From Atomistic to Continuum Theory for Brittle Materials: A Two-Dimensional Model Problem

Manuel Friedrich, Bernd Schmidt

Angaben zur Veröffentlichung / Publication details:

Friedrich, Manuel, and Bernd Schmidt. 2011. "From Atomistic to Continuum Theory for Brittle Materials: A Two-Dimensional Model Problem." Augsburg: Universität Augsburg.

Universität Augsburg

Institut für Mathematik

Manuel Friedrich, Bernd Schmidt

From Atomistic to Continuum Theory for Brittle Materials: A Two-Dimensional Model Problem

Preprint Nr. 13/2011 — 02. September 2011

Institut für Mathematik, Universitätsstraße, D-86135 Augsburg http://www.math.uni-augsburg.de/

Impressum:

Herausgeber:

Institut für Mathematik
Universität Augsburg
86135 Augsburg
http://www.math.uni-augsburg.de/pages/de/forschung/preprints.shtml

ViSdP:

Bernd Schmidt
Institut für Mathematik
Universität Augsburg
86135 Augsburg

From atomistic to continuum theory for brittle materials: A two-dimensional model problem

Manuel Friedrich[1] and Bernd Schmidt[2]

August 18, 2011

Abstract

A two-dimensional atomic mass spring system is investigated for critical fracture loads and its crack path geometry. We rigorously prove that in the discrete-to-continuum limit, the minimal energy leads to a universal cleavage law and energy minimizers are either homogeneous elastic deformations or configurations that are cracked along specific crystallographic hyperplanes. Furthermore, we identify an effective continuum model through Γ-convergence.

Keywords. Brittle materials, variational fracture, atomistic models, discrete-to-continuum limits, free discontinuity problems.
AMS classification. 74R10, 49J45, 70G75

Contents

[1]Zentrum Mathematik, Technische Universität München, Boltzmannstr. 3, 85747 Garching, Germany. `manuel.friedrich@mytum.de`

[2]Universität Augsburg, Institut für Mathematik, Universitätsstr. 14, 86159 Augsburg, Germany. `bernd.schmidt@math.uni-augsburg.de`

1 Introduction

The behavior of brittle materials is of great interest in applications as well as from a theoretical point of view. Such materials show an elastic response to very small displacements and develop cracks already at moderately large strains. In particular, there is typically no plastic regime in between the restorable elastic deformations and complete failure due to fracture. Major challenges in the experimental sciences and theoretical studies are to identify critical loads at which such a body fails and to determine the geometry of crack paths that occur in the fractured regime.

In variational fracture mechanics displacements and crack paths are determined from an energy minimization principle. Following the pioneering work of Griffith [26], Francfort and Marigo [23] have introduced an energy functional comprising elastic bulk contributions for the unfractured regions of the body and surface terms that assign energy contributions on the crack paths comparable to the size of the crack of codimension one. Subsequently these models have been investigated and extended in various directions. Among the vast body of literature we only mention the work of Dal Maso and Toader [20]; Francfort and Larsen [22]; Dal Maso, Francfort and Toader [19]. Determining energy minimizers of such functionals leads to solving a free discontinuity problem in the language of Ambrosio and De Giorgi [21] as the crack path, i.e., the set of discontinuity of the diplacement field is not pre-assigned but has to be found as a solution to the variational problem. In particular, these models also lead to efficient numerical approximation schemes, cf., e.g., [4, 6, 28, 29, 32].

Due to the crystalline structure of matter, under tensile boundary loads fracture typically occurs in the form of cleavage along crystallographic hyperplanes of the atomic lattice. On the continuum side such behavior can be modelled by anisotropic surface terms which are locally minimized for these crack geometries, see e.g. [1, 15, 28]. A discrete model has been investiged by Braides, Lew and Ortiz [13], who assume that fracture can only occur in these directions and then calculate a limiting continuum energy: a cleavage law. This assumption leads to an effective one-dimenional problem which is much easier to analyze. Indeed in the one-dimensional setting, where discrete models describe the behavior of atom chains, a number of results have appeared rather recently on the literature, including [8, 9, 10, 11]. While by now for many atomistic models the passage to effective continuum models is well understood in the regime of purely elastic interactions, see [5, 14, 31], not much is known on discrete-to-continuum limits for models allowing for fracture in more than one dimension. The farthest reaching results in that direction seem to be due to Braides and Gelli [12], who prove Γ-convergence results for scalar valued free discontinuity problems.

However, all these ansatzes fall short of rigorous arguments that indeed in more than one dimension, if fracture occurs at all, then it is energetically favorable to cleave the specimen along particular crystallographic hyperplanes. The main goal of this paper is to provide a rigorous and rather complete study of a two dimensional model problem. We assume that the body is a rectangular strip subject to uniaxial tensile boundary conditions. The atoms in their reference configuration shall be given by the portion of a triangular lattice in that strip that interact via next neighbor Lennard-Jones type potentials. This model seems to be the simplest model problem which (1) is frame indifferent in its vector-valued arguments in more than one dimension, (2) gives rise to non-degenerate elastic bulk terms and (3) leads to surface contributions sensitive to the crack geometry with competing crystallographic hyperplanes. Moreover, two-dimensional lattice surfaces naturally appear in the analysis of thin structures. In particular we will also discuss consequences of our analysis on the stability of brittle nanotubes under interior expansive pressure.

Indeed we will prove that under uniaxial tension in the continuum limit the energy satisfies a particular cleavage law with quadratic response to small boundary displacements followed by a sharp constant cut-off beyond some critical value. Moreover, we will see that any sequence of minimizers converges (up to subsequences) to a homogeneous continuum deformation for subcritical boundary values, while it converges to a continuum deformation which is cracked along a crystallographic line and does not store elastic energy in the supercritical case, whenever the optimal crystallygraphic line is unique. The model under investigation leads, in particular, to configurations respecting the Poisson effect, which would not be possible in scalar models. These results justify rigorously the aforementioned assumptions in the derivation of cleavage laws as, e.g., in [13]. Finally we relate the sequence of discrete energy functionals to a limiting functional through a Γ-convergence result.

The paper is organized as follows. We first introduce our discrete model and state our main results in Section 2. Here we already discuss different scalings of the boundary data and find the interesting regime where both energy contributions, the elastic and the crack energy, are of the same order.

In Section 3 we collect some elementary properties of the cell energy. In particular, we introduce a lower-bound comparison energy, called *reduced energy*, providing the optimal cell energy in dependence of the cell expansion in the space direction where tensile boundary conditions are imposed.

Section 4 is devoted to the derivation of cleavage laws for the limiting minimal energy. Using an elementary slicing argument we reduce the problem to one-dimensional segments and show that the limiting energy has a universal form independent of the interatomic potential. Our result is similar to the effective one-dimesional law discussed in [13]. We obtain that the crack energy is anisotropic and depends explicitly on the lattice orientation. We then give finer estimates on the limiting energy by deriving higher order terms for the discrete minimal energies and show that in contrast to the limiting behavior anisotropic contributions also occur in the elastic regime. Moreover, our proof illustrates the typical behavior of brittle materials already seen in the continuum cleavage law also in a discrete framework: There is essentially no plastic regime besides the elastic and the crack regime. More precisely, we see that for almost minimizer the deformation is either $\sqrt{\varepsilon}$-close to the identity mapping (representing elastic response) or springs between adjacent atoms are elongated by a factor scaling like $\frac{1}{\sqrt{\varepsilon}}$ (leading to fracture in the limit description) where ε denotes the typical interatomic distance. In particular, here we can already see that homogeneous deformations or cleavage along specific lines are asymptotically optimal.

In Section 5 we proceed to show that, under appropriate assumptions, in terms of suitably rescaled displacement fields indeed all discrete energy minimizers converge strongly to such continuum deformations. We provide a fine characterization of the crack, i.e. of the number and position of largely elongated springs. In the subcritical case the contribution of such springs is abitrarily small such that the purely elastic theory applies. For supercritical boundary values largely deformed springs lie in a small stripe in direction of the optimal cristallographic line.

The last Section 6 finally contains our results on the limiting variational problem. We first show that the discrete energy functionals converge to an energy functional defined on the continuum in the sense of Γ-convergence. We finally analyze the continuum problem under the same tensile boundary values and in that way we re-derive the results of Section 4 and Section 5.

2 The model and main results

The discrete model

Let $\mathcal{L}$ denote the rotated triangular lattice

$$\mathcal{L} = R_{\mathcal{L}} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \mathbb{Z}^2 = \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 : \lambda_1, \lambda_2 \in \mathbb{Z}\},$$

where $R_{\mathcal{L}} \in SO(2)$ is some rotation and $\mathbf{v}_1$, $\mathbf{v}_2$ are the lattice vectors $\mathbf{v}_1 = R_{\mathcal{L}}\mathbf{e}_1$ and $\mathbf{v}_2 = R_{\mathcal{L}}(\frac{1}{2}\mathbf{e} + \frac{\sqrt{3}}{2}\mathbf{e}_2)$, respectively. We collect the basic lattice vectors in the set $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_1\}$. The region $\Omega = (0,l) \times (0,1) \subset \mathbb{R}^2$, $l > 0$, is considered the macroscopic region occupied by the body under investigation. In the reference configuration the positions of the specimen's atoms are given by the points of the scaled lattice $\varepsilon\mathcal{L}$ that lie within Ω. Here ε is a small parameter defining the length scale of the typical interatomic distances.

The deformations of our system are mappings $y : \varepsilon\mathcal{L} \cap \Omega \to \mathbb{R}^2$. The energy associated to such a deformation y is assumed to be given by nearest neighbor interactions as

$$E_\varepsilon(y) = \frac{1}{2} \sum_{\substack{x,x' \in \varepsilon\mathcal{L}\cap\Omega \\ |x-x'|=\varepsilon}} W\left(\frac{|y(x) - y(x')|}{\varepsilon}\right). \tag{1}$$

Note that the scaling factor $\frac{1}{\varepsilon}$ in the argument of W takes account of the scaling of the interatomic distances with ε. The pair interaction potential $W : [0,\infty) \to [0,\infty]$ is supposed to be of 'Lennard-Jones-type':

(i) $W \geq 0$ and $W(r) = 0$ if and only if $r = 1$.

(ii) W is continuous on $[0,\infty)$ and C^2 in a neighborhood of 1 with $\alpha := W''(1) > 0$.

(iii) $\lim_{r\to\infty} W(r) = \beta$.

In order to obtain fine estimates on limiting energies and configurations we will also consider the following stronger versions of hypotheses (ii) and (iii):

(ii') W is continuous on $[0,\infty)$ and C^4 in a neighborhood of 1 with $\alpha := W''(1) > 0$ and arbitrary $\alpha' := W'''(1)$.

(iii') $W(r) = \beta + O(r^{-2})$ as $r \to \infty$,

which is still satisfied, e.g., by the classical Lennard-Jones potential.

In order to analyze the passage to the limit as $\varepsilon \to 0$ it will be useful to interpolate and rewrite the energy as an integral functional. Let $\mathcal{C}_\varepsilon$ be the set of equilateral triangles $\triangle \subset \Omega$ of sidelength ε with vertices in $\varepsilon\mathcal{L}$ and define $\Omega_\varepsilon = \bigcup_{\triangle\in\mathcal{C}_\varepsilon} \triangle$. By $\tilde{y} : \Omega_\varepsilon \to \mathbb{R}^2$ we denote the interpolation of y, which is affine on each $\triangle \in \mathcal{C}$. The derivative of $\tilde{y}$ is denoted by $\nabla\tilde{y}$, whereas we write $(y)_\triangle$ for the (constant) value of the derivative on a triangle $\triangle \in \mathcal{C}_\varepsilon$. Then (1) can be rewritten as

$$\begin{aligned} E_\varepsilon(y) &= \sum_{\triangle\in\mathcal{C}_\varepsilon} W_\triangle((\tilde{y})_\triangle) + E_\varepsilon^{\text{boundary}}(y) \\ &= \frac{4}{\sqrt{3}\varepsilon^2} \int_{\Omega_\varepsilon} W_\triangle(\nabla\tilde{y})\,dx + E_\varepsilon^{\text{boundary}}(y), \end{aligned} \tag{2}$$

where

$$W_\triangle(F) = \frac{1}{2}\Big(W(|F\mathbf{v}_1|) + W(|F\mathbf{v}_2|) + W(|F(\mathbf{v}_2 - \mathbf{v}_1)|)\Big). \tag{3}$$

(Note that $|\triangle| = \sqrt{3}\varepsilon^2/4$.) Here the boundary term is the sum of pair interaction energies $\frac{1}{4}W(\frac{|y(x)-y(x')|}{\varepsilon})$ over nearest neighbor pairs which form the side of only one triangle in $\mathcal{C}_\varepsilon$.

Boundary values and scaling

We are interested in the behavior of the system when applying tensile boundary conditions, say in $\mathbf{e}_1$-direction. In particular, we would like to investigate when and how the body breaks, i.e.,

(1) at which value of the boundary displacement energetic minimizers are no longer elastic deformations but exhibit cracks and

(2) if indeed it is most favorable for the cracks to separate the body along crystallographic lines.

In order to avoid geometric artefacts, we will therefore assume that $l > \frac{1}{\sqrt{3}}$, so that it is possible for the body to completely break apart along lines parallel to $\mathbb{R}\mathbf{v}_1$, $\mathbb{R}\mathbf{v}_2$ or $\mathbb{R}(\mathbf{v}_2 - \mathbf{v}_1)$ not passing through the left or right boundaries.

Due to the discreteness of the underlying atomic lattice we have to impose the boundary conditions of uniaxial extension in small neighborhoods of $\{0\} \times (0,1)$ and $\{l\} \times (0,1)$, respectively: For $a_\varepsilon > 0$ we set

$$\mathcal{A}(a_\varepsilon) = \big\{y = (y_1, y_2) : \varepsilon\mathcal{L} \cap \Omega \to \mathbb{R}^2 : \\ y_1(x) = (1 + a_\varepsilon)x_1 \text{ for } x_1 \le \varepsilon \text{ and } x_1 \ge l - \varepsilon\big\}.$$

Note that there is some arbitrariness in this implementation of boundary values as one might, e.g., equally well ask that

$$y_1(x) = x_1 \text{ for } x_1 \le \varepsilon \quad \text{and} \quad y_1(x) = x_1 + a_\varepsilon l \text{ for } x_1 \ge l - \varepsilon. \tag{4}$$

Such different choices will, however, not change the results of the analysis.

Also note that there is no assumption on the second component of the boundary displacement, i.e., the atoms may 'slide along the boundary lines'. Besides describing a basic experiment on elastic bodies, this assumption allows for a direct application of our results to the stability analysis of nanotubes:

If the rotation $R_\mathcal{L}$ and the length l are such that for a sequence $\varepsilon_k \to 0$ the translated lattice $\varepsilon_k\mathcal{L} + (l, 0)$ concides with the original lattice $\varepsilon_k\mathcal{L}$, we may view the system as an atomistic nanotube with macroscopic region $\frac{l}{2\pi}S^1 \times (0,1)$. (Note that for small ε_k the bending energy contributions when rolling up $(0,l) \times (0,1)$ into a cylinder are negligible as this mapping is an isometric immersion and thus infinitesimally rigid.) Imposing periodic boundary conditions, for arbitrary $l > 0$ our system then models deformations of a nanotube subject to expansion of the diameter.

There are two obvious choices for deformations satisfying the boundary conditions: The homogeneous elastic deformation $y^{\mathrm{el}}(x) = (1+a_\varepsilon)x$ and a cracked body deformation y^{cr}, which, up to a boundary layer of negligible energy, is the identity to the left and a translation by $a_\varepsilon l\mathbf{e}_1$ to the right of some segment (or curve) passing through Ω that connects a point on the lower boundary $(\varepsilon, l - \varepsilon) \times \{0\}$ and a point on the upper boundary $(\varepsilon, l - \varepsilon) \times \{1\}$. It is not hard to see that

$$E_\varepsilon(y^{\mathrm{el}}) \sim \varepsilon^{-2} W(1 + a_\varepsilon), \qquad E_\varepsilon(y^{\mathrm{cr}}) \sim \varepsilon^{-1}.$$

In particular, we are interested in the most interesting regime where both of these energy values are of the same order, i.e., a_ε is small and

$$\varepsilon^{-2}a_\varepsilon^2 \sim \varepsilon^{-2}W(1 + a_\varepsilon) \sim \varepsilon^{-1} \qquad \Longrightarrow \qquad a_\varepsilon \sim \sqrt{\varepsilon}.$$

In order to obtain finite and nontrivial energies in the limit $\varepsilon \to 0$, we accordingly rescale E_ε to $\mathcal{E}_\varepsilon := \varepsilon E_\varepsilon$.

Conceivable alternative implementations of the boundary conditions as alluded to above will then result in energy changes of order $O(\varepsilon)$. We will account for all such possibilities by characterizing not only energy minimizing configurations, but more generally all configurations which are energy minimizing up to an error term of order $O(\varepsilon)$.

