \mathbf{C}_2=\begin{pmatrix}-1&1\\1&1\end{pmatrix},\quad \mathbf{D}_2=\begin{pmatrix}1&2\\2&4\end{pmatrix}, \tag{2.10}$$

and

$$\mathbf{A}_3 = \mathbf{C}_1, \qquad \mathbf{B}_3 = \mathbf{D}_1, \qquad \mathbf{C}_3 = \mathbf{A}_1, \qquad \mathbf{D}_3 = \mathbf{B}_1 \tag{2.11}$$

constitute appropriate examples.

Since the operation of taking principal submatrices is isotonic with respect to the Löwner ordering and since the Hadamard product, $\mathbf{K} * \mathbf{L}$, of $\mathbf{K}, \mathbf{L} \in \mathbb{C}_{m,m}$ is just a principal submatrix of the corresponding direct product, Theorem 2.5 leads to Corollary 2.4 below, which is comparable with Theorem 17 of Johnson [22].

COROLLARY 2.4. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{C}_{m,m}$, *and let* $\mathbf{A}$ *with* $\mathbf{C}$, $\mathbf{A}$ *with* $\mathbf{D}$, *or* $\mathbf{B}$ *with* $\mathbf{C}$ *be Hermitian nonnegative definite. Then*

$$\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \text{ and } \mathbf{C} \overset{\mathrm{L}}{\leqslant} \mathbf{D} \quad \Rightarrow \quad \mathbf{A} * \mathbf{C} \overset{\mathrm{L}}{\leqslant} \mathbf{B} * \mathbf{D}. \tag{2.12}$$

The matrices in (2.9), (2.10), and (2.11) can again be utilized to show that the assumption in Corollary 2.4 is essential.

In view of the above and Theorem 2.4, we may ask whether an analogue to (2.12) holds under the star ordering or minus ordering. The answer is in both cases negative, as can be seen by taking

$$\mathbf{A} = \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \mathbf{D} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

This example also shows that the operation of taking principal submatrices is not isotonic with respect to the star ordering or the minus ordering.

3. ORDERINGS AMONG GENERALIZED INVERSES OF A GIVEN MATRIX

Theorem 2 in Drazin [12] asserts that $\mathbf{A}^+$ is the least element in the set $\mathbf{A}\{1,3,4\}$ and the greatest element in the set $\mathbf{A}\{2,3,4\}$ with respect to the star ordering, that is

$$\mathbf{H} \overset{*}{\leqslant} \mathbf{A}^+ \overset{*}{\leqslant} \mathbf{G} \qquad \text{for every } \mathbf{G} \in \mathbf{A}\{1,3,4\} \text{ and every } \mathbf{H} \in \mathbf{A}\{2,3,4\}. \tag{3.1}$$

The part $\mathbf{A}^+ \overset{*}{\leqslant} \mathbf{G}$ is an immediate consequence of Corollary 2.6 in Drazin

[11], stating that

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^{+}\mathbf{B}\mathbf{A}^{+} = \mathbf{A}^{+}, \quad \mathbf{A}^{+}\mathbf{B} = (\mathbf{A}^{+}\mathbf{B})^{*}, \quad \mathbf{B}\mathbf{A}^{+} = (\mathbf{B}\mathbf{A}^{+})^{*},$$

while the part $\mathbf{H} \overset{*}{\leqslant} \mathbf{A}^{+}$ follows by the dual characterization

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}\mathbf{B}^{+}\mathbf{A} = \mathbf{A}, \quad \mathbf{A}\mathbf{B}^{+} = (\mathbf{A}\mathbf{B}^{+})^{*}, \quad \mathbf{B}^{+}\mathbf{A} = (\mathbf{B}^{+}\mathbf{A})^{*};$$

cf. Hartwig and Styan [18, Theorem 2(c)]. In the first part of this section, a number of new relationships among star-ordered inner and outer inverses of a given $\mathbf{A} \in \mathbb{C}_{m,n}$ are established. In particular, it is shown that $\mathbf{A}\{1,3,4\}$ and $\mathbf{A}\{2,3,4\}$ are actually the sets of all star-successors and all star-predecessors, respectively, of $\mathbf{A}^{+}$.

THEOREM 3.1. *Let $\mathbf{A} \in \mathbb{C}_{m,n}$. Then, for $i = 3$ or 4,*

$$\mathbf{G}_0 \in \mathbf{A}\{1,i\}, \quad \mathbf{G}_0 \overset{*}{\leqslant} \mathbf{G} \quad \Rightarrow \quad \mathbf{G} \in \mathbf{A}\{1\}. \tag{3.2}$$

Proof. If $i = 3$, then $\mathbf{A} = \mathbf{G}_0^{*}\mathbf{A}^{*}\mathbf{A}$, cf. (1.1) and (1.3), and hence

$$\mathbf{AGA} = \mathbf{AGG}_0^{*}\mathbf{A}^{*}\mathbf{A} = \mathbf{AG}_0\mathbf{G}_0^{*}\mathbf{A}^{*}\mathbf{A} = \mathbf{AG}_0\mathbf{A} = \mathbf{A}.$$

For $i = 4$, the result follows similarly using the equality $\mathbf{A} = \mathbf{AA}^{*}\mathbf{G}_0^{*}$; cf. (1.1) and (1.4). ■

Notice that the implication (3.2) is no longer true when the condition $\mathbf{G}_0 \in \mathbf{A}\{1,i\}$ is weakened to the form $\mathbf{G}_0 \in \mathbf{A}\{1\}$. A counterexample is the triplet

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{G}_0 = \begin{pmatrix} t_0 & u_0 \\ v_0 & w_0 \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} t & u \\ v & w \end{pmatrix}, \tag{3.3}$$

with $t_0 = u_0 = v_0 = w_0 = 1$ and $t = w = 0$, $u = v = 2$. Moreover, $\mathbf{G}$ need not be an inner inverse of $\mathbf{A}$ either in the case where the star ordering on the left-hand side of (3.2) is reversed, a counterexample being (3.3) with $t_0 = w_0 = 1$, $u_0 = v_0 = 0$ and $t = u = v = 0$, $w = 1$. However, if the reversed ordering holds and $\mathbf{G}$ is known to be an inner inverse of $\mathbf{A}$, then $\mathbf{G}$ has necessarily the same additional property as $\mathbf{G}_0$.

