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FRONT MOTION IN THE ONE-DIMENSIONAL STOCHASTIC CAHN-HILLIARD EQUATION

D.C. ANTONOPOULOU[†¶], D. BLÖMKER[‡], G.D. KARALI[†¶]

Abstract. In this paper, we consider the one-dimensional Cahn-Hilliard equation perturbed by additive noise and study the dynamics of interfaces for the new stochastic model. The noise is smooth in space and is defined as a Fourier series with independent Brownian motions in time. Motivated by the work of Bates & Xun on slow manifolds for the integrated Cahn-Hilliard equation, our analysis reveals the significant difficulties and differences in comparison with the deterministic problem. New higher order terms, that we estimate, appear due to Itô calculus and stochastic integration dominating the exponentially slow deterministic dynamics of the interfaces. We derive a first order linear system of stochastic ordinary differential equations approximating the equations of front motion. Furthermore, we prove stochastic stability for the approximate slow manifold of solutions on a very long time scale and evaluate the noise effect.

Keywords: 1-D Stochastic Cahn-Hilliard, slow manifold, interface motion, additive noise, dynamics.

1. Introduction

1.1. **The problem.** The standard Cahn-Hilliard equation is a simple model for the phase separation of a binary alloy at a fixed temperature, proposed in [17, 18]. This model was extended by Cook [23, 40] in order to incorporate thermal fluctuations in the form of an additive noise. In this paper, we consider the one-dimensional Cahn-Hilliard equation posed on $(0,1)$ with an additive stochastic term:

$$u_t = (-\varepsilon^2 u_{xx} + f(u))_{xx} + \partial_x \dot{W}, \quad 0 < x < 1, \quad t > 0, \tag{SC-H}$$

with no-flux boundary conditions of Neumann type:

$$u_x = u_{xxx} = 0 \ \text{ at } \ x = 0, 1. \tag{1.1}$$

Here, $\dot{W}$ is a smooth in space space-time noise defined as the formal derivative of a Wiener process W. The nonlinearity $f = f(u)$ is the derivative of a smooth double equal-well potential F taking its global minimum value 0 at $u = \pm 1$ [1], with non-degenerate minima. A typical example is $F(u) := \frac{1}{4}(u^2-1)^2$ with $f(u) := u^3 - u$. The parameter $\varepsilon > 0$ is a small atomistic interaction length modeling the width of layers that develop during the initial phase separation of spinodal decomposition (cf. [12, 13]). In the later stages of the separation process ε measures the width of interfacial regions between the pure phases $u = \pm 1$.

A characteristic feature of the Cahn-Hilliard model is the conservation of total mass $\int_0^1 u(t,x)dx$, which we now fix to be $M \in (-1,1)$. Substituting $\tilde{u}(t,x) := \int_0^x u(t,y)dy$ we obtain the equivalent integrated stochastic Cahn-Hilliard equation:

$$\tilde{u}_t = -\varepsilon^2 \tilde{u}_{xxxx} + (f(\tilde{u}_x))_x + \dot{W}, \quad 0 < x < 1, \quad t > 0, \tag{ISC-H}$$

associated with the boundary conditions:

$$\begin{aligned} &\tilde{u}(0,t) = 0, \quad \tilde{u}(1,t) = M, \\ &\tilde{u}_{xx}(0,t) = \tilde{u}_{xx}(1,t) = 0. \end{aligned} \tag{1.2}$$

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J. Carr and R. Pego in [21, 22] presented a detailed analysis of the slow evolution of patterns of the singularly perturbed Ginzburg-Landau equation. They proved existence and persistence of metastable patterns and analyzed the equations governing their motion. These metastable states have been characterized in terms of the global unstable manifolds of equilibria. In [7, 8], P.W. Bates and J. Xun extended their argument and studied the dynamics of the one-dimensional C-H equation in a neighborhood of an equilibrium having $N+1$ transition layers, using several estimates presented in [21, 22]. They determined the exponentially slow speed of the layer motion and described precisely the layer motion directions. In addition, they established existence of an N-dimensional unstable invariant manifold attracting solutions exponentially fast uniformly in ε. Related work in this direction are [9, 33, 41].

Motivated by the work of Bates and Xun for the deterministic problem, we study dynamics for the stochastic model. Due to stochastic integration, new higher order terms appear that we estimate using techniques and ideas of [7, 8, 21, 22]. In the sequel we shall refer frequently to some important definitions and results presented in the aforementioned articles, therefore, we give some details concerning our notation. Following [21, 22], we use the letter f for the nonlinearity in (SC-H), and denote by F the double equal well potential. In [7, 8] the symbol W' is used in place of f; we avoided such a notation since we name by the standard symbol $\dot{W}$ the additive noise.

1.2. **The effect of noise.** The stochastic Cahn-Hilliard equation being one of the important examples of the nonlinear Langevin equations is based on a field-theoretic approach to the non-equilibrium dynamics of metastable states (see for example [23, 37, 40]). The multi-dimensional generalized stochastic Cahn-Hilliard equation associated with Neumann boundary conditions posed on bounded domains contains a time dependent noise into the chemical potential and an additive multiplicative noise defined as the formal derivative of a Wiener process. The chemical potential noise describes external fields [37, 35, 39], while the free-energy independent noise may describe thermal fluctuations or external mass supply [23, 40, 37, 35].

Existence and uniqueness of solution for the stochastic problem was first studied in [24], where the nonlinearity f is a polynomial of odd degree and the problem is posed on multi-dimensional rectangular domains. Further, in [19], the author proved existence of solution and of its density for the stochastic Cahn-Hilliard with additive noise (in the sense of Walsh, cf. [44]) posed on cubic domains. When the trace of the Wiener process is finite, existence was analyzed in [28]. In [4], existence for the generalized stochastic Cahn-Hilliard equation was derived for general convex or Lipschitz domains; the main novelty was the derivation of space-time Hölder estimates for the Green's kernel of the stochastic problem, by using the domain's geometry, which can be very useful in many other circumstances. The polynomial nonlinearity which forces the solution to stay between the pure phases ± 1 has been analyzed in [12, 13, 19, 20, 24, 28], while in [27, 26, 34] a stochastic Cahn-Hilliard with reflection was considered.

In [12, 13] (see [14] for a review), the effect of noise on evolving interfaces during the initial stage of phase separation is analyzed. The evolution of these interfaces is stochastic and not yet fully understood. In [12], the authors show that for a solution starting at the homogeneous state, the probability of staying near a certain finite-dimensional space of pattern is high as long the solution stays within the distance of the size of the homogeneous state. Further, in [13], the dynamics of a nonlinear partial differential equation perturbed by additive noise are considered. Under the assumption that the underlying deterministic equation has an unstable equilibrium, the authors show that the nonlinear stochastic partial differential equation exhibits essentially linear dynamics far from equilibrium.

On the other hand interface motion has been studied for many related models like Allen-Cahn or Ginzburg Landau and phase-field models, cf. for example [15, 11] for a rigorous analysis or the results of [30] for formal arguments, which describe the interfaces as interacting Brownian motions. Numerical results for interface motion are [43, 36]. The problem of singular perturbation for a reaction-diffusion stochastic partial differential equation of Ginzburg-Landau type is investigated in [32]. The motion of interfaces for Cahn-Hilliard was only studied in an unpublished note by S. Brassesco in 2003, where she studied a solution with a single interface on $\mathbb{R}$. When properly rescaled the interface is driven by non-Markovian dynamics. A similar result is [11]. In [42], the authors present a numerical study of the late stages of spinodal decomposition with noise.

The deterministic Cahn-Hilliard equation was proposed by Cahn and Hilliard ([17, 16]) as a model for the phase separation of a binary alloy at a fixed temperature, with $u(x,t)$ defining the mass concentration of one of the phases at a point x at time t. For more physical background, derivation and discussion of the deterministic Cahn-Hilliard equation and related equations we refer to [6, 16, 17, 29, 31] and the references therein. Results for the noisy Cahn-Hilliard equation are of great interest for the studying of Ostwald ripening [2, 3, 38] and nucleation [10]. For a survey, including numerical results and conjectures concerning the nucleation problem, see [14].

1.3. **The approximate slow manifold.** The space-time noise that we introduce is smooth in space, so, integration in space is deterministic. Therefore, in order to study the transition layers dynamics for the stochastic model in the finite interval $(0,1)$, we closely follow the approach of Bates & Xun and Carr & Pego that is based on the analysis of an approximate invariant manifold $\mathcal{M}$. Although constructed in a different way, it can be thought of as piecing together a rescaled one kink (or front) steady state solution on the whole real-line. The elements of the manifold are parametrized by the position of the fronts given by $h \in \mathbb{R}^{N+1}$. Nevertheless, in our case the dependency on time is stochastic. This fact arises the very interesting and difficult problem of investigating further the properties of $\mathcal{M}$ by means of deriving higher order estimates related to the stationary problem.

Let us present first the details necessary for the steady state solutions ϕ, the parameters h and the manifold $\mathcal{M}$. Given $\varepsilon > 0$, we consider a such that $f'(u) > 0$ for all u satisfying $|u \pm 1| < a$. Then, cf. [21], there exists $\rho > 0$ such that if ℓ satisfies $\frac{\varepsilon}{\ell} < \rho$ then a unique solution $\phi = \phi(x, \ell, \pm 1)$ exists for the following stationary Dirichlet problem

$$\begin{aligned} &\varepsilon^2 \phi_{xx} - f(\phi) = 0, \quad -\ell/2 < x < \ell/2, \\ &\phi = 0, \quad x = \pm \ell/2, \end{aligned} \tag{1.3}$$

that satisfies: (a) $\phi(x, \ell, +1) > 0$ for $|x| < \ell/2$, and $|\phi(0) - 1| < a$, (b) $\phi(x, \ell, -1) < 0$ for $|x| < \ell/2$, and $|\phi(0) + 1| < a$. For $\varepsilon > 0$ small, it is known that $\phi \approx \pm 1$ with transition layers of order $\mathcal{O}(\varepsilon)$ near $x = \pm \ell/2$.

Following [8], we consider the slowly evolving solutions with $N+1$ layers well separated and bounded away from the boundary $x = 0, 1$ and define the set of admissible positions h of the interfaces

$$\Omega_\rho := \Big\{ h \in R^{N+1} : 0 < h_1 < \cdots < h_{N+1} < 1, \text{ and } \frac{\varepsilon}{\rho} < h_j - h_{j-1}, \;\; j = 1, \ldots, N+2 \Big\}, \tag{1.4}$$

with $h_0 := -h_1$, $h_{N+2} := 2 - h_{N+1}$. These interfaces evolve in time, and we expect them to have a width of order ε. Thus, the distance is bounded below by ε/ρ for some small ρ. Later we fix $\rho = \varepsilon^\kappa$ for any small $\kappa > 0$.

Let $h \in \Omega_\rho$ be given as above, and denote the mid points between interfaces by $m_j := \frac{h_{j-1} + h_j}{2}$ for $j = 1, \ldots, N+2$ with $m_0 = 0$ and $m_{N+1} = 1$. Moreover, we define the function $u^h : I_j := [m_j, m_{j+1}] \to \mathbb{R}$ for the interfaces h by

$$\begin{aligned} u^h(x) = &\Big[1 - \chi\left(\tfrac{x - h_j}{\varepsilon}\right)\Big] \cdot \phi\left(x - m_j, h_j - h_{j-1}, (-1)^j\right) \\ &+ \chi\left(\tfrac{x - h_j}{\varepsilon}\right) \cdot \phi\left(x - m_{j+1}, h_{j+1} - h_j, (-1)^{j+1}\right), \end{aligned} \tag{1.5}$$

where $\chi : \mathbb{R} \to [0,1]$ is a C^∞ cut-off function such that $\chi = 1$ on $[1, \infty)$ and $\chi = 0$ on $(-\infty, -1]$.

Definition 1.1 (approximate slow manifold). The first approximate manifold of the stochastic Cahn-Hilliard solution is defined by

$$\mathcal{M}_1 := \Big\{ u^h : h \in \Omega_\rho \Big\}.$$

Fixing a mass $M \in (-1,1)$, we define as the second approximate manifold the submanifold $\mathcal{M}$ of $\mathcal{M}_1$ where mass conservation holds i.e.

$$\mathcal{M} := \Big\{ u^h \in \mathcal{M}_1 : \int_0^1 u^h dx = M \Big\}.$$

For the integrated equation, we consider the manifold

$$\tilde{\mathcal{M}} := \Big\{ \tilde{u}^h \ : \ u^h \in \mathcal{M}, \ \tilde{u}^h(x) = \int_0^x u^h dx \Big\}.$$

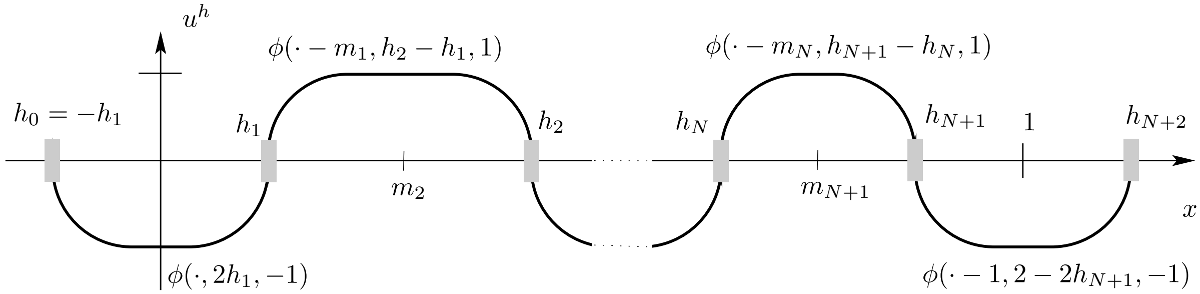

FIGURE 1.1. Gluing together positive and negative solutions of (1.3) to obtain $u^h \in \mathcal{M}$. Note that $m_1 = 0$, $m_{N+2} = 1$, and $I_j = [m_j, m_{j+1}]$.

Remark 1.2. In view of the initial stochastic equation (SC-H), conservation of mass holds if and only if formally

$$\int_0^1 \partial_x \dot{W} dy = \dot{W}(1) - \dot{W}(0) = 0. \tag{1.6}$$

This is later assured by our assumptions on W, which impose Dirichlet-boundary conditions (cf. Definition 2.2 and Assumption 2.3). A very simple rigorous example is the following: consider $\dot{W} := g(x)\dot{V}(t)$, where $\dot{V}(t)$ is a white noise in time and g a smooth function satisfying $g(1) = g(0)$, then by integrating in space the equation (SC-H) and using the fact that

$$\int_0^1 \partial_x \dot{W} dy = \dot{V}(t) \int_0^1 g_x(y) dy = 0,$$

we obtain mass conservation even with the noise. We can extend this example to infinite series of terms of these type.

Throughout the entire paper we will assume that the additive noise in (SC-H) satisfies (1.6), and therefore *the proposed stochastic model exhibits mass conservation.*

1.4. **The new coordinate system.** Along $\tilde{\mathcal{M}}$ the natural coordinate system would be to use the parameters $h \in \Omega_\rho$ for the position in $\tilde{\mathcal{M}}$ (where N of them are sufficient due to mass conservation), together with the orthogonal projection onto $\tilde{\mathcal{M}}$. In order to relate the coordinate system to the deterministic flow of (ISC-H), one approximates the tangential space of $\tilde{\mathcal{M}}$ by the span of some functions E_i^h, $i = 1, \dots, N$ to be defined in the sequel; here, we follow [7].

We denote the $L^2(0,1)$ inner product by $\langle u, v\rangle := \int_0^1 uvdx$, the induced L^2-norm by $\|\cdot\|$ and introduce the symbol $\tilde{g}(x,t) := \int_0^x g(y,t)dy$, for any g smooth in space.