Limiting minimal energy and cleavage laws

We begin our analysis with an elementary argument which yields the limiting minmal energy as $\varepsilon \to 0$ when $a_\varepsilon/\sqrt{\varepsilon} \to a \in [0, \infty]$. We first establish a lower bound for this energy by considering slices of the form $(0, l) \times \{x_2\}$ for $x_2 \in (0, 1)$ and using the *reduced energy* $\tilde{W}$ defined by

$$\tilde{W}(r) = \inf\{W_\triangle(F) : \mathbf{e}_1^T F \mathbf{e}_1 = r\}. \tag{5}$$

In a second step we show that this bound is attained. In particular, it turns out that the limiting minimal energy is given by elastic deformations up to some critical value a_{crit} of the boundary displacements and by cleavage along a specific crystallographic line beyond a_{crit}.

Let $\gamma = \max\{|\mathbf{v}_1 \cdot \mathbf{e}_2|, |\mathbf{v}_2 \cdot \mathbf{e}_2|, |(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{e}_2|\}$ and $\mathbf{v}_\gamma \in \mathcal{V}$ such that $\gamma = |\mathbf{v}_\gamma \cdot \mathbf{e}_2|$. We note that γ takes values in $[\sqrt{3}/2, 1]$ and that $\mathbf{v}_\gamma$ is unique if $\gamma > \sqrt{3}/2$.

Theorem 2.1 *Suppose $a_\varepsilon/\sqrt{\varepsilon} \to a \in [0, \infty]$. The limiting minimal energy is given by*

$$\lim_{\varepsilon \to 0} \inf\{\mathcal{E}_\varepsilon(y) : y \in \mathcal{A}(a_\varepsilon)\} = \min\left\{\frac{\alpha l}{\sqrt{3}} a^2, \frac{2\beta}{\gamma}\right\}. \tag{6}$$

As already motivated above, only one of the regimes is energetically favorable if $a \in \{0, \infty\}$. In the interesting case $a \in (0, \infty)$ we indeed will see that in terms of the critical boundary displacement

$$a_{\text{crit}} = \sqrt{\frac{2\sqrt{3}\beta}{\alpha \gamma l}}$$

the limit is attained for homogeneously deformed configurations if $a \le a_{\text{crit}}$ and for configurations cracked along lines parallel to $\mathbb{R}\mathbf{v}_\gamma$, if $a \ge a_{\text{crit}}$. In the special case that $\mathbf{v}_\gamma$ is not unique the limit is also attained if the crack takes a serrated course parallel to $\mathbb{R}(\frac{1}{2}, \frac{\sqrt{3}}{2})^T$ or $\mathbb{R}(-\frac{1}{2}, \frac{\sqrt{3}}{2})^T$.

For the sake of simplicity we specialize to sequences $a_\varepsilon = \sqrt{\varepsilon} a$. Without loss of generality we assume that $R_{\mathcal{L}} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ for $\phi \in [0, \frac{\pi}{3})$, so that $\gamma = \sin(\phi + \frac{\pi}{3}) = \mathbf{v}_\gamma \cdot \mathbf{e}_2$. If the assumptions (ii') and (iii') on W hold, we have the following sharp estimate on the discrete minimal energies up to error terms of the order of surface contributions.

Theorem 2.2 *For ε small the discrete minimal energy is given by*

$$\inf \mathcal{E}_\varepsilon = \min\left\{ \frac{\alpha l}{\sqrt{3}} a^2 + \frac{[6\alpha + 7\alpha' - 2(3\alpha - \alpha')\cos(6\phi)]l}{27\sqrt{3}} \sqrt{\varepsilon} a^3, \frac{2\beta}{\gamma} \right\} + O(\varepsilon).$$

Thus, while the zeroth order contributions in the elastic regime are isotropic, the higher order contributions as well as the fracture energy explicitly depend on the lattice orientation angle ϕ.

Detailed proofs of these results will be given in Section 4.

Limiting minimal configurations

Our analysis of the limiting minimal energy so far showed that, depending on the boundary data, homogeneous deformations or completely cracked configurations are energy minimizing in the limit $\varepsilon \to 0$. However, it falls short of showing that in fact these configurations are the only possibilities to obtain asymptotically optimal energies. Indeed, if $\mathbf{v}_\gamma$ is not unique, then we have already seen that the crack path can become geometrically much more complicated. Our next result shows that if $\mathbf{v}_\gamma$ is unique, energy minimizing configurations converge to a homogeneous continuum deformation for subcritical boundary values, while in the supercritical case they converge to a continuum deformation which is cracked along a crystallographic line and does not store elastic energy.

The basic idea behind our reasoning will be to 'count' the number of 'broken' springs, i.e. the springs intersected transversally by the crack path. We see that the springs broken by a crack line $(p, 0) + \mathbb{R}\mathbf{v}_\gamma$ do not overlap in the projection onto the x_2-axis and the length of the projection of two adjacent broken springs equals $\varepsilon\gamma$. This leads to a fracture energy of approximately $\frac{2\beta}{\gamma}$. If we assume that the cleavage is not parallel to $\mathbb{R}\mathbf{v}_\gamma$ we conclude that some springs in $\mathbf{v}_\gamma$ direction must be broken, too. If we consider the adjacent triangles of such a spring and their neighbors we find that the projection onto the x_2-axis of broken springs overlap. A careful analysis of this phenomenon then shows that every broken spring in $\mathbf{v}_\gamma$ direction 'costs' an additional energy of $\approx 2\varepsilon\beta\frac{P(\gamma)}{\gamma}$, where $P(\gamma)$ is the geometrical factor

$$P(\gamma) = \frac{1}{2}\left(1 - \sqrt{3}\frac{\sqrt{1-\gamma^2}}{\gamma}\right). \qquad (7)$$

(Note that $P(\gamma) = 0$ if and only if $\gamma = \frac{\sqrt{3}}{2}$ in accordance to the above considerations.)

In order to give a precise meaning to the convergence of discrete to continuum deformations, to each discrete deformation $y : \varepsilon\mathcal{L} \to \mathbb{R}^2$ we assign – as mentioned above – the affine interpolation $\tilde{y}$ on each triangle $\triangle \in \mathcal{C}_\varepsilon$. Accordingly, to the rescaled discrete displacements $u : \varepsilon\mathcal{L} \to \mathbb{R}^2$ with $y = \mathbf{id} + \sqrt{\varepsilon}u$ ($\mathbf{id}$ denoting the identity mapping $\mathbf{id}(x) = x$) we define $\tilde{u}$ to be its affine interpolation on each triangle $\triangle \in \mathcal{C}_\varepsilon$.

In the cracked regime we may of course only hope for a unique limiting deformation up to translation of the crack path. However, without an additional mild extra assumption on the admissible discrete configurations or their energy even this cannot hold true, as apart from the crack, parts of the specimen could flip their orientation and fold onto other parts on the body at zero energy. In order to avoid such unphysical behavior we add a frame indifferent penalty term $\chi \geq 0$ to $W_\triangle$ with $\chi \geq c_\chi$ in a neighborhood of $O(2) \setminus SO(2)$ and $\chi \equiv 0$ in a neighborhood of $SO(2)$:

$$W_{\triangle,\chi} = W_\triangle + \chi. \tag{8}$$

We set

$$\mathcal{E}_\varepsilon^\chi(y) = \frac{4}{\sqrt{3}\varepsilon} \int_{\Omega_\varepsilon} W_{\triangle,\chi}(\nabla \tilde{y})\, dx + \varepsilon E_\varepsilon^{\text{boundary}}(y),$$

for $u \in \mathcal{A}(a_\varepsilon)$. More generally than a sequence of minimizers we will consider sequences (y_ε) of almost minimizers that satisfy

$$\mathcal{E}_\varepsilon^\chi(y_\varepsilon) = \inf\{\mathcal{E}_\varepsilon^\chi(y) : y \in \mathcal{A}(a_\varepsilon)\} + O(\varepsilon). \tag{9}$$

For those deformations we will show in Section 5:

Theorem 2.3 *Assume that W satisfies (i), (ii') and (iii'). Let $\mathbf{v}_\gamma$ be unique, $a_\varepsilon = \sqrt{\varepsilon}a$, $a \neq a_{\text{crit}}$ and suppose (y_ε) satisfies (9). Let u_ε such that $y_\varepsilon = \mathbf{id} + \sqrt{\varepsilon}u_\varepsilon$. Then there exist $\bar{u}_\varepsilon : \Omega \to \mathbb{R}^2$ with $|\{x \in \Omega_\varepsilon : \bar{u}_\varepsilon(x) \neq \tilde{u}_\varepsilon(x)\}| = O(\varepsilon)$ such that:*

(i) If $a < a_{\text{crit}}$, then there is a sequence $s_\varepsilon \in \mathbb{R}$ such that

$$\|\bar{u}_\varepsilon - (0, s_\varepsilon) - F^a \cdot\|_{H^1(\Omega)} \to 0,$$

where $F^a = \begin{pmatrix} a & 0 \\ 0 & -\frac{a}{3} \end{pmatrix}$.

(ii) If $a > a_{\text{crit}}$, then there exist sequences $p_\varepsilon \in (0, l)$, $s_\varepsilon, t_\varepsilon \in \mathbb{R}$ such that $(p_\varepsilon, 0) + \mathbb{R}\mathbf{v}_\gamma$ intersects both the segments $(0, l) \times \{0\}$ and $(0, l) \times \{1\}$ and, for the parts to the left and right of $(p_\varepsilon, 0) + \mathbb{R}\mathbf{v}_\gamma$

$$\Omega^{(1)} := \{x \in \Omega : 0 < x_1 < p_\varepsilon + (\mathbf{v}_\gamma \cdot e_1)x_2\} \text{ and}$$
$$\Omega^{(2)} := \{x \in \Omega : p_\varepsilon + (\mathbf{v}_\gamma \cdot e_1)x_2 < x_1 < l\},$$

respectively, we have – possibly after rotating y_ε by π on $\Omega^{(1)}$ or $\Omega^{(2)}$ –

$$\|\bar{u}_\varepsilon - (0, s_\varepsilon)\|_{H^1(\Omega^{(1)})} + \|\bar{u}_\varepsilon - (al, t_\varepsilon)\|_{H^1(\Omega^{(2)})} \to 0.$$

Note that a rotation by π on $\Omega^{(i)}$, $i = 1, 2$, might be necessary as on each of these sets there are, up to translation in x_2-direction and a small correction in a boundary layer of negligible energy, two deformations respecting the boundary conditions that do not store elastic energy: $y(x) = x$ and $y(x) = -x$ in $\Omega^{(1)}$, respectively, $y(x) = x + a_\varepsilon l \mathbf{e}_1$ and $y(x) = -x + (2 + a_\varepsilon) l \mathbf{e}_1$ in $\Omega^{(2)}$.

The limiting variational problem

We finally address the more general question if not only the minimal values or the minimizers but the whole energy functionals converge to a continuum energy functional in a variational sense. Furthermore, we analyze the limiting problem independent of its discrete approximations. The results announced here are proved in Section 6.

Our convergence analysis applies to discrete deformations which may elongate a number scaling with $\frac{1}{\varepsilon}$ of springs very largely, leading to cracks of finite length in the continuum limit. On triangles not adjacent to such essentially broken springs, the defomations are $\sqrt{\varepsilon}$-close to the identity mapping, so that the accordingly rescaled displacements are of bounded L^2-norm. Note that the first of these assumptions can be inferred from suitable energy bounds. By way of example, however, we will see that this cannot be true for the displacement estimates in the bulk: The sequence of functionals $(\mathcal{E}_\varepsilon)$ is not equicoercive. Nevertheless, it is interesting to investigate this regime in order to identify a corresponding continuum functional which describes the system in the realm of Griffith models with linearized elasticity. As a particular case of Theorem 2.3 we mention that $(\mathcal{E}_\varepsilon)$ is still mildly equicoercive.

Recall that the space $SBV(\Omega;\mathbb{R}^2)$, abbreviated as $SBV(\Omega)$ hereafter, of special functions of bounded variation consists of functions $u \in L^1(\Omega;\mathbb{R}^2)$ whose distributional derivative Du is a finite Radon measure, which splits into an absolutely continuous part with density ∇u with respect to Lebesgue measure and a singular part $D^j u$ whose Cantor part vanishes and thus is of the form

$$D^j u = [u] \otimes \nu_u \mathcal{H}^1 \lfloor J_u,$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, J_u (the 'crack path') is an $\mathcal{H}^1$-rectifiable set in Ω, ν_u is a normal of J_u and $[u] = u^+ - u^-$ (the 'crack opening') with $u^\pm$ being the one-sided limits of u at J_u. If in addition $\nabla u \in L^2(\Omega)$ and $\mathcal{H}^1(J_u) < \infty$, we write $u \in SBV^2(\Omega)$. See [3] for the basic properties of these function spaces.

The sense in which discrete displacements are considered convergent to a limiting displacement in SBV is made precise in the following definition.

Definition 2.4 *Suppose u_ε is a sequence of discrete displacements such that the corresponding deformations $y_\varepsilon = \mathbf{id} + \sqrt{\varepsilon} u_\varepsilon$ are uniformly bounded in L^∞. We say that u_ε converges to some $u \in SBV^2(\Omega)$: $u_\varepsilon \to u$, if*

(i) $\chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon \to u$ *in* $L^1(\Omega)$

and there exists a sequence $\mathcal{C}_\varepsilon^* \subset \mathcal{C}_\varepsilon$ with $\#\mathcal{C}_\varepsilon^* \le \frac{C}{\varepsilon}$ for a constant C independent of ε such that

(ii) $\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \cup_{\triangle \in \mathcal{C}_\varepsilon^*} \triangle)} \le C.$

The main idea will be to separate the energy into elastic and crack surface contributions by introducing a threshold such that triangles $\triangle$ with $(y)_\triangle$ beyond that threshold are considered as cracked and $\tilde{y}$ is modified there to a discontinuous function.

Consider the limiting functional

$$\mathcal{E}(u) = \frac{4}{\sqrt{3}} \int_\Omega \frac{1}{2} Q(e(u))\, dx + \int_{J_u} \sum_{\mathbf{v} \in \mathcal{V}} \frac{2\beta}{\sqrt{3}} |\mathbf{v} \cdot \nu_u| \, d\mathcal{H}^1 \tag{10}$$

for $u \in SBV^2(\Omega)$, where $e(u) = \frac{1}{2}\left(\nabla u^T + \nabla u\right)$ denotes the symmetric part of the gradient. Q is the linearization of $W_\triangle$ about the identity matrix $\mathbf{Id}$ (see Section 4 for its explicit form). For the sake of simplicity we again suppose that $a_\varepsilon = \sqrt{\varepsilon} a$ for all ε. We then have the following Γ-convergence result:

Theorem 2.5 *Let $a_\varepsilon = \sqrt{\varepsilon} a$ and $a \in [0, \infty)$.*

(i) If (u_ε) is a sequence of discrete displacements such that $y_\varepsilon = \mathbf{id} + \sqrt{\varepsilon} u_\varepsilon \in \mathcal{A}(a_\varepsilon)$ and $u_\varepsilon \to u \in SBV^2(\Omega)$ with $u_1(0, \cdot) = 0$ and $u_1(l, \cdot) = la$ (in the sense of traces), then

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}(u).$$

(ii) For every $u \in SBV^2(\Omega)$ with $u_1(0, \cdot) = 0$ and $u_1(l, \cdot) = la$ (in the trace sense) there is a sequence (u_ε) of discrete displacements such that $y_\varepsilon = \mathbf{id} + \sqrt{\varepsilon} u_\varepsilon \in \mathcal{A}(a_\varepsilon)$, $u_\varepsilon \to u \in SBV^2(\Omega)$ and

$$\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}(u).$$

We note that the sequence $(\mathcal{E}_\varepsilon)$ is mildly equicoercive in the sense that low energy sequences (satisfying (9)) admit a subsequence converging in the sense of Definition 2.4 by Theorem 2.3 (the convergence is even stronger in this case). Due to the frame indifference of W, $(\mathcal{E}_\varepsilon)$ is not equicoercive as the following example shows.

Example. Assume that the specimen satisfying the boundary conditions is broken into three parts by two even cracks where the middle part is subject to a rotation $R \neq \mathbf{Id}$ so that

$$\nabla \tilde{y}_\varepsilon = R \text{ for } p \leq x_1 \leq q, \ 0 < p < q < l.$$

In particular, the energy of the configuration is of order 1. But for $p \leq x_1 \leq q$

$$|\nabla \tilde{u}_\varepsilon(x)| = \left| \frac{1}{\sqrt{\varepsilon}} (R - \mathbf{Id}) \right| \to \infty \text{ for } \varepsilon \to 0.$$

Thus, $\nabla \tilde{u}_\varepsilon$ is not bounded in L^1 and so u_ε does not converge.