THEOREM 3.2. *Let* $\mathbf{A} \in \mathbb{C}_{m,n}$. *Then, for* $i = 3$ *or* 4,

$$\mathbf{G} \in \mathbf{A}\{1\}, \quad \mathbf{G}_0 \in \mathbf{A}\{1,i\}, \quad \mathbf{G} \overset{*}{\leqslant} \mathbf{G}_0 \quad \Rightarrow \quad \mathbf{G} \in \mathbf{A}\{1,i\}. \tag{3.4}$$

Proof. If $i = 3$, then

$$\mathbf{A}(\mathbf{G} - \mathbf{G}_0)(\mathbf{G} - \mathbf{G}_0)^*\mathbf{A}^* = \mathbf{A}(\mathbf{G}_0\mathbf{G}_0^* - \mathbf{G}\mathbf{G}^*)\mathbf{A}^* = \mathbf{A}\mathbf{G}_0 - \mathbf{A}\mathbf{G}\mathbf{G}^*\mathbf{A}^*. \tag{3.5}$$

But

$$\mathbf{A}\mathbf{G}_0 = \mathbf{A}\mathbf{G}\mathbf{A}\mathbf{G}_0 = \mathbf{A}\mathbf{G}\mathbf{G}_0^*\mathbf{A}^* = \mathbf{A}\mathbf{G}\mathbf{G}^*\mathbf{A}^*, \tag{3.6}$$

and combining (3.5) with (3.6) yields $\mathbf{AG} = \mathbf{AG}_0$. If $i = 4$, then similar arguments lead to the equality $\mathbf{GA} = \mathbf{G}_0\mathbf{A}$. ■

If the star ordering on the left-hand side of (3.4) is reversed, then $\mathbf{G}$ need not have the additional property of $\mathbf{G}_0$. A counterexample for $i = 3$ is (3.3), with $t_0 = v_0 = 1$, $u_0 = w_0 = 0$ and $t = v = w = 1$, $u = -1$. However, the situation changes when $\mathbf{G}_0$ has the two additional properties simultaneously.

THEOREM 3.3. *Let* $\mathbf{A} \in \mathbb{C}_{m,n}$. *Then*

$$\mathbf{G}_0 \in \mathbf{A}\{1,3,4\}, \quad \mathbf{G}_0 \overset{*}{\leqslant} \mathbf{G} \quad \Rightarrow \quad \mathbf{G} \in \mathbf{A}\{1,3,4\}.$$

Proof. Theorem 3.1 implies that $\mathbf{G} \in \mathbf{A}\{1\}$. Moreover,

$$\mathbf{AG} = \mathbf{AG}_0\mathbf{AG} = \mathbf{AA}^*\mathbf{G}_0^*\mathbf{G} = \mathbf{AA}^*\mathbf{G}_0^*\mathbf{G}_0 = \mathbf{AG}_0$$

and, similarly, $\mathbf{GA} = \mathbf{G}_0\mathbf{A}$. ■

Combining Theorem 3.3 with the latter part of (3.1) yields the following:

COROLLARY 3.1. *Let* $\mathbf{A} \in \mathbb{C}_{m,n}$. *Then*

$$\mathbf{A}\{1,2,3\} = \left\{\mathbf{G} \in \mathbb{C}_{m,n} : \mathbf{A}^+ \overset{*}{\leqslant} \mathbf{G}\right\},$$

that is,

$$\mathbf{A}^+ \overset{*}{\leqslant} \mathbf{G} \quad \Leftrightarrow \quad \mathbf{G} \in \mathbf{A}\{1,3,4\}.$$

In view of (1.15), an obvious consequence of (3.1) is that $\mathbf{A}^+$ is the least element of $\mathbf{A}\{1,3,4\}$ also with respect to the minus ordering. However, a characterization similar to that in Corollary 3.1 is not valid in this case. Actually, none of Theorems 3.1, 3.2, and 3.3 remains true when the star ordering involved is replaced by the corresponding minus ordering. Counterexamples are obtained from (3.3) taking $t_0 = 1$, $u_0 = v_0 = w_0 = 0$ and $t = 2$, $u = v = w = 1$ in the first and third cases, and $t_0 = w_0 = 1$, $u_0 = v_0 = 0$ and $t = u = 1$, $v = w = 0$ in the second case ($i = 3$).

A similar series of results will now be given for outer inverses of $\mathbf{A}$. The first may be formulated using the minus ordering.

THEOREM 3.4. *Let* $\mathbf{A} \in \mathbf{C}_{m,n}$. *Then*

$$\mathbf{H}_0 \in \mathbf{A}\{2\}, \quad \mathbf{H} \overset{\mathrm{rs}}{\leqslant} \mathbf{H}_0 \quad \Rightarrow \quad \mathbf{H} \in \mathbf{A}\{2\}.$$

Proof. In view of (1.13), it follows that, for any $\mathbf{H}_0^- \in \mathbf{H}_0\{1\}$,

$$\mathbf{HAH} = \mathbf{HH}_0^- \mathbf{H}_0\mathbf{AH}_0\mathbf{H}_0^- \mathbf{H} = \mathbf{HH}_0^- \mathbf{H} = \mathbf{H}. \qquad \blacksquare$$

THEOREM 3.5. *Let* $\mathbf{A} \in \mathbf{C}_{m,n}$. *Then, for* $i = 3$ *or* 4,

$$\mathbf{H} \in \mathbf{A}\{2\}, \quad \mathbf{H}_0 \in \mathbf{A}\{2,i\}, \quad \mathbf{H}_0 \overset{*}{\leqslant} \mathbf{H} \quad \Rightarrow \quad \mathbf{H} \in \mathbf{A}\{2,i\}.$$

Proof. If $i = 3$, then on account of (1.11) and (1.10), it follows that

$$\mathbf{AH} = \mathbf{AH}_0\mathbf{H}_0^+ \mathbf{H} = \mathbf{H}_0^*\mathbf{A}^*\mathbf{H}_0^+ \mathbf{H} = \mathbf{H}_0^+ \mathbf{H} = \mathbf{H}_0^+ \mathbf{H}_0. \qquad \blacksquare$$

THEOREM 3.6. *Let* $\mathbf{A} \in \mathbf{C}_{m,n}$. *Then*

$$\mathbf{H}_0 \in \mathbf{A}\{2,3,4\}, \quad \mathbf{H} \overset{*}{\leqslant} \mathbf{H}_0 \quad \Rightarrow \quad \mathbf{H} \in \mathbf{A}\{2,3,4\}.$$

Proof. Theorem 3.4 implies that $\mathbf{H} \in \mathbf{A}\{2\}$. Moreover, on account of (1.11),

$$\mathbf{AH} = \mathbf{AH}_0\mathbf{H}^+ \mathbf{H} = (\mathbf{H}^+ \mathbf{HAH}_0)^* = \mathbf{H}^+ \mathbf{H}$$

and, similarly, $\mathbf{HA} = \mathbf{HH}^+$. $\blacksquare$

Combining Theorem 3.6 with the first part of (3.1) yields the dual of Corollary 3.1:

COROLLARY 3.2. *Let* $\mathbf{A} \in \mathbb{C}_{m,n}$. *Then*

$$\mathbf{A}\{2,3,4\} = \left\{\mathbf{H} \in \mathbb{C}_{n,m} : \mathbf{H} \overset{*}{\leqslant} \mathbf{A}^{+}\right\},$$

that is,

$$\mathbf{H} \overset{*}{\leqslant} \mathbf{A}^{+} \quad \Leftrightarrow \quad \mathbf{H} \in \mathbf{A}\{2,3,4\}.$$

The matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{H}_0 = \begin{pmatrix} 0 & 0 & y \\ 0 & z & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{H} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.7}$$

with $y = z = 0$, constitute an example that Theorem 3.5 is no longer true when the star ordering involved is replaced by the corresponding minus ordering. Setting $y = z = 1$ in (3.7) shows that the same conclusion may be made about Theorem 3.6.

The second part of this section refers to Theorems 3 and 4 of Wu [37], which are restated in Theorem 3.7 below. Henceforth, $\mathbf{A}\{1, \geqslant\}$ and $\mathbf{A}\{2, \geqslant\}$ will denote the sets of all Hermitian nonnegative definite inner and outer inverses, respectively, of a Hermitian nonnegative definite $\mathbf{A} \in \mathbb{C}_{m,m}$.

THEOREM 3.7. *Let* $\mathbf{A} \in \mathbb{C}_{m,m}$ *be Hermitian nonnegative definite, and let* $\mathrm{r}(\mathbf{A}) = p$. *Then, for any fixed* $\mathbf{G}_r^0 \in \mathbf{A}\{1, \geqslant\}$ *of rank* r *and any* $p \leqslant q \leqslant r \leqslant s$, *there exist* $\mathbf{G}_q^0, \mathbf{G}_s^0 \in \mathbf{A}\{1, \geqslant\}$ *of ranks* q *and* s, *respectively, such that*

$$\mathbf{G}_q^0 \overset{\mathrm{L}}{\leqslant} \mathbf{G}_r^0 \overset{\mathrm{L}}{\leqslant} \mathbf{G}_s^0, \tag{3.8}$$

and for any fixed $\mathbf{H}_r^0 \in \mathbf{A}\{2, \geqslant\}$ *of rank* r *and any* $s \leqslant r \leqslant q \leqslant p$ *there exist* $\mathbf{H}_q^0, \mathbf{H}_s^0 \in \mathbf{A}\{2, \geqslant\}$ *of ranks* q *and* s, *respectively, such that*

$$\mathbf{H}_s^0 \overset{\mathrm{L}}{\leqslant} \mathbf{H}_r^0 \overset{\mathrm{L}}{\leqslant} \mathbf{H}_q^0. \tag{3.9}$$

Precise inspection of the arguments used by Wu [37, pp. 53–54] in establishing the results quoted in Theorem 3.7 shows that he actually proved stronger relations than (3.8) and (3.9), namely $G_q^0 \overset{\text{rs}}{\leqslant} G_r^0 \overset{\text{rs}}{\leqslant} G_s^0$ and $H_s^0 \overset{\text{rs}}{\leqslant} H_q^0 \leqslant H_q^0$, respectively; cf. (1.18). It appears that similar relations can be established for inner and outer inverses of any $A \in \mathbb{C}_{m,n}$.

THEOREM 3.8. *Let* $A \in \mathbb{C}_{m,n}$ *and let* $r(A) = p$. *Then for any fixed* $G_r \in A\{1\}$ *of rank* r *and any* $p \leqslant q \leqslant r \leqslant s$ *there exist* $G_q, G_s \in A\{1\}$ *of ranks* q *and* s, *respectively, such that*

$$G_q \overset{\text{rs}}{\leqslant} G_r \overset{\text{rs}}{\leqslant} G_s, \tag{3.10}$$

and for any fixed $H_r \in A\{2\}$ *of rank* r *and any* $s \leqslant r \leqslant q \leqslant p$ *there exist* $H_q, H_s \in A\{2\}$ *of ranks* q *and* s, *respectively, such that*

$$H_s \overset{\text{rs}}{\leqslant} H_r \overset{\text{rs}}{\leqslant} H_q. \tag{3.11}$$

Proof. If $q = r = s$, which is necessarily the case when $p = \min(m, n)$, then the only solution to (3.10) is $G_q = G_r = G_s$. Now, let $\min(m, n) < p < q < r < s$, and let $P \in \mathbb{C}_{m,m}$ and $Q \in \mathbb{C}_{n,n}$ be nonsingular and such that $PAQ = J_p$, where J_p denotes the matrix, of size clear from the context, with I_p in the northwest corner and zeros elsewhere. Consequently, G_r admits the representation

$$G_r = Q \begin{pmatrix} I_p & K \\ L & M_r \end{pmatrix} P, \tag{3.12}$$

with some fixed $K \in \mathbb{C}_{p,m-p}$, $L \in \mathbb{C}_{n-p,p}$, and $M_r \in \mathbb{C}_{n-p,m-p}$ such that $r(M_r - LK) = r - p$. Let $S \in \mathbb{C}_{n-p,n-p}$ and $T \in \mathbb{C}_{m-p,m-p}$ be nonsingular and such that

$$S(M_r - LK)T = J_{r-p}.$$

Then it can easily be verified that G_q and G_s of the form (3.12) but with M_r replaced by

$$M_q = LK + S^{-1}J_{q-p}T^{-1} \quad \text{and} \quad M_s = LK + S^{-1}J_{s-p}T^{-1},$$

respectively, satisfy the conditions $r(G_q) = q$, $r(G_s) = s$, and (3.10), thus establishing the first part of the theorem.