Due to mass conservation, we reduce the parameter space Ω_ρ by one dimension, define

$$\xi := (\xi_1, \dots, \xi_N) = (h_1, \dots, h_N),$$

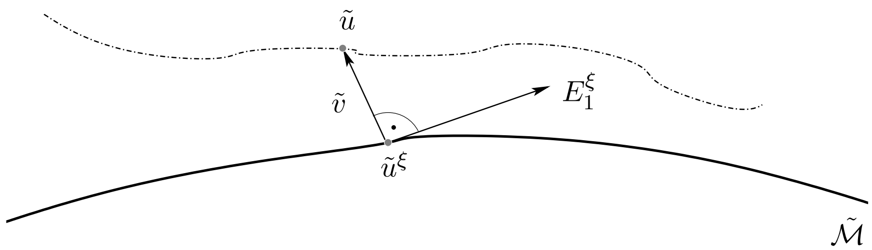

FIGURE 1.2. The local coordinate system $\tilde{u} = \tilde{u}^\xi + \tilde{v}$ around $\tilde{\mathcal{M}}$ for $N = 1$ (two interfaces). Note that $E_1^\xi \approx \tilde{u}_1^\xi$, which is the tangential vector along the manifold.

and consider h_{N+1} as a function of ξ. Thus, for $\tilde{u}_j^h := \frac{\partial \tilde{u}^h}{\partial h_j}$ and $\tilde{u}_j^\xi := \frac{\partial \tilde{u}^\xi}{\partial \xi_j}$ we obtain that

$$\tilde{u}_j^\xi = \frac{\partial \tilde{u}^h}{\partial h_{N+1}} \cdot \frac{\partial h_{N+1}}{\partial h_j} + \frac{\partial \tilde{u}^h}{\partial h_j} \ .$$

We use the following coordinate system around $\tilde{\mathcal{M}}$: $\tilde{u} \to (\xi, \tilde{v})$, where we write the stochastic solution $\tilde{u}$ of (ISC-H) as a sum of stochastic processes

$$\tilde{u}(t) := \tilde{u}^{\xi(t)} + \tilde{v}(t). \tag{1.7}$$

Here the position on $\tilde{\mathcal{M}}$ is given by $\tilde{u}^\xi \in \tilde{\mathcal{M}}$ while the distance from $\tilde{\mathcal{M}}$ is given by $\tilde{v}$ which is defined as the following projection such that

$$\langle \tilde{v}, E_j^\xi \rangle = 0 \quad \text{for} \quad j = 1, \ldots, N \ . \tag{1.8}$$

It turns out that the functions E_j^ξ are good approximations to the first eigenfunctions of the linearized integrated Cahn-Hilliard operator, which in turn are good approximations to the tangential space of $\tilde{\mathcal{M}}$. They are defined as follows:

$$E_j^\xi := \tilde{w}_j(x) - Q_j(x), \qquad \tilde{w}_j := \tilde{u}_j^h(x) + \tilde{u}_{j+1}^h(x),$$
$$Q_j(x) := (-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x)\tilde{w}_{jxx}(0) + \frac{1}{6}(x^3 - x)\tilde{w}_{jxx}(1) + x\tilde{w}_j(1), \quad j = 1, \ldots, N,$$

where the Q_j later turn out to be exponentially small terms (cf. [7]), that only takes care of the boundary values of E_j^ξ.

For short-hand notation, we also define higher derivatives using indices

$$E_{il}^\xi := \frac{\partial E_i^\xi}{\partial \xi_l}, \;\; E_{ilk}^\xi := \frac{\partial^2 E_i^\xi}{\partial \xi_l \partial \xi_k}, \;\; \tilde{u}_{kl}^\xi := \frac{\partial^2 \tilde{u}^\xi}{\partial \xi_k \partial \xi_l}. \tag{1.9}$$

The rest of the paper is organized as follows: In Section 2, we present the main results including a proper definition of the noise, the equations for the motion of the interfaces, the stability of the approximate manifold $\tilde{\mathcal{M}}$, and the approximation of the front motion in a neighborhood of $\tilde{\mathcal{M}}$. The proofs of the main results appear in Sections 3 and 4, while the final Section 5 collects all the estimates of the higher order terms appearing in the stochastic dynamics. Here, we consider the stationary problem (1.3) and analyze the properties of its solution by deriving bounds for higher order derivatives, extending some of the results of [21, 22, 7, 8].

2. Main results

The SDE (Stochastic Differential Equation) system for the motion of fronts is given by the projection onto the manifold $\tilde{\mathcal{M}}$, using the coordinate system of Section 1.4. We then prove that $\tilde{\mathcal{M}}$ is locally exponentially attracting and show that solutions stay with high probability in a small slow tube around $\tilde{\mathcal{M}}$, until large times or until one of the layers becomes small. The flow along $\tilde{\mathcal{M}}$ is well described by the SDE for the interfaces ξ. Depending on the strength of the noise we investigate how the equation of motion of the fronts looks like and evaluate the noise effect. In addition, we investigate extensively the case $N = 1$ where the motion of the second interface is determined by the first. Finally, the case of space-time white noise is discussed. In the final section we present the proofs of the estimates used in our analysis concerning all the higher order terms that appear in the to stochastic setting. These are technical results that are independent of the other section.

Let us first explain briefly how the equations of motions along $\tilde{\mathcal{M}}$ are derived in Section 3; for details we refer to Subsection 3.2. If $\tilde{u}$ is the solution of (ISC-H), then using the Itô-formula to differentiate $\tilde{u}^\xi$ in t we get

$$d\tilde{u} = \sum_{j=1}^{N} \tilde{u}_j^\xi d\xi_j + \frac{1}{2} \sum_{1\le k,l\le N} \tilde{u}_{kl}^\xi d\xi_k d\xi_l + d\tilde{v}. \tag{2.1}$$

We take the inner product in space of (ISC-H) with E_i^ξ to obtain for any $i = 1, \dots, N$:

$$\langle E_i^\xi, d\tilde{u}\rangle = \langle -\varepsilon^2 \tilde{u}_{xxxx} + (f(\tilde{u}_x))_x, E_i^\xi\rangle dt + \langle E_i^\xi, dW\rangle\ . \tag{2.2}$$

The inner product of (2.1) with E_i^ξ now gives

$$\langle E_i^\xi, d\tilde{u}\rangle = \sum_{j=1}^{N} \langle \tilde{u}_j^\xi, E_i^\xi\rangle d\xi_j + \tfrac{1}{2} \sum_{1\le k,l\le N} \langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle d\xi_k d\xi_l + \langle E_i^\xi, d\tilde{v}\rangle\ . \tag{2.3}$$

Applying the Itô-formula in differentiating in t the term $\langle \tilde{v}, E_i^\xi\rangle = 0$, using $d\tilde{u} = d\tilde{u}^\xi + d\tilde{v}$ and combining (2.2) with (2.3), we get for $i = 1, \cdots, N$ the following system in $d\xi_1, \cdots, d\xi_N$ for the stochastic Cahn-Hilliard equation:

$$\begin{aligned}\sum_j \Big[\langle \tilde{u}_j^\xi, E_i^\xi\rangle - \langle \tilde{v}, E_{ij}^\xi\rangle\Big] d\xi_j =& \langle -\varepsilon^2 (\tilde{u}_{xxxx}^\xi + \tilde{v}_{xxxx}) + (f(\tilde{u}_x^\xi + \tilde{v}_x))_x, E_i^\xi\rangle dt \\ &+ \sum_{l,k} \Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi\rangle\Big] d\xi_l d\xi_k \\ &+ \sum_j \langle dW, E_{ij}^\xi\rangle d\xi_j \\ &+ \langle E_i^\xi, dW\rangle\ .\end{aligned} \tag{2.4}$$

In the above, we denote that the last three additive terms at the right-hand side give the difference from the deterministic Cahn-Hilliard system of [8].

Remark 2.1. In view of (2.4), we observe that the study of dynamics for the stochastic Cahn-Hilliard, even in one dimension, arises a much more complicated and difficult problem in comparison with the deterministic one.

(1) *Deterministic case:* The system is linear in $d\xi_j$, therefore by estimating the inverse matrix on the left-hand side (which is possible close to $\tilde{\mathcal{M}}$) and the right-hand side terms, the motion of interfaces is obtained, see [8].

(2) *Stochastic case:* Obviously, for a general noise definition the system is non-linear due to the appearance of $d\xi_l d\xi_k$. In the sequel, we make an ansatz for ξ in order to get a linear system, which then justifies the ansatz. Further, we need estimates for the additional higher order terms E_{ij}^ξ, E_{ilk}^ξ, and $\tilde{u}_{kl}^\xi$. Here we need to improve the estimates of [7].

The sufficiently regular noise $\dot{W}$ is the formal derivative of a Wiener process W defined as follows.

Definition 2.2 (The Wiener process W). Let W be a $\mathcal{Q}$-Wiener process in the underlying Hilbert-space $H = L^2(0,1)$, $\mathcal{Q}$ a symmetric operator and $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis with corresponding eigenvalues α_k^2, such that

$$\mathcal{Q}e_k = \alpha_k^2 e_k \quad \text{and} \quad W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) e_k,$$

for a sequence of independent real-valued standard Brownian motions $\{\beta_k(t)\}_{t\geq 0}$ (cf. DaPrato, Zabzcyck [25]).

We will always use the following assumption, which is an assumption for mass conservation and regularity.

Assumption 2.3. Suppose that the e_k are also the eigenfunctions of the Dirichlet-Laplacian. Moreover, we assume that for some $0 < \delta_\varepsilon$

(1) $\|\mathcal{Q}\| < C\delta_\varepsilon^2$,
(2) $\sum_{k=1}^{\infty} \alpha_k^2 B_\varepsilon(e_k) < C\delta_\varepsilon^2$.

where for some small $\kappa > 0$ we have $\delta_\varepsilon < \varepsilon^{(8+\kappa)/(2-\kappa)}$.

The first assumption on the norm of $\mathcal{Q}$ as an operator in H means that the strength of the noise is bounded by $\mathcal{O}(\delta_\varepsilon)$, while the second one is an assumption on the noise regularity. Note that

$$B_\varepsilon(e) = \varepsilon^2\|e_{xx}\|^2 + \|e_x\|^2,$$

which is equivalent to the standard H^2-norm (see (3.14)).

The next crucial assumption considered in order to obtain the equation for the interfaces ξ is the following. Let $\tilde{u}$ be a solution of (ISC-H), then let $\xi(t)$ be a diffusion process in $\mathbb{R}^N$ defined for any $k = 1, \dots, N$ by

$$d\xi_k = b_k(\xi)dt + \langle \sigma_k(\xi), dW\rangle,$$

for some vector field $b : \mathbb{R}^N \to \mathbb{R}^N$ and some variance $\sigma : \mathbb{R}^N \to H^N$. Let us define as in [8] the matrix

$$A_{ij}(\xi) = \langle \tilde{u}_j^\xi, E_i^\xi\rangle - \langle \tilde{v}, E_{ij}^\xi\rangle,$$

which is invertible, provided that we are near the slow manifold. The previous noise definition combined with (2.4), gives the following SDE system for the interfaces motion for the stochastic C-H:

$$\begin{aligned}\sum_j A_{ij}(\xi)d\xi_j =& \langle -\varepsilon^2(\tilde{u}_{xxxx}^\xi + \tilde{v}_{xxxx}) + (f(\tilde{u}_x^\xi + \tilde{v}_x))_x, E_i^\xi\rangle dt \\ &+ \sum_{l,k}\Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi\rangle\Big]\langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi)\rangle dt \\ &+ \sum_j \langle \mathcal{Q}E_{ij}^\xi, \sigma_j(\xi)\rangle dt \\ &+ \langle E_i^\xi, dW\rangle.\end{aligned} \tag{2.5}$$

(cf. also the equivalent presentation (3.11)). We can easily read off b and σ from this equation for ξ. Moreover, it gives the flow along $\tilde{\mathcal{M}}$ by describing the interface positions. It is now easy to check by construction that the difference $\tilde{v} = \tilde{u} - \tilde{u}^\xi$ is actually the $\tilde{v}$ of the coordinate system (see Sec. 1.4). In addition, a solution of (2.5) together with a corresponding equation for $\tilde{v}$ (see (3.15), later) describes a solution $\tilde{u}$ of (ISC-H).

Further, in Section 3 the variance σ of the multi-dimensional diffusion process ξ of the interfaces is computed first explicitly and then estimated in terms of ε. A main result of grate importance is the stochastic analysis of the stability of the second approximate manifold which is presented in Theorem 3.6 of this section. Over a long time-scale of order $\mathcal{O}(\varepsilon^{-q})$ for any $q > 0$, we show that with high probability the solution of the stochastic Cahn-Hilliard stays in a small neighborhood Γ of the integrated manifold $\tilde{\mathcal{M}}$, unless an interface breaks down.

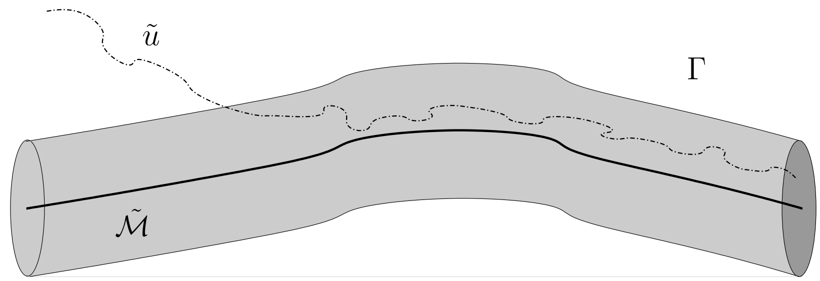

FIGURE 2.1. The stability of the *slow manifold* $\tilde{\mathcal{M}}$ for two interfaces ($N = 1$). A small tubular neighbourhood Γ, the *slow channel*, is attracting over long time-scales. Solutions tend to exit at the end of Γ by loosing an interface.

In Section 4, we present first Theorem 4.1 in which we approximate (2.5) and derive the equations of interfaces motion. Further, we consider several examples where Theorem 4.1 is simplified. If the noise is exponentially small, then we recover the slow motion results of [7, 8]. There is a slow channel as a neighborhood of $\tilde{\mathcal{M}}$, in which with high probability the motion of the interfaces is described by the deterministic regime. There is also an interesting intermediate regime of still exponentially small noise, which for simplicity of presentation we do not consider in this article. Here, due to the presence of noise, additional deterministic and stochastic terms appear in the deterministic equation of Bates & Xun [8]. An interesting case from the point of applications is the case where the noise strength is a power of ε. As the general case is quite involved in presentation, we consider only two interfaces (i.e. $N = 1$). Here, obviously the motion of the second interface is determined by the first which is approximated by the following SDE (cf. (4.10)):

$$d\xi_1 = \frac{1}{32\ell_2^2}\frac{\partial}{\partial\xi_1}\|\mathcal{Q}^{1/2}E_1^\xi\|^2 dt + \frac{1}{4\ell_2}\langle E_1^\xi, dW\rangle \,, \tag{2.6}$$

where ℓ_2 is the distance between the two interfaces. Finally in this section, we also discuss the case of non-smooth in space space-time white noise ($\mathcal{Q} = Id$), which we do not treat by our assumptions; here ξ_1 would be close to a Brownian motion with variance $\delta_\varepsilon^2/(16\ell_2^2)$.

Section 5 provides estimates for the second order derivatives $\frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j}$, for the higher order derivatives of E_j^ξ and $\tilde{u}^\xi$, and a bound for the quantity $\langle L^c\tilde{v}, \tilde{u}_{kl}^\xi\rangle$ (needed in the proof of the stability Theorem). Here the operator L^c acting on a general smooth in space function ϕ is given by

$$L^c(\phi) := -\varepsilon^2\phi_{xxxx} + (f'(u^h)\phi_x)_x.$$

The results of this section are quite technical since their proof involves extensive computations related to the stationary problem (1.3) properties. The new estimated terms appear only in the stochastic setting due to the presence of noise, and where therefore not treated in the work of Bates & Xun [7, 8] or Carr & Pego [21, 22]. A main result of this paper is that the stochastic treatment of the very important deterministic result of Bates & Xun gives new insights on the analysis of the deterministic stationary problem by means of a higher order regularity point of view.

3. FRONT MOTION

In this section, we derive the equations of motions of the fronts and show that the approximate manifold is locally attracting.

3.1. **Preliminaries and definitions.** Let us first recall some notation. If u is the solution of (SC-H), then $\tilde{u}(x,t) := \int_0^x u(y,t)dy$ is the solution of the integrated one i.e. of (ISC-H). Let a, ε, ρ, N be given; for some ℓ such that $\varepsilon/\ell < \rho$, we consider the unique solution ϕ of (1.3) which satisfies the properties (a) and (b). Let also $(h_1, \dots, h_{N+1}) \in \Omega_\rho$ be the admissible interfaces positions and take $h_0 := -h_1$, $h_{N+2} := 2 - h_{N+1}$.