Finally, the limiting functional $\mathcal{E}$ can also be analyzed directly without recourse to the approximating functionals. We determine the minimizers and prove uniqueness up to translation of the specimen and the crack line for the boundary conditions

$$u_1 = 0 \text{ for } x_1 = 0 \quad \text{and} \quad u_1 = al \text{ for } x_1 = l. \tag{11}$$

Theorem 2.6 *Let $\mathbf{v}_\gamma$ be unique and $a \neq a_{\mathrm{crit}}$. Then* $\min \mathcal{E} = \min \left\{ \frac{\alpha l}{\sqrt{3}} a^2, \frac{2\beta}{\gamma} \right\}$ *and all minimizers of $\mathcal{E}$ subject to* (11) *are of the following form:*

(i) If $a < a_{\mathrm{crit}}$, then

$$u^{\mathrm{el}}(x) = (0, s) + F^a x$$

for some $s \in \mathbb{R}$

(ii) If $a > a_{\mathrm{crit}}$, then

$$u^{\mathrm{cr}}(x) = \begin{cases} (0, s) & \text{for } x \text{ to the left of } (p, 0) + \mathbb{R}\mathbf{v}_\gamma, \\ (al, t) & \text{for } x \text{ to the right of } (p, 0) + \mathbb{R}\mathbf{v}_\gamma, \end{cases}$$

for some $s, t \in \mathbb{R}$ and $p \in (0, l)$ such that $(p, 0) + \mathbb{R}\mathbf{v}_\gamma$ intersects both the segments $(0, l) \times \{0\}$ and $(0, l) \times \{1\}$.

An analogous result for nonlinear but isotropic energy functionals has been obtained recently by Mora-Corral [27].

We close this introductory chapter emphasizing that all the optimal configurations found in Theorem 2.3 and Theorem 2.6 by minimizing the energy without a priori assumptions show purely elastic behavior in the subcritical case and complete fracture in the supercritical regime. In particular, the elastic minimizer u^{el} shows elongation a in $\mathbf{e}_1$-direction and compression $-\frac{a}{3}$ in the perpendicular $\mathbf{e}_2$-direction, a manifestation of the Poisson effect (with Poisson ratio $\frac{1}{3}$), which cannot be derived in scalar valued models. On the other hand, the crack minimizer u^{cr} is broken parallel to $\mathbb{R}\mathbf{v}_\gamma$ which proves that cleavage occurs along crystallographic lines.

3 Elementary properties of the cell energy

We collect some properties of the cell energy $W_\triangle$ and the reduced energy defined in (5) for W satisfying the assumptions (i), (ii) and (iii).

Lemma 3.1 *$W_\triangle$ is*

(i) frame indifferent: $W_\triangle(QF) = W_\triangle(F)$ for all $F \in \mathbb{R}^{2\times 2}$, $Q \in SO(2)$,

(ii) non-negative and satisfies $W_\triangle(F) = 0$ if and only if $F \in O(2)$ and

(iii) $\liminf_{|F|\to\infty} W_\triangle(F) = \beta$.

Proof. (i) is clear. For (ii) it suffices to note that $v^T F^T F v = 1$ for three vectors v, no two of which are collinear, implies that $F^T F = \mathbf{Id}$. (iii) can be seen by noting that if $|F| \to \infty$, then for at least two vectors $\mathbf{v} \in \mathcal{V}$ one has $|F\mathbf{v}| \to \infty$. $\square$

We compute the linearization about the identity matrix $\mathbf{Id}$:

Lemma 3.2 *Let $F = \mathbf{Id} + G$ for $G \in \mathbb{R}^{2\times 2}$. Then for $|G|$ small*

$$W_\triangle(F) = \frac{1}{2} Q(G) + o(|G|^2),$$

where $Q(G) = \frac{3\alpha}{16}\left(3g_{11}^2 + 3g_{22}^2 + 2g_{11}g_{22} + 4\left(\frac{g_{12}+g_{21}}{2}\right)^2\right)$.

In particular, $Q(G)$ only depends on the symmetric part $\left(G^T + G\right)/2$ of G. Q is positive semidefinite and thus convex on $\mathbb{R}^{2\times 2}$ and positive definite and strictly convex on the subspace $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ of symmetric matrices.

Proof. Let $\mathbf{v} \in \mathcal{V}$ and $G \in \mathbb{R}^{2\times 2}$ small. We Taylor expand the contributions $W(|F\mathbf{v}|)$ to the energy $W_\triangle$:

$$\begin{aligned} W(|(\mathbf{Id} + G)\mathbf{v}|) &= W\left(\sqrt{\langle \mathbf{v}, (\mathbf{Id} + G^T)(\mathbf{Id} + G)\mathbf{v}\rangle}\right) \\ &= \frac{W''(1)}{2}\left\langle \mathbf{v}, \frac{G^T + G}{2}\mathbf{v}\right\rangle^2 + o(|G|^2). \end{aligned}$$

Now using the elementary identity

$$\begin{aligned} &\langle \mathbf{v}_1, H\mathbf{v}_1\rangle^2 + \langle \mathbf{v}_2, H\mathbf{v}_2\rangle^2 + \langle (\mathbf{v}_2 - \mathbf{v}_1), H(\mathbf{v}_2 - \mathbf{v}_1)\rangle^2 \\ &= \frac{3}{8}\left(2\,\mathrm{trace}(H^2) + (\mathrm{trace}\, H)^2\right) \end{aligned} \tag{12}$$

for any symmetric matrix $H \in \mathbb{R}^{2\times 2}$, we obtain by summing over $\mathbf{v} \in \mathcal{V}$

$$\begin{aligned} W_\triangle(F) &= \frac{1}{2}\cdot\frac{\alpha}{2}\cdot\frac{3}{8}\cdot\left(2\,\mathrm{trace}\left(\left(\frac{G^T + G}{2}\right)^2\right) + \left(\mathrm{trace}\,\frac{G^T + G}{2}\right)^2\right) + o(|G|^2) \\ &= \frac{1}{2}Q(G) + o(|G|^2). \end{aligned}$$

As $Q(G) \geq \frac{3\alpha}{16}(2g_{11}^2 + 2g_{22}^2 + (g_{12} + g_{21})^2)$, Q is positive semidefinite on $\mathbb{R}^{2\times 2}$ and positive definite on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$. $\square$

As a consequence, we have the following properties of the reduced energy $\tilde{W}$.

Lemma 3.3 *The reduced energy satisfies* *(i)* $\tilde{W}(r) = 0 \iff |r| \leq 1$.

(ii) For $r \geq 1$ *one has*

$$\tilde{W}(r) = W_\triangle\left(\begin{pmatrix} r & 0 \\ 0 & \frac{4-r}{3} \end{pmatrix}\right) + o((r-1)^2) = \frac{\alpha}{4}(r-1)^2 + o((r-1)^2).$$

(iii) $\lim_{|r|\to\infty} \tilde{W}(r) = \beta$.

Proof. (i) If $r \leq 1$, then one can choose $Q \in SO(2)$ with $\mathbf{e}_1^T Q \mathbf{e}_1 = r$ and so $0 \leq \tilde{W}(r) \leq W_\triangle(Q) = 0$. If $|r| > 1$, then $\tilde{W}(r) > 0$ for otherwise there would be a sequence $F_k \in \mathbb{R}^{2\times 2}$ with $\mathbf{e}_1^T F_k \mathbf{e}_1 = r$ and $W_\triangle(F_k) \to 0$. But then $\operatorname{dist}(F_k, O(2)) \to 0$ by (ii) and (iii) of Lemma 3.1 and thus, up to subsequences, $F_k \to F \in O(2)$ with $\mathbf{e}_1^T F \mathbf{e}_1 = r$, which is impossible.

(ii) This discussion shows that in fact for any $\delta > 0$ there exists $\eta > 0$ such that $W_\triangle(F) > \delta$ whenever $\operatorname{dist}(F, O(2)) \geq \eta$. Now since $\tilde{W}(r) \to 0$ as $r \searrow 1$, we obtain that, for sufficiently small $r > 1$ and $\delta > 0$, any F with $W_\triangle(F) < \tilde{W}(r) + \delta$ is contained in a small neighborhood of $O(2)$. If in addition $\mathbf{e}_1^T F \mathbf{e}_1 = r$ holds, then in fact, F must be close to $\mathbf{Id}$ or to $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In particular, by continuity of W, the infimum on the right hand side in the definition of $\tilde{W}$ is attained for those r.

We now fix such an $r > 1$ near 1 and choose $F = \mathbf{Id} + G$ such that $\tilde{W}(r) = W_\triangle(F)$ and $\mathbf{e}_1^T F \mathbf{e}_1 = r$. As $W_\triangle$ is invariant under the reflection P, we may without loss of generality assume that G is small. Then Lemma 3.2 yields

$$W_\triangle(F) = \frac{3\alpha}{32}\left(3g_{11}^2 + 3g_{22}^2 + 2g_{11}g_{22} + 4\left(\frac{g_{12}+g_{21}}{2}\right)^2\right) + o(|G|^2).$$

We find that $g_{11} = r - 1$, $g_{12} + g_{21} = o(r-1)$ and $g_{22} = -\frac{1}{3}g_{11} + o(r-1)$ and F satisfies

$$\frac{F^T + F}{2} = \begin{pmatrix} r & 0 \\ 0 & \frac{4-r}{3} \end{pmatrix} + o(r-1)$$

with energy

$$\begin{aligned} W_\triangle(F) &= W_\triangle\left(\frac{F^T + F}{2}\right) + o((r-1)^2) \\ &= \frac{\alpha}{4}(r-1)^2 + o((r-1)^2). \end{aligned}$$

(iii) This is immediate from Lemma 3.1(iii). $\square$

Under strengthened hypotheses on W we have the following expansion:

Lemma 3.4 *If W in addition satisfies the assumptions (ii') and (iii'), then for $r > 1$ close to 1 we have*

$$\tilde{W}(r) = \frac{\alpha(r-1)^2}{4} + \frac{1}{108}\Big(6\alpha + 7\alpha' - 2(3\alpha - \alpha')\cos(6\phi)\Big)(r-1)^3 + O((r-1)^4),$$

where ϕ is such that $R_{\mathcal{L}} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

Proof. Let $s = r - 1$. By definition,

$$\tilde{W}(r) = \min\left\{W_{\triangle}(F(s,x,y,z)) : x,y,z \in \mathbb{R}\right\},$$

where $F(s,x,y,z) = \begin{pmatrix} 1+s & z+y \\ z-y & 1+x \end{pmatrix}$. Due to the quadratic energy growth near $SO(2)$, we need to minimize only over x, y, z with $|x|, |z|, \sqrt{s}|y| \leq Cs$ for a constant C large enough. Indeed, as $W_{\triangle}(F(s,0,0,0)) = O(s^2)$, for a minimizer one has without loss of generality $\mathrm{dist}(F(s,x,y,z), SO(2)) = O(s)$. But then $\sqrt{(1+s)^2 + (z \pm y)^2} = 1 + O(s)$, which implies $|z \pm y| = O(\sqrt{s})$ and so $|z|, |y| = O(\sqrt{s})$, and also $\sqrt{(1+x)^2 + (z \pm y)^2} = 1 + O(s)$, which then implies $\pm(1+x) = 1 + O(s)$ and thus without loss of generality $x = O(s)$. Finally using that the scalar product $(1+s)(z+y) + (1+x)(z-y) = 2z + O(s^{3/2})$ of the two columns of $F(s,x,y,z)$ in absolute value is also bounded by $O(s)$, we obtain that $|z| = O(s)$.

Set $x = -\frac{s}{3} + sx_1$, $y = \sqrt{s}y_1$, $z = sz_1$ with $|x_1|, |y_1|, |z_1| \leq C$. Explicit calculation gives

$$W_{\triangle}(F(s,x,y,z)) = \frac{\alpha}{32}\left(8 + 3x_1^2 + 8y_1^2 + 12z_1^2 + 6(x_1 + y_1^2)^2\right)s^2 + O(s^3).$$

Since $\alpha > 0$, we thus obtain that this expression is minimized in x_1, y_1, z_1 with $x_1^2, y_1^2, z_1^2 = O(s)$ and we may set $x_1 = \sqrt{s}x_2$, $y_1 = \sqrt{s}y_2$ and $z_1 = \sqrt{s}z_2$ with $|x_2|, |y_2|, |z_2| \leq C$ for some $C > 0$. Explicit expansion in powers of s then yields

$$\begin{aligned}
&W_{\triangle}(F(s,x,y,z)) \\
&= \frac{\alpha s^2}{4} + \frac{1}{864}\Big(48\alpha + 56\alpha' - 16(3\alpha - \alpha')\cos(6\phi) \\
&\qquad\qquad + 3\alpha\left(81x_2^2 + 72y_2^2 + 108z_2^2\right)\Big)s^3 \\
&\quad + \frac{1}{24}\Big(\left(9\alpha y^2 + \alpha' + (3\alpha - \alpha')\cos(6\phi)\right)x_2 \\
&\qquad\qquad + 2(3\alpha - \alpha')\sin(6\phi)z_2)\Big)s^{7/2} + O(s^4) \\
&= \frac{\alpha s^2}{4} + \frac{1}{108}\Big(6\alpha + 7\alpha' - 2(3\alpha - \alpha')\cos(6\phi)\Big)s^3 \\
&\quad + \frac{9\alpha}{32}\Big(x_2^2 + 2A\sqrt{s}x_2\Big)s^3 + \frac{\alpha y_2^2 s^3}{4} + \frac{3\alpha}{8}\Big(z_2^2 + 2B\sqrt{s}z_2\Big)s^3 + O(s^4)
\end{aligned}$$

for A and B bounded uniformly in s and so

$$\begin{aligned}
&W_\triangle(F(s,x,y,z)) \\
&= \frac{\alpha s^2}{4} + \frac{1}{108}\Big(6\alpha + 7\alpha' - 2(3\alpha - \alpha')\cos(6\phi)\Big)s^3 \\
&\quad + \frac{9\alpha}{32}\Big(x_2 + A\sqrt{s}\Big)^2 s^3 + \frac{\alpha y_2^2 s^3}{4} + \frac{3\alpha}{8}\Big(z_2 + B\sqrt{s}\Big)^2 s^3 + O(s^4).
\end{aligned}$$

Minimizing with respect to x_2, y_2 and z_2 we finally obtain that

$$\tilde{W}(1+s) = \frac{\alpha s^2}{4} + \frac{1}{108}\Big(6\alpha + 7\alpha' - 2(3\alpha - \alpha')\cos(6\phi)\Big)s^3 + O(s^4).$$

□

The following lemma provides useful lower bounds for the energy $W_\triangle$ and the reduced energy $\tilde{W}$.

Lemma 3.5 *For all $T > 1$ one has:*

(i) There exists some $c > 0$ such that $c\,\mathrm{dist}^2(F, O(2)) \leq W_\triangle(F)$ for all $F \in \mathbb{R}^{2\times 2}$ satisfying $|F| \leq T$.

(ii) For $\delta > 0$ small enough, there is a convex function $V \geq 0$ with $V(r) \leq \tilde{W}(r)$ for $r \leq T$ and such that the second derivative $V_+''(1)$ from the right at 1 exists and satisfies $V_+''(1) = \frac{\alpha}{2} - 2\delta$.

(iii) If in addition W satisfies assumptions (ii') and (iii'), then there exists a convex function $V \geq 0$ with $V(r) \leq \tilde{W}(r) \leq V(r) + O((r-1)^4)$ for $r \leq T$.

(iv) For $\rho > 0$ there is an increasing, subadditive function $\psi^\rho : [0,\infty) \to (0,\infty)$ which satisfies $\psi^\rho(r) - \rho \leq W(r+1)$ for all $r \geq 0$ and $\psi(r) = \beta$ for all $r \geq c_\rho$ for some constant c_ρ only depending on ρ.

Proof. (i) Let $F \in \mathbb{R}^{2\times 2}$ satisfying $|F| \leq T$. By polar decomposition we find $R \in O(2)$ and $U = \sqrt{F^T F}$ symmetric and positiv definite such that $F = RU$. A short computation yields $|U - \mathbf{Id}| = \mathrm{dist}(F, O(2))$. Assume first $|U - \mathbf{Id}| < \eta$ for $\eta > 0$ small enough. Since $W_\triangle(F)$ is invariant under rotation and reflection we obtain applying Lemma 3.2:

$$W_\triangle(F) = W_\triangle(R^T R U) \geq \frac{1}{2} Q(U - \mathbf{Id}) + o(|U - \mathbf{Id}|^2).$$

Noting that Q grows quadratically on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ (see Lemma 3.2) we obtain a constant $c_1 > 0$ such that for $|U - \mathbf{Id}| < \eta$

$$W_\triangle(F) \geq c_1 |U - \mathbf{Id}|^2 = c_1\,\mathrm{dist}^2(F, O(2)).$$

Consider the compact set $M := \{F \in \mathbb{R}^{2\times 2}, \operatorname{dist}(F, O(2)) \geq \eta, |F| \leq T\}$. $W_\triangle$ attains its minimum on M, which is strictly positiv by Lemma 3.1(ii). This provides a second constant $c_2 > 0$ such that for all $F \in M$

$$W_\triangle(F) \geq c_2|U - \mathbf{Id}|^2 = c_2 \operatorname{dist}^2(F, O(2)).$$

Taking $c = \min\{c_1, c_2\}$ yields the claim.