To prove the second part, let

$$\mathbf{H}_r = \mathbf{V}_r \boldsymbol{\Delta}_r \mathbf{U}_r^*, \tag{3.13}$$

with $\mathbf{U}_r \in \mathbb{C}_{m,r}$, $\mathbf{V}_r \in \mathbb{C}_{n,r}$, and positive definite diagonal $\boldsymbol{\Delta}_r \in \mathbb{C}_{r,r}$, be a singular-value decomposition of $\mathbf{H}_r$. Then since $\mathbf{H}_r \in \mathbf{A}\{2\}$, it follows that

$$\mathbf{U}_r^* \mathbf{A} \mathbf{V}_r = \boldsymbol{\Delta}_r^{-1}. \tag{3.14}$$

Let the matrices $\mathbf{U}_s$ and $\mathbf{V}_s$ comprise the first s columns of $\mathbf{U}_r$ and $\mathbf{V}_r$, respectively, and let $\boldsymbol{\Delta}_s$ be the $s \times s$ northwest submatrix of $\boldsymbol{\Delta}_r$. From (3.14), it is seen that $\mathbf{U}_s^* \mathbf{A} \mathbf{V}_s^* = \boldsymbol{\Delta}_s^{-1}$, and thus $\mathbf{H}_s = \mathbf{V}_s \boldsymbol{\Delta}_s \mathbf{U}_s^*$ is an outer inverse of $\mathbf{A}$ with rank s. Moreover,

$$\mathrm{r}(\mathbf{H}_r - \mathbf{H}_s) = s - r = \mathrm{r}(\mathbf{H}_r) - \mathrm{r}(\mathbf{H}_s). \tag{3.15}$$

To complete the proof of (3.11) let $\mathbf{W} \in \mathbb{C}_{m,m-r}$ and $\mathbf{Z} \in \mathbb{C}_{n,n-r}$ be of ranks $m - r$ and $n - r$, respectively, and such that

$$\mathbf{W}^* \mathbf{A} \mathbf{V}_r = \mathbf{0} \quad \text{and} \quad \mathbf{U}_r^* \mathbf{A} \mathbf{Z} = \mathbf{0}. \tag{3.16}$$

From (3.14) and (3.16) it follows that $\mathcal{R}(\mathbf{U}_r) \cap \mathcal{R}(\mathbf{W}) = \{\mathbf{0}\}$ and $\mathcal{R}(\mathbf{V}_r) \cap \mathcal{R}(\mathbf{Z}) = \{\mathbf{0}\}$, and hence both $(\mathbf{U}_r : \mathbf{W})$ and $(\mathbf{V}_r : \mathbf{Z})$ are nonsingular. Consequently, $\mathrm{r}(\mathbf{W}^* \mathbf{A} \mathbf{Z}) = p - r$, and thus there exist $\mathbf{S} \in \mathbb{C}_{m-r,q-r}$ and $\mathbf{T} \in \mathbb{C}_{n-r,q-r}$ such that

$$\mathbf{S}^* \mathbf{W}^* \mathbf{A} \mathbf{Z} \mathbf{T} = \mathbf{I}_{q-r}. \tag{3.17}$$

Using (3.16) and (3.17), it can easily be verified that

$$\mathbf{H}_q = \mathbf{V}_r \boldsymbol{\Delta}_r \mathbf{U}_r^* + \mathbf{Z} \mathbf{T} \mathbf{S}^* \mathbf{W}^*$$

is an outer inverse of $\mathbf{A}$ such that $\mathrm{r}(\mathbf{H}_q) = q$ and

$$\mathrm{r}(\mathbf{H}_q - \mathbf{H}_r) = \mathrm{r}(\mathbf{Z} \mathbf{T} \mathbf{S}^* \mathbf{W}^*) = q - r = \mathrm{r}(\mathbf{H}_q) - \mathrm{r}(\mathbf{H}_r),$$

which concludes the proof. ∎

It is interesting to remark that three of the four results in Theorem 3.8 do not hold if the minus orderings involved are replaced by the corresponding star orderings. For example, if

$$\mathbf{A}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad \mathbf{G}_1=\begin{pmatrix}1&1\\1&1\end{pmatrix},\qquad \mathbf{G}_2=\begin{pmatrix}1&1\\1&0\end{pmatrix},$$

then there does not exist $\mathbf{G}\in\mathbf{A}\{1\}$ with rank 1 such that $\mathbf{G}\overset{*}{\leqslant}\mathbf{G}_2$ and there does not exist $\mathbf{G}\in\mathbf{A}\{1\}$ with rank 2 such that $\mathbf{G}_1\overset{*}{\leqslant}\mathbf{G}$. Further, if

$$\mathbf{A}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \mathbf{H}_1=\begin{pmatrix}0&0\\1&1\end{pmatrix},$$

then there does not exist $\mathbf{H}\in\mathbf{A}\{2\}$ with rank 1 such that $\mathbf{H}_1\overset{*}{\leqslant}\mathbf{H}$. However, an outer inverse of $\mathbf{A}$ preceding a given outer inverse of $\mathbf{A}$ can always be found.

THEOREM 3.9. *Let* $\mathbf{A}\in\mathbb{C}_{m,n}$ *and let* $\mathrm{r}(\mathbf{A})=p$. *Then for any fixed* $\mathbf{H}_r\in\mathbf{A}\{2\}$ *of rank* r *and any* $s\leqslant r\leqslant p$, *there exists* $\mathbf{H}_s\in\mathbf{A}\{2\}$ *of rank* s *such that* $\mathbf{H}_s\overset{*}{\leqslant}\mathbf{H}_r$.

Proof. Let $\mathbf{H}_r$ be decomposed as in (3.13), and let $\mathbf{H}_s$ be specified as in (3.15). Then, $\mathbf{H}_s^*\mathbf{H}_r=\mathbf{U}_s\Delta_s^2\mathbf{U}_s^*=\mathbf{H}_s^*\mathbf{H}_s$ and $\mathbf{H}_r\mathbf{H}_s^*=\mathbf{V}_s\Delta_s^2\mathbf{V}_s^*=\mathbf{H}_s\mathbf{H}_s^*$, as desired. ∎

4. PRESERVING OR REVERSING MATRIX ORDERINGS UNDER GENERALIZED INVERSIONS

Drazin [12, Corollary 1] pointed out that

$$\mathbf{A}\overset{*}{\leqslant}\mathbf{B}\quad\Leftrightarrow\quad\mathbf{A}^+\overset{*}{\leqslant}\mathbf{B}^+,\tag{4.1}$$

which means that the Moore-Penrose inverse is isotonic (cf. Marshall and Olkin [25, p. 13]) with respect to the star ordering. Combining (4.1) with our Corollaries 3.1 and 3.2 shows that

$$\mathbf{A}\overset{*}{\leqslant}\mathbf{B}\quad\Rightarrow\quad\mathbf{H}_\mathbf{A}\overset{*}{\leqslant}\mathbf{A}^+\overset{*}{\leqslant}\mathbf{B}^+\overset{*}{\leqslant}\mathbf{G}_\mathbf{B}\quad\text{for every }\mathbf{H}_\mathbf{A}\in\mathbf{A}\{2,3,4\}$$
$$\text{and every }\mathbf{G}_\mathbf{B}\in\mathbf{B}\{1,3,4\}.\tag{4.2}$$

A part of (4.2) may be used to generalize (4.1).