Let $\ell_j = h_j - h_{j-1}$ be the distance between interfaces and $\ell := \min\{\ell_1, \dots, \ell_N\}$ the lower bound on them. Note that by the construction of Ω_ρ the functions ϕ are always well defined. Let

$$r := \varepsilon/\ell, \quad \beta_\pm(r) := 1 \mp \phi(0, \ell, \pm) \quad \text{and } \alpha_\pm(r) := F(\phi(0, \ell, \pm)) \ .$$

In view of (1.5), we also define

$$\phi^j(x) := \phi\left(x - m_j, \ell_j, (-1)^j\right),$$

and $u_j^h := \frac{\partial u^h}{\partial h_j}$ for $j = 1, \dots, N+1$. Considering $r_j := \varepsilon/\ell_j$, let

$$\beta^j(r) := \begin{cases} \beta_+(r_j) & \text{for } j \text{ even} \\ \beta_-(r_j) & \text{for } j \text{ odd,} \end{cases} \quad \text{and} \quad \beta(r) := \max_j \beta^j(r) \ .$$

We denote that in [8], as an application of the implicit function Theorem,

$$\frac{\partial h_{N+1}}{\partial h_j} = (-1)^{N-j} + \mathcal{O}(\varepsilon^{-1}\beta(r)). \tag{3.1}$$

In addition, let

$$\alpha^j(r) := \begin{cases} \alpha_+(r_j) & \text{for } j \text{ even} \\ \alpha_-(r_j) & \text{for } j \text{ odd} \end{cases} \quad \text{and} \quad \alpha(r) := \max_j \alpha^j(r) \ .$$

We see later, that both α and β are exponentially small in ε, if we consider $r_j \le \rho \le \varepsilon^\kappa$ for some small positive κ.

3.2. **The general SDE for the front motion.** Let $\tilde{u}$ be a solution of (ISC-H). We assume that the N front positions, i.e. the coordinates of $\xi(t) = (\xi_1(t), \dots, \xi_N(t))$, define a multi-dimensional diffusion process which is given by

$$d\xi_k = b_k(\xi)dt + \langle \sigma_k(\xi), dW \rangle, \quad k = 1, \dots, N, \tag{3.2}$$

for some vector field $b : \mathbb{R}^N \to \mathbb{R}^N$ and some variance $\sigma : \mathbb{R}^N \to H^N$. The main aim of this paragraph is to identify b and σ, which might also depend on $\tilde{v}$, i.e. on the distance from the manifold.

We use Itô-formula, in order to differentiate $\tilde{u}^\xi$ with respect to t, and get

$$d\tilde{u} = \sum_{j=1}^N \tilde{u}_j^\xi d\xi_j + \tfrac{1}{2} \sum_{1 \le k,l \le N} \tilde{u}_{kl}^\xi d\xi_k d\xi_l + d\tilde{v} \ , \qquad \text{with } \ \tilde{u}_{kl}^\xi = \frac{\partial^2 \tilde{u}^\xi}{\partial \xi_k \partial \xi_l}. \tag{3.3}$$

We take as in [8], p. 175, the inner product in space of equation (ISC-H) with E_i^ξ, to get for any $i = 1, \dots, N$

$$\langle E_i^\xi, d\tilde{u} \rangle = \langle \mathcal{L}^c(\tilde{u}), E_i^\xi \rangle dt + \langle E_i^\xi, dW \rangle \ , \tag{3.4}$$

where we defined the nonlinear ICH-operator as

$$\mathcal{L}^c(u) := -\varepsilon^2 u_{xxxx} + (f(u_x))_x$$

for short-hand notation.

On the other hand, if we take the inner product of (3.3) with E_i^ξ, we derive

$$\langle E_i^\xi, d\tilde{u} \rangle = \sum_{j=1}^N \langle \tilde{u}_j^\xi, E_i^\xi \rangle d\xi_j + \tfrac{1}{2} \sum_{1 \le k,l \le N} \langle \tilde{u}_{kl}^\xi, E_i^\xi \rangle d\xi_k d\xi_l + \langle E_i^\xi, d\tilde{v} \rangle \ . \tag{3.5}$$

Throughout the rest of this paper, any summation is on $1, 2, \dots, N$ for any index.

In order to eliminate $d\tilde{v}$, we apply Itô-formula to the orthogonality condition $\langle \tilde{v}, E_i^\xi \rangle = 0$, and arrive at

$$\begin{aligned} \langle E_i^\xi, d\tilde{v} \rangle &= -\langle \tilde{v}, dE_i^\xi \rangle - \langle d\tilde{v}, dE_i^\xi \rangle \\ &= -\sum_j \langle \tilde{v}, E_{ij}^\xi \rangle d\xi_j - \tfrac{1}{2} \sum_{j,k} \langle \tilde{v}, E_{ijk}^\xi \rangle d\xi_j d\xi_k - \sum_j \langle E_{ij}^\xi, d\tilde{v} \rangle d\xi_j. \end{aligned}$$

Now, we use that $d\tilde{v} = d\tilde{u} - d\tilde{u}^{\xi}$ and the fact that $dtdt = 0$ and $dWdt = 0$. In details,

$$\begin{aligned} -\sum_j \langle E^{\xi}_{ij}, d\tilde{v}\rangle d\xi_j &= -\sum_j \langle E^{\xi}_{ij}, d\tilde{u}\rangle d\xi_j + \sum_j \langle E^{\xi}_{ij}, d\tilde{u}^{\xi}\rangle d\xi_j \\ &= -\sum_j \langle E^{\xi}_{ij}, \mathcal{L}^c(\tilde{u})\rangle dt d\xi_j - \sum_j \langle E^{\xi}_{ij}, dW\rangle d\xi_j \ + \sum_{j,k} \langle E^{\xi}_{ij}, \tilde{u}^{\xi}_k\rangle d\xi_k d\xi_j \\ &= -\sum_j \langle E^{\xi}_{ij}, dW\rangle d\xi_j \ , + \sum_{j,k} \langle E^{\xi}_{ij}, \tilde{u}^{\xi}_k\rangle d\xi_k d\xi_j , \end{aligned} \tag{3.6}$$

where we took the inner product in space of equation (ISC-H) with E^{ξ}_{ij}, and used that

$$d\xi_j dt = b_j(\xi) dt dt + \langle \sigma_j(\xi), dW\rangle dt = 0.$$

Therefore, by (3.6) it follows that

$$\langle E^{\xi}_i, d\tilde{v}\rangle = -\sum_j \langle \tilde{v}, E^{\xi}_{ij}\rangle d\xi_j - \tfrac{1}{2}\sum_{j,k} \langle \tilde{v}, E^{\xi}_{ijk}\rangle d\xi_j d\xi_k - \sum_j \langle dW, E^{\xi}_{ij}\rangle d\xi_j + \sum_{j,k} \langle \tilde{u}^{\xi}_k, E^{\xi}_{ij}\rangle d\xi_j d\xi_k \ . \tag{3.7}$$

Combining (3.4) with (3.5) and (3.7) we arrive at

$$\begin{aligned} \sum_j \left[\langle \tilde{u}^{\xi}_j, E^{\xi}_i\rangle - \langle \tilde{v}, E^{\xi}_{ij}\rangle \right] d\xi_j =& \langle \mathcal{L}^c(\tilde{u}), E^{\xi}_i\rangle dt \\ & + \sum_{l,k} \left[\tfrac{1}{2}\langle \tilde{v}, E^{\xi}_{ilk}\rangle - \tfrac{1}{2}\langle \tilde{u}^{\xi}_{kl}, E^{\xi}_i\rangle - \langle \tilde{u}^{\xi}_k, E^{\xi}_{il}\rangle \right] d\xi_l d\xi_k \\ & + \sum_j \langle dW, E^{\xi}_{ij}\rangle d\xi_j + \langle E^{\xi}_i, dW\rangle \ . \end{aligned} \tag{3.8}$$

Lemma 3.1. *For all $1 \le k,\ l \le N$ it holds that*

$$\langle \sigma_k(\xi), dW\rangle \langle \sigma_l(\xi), dW\rangle = \langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi)\rangle dt.$$

Proof. Since $d\beta_j d\beta_i = \delta_{ij} dt$ and $W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) e_k$ we obtain, using Parcevals identity,

$$\begin{aligned} \langle \sigma_k(\xi), dW\rangle \langle \sigma_l(\xi), dW\rangle &= \sum_{i,j} \alpha_i \alpha_j \langle \sigma_k(\xi), e_i\rangle \langle \sigma_l(\xi), e_j\rangle d\beta_j d\beta_i = \sum_j \alpha_j^2 \langle \sigma_k(\xi), e_j\rangle \langle \sigma_l(\xi), e_j\rangle dt \\ &= \sum_j \langle \mathcal{Q}\sigma_k(\xi), e_j\rangle \langle \sigma_l(\xi), e_j\rangle dt = \langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi)\rangle dt \ . \end{aligned}$$

□

Analogously to this Lemma we easily obtain (using $dtdW = 0$)

$$\langle E^{\xi}_{ij}, dW\rangle d\xi_j = \langle E^{\xi}_{ij}, dW\rangle \langle \sigma_j(\xi), dW\rangle = \langle \mathcal{Q}E^{\xi}_{ij}, \sigma_j(\xi)\rangle dt.$$

Moreover, for short-hand notation, as in [7], we define the matrix $A(\xi) = (A_{ij}(\xi)) \in \mathbb{R}^{N\times N}$ by

$$A_{ij}(\xi) = \langle \tilde{u}^{\xi}_j, E^{\xi}_i\rangle - \langle \tilde{v}, E^{\xi}_{ij}\rangle \ , \tag{3.9}$$

which is invertible, provided that we are near the slow manifold (cf. Lemma 3.4 later). Let us denote the inverse matrix of A by $A^{-1}(\xi) = (A^{-1}_{ij}(\xi)) \in \mathbb{R}^{N\times N}$.

Therefore, for all $i \in \{1, \dots, N\}$ we arrive at

$$\begin{aligned}\sum_j A_{ij}(\xi)d\xi_j = \ & \langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi \rangle dt \\ & + \sum_{l,k} \Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi \rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi \rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi \rangle \Big] \langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi) \rangle dt \\ & + \sum_j \langle \mathcal{Q}E_{ij}^\xi, \sigma_j(\xi) \rangle dt + \langle E_i^\xi, dW \rangle \ . \end{aligned} \tag{3.10}$$

To obtain the equation for $d\xi$ we use that $d\xi = A(\xi)^{-1}A(\xi)d\xi$.

Thus, the final equation for ξ (as long as $\tilde{u}$ is near the manifold) is given for any $r = 1, \dots, N$ by

$$\begin{aligned} d\xi_r = \ & \sum_i A_{ri}^{-1}(\xi) \langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi \rangle dt \\ & + \sum_{i,l,k} A_{ri}^{-1}(\xi) \Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi \rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi \rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi \rangle \Big] \langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi) \rangle dt \\ & + \sum_i A_{ri}^{-1}(\xi) \sum_j \langle \mathcal{Q}E_{ij}^\xi, \sigma_j(\xi) \rangle dt + \sum_i A_{ri}^{-1}(\xi) \langle E_i^\xi, dW \rangle \ . \end{aligned} \tag{3.11}$$

We can now recover σ and b from (3.11). The only term that does involve noise is the last one. Thus, in view of (3.2) we get

$$\sigma_r(\xi) = \sum_i A_{ri}^{-1}(\xi) E_i^\xi \ . \tag{3.12}$$

After we obtained σ, we can proceed, in order to determine $b(\xi)$ from the remaining terms (cf. (3.2)). So, we get for $r = 1, \dots, N$ that

$$\begin{aligned} b_r(\xi) = & \sum_i A_{ri}^{-1}(\xi) \langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi \rangle \\ & + \sum_{i,l,k} A_{ri}^{-1}(\xi) \Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi \rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi \rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi \rangle \Big] \langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi) \rangle \\ & + \sum_i A_{ri}^{-1}(\xi) \sum_j \langle \mathcal{Q}E_{ij}^\xi, \sigma_j(\xi) \rangle. \end{aligned} \tag{3.13}$$

Remark 3.2. (Well defined coordinates) It is easy to check from the construction, that given our $\xi(t)$ from Equation (3.2) with b and σ defined as above, then there exists a corresponding solution $\tilde{u}$ of (ISC-H). If the distance $\tilde{v}$ from the manifold $\tilde{\mathcal{M}}$ is sufficiently small, then ξ describes the motion of the interfaces of $\tilde{u}$.

3.3. **Stability and Attractivity of the manifold.** In this paragraph, we prove the stability and discuss the attractivity of $\tilde{\mathcal{M}}$. Considering the stability, we show that with high probability (over a long time-scale) the solution stays close to $\tilde{\mathcal{M}}$, unless an interface breaks down.

In [7, Theorem B], Bates and Xun show that in the deterministic setting the slow manifold is exponentially attracting in a $\mathcal{O}(\varepsilon^{7/2})$-neighborhood in H^2, until the solution reaches an exponentially small neighborhood, where the motion of the solution along the manifold is exponentially slow. Using large deviation estimates, it is straightforward to verify for small noise, that the stochastic solution follows the deterministic one up to error terms of the order of the noise strength. Hence, the exponential attraction of $\tilde{\mathcal{M}}$ still holds for (ISC-H), until the solution reaches a neighborhood of the manifold that is determined by the strength of the noise.

Here, for simplicity of presentation we will focus only on the stability of $\tilde{\mathcal{M}}$. The proof can be easily modified to show attraction, too. Once, we are in the slow channel around $\tilde{\mathcal{M}}$, with high probability we cannot exit for a long time-scale T_ϵ, unless one of the interfaces breakes down.

We define the metrics A_ε and B_ε as

$$A_\varepsilon(\tilde v) = \int_0^1 [\varepsilon^2 \tilde v_{xx}^2 + f'(u^\xi)\tilde v_x^2]dx \qquad \text{and} \qquad B_\varepsilon(\tilde v) = \int_0^1 [\varepsilon^2 \tilde v_{xx}^2 + \tilde v_x^2]dx. \tag{3.14}$$

Note that it is easy to check that

$$\|\partial_x \tilde v\|^2 \le C\varepsilon^{-1} B_\varepsilon(\tilde v) \le C\varepsilon^{-3} A_\varepsilon(\tilde v) \le C\varepsilon^{-3} B_\varepsilon(\tilde v) \le C\varepsilon^{-3}\|\tilde v\|_{H^2}^2,$$

and

$$\|\tilde v\|_\infty^2 \le B_\varepsilon(\tilde v)\ , \qquad \|\tilde v_x\|_\infty^2 \le \tfrac{1+\varepsilon}{\varepsilon} B_\varepsilon(\tilde v)\ .$$

Definition 3.3. (cf. [7], p. 452) Define a neighborhood Γ' of $\tilde{\mathcal{M}}$ by

$$\Gamma' = \{\tilde u^\xi + \tilde v \ : \ \xi \in \Omega_\rho,\ B_\varepsilon(\tilde v) < \varepsilon^3\},$$

and we define the slow tube Γ by

$$\Gamma := \{\tilde u^\xi + \tilde v \ : \ \xi \in \Omega_\rho,\ A_\varepsilon(\tilde v) < \delta_\varepsilon^{2-\kappa}\}\ ,$$

where $\kappa > 0$ is presented in the definition of the noise (cf. Assumption 2.3) and δ_ε estimates the noise strength.

The small tube Γ' is a neighborhood of the slow manifold, where the coordinate system (cf. (1.7)) is well defined, while the slow tube Γ is a neighborhood in which solutions with high probability do not exit for long times unless one of the interfaces breakes down. Recall that $\Gamma \subset \Gamma'$ by definition of δ_ε. We even have $B_\varepsilon(\tilde v(t)) < C\delta_\varepsilon^2\varepsilon^{-2} \le C\varepsilon^{6+\kappa}$, which we need in the proof of stability.

As indicated in the introduction, the first term at the right-hand side of the flow given by (3.11), is identical to the right-hand side of the deterministic flow and has been estimated in [7]. In our stochastic case, in order to approximate the flow, we need to bound also the additional higher order terms and estimate the contribution of the noise. Later, in the next Section 4, we will identify the dominant terms in (3.11).

Using (4.27) of [8] and the fact that $\|E_{ij}^\xi\| = \mathcal{O}(\varepsilon^{-1/2})$ ([8] p. 187), we obtain in Γ' considering the matrix A the following invertibility result:

Lemma 3.4. *Suppose that $h \in \Omega_\rho$ and $\|\tilde v\| = \mathcal{O}(\varepsilon^{3/2})$, then*

$$A_{ij}(\xi) = \mathcal{O}(\varepsilon) + \begin{cases} (-1)^{i+j} 4\ell_{j+1} & \text{if} \quad i \ge j \\ 0 & \text{if} \quad i < j \end{cases}$$

and the matrix is invertible, with

$$A_{ij}^{-1}(\xi) = \mathcal{O}(\varepsilon) + \begin{cases} \frac{1}{4\ell_{j+1}} & \text{if} \quad i = j, j-1 \\ 0 & \text{otherwise} \end{cases}$$

where $1 > \ell_i > \varepsilon/\rho$ denotes the length of the i-th interface.