(ii) We construct such a function directly applying Lemma 3.3.

$$V(r) = \begin{cases} 0 & \text{for } r \leq 1, \\ \left(\frac{\alpha}{4} - \delta\right)(r-1)^2 & \text{for } 1 \leq r \leq 1+\eta, \\ \left(\frac{\alpha}{4} - \delta\right)\eta\,(2r - 2 - \eta) & \text{for } r \geq 1+\eta, \end{cases}$$

when $\eta > 0$ is sufficiently small.

(iii) With $f(r) := \frac{\alpha(r-1)^2}{4} + \frac{1}{108}\Big(6\alpha + 7\alpha' - 2(3\alpha - \alpha')\cos(6\phi)\Big)(r-1)^3 - C(r-1)^4$ for sufficiently large C, Lemma 3.4 shows that we can choose

$$V(r) = \begin{cases} 0 & \text{for } r \leq 1, \\ f(r) & \text{for } 1 \leq r \leq 1+\eta, \\ f(1+\eta) + f'(1+\eta)(r-1-\eta) & \text{for } r \geq 1+\eta, \end{cases}$$

when $\eta > 0$ is sufficiently small.

(iv) We define

$$\bar{\psi}(r) = \begin{cases} \eta r & \text{for } 0 \leq r \leq \frac{\beta}{\eta}, \\ \beta & \text{for } r \geq \frac{\beta}{\eta}, \end{cases}$$

for some $\eta > 0$ (depending on ρ) such that $\bar{\psi} - \rho \leq W$. Then we set $\psi^\rho(r) = \bar{\psi}(r+1)$. As ψ^ρ is a concave function with $\psi^\rho(0) > 0$, it is subadditive. $\square$

4 Limiting minimal energy and cleavage laws

We now prove Theorems 2.1 and 2.2 on cleavage laws and fine energy estimates.

Limiting minimal energy

We can classify (or 'color') all triangles in $\mathcal{C}_\varepsilon$ into two types, say 'type one' and 'type two', such that all triangles of the same type are translates of each other. Then only triangles of different type can share a common side. Denote the sets by $\mathcal{C}_\varepsilon^{(1)}$ and $\mathcal{C}_\varepsilon^{(2)}$, respectively.

Proof of Theorem 2.1. We first show that the expression on the right hand side is a lower bound for the limiting minimal energy. For every deformation $y \in \mathcal{A}(a_\varepsilon)$

we have by (2) and (3)

$$\mathcal{E}_\varepsilon(y) \geq \frac{4}{\sqrt{3}\varepsilon} \int_{\Omega_\varepsilon \cap (0,l)\times(\varepsilon,1-\varepsilon)} W_\triangle\big(\nabla\tilde{y}\big)\,dx.$$

Let $0 < \delta < \frac{\alpha}{4}$ and choose R so large that $W(r) > \beta - \delta$ if $r \geq R$. Define $\bar{\mathcal{C}}_\varepsilon^{(1)}$ to be the set of those triangles $\triangle$ of type one for which at least one side in the deformed configuration $y(\triangle)$ is larger than $2R\varepsilon$. By $I \subset (\varepsilon, 1-\varepsilon)$ we denote the set of those points x_2 for which there exists $x_1 \in (0,l)$ such that (x_1, x_2) lies in one of these triangles.

We can then estimate the energy integral by splitting the x_2-integration into a first part where $x_2 \notin I$ and a second part with $x_2 \in I$.

1. If $x_2 \notin I$, then all sidelengths of $y(\triangle)$ for a triangle $\triangle$ whose interior intersects the segment $(0,l) \times \{x_2\}$ are less or equal to $4R\varepsilon$. This is clear for triangles of type one by construction. For triangles of type two it follows from the fact that the two sides of $\triangle$ intersecting $(0,l)\times\{x_2\}$ are also sides of triangles of type one and therefore bounded by $2R\varepsilon$. The third side is thus less than $4R\varepsilon$, too.

It is elementary to see that for $F \in \mathbb{R}^{2\times 2}$

$$|\mathbf{e}_1^T F \mathbf{e}_1| \leq 8R, \quad \text{if } |\mathbf{v}^T F \mathbf{v}| \leq 4R \text{ for all } \mathbf{v} \in \mathcal{V}. \tag{13}$$

Indeed, if λ_1, λ_2 are the eigenvalues of $\frac{1}{2}(F^T+F)$, then by (12) one has $\frac{3}{4}(\lambda_1^2+\lambda_2^2) = \frac{3}{4}\,\mathrm{trace}\left(\frac{1}{2}(F^T+F)\right)^2 \leq 3\cdot(4R)^2$ and thus $|\mathbf{e}_1^T F \mathbf{e}_1| \leq \max\{|\lambda_1|, |\lambda_2|\} \leq 8R$. Consequently, for almost every $x_2 \notin I$ we have $\mathbf{e}_1^T \nabla\tilde{y}(x_1,x_2)\mathbf{e}_1 \leq 8R$ for all $x_1 \in (0,l)$.

By Lemma 3.5(ii) choose a convex function with $V(r) \leq \tilde{W}(r)$ for $r \leq 8R$ and $V_+''(1) = \frac{\alpha}{2} - 2\delta$. For $x_2 \in (\varepsilon, 1-\varepsilon)$ define $\Omega_\varepsilon^{x_2} \subset (0,l)$ such that $\Omega_\varepsilon^{x_2} \times \{x_2\} = \Omega_\varepsilon \cap (0,l) \times \{x_2\}$. Then for the first part one obtains, if $a < \infty$, by convexity of V

$$\begin{aligned}
\frac{4}{\sqrt{3}\varepsilon} \int_{(\varepsilon,1-\varepsilon)\setminus I} \int_{\Omega_\varepsilon^{x_2}} W_\triangle\big(\nabla\tilde{y}\big)\,dx_1\,dx_2 &\geq \frac{4}{\sqrt{3}\varepsilon} \int_{(\varepsilon,1-\varepsilon)\setminus I} \int_{\Omega_\varepsilon^{x_2}} V\big(\mathbf{e}_1^T \nabla\tilde{y}\,\mathbf{e}_1\big)\,dx_1\,dx_2 \\
&\geq \frac{4}{\sqrt{3}\varepsilon} \int_{(\varepsilon,1-\varepsilon)\setminus I} |\Omega_\varepsilon^{x_2}| V(1+a_\varepsilon)\,dx_2 \\
&= \frac{2}{\sqrt{3}\varepsilon}(1-2\varepsilon-|I|)(l-2\varepsilon)(V_+''(1)a_\varepsilon^2 + o(\varepsilon)) \\
&\to \frac{2}{\sqrt{3}}(1-|I|)lV_+''(1)a^2
\end{aligned} \tag{14}$$

as $\varepsilon \to 0$. It is not hard to see that this asymptotic estimate remains true also for $a = \infty$.

2. On the other hand, the energy of the second part can be estimated by the energy of all springs lying on the side of a triangle in $\bar{\mathcal{C}}_\varepsilon^{(1)}$, which yields

$$\frac{4}{\sqrt{3}\varepsilon}\int_I\int_{\Omega_\varepsilon^{x_2}} W_\triangle\big(\nabla\tilde{y}\big)\,dx_1\,dx_2 \geq 2(\beta-\delta)\varepsilon\#\bar{\mathcal{C}}_\varepsilon^{(1)}, \tag{15}$$

as the length of at least two springs in each of these triangles is larger than $R\varepsilon$ in the deformed configuration. Now the projection of any triangle onto the x_2-axis is an interval of length $\varepsilon\gamma$, and so $\varepsilon\gamma\#\bar{\mathcal{C}}_\varepsilon^{(1)} \geq |I|$, i.e.,

$$\frac{4}{\sqrt{3}\varepsilon}\int_I\int_{\Omega_\varepsilon^{x_2}} W_\triangle\big(\nabla\tilde{y}\big)\,dx_1\,dx_2 \geq 2(\beta-\delta)\gamma^{-1}|I|. \tag{16}$$

Summarizing (14) and (16) we find

$$\begin{aligned}
&\liminf_{\varepsilon\to\infty}\min\{\mathcal{E}_\varepsilon(y) : y\in\mathcal{A}(a_\varepsilon)\}\\
&\geq \min\left\{\frac{2}{\sqrt{3}}\left(\frac{\alpha}{2}-2\delta\right)la^2(1-|I|)+2(\beta-\delta)\gamma^{-1}|I| : |I|\in[0,1]\right\}\\
&= \min\left\{\frac{2}{\sqrt{3}}\left(\frac{\alpha}{2}-2\delta\right)la^2, \frac{2(\beta-\delta)}{\gamma}\right\}.
\end{aligned}$$

Now $\delta\to 0$ shows

$$\liminf_{\varepsilon\to\infty}\min\{\mathcal{E}_\varepsilon(y) : y\in\mathcal{A}(a_\varepsilon)\} \geq \min\left\{\frac{\alpha l}{\sqrt{3}}a^2, \frac{2\beta}{\gamma}\right\}.$$

This establishes the lower bound.

It remains to prove that the right hand side in Theorem 2.1 is attained for some sequence of deformations. In order to do so, we consider two specific sequences of deformations. First, for $a<\infty$ let

$$y_\varepsilon^{\mathrm{el}}(x) = (\mathbf{Id}+F^{a_\varepsilon})x = \begin{pmatrix} 1+a_\varepsilon & 0\\ 0 & 1-\frac{a_\varepsilon}{3}\end{pmatrix}x. \tag{17}$$

By Lemma 3.3(ii) we have that $W_\triangle(F) = \frac{\alpha}{4}a_\varepsilon^2 + o(\varepsilon)$ and so

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y_\varepsilon^{\mathrm{el}}) = \frac{\alpha l}{\sqrt{3}}a^2$$

by (2).

To define y^{cr} we choose any line $(s,0)+\mathbb{R}\mathbf{v}_\gamma$ intersecting both the segments $(0,l)\times\{0\}$ and $(0,l)\times\{1\}$ (as in Theorem 2.6). This is possible since $l>\frac{1}{\sqrt{3}}$. Let $a>0$ and set

$$y_\varepsilon^{\mathrm{cr}}(x) = \begin{cases} x & \text{for } x \text{ to the left of } (s,0)+\mathbb{R}\mathbf{v}_\gamma,\\ x+a_\varepsilon l\mathbf{e}_1 & \text{for } x \text{ to the right of } (s,0)+\mathbb{R}\mathbf{v}_\gamma \end{cases} \tag{18}$$

for atoms x with $\varepsilon < x_1 < l - \varepsilon$. Except for a negligible contribution from the boundary layers, the energy of this configuration can be estimated as in Step 2 of the proof of the lower bound: It is given by the energy of springs intersecting $(s,0) + \mathbb{R}\mathbf{v}_\gamma$, i.e., by the two springs lying on the boundary of the triangles of type one which are intersected by $(s,0)+\mathbb{R}\mathbf{v}_\gamma$. These springs are elongated by a factor scaling with $a_\varepsilon/\varepsilon$, thus yielding a contribution β in the limit $\varepsilon \to 0$. $\square$

Fine estimates on the limiting minimal energy

Assume now that W in addition satisfies assumptions (ii') and (iii'). In order to investigate a deformation y again we let $\bar{\mathcal{C}}_\varepsilon$ and $\bar{\mathcal{C}}_\varepsilon^{(1)}$ denote the set of triangles $\triangle$ (of type one respectively) for which at least one side in $y(\triangle)$ is larger than $2R\varepsilon$, where now the threshold value $R > 1$ is chosen in such a way that $c_R := \inf\{W(r) : r \geq R\} \geq \frac{\beta}{2}$. According to Lemma 3.5(iii) we may choose a convex function V such that

$$0 \leq V(r) \leq \tilde{W}(r) \leq V(r) + O((r-1)^4) \text{ for } r \leq 8R. \tag{19}$$

As in (13) we observe that $|\mathbf{e}_1^T (y)_\triangle \mathbf{e}_1|$ is bounded by $8R$ on triangles with bond length not exceeding $4R\varepsilon$ and thus lies in the convex regime of V. Moreover, we find that every triangle in $\bar{\mathcal{C}}_\varepsilon$ provides at least the energy $\frac{4}{\sqrt{3}\varepsilon}\int_\triangle W_\triangle(\nabla\tilde{y}) \geq c_R\varepsilon$.

For given $0 < \eta < a$ we also define $R_{\varepsilon,\eta} = \frac{a-\eta}{\sqrt{\varepsilon}}$ as a threshold for triangles we consider 'essentially broken':

$$\bar{\mathcal{C}}_{\varepsilon,\eta} = \left\{\triangle \in \bar{\mathcal{C}}_\varepsilon, |\nabla y_\varepsilon \mathbf{v}| > R_{\varepsilon,\eta} \text{ for at least two } \mathbf{v} \in \mathcal{V}\right\}. \tag{20}$$

The minimal energy contribution of all the springs on such a triangle in $\bar{\mathcal{C}}_{\varepsilon,\eta}$ is given by

$$2\beta^\eta\varepsilon = 2\inf\left\{W(r) : r \geq \frac{a-\eta}{\sqrt{\varepsilon}}\right\}\varepsilon = (2\beta + O(\varepsilon))\varepsilon$$

by the assumption (iii') on W. By $I \subset (\varepsilon, 1-\varepsilon)$ we denote the set of points x_2 for which the segment $(0,l) \times \{x_2\}$ intersects a broken triangle (of type one) in $\bar{\mathcal{C}}_\varepsilon^{(1)}$. In addition, we say $x_2 \in I^\eta \subset I$ if one of the intersected triangles lies in $\bar{\mathcal{C}}_{\varepsilon,\eta} \cap \bar{\mathcal{C}}_\varepsilon^{(1)}$.

With these preparations we can now proceed to prove Theorem 2.2:

Proof of Theorem 2.2. Let $\mathcal{E}_\varepsilon(y) = \inf \mathcal{E}_\varepsilon + O(\varepsilon)$. Inspired by (14) and (15) we establish a lower bound for the energies additionally taking the set $I \setminus I^\eta$ into account. Since the sidelength of any triangle whose interior intersects $(0,l) \times (I \setminus I^\eta)$ is bounded by $4R_{\varepsilon,\eta}$, we find

$$|\mathbf{e}_1^T \nabla\tilde{y}(x_1,x_2)\,\mathbf{e}_1| \leq 8R_{\varepsilon,\eta}$$

for all $(x_1, x_2) \in (0,l) \times (I \setminus I^\eta)$ as in (13). Let $k = k(x_2)$ count the number of triangles in $\bar{\mathcal{C}}_\varepsilon$ on the slice $(0,l) \times \{x_2\}$, $x_2 \in I \setminus I^\eta$ and define $\bar{\mathcal{C}}_\varepsilon^{x_2} \subset (0,l)$ such that $((0,l) \times \{x_2\}) \cap \bigcup_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \triangle = \bar{\mathcal{C}}_\varepsilon^{x_2} \times \{x_2\}$. Then

$$\int_{\bar{\mathcal{C}}_\varepsilon^{x_2}} \mathbf{e}_1^T \nabla \tilde{y}(x_1, x_2)\, \mathbf{e}_1 \, dx_1 \le 8k\varepsilon R_{\varepsilon,\eta}.$$

and so

$$\begin{aligned}\int_{\Omega_\varepsilon^{x_2} \setminus \bar{\mathcal{C}}_\varepsilon^{x_2}} \mathbf{e}_1^T \nabla \tilde{y}(x_1, x_2)\, \mathbf{e}_1 &\ge (1 + \sqrt{\varepsilon} a)(l + O(\varepsilon)) - 8k\varepsilon R_{\varepsilon,\eta} \\ &= \left(1 + \sqrt{\varepsilon}\left(a - \frac{8k(a-\eta)}{l} + O(\sqrt{\varepsilon})\right)\right) l.\end{aligned}$$

A convexity argument as in the proof of Theorem 2.1 on slices $(0,l) \times \{x_2\}$ with $x_2 \in (\varepsilon, 1-\varepsilon) \setminus I$ and on the unbroken part $\left(\Omega_\varepsilon^{x_2} \setminus \bar{\mathcal{C}}_\varepsilon^{x_2}\right) \times \{x_2\}$ of slices with x_2 in $I \setminus I^\eta$ then shows that

$$\mathcal{E}_\varepsilon(y) \ge \frac{4(l-2\varepsilon)}{\sqrt{3}\varepsilon} V(1 + \sqrt{\varepsilon} a)(1 - 2\varepsilon - |I|) + G_{\eta,\varepsilon} |I \setminus I^\eta| + \frac{2\beta^\eta}{\gamma} |I^\eta| + O(\varepsilon), \tag{21}$$

where

$$G_{\eta,\varepsilon} = \min_{k \in \mathbb{N}} \left(\frac{4l}{\sqrt{3}\varepsilon} V\left(1 + \sqrt{\varepsilon}\left(a - \frac{8k(a-\eta)}{l} + O(\sqrt{\varepsilon})\right)\right) + \frac{kc_R}{\gamma} \right).$$

We note that this minimum exists and can be taken over $1 \le k \le K_0$ for some $K_0 \in \mathbb{N}$ large enough and independent of η as $\frac{kc_R}{\gamma} \to \infty$ for $k \to \infty$. We choose $0 < \eta < a$ large enough such that

$$\frac{l\alpha}{\sqrt{3}} a^2 < \min_{1 \le k \le K_0} \left(\frac{\alpha l}{\sqrt{3}} \left(a - \frac{8k}{l}(a - \eta)\right)^2 + \frac{kc_R}{\gamma} \right).$$

Recalling that, by (19) and Lemma 3.3, $\frac{4l}{\sqrt{3}\varepsilon} V(1+\sqrt{\varepsilon} r) = \frac{4l}{\sqrt{3}\varepsilon} \tilde{W}(1+\sqrt{\varepsilon} r) + O(\varepsilon) \to \frac{l\alpha}{\sqrt{3}} r^2$ uniformly in r on bounded sets in $\mathbb{R}$, we see that thus $G_{\eta,\varepsilon}$ exceeds the elastic term $\frac{4l}{\sqrt{3}\varepsilon} V(1 + \sqrt{\varepsilon} a)$ for ε sufficiently small. So from (21) we obtain

$$\mathcal{E}_\varepsilon(y) \ge \frac{4l}{\sqrt{3}\varepsilon} V(1 + \sqrt{\varepsilon} a)(1 - 2\varepsilon - |I^\eta|) + \frac{2\beta^\eta}{\gamma} |I^\eta| + O(\varepsilon). \tag{22}$$

As $\frac{4l}{\sqrt{3}\varepsilon} V(1 + \sqrt{\varepsilon} a) \to \frac{l\alpha}{\sqrt{3}} a^2$ and $\beta^\eta \to \beta$ for all $\eta > 0$, for ε small enough we thus obtain $\inf \mathcal{E}_\varepsilon \ge \frac{4}{\sqrt{3}} \frac{l}{\varepsilon} V(1 + \sqrt{\varepsilon} a)(1 - 2\varepsilon) + O(\varepsilon) = \frac{4}{\sqrt{3}} \frac{l}{\varepsilon} \tilde{W}(1 + \sqrt{\varepsilon} a) + O(\varepsilon)$ or $\inf \mathcal{E}_\varepsilon \ge \frac{2\beta + O(\varepsilon)}{\gamma}(1 - 2\varepsilon) = \frac{2\beta}{\gamma} + O(\varepsilon)$, respectively, depending on a.