THEOREM 4.1. *Let* $\mathbf{A}, \mathbf{B} \in \mathbf{C}_{m,n}$. *Then the following statements are equivalent*:

(a) $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$,

(b) $\mathbf{A}^{+} \overset{*}{\leqslant} \mathbf{G}_{\mathbf{B}}$ *for every* $\mathbf{G}_{\mathbf{B}} \in \mathbf{B}\{1,3,4\}$,

(c) $\mathbf{A} \underset{s}{\prec} \mathbf{B}$ *and* $\mathbf{A}^{+} \overset{*}{\leqslant} \mathbf{G}_{\mathbf{B}}$ *for some* $\mathbf{G}_{\mathbf{B}} \in \mathbf{B}\{1,3,4\}$.

Proof. The part (a) $\Rightarrow$ (b) is inherent in (4.2). The condition (b) implies, in particular, that $\mathbf{A}^{+} \overset{*}{\leqslant} \mathbf{B}^{+}$. Then (1.15) entails $\mathbf{A}^{+} \underset{s}{\prec} \mathbf{B}^{+}$, which is clearly equivalent to $\mathbf{A} \underset{s}{\prec} \mathbf{B}$. If (c) holds, then from (1.14) and (1.11) it follows that $\mathbf{A}\mathbf{G}_{\mathbf{B}}\mathbf{B} = \mathbf{A} = \mathbf{B}\mathbf{G}_{\mathbf{B}}\mathbf{A}$ and

$$\mathbf{A}^{+}\mathbf{A}\mathbf{G}_{\mathbf{B}} = \mathbf{A}^{+} = \mathbf{G}_{\mathbf{B}}\mathbf{A}\mathbf{A}^{+}. \tag{4.3}$$

Consequently, postmultiplying the first and premultiplying the second equality in (4.3) by $\mathbf{B}$ yields the two equalities in (1.10). ■

It is interesting to remark that the Moore-Penrose inverse is not, in general, isotonic with respect to the minus ordering. A counterexample is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The problem of characterizing the cases in which the isotonicity property holds was considered by Hartwig and Styan [18, Theorem 3]. They showed that if $\mathbf{A} \overset{rs}{\leqslant} \mathbf{B}$, then $\mathbf{A}^{+} \overset{rs}{\leqslant} \mathbf{B}^{+}$ if and only if $\mathbf{A}^{+}\mathbf{B}\mathbf{A}^{+} = \mathbf{A}^{+}$, and also pointed out that even if the orderings $\mathbf{A} \overset{rs}{\leqslant} \mathbf{B}$ and $\mathbf{A}^{+} \overset{rs}{\leqslant} \mathbf{B}^{+}$ hold simultaneously, then $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ need not hold.

On the other hand, there is no nontrivial case in which the Moore-Penrose inverse is antitonic with respect to the star ordering or minus ordering. This is a direct consequence of the following more general statement:

THEOREM 4.2. *Let* $\mathbf{A}, \mathbf{B} \in \mathbf{C}_{m,n}$, *and let* $\mathbf{H}_{\mathbf{A}} \in \mathbf{A}\{2\}$ *and* $\mathbf{G}_{\mathbf{B}} \in \mathbf{B}\{1\}$. *Then the orderings* $\mathbf{A} \overset{rs}{\leqslant} \mathbf{B}$ *and* $\mathbf{G}_{\mathbf{B}} \overset{rs}{\leqslant} \mathbf{H}_{\mathbf{A}}$ *cannot hold simultaneously except for the trivial case where* $\mathbf{A} = \mathbf{B}$ *and* $\mathbf{H}_{\mathbf{A}} = \mathbf{G}_{\mathbf{B}}$.

Proof. If $\mathbf{A} \overset{rs}{\leqslant} \mathbf{B}$ and $\mathbf{G}_{\mathbf{B}} \overset{rs}{\leqslant} \mathbf{H}_{\mathbf{A}}$, then (1.15) implies that $r(\mathbf{A}) \leqslant r(\mathbf{B})$ and $r(\mathbf{G}_{\mathbf{B}}) \leqslant r(\mathbf{H}_{\mathbf{A}})$. But, in view of (1.1) and (1.2), we have $r(\mathbf{H}_{\mathbf{A}}) \leqslant r(\mathbf{A})$ and

$r(\mathbf{B}) \leqslant r(\mathbf{G}_\mathbf{B})$, and hence $r(\mathbf{A}) = r(\mathbf{B})$ and $r(\mathbf{H}_\mathbf{A}) = r(\mathbf{G}_\mathbf{B})$. Consequently, (1.12) yields $\mathbf{A} = \mathbf{B}$ and $\mathbf{H}_\mathbf{A} = \mathbf{G}_\mathbf{B}$. ∎

Clearly, in view of (1.15), Theorem 4.2 also holds for the star ordering. Contrary to the above, however, the Moore-Penrose inverse proves to be antitonic with respect to the Löwner ordering, although within the set of Hermitian nonnegative definite matrices of equal ranks only. Generalizing the well-known result (cf. Roy and Shah [30, p. 140]), that if Hermitian nonnegative definite $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,m}$ are both nonsingular, then $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$ is equivalent to $\mathbf{B}^{-1} \overset{\mathrm{L}}{\leqslant} \mathbf{A}^{-1}$, Milliken and Akdeniz [26], Hartwig [14], and Werner [36] contributed to establishing the following:

THEOREM 4.3. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,m}$ be Hermitian nonnegative definite. Then any two of the conditions*

(a) $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$,
(b) $r(\mathbf{A}) = r(\mathbf{B})$,
(c) $\mathbf{B}^+ \overset{\mathrm{L}}{\leqslant} \mathbf{A}^+$,

imply the third condition.

For a quick proof of Theorem 4.3 see Styan [34, Theorem 1.2].

From (1.19) it is clear that the condition (b) in Theorem 4.3 may be replaced by $\mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{B})$; cf. Hartwig [14, Theorem 1]. Also notice that the restriction to the case of Hermitian nonnegative definite matrices is essential, as may be seen taking

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Theorem 4.3 can be generalized by considering wider classes of the generalized inverses involved. Henceforth, $\mathbf{A}\{1,2,\mathrm{H}\}$ will denote the set of all Hermitian reflexive generalized inverses of a Hermitian nonnegative definite $\mathbf{A} \in \mathbb{C}_{m,m}$. Observe that all matrices in $\mathbf{A}\{1,2,\mathrm{H}\}$ are Hermitian nonnegative definite (cf. Wu [37, Theorem 1]) and also that $\mathbf{A}^+ \in \mathbf{A}\{1,2,\mathrm{H}\}$.