As the equation is deterministically stable, we should be able to show that $\tilde v$ stays small for a long time (depending on the noise strength). To be more precise, we show a bound on $A_\varepsilon(\tilde v)$ for solutions near $\tilde{\mathcal{M}}$. Following [7] p. 449, we consider equation (3.3)

$$d\tilde v = d\tilde u - \sum_{j=1}^N \tilde u_j^\xi d\xi_j - \frac{1}{2}\sum_{kl} \tilde u_{kl}^\xi d\xi_k d\xi_l,$$

and thus the key equation for the distance from the manifold $\tilde{\mathcal{M}}$ is described by

$$d\tilde v = \mathcal{L}^c(\tilde u)dt - \sum_j \tilde u_j^\xi b_j(\xi)dt - \sum_j \tilde u_j^\xi \langle \sigma_j(\xi), dW\rangle - \frac{1}{2}\sum_{kl} \tilde u_{kl}^\xi \langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi)\rangle dt + dW\ . \tag{3.15}$$

We can now proceed (cf. also (86) of [7]) and show a bound on $\tilde v$ in terms of A_ε.

Fix some large time T_ε and define $\tau^* > 0$ as the first exit time (below the threshold T_ε) of $\tilde{u}$ from Γ'. This is the stopping time

$$\tau^* = T_\varepsilon \wedge \inf\{t > 0 : \xi(t) \notin \Omega_\rho \text{ or } A_\varepsilon(\tilde{v}(t)) \geq \delta_\varepsilon^{2-\kappa}\} .$$

Note that for $t \leq \tau^*$ also $B_\varepsilon(\tilde{v}(t)) \leq C\varepsilon^{6+\kappa}$.

Definition 3.5. We say that a term is $\mathcal{O}(e_\varepsilon)$, if it is asymptotically smaller than any polynomial uniformly for times $t \leq \tau^*$.

Note that α, β are $\mathcal{O}(e_\varepsilon)$, if $\rho = \varepsilon^\kappa$.

Theorem 3.6. *Suppose $\rho = \epsilon^\kappa$ for some small $\kappa > 0$, $\delta_\varepsilon \geq C\varepsilon^{-q}$ for any $q > 0$, and suppose that for all $p > 0$ there exists a constant $c_p > 0$ such that $\mathbb{E}A_\varepsilon(\tilde{v}(0))^p \leq c_p\delta_\varepsilon^{2p}$. Then for all $p > 0$ there exists a constant $C_p > 0$ such that*

$$\mathbb{E}A_\varepsilon(\tilde{v}(\tau^*))^p \leq C_p(T_\epsilon + 1)\delta_\varepsilon^{2p} .$$

Therefore, we can show that the probability that the solution exits from the slow tube before T_ε (i.e. $\tau^* = T_\varepsilon$) or an interface is breaking down (i.e. $\xi(\tau^*) \notin \Omega_\rho$) is bounded above by

$$\mathbb{P}\left(A_\varepsilon(\tilde{v}(\tau^*)) \geq \delta_\varepsilon^{2-\kappa}\right) \leq \mathbb{E}A_\varepsilon(\tilde{v}(\tau^*))^p \delta_\varepsilon^{-p(2-\kappa)} \leq C_p(T_\varepsilon + 1)\delta_\varepsilon^{\kappa p}$$

for any $p > 0$. Thus the probability that the solution exits from the slow tube before T_ε is of order $\mathcal{O}(e_\varepsilon)$ provided $T_\varepsilon \ll \delta_\varepsilon^{-q}$ for some large $q > 0$. The typical case for applications would be to consider a noise strength polynomial in ε, where we can take $T_\varepsilon = \varepsilon^{-q}$ for any $q > 0$.

Remark 3.7. (Exponentially small noise-strength δ_ε) If we want to have exponentially long times T_ε, then we need to take exponentially small noise strength δ_ε and look closer at the various error terms in the proof of Theorem 3.6. This is straightforward, but for simplicity of presentation, we refrain from stateing details here.

On the other hand, assuming that δ_ε is exponentially small, the probability of the solution exiting the slow tube Γ before T_ε, without an interface breaking down, is exponentially small, even for exponentially large times T_ε.

3.4. **Bounds on the sde.** The following Lemmas replace the bound on $\dot{\xi}$, which is used in the deterministic setting (cf. Lemma 4.3. in [7]).

Lemma 3.8. *Let $\tilde{u}^\xi + \tilde{v} \in \Gamma'$ and $r = 1, \ldots, N$, then (with $E_{N+1}^\xi = 0$ for shorthand notation)*

$$\sigma_r(\xi) = \frac{1}{4\ell_{r+1}}(E_r^\xi + E_{r+1}^\xi) + \mathcal{O}(\varepsilon),$$

and

$$\|\sigma_r(\xi)\| \leq C/\ell < C\rho/\varepsilon.$$

Proof. Note that $\|\tilde{v}\| \leq B_\varepsilon(\tilde{v})^{1/2}$. Thus from the definition of σ (cf. (3.12)), Lemma 3.4, and the bound on E_i^ξ one has

$$\|\sigma_r(\xi)\| \leq \sum_i |A_{ri}^{-1}(\xi)| \|E_i^\xi\| \leq C/\ell .$$

Moreover

$$\sigma_r(\xi) = A_{r,r}^{-1}E_r^\xi + A_{r,r+1}^{-1}E_{r+1}^\xi + \mathcal{O}(\varepsilon),$$

and the claim follows from Lemma 3.4. ☐

The next Lemma estimates the vector field b of the diffusion process ξ.

Lemma 3.9. *Let $\tilde{u}^\xi + \tilde{v} \in \Gamma'$ and assume that $\rho = \varepsilon^\kappa$ for some small $\kappa > 0$, then there is a constant $c > 0$ such that*

$$|b_r(\xi)| \leq c\|\mathcal{Q}\|\Big\{\varepsilon^{3\kappa-7/2} + \varepsilon^{2\kappa-5/2}\Big\} + \mathcal{O}(e_\varepsilon), \tag{3.16}$$

for any $r = 1, \dots, N$.

Proof. We recall first b_r

$$\begin{aligned} b_r(\xi) = &\sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi\rangle \\ &+ \sum_{i,l,k} A_{ri}^{-1}(\xi)\Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi\rangle\Big]\langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi)\rangle \\ &+ \sum_i A_{ri}^{-1}(\xi)\sum_j \langle \mathcal{Q}E_{ij}^\xi, \sigma_j(\xi)\rangle. \end{aligned} \tag{3.17}$$

Then we use Lemma 3.4 and the bound on σ. Moreover, in Section 5, after tedious computations the next estimates are derived (cf. (5.40), (5.41), (5.42), (5.37) and (5.38), respectively):

$$\begin{aligned} &|\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle| \leq \mathcal{O}(\varepsilon^{-1/2})\Big[4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta)\Big], \\ &|\langle \tilde{u}_k^\xi, E_{il}^\xi\rangle| \leq \mathcal{O}(\varepsilon^{-1/2} + \varepsilon^{-4}r^{-1}\beta), \\ &|\langle \tilde{v}, E_{ilk}^\xi\rangle| \leq \mathcal{O}(\varepsilon^{-3/2} + \varepsilon^{-5}r^{-1}\beta)\|\tilde{v}\| \leq c + \mathcal{O}(\varepsilon^{-7/2}r^{-1}\beta)\ , \end{aligned}$$

since in the slow channel $\|\tilde{v}\| \leq \|\tilde{v}\|_\infty \leq cB_\varepsilon(\tilde{v})^{1/2} \leq c\varepsilon^{3/2}$. Moreover,

$$\|E_i^\xi\| \leq 4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta), \qquad \|E_{ij}^\xi\| \leq \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(\varepsilon^{-4}r^{-1}\beta)\ .$$

In addition, we observe that (cf. [8])

$$|\sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi\rangle| = \mathcal{O}(\alpha/\ell) + \mathcal{O}(\varepsilon\alpha) = \mathcal{O}(e_\varepsilon)\ .$$

In this way, since $\sigma = \mathcal{O}(\rho\varepsilon^{-1})$ and $A_{ij}^{-1} = \mathcal{O}(\rho\varepsilon^{-1})$, we obtain

$$|b_r(\xi)| \leq c\|\mathcal{Q}\|\rho^3\varepsilon^{-3-1/2} + c\|\mathcal{Q}\|\rho^2\varepsilon^{-5/2} + \mathcal{O}(e_\varepsilon) \leq c\|\mathcal{Q}\|\Big\{\varepsilon^{3k-7/2} + \varepsilon^{2k-5/2}\Big\} + \mathcal{O}(e_\varepsilon)\ .$$

□

3.5. **Proof of Stability.** Now let us turn to the proof of the Theorem 3.6. Considering the linearized C-H-operator and using Itô-formula we arrive at

$$dA_\varepsilon(\tilde{v}) = d\langle -L^c\tilde{v}, \tilde{v}\rangle = 2\langle -L^c\tilde{v}, d\tilde{v}\rangle + \langle -L^c d\tilde{v}, d\tilde{v}\rangle$$

and therefore, Lemma 3.1 gives

$$dA_\varepsilon(\tilde v) = 2\langle -L^c\tilde v, \mathcal{L}^c(\tilde u)\rangle dt \tag{3.18}$$

$$- \sum_j 2\langle -L^c\tilde v, \tilde u_j^\xi\rangle b_j(\xi)dt \tag{3.19}$$

$$- \sum_j 2\langle -L^c\tilde v, \tilde u_j^\xi\rangle \langle \sigma_j(\xi), dW\rangle$$

$$- \sum_{kl} \langle -L^c\tilde v, \tilde u_{kl}^\xi\rangle \langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi)\rangle dt \tag{3.20}$$

$$+ \sum_{ij} \langle -L^c\tilde u_i^\xi, \tilde u_j^\xi\rangle \langle \mathcal{Q}\sigma_i(\xi), \sigma_j(\xi)\rangle dt \tag{3.21}$$

$$+ \sum_{i} \langle -L^c\tilde u_i^\xi, \mathcal{Q}\sigma_i(\xi)\rangle dt \tag{3.22}$$

$$- 2\langle L^c\tilde v, dW\rangle$$

$$+ \mathrm{trace}(\mathcal{Q}^{1/2}L^c\mathcal{Q}^{1/2})dt. \tag{3.23}$$

For the term in (3.18) we follow [7] pages 449/450, where

$$\mathcal{L}^c(\tilde u) = \mathcal{L}^c(\tilde u^\xi + \tilde v) = L^c\tilde v + \mathcal{L}^c(\tilde u^\xi) + \partial_x(f_2\partial_x\tilde v)$$

with

$$\|\partial_x(f_2\partial_x\tilde v)\| \le C\varepsilon^{-2}B_\varepsilon(\tilde v).$$

Moreover, note that by Lemma 5.1 in [7] we have

$$\|\mathcal{L}^c(\tilde u^\xi)\|_\infty = \|\partial_x\mathcal{L}^b(u^\xi)\|_\infty \le C\varepsilon^{-1}\alpha(r)\ ,$$

and thus

$$\begin{aligned}\langle -L^c\tilde v, \mathcal{L}^c(\tilde u)\rangle &\le -\|L^c\tilde v\|^2 + C(\varepsilon^{-2}B_\varepsilon(\tilde v) + \varepsilon^{-1}\alpha(r))\|L^c\tilde v\| \\ &\le -\tfrac{2}{3}\|L^c\tilde v\|^2 + C\varepsilon^{-2}B_\varepsilon(\tilde v)\|L^c\tilde v\| + C\varepsilon^{-2}\alpha(r)^2 \\ &\le -\tfrac{1}{2}\|L^c\tilde v\|^2 + C\varepsilon^{-2}\alpha(r)^2,\end{aligned} \tag{3.24}$$

where we used that for some constant $a > 0$ independent of ε and r (cf. [7], Lemma 3.2 at p. 434, and Lemma 4.2 at p. 446)

$$B_\varepsilon(\tilde v) < C\varepsilon^{-2}A_\varepsilon(\tilde v) < \frac{C}{2a}\varepsilon^{-2}\|L^c\tilde v\|^2\ .$$

This is the crucial and only point, where we need $B_\varepsilon(\tilde v) = \mathcal{O}(\varepsilon^{6+\kappa})$. Thus, we obtain

$$2\langle -L^c\tilde v, \mathcal{L}^c(\tilde u)\rangle \le -\tfrac{1}{2}\|L^c\tilde v\|^2 - aA_\varepsilon(\tilde v) + C\varepsilon^{-2}\alpha(r)^2.$$

Now consider the remaining four deterministic integrals. For the term in (3.19), notice that

$$\langle L^c\tilde v, \tilde u_j^\xi\rangle = \langle \tilde v, L^c\tilde u_j^\xi\rangle = \langle \tilde v, \partial_x\partial_j\mathcal{L}^b(u^\xi)\rangle\ .$$

Thus using integration by parts and Lemma 5.2 of [7] yields

$$|\langle L^c\tilde v, \tilde u_j^\xi\rangle| \le C\|\partial_x\tilde v\|\varepsilon^{-2}\beta(r) = \mathcal{O}(e_\varepsilon). \tag{3.25}$$

We use now (3.25) to arrive at

$$|\sum_j \langle -L^c\tilde v, \tilde u_j^\xi\rangle b_j(\xi)| \le C\varepsilon^{-5/2}\beta(r)B_\varepsilon(\tilde v)^{1/2}\sup_j\{|b_j(\xi)|\} = \mathcal{O}(e_\varepsilon), \tag{3.26}$$

which is exponentially small in ε by Lemma 3.9. By Definition 3.5, a term is $\mathcal{O}(e_\varepsilon)$, if it is asymptotically smaller than any polynomial in ε uniformly for times $t \le \tau^*$.

Now let us turn to (3.21). Similarly, we get

$$|\langle -L^c\tilde{u}_i^\xi, \tilde{u}_j^\xi\rangle| = |\langle \tilde{u}_i^\xi, \partial_x\partial_j\mathcal{L}^b(u^\xi)\rangle| \leq \|\tilde{u}_i^\xi\|_{L^1}\|\partial_x\partial_j\mathcal{L}^b(u^\xi)\|_\infty \leq C\varepsilon^{-4}\beta(r) ,$$

where we used Lemma 5.1 of [7] and the bound $\|\tilde{u}_i^\xi\|_{L^1} = \mathcal{O}(1)$ (cf. (5.34), for β bounded). Thus we obtain for the term in (3.21)

$$|\sum_{ij}\langle -L^c\tilde{u}_i^\xi, \tilde{u}_j^\xi\rangle\langle \mathcal{Q}\sigma_i(\xi), \sigma_j(\xi)\rangle| \leq C\varepsilon^{-4}\beta(r)\|\mathcal{Q}\|\ell^{-2} = \mathcal{O}(e_\varepsilon) . \tag{3.27}$$

For the term in (3.20) we use the bounds on $\langle -L^c\tilde{v}, \tilde{u}_{kl}^\xi\rangle$ provided by Theorem 5.43. Thus, we get

$$|\langle L^c\tilde{v}, \tilde{u}_{kl}^\xi\rangle\langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi)\rangle| \leq C\|\mathcal{Q}\|\varepsilon^{-2}C\varepsilon^{-2}\beta(r)\|\tilde{v}\| = \mathcal{O}(e_\varepsilon) .$$

Using similar estimates and Lemma 3.8 the term in (3.22) is also $\mathcal{O}(e_\varepsilon)$.

For the term in (3.23), we use the eigenfunctions e_k of $\mathcal{Q}$ and the uniform bound on $f'(u^\xi)$, in order to obtain

$$\text{trace}(\mathcal{Q}^{1/2}L^c\mathcal{Q}^{1/2}) = \sum_{k=1}^\infty \alpha_k^2\langle L^c e_k, e_k\rangle \leq C\sum_{k=1}^\infty \alpha_k^2 B_\varepsilon(e_k) \leq C\delta_\varepsilon^2 .$$

This is the largest deterministic term, as the other ones are all $\mathcal{O}(e_\varepsilon)$. This term comes directly from the Itô-correction of the additive noise.