Applying (17) and (18) we then get indeed $\inf \mathcal{E}_\varepsilon = \frac{4}{\sqrt{3}} \frac{l}{\varepsilon} \tilde{W}(1 + \sqrt{\varepsilon} a) + O(\varepsilon)$ or $\inf \mathcal{E}_\varepsilon = \frac{2\beta}{\gamma} + O(\varepsilon)$, respectively. The claim now follows from Lemma 3.4 . $\square$

Remark. From the proof of Theorem 2.2, especially taking (17) and (18) into account, it follows that Theorem 2.2 still holds if $\mathcal{E}_\varepsilon$ is replaced by $\mathcal{E}_\varepsilon^\chi$.

5 Limiting minimal energy configurations

Throughout this section we will assume that $a_\varepsilon = \sqrt{\varepsilon}a$, y_ε is a sequence of deformations satisfying (9), the threshold value R is chosen as above Equation (19) and that $\bar{\mathcal{C}}_\varepsilon$ is defined accordingly.

For a rescaled displacement $\tilde{u}$ we denote by $D^\mu \subset (0,1)$ for $\mu > 0$ the set of x_2 such that there is precisely one triangle $\triangle_{x_2} \in \bar{\mathcal{C}}_\varepsilon^{(1)}$ with $\mathrm{int}(\triangle_{x_2}) \cap ((0,l)\times\{x_2\}) \neq \emptyset$ and

$$\int_{\Omega_\varepsilon^{x_2}\setminus\bar{\mathcal{C}}_\varepsilon^{x_2}} \mathbf{e}_1^T \nabla\tilde{u}(x_1,x_2)\mathbf{e}_1\,dx_1 \leq l\mu. \tag{23}$$

Note that $D^\mu \subset I^\eta$ for μ small enough: For $x_2 \in D^\mu$ we have

$$\int_{\bar{\mathcal{C}}_\varepsilon^{x_2}} \mathbf{e}_1^T \nabla\tilde{y}(x_1,x_2)\mathbf{e}_1\,dx_1 \geq \sqrt{\varepsilon}l(a-\mu) + O(\varepsilon)$$

and using the arguments in (13) we see that for given η we can choose μ small enough such that $\triangle_{x_2} \in \bar{\mathcal{C}}_{\varepsilon,\eta}$ and thus $x_2 \in I^\eta$. We also define $\bar{\mathcal{C}}_{\varepsilon,\eta}^\mu \subset \bar{\mathcal{C}}_{\varepsilon,\eta}$ as the set of those essentially broken triangles $\triangle$ for which there exists some $x_2 \in D^\mu$ such that $\mathrm{int}\,(\triangle) \cap ((0,l)\times\{x_2\}) \neq \emptyset$. The projection of a triangle $\triangle$ onto the linear subspace spanned by $\mathbf{v}_\gamma^\perp$ is an interval of length $\frac{\sqrt{3}}{2}\varepsilon$. We denote the center of this interval by $m_\triangle$.

The following lemma gives sharp estimates on the number of broken triangles and their position.

Lemma 5.1 *Let $\mathbf{v}_\gamma$ be unique and $a \neq a_{\mathrm{crit}}$. Let $\tilde{u}_\varepsilon$ be a minimizing sequence satisfying*

$$\mathcal{E}_\varepsilon(\mathbf{id} + \sqrt{\varepsilon}u_\varepsilon) = \inf\mathcal{E}_\varepsilon + O(\varepsilon).$$

(i) If $a < a_{\mathrm{crit}}$, then $\varepsilon\#\bar{\mathcal{C}}_\varepsilon = O(\varepsilon)$.

(ii) If $a > a_{\mathrm{crit}}$, then $|I^\eta| = 1 - O(\varepsilon)$ for $0 < \eta < a$. Furthermore, for μ sufficiently small, $\varepsilon\#\left(\bar{\mathcal{C}}_\varepsilon \setminus \bar{\mathcal{C}}_{\varepsilon,\eta}^\mu\right) = O(\varepsilon)$ and

$$\sup\left\{|m_{\triangle_1} - m_{\triangle_2}|, \triangle_1, \triangle_2 \in \bar{\mathcal{C}}_{\varepsilon,\eta}^\mu\right\} = O(\varepsilon).$$

Proof. (i) Using (21) we find

$$\begin{aligned}
\mathcal{E}_\varepsilon(y_\varepsilon) &= \frac{4l}{\sqrt{3}\varepsilon}\tilde{W}(1+\sqrt{\varepsilon}a) + O(\varepsilon) \\
&\geq \frac{4(l-2\varepsilon)}{\sqrt{3}\varepsilon}\tilde{W}(1+\sqrt{\varepsilon}a)(1-2\varepsilon-|I|) + \min\left\{G_{\eta,\varepsilon}, \frac{2\beta^\eta}{\gamma}\right\}|I| + O(\varepsilon) \\
&= \frac{4l}{\sqrt{3}\varepsilon}\tilde{W}(1+\sqrt{\varepsilon}a)(1-|I|) + \min\left\{G_{\eta,\varepsilon}, \frac{2\beta^\eta}{\gamma}\right\}|I| + O(\varepsilon).
\end{aligned}$$

An elementary computation yields, whenever ε is small enough,

$$\begin{aligned}|I| &\leq \left(\min\left\{G_{\eta,\varepsilon}, \frac{2\beta^\eta}{\gamma}\right\} - \frac{4l}{\sqrt{3}\varepsilon}\tilde{W}(1+\sqrt{\varepsilon}a)\right)^{-1} \cdot O(\varepsilon)\\ &= \left(\min\left\{G_{\eta,\varepsilon}, \frac{2\beta}{\gamma}\right\} - \frac{\alpha l}{\sqrt{3}}a^2 + o(1)\right)^{-1} \cdot O(\varepsilon) = O(\varepsilon),\end{aligned}$$

(The argument leading to (22) together with $a < a_{\text{crit}}$ shows that the term in parentheses is bounded from below by a positive constant independent of ε). Then the elastic energy is $\frac{4l}{\sqrt{3}\varepsilon}\tilde{W}(1+\sqrt{\varepsilon}a) + O(\varepsilon)$ and consequently, the crack energy coming from triangles in $\bar{\mathcal{C}}_\varepsilon$ is of order $O(\varepsilon)$. As every broken triangle in $\bar{\mathcal{C}}_\varepsilon$ provides at least energy εc_R we conclude $\varepsilon\#\bar{\mathcal{C}}_\varepsilon^{(1)} = O(\varepsilon)$. But then, possibly after replacing R by $2R$, also $\varepsilon\#\bar{\mathcal{C}}_\varepsilon^{(2)} = O(\varepsilon)$ as those triangles are neighbors of broken triangles of type 1.

(ii) Using (22) we find after without loss of generality choosing η sufficiently large

$$\mathcal{E}_\varepsilon(y_\varepsilon) = \frac{2\beta}{\gamma} + O(\varepsilon) \geq \frac{4l}{\sqrt{3}\varepsilon}\tilde{W}(1+\sqrt{\varepsilon}a)(1-2\varepsilon-|I^\eta|) + \frac{2\beta^\eta}{\gamma}|I^\eta|.$$

So for ε small enough we obtain

$$1 - |I^\eta| \leq \left(\frac{\alpha l}{\sqrt{3}}a^2 + o(1) - \frac{2\beta}{\gamma}\right)^{-1} \cdot O(\varepsilon) = O(\varepsilon)$$

since $a > a_{\text{crit}}$. Consequently, the crack energy from triangles in $\bar{\mathcal{C}}_{\varepsilon,\eta}$ is given by $\frac{2\beta}{\gamma} + O(\varepsilon)$ and thus the energy contribution from $\bar{\mathcal{C}}_\varepsilon \setminus \bar{\mathcal{C}}_{\varepsilon,\eta}$ is of order $O(\varepsilon)$. As in (i) we find $\varepsilon\#\left(\bar{\mathcal{C}}_\varepsilon \setminus \bar{\mathcal{C}}_{\varepsilon,\eta}\right) = O(\varepsilon)$. A convexity argument yields that the energy of a slice in $I^\eta \setminus D^\mu$ is larger or equal to

$$\frac{2\beta}{\gamma} + \min\left\{c_R, \frac{4l}{\sqrt{3}\varepsilon}\tilde{W}(1+\sqrt{\varepsilon}\mu)\right\} + O(\varepsilon) = \frac{2\beta}{\gamma} + \min\left\{c_R, \frac{\alpha l}{\sqrt{3}}\mu^2\right\} + o(1).$$

It follows $|I^\eta \setminus D^\mu| = O(\varepsilon)$. We conclude $|D^\mu| = 1 - O(\varepsilon)$, whence the crack energy from triangles in $\bar{\mathcal{C}}^\mu_{\varepsilon,\eta}$ is given by $\frac{2\beta}{\gamma} + O(\varepsilon)$ and then also $\varepsilon\#\left(\bar{\mathcal{C}}_\varepsilon \setminus \bar{\mathcal{C}}^\mu_{\varepsilon,\eta}\right) = O(\varepsilon)$.

Finally, we concern ourselves with the projected distance of triangles in $\bar{\mathcal{C}}^\mu_{\varepsilon,\eta}$. We first note that it suffices to show

$$\sup\left\{|m_{\triangle_1} - m_{\triangle_2}| : \triangle_1, \triangle_2 \in \bar{\mathcal{C}}^\mu_{\varepsilon,\eta} \cap \bar{\mathcal{C}}_\varepsilon^{(1)}\right\} = O(\varepsilon)$$

since for a suitable $\tilde{\eta} \geq \eta$ for any $\triangle \in \bar{\mathcal{C}}^\mu_{\varepsilon,\eta} \cap \bar{\mathcal{C}}_\varepsilon^{(2)}$ there is a $\tilde{\triangle} \in \bar{\mathcal{C}}^\mu_{\varepsilon,\tilde{\eta}} \cap \bar{\mathcal{C}}_\varepsilon^{(1)}$ with $|m_\triangle - m_{\tilde{\triangle}}| \leq \varepsilon$. Let $x_2, z_2 \in D^\mu$, $x_2 < z_2$ with $z_2 - x_2 \leq C\varepsilon$ and $|m_{\triangle_1} - m_{\triangle_2}| > 0$ for the corresponding broken triangles $\triangle_1, \triangle_2 \in \bar{\mathcal{C}}_\varepsilon^{(1)}$. We may assume if a triangle intersects $(0,l)\times\{z_2\}$ or $(0,l)\times\{x_2\}$ then its interior does so, too. Denote by $\bar{d}=\gamma^{-1}|m_{\triangle_1}-m_{\triangle_2}|$ the distances of the centers in $\mathbf{v}_\gamma$-projection onto the x_1-axis.

Let $x_1,z_1\in(0,l)$ be the points on the slices $(0,l)\times\{x_2\}$ and $(0,l)\times\{z_2\}$ satisfying $\pi_{\mathbf{v}_\gamma^\perp}(x_1,x_2)=m_{\triangle_1}$ and $\pi_{\mathbf{v}_\gamma^\perp}(z_1,z_2)=m_{\triangle_2}$, respectively, where $\pi_{\mathbf{v}_\gamma^\perp}$ denotes the orthogonal projection onto the linear subspace spanned by $\mathbf{v}_\gamma^\perp$. Let $w=\mathbf{e}_1\cdot\mathbf{v}_\gamma|x_2-z_2|/\gamma$. Then the $\mathbf{v}_\gamma$-projection of $z=(z_1,z_2)$ onto the x-slice is given by $(\tilde{z}_1,x_2)$ with $\tilde{z}_1=z_1-w$. Then $\bar{d}=|x_1-\tilde{z}_1|$ and without restriction we may assume $x_1>\tilde{z}_1$.

Let $s_\varepsilon=\frac{\sqrt{3}\varepsilon}{4\gamma}$. We now consider the area bounded by the parallelogram with corners $(\tilde{z}_1+s_\varepsilon,x_2)$, (x_1-s_ε,x_2), $(z_1+\bar{d}-s_\varepsilon,z_2)$, (z_1+s_ε,z_2). It is covered by $\frac{2\gamma\bar{d}}{\sqrt{3}\varepsilon}-1$ stripes of width $\frac{\sqrt{3}}{2}\varepsilon$ in $\mathbf{v}_\gamma$-direction consisting of lattice triangles intersecting the parallelogram, the first of these stripes touching $\triangle_1$, the last one touching $\triangle_2$ (note that if $\gamma\bar{d}=\frac{\sqrt{3}}{2}\varepsilon$ the parallelogram is degenerated to a segment). For the intermediate stripes (23) shows that

$$\begin{aligned}&y_1(t,x_2)\le t+\sqrt{\varepsilon}l\mu\quad\forall\,t<x_1-s_\varepsilon\qquad\text{and}\\&y_1(t,z_2)\ge t+\sqrt{\varepsilon}l(a-\mu)\quad\forall\,t>z_1+s_\varepsilon.\end{aligned}$$

This shows that if (t,x_2) and $(t+w,z_2)$, $x_1-\bar{d}+s_\varepsilon<t<x_1-s_\varepsilon$ lie in the bottom and top triangles of some intermediate stripe, respectively, which are unbroken by construction of D^μ, then

$$|y(t+w,z_2)-y(t,x_2)|\ge y_1(t+w,z_2)-y_1(t,x_2)\ge w+\sqrt{\varepsilon}l(a-2\mu)\sim\sqrt{\varepsilon}.$$

Consider the $\frac{2\gamma\bar{d}}{\sqrt{3}\varepsilon}$ atomic chains in $\mathbf{v}_\gamma$ direction that lie on the boundary of these stripes. They are of length $\gamma^{-1}(z_2-x_2)+O(\varepsilon)\le C\varepsilon\ll\sqrt{\varepsilon}$. So there is a constant $c>0$ such that each of these chains contains at least one spring elongated by a factor of more than $\frac{c}{\sqrt{\varepsilon}}$. By passing, if necessary, to a lower threshold $\tilde{\eta}\ge\eta$, we obtain that the triangles sharing such a spring are broken and additionally one neighbor of each. As broken triangles for such springs on neighboring chains might overlap, we only consider every second atom chain and denote the set of type one triangles adjacent to such a spring on atom chains of odd numbers by $\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}(\triangle_1,\triangle_2)$. We note that

$$\gamma\bar{d}\le\sqrt{3}\varepsilon\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}(\triangle_1,\triangle_2).\tag{24}$$

The projection onto the x_2-axis of the spring in $\mathbf{v}_\gamma$-direction is an interval J of length $\gamma\varepsilon$. Counting broken springs, it is elementary to see that the energy contribution $\frac{4}{\sqrt{3}\varepsilon}\int_{(\varepsilon,l-\varepsilon)\times J}W_\triangle(\nabla\tilde{y}_\varepsilon)$ of the part of these broken triangles that lies in the stripe $(0,l)\times J$ is bounded from below by

$$2\varepsilon(1+P(\gamma))\beta^{\tilde{\eta}},\tag{25}$$

where $P(\gamma)$ is the projection coefficient from (7) satisfying $P(1)=\frac{1}{2}$ and in particular $P(\gamma)=0$ if and only if $\gamma=\frac{\sqrt{3}}{2}$. On the other hand, the energy within stripes $(0,l)\times J'$ when J' is the projection of an arbitrary broken triangle is still bounded from below by $2\varepsilon\beta^{\tilde\eta}$.