THEOREM 4.4. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,m}$ be Hermitian nonnegative definite. Then, for any $\mathbf{G}_\mathbf{A} \in \mathbf{A}\{1,2,\mathrm{H}\}$ and $\mathbf{G}_\mathbf{B} \in \mathbf{B}\{1,2,\mathrm{H}\}$,*

$$\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \text{ and } \mathbf{G}_\mathbf{B} \overset{\mathrm{s}}{\prec} \mathbf{G}_\mathbf{A} \quad \Leftrightarrow \quad \mathbf{A} \overset{\mathrm{s}}{\prec} \mathbf{B} \text{ and } \mathbf{G}_\mathbf{B} \overset{\mathrm{L}}{\leqslant} \mathbf{G}_\mathbf{A}.$$

Proof. Since $\mathbf{A} \in \mathbf{G}_{\mathbf{A}}\{1\}$, it follows from (1.20) that the proof of the $\Rightarrow$ part reduces to establishing that

$$\mathbf{G}_{\mathbf{B}}\mathbf{A}\mathbf{G}_{\mathbf{B}} \overset{\mathrm{L}}{\leqslant} \mathbf{G}_{\mathbf{B}}. \tag{4.4}$$

But (4.4) is equivalent to

$$\mathbf{G}_{\mathbf{B}}(\mathbf{A}-\mathbf{B})\mathbf{G}_{\mathbf{B}} \overset{\mathrm{L}}{\leqslant} \mathbf{0},$$

which is a straightforward consequence of $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$ and $\mathbf{G}_{\mathbf{B}} = \mathbf{G}_{\mathbf{B}}^{*}$. The reverse implication follows similarly. ■

A somewhat different approach to the problem of reversing the ordering $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$ results in the following:

THEOREM 4.5. *Let* $\mathbf{A},\mathbf{B} \in \mathbf{C}_{m,m}$ *be Hermitian nonnegative definite. If* $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$, *then for any* $\mathbf{G}_{\mathbf{A}} \in \mathbf{A}\{1,2,\mathrm{H}\}$ *and* $\mathbf{G}_{\mathbf{B}} \in \mathbf{B}\{1,2,\mathrm{H}\}$ *the following statements are equivalent*:

(a) $\mathbf{G}_{\mathbf{B}} \overset{\mathrm{L}}{\leqslant} \mathbf{G}_{\mathbf{A}}$,
(b) $\mathbf{G}_{\mathbf{B}} \overset{\mathrm{s}}{\prec} \mathbf{G}_{\mathbf{A}}$,
(c) $\mathscr{R}(\mathbf{G}_{\mathbf{A}}) = \mathscr{R}(\mathbf{G}_{\mathbf{B}})$,
(d) $\mathbf{A}\mathbf{G}_{\mathbf{A}} = \mathbf{B}\mathbf{G}_{\mathbf{B}}$.

Proof. The result that if $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$ then (a) $\Leftrightarrow$ (d) is due to Styan and Pukelsheim [35]; for a quick proof see Styan [34, Theorem 1.1]. That (a) $\Rightarrow$ (b) follows by (1.19). The part (b) $\Rightarrow$ (c) is a consequence of the equalities $\mathrm{r}(\mathbf{G}_{\mathbf{A}}) = \mathrm{r}(\mathbf{A})$ and $\mathrm{r}(\mathbf{G}_{\mathbf{B}}) = \mathrm{r}(\mathbf{B})$ and the inequality $\mathrm{r}(\mathbf{A}) \leqslant \mathrm{r}(\mathbf{B})$. Further, since $\mathbf{A} \in \mathbf{G}_{\mathbf{A}}\{1\}$, the condition (c) implies that $\mathbf{G}_{\mathbf{B}} = \mathbf{G}_{\mathbf{B}}\mathbf{A}\mathbf{G}_{\mathbf{A}}$. Premultiplying this equality by $\mathbf{B}$ and using $\mathbf{B}\mathbf{G}_{\mathbf{B}}\mathbf{A} = \mathbf{A}$ yields (d). ■

It is clear that the condition (c) in Theorem 4.5 implies $\mathrm{r}(\mathbf{A}) = \mathrm{r}(\mathbf{B})$. Contrary to the case when $\mathbf{G}_{\mathbf{A}} = \mathbf{A}^{+}$ and $\mathbf{G}_{\mathbf{B}} = \mathbf{B}^{+}$ (cf. Theorem 4.3), this rank equality is not sufficient for the ordering

$$\mathbf{G}_{\mathbf{B}} \overset{\mathrm{L}}{\leqslant} \mathbf{G}_{\mathbf{A}} \tag{4.5}$$

to hold for any Hermitian reflexive generalized inverses of $\mathbf{A}$ and $\mathbf{B}$. A counterexample is obtained by taking

$$\mathbf{A} = \mathbf{G}_{\mathrm{a}} = \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{G}_{\mathbf{B}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

A simple consequence of Theorem 4.5 is the result originally given in Theorem 7 of Wu [37].

THEOREM 4.6. *Let* $\mathbf{A},\mathbf{B} \in \mathbf{C}_{m,m}$ *be Hermitian nonnegative definite. If* $\mathrm{r}(\mathbf{A}) = \mathrm{r}(\mathbf{B})$ *and* $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$, *then for any fixed* $\mathbf{G}_{\mathbf{A}} \in \mathbf{A}\{1,2,\mathrm{H}\}$ *there exists a* $\mathbf{G}_{\mathbf{B}} \in \mathbf{B}\{1,2,\mathrm{H}\}$, *and for any fixed* $\mathbf{G}_{\mathbf{B}} \in \mathbf{B}\{1,2,\mathrm{H}\}$ *there exists a* $\mathbf{G}_{\mathbf{A}} \in \mathbf{A}\{1,2,\mathrm{H}\}$, *such that* $\mathbf{G}_{\mathbf{B}} \overset{\mathrm{L}}{\leqslant} \mathbf{G}_{\mathbf{A}}$.