Consider now Equations (3.18) - (3.23), with all deterministic integrals already estimated. For $t \leq \tau^*$

$$dA_\varepsilon(\tilde{v}(t)) \leq C\delta_\varepsilon^2 dt - (\tfrac{1}{2}\|L^c\tilde{v}\|^2 + aA_\varepsilon(\tilde{v}))dt + < I, dW > , \tag{3.28}$$

where

$$I = \sum_j 2\langle -L^c\tilde{v}, \tilde{u}_j^\xi\rangle\sigma_j(\xi) - 2L^c\tilde{v} .$$

In order to bound I, we use (3.25), and the asymptotic formula for $\sigma_j(\xi)$ of Lemma 3.8 combined with (54)-(55) of [7] to obtain that $\langle L^c\tilde{v}, \tilde{u}_j^\xi\rangle\sigma_j(\xi) = \mathcal{O}(e_\varepsilon)$ and thus

$$|\langle I, QI\rangle| \leq \mathcal{O}(e_\varepsilon) + 2\|Q\|\|L^c\tilde{v}\|^2 .$$

Now we can bound powers of A_ε for $t \leq \tau^*$

$$\begin{aligned} \tfrac{1}{p}dA_\varepsilon(\tilde{v})^p &= A_\varepsilon(\tilde{v})^{p-1}dA_\varepsilon(\tilde{v}) + \tfrac{p-1}{2}A_\varepsilon(\tilde{v})^{p-2}(dA_\varepsilon(\tilde{v}))^2 \\ &\leq C\varepsilon^{2\delta_s}A_\varepsilon(\tilde{v})^{p-1}dt - (\tfrac{1}{2}\|L^c\tilde{v}\|^2 + aA_\varepsilon(\tilde{v}))A_\varepsilon(\tilde{v})^{p-1}dt \\ &\quad + A_\varepsilon(\tilde{v})^{p-1}\langle I, dW\rangle + \tfrac{p-1}{2}A_\varepsilon(\tilde{v})^{p-2}\langle I, QI\rangle dt . \end{aligned} \tag{3.29}$$

Taking integrals up to τ^* and expectation, we easily obtain from (3.28) and (3.29) (using that the expectation of a stochastic integral is 0)

$$\mathbb{E}A_\varepsilon(\tilde{v}(\tau^*)) + \tfrac{1}{2}\mathbb{E}\int_0^{\tau^*}\|L^c\tilde{v}\|^2 dt + a\mathbb{E}\int_0^{\tau^*}A_\varepsilon(\tilde{v})dt \leq A_\varepsilon(\tilde{v}(0)) + CT_\varepsilon\delta_\varepsilon^2 ,$$

and for $p \geq 2$

$$\begin{aligned} &\tfrac{1}{p}\mathbb{E}A_\varepsilon(\tilde{v}(\tau^*))^p + \tfrac{1}{2}\mathbb{E}\int_0^{\tau^*}\|L^c\tilde{v}\|^2A_\varepsilon(\tilde{v})^{p-1}dt + a\mathbb{E}\int_0^{\tau^*}A_\varepsilon(\tilde{v})^p dt \\ &\leq \tfrac{1}{p}\mathbb{E}A_\varepsilon(\tilde{v}(0))^p + C\delta_\varepsilon^2\mathbb{E}\int_0^{\tau^*}A_\varepsilon(\tilde{v})^{p-1}dt + \mathcal{O}(e_\varepsilon)\cdot\mathbb{E}\int_0^{\tau^*}A_\varepsilon(\tilde{v})^{p-2}dt + 2\|Q\|\cdot\mathbb{E}\int_0^{\tau^*}A_\varepsilon(\tilde{v})^{p-2}\|L^c\tilde{v}\|^2 dt . \end{aligned}$$

Now (using $\delta_\varepsilon \geq C\varepsilon^q$) it is easy to verify by induction on p that

$$\tfrac{1}{p}\mathbb{E}A_\varepsilon(\tilde{v}(\tau^*))^p + \tfrac{1}{2}\mathbb{E}\int_0^{\tau^*}\|L^c\tilde{v}\|^2A_\varepsilon(\tilde{v})^{p-1}dt + a\mathbb{E}\int_0^{\tau^*}A_\varepsilon(\tilde{v})^p dt \leq C(T_\varepsilon + 1)\delta_\varepsilon^{2p} .$$

This implies the claim.

4. Motion of the interfaces

In this section, we investigate in detail what the SDE (2.5) for ξ actually implies for the motion of the interfaces considering some important special cases, where the equation simplifies a lot. Let us assume first that the noise is exponentially small. Then in the case of two interfaces (i.e., $N = 1$) we discuss the case of noise strength being polynomial in ε. Finally, although not covered by our theorems, we present some comments on how the equation would look like for non-smooth in space space-time white noise, which means that $\mathcal{Q}$ is the identity.

Let us first state the result we achieved so far. The motion of the interfaces for the stochastic model is given by the following theorem.

Theorem 4.1. *Let $\tilde{u}^\xi + \tilde{v} \in \Gamma'$ and assume that ρ is small, then the equations dominating the flow of the Stochastic Cahn-Hilliard equation within the slow channel are given by*

$$
\begin{aligned}
d\xi_1 &= \frac{1}{4\ell_2}(\alpha^3 - \alpha^1)dt + \mathcal{O}(\varepsilon\alpha)dt + d\mathcal{A}_s^{(1)} \\
d\xi_2 &= \frac{1}{4\ell_2}(\alpha^3 - \alpha^1)dt + \frac{1}{4\ell_3}(\alpha^4 - \alpha^2)dt + \mathcal{O}(\varepsilon\alpha)dt + d\mathcal{A}_s^{(2)} \\
d\xi_3 &= \frac{1}{4\ell_3}(\alpha^4 - \alpha^2)dt + \frac{1}{4\ell_4}(\alpha^5 - \alpha^3)dt + \mathcal{O}(\varepsilon\alpha)dt + d\mathcal{A}_s^{(3)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
d\xi_N &= \frac{1}{4\ell_N}(\alpha^{N+1} - \alpha^{N-1})dt + \frac{1}{4\ell_{N+1}}(\alpha^{N+2} - \alpha^N)dt + \mathcal{O}(\varepsilon\alpha)dt + d\mathcal{A}_s^{(N)},
\end{aligned}
\tag{4.1}
$$

where

$$
\alpha^j = \frac{1}{2}K_\pm^2 A_\pm^2 \exp(-A_\pm \ell_j/\varepsilon)\Big[1 + \mathcal{O}\Big(\frac{\ell_j}{\varepsilon}\exp\Big(\frac{-A_\pm \ell_j}{2\varepsilon}\Big)\Big)\Big] \quad j = 1, 2, \ldots, N+2,
\tag{4.2}
$$

for

$$
A_\pm := f'(\pm 1) \quad \text{and} \quad K_\pm := 2\exp\Big[\int_0^1 \Big[\frac{A_\pm}{2F(\pm t)^{1/2}} - \frac{1}{1-t}\Big]dt\Big].
\tag{4.3}
$$

Here, the stochastic processes $\mathcal{A}_s^{(r)}$, $r = 1, \ldots, N$ are related to the noise; they depend on the symmetric operator $\mathcal{Q}$ and the variance σ, and are given by the formula

$$
\begin{aligned}
d\mathcal{A}_s^{(r)} := &\sum_{i,l,k} A_{ri}^{-1}(\xi)\Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi\rangle\Big]\langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi)\rangle dt \\
&+ \sum_i A_{ri}^{-1}(\xi)\sum_j \langle \mathcal{Q}E_{ij}^\xi, \sigma_j(\xi)\rangle dt + \sum_i A_{ri}^{-1}(\xi)\langle E_i^\xi, dW\rangle\ .
\end{aligned}
\tag{4.4}
$$

Proof. Remind that as long as $\tilde{u}$ is near the manifold, then by (3.11) we obtained for any $r = 1, \ldots, N$

$$
d\xi_r = \sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi\rangle dt + d\mathcal{A}_s^{(r)}.
$$

Lemma 3.4 gives that the matrix A^{-1} and therefore the terms $\sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi\rangle$ are identical to those presented in [7, 8] for the deterministic case (i.e. when $d\mathcal{A}_s^{(r)} = 0$ for any r). Hence, using (4.32) of [8] we obtain the result. □

We observe that

$$
d\mathcal{A}_s^{(r)} := \mathcal{A}_{\mathcal{Q}}^{(r)}dt + \sum_i A_{ri}^{-1}(\xi)\langle E_i^\xi, dW\rangle\ ,
\tag{4.5}
$$

for

$$\begin{aligned}\mathcal{A}_{\mathcal{Q}}^{(r)} := &\sum_{i,l,k} A_{ri}^{-1}(\xi)\Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^{\xi}\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^{\xi}, E_{i}^{\xi}\rangle - \langle \tilde{u}_{k}^{\xi}, E_{il}^{\xi}\rangle\Big]\langle \mathcal{Q}\sigma_k(\xi), \sigma_l(\xi)\rangle \\ &+ \sum_{i} A_{ri}^{-1}(\xi) \sum_{j}\langle \mathcal{Q}E_{ij}^{\xi}, \sigma_j(\xi)\rangle\end{aligned} \tag{4.6}$$

Following Lemma 3.9 we obtain in the slow channel that

$$|\mathcal{A}_{\mathcal{Q}}^{(r)}| \leq c\|\mathcal{Q}\|\rho^2(\rho\varepsilon^{-3-1/2} + \varepsilon^{-5/2}), \quad \text{for all } r = 1,\dots,N\ . \tag{4.7}$$

Thus, in case of $\|\mathcal{Q}\| = \mathcal{O}(\varepsilon^{4+1/2}\alpha)$, since ρ is at least bounded, we can show that $\mathcal{A}_{\mathcal{Q}}^{(r)} = \mathcal{O}(\varepsilon\alpha)$. It is not hard to show that we can also neglect the stochastic term from (4.1), in order to recover the result of Bates & Xun on metastable slow motion, at least with high probability.

An interesting case arises, when the additional terms in $\mathcal{A}_s^{(r)}$ are of order $\mathcal{O}(\alpha)$. Then we obtain additional terms in (4.1). Nevertheless, for simplicity of presentation, we refrain from stating details here.

4.1. **Polynomial noise strength.** For the remainder of this section we fix $N = 1$, which is the case of two interfaces, and a noise strength $\delta_\varepsilon = \varepsilon^\delta$ for some $\delta > 4$. To be more precise suppose $\mathcal{Q} = \mathcal{Q}_0\varepsilon^\delta$ with $\mathcal{Q}_0 = \mathcal{O}(1)$.

Using (4.1), we notice that the equation of motion for the first interface is given by

$$d\xi_1 = \mathcal{O}(\alpha)dt + d\mathcal{A}_s^{(1)}\ ,$$

and the motion of the second interface is fixed due to mass conservation.

Recall that ℓ_2 is the distance between the two interfaces, and fix $\rho = \varepsilon^\kappa$, which means that the lower bound on ℓ_2 is $\varepsilon^{1-\kappa}$. Let us now first look at (3.12)

$$\sigma_1(\xi) = A_{11}^{-1}E_1^{\xi}.$$

Since $\tilde{u_1}^{\xi} = \tilde{u}_2^h \frac{\partial h_2}{\partial h_1} + \tilde{u}_1^h$ while $\frac{\partial h_2}{\partial h_1} = 1 + \mathcal{O}(e_\varepsilon)$ and $E_1^\xi = \tilde{u}_1^h + \tilde{u}_2^h + \mathcal{O}(e_\varepsilon)$, it follows that

$$E_1^\xi = \tilde{u}_1^\xi + \mathcal{O}(e_\varepsilon)\ ,$$

and again the error term remains of the same order under differentiation w.r.t. ξ_1. Secondly, from (4.24) in [8] there is a constant $c_\star$ such that $\|\tilde{u}_1^\xi\|^2 = 4\ell_2 + c_\star\varepsilon + \mathcal{O}(e_\varepsilon)$, and the error term remains $\mathcal{O}(e_\varepsilon)$ under differentiation. (In our case $N = 1$ we have that $\tilde{w}_1$ used in [8] is up to errors of order $\mathcal{O}(e_\varepsilon)$ equal to $\tilde{u}_1^\xi$.). Moreover, by definition

$$A_{11} = \langle \tilde{u}_1^\xi, E_1^\xi\rangle - \langle \tilde{v}, E_{11}^\xi\rangle = \|\tilde{u}_1^\xi\|^2 + \|\tilde{v}\|_\infty \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(e_\varepsilon)$$

where we used (5.38) (cf. also [8], where the same estimate is used, though never presented analytically) for $E_{11}^\xi = \mathcal{O}(\varepsilon^{-1/2})$. Recall that in the slow channel Γ we have

$$\|v\|_\infty \leq (B_\varepsilon(v))^{1/2} \leq C\varepsilon^{-1}(A_\varepsilon(v))^{1/2} \leq C\varepsilon^{-1}(\delta_\varepsilon^{2-\kappa})^{1/2} \leq C\varepsilon^{-1+\delta(1-\kappa/2)}\ . \tag{4.8}$$

Thus we proved

$$A_{11} = 4\ell_2 + c_\star\varepsilon + \mathcal{O}(\varepsilon^{\delta(1-\kappa/2)-\frac{3}{2}}) \quad \text{and} \quad \sigma_1(\xi) = \frac{1}{4\ell_2 + c_\star\varepsilon + \mathcal{O}(\varepsilon^{\delta(1-\kappa/2)-\frac{3}{2}})}E_1^\xi + \mathcal{O}(e_\varepsilon)\ . \tag{4.9}$$

Now we can consider the deterministic drift

$$\begin{aligned}\mathcal{A}_{\mathcal{Q}}^{(1)} &= A_{11}^{-1}(\xi)\Big[\tfrac{1}{2}\langle \tilde{v}, E_{111}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{11}^\xi, E_1^\xi\rangle - \langle \tilde{u}_1^\xi, E_{11}^\xi\rangle\Big]\langle \mathcal{Q}\sigma_1(\xi), \sigma_1(\xi)\rangle + A_{11}^{-1}(\xi)\langle \mathcal{Q}E_{11}^\xi, \sigma_j(\xi)\rangle \\ &= A_{11}^{-3}\Big[\mathcal{O}(\varepsilon^{-3/2})\|\tilde{v}\| - \tfrac{3}{4}\tfrac{\partial}{\partial\xi_1}\|E_1^\xi\|^2\Big]\|\mathcal{Q}^{1/2}E_1^\xi\|^2 + A_{11}^{-2}\tfrac{1}{2}\tfrac{\partial}{\partial\xi_1}\|\mathcal{Q}^{1/2}E_1^\xi\|^2 + \mathcal{O}(e_\varepsilon)\end{aligned}$$

Thus in the slow channel Γ (cf. (4.8)) the equation of motion for the interface is reduced to

$$\begin{aligned} d\xi_1 =& A_{11}^{-3}\mathcal{O}(\varepsilon^{\delta(1-\kappa/2)-5/2})\|\mathcal{Q}^{1/2}E_1^\xi\|^2dt - \tfrac{3}{4}A_{11}^{-3}\left(\tfrac{\partial}{\partial\xi_1}\|E_1^\xi\|^2\right)\|\mathcal{Q}^{1/2}E_1^\xi\|^2dt \\ &+ A_{11}^{-2}\tfrac{1}{2}\tfrac{\partial}{\partial\xi_1}\|\mathcal{Q}^{1/2}E_1^\xi\|^2dt + A_{11}^{-1}\langle E_1^\xi, dW\rangle + \mathcal{O}(e_\varepsilon)dt\ . \end{aligned}$$

By (45) of [7] we know that

$$\tilde{u}_1^\xi = 1 - u^\xi + \mathcal{O}(e_\varepsilon) \quad \text{and} \quad u_1^\xi = -u_x^\xi + \mathcal{O}(e_\varepsilon)\ ,$$

(as $[0,1] = I_1 \cup I_2$ and $u^\xi(m_1) = u^\xi(0) = -1 + \mathcal{O}(e_\varepsilon)$). They also proved, that the error terms remain $\mathcal{O}(e_\varepsilon)$, under differentiation w.r.t. ξ. Thus, we obtain

$$\|\tilde{u}_1^\xi\|^2 = \|1 - u^\xi\|^2 + \mathcal{O}(e_\varepsilon) = 1 - 2M + \|u^\xi\|^2 + \mathcal{O}(e_\varepsilon)$$

Taking again a derivative yields

$$\tfrac{\partial}{\partial\xi_1}\|\tilde{u}_1^\xi\|^2 = 2\langle u_1^\xi, u^\xi\rangle + \mathcal{O}(e_\varepsilon) = -2\langle u_x^\xi, u^\xi\rangle + \mathcal{O}(e_\varepsilon) = u^\xi(0)^2 - u^\xi(1)^2 + \mathcal{O}(e_\varepsilon) = \mathcal{O}(e_\varepsilon)\ .$$

And thus we verified that

$$\tfrac{\partial}{\partial\xi_1}\|E_1^\xi\|^2 = \mathcal{O}(e_\varepsilon),$$

therefore, the equation of motion for ξ simplifies to

$$d\xi_1 = \mathcal{O}(\varepsilon^{\delta(3-\kappa/2)-11/2})dt + A_{11}^{-2}\tfrac{1}{2}\tfrac{\partial}{\partial\xi_1}\|\mathcal{Q}^{1/2}E_1^\xi\|^2dt + A_{11}^{-1}\langle E_1^\xi, dW\rangle\ . \tag{4.10}$$

Although this is not covered by our assumptions, as a final example we consider space-time white noise with $\mathcal{Q} = \varepsilon^\delta Id$. In this case

$$d\xi = \mathcal{O}(\varepsilon^{3\delta-7/2})dt + \varepsilon^\delta A_{11}^{-1}\langle E_1^\xi, d\hat{W}\rangle,$$

which is a rescaled equation valid on the timescale $\mathcal{O}(\varepsilon^{-\delta})$. Up to the small deterministic error terms, ξ is a stochastic process with mean zero and quadratic variation

$$\begin{aligned} \int_0^t \varepsilon^{2\delta}A_{11}^{-2}\langle E_1^\xi, E_1^\xi\rangle dt &= \varepsilon^{2\delta}\int_0^t A_{11}^{-2}\|\tilde{u}_1^\xi\|^2dt + \mathcal{O}(e_\varepsilon)t \\ &= \varepsilon^{2\delta}\int_0^t A_{11}^{-1}dt + \mathcal{O}(\varepsilon^{\delta-3/2+\kappa})\frac{t}{\ell_2^2} = \frac{\varepsilon^{2\delta}}{4\ell_2}t + \mathcal{O}(\varepsilon^{2\delta+1})t + \mathcal{O}(\varepsilon^{3\delta-7/2+\kappa})t\ , \end{aligned}$$

which means (compare to Levy's characterization of Brownian motion) that in first approximation for times not too large the interface behaves similar to a Brownian motion with variance $\varepsilon^{2\delta}/(4\ell_2)$.