Now let $\triangle_i$, $i=1,\dots,M_\varepsilon$, denote all triangles $\triangle$ in $\bar{\mathcal{C}}^\mu_{\varepsilon,\tilde\eta}\cap\bar{\mathcal{C}}^{(1)}_\varepsilon$ such that there exists $x_2^{(i)}\in D^\mu$ with $(0,l)\times\{x_2^{(i)}\}$ intersecting with the interior of $\triangle$. The numbering shall be chosen so as to satisfy $x_2^{(1)}<\ldots<x_2^{(M_\varepsilon)}$. As $1-|D^\mu|=O(\varepsilon)$, there exists a constant $C>0$ such that $x_2^{(i+1)}-x_2^{(i)}<C\varepsilon$, $i=1,\dots,M_\varepsilon-1$. We define the subset $\{x_2^{(i_j)}\}_{j=1,\dots N_\varepsilon}$ of $\{x_2^{(i)}\}_{i=1,\dots,M_\varepsilon}$ such that $x_2^{(i)}=x_2^{(i_j)}$ for a $j=1,\dots N_\varepsilon$ if and only if $|m_{\triangle_i}-m_{\triangle_{i+1}}|>0$. According to our previous considerations, if $I^{\tilde\eta}_{\mathbf{v}_\gamma}$ is the projection of $\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}:=\bigcup_{j=1}^{N_\varepsilon}\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}(\triangle_{i_j},\triangle_{i_j+1})$ onto the x_2-axis, then

$$|I^{\tilde\eta}_{\mathbf{v}_\gamma}|\le\gamma\varepsilon\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}. \tag{26}$$

As before using (25) and (26) we see that the total energy is greater or equal to

$$\begin{aligned}
&\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}2\varepsilon(1+P(\gamma))\beta^{\tilde\eta}+|I^{\tilde\eta}\setminus I^{\tilde\eta}_{\mathbf{v}_\gamma}|\frac{2\beta^{\tilde\eta}}{\gamma}+O(\varepsilon)\\
&=|I^{\tilde\eta}|\frac{2\beta^{\tilde\eta}}{\gamma}+2\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}\varepsilon P(\gamma)\beta^{\tilde\eta}+2\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}\varepsilon\beta^{\tilde\eta}-|I^{\tilde\eta}_{\mathbf{v}_\gamma}|\frac{2\beta^{\tilde\eta}}{\gamma}+O(\varepsilon)\\
&\ge\frac{2\beta}{\gamma}+2\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}\varepsilon P(\gamma)\beta^{\tilde\eta}+O(\varepsilon),
\end{aligned}$$

and so $\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}=O(1)$. As every $\triangle\in\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}$ is in at most two different $\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}(\triangle_{i_j},\triangle_{i_j+1})$, this also yields $\sum_{j=1}^{N_\varepsilon}\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}(\triangle_{i_j},\triangle_{i_j+1})=O(1)$.

Applying (24) we find that

$$O(1)=\sum_{j=1}^{N_\varepsilon}\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}(\triangle_{i_j},\triangle_{i_j+1})\ge\sum_{j=1}^{N_\varepsilon}\frac{\gamma\bar d_{i_j}}{\sqrt3\varepsilon}\ge\frac{c}{\varepsilon}\sum_{j=1}^{N_\varepsilon}|m_{\triangle_{i_j}}-m_{\triangle_{i_j+1}}|$$

for a constant $c>0$, when $\bar d_i=\gamma^{-1}|m_{\triangle_i}-m_{\triangle_{i+1}}|$. This concludes the proof. $\square$

Remark. We recall that the last claim in Lemma 5.1 does not hold if $\mathbf{v}_\gamma$ is not unique ($\gamma=\frac{\sqrt3}{2}$) as the crack can take a serrated course. Indeed, if $P(\gamma)$ vanishes, we cannot conlude that $\#\bar{\mathcal{C}}^{(1)}_{\mathbf{v}_\gamma}=O(1)$ in the above proof.

The above Lemma 5.1 shows that for a sequence of almost minimizers $(\tilde y_\varepsilon)$ satisfying (9), the number $\#\bar{\mathcal{C}}_\varepsilon$ of largely deformed triangles is bounded independently of ε for $a<a_{\mathrm{crit}}$, while in the supercritical case there are two subsets

$$\begin{aligned}
\Omega^{(1)}_\varepsilon&:=\{x\in\Omega_\varepsilon:0\le x_1\le p_\varepsilon-c\varepsilon+(\mathbf{v}_\gamma\cdot\mathbf{e}_1)x_2\}\,,\\
\Omega^{(2)}_\varepsilon&:=\{x\in\Omega_\varepsilon:p_\varepsilon+c\varepsilon+(\mathbf{v}_\gamma\cdot\mathbf{e}_1)x_2\le x_1\le l\}\,,
\end{aligned}\tag{27}$$

$c>0$ independent of ε and p_ε to be chosen appropriately, such that the number of triangles in $\bar{\mathcal{C}}_\varepsilon$ intersecting $\Omega^{(1)}_\varepsilon\cup\Omega^{(2)}_\varepsilon$ is bounded uniformly in ε. The following lemma shows that broken triangles in these sets can be 'healed'. In order to treat these cases simultaneously in the following we will call these sets the 'good set'

$$\Omega_{\mathrm{good}} = \begin{cases} \Omega_\varepsilon & \text{for } a < a_{\mathrm{crit}} \text{ and} \\ \Omega_\varepsilon^{(1)} \cup \Omega_\varepsilon^{(2)} & \text{for } a > a_{\mathrm{crit}}. \end{cases}$$

Lemma 5.2 *There exists $\bar{y}_\varepsilon \in W^{1,\infty}(\Omega_{\mathrm{good}}; \mathbb{R}^2)$ with $\nabla \bar{y}_\varepsilon$ bounded in $L^\infty(\Omega_{\mathrm{good}})$ uniformly in ε such that*

$$|\{x \in \Omega_{\mathrm{good}} : \bar{y}_\varepsilon(x) \neq \tilde{y}_\varepsilon(x)\}| = O(\varepsilon^2)$$

and

$$\int_{\Omega_{\mathrm{good}}} \mathrm{dist}^2(\nabla \bar{y}_\varepsilon(x), SO(2))\, dx \leq C \int_{\Omega_{\mathrm{good}} \setminus \bigcup_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \triangle} \mathrm{dist}^2(\nabla \tilde{y}_\varepsilon, SO(2))\, dx.$$

Proof. For notational convenience we drop the subscript ε in the following proof. We can partition the area covered by the (closed) triangles in $\bar{\mathcal{C}}$ intersecting Ω_{good} into connected components $C_1, \ldots, C_N$ such that

$$\bigcup_{\triangle \in \bar{\mathcal{C}} : \triangle \cap \Omega_{\mathrm{good}} \neq \emptyset} \triangle = C_1 \dot{\cup} \ldots \dot{\cup} C_N,$$

where N is bounded uniformly in ε. Then the maximal diameter of each sets C_i is bounded by a term $O(\varepsilon)$. For each i, the largest connected component D_i of the complement $\Omega_{\mathrm{good}} \setminus C_i$ lying in the same component of Ω_{good} is unique (with area of the order 1 while all the other components of the complement are of size $O(\varepsilon^2)$). Let V_i be the union of triangles whose interior is contained in D_i that touch the boundary of C_i.

We now proceed to define $\bar{y}$ by modifying $\tilde{y}$ on all the triangles not contained in $\overline{D}_i$, successively for $i = 1, \ldots, N$. For each i this modification is done iteratively on triangles $\triangle$ which share at least one side with a triangle that has been modified previously or with a triangle lying in V_i in such a way that $\bar{y}$ is continuous along such sides and $\bar{y}|_\triangle$ is affine and minimizes $\mathrm{dist}((\bar{y})_\triangle, SO(2))$.

In order to estimate $\mathrm{dist}(\nabla \bar{y}, SO(2))$ we recall the following geometric rigidity result proved in [24]: If $U \subset \mathbb{R}^d$ is a (connected) Lipschitz domain, then there exists a constant $C = C(U)$ such that for any $f \in H^1(U, \mathbb{R}^d)$ there is a rotation $R \in SO(d)$ with

$$\int_U |\nabla f(x) - R|^2\, dx \leq C \int_U \mathrm{dist}^2(\nabla f(x), SO(d))\, dx. \tag{28}$$

The constant $C(U)$ is invariant under rescaling of the domain. For later use we mention that if $\mathrm{dist}^2(\nabla f(x), SO(d))$ is equiintegrable, then R can be chosen in such a way that also $|\nabla f(x) - R|^2$ is equiintegrable, cf. [25].

Consider a single step in the modification process, when $\tilde{y}$ is modified to $\bar{y}$ on $\triangle$, and let U be the union of triangles that have been modified previously or lie in V_i. By the geometric rigidity estimate (28), there is a rotation $R \in SO(2)$ such that (28) holds for $f = \bar{y}$. Since $\nabla\bar{y}$ is piecewise constant, this means

$$\sum_{\triangle' \subset U} |(\bar{y})_{\triangle'} - R|^2 \le C \sum_{\triangle' \subset U} \operatorname{dist}^2((\bar{y})_{\triangle'}, SO(2)).$$

It is not hard to see that there exists an extension w of $\bar{y}$ from U to $U \cup \triangle$ such that

$$|(w)_{\triangle} - R|^2 \le C \sum_{\triangle' \subset U} |(\bar{y})_{\triangle'} - R|^2.$$

(If there is only one side of $\triangle$ on the boundary of U, say adjacent to $\triangle' \subset U$, then one can take w with $(w)_{\triangle} = (\bar{y})_{\triangle'}$. If at least two sides, say in $\mathbf{v}_1$ and $\mathbf{v}_2$ direction, are shared by triangles $\triangle_1, \triangle_2 \subset U$, respectively, then these sides have a common corner and the unique extension w satisfies $(w)_{\triangle}\mathbf{v}_i = (\bar{y})_{\triangle_i}\mathbf{v}_i = R\mathbf{v}_i + ((\bar{y})_{\triangle_i} - R)\mathbf{v}_i$, $i = 1, 2$.) Now by construction of $\bar{y}$ on $\triangle$ we see that

$$\operatorname{dist}^2((\bar{y})_{\triangle}, SO(2)) \le C \sum_{\triangle' \subset U} |(\bar{y})_{\triangle'} - R|^2$$

and so

$$\int_{U \cup \triangle} \operatorname{dist}^2(\nabla\bar{y}(x), SO(2))\, dx \le C \int_U \operatorname{dist}^2(\nabla\bar{y}(x), SO(2))\, dx.$$

Iterating this estimate we finally arrive at

$$\int_{\Omega_{\text{good}}} \operatorname{dist}^2(\nabla\bar{y}(x), SO(2))\, dx \le C \int_{\Omega_{\text{good}} \setminus \bigcup_i C_i} \operatorname{dist}^2(\nabla\tilde{y}, SO(2))\, dx.$$

Here the constant C can be chosen independently of ε. This is due to the facts that the number of modification steps is bounded uniformly in ε and – after rescaling the shapes U with $\frac{1}{\varepsilon}$ – there is also only a uniformly bounded number of shapes U involved in the previous rigidity estimates. Moreover, each triangle is covered by no more than three of the sets V_i.

The uniform boundedness of the number of modification steps also shows that $|\{x \in \Omega_{\text{good}} : \bar{y}(x) \neq \tilde{y}(x)\}| = O(\varepsilon^2)$ and, by definition of $\bar{\mathcal{C}}$ and construction of $\bar{y}$, that $\|\nabla\bar{y}\|_{L^\infty(\Omega_{\text{good}})} = O(1)$. $\square$

Note that up to a set of small size $\bar{y}_\varepsilon$ satisfies the same boundary conditions as $\tilde{y}_\varepsilon$ on the lateral boundary. More precisely, there are $\Gamma_\varepsilon^{(i)} \subset (0,1)$, $|\Gamma_\varepsilon^{(i)}| = O(\varepsilon)$, $i = 1, 2$, such that $\bar{y}_\varepsilon$ and $\tilde{y}_\varepsilon$ coincide on $(0, \varepsilon) \times ((0,1) \setminus \Gamma_\varepsilon^{(1)})$ and $(l - \varepsilon, l) \times ((0,1) \setminus \Gamma_\varepsilon^{(2)})$. With these boundary conditions and the geometric rigidity estimate (28) we can now derive strong convergence results for $\bar{y}_\varepsilon$ and even the corresponding rescaled displacement $\bar{u}_\varepsilon = \frac{1}{\sqrt{\varepsilon}}(\bar{y}_\varepsilon - \mathbf{id})$ on Ω_{good}. We first consider the supercritical case.

Lemma 5.3 *If $a > a_{\rm crit}$, then – possibly after a rotation by π on the components of $\Omega_{\rm good}$ – there exist sequences $s_\varepsilon, t_\varepsilon \in \mathbb{R}$ such that*

$$\|\bar{u}_\varepsilon - (0, s_\varepsilon)\|_{H^1(\Omega_\varepsilon^{(1)})} + \|\bar{u}_\varepsilon - (al, t_\varepsilon)\|_{H^1(\Omega_\varepsilon^{(2)})} \to 0.$$

Proof. We again drop the subscript ε. Applying the geometric rigidity estimate (28) to $\Omega^{(1)}$ and to $\Omega^{(2)}$, we obtain rotations $R^{(1)}, R^{(2)} \in SO(2)$ such that

$$\|\nabla \bar{y} - R^{(i)}\|_{L^2(\Omega_\varepsilon^{(i)})} \le C\| \operatorname{dist}(\nabla \bar{y}, SO(2))\|_{L^2(\Omega_\varepsilon^{(i)})}, \quad i = 1, 2.$$

Here C can be chosen independently of ε as all the possible shapes of $\Omega^{(i)}$ are related through bi-Lipschitzian homeomorphisms with Lipschitz constants of both the homeomorphism itself and its inverse bounded uniformly in ε, cf. [24]. Now using that $\nabla \bar{y}$ is uniformly bounded in L^∞, we obtain from Lemmas 5.2 and 3.5(i)

$$\begin{aligned}
\sum_{i=1}^{2} \|\nabla \bar{y} - R^{(i)}\|^2_{L^2(\Omega_\varepsilon^{(i)})} &\le C \int_{\Omega_{\rm good} \setminus \bigcup_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \triangle} \operatorname{dist}^2(\nabla \tilde{y}, SO(2))\, dx \\
&\le C \int_{\Omega_{\rm good} \setminus \bigcup_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \triangle} \operatorname{dist}^2(\nabla \tilde{y}, O(2)) + \chi(\nabla \tilde{y})\, dx \\
&\le C \int_{\Omega_{\rm good} \setminus \bigcup_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \triangle} W_{\triangle,\chi}(\nabla \tilde{y})\, dx.
\end{aligned}$$

But, as seen before,

$$\frac{4}{\sqrt{3}\varepsilon} \int_{\Omega_{\rm good} \setminus \bigcup_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \triangle} W_{\triangle,\chi}(\nabla \tilde{y})\, dx \le \mathcal{E}^\chi(y) - \frac{2\beta^\eta}{\gamma}|I^\eta| = O(\varepsilon)$$

by Lemma 5.1, and so

$$\sum_{i=1}^{2} \|\nabla \bar{y} - R^{(i)}\|^2_{L^2(\Omega_\varepsilon^{(i)})} = O(\varepsilon^2).$$

By Poincaré's inequality we then deduce that there are $\zeta^{(i)} \in \mathbb{R}^2$ such that

$$\sum_{i=1}^{2} \|\bar{y} - R^{(i)} \cdot - \zeta^{(i)}\|_{H^1(\Omega_\varepsilon^{(i)})} = O(\varepsilon). \tag{29}$$

We extend $\bar{y}$ as an H^1 function from $\Omega_\varepsilon^{(i)}$ to $\Omega^{(i)}$ (as defined in Theorem 2.3), $i = 1, 2$, such that (29) still holds and $\bar{y}_1(0, x_2) = 0$ for $x_2 \in (0, 1) \setminus \Gamma_\varepsilon^{(1)}$, $\bar{y}_1(l, x_2) = l(1 + a_\varepsilon)$ for $x_2 \in (0, 1) \setminus \Gamma_\varepsilon^{(2)}$. The trace theorem for Sobolev functions with $x_1 = 0$ or $x_1 = l$ according to $i = 1$ and $i = 2$, respectively, gives

$$\sum_{i=1}^{2} \|\bar{y}(x_1, \cdot) - R^{(i)}(x_1, \cdot) - \zeta^{(i)}\|_{L^2(0,1)} = O(\varepsilon).$$

In particular, setting $\tilde{\zeta}^{(1)} = \zeta^{(1)}$ and $\tilde{\zeta}^{(2)} = \zeta^{(2)} - la_\varepsilon \mathbf{e}_1$, the first components satisfy

$$\sum_{i=1}^{2} \|x_1 - R_{11}^{(i)} x_1 - R_{12}^{(i)} \cdot - \tilde{\zeta}_1^{(i)}\|_{L^2((0,1)\setminus\Gamma_\varepsilon^{(i)})} = O(\varepsilon). \tag{30}$$

But then also the constant function

$$\frac{1}{2} R_{12}^{(i)} = \left(x_1 - R_{11}^{(i)} x_1 - R_{12}^{(i)} \left(\cdot - \frac{1}{2} \right) - \tilde{\zeta}_1^{(i)} \right) - \left(x_1 - R_{11}^{(i)} x_1 - R_{12}^{(i)} \cdot - \tilde{\zeta}_1^{(i)} \right)$$

is of order ε in $L^2((0, \frac{1}{2}) \setminus \Gamma^{(i)})$ and thus $R_{12}^{(i)} \leq C\varepsilon$. An elementary argument now yields

$$|R^{(i)} - \mathbf{Id}| = O(\varepsilon) \qquad \text{or} \qquad |R^{(i)} + \mathbf{Id}| = O(\varepsilon)$$

and, possibly after rotating by π, we may assume that $|R^{(i)} - \mathbf{Id}| = O(\varepsilon)$.