Proof. In view of (1.20), the assumptions $\mathrm{r}(\mathbf{A}) = \mathrm{r}(\mathbf{B})$ and $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$ imply $\mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{B})$ and $\mathbf{A}\mathbf{B}^{+}\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{A}$. Given $\mathbf{G}_{\mathbf{A}} \in \mathbf{A}\{1,2\}$, let $\mathbf{G}_{\mathbf{B}} = \mathbf{G}_{\mathbf{A}}\mathbf{A}\mathbf{B}^{+}\mathbf{A}\mathbf{G}_{\mathbf{A}}^{*}$, and notice that $\mathbf{B}\mathbf{G}_{\mathbf{B}}\mathbf{B} = \mathbf{B}\mathbf{B}^{+}\mathbf{B} = \mathbf{B}$, $\mathrm{r}(\mathbf{G}_{\mathbf{B}}) = \mathrm{r}(\mathbf{A}\mathbf{B}^{+}\mathbf{A}) = \mathrm{r}(\mathbf{B})$, and

$$\mathbf{G}_{\mathbf{A}} - \mathbf{G}_{\mathbf{B}} = \mathbf{G}_{\mathbf{A}}(\mathbf{A} - \mathbf{A}\mathbf{B}^{+}\mathbf{A})\mathbf{G}_{\mathbf{A}}^{*} \overset{\mathrm{L}}{\geqslant} \mathbf{0},$$

as desired. The proof of the second statement follows similarly. ■

Theorem 4.5 obviously implies that there are no $\mathbf{G}_{\mathbf{A}} \in \mathbf{A}\{1,2,\mathrm{H}\}$ and $\mathbf{G}_{\mathbf{B}} \in \mathbf{B}\{1,2,\mathrm{H}\}$ satisfying (4.4) when Hermitian nonnegative definite $\mathbf{A},\mathbf{B} \in \mathbf{C}_{m,m}$ are such that $\mathrm{r}(\mathbf{A}) < \mathrm{r}(\mathbf{B})$. However, the ordering (4.4) may hold when the generalized inverses involved need not be reflexive. This was shown by Werner [36, Theorem 2] and, in a much stronger form, by Wu [37, Theorem 5]. The first part of Wu's result is restated below as Theorem 4.7 with a new, shorter proof.

THEOREM 4.7. *Let* $\mathbf{A},\mathbf{B} \in \mathbf{C}_{m,m}$ *be such that* $\mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$, *and let* a, b *be positive integers such that* $\mathrm{r}(\mathbf{B}) \leqslant b \leqslant a \leqslant m$. *Then for any fixed* $\mathbf{G}_{\mathbf{B}} \in \mathbf{B}\{1, \geqslant\}$ *of rank* b *there exists a* $\mathbf{G}_{\mathbf{A}} \in \mathbf{A}\{1, \geqslant\}$ *of rank* a *such that* $\mathbf{G}_{\mathbf{B}} \overset{\mathrm{L}}{\leqslant} \mathbf{G}_{\mathbf{A}}$.

Proof. In view of (1.20), the assumption $\mathbf{0} \overset{L}{\leqslant} \mathbf{A} \overset{L}{\leqslant} \mathbf{B}$ implies that $\mathbf{AG_BB} = \mathbf{A}$ and $\mathbf{AG_BA} \overset{L}{\leqslant} \mathbf{A}$. Consequently, the matrix

$$\mathbf{G} = \mathbf{G_B} + \mathbf{G_BBA^+(A - AG_BA)A^+BG_B}$$

satisfies the conditions $\mathbf{G_B} \overset{L}{\leqslant} \mathbf{G}$, $r(\mathbf{G}) = r(\mathbf{G_B}) = b$, and $\mathbf{G} \in \mathbf{A}\{1\}$. This actually concludes the proof, for Theorem 3.7 assures the existence of $\mathbf{G_A}$ such that $r(\mathbf{G_A}) = a$ and $\mathbf{G} \overset{L}{\leqslant} \mathbf{G_A}$, which in turn entails $\mathbf{G_B} \overset{L}{\leqslant} \mathbf{G_A}$. ■

An immediate consequence of Theorem 4.7 and the fact that the Löwner ordering of Hermitian nonnegative definite matrices entails the corresponding space preordering [cf. (1.19) and (1.20)] is that if $\mathbf{0} \overset{L}{\leqslant} \mathbf{A} \overset{L}{\leqslant} \mathbf{B}$, then for any Hermitian nonnegative definite $\mathbf{C} \in \mathbb{C}_{m,m}$ and for any $(\mathbf{A}+\mathbf{C})^- \in (\mathbf{A}+\mathbf{C})\{1\}$ and $(\mathbf{B}+\mathbf{C})^- \in (\mathbf{B}+\mathbf{C})\{1\}$ we have

$$\mathbf{A_1^*(B+C)^-A_1} \overset{L}{\leqslant} \mathbf{A_1^*(A+C)^-A_1} \quad \text{and} \quad \mathbf{C_1^*(B+C)^-C_1} \overset{L}{\leqslant} \mathbf{C_1^*(A+C)^-C_1},$$

where $\mathbf{A_1A_1^*} = \mathbf{A}$ and $\mathbf{C_1C_1^*} = \mathbf{C}$. Consequently,

$$\operatorname{trace}\left[(\mathbf{B}+\mathbf{C})^-\mathbf{A}\right] \leqslant \operatorname{trace}\left[(\mathbf{A}+\mathbf{C})^-\mathbf{A}\right]$$

and

$$\operatorname{trace}\left[(\mathbf{B}+\mathbf{C})^-\mathbf{C}\right] \leqslant \operatorname{trace}\left[(\mathbf{A}+\mathbf{C})^-\mathbf{C}\right]. \tag{4.6}$$

The inequality (4.6) may be applied to establish a generalization of a result in Patel and Toda [28, Inequality V].

COROLLARY 4.1. *Let* $\mathbf{A},\mathbf{B} \in \mathbb{C}_{m,m}$ *be such that* $\mathbf{0} \overset{L}{\leqslant} \mathbf{A} \overset{L}{\leqslant} \mathbf{B}$. *Then, for any Hermitian nonnegative definite* $\mathbf{C} \in \mathbb{C}_{m,m}$ *and any* $(\mathbf{A}+\mathbf{C})^- \in (\mathbf{A}+\mathbf{C})\{1\}$ *and* $(\mathbf{B}+\mathbf{C})^- \in (\mathbf{B}+\mathbf{C})\{1\}$,

$$\operatorname{trace}\left[(\mathbf{A}+\mathbf{C})^-\mathbf{A}\right] \leqslant \operatorname{trace}\left[(\mathbf{B}+\mathbf{C})^-\mathbf{B}\right].$$