5. Higher order estimates

5.1. **Preliminaries.** This section deals with the estimation of all the following higher order terms that appear due to stochastic integration when deriving the equations of motion in the slow channel:

$$\langle\tilde{v}, E_{ilk}^\xi\rangle, \quad \langle\tilde{u}_{kl}^\xi, E_i^\xi\rangle, \quad \langle\tilde{u}_k^\xi, E_{il}^\xi\rangle.$$

In addition, we bound the quantity $\langle L^c\tilde{v}, \tilde{u}_{kl}^\xi\rangle$. Considering a general smooth in space function ϕ, the operator L^c is given by

$$L^c(\phi) := -\varepsilon^2\phi_{xxxx} + (f'(u^h)\phi_x)_x.$$

In order to achieve this aim, we investigate the properties of the stationary problem (1.3). Our analysis admits extensive computations and is based on the ideas and technics presented in [21, 22, 7, 8] for the deterministic case where analogous terms of lower order have been estimated.

Denote first, that for the construction of the approximate manifold of solutions for the stochastic Cahn-Hilliard we use a local coordinate system when presenting the admissible interface positions. The h_{N+1} variable depends on $h_i = \xi_i$, $i = 1, \ldots, N$, therefore, when differentiating two times in ξ variables and applying the chain rule the second order term $\frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j}$ appears. More specifically, for a general function f smooth in space and any $i, j = 1, \ldots, N$, we obtain

$$\frac{\partial f}{\partial \xi_i} = \frac{\partial f}{\partial h_i} + \frac{\partial f}{\partial h_{N+1}} \frac{\partial h_{N+1}}{\partial h_i}, \quad \text{and}$$
$$\frac{\partial^2 f}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 f}{\partial h_i \partial h_j} + \Big(\frac{\partial^2 f}{\partial h_{N+1} \partial h_j} + \frac{\partial^2 f}{\partial h_{N+1}^2} \frac{\partial h_{N+1}}{\partial h_j} \Big) \frac{\partial h_{N+1}}{\partial h_i} + \frac{\partial f}{\partial h_{N+1}} \Big(\frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j} + \frac{\partial^2 h_{N+1}}{\partial h_i \partial h_{N+1}} \frac{\partial h_{N+1}}{\partial h_j} \Big). \tag{5.1}$$

By the next lemma considering $\rho = \varepsilon^\kappa$ for some small $\kappa > 0$ and thus α, β are exponentially small, we estimate $\left| \frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j} \right|$. As in [7], where the analogous first order estimate has been derived, we use an implicit function theorem argument combined with the mass conservation constraint. If u^h is in the second approximate manifold $\mathcal{M}$ then by definition mass conservation holds i.e.

$$M = M(h) = \int_0^1 u^h(x) dx.$$

Differentiating two times in h variables, we get

$$\frac{d^2}{dh_i dh_j} M(h) = \int_0^1 u^h_{ij} dx,$$

where $u^h_{ij} := \frac{\partial^2 u^h}{\partial h_i \partial h_j} = \frac{\partial u^h_i}{\partial h_j}$.

Lemma 5.1. *For any $i, j = 1, \ldots, N$ the next inequality follows*

$$\Big| \frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j} \Big| \leq \mathcal{O}(e_\varepsilon).$$

Proof. Consider ℓ a generic positive variable. According to the analysis presented in [21], when comparing the x and ℓ derivatives of the solution ϕ of the stationary problem (1.3), we obtain a residual function w given by the following relation

$$2\phi_\ell(x, \ell, \pm 1) = -(\operatorname{sgn} x)\phi_x(x, \ell, \pm 1) + 2w(x, \ell, \pm 1). \tag{5.2}$$

Let us define $I_j := [m_j, m_{j+1}]$, $\chi^j(x) := \chi\Big(\frac{x-h_j}{\varepsilon}\Big)$. If $w^j(x) := w(x - m_j, h_j - h_{j-1}, (-1)^j)$, then the interval $[h_{j-1} - \varepsilon, h_{j+1} + \varepsilon]$ contains the support of u^h_j and

$$u^h_j(x) = \begin{cases} \chi^{j-1} w^j & \text{for } x \in I_{j-1} \\ (1 - \chi^j)(-\phi^j_x + w^j) + \chi^j(-\phi^{j+1}_x - w^{j+1}) + \chi^j_x(\phi^j - \phi^{j+1}) & \text{for } x \in I_j \\ -(1 - \chi^{j+1}) w^{j+1} & \text{for } x \in I_{j+1} \end{cases} \tag{5.3}$$

where $\chi^j_x = \partial_x \Big(\chi\Big(\frac{x-h_j}{\varepsilon}\Big)\Big)$ and $\phi^j_x = \phi_x(x - m_j, l_j - l_{j-1}, (-1)^j)$ (cf. [21], p. 561). We denote that in I_j (cf. [7] p. 430)

$$u^h_j = -u^h_x + (1 - \chi^j) w^j - \chi^j w^{j+1}$$

and thus

$$u^h_{ji} = -\frac{\partial u^h_x}{\partial h_i} + (-\delta_{j,i}\chi^j_x) w^j + (1 - \chi^j)(A_{j,i} w^j_x + B_{j,i} w^j_\ell) - \delta_{j,i}\chi^j_x w^{j+1} - \chi^j (A_{j+1,i} w^{j+1}_x + B_{j+1,i} w^{j+1}_\ell), \text{ in } I_j \tag{5.4}$$

where $w_x^j = w_x(x-m_j, l_j - l_{j-1}, (-1)^j)$, $w_\ell^j = w_l(x-m_j, l_j-l_{j-1}, (-1)^j)$, $\delta_{j,i}$ is the Kronecher delta, while

$$A_{j,i} := \frac{\partial(x-m_j)}{\partial h_i} = \begin{cases} 0 & \text{for } i \neq j, j-1 \\ -1/2 & \text{for } i = j, j-1 \end{cases}$$

and

$$B_{j,i} := \frac{\partial(h_j-h_{j-1})}{\partial h_i} = \begin{cases} 0 & \text{for } i \neq j, j-1 \\ 1 & \text{for } i = j \\ -1 & \text{for } i = j-1. \end{cases}$$

In a similar way we obtain

$$u^h_{ji} = \delta_{j-1,i}\chi_x^{j-1} w^j + \chi^{j-1}(A_{j,i}w_x^j + B_{j,i}w_\ell^j), \text{ in } I_{j-1}, \tag{5.5}$$

$$u^h_{ji} = \delta_{j+1,i}\chi_x^{j+1} w^{j+1} - (1-\chi^{j+1})(A_{j+1,i}w_x^{j+1} + B_{j+1,i}w_\ell^{j+1}), \text{ in } I_{j+1}. \tag{5.6}$$

Using now the estimates of w^j, w_x^j, w_ℓ^j (cf. [21], or [7] at p. 172), then for $r > 0$ sufficiently small, we obtain

$$\Big|\int_{I_{j-1}\cup I_{j+1}} u^h_{ji}(x)dx\Big| \leq C\varepsilon^{-2}(r^{-1}+1)\beta(r)\mathcal{K}_{j,i} + \mathcal{O}(e_\varepsilon)(\delta_{j-1,i}+\delta_{j+1,i}),$$

with $\mathcal{K}_{j,i} = |A_{j,i}| + |A_{j+1,i}| + |B_{j,i}| + |B_{j+1,i}|$ and

$$\begin{aligned} \Big|\int_{I_j} \Big[&(-\delta_{j,i}\chi_x^j)w^j + (1-\chi^j)(A_{j,i}w_x^j + B_{j,i}w_\ell^j) \\ &- \delta_{j,i}\chi_x^j w^{j+1} - \chi^j(A_{j+1,i}w_x^{j+1} + B_{j+1,i}w_\ell^{j+1})\Big]dx\Big| \\ &\leq C\varepsilon^{-2}(r^{-1}+1)\beta(r)\mathcal{K}_{j,i} + \mathcal{O}(e_\varepsilon)\delta_{j,i}. \end{aligned}$$

Therefore, using the estimates for w^i it follows that

$$\begin{aligned} &\frac{d^2}{dh_j dh_i}M(h) = \int_0^1 u^h_{ji}dx = \int_{I_j} -\frac{\partial^2 u^h}{\partial x \partial h_i}dx + \mathcal{O}(\varepsilon^{-2}(r^{-1}+1)\beta(r))\mathcal{K}_{j,i} \\ &+ \mathcal{O}(e_\varepsilon)(\delta_{j-1,i}+\delta_{j,i}+\delta_{j+1,i}) \\ &= \int_{I_j}(-\frac{\partial u_i^h}{\partial x})dx + \mathcal{O}(\varepsilon^{-2}(r^{-1}+1)\beta(r))\mathcal{K}_{j,i} + \mathcal{O}(e_\varepsilon)(\delta_{j-1,i}+\delta_{j,i}+\delta_{j+1,i}) \\ &= -(u_i^h(m_{j+1}) - u_i^h(m_j)) + \mathcal{O}(\varepsilon^{-2}(r^{-1}+1)\beta(r))\mathcal{K}_{j,i} \\ &+ \mathcal{O}(e_\varepsilon)(\delta_{j-1,i}+\delta_{j,i}+\delta_{j+1,i}). \end{aligned}$$

Since the support of u_i^h is $I_{i-1}\cup I_i \cup I_{i+1} \ni m_{i-1}, m_i, m_{i+1}, m_{i+2}$ then we get that $\frac{d^2}{dh_i dh_j}M = 0$ if $j \neq i-1, i, i+1, i+2$, while for example $u_i^h(m_i) = \chi^{i-1}w^i|_{m_i} = \chi^{i-1}|_{m_i} w(0, l_i, \pm 1)$ and $u_i^h(m_{i+1}) = -(1-\chi^{i+1})w^{i+1}|_{m_{i+1}} = -(1-\chi^{i+1})|_{m_{i+1}} w(0, l_{i+1}, \pm 1)$. But $w(0) = \mathcal{O}(\varepsilon^{-1})\alpha'_\pm(r)$, [21] p. 558, since $\phi_{xx}(0)^{-1} = \varepsilon^2/W'(\phi(0))$ and ε/l is uniformly bounded, while χ is C^∞.

Let us now for simplicity consider $N = 1$ then $M(h_1, y) =$ constant, when $y = h_2$ where h_2 is a function of h_1, so

$$\frac{\partial M}{\partial h_1} + \frac{\partial M}{\partial y}\frac{\partial y}{h_1} = 0$$

and thus

$$\frac{\partial^2 M}{\partial h_1 \partial h_1} + (\frac{\partial M}{\partial y})_y \frac{\partial y}{\partial h_1}\frac{\partial y}{\partial h_1} + \frac{\partial M}{\partial h_1}\frac{\partial^2 y}{\partial h_1^2} = 0.$$

We set $y = h_2$ to get using the estimate $\frac{\partial h_{N+1}}{\partial h_j} = \mathcal{O}(1)$

$$\mathcal{O}(e_\varepsilon) + \mathcal{O}(e_\varepsilon)\mathcal{O}(1) + \mathcal{O}(1)\frac{\partial^2 h_2}{\partial h_1^2} = 0$$

and thus

$$\frac{\partial^2 h_2}{\partial h_1^2} = O(e_\varepsilon).$$

The same follows when $N > 1$. Therefore, we obtain the result. □

5.2. **The estimates.** We define $I_s := [-\ell/2 - \varepsilon, \ell/2 + \varepsilon]$, then for any $x \in I_s$ it holds that ([8, 21, 22])

$$\begin{aligned} &|w| \le c\varepsilon^{-1}\beta_\pm(r), \\ &|w_x| \le c\varepsilon^{-2}r^{-1}\beta_\pm(r), \\ &|w_\ell| \le c\varepsilon^{-2}\beta_\pm(r), \\ &|w_{x\ell}| \le c\varepsilon^{-3}r^{-1}\beta_\pm(r), \\ &|w_{xx}| \le c\varepsilon^{-3}\beta_\pm(r). \end{aligned} \tag{5.7}$$

For the purposes of our proof we will need estimates for the terms

$$|w_{\ell\ell}|, \ \ |w_{xxx}|, \ \ |w_{xx\ell}|, \ \ |w_{x\ell\ell}|, \ \ |w_{xxxxx}|, \ \ |w_{xxx\ell}|, \ \ |w_{xx\ell\ell}|.$$

It is sufficient to estimate the above terms in $I := [0, \ell/2 + \varepsilon]$ or in $(0, \ell/2 + \varepsilon]$. We write $I = [0, \ell/2 - \varepsilon H] \cup [\ell/2 - \varepsilon H, \ell/2 + \varepsilon]$, for a positive H to be defined in the sequel. We set

$$I_H := [0, \ell/2 - \varepsilon H], \quad \text{and} \quad J := [\ell/2 - \varepsilon H, \ell/2 + \varepsilon],$$

and prove the next lemma related to the second derivative of w in ℓ.

Lemma 5.2. *For any $x \in I_s$ it holds that*

$$|w_{\ell\ell}| \le c\varepsilon^{-3}\beta_\pm(r). \tag{5.8}$$

Proof. Motivated by the proof of [22] for the estimate of $|w_\ell|$, we use that

$$\varepsilon^2 w_{xx} = f'(\phi(x))w \ \text{ in } \ (0, \ell/2 + \varepsilon) \supset I_H^\circ,$$

and differentiate two times in ℓ to obtain

$$\varepsilon^2 (w_{\ell\ell})_{xx} - f'(\phi)w_{\ell\ell} = \mathcal{F}$$

for $\mathcal{F} := f'''(\phi)\phi_\ell^2 w + f''(\phi)\phi_{\ell\ell}w + 2f''(\phi)\phi_\ell w_\ell$. By maximum principle it follows that

$$|w_{\ell\ell}(x)| \le \max\Big\{|w_{\ell\ell}(0)|, \ \ |w_{\ell\ell}(\ell/2 - \varepsilon H)|, \ \ \sup_{x\in I_H}\Big|\mathcal{F}/f'(\phi)\Big|\Big\} \ \text{ for any } x \in I_H. \tag{5.9}$$

Following Carr and Pego (cf. [21] p. 560), we choose α and H such that $f'(\phi(x)) \ge c_0 > 0$ for $0 < x < \ell/2 - \varepsilon H$. Since $\varepsilon^2\phi_x^2 = 2(F(\phi) - \alpha)$, then there exists $C > 0$ such that $\frac{1}{|\phi_x|} \le \frac{\varepsilon}{C}$ for any $x \in J = [\ell/2 - \varepsilon H, \ell/2 + \varepsilon]$ (cf. [21] p. 560, and p. 557).