Returning to (30) and (29), it now follows that $|\tilde{\zeta}_1^{(i)}| = O(\varepsilon)$ and then

$$\|\bar{u} - (0, \zeta_2^{(1)})\|_{H^1(\Omega_\varepsilon^{(1)})} + \|\bar{u} - (al, \zeta_2^{(2)})\|_{H^1(\Omega_\varepsilon^{(2)})} = O(\sqrt{\varepsilon}).$$

□

Strong convergence in the subcritical case can be shown along the lines of the proofs of the main linearization results in [30] and [31]. We include a simplified proof adapted to the present situation here for the sake of completeness.

Lemma 5.4 *If $a < a_{\text{crit}}$, then there is a sequence $s_\varepsilon \in \mathbb{R}$ such that*

$$\|\bar{u}_\varepsilon - (0, s_\varepsilon) - F^a \cdot\|_{H^1(\Omega_{\text{good}})} \to 0.$$

where $F^a = \begin{pmatrix} a & 0 \\ 0 & -\frac{a}{3} \end{pmatrix}$.

Proof. We again drop subscripts ε if no confusion arises. With the help of the geometric rigidity estimate (28) we find by arguing as in the proof of Lemma 5.3 that

$$\|\nabla \bar{y} - R\|_{L^2(\Omega_\varepsilon)}^2 \leq C \int_{\Omega_\varepsilon \setminus \bigcup_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \triangle} W_{\triangle,\chi}(\nabla \tilde{y})\, dx = O(\varepsilon)$$

for a suitable rotation $R \in SO(2)$ with

$$|R \pm \mathbf{Id}| = O(\sqrt{\varepsilon}) \tag{31}$$

and

$$\|\bar{y} \pm \mathbf{id} - \zeta\|_{H^1(\Omega_\varepsilon)} = O(\sqrt{\varepsilon})$$

for some $\zeta \in \mathbb{R}^2$ with $\zeta_1 = O(\sqrt{\varepsilon})$ and thus, due to the boundary conditions,

$$\|\bar{u} - (0, \zeta_2)\|_{H^1(\Omega_\varepsilon)} = O(1).$$

In particular, $\bar{u}_\varepsilon - (\zeta_\varepsilon)_2 \mathbf{e}_2$ converges – up to passing to a subsequence – weakly. It now suffices to prove that $\|e(\bar{u}_\varepsilon) - F^a\|_{L^2(\Omega_\varepsilon)} \to 0$, where $e(u) = \frac{(\nabla u)^T + \nabla u}{2}$ denotes the symmetrized gradient, for then the assertion follows from Korn's inequality.

To this end, we let $V_\varepsilon(F) = \frac{1}{\varepsilon} W_\triangle(\mathbf{Id} + \sqrt{\varepsilon} F)$ and $V_{\varepsilon,\chi}(F) = V_\varepsilon(F) + \frac{1}{\varepsilon}\chi(\mathbf{Id} + \sqrt{\varepsilon}F)$, so that $V_{\varepsilon,\chi}(F) \to \frac{1}{2} D^2 W_\triangle(\mathbf{Id})[F,F] = \frac{1}{2} Q(F)$ uniformly on compact subsets of $\mathbb{R}^{2\times 2}$. Then by frame indifference (see Lemma 3.1)

$$\begin{aligned} W_{\triangle,\chi}(\mathbf{Id} + \sqrt{\varepsilon} F) &= W_{\triangle,\chi}\left(\sqrt{(\mathbf{Id} + \sqrt{\varepsilon}F)^T(\mathbf{Id} + \sqrt{\varepsilon}F)}\right) \\ &= \varepsilon V_{\varepsilon,\chi}\left(\frac{F^T + F}{2} + \frac{1}{\sqrt{\varepsilon}} f(\sqrt{\varepsilon}F)\right) \end{aligned} \tag{32}$$

with $f(F) = \sqrt{(\mathbf{Id} + F)^T(\mathbf{Id} + F)} - \mathbf{Id} - \frac{F^T+F}{2}$, so that $|f(F)| \le C \min\{|F|, |F|^2\}$. Then by Lemma 3.5(i) and (32) $V_{\varepsilon,\chi}$ satisfies

$$\begin{aligned} V_{\varepsilon,\chi}\left(\frac{F^T + F}{2} + \frac{1}{\sqrt{\varepsilon}} f(\sqrt{\varepsilon}F)\right) &\ge \frac{c}{\varepsilon} \operatorname{dist}^2(\mathbf{Id} + \sqrt{\varepsilon}F, O(2)) + \frac{1}{\varepsilon}\chi(\mathbf{Id} + \sqrt{\varepsilon}F) \\ &\ge \frac{c}{\varepsilon} \operatorname{dist}^2(\mathbf{Id} + \sqrt{\varepsilon}F, SO(2)) \\ &\ge \frac{c}{\varepsilon} \left|\sqrt{(\mathbf{Id} + \sqrt{\varepsilon}F)^T(\mathbf{Id} + \sqrt{\varepsilon}F)} - \mathbf{Id}\right|^2 \\ &= c\left|\frac{F^T + F}{2} + \frac{1}{\sqrt{\varepsilon}} f(\sqrt{\varepsilon}F)\right|^2. \end{aligned} \tag{33}$$

In the sequel we set $A_\varepsilon(F) = \frac{F^T+F}{2} + \frac{1}{\sqrt{\varepsilon}} f(\sqrt{\varepsilon}F)$. Choose convex functions $\psi_k : \mathbb{R}^{2\times 2} \to \mathbb{R}$ with linear growth at infinity such that $\psi_1 \le \psi_2 \le \ldots$ and $\psi_k(F) \to \frac{1}{2} Q(F)$ uniformly on compact subsets of $\mathbb{R}^{2\times 2}$. The previous quadratic estimate on $V_{\varepsilon,\chi}(A_\varepsilon(F))$ from below and the fact that $V_{\varepsilon,\chi} \to \frac{1}{2} Q$ uniformly on compacts then shows that we can also choose $\delta > 0$ and a sequence $r_k \to \infty$ such that

$$V_{\varepsilon,\chi}\left(A_\varepsilon(F)\right) - \delta \chi_{\{|A_\varepsilon(F)| \ge r_k\}} |A_\varepsilon(F)|^2 \ge \psi_k\left(A_\varepsilon(F)\right) - \frac{1}{k},$$

whenever ε (depending on k) is sufficiently small.

With (32) we now obtain that

$$\begin{aligned} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W_{\triangle,\chi}(\bar{y})\, dx &= \int_{\Omega_\varepsilon} V_{\varepsilon,\chi}\left(A_\varepsilon(\nabla \bar{u})\right)\, dx \\ &\ge \int_{\Omega_\varepsilon} \psi_k\left(A_\varepsilon(\nabla \bar{u})\right)\, dx + \delta \int_{\Omega_\varepsilon} \chi_{\{|A_\varepsilon(\nabla \bar{u})| \ge r_k\}} |A_\varepsilon(\nabla \bar{u})|^2\, dx - \frac{1}{k}. \end{aligned}$$

As ψ_k has linear growth at infinity and $\frac{1}{\sqrt{\varepsilon}} f(\sqrt{\varepsilon}\nabla \bar{u}_\varepsilon) \le C \min\{|\nabla \bar{u}_\varepsilon|, \sqrt{\varepsilon}|\nabla \bar{u}_\varepsilon|^2\}$, $\nabla \bar{u}_\varepsilon$ bounded in L^2, by splitting the integration into two parts according to $|\nabla \bar{u}_\varepsilon| \le M$ or $|\nabla \bar{u}_\varepsilon| > M$ and eventually sending M to infinity, we find

$$\liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \psi_k \left(A_\varepsilon(\nabla \bar{u}_\varepsilon) \right) \, dx = \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \psi_k \left(e(\bar{u}_\varepsilon) \right) \, dx.$$

When $\bar{u}_\varepsilon - (\zeta_\varepsilon)_2 \mathbf{e}_2 \rightharpoonup u$ in H^1, by Theorem 2.1 it then follows that

$$\begin{aligned} \frac{\alpha l a^2}{\sqrt{3}} &= \lim_{\varepsilon \to 0} \frac{4}{\sqrt{3}} \int_{\Omega_\varepsilon} V_{\varepsilon,\chi} \left(A_\varepsilon(\nabla \bar{u}_\varepsilon) \right) \, dx \\ &\ge \liminf_{\varepsilon \to 0} \frac{4}{\sqrt{3}} \int_{\Omega} \chi_{\{\mathrm{dist}(x,\partial\Omega) \ge k^{-1}\}} \psi_k \left(e(\bar{u}_\varepsilon) \right) \, dx \\ &\qquad + \limsup_{\varepsilon \to 0} \frac{4\delta}{\sqrt{3}} \int_{\Omega_\varepsilon} \chi_{\{|A_\varepsilon(\nabla \bar{u}_\varepsilon)| \ge r_k\}} |A_\varepsilon(\nabla \bar{u}_\varepsilon)|^2 \, dx - \frac{4}{\sqrt{3}k}. \end{aligned}$$

Using that by convexity of ψ_k the first term on the right hand side is lower semicontinuous in $\nabla \bar{u}_\varepsilon$ and that $\chi_{\{\mathrm{dist}(\cdot,\partial\Omega) \ge k^{-1}\}} \psi_k \to \frac{1}{2} Q$ monotonically, we finally find by letting $k \to \infty$

$$\begin{aligned} \frac{\alpha l a^2}{\sqrt{3}} &\ge \frac{2}{\sqrt{3}} \int_{\Omega} Q\left(e(u) \right) \\ &\qquad + \lim_{k \to \infty} \limsup_{\varepsilon \to 0} \frac{4\delta}{\sqrt{3}} \int_{\Omega_\varepsilon} \chi_{\{|A_\varepsilon(\nabla \bar{u}_\varepsilon)| \ge r_k\}} |A_\varepsilon(\nabla \bar{u}_\varepsilon)|^2 \, dx. \end{aligned} \tag{34}$$

A slicing and convexity argument similar to (14) now shows that $\frac{2}{\sqrt{3}} \int_\Omega Q(e(w)) \ge \frac{\alpha l a^2}{\sqrt{3}}$ for all $w \in H^1$ subject to $w_1(0, x_2) = 0$ and $w_1(l, x_2) = al$ and thus

$$\lim_{k \to \infty} \limsup_{\varepsilon \to 0} \frac{4\delta}{\sqrt{3}} \int_{\Omega_\varepsilon} \chi_{\{|A_\varepsilon(\nabla \bar{u}_\varepsilon)| \ge r_k\}} |A_\varepsilon(\nabla \bar{u}_\varepsilon)|^2 \, dx = 0,$$

or, in other words, $|A_\varepsilon(\nabla \bar{u}_\varepsilon)|^2$ is equiintegrable. By the estimate $|V_{\varepsilon,\chi}(F)| = |\frac{1}{\varepsilon} W_{\triangle,\chi}(\mathbf{Id} + \sqrt{\varepsilon} F)| \le C(1 + |F|^2)$, (33) shows that also

$$\frac{c}{\varepsilon} \, \mathrm{dist}^2(\nabla \bar{y}_\varepsilon, SO(2)) \le V_{\varepsilon,\chi}(A_\varepsilon(\nabla \bar{u}_\varepsilon))$$

is equiintegrable, so that by the discussion following Equation (28) in fact we may assume that $\frac{1}{\varepsilon} \|\nabla \bar{y}_\varepsilon - R\|^2_{L^2(\Omega_\varepsilon)}$ is equiintegrable, too, and $|R - \mathbf{Id}| = O(\sqrt{\varepsilon})$ by (31). But then also $|\nabla \bar{u}_\varepsilon|^2$ is equiintegrable and this together with (34) yields

$$\lim_{\varepsilon \to 0} \frac{2}{\sqrt{3}} \int_{\Omega_\varepsilon} Q(e(\bar{u}_\varepsilon)) = \frac{2}{\sqrt{3}} \int_{\Omega} Q(e(u)) = \frac{\alpha l a^2}{\sqrt{3}}.$$

For some $\delta > 0$ small enough we finally obtain that

$$
\begin{aligned}
\frac{\alpha l a^2}{\sqrt{3}} &= \frac{2}{\sqrt{3}} \int_\Omega Q(F^a)\,dx \\
&= \inf \Bigg\{ \frac{2}{\sqrt{3}} \int_\Omega Q(e(w)) - \delta |e(w) - F^a|^2 \,dx : \\
&\qquad\qquad w \in H^1(\Omega), w(0,x_2) = 0, w(l,x_2) = al \Bigg\} \\
&\le \liminf_{\varepsilon \to 0} \frac{2}{\sqrt{3}} \int_{\Omega_\varepsilon} Q(e(\bar{u}_\varepsilon)) - \delta |e(\bar{u}_\varepsilon) - F^a|^2 \,dx \\
&= \frac{\alpha l a^2}{\sqrt{3}} - \delta \limsup_{\varepsilon \to 0} \|e(\bar{u}_\varepsilon) - F^a\|^2_{L^2(\Omega_\varepsilon)}
\end{aligned}
$$

and therefore $\lim_{\varepsilon \to 0} \|e(\bar{u}_\varepsilon) - F^a\|^2_{L^2(\Omega_\varepsilon)} = 0$ indeed. □

After all these preparatory lemmas, the proof of our main limiting result Theorem 2.3 is now straightforward.

Proof of Theorem 2.3. Choose p_ε as in (27) and s_ε, respectively, s_ε and t_ε, as in Lemmas 5.4, respectively, 5.3. By Lemmas 5.4 and 5.3, $\bar{u}_\varepsilon$ can be extended as an H^1-function from Ω_ε to Ω, respectively, from $\Omega_\varepsilon^{(i)}$ to $\Omega^{(i)}$, $i = 1, 2$, such that still

$$\|\bar{u}_\varepsilon - (0, s_\varepsilon) - F^a \cdot\|_{H^1(\Omega)} \to 0,$$

respectively,

$$\|\bar{u}_\varepsilon - (0, s_\varepsilon)\|_{H^1(\Omega^{(1)})} + \|\bar{u}_\varepsilon - (al, t_\varepsilon)\|_{H^1(\Omega^{(2)})} \to 0.$$

This completes the proof as by Lemma 5.2 we also still have $|\{x \in \Omega_\varepsilon : \bar{u}_\varepsilon(x) \neq \tilde{u}_\varepsilon(x)\}| = O(\varepsilon)$. □

6 The limiting variational problem

Convergence of the variational problems

We first address the question of Γ-convergence of $\mathcal{E}_\varepsilon$. The main idea in the proof is a separation of the discrete energy into bulk and surface contributions. The treatment of the elastic part draws ideas from [31] and [24]. To derive the crack energy, one could use a slicing technique, see, e.g., [12]. Although also possible in our framework, we follow a different approach here: We carefully construct crack shapes of discrete configurations in an explicit way which allows us to directly appeal to lower semicontinuity results for SBV functions in order to derive the main energy estimates.

As a preparation we modify the interpolation $\tilde{y}$ on triangles with large deformation: We fix a threshold explicitly as $R = 4\sqrt{2}$ and let $\bar{\mathcal{C}}_\varepsilon$ be the set of those triangles where $|(\tilde{y})_\triangle| > R$. We introduce another interpolation y' which leaves $\tilde{y}$ unchanged on $\triangle \in \mathcal{C}_\varepsilon \setminus \bar{\mathcal{C}}_\varepsilon$ and replaces $\tilde{y}$ on $\triangle \in \bar{\mathcal{C}}_\varepsilon$ by a discontinuous function with constant derivative satisfying $|(y')_\triangle| \le R$. In fact, by introducing jumps we achieve a release of the elastic energy. Note that $y' \in SBV(\Omega)$.