Proof. It is clear that $\mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{A}+\mathbf{C} \overset{\mathrm{L}}{\leqslant} \mathbf{B}+\mathbf{C}$. Hence, $\mathrm{r}(\mathbf{A}+\mathbf{C}) \leqslant \mathrm{r}(\mathbf{B}+\mathbf{C})$ [cf. (1.19)] and using this inequality along with (4.5) yields

$$\begin{aligned}
\operatorname{trace}\left[(\mathbf{A}+\mathbf{C})^{-}\mathbf{A}\right] &= \operatorname{trace}\left[(\mathbf{A}+\mathbf{C})^{-}(\mathbf{A}+\mathbf{C}-\mathbf{C})\right] \\
&= \mathrm{r}(\mathbf{A}+\mathbf{C}) - \operatorname{trace}\left[(\mathbf{A}+\mathbf{C})^{-}\mathbf{C}\right] \\
&\leqslant \mathrm{r}(\mathbf{B}+\mathbf{C}) - \operatorname{trace}\left[(\mathbf{B}+\mathbf{C})^{-}\mathbf{C}\right] \\
&= \operatorname{trace}\left[(\mathbf{B}+\mathbf{C})^{-}(\mathbf{B}+\mathbf{C}-\mathbf{C})\right] \\
&= \operatorname{trace}\left[(\mathbf{B}+\mathbf{C})^{-}\mathbf{B}\right],
\end{aligned}$$

as desired. ■

Hartwig [14, Theorem 1] proved that if Hermitian nonnegative definite $\mathbf{A},\mathbf{B} \in \mathbb{C}_{m,m}$ are ordered as $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$, then their Moore-Penrose inverses cannot be ordered as $\mathbf{A}^{+} \overset{\mathrm{L}}{\leqslant} \mathbf{B}^{+}$ unless $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$, which in view of (1.9) is equivalent to the equality $\mathbf{AB}^{+} = \mathbf{AA}^{+}$. The following theorem generalizes this result to Hermitian reflexive generalized inverses.

THEOREM 4.8. *Let* $\mathbf{A},\mathbf{B} \in \mathbb{C}_{m,m}$ *be such that* $\mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$, *and let* $\mathbf{G_A} \in \mathbf{A}\{1,2,\mathrm{H}\}$ *and* $\mathbf{G_B} \in \mathbf{B}\{1,2,\mathrm{H}\}$. *Then the following statements are equivalent*:

(a) $\mathbf{G_A} \overset{\mathrm{L}}{\leqslant} \mathbf{G_B}$,
(b) $\mathbf{AG_B} = \mathbf{AG_A}$,
(c) $\mathbf{G_A} \overset{\mathrm{rs}}{\leqslant} \mathbf{G_B}$.

Proof. In view of (1.20), it follows that

$$\mathbf{0} \overset{\mathrm{L}}{\leqslant} \mathbf{A} - \mathbf{AG_BA} = \mathbf{A}(\mathbf{G_A} - \mathbf{G_B})\mathbf{A}.$$

Combining this with (a) yields the equality $\mathbf{A}(\mathbf{G_B} - \mathbf{G_A})\mathbf{A} = \mathbf{0}$, and hence (b). Since (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) follow directly from (1.6) and (1.18), respectively, the proof is complete. ■

The condition $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ (cf. Hartwig [14]) for Hermitian nonnegative definite $\mathbf{A},\mathbf{B} \in \mathbb{C}_{m,m}$ to satisfy simultaneously $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$ and $\mathbf{A}^+ \overset{\mathrm{L}}{\leqslant} \mathbf{B}^+$ may also be expressed as the equality $\mathbf{BA}^+ = \mathbf{AA}^+$; cf. (1.10). Its counterpart $\mathbf{BG_A} = \mathbf{AG_A}$, however, proves to be insufficient for $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$ to hold along with $\mathbf{G_A} \overset{\mathrm{L}}{\leqslant} \mathbf{G_B}$ when $\mathbf{G_A}$ and $\mathbf{G_B}$ are any Hermitian reflexive generalized inverses. The reason is that the preordering $\mathbf{G_A} \overset{\mathrm{s}}{\prec} \mathbf{G_B}$ is then not assured as seen by the following example:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{G_A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{G_B} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4.8 solves the problem of isotonicity of Hermitian reflexive generalized inverses with respect to the Löwner and minus partial orderings. When considering the same problem in the context of the star ordering, the first observation is that (4.1) fails to be true if the Moore-Penrose inverses $\mathbf{A}^+$ and $\mathbf{B}^+$ are replaced therein by any $\mathbf{G_A} \in \mathbf{A}\{1,2,\mathrm{H}\}$ and any $\mathbf{G_B} \in \mathbf{B}\{1,2,\mathrm{H}\}$, respectively. An example is obtained by taking

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{G_A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{G_B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

in which case $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ while $\mathbf{G_A}$ is not a predecessor of $\mathbf{G_B}$ even under the Löwner ordering. A general solution is given in the following:

THEOREM 4.9. *Let Hermitian nonnegative definite* $\mathbf{A},\mathbf{B} \in \mathbb{C}_{m,m}$ *be such that* $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$, *and let* $\mathbf{G_A} \in \mathbf{A}\{1,2,\mathrm{H}\}$ *and* $\mathbf{G_B} \in \mathbf{B}\{1,2,\mathrm{H}\}$. *Then* $\mathbf{G_A} \overset{*}{\leqslant} \mathbf{G_B}$ *if and only if* $\mathbf{G_A} \overset{\mathrm{L}}{\leqslant} \mathbf{G_B}$ *and* $\mathbf{G_AG_B} = \mathbf{G_BG_A}$.

Proof. The necessity is obvious in view of (1.18), (1.15), and (1.5). Conversely, according to Theorem 4.8, $\mathbf{G_A} \overset{\mathrm{L}}{\leqslant} \mathbf{G_B}$ implies that $\mathbf{AG_B} = \mathbf{AG_A}$, and then

$$\mathbf{G_A G_B} = \mathbf{G_A A G_A G_B} = \mathbf{G_A A G_B G_A} = \mathbf{G_A A G_A^2} = \mathbf{G_A^2},$$

as desired. ■

The commuting condition in Theorem 4.9 may be deleted when $\mathbf{G_A} = \mathbf{A}^+$. In view of Theorem 4.8, $\mathbf{A}^+ \overset{L}{\leqslant} \mathbf{G_B}$ implies that $\mathbf{AG_B} = \mathbf{AA}^+$. Premultiplying this equality by $(\mathbf{A}^+)^2$ yields $\mathbf{A}^+\mathbf{G_B} = (\mathbf{A}^+)^2 = \mathbf{G_B A}^+$. This establishes the following:

COROLLARY 4.2. *Let Hermitian nonnegative definite* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,m}$ *be such that* $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$, *and let* $\mathbf{G_B} \in \mathbf{B}\{1,2,\mathrm{H}\}$. *Then* $\mathbf{A}^+ \overset{*}{\leqslant} \mathbf{G_B}$ *if and only if* $\mathbf{A}^+ \overset{L}{\leqslant} \mathbf{G_B}$.

This concludes our results on some properties of matrix partial orderings.

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