We will estimate first, $|w_{\ell\ell}(x, \ell, -1)|$ in J. It holds that (cf. [21] p. 558)

$$w(x, \ell, -1) = \varepsilon^{-1}\ell^{-2}\alpha_-'(r)\phi_x(|x|, \ell, -1)\int_{\ell/2}^{|x|}\frac{ds}{\phi_x(s, \ell, -1)^2}. \tag{5.10}$$

Let us define $\mathcal{A} := \int_{\ell/2}^{|x|}\frac{ds}{\phi_x(s,\ell,-1)^2}$; for simplicity we shall refer to α_- by using the symbol α. We differentiate relation (5.10) and arrive at

$$\begin{aligned} w_{\ell\ell} = \varepsilon^{-1}\Big\{&(\ell^{-2}\alpha'(r))_{\ell\ell}\phi_x\mathcal{A} + 2(\ell^{-2}\alpha'(r))_\ell\phi_{x\ell}\mathcal{A} + 2(\ell^{-2}\alpha'(r))_\ell\phi_x\mathcal{A}_\ell \\ &+ (\ell^{-2}\alpha'(r))\phi_{x\ell\ell}\mathcal{A} + 2(\ell^{-2}\alpha'(r))\phi_{x\ell}\mathcal{A}_\ell + (\ell^{-2}\alpha'(r))\phi_x\mathcal{A}_{\ell\ell}\Big\}. \end{aligned} \tag{5.11}$$

According to [21, 22] it follows that

$$|\alpha'| \leq cr^{-2}\alpha, \ \ |\alpha''| \leq cr^{-4}\alpha,$$

analogously we obtain

$$|\alpha'''| \leq cr^{-6}\alpha.$$

So, observing that that $r = \varepsilon/\ell$ is bounded, i.e. $\ell^{-1} \leq c\varepsilon^{-1}$, we get

$$|\ell^{-2}\alpha'(r)| \leq c\varepsilon^{-2}\alpha, \ \ |(\ell^{-2}\alpha'(r))_\ell| \leq c\varepsilon^{-3}\alpha, \ \ |(\ell^{-2}\alpha'(r))_{\ell\ell}| \leq c\varepsilon^{-4}\alpha. \tag{5.12}$$

Obviously since $x \in J$ then $|\mathcal{A}| \leq c\varepsilon^{2+1}$. Denote that

$$\varepsilon^2\phi_x^2 = 2(F(\phi) - \alpha) \tag{5.13}$$

(cf. [21] p. 552), while

$$\varepsilon^2\phi_{xx} = f(\phi). \tag{5.14}$$

Since $\int_{-\ell/2}^{\ell/2} |\phi_x| \leq 2$ (cf. [21] p. 558), and ϕ satisfies a Dirichlet problem then by trace inequality we get that ϕ is uniformly bounded. Therefore, we obtain

$$|\phi_x| \leq c\varepsilon^{-1}, \ \ |\phi_{xx}| \leq c\varepsilon^{-2}, \ \ |\phi_{xxx}| \leq c\varepsilon^{-3}.$$

Using now the definition (5.2) of w, and the fact that $|w| + |\phi_x| \leq c\varepsilon^{-1}$, we arrive at

$$|\phi_\ell| \leq c\varepsilon^{-1},$$

while $|\phi_{xl}| \leq c|\phi_{xx}| + c|w_x|$. So, using that $|w_x| \leq c\varepsilon^{-2}$, [8], we get

$$|\phi_{x\ell}| \leq c\varepsilon^{-2}.$$

By (5.14) it follows that

$$|\phi_{xx\ell}| \leq c\varepsilon^{-3}.$$

Finally, we will also need an estimate for the term $\phi_{x\ell\ell}$. We differentiate two times in ℓ the equation (5.13) and obtain

$$|\varepsilon^2\phi_x\phi_{x\ell\ell}| \leq c\varepsilon^{-2},$$

hence using that in J it holds that $\frac{1}{|\phi_x|} \leq c\varepsilon$ we get

$$|\phi_{x\ell\ell}| \leq c\varepsilon^{-3} \ \text{ in } \ J.$$

In order to compute the derivatives of $\mathcal{A}$ in (5.11), we apply the formulae

$$\frac{d}{d\ell}\int_{s(\ell)}^b g(s,\ell)ds = \int_{s(\ell)}^b g_\ell(s,\ell)ds - s'(\ell)g(s(\ell),\ell),$$

$$\frac{d^2}{d\ell^2}\int_{s(\ell)}^b g(s,\ell)ds = \int_{s(\ell)}^b g_{\ell\ell}(s(\ell),\ell)ds - s'(\ell)g_\ell(s(\ell),\ell)$$
$$- s''(\ell)g(s(\ell),\ell) - s'(\ell)^2 g_x(s(\ell),l) - s'(\ell)g_\ell(s(\ell),\ell).$$

After tedious computations, using the above estimates and the fact that the interval's length is of order $\mathcal{O}(\varepsilon)$ we arrive at

$$|\mathcal{A}_\ell| \leq c\varepsilon^2, \ \ |\mathcal{A}_{\ell\ell}| \leq c\varepsilon.$$

We denote that ε/ℓ is bounded i.e. $\ell^{-1} \leq c\varepsilon^{-1}$, thus by (5.11) and (5.12) we obtain

$$|w_{\ell\ell}| \leq c\varepsilon^{-3}\alpha \ \text{ in } \ J. \tag{5.15}$$

So by (5.15), since $\ell/2 - \varepsilon H \in J$, it follows that

$$|w_{\ell\ell}(\ell/2 - \varepsilon H)| \leq c\varepsilon^{-3}\alpha. \tag{5.16}$$

By the definition of $\mathcal{F}$, the fact that $f' \geq c_0 > 0$ in I_H and the first and third estimate of (5.7) we get for $\beta := \beta_-$ that

$$\sup_{x\in I_H}\Big|\mathcal{F}/f'(\phi)\Big| \leq c\Big[|\phi_\ell|^2|w| + |\phi_{\ell\ell}||w| + |\phi_\ell||w_\ell|\Big] \leq c\varepsilon^{-1}\beta\Big[|\phi_\ell|^2 + |\phi_{\ell\ell}| + \varepsilon^{-1}|\phi_\ell|\Big].$$

In addition, since $|w_\ell| + |\phi_{x\ell}| \leq c\varepsilon^{-2}$ [21, 8], then it follows that

$$|\phi_{\ell\ell}| \leq c\varepsilon^{-2},$$

while, as we proved, $|\phi_\ell| \leq c\varepsilon^{-1}$, so

$$\sup_{x\in I_H}\Big|\mathcal{F}/f'(\phi)\Big| \leq c\varepsilon^{-3}\beta. \tag{5.17}$$

What is missing is the estimate of $|w_{\ell\ell}(0)|$; in [22] by use of the relation $w(0) = -\frac{\partial\beta}{\partial\ell}(\varepsilon/\ell)$, it was demonstrated that $|w_\ell(0)| \leq c\varepsilon^{-2}\beta$, analogously by differentiating in ℓ it follows that

$$|w_{\ell\ell}(0)| \leq c\varepsilon^{-3}\beta. \tag{5.18}$$

Using now (5.9), (5.15), (5.16), (5.17) and (5.18) we obtain that $|w_{\ell\ell}| \leq c\varepsilon^{-3}\beta$ for any x in $I = I_H \cup J$. By symmetry we prove finally that $|w_{\ell\ell}| \leq c\varepsilon^{-3}\beta_\pm(r)$ in I_s. □

The next three lemmas present bounds for the third or higher order terms.

Lemma 5.3. *For any $x \in I_s^\circ - \{0\}$ it holds that*

$$|w_{xxx}| \leq c\varepsilon^{-4}r^{-1}\beta_\pm(r), \tag{5.19}$$

$$|w_{xx\ell}| \leq c\varepsilon^{-4}\beta_\pm(r). \tag{5.20}$$

Proof. We consider $x \in (0, \ell/2 + \varepsilon)$ so $\varepsilon^2 w_{xx} = f'(\phi)w$. By differentiating the previous in x and using (5.7) and the $|\phi_x|$ estimate, or by differentiating in ℓ and using (5.7) and the $|\phi_\ell|$ estimate we get the following

$$\begin{aligned}|w_{xxx}| &\leq c\varepsilon^{-2}\Big[|f'(\phi)||w_x| + |f''(\phi)||\phi_x||w|\Big]\\ &\leq c\varepsilon^{-2}\Big[c\varepsilon^{-2}r^{-1}\beta + c\varepsilon^{-1}\varepsilon^{-1}\beta\Big] \leq c\varepsilon^{-4}r^{-1}\beta,\end{aligned}$$

and

$$\begin{aligned}|w_{xx\ell}| &\leq c\varepsilon^{-2}\Big[|f'(\phi)||w_\ell| + |f''(\phi)||\phi_\ell||w|\Big]\\ &\leq c\varepsilon^{-2}\Big[c\varepsilon^{-2}\beta + c\varepsilon^{-1}\varepsilon^{-1}\beta\Big] \leq c\varepsilon^{-4}\beta,\end{aligned}$$

for $\beta = \beta_-$. Therefore, we obtain the results in $I_s^\circ - \{0\}$. □

Lemma 5.4. *For any $x \in I_s - \{0\}$ it holds that*

$$|w_{x\ell\ell}| \leq c\varepsilon^{-4}r^{-1}\beta_\pm(r). \tag{5.21}$$

Proof. We consider $x \in (0, \ell/2 + \varepsilon]$, write $w_{x\ell\ell}(\ell/2) - w_{x\ell\ell}(x) = \int_x^{\ell/2} w_{xx\ell\ell}(s)ds$ and get

$$|w_{x\ell\ell}(x)| \leq |w_{x\ell\ell}(\ell/2)| + \int_x^{\ell/2}|w_{xx\ell\ell}(s)|ds. \tag{5.22}$$

We use the definition of w given in (5.10), set $p = \varepsilon^{-1}\ell^2\alpha'$, and remind that $\mathcal{A} = \int_{\ell/2}^{|x|}\frac{ds}{\phi_x^2}$. We take first the x derivative and then the $\ell\ell$ derivative to obtain

$$\begin{aligned}w_{x\ell\ell} =& p_{\ell\ell}\phi_{xx}\mathcal{A} + p_\ell\phi_{xx\ell}\mathcal{A} + 2p_\ell\phi_{xx}\mathcal{A}_\ell + p_\ell\phi_{xx\ell}\mathcal{A} + p\phi_{xx\ell\ell}\mathcal{A}\\ &+ 2p\phi_{xx\ell}\mathcal{A}_\ell + p\phi_{xx}\mathcal{A}_{\ell\ell} - \frac{p_\ell\phi_{x\ell}}{\phi_x^2} - p\frac{(\phi_{x\ell\ell}\phi_x^2 - 2\phi_{x\ell}^2\phi_x)}{\phi_x^4} + \frac{p_{\ell\ell}}{\phi_x} - \frac{p_\ell\phi_{x\ell}}{\phi_x^2}.\end{aligned}$$

Observe that $\mathcal{A}=0$ at $x=\ell/2$, while

$$\mathcal{A}_\ell(\ell/2)=-\frac{1}{2}\phi_x(\ell/2)^{-2},\ \ \mathcal{A}_{\ell\ell}(\ell/2)=\phi_x(\ell/2)^{-3}\phi_{x\ell}(\ell/2)+\phi_{x\ell}(\ell/2)\phi_x(\ell/2)^{-3}.$$

We also denote that $\ell/2\in J$, so by the estimates of Lemma 5.2 we obtain $|\phi_{x\ell}(\ell/2)|\leq c\varepsilon^{-2}$ and $|\phi_x(\ell/2)|^{-1}\leq c\varepsilon$. Thus, as in Lemma 5.2 for general $x\in J$, we get that $\varepsilon^{-1}|\mathcal{A}_\ell(\ell/2)|+|\mathcal{A}_{\ell\ell}(\ell/2)|\leq c\varepsilon$.

In addition using the last estimate of (5.12) we obtain that $|p_{\ell\ell}(\ell/2)|\leq c\varepsilon^{-5}\alpha$. Further, we use that $\ell/2\in J$, so by the proof of Lemma 5.2 we have that $|\phi_{xx}(\ell/2)|\leq c\varepsilon^{-2}$, while $|\phi_{x\ell\ell}(\ell/2)|\leq c\varepsilon^{-3}$, and therefore, we obtain finally

$$|w_{x\ell\ell}(\ell/2)|\leq c\varepsilon^{-4}\alpha, \tag{5.23}$$

for $\alpha=\alpha_-$.

Since $\varepsilon^2 w_{xx}=f'(\phi)w$ in $(0,\ell/2+\varepsilon)$, then taking the $\ell\ell$ derivative we arrive at

$$|w_{xx\ell\ell}(x)|\leq c\varepsilon^{-2}\Big[|\phi_\ell|^2|w|+|\phi_\ell||w_\ell|+|w_{\ell\ell}|\Big]\leq c\varepsilon^{-5}\beta \tag{5.24}$$

for $\beta=\beta_-$. Here, we used the estimates of the proof of Lemma 5.2 i.e. that $|\phi_\ell|\leq c\varepsilon^{-1}$, the first and third estimate of (5.7), the fact that $|w|\leq c\varepsilon^{-1}\beta$ while $|w_\ell|\leq c\varepsilon^{-2}\beta$, and the result of Lemma 5.2 i.e. that $|w_{\ell\ell}|\leq c\varepsilon^{-3}\beta$.

Since $x\in(0,\ell/2)$ then using that $r=\varepsilon/\ell$, we get that $|x-\ell/2|\leq c(\ell/2+\varepsilon)\leq c\varepsilon r^{-1}$, and therefore, (5.22), (5.23) and (5.24) give

$$|w_{x\ell\ell}(x)|\leq c\varepsilon^{-4}r^{-1}\beta,\ \ x\in(0,\ell/2+\varepsilon].$$

By symmetry the analogous result holds for any $x\in[-\ell/2-\varepsilon,0)$. □

Analogously the next lemma follows:

Lemma 5.5. *For any $x\in I_s-\{0\}$ it holds that*

$$|w_{xxxxx}|+|w_{xxx\ell}|+|w_{xx\ell\ell}|\leq c\varepsilon^{-5}r^{-1}\beta_\pm(r). \tag{5.25}$$

According to the definition of E_i^ξ, in order to estimate E_i^ξ, E_{ij}^ξ and E_{ijk}^ξ we need first the next result.

Lemma 5.6. *For any i,j,k it follows that*

$$\begin{aligned}&|Q_j|\leq c\varepsilon^{-3}\beta,\\&|Q_{ij}|\leq c\varepsilon^{-4}r^{-1}\beta,\\&|Q_{ijk}|\leq c\varepsilon^{-5}r^{-1}\beta.\end{aligned} \tag{5.26}$$

Proof. We remind that

$$u_j^h(x)=\begin{cases}\chi^{j-1}w^j & \text{for } x\in I_{j-1}\\(1-\chi^j)(-\phi_x^j+w^j)+\chi^j(-\phi_x^{j+1}-w^{j+1})+\chi_x^j(\phi^j-\phi^{j+1}) & \text{for } x\in I_j\\-(1-\chi^{j+1})w^{j+1} & \text{for } x\in I_{j+1}.\end{cases}$$

Consider $x=0,1$ (i.e. in the first and last set of the support). Using the estimates of $|w|,|wxx|$ we arrive at

$$|\tilde{u}_j^h|\leq c\varepsilon^{-1}\beta\ \text{ and thus }\ |\tilde{w}_j|\leq c\varepsilon^{-1}\beta,$$
$$|\tilde{u}_{jxx}^h|\leq c\varepsilon^{-3}\beta\ \text{ and thus }\ |\tilde{w}_{jxx}|\leq c\varepsilon^{-3}\beta.$$

The estimates of $|w_x|,|w_\ell|$ and of $|w_{xxx}|,|w_{\ell xx}|$ respectively, now give

$$|\tilde{u}_{ji}^h|\leq c\varepsilon^{-2}r^{-1}\beta\ \text{ and thus }\ |\tilde{w}_{ji}|\leq c\varepsilon^{-2}r^{-1}\beta,$$
$$|\tilde{u}_{jixx}^h|\leq c\varepsilon^{-4}r^{-1}\beta\ \text{ and thus }\ |\tilde{w}_{jixx}|\leq c\varepsilon^{-4}r^{-1}\beta.$$

Finally, using the estimates of $|w_{xx}|, |w_{x\ell}|, |w_{\ell\ell}|$ and of $|w_{xxxx}|, |w_{xxx\ell}|, |w_{xx\ell\ell}|$ respectively we obtain

$$|\tilde{u}^h_{jik}| \leq c\varepsilon^{-3}r^{-1}\beta \text{ and thus } |\tilde{w}_{jik}| \leq c\varepsilon^{-3}r^{-1}\beta,$$
$$|\tilde{u}^h_{jikxx}| \leq c\varepsilon^{-5}r^{-1}\beta \text{ and thus } |\tilde{w}_{jikxx}| \leq c\varepsilon^{-5}r^{-1}\beta.$$

Remind also that

$$\tilde{w}_j := \tilde{u}^h_j(x) + \tilde{u}^h_{j+1}(x),$$
$$Q_j(x) := (-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x)\tilde{w}_{jxx}(0) + \frac{1}{6}(x^3 - x)\tilde{w}_{jxx}(1) + x\tilde{w}_j(1), \quad j = 1, \dots, N,$$

thus, the definition of Q_j combined with the above estimates on $\tilde{w}_j$ give the result. □

Remark 5.7. By [21] p. 557-556, the next estimates hold true

$$\int_{-\ell/2}^{0} \phi_x(x, \ell, -1)^2 + \int_0^{\ell/2} \phi_x(x, \ell, +1)^2 \leq \varepsilon^{-1}S_\infty + E(r), \tag{5.27}$$

where $|E| \leq c\varepsilon^{-1}\beta$ and $S_\infty = \int_{-1}^1 \sqrt{2F(u)}du$, and

$$\int_{-\ell/2}^{\ell/2} |\phi_x| dx \leq 2, \tag{5.28}$$

and

$$\int_{-\ell/2}^{\ell/2} |\phi_{xx}|^2 + dx \leq c\varepsilon^{-3}. \tag{5.29}$$

In addition, there exists constant $c > 0$ such that for $x \in [h_j - \varepsilon, h_j + \varepsilon], j = 0, \dots, N+1$ we have

$$|\phi^j(x) - \phi^{j+1}(x)| \leq c|a^j - a^{j+1}|, \tag{5.30}$$

$$|\phi^j_x(x) - \phi^{j+1}_x(x)| \leq c\varepsilon^{-1}|a^j - a^{j+1}|, \tag{5.31}$$

$$|\phi^j_{xx}(x) - \phi^{j+1}_{xx}(x)| \leq c\varepsilon^{-2}|a^j - a^{j+1}|, \tag{5.32}$$

provided $\varepsilon/\ell_j, \varepsilon/\ell_{j+1} < r_0$ with r_0 small (cf. [7]).

Now, we are able to compute bounds for the term $\tilde{u}^h$ which are presented in the next theorem.

Theorem 5.8. *For any i, j, k it holds that*

$$\begin{aligned} &|\tilde{u}^h_j| \leq \mathcal{O}(1) + \mathcal{O}(\|w\|), \\ &\|\tilde{u}^h_{ji}\| \leq c\varepsilon^{-1/2}(1 + S_\infty^{1/2} + \max(r_j a^j, r_{j+1}a^{j+1})^{1/2}) + c\|w_x\| + c\|w_\ell\|, \\ &\|\tilde{u}^h_{jik}\| \leq c\varepsilon^{-3/2} + c\|w_x\| + c\|w_\ell\| + c\|w_{xx}\| + c\|w_{x\ell}\| + c\|w_{\ell\ell}\|. \end{aligned} \tag{5.33}$$

Proof. We use the definition of u^h_j and get by (5.28) that

$$|\tilde{u}^h_j| \leq c\int_0^x |\phi_x| dx + c\|w\| \leq c + c\|w\|.$$

By [7] p. 38 it holds that

$$u^h_j(x) = -u^h_x(x) + (1 - \chi^j)w^j - \chi^j w^{j+1} \quad x \in I_j,$$

so using the above and (5.3) we obtain

$$u^h_{ji}(x) = \begin{cases} \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j-1} \\ -u^h_{xx}(x) - u^h_{xi}(x) + \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_j \\ \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j+1}, \end{cases}$$

and therefore we arrive at

$$\tilde{u}^h_{ji}(x) = \int_0^x u^h_{ji}(y)dy = \begin{cases} \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j-1} \\ \mathcal{O}(u^h_x + u^h_i + w_x + w_\ell) & \text{for } x \in I_j \\ \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j+1}. \end{cases}$$

The argument of [21] p. 562 of Lemma 8.3 applied for u^h in I_j and the support of $|\phi^j_x - \phi^{j+1}_x|$ combined with (5.31) and (5.28), since

$$u^h_x = \mathcal{O}(|\phi_x|) + \mathcal{O}(|\phi^j_x - \phi^{j+1}_x|),$$

now gives that

$$\|u^h_x\| \le \|\phi_x\| + \sqrt{\mathcal{O}(\varepsilon^{-2}\varepsilon)} \le c\varepsilon^{-1/2},$$

while by [21] (cf. p. 563, relation (8.6)) it holds that

$$\|u^h_i\| \le \varepsilon^{-1/2}(S^{1/2}_\infty + \max(r_j a^j, r_{j+1}a^{j+1})^{1/2}).$$

Using the above estimates we obtain

$$\|\tilde{u}^h_{ji}\| \le c\varepsilon^{-1/2}(1 + S^{1/2}_\infty + \max(r_j a^j, r_{j+1}a^{j+1})^{1/2}) + c\|w_x\| + c\|w_\ell\|.$$

Observe now that

$$\tilde{u}^h_{jik}(x) = \int_0^x u^h_{jik}(y)dy = \begin{cases} \mathcal{O}(w_{xx} + w_{x\ell} + w_{\ell\ell}) & \text{for } x \in I_{j-1} \\ \mathcal{O}(u^h_{xx} + u^h_{xi} + w_{xx} + w_{x\ell} + w_{\ell\ell}) & \text{for } x \in I_j \\ \mathcal{O}(w_{xx} + w_{x\ell} + w_{\ell\ell}) & \text{for } x \in I_{j+1}. \end{cases}$$

In addition, since $u^h_h = -u^h_x + (1-\chi^j)w^j - \chi^j w^{j+1}$ in I_j, then we obtain that

$$\|u^h_{xi}\| \le \|u^h_{xx}\| + c\|w_x\|.$$

Differentiating two times in x the function u^h and using the estimate (5.32) and (5.29) and the support of $|\phi^j_{xx} - \phi^{j+1}_{xx}|$ we get

$$u^h_{xx} = \mathcal{O}(|\phi_{xx}|) + \mathcal{O}(|\phi^j_{xx} - \phi^{j+1}_{xx}|),$$

and thus

$$\|u^h_{xx}\| \le \|\phi_{xx}\| + \sqrt{\mathcal{O}(\varepsilon^{-4}\varepsilon)} \le c\varepsilon^{-3/2}.$$

So, it follows that

$$\|u^h_{xi}\| \le c\varepsilon^{-3/2} + c\|w_x\|.$$

The previous estimates give finally

$$\|\tilde{u}^h_{jik}\| \le c\varepsilon^{-3/2} + c\|w_x\| + c\|w_\ell\| + c\|w_{xx}\| + c\|w_{x\ell}\| + c\|w_{\ell\ell}\|.$$

□

Using now the estimate $|\tilde{u}^h_j| \le \mathcal{O}(1) + \mathcal{O}(\|w\|)$ combined with the implicit function result for change of variables we get that

$$|\tilde{u}^\xi_j| \le (\mathcal{O}(1) + \mathcal{O}(\|w\|))[\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-1}\beta)], \tag{5.34}$$

while the second derivative in ξ variables gives

$$\tilde{u}^\xi_{jk} \le |\tilde{u}^h_{jk}|[\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-1}\beta)]^2 + |\tilde{u}^h_{jk}|[\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-1}\beta)] + |\tilde{u}^h_j|\mathcal{O}(e_\varepsilon).$$

So, the next lemma follows.

Lemma 5.9. *For any j, k it holds*

$$\|\tilde{u}^\xi_j\| \le (\mathcal{O}(1) + \mathcal{O}(\|w\|))[\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-1}\beta)], \tag{5.35}$$

$$\|\tilde{u}^\xi_{jk}\| \le [\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-2}\beta^2) + \mathcal{O}(\varepsilon^{-1}\beta)][\mathcal{O}(w_x + w_\ell) + \varepsilon^{-1/2} + \varepsilon^{-1/2}A] + \mathcal{O}(e_\varepsilon)[\mathcal{O}(1) + \mathcal{O}(\|w\|)], \tag{5.36}$$

for $A := S^{1/2}_\infty + \max_j(r_j a^j, r_{j+1}a^{j+1})^{1/2}$.

The following theorem gives the final estimates concerning the term E_i^ξ in the L^2 norm.

Theorem 5.10. *For any i, j, k the next inequalities hold:*

$$\|E_i^\xi\| \leq 4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta), \tag{5.37}$$

$$\|E_{ij}^\xi\| \leq \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(\varepsilon^{-4}r^{-1}\beta), \tag{5.38}$$

$$\|E_{ijk}^\xi\| \leq \mathcal{O}(\varepsilon^{-3/2}) + \mathcal{O}(\varepsilon^{-5}r^{-1}\beta). \tag{5.39}$$

Proof. Using that $\|E_j^\xi\| \leq \|\tilde{w}_j\| + \|Q_j\|$, the estimate of $\|\tilde{w}_j\|$ presented in [8] (cf. p. 186, relation (4.24)) and Lemma 5.6, we obtain (5.37). Also, observe that

$$\begin{aligned} E_{ji}^\xi &= \tilde{w}_{ji} + \mathcal{O}(Q_{ji}) + \mathcal{O}(Q_{ijx}) = \mathcal{O}(w_x + w_\ell) + \int_0^x (-u_{xx}^h - u_{xi})dy + \mathcal{O}(Q_{ji}) + \mathcal{O}(Q_{ijx}) \\ &\leq \mathcal{O}(w_x + w_\ell) + \mathcal{O}(u_x^h + u_i^h) + \mathcal{O}(Q_{ji}), \end{aligned}$$

so,

$$\|E_{ji}^\xi\| \leq \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(\varepsilon^{-4}r^{-1}\beta).$$

Further, we obtain

$$\begin{aligned} E_{jik}^\xi &= \tilde{w}_{jik} + \mathcal{O}(Q_{jik}) + \mathcal{O}(Q_{jikx}) = \mathcal{O}(w_{xx} + w_{\ell\ell} + w_{x\ell}) + \int_0^x (-u_{xxx}^h - u_{xxk})dy + \mathcal{O}(Q_{jik}) + \mathcal{O}(Q_{jikx}) \\ &\leq \mathcal{O}(w_{xx} + w_{\ell\ell} + w_{x\ell}) + \mathcal{O}(u_{xx}^h + u_{xk}^h) + \mathcal{O}(Q_{jik}), \end{aligned}$$

so, by Lemma 5.6 we get

$$\|E_{ijk}^\xi\| \leq \mathcal{O}(\varepsilon^{-3/2}) + \mathcal{O}(\varepsilon^{-5}r^{-1}\beta).$$

□

Remark 5.11. We denote that the estimate of $\|E_{ij}^\xi\|$ presented in the previous Theorem coincides in the main order term with the estimate that was used but not presented analytically in [8].

Using the results of the previous analysis we derive finally by Cauchy-Schwarz inequality all the desired estimates involving the higher order derivatives which are presented at the next main theorem of this section.

Theorem 5.12. *The next inequalities hold for any i, l, k:*

$$|\langle \tilde{u}_{kl}^\xi, E_i^\xi \rangle| \leq \mathcal{O}(\varepsilon^{-1/2})\Big[4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta)\Big], \tag{5.40}$$

$$|\langle \tilde{u}_k^\xi, E_{il}^\xi \rangle| \leq \mathcal{O}(\varepsilon^{-1/2} + \varepsilon^{-4}r^{-1}\beta), \tag{5.41}$$

and

$$|\langle \tilde{v}, E_{ilk}^\xi \rangle| \leq \mathcal{O}(\varepsilon^{-3/2} + \varepsilon^{-5}r^{-1}\beta)\|\tilde{v}\|. \tag{5.42}$$

The last term to be analyzed is $\langle L^c\tilde{v}, \tilde{u}_{kl}^\xi \rangle$. Therefore, we prove the following main result.

Theorem 5.13. *For any k, l, it holds that*

$$|\langle L^c\tilde{v}, \tilde{u}_{kl}^\xi \rangle| \leq \varepsilon^{-5}\beta(r)\Big(\mathcal{O}(1) + \varepsilon^{-2}\beta(r)^2\Big)\|\tilde{v}\|. \tag{5.43}$$

Proof. We denote that

$$\langle L^c\tilde{v}, \tilde{u}_{kl}^\xi \rangle = -\langle \tilde{v}, \partial_x \partial_{\xi_k} \partial_{\xi_l} \mathcal{L}^b(u^\xi) \rangle,$$

where $\mathcal{L}^b(\phi) := \varepsilon^2\phi_{xx} - f(\phi)$. As in [7] (cf. p. 452-453) we write for $x \in [h_j - \varepsilon, h_j + \varepsilon]$, $j = 1, 2, \ldots, N+1$

$$\mathcal{L}^b(u^h) = f_1 + f_2 + G, \tag{5.44}$$

for

$$f_1 := \varepsilon^2\chi^j_{xx}(\phi^{j+1}-\phi^j),\ \ f_2 := 2\varepsilon^2\chi^j_x(\phi^{j+1}_x-\phi^j_x),$$
$$G := (\phi^{j+1}-\phi^j)^2\Big\{(1-\chi^j)\int_0^{\chi^j} sf''(\theta)ds + \chi^j\int_{\chi^j}^1(1-s)f''(\theta)ds\Big\},$$

with $\theta=\theta(s) := (1-s)\phi^j(x)+s\phi^{j+1}(x)$. For other x, we use $\mathcal{L}^b(u^h)=0$.

In Lemma 5.2 of [7] at p. 454, after differentiating f_1, f_2, G in h_j is derived that

$$\Big|\frac{\partial}{\partial h_j}\mathcal{L}^b u^h\Big| \leq c\varepsilon^{-2}\beta(r).$$

Applying the analogous computation, i.e. differentiating in h_j, h_i, we may derive

$$\Big|\frac{\partial^2}{\partial h_j\partial h_i}\mathcal{L}^b u^h\Big| \leq c\varepsilon^{-3}\beta(r). \tag{5.45}$$

Denote that in the above computation the worst term is $|\phi^j_{xxx}(x)-\phi^{j+1}_{xxx}(x)|$. But $\varepsilon^2\phi_{xxx}=f'(\phi)\phi_x$, where $f(\phi)=\phi^3-\phi$ and $f'(\phi)=3\phi^2-1$, so using the estimates of ϕ, ϕ_x and the results for the differences presented at p. 453 of [7], we get

$$\begin{aligned}
&|\phi^j_{xxx}(x)-\phi^{j+1}_{xxx}(x)| = \varepsilon^{-2}|f'(\phi^j)\phi^j_x(x)-f'(\phi^{j+1})\phi^{j+1}_x(x)|\\
&=\varepsilon^{-2}|f'(\phi^j)\phi^j_x(x)-f'(\phi^{j+1})\phi^{j+1}_x(x)-f'(\phi^j)\phi^{j+1}_x(x)+f'(\phi^j)\phi^{j+1}_x(x)|\\
&\leq \varepsilon^{-2}|f'(\phi^j)||\phi^j_x(x)-\phi^{j+1}_x(x)|+\varepsilon^{-2}|\phi^{j+1}_x(x)||f'(\phi^j)-f'(\phi^{j+1})|\\
&\leq c\varepsilon^{-2}|\phi^j_x(x)-\phi^{j+1}_x(x)|+c\varepsilon^{-2}\varepsilon^{-1}|f'(\phi^j)-f'(\phi^{j+1})|\\
&\leq c\varepsilon^{-3}|a^j-a^{j+1}|+c\varepsilon^{-3}|3\phi^j(x)^2-1-3\phi^{j+1}(x)^2+1|\\
&\leq c\varepsilon^{-3}|a^j-a^{j+1}|+c\varepsilon^{-3}|\phi^j(x)+\phi^{j+1}||\phi^j(x)-\phi^{j+1}|\\
&\leq c\varepsilon^{-3}|a^j-a^{j+1}|+c\varepsilon^{-3}|a^j-a^{j+1}|\\
&\leq c\varepsilon^{-3}|a^j-a^{j+1}|.
\end{aligned}$$

Again as in [7] (cf. p. 456), by using that $\varepsilon^2 w_{xx}=f'(\phi(x))w$ and differentiating it in x, we may obtain that

$$\Big|\frac{\partial^2}{\partial h_j\partial h_i}\frac{\partial}{\partial x}\mathcal{L}^b u^h\Big| \leq c\varepsilon^{-5}\beta(r). \tag{5.46}$$

Returning now in ξ variables, since the second derivative appears, then by use of the formula (5.1) in (5.46) and since (cf. [7] p. 454) it holds that

$$\Big|\frac{\partial}{\partial h_j}\frac{\partial}{\partial x}\mathcal{L}^b u^h\Big| \leq c\varepsilon^{-4}\beta(r), \tag{5.47}$$

we obtain finally

$$\begin{aligned}
\Big|\frac{\partial^2}{\partial\xi_k\partial\xi_l}\frac{\partial}{\partial x}\mathcal{L}^b u^h\Big| \leq&\varepsilon^{-5}\beta(r)\Big\{(\mathcal{O}(1)+\varepsilon^{-1}\beta(r))^2+(\mathcal{O}(1)+\varepsilon^{-1}\beta(r))\Big\}+\varepsilon^{-4}\beta(r)\mathcal{O}(e_\varepsilon)\\
\leq&\varepsilon^{-5}\beta(r)\Big(\mathcal{O}(1)+\varepsilon^{-2}\beta(r)^2\Big).
\end{aligned} \tag{5.48}$$

So, the result follows. □

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