More precisely, note that on $\triangle \in \bar{\mathcal{C}}_\varepsilon$ we have $|(\tilde{y})_\triangle \mathbf{v}| \ge 2$ for at least two springs $\mathbf{v} \in \mathcal{V}$. Indeed, for $F \in \mathbb{R}^{2\times 2}$ with $32 < |F|^2 = |F^T F|$ using (12) we find $\frac{3}{4}|F^T F|^2 = \frac{3}{4}\,\mathrm{trace}(F^T F)^2 \le \sum_{\mathbf{v}\in\mathcal{V}} \langle \mathbf{v}, F^T F \mathbf{v}\rangle^2$ and so $\frac{32^2}{4} < \max_{\mathbf{v}\in\mathcal{V}} |F\mathbf{v}|^4$. Hence, $|F\mathbf{v}| > 4$ for at least one $\mathbf{v} \in \mathcal{V}$ and at least two springs are elongated by a factor larger than 2. For $m = 2, 3$ let $\bar{\mathcal{C}}_{\varepsilon,m} \subset \bar{\mathcal{C}}_\varepsilon$ be the set of triangles where $|(\tilde{y})_\triangle \mathbf{v}| \ge 2$ holds for exactly m springs $\mathbf{v} \in \mathcal{V}$. Set $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$. For $i, j, k = 1, 2, 3$ pairwise distinct let h_i denote the segment between the centers of the sides in $\mathbf{v}_j$ and $\mathbf{v}_k$ direction and define the set $V_i = h_j \cup h_k$.

We now construct $y' \in SBV^2(\Omega_\varepsilon)$. On $\triangle \in \mathcal{C}_\varepsilon \setminus \bar{\mathcal{C}}_\varepsilon$ we simply set $y' = \tilde{y}$. On $\triangle \in \bar{\mathcal{C}}_{\varepsilon,2}$, assuming $|(\tilde{y})_\triangle \mathbf{v}_i| \le 2$, we choose y' such that $\nabla y'$ assumes the constant value $(y')_\triangle$ on $\triangle$ with $(y')_\triangle \mathbf{v}_i = (\tilde{y})_\triangle \mathbf{v}_i$ and $|(y')_\triangle \mathbf{v}| = 1$ for $\mathbf{v} \in \mathcal{V} \setminus \{\mathbf{v}_i\}$. Moreover, we ask that $y' = \tilde{y}$ at the three vertices and on the side orientated in $\mathbf{v}_i$ direction. This can and will be done in such a way that y' is continuous on $\mathrm{int}(\triangle) \setminus h_i$. We note that the definition of $(y')_\triangle$ is unique up to a reflection, unless $(\tilde{y})_\triangle \mathbf{v}_i = 0$. We may and will assume that

$$\mathrm{dist}\left((y')_\triangle, SO(2)\right) \le \mathrm{dist}\left((y')_\triangle, O(2) \setminus SO(2)\right). \tag{35}$$

For $\triangle \in \bar{\mathcal{C}}_{\varepsilon,3}$ we set $(y')_\triangle = \mathbf{Id}$ and $y' = \tilde{y}$ at the three vertices such that y' is continuous on $\mathrm{int}(\triangle) \setminus V_i$. Here, the set V_i can be taken arbitrarily at first.

We define the interpolation u' for the rescaled displacement field by $u' = \frac{1}{\sqrt{\varepsilon}}(y' - \mathbf{id})$. For future reference we define y'_{V_i} as 'variants' of y' satisfying that for the jump set in some $\triangle \in \bar{\mathcal{C}}_{\varepsilon,3}$ we always choose V_i. We note that by construction also on an edge $[p, q] \subset \partial\triangle$ for $\triangle \in \bar{\mathcal{C}}_\varepsilon$ jumps may occur. There, however, the jump height $|[u_\varepsilon]|$ can be bounded by

$$|[u'_\varepsilon](x)| \le \varepsilon \left\|\nabla u'_\varepsilon\right\|_\infty \le \varepsilon \cdot c\varepsilon^{-\frac{1}{2}} = c\sqrt{\varepsilon} \tag{36}$$

for a constant $c > 0$ independent of ε and $x \in [p, q]$. This holds since the interpolations are continuous at the vertices.

The following lemma shows that we may pass from $\tilde{u}_\varepsilon$ to u'_ε without changing the limit.

Lemma 6.1 *If $u_\varepsilon \to u$ in the sense of Definition 2.4 and $\mathcal{E}(u_\varepsilon)$ is uniformly bounded, then $\chi_{\Omega_\varepsilon} u'_\varepsilon \to u$ in $L^1(\Omega)$, $\chi_{\Omega_\varepsilon} \nabla u'_\varepsilon \rightharpoonup \nabla u$ in $L^2(\Omega)$ and $\mathcal{H}^1(J_{u'_\varepsilon})$ is uniformly bounded.*

Proof. We first note that there is some $M > 0$ such that

$$\#\bar{\mathcal{C}}_\varepsilon \le \frac{M}{\varepsilon} \tag{37}$$

for all $\varepsilon > 0$. To see this, we just recall that every triangle $\triangle \in \bar{\mathcal{C}}_\varepsilon$ provides at least the energy $\varepsilon \inf \{W(r) : r \geq 2\}$. In fact we may assume that $\mathcal{C}_\varepsilon^* = \bar{\mathcal{C}}_\varepsilon$ in Definition 2.4 as for $\Delta \in \mathcal{C}_\varepsilon^* \setminus \bar{\mathcal{C}}_\varepsilon$ we have $|(\tilde{u}_\varepsilon)_\triangle| \leq \frac{C}{\sqrt{\varepsilon}}|(\tilde{y}_\varepsilon)_\triangle - \mathbf{Id}| \leq \frac{C}{\sqrt{\varepsilon}}$ and so

$$\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \cup_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \triangle)} \leq \|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \cup_{\triangle \in \mathcal{C}_\varepsilon^*} \triangle)} + \|\nabla \tilde{u}_\varepsilon\|_{L^2(\cup_{\triangle \in \mathcal{C}_\varepsilon^* \setminus \bar{\mathcal{C}}_\varepsilon} \triangle)}$$
$$C + \left(\#(\mathcal{C}_\varepsilon^* \setminus \bar{\mathcal{C}}_\varepsilon) \frac{\sqrt{3}\varepsilon^2}{4} \cdot \frac{C}{\varepsilon} \right)^{\frac{1}{2}} \leq C.$$

It follows that $\chi_{\Omega_\varepsilon} \nabla u_\varepsilon'$ is bounded uniformly in L^2 and, in particular, equiintegrable. Finally, the jump lengths $\mathcal{H}^1(J_{u_\varepsilon'})$ are readlily seen to be bounded by $C\varepsilon \# \bar{\mathcal{C}}_\varepsilon \leq C$. But then Ambrosio's compactness Theorem for GSBV [2, Theorem 2.2] shows that indeed $\chi_{\Omega_\varepsilon} \nabla u_\varepsilon' \rightharpoonup \nabla u$ in $L^2(\Omega)$. □

Proof of Theorem 2.5. (i) Let $\varepsilon^{-\frac{1}{2}} a_\varepsilon = a \in [0, \infty)$ for all ε. Let $u \in SBV^2(\Omega)$ and consider a sequence $u_\varepsilon \subset SBV^2(\Omega_\varepsilon)$ with $y_\varepsilon = \mathbf{id} + \sqrt{\varepsilon} u_\varepsilon \in \mathcal{A}(a_\varepsilon)$ converging to u in SBV^2 in the sense of Definition 2.4. We split up the energy into bulk and crack part neglecting the contribution $\varepsilon E_\varepsilon^{\text{boundary}}$ from the boundary layers:

$$\begin{aligned}
\mathcal{E}_\varepsilon(u_\varepsilon) &\geq \varepsilon \sum_{\triangle \notin \bar{\mathcal{C}}_\varepsilon} W_\triangle((\tilde{y}_\varepsilon)_\triangle) + \varepsilon \sum_{\triangle \in \bar{\mathcal{C}}_\varepsilon} W_\triangle((\tilde{y}_\varepsilon)_\triangle) \\
&= \frac{4}{\sqrt{3}\varepsilon} \int_{\Omega_\varepsilon} W_\triangle \left(\mathbf{Id} + \sqrt{\varepsilon} \nabla u_\varepsilon' \right) + \varepsilon \sum_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \sum_{\substack{\mathbf{v} \in \mathcal{V}, \\ |(\tilde{y}_\varepsilon)_\triangle \mathbf{v}| > 2}} \frac{1}{2} W \left(|(\tilde{y}_\varepsilon)_\triangle \mathbf{v}| \right) \qquad (38) \\
&=: \mathcal{E}_\varepsilon^{\text{elastic}}(u_\varepsilon) + \mathcal{E}_\varepsilon^{\text{crack}}(u_\varepsilon).
\end{aligned}$$

We note that by contruction of the interpolation u_ε' we may take the integral over Ω_ε. As both parts separate completely in the limit, we discuss them individually.

Elastic energy. We first concern ourselves with the elastic part of the energy. We recall $W_\triangle(\mathbf{Id} + G) = \frac{1}{2}Q(G) + \omega(G)$ with $\sup \left\{ \frac{\omega(F)}{|F|^2}, |F| \leq \rho \right\} \to 0$ as $\rho \to 0$. Let $\chi_\varepsilon(x) := \chi_{[0, \varepsilon^{-1/4})}(|\nabla u_\varepsilon(x)|)$. Note that for $F \in \mathbb{R}^{2\times 2}$, $r > 0$ one has $Q(rF) = r^2 Q(F)$. We compute

$$\mathcal{E}_\varepsilon^{\text{elastic}}(u_\varepsilon) \geq \frac{4}{\sqrt{3}} \int_{\Omega_\varepsilon} \chi_\varepsilon(x) \left(\frac{1}{2} Q(\nabla u_\varepsilon') + \frac{1}{\varepsilon} \omega \left(\sqrt{\varepsilon} \nabla u_\varepsilon'(x) \right) \right) dx.$$

The second term of the integral can be bounded by

$$\chi_\varepsilon |\nabla u_\varepsilon'|^2 \frac{\omega \left(|\sqrt{\varepsilon} \nabla u_\varepsilon'| \right)}{|\sqrt{\varepsilon} \nabla u_\varepsilon'|^2}.$$

Since $\nabla u_\varepsilon'$ is bounded in L^2 and $\chi_\varepsilon \frac{\omega\left(\sqrt{\varepsilon} \nabla u_\varepsilon'\right)}{|\sqrt{\varepsilon} \nabla u_\varepsilon'|^2}$ converges uniformly to 0 as $\varepsilon \to 0$ it follows that

$$\liminf_{\varepsilon\to 0}\mathcal{E}^{\text{elastic}}_{\varepsilon}(u_\varepsilon)\geq \liminf_{\varepsilon\to 0}\frac{4}{\sqrt{3}}\int_{\Omega_\varepsilon}\chi_\varepsilon(x)\frac{1}{2}Q(\nabla u'_\varepsilon(x))\,dx$$
$$\geq \liminf_{\varepsilon\to 0}\frac{4}{\sqrt{3}}\int_{\Omega}\frac{1}{2}Q(\chi_{\Omega_\varepsilon}\chi_\varepsilon(x)\nabla u'_\varepsilon(x))\,dx.$$

By assumption $\chi_{\Omega_\varepsilon}\nabla u'_\varepsilon \rightharpoonup \nabla u$ weakly in L^2. As $\chi_\varepsilon \to 1$ boundedly in measure on Ω, it follows $\chi_{\Omega_\varepsilon}\chi_\varepsilon\nabla u'_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega)$. By lower semicontinuity (Q is convex by Lemma 3.2) we conclude recalling that Q only depends on the symmetric part of the gradient:

$$\liminf_{\varepsilon\to 0}\mathcal{E}^{\text{elastic}}_{\varepsilon}(u_\varepsilon)\geq \frac{4}{\sqrt{3}}\int_{\Omega}\frac{1}{2}Q(e(u(x)))\,dx.$$

Crack energy. By construction the functions u'_ε have jumps on destroyed triangles $\triangle\in\bar{\mathcal{C}}_\varepsilon$. We now write the energy of such a triangle in terms of the jump height $[u]=u^+-u^-$. We first concern ourselves with some $\triangle\in\bar{\mathcal{C}}_{\varepsilon,3}$. For the variant u'_{ε,V_i}, $i=1,2,3$ we consider the springs in $\mathbf{v}_j,\mathbf{v}_k$ direction for $j,k\neq i$. We see that the two parts of the jump set, $h_{\mathbf{v}_j},h_{\mathbf{v}_k}$ do not overlap in the projection onto the one-dimensional hyperplanes spanned by $\mathbf{v}_j^\perp$ and $\mathbf{v}_k^\perp$, respectively. Thus, we compute

$$\varepsilon(\tilde{y}_\varepsilon)_\triangle\,\mathbf{v}_j=\varepsilon(y'_\varepsilon)_\triangle\,\mathbf{v}_j+[y'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}=\varepsilon\mathbf{v}_j+\sqrt{\varepsilon}[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}},\tag{39}$$

where $[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}$ denotes the jump height on the set $h_{\mathbf{v}_k}$. Here and in the following equations, the same holds true if we interchange the roles of j and k. We claim that

$$|(\tilde{y}_\varepsilon)_\triangle\,\mathbf{v}_j|\geq\varepsilon^{\frac{1}{4}}\left|\frac{1}{\sqrt{\varepsilon}}[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}\right|+1.\tag{40}$$

Indeed, for $|\frac{1}{\sqrt{\varepsilon}}[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}|\leq\varepsilon^{-\frac{1}{4}}$ this is clear since $|(\tilde{y}_\varepsilon)_\triangle\,\mathbf{v}_j|\geq 2$. Otherwise, applying (39) we compute for ε small enough:

$$\begin{aligned}|(\tilde{y}_\varepsilon)_\triangle\,\mathbf{v}_j|&=\left|\frac{1}{\sqrt{\varepsilon}}[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}+\mathbf{v}_j\right|\geq\left|\frac{1}{\sqrt{\varepsilon}}[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}\right|-1\\&\geq\varepsilon^{\frac{1}{4}}\left|\frac{1}{\sqrt{\varepsilon}}[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}\right|+\left(1-\varepsilon^{\frac{1}{4}}\right)\varepsilon^{-\frac{1}{4}}-1\geq\varepsilon^{\frac{1}{4}}\left|\frac{1}{\sqrt{\varepsilon}}[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}\right|-2+\varepsilon^{-\frac{1}{4}}\\&\geq\varepsilon^{\frac{1}{4}}\left|\frac{1}{\sqrt{\varepsilon}}[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}\right|+1.\end{aligned}$$

Let $\rho>0$ sufficiently small. Applying Lemma 3.5(iv) there is an increasing subadditive function ψ^ρ with $\psi^\rho(r-1)-\rho\leq W(r)$ for $r\geq 1$. We define $\tilde{\psi}^\rho=\psi^\rho-\rho$. The monotonicity of ψ^ρ and (40) yield

$$W(|(\tilde{y}_\varepsilon)_\triangle\,\mathbf{v}_j|)\geq\tilde{\psi}^\rho(|(\tilde{y}_\varepsilon)_\triangle\,\mathbf{v}_j|-1)\geq\tilde{\psi}^\rho\left(\left|\varepsilon^{-\frac{1}{4}}[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}\right|\right).\tag{41}$$

Now for $\triangle \in \bar{\mathcal{C}}_{\varepsilon,3}$ we may estimate the energy as follows:

$$\begin{aligned} W_{\triangle}\left((\tilde{y}_\varepsilon)_{\triangle}\right) &= \frac{1}{2}\sum_{l=1}^{3} W(|(\tilde{y}_\varepsilon)_{\triangle}\,\mathbf{v}_l|) \\ &\geq \frac{1}{4}\sum_{i=1}^{3}\left\{\tilde{\psi}^{\rho}\left(\varepsilon^{-\frac{1}{4}}|[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_k}}|\right) + \tilde{\psi}^{\rho}\left(\varepsilon^{-\frac{1}{4}}|[u'_{\varepsilon,V_i}]_{h_{\mathbf{v}_j}}|\right)\right\} =: W_{\triangle,3}\left((\tilde{y}_\varepsilon)_{\triangle}\right), \end{aligned}$$

where $i,j,k = 1,2,3$ are pairwise distinct. With $\nu_u^{(i)} = \nu_{u'_{\varepsilon,V_i}}$ we can also write

$$W_{\triangle,3}\left((\tilde{y}_\varepsilon)_{\triangle}\right) = \frac{1}{4}\cdot\frac{2}{\varepsilon}\cdot\frac{2}{\sqrt{3}}\sum_{i=1}^{3}\int_{h_{\mathbf{v}_j}\cup h_{\mathbf{v}_k}} \tilde{\psi}^{\rho}\left(\varepsilon^{-\frac{1}{4}}|[u'_{\varepsilon,V_i}]|\right)\left(|\mathbf{v}_j\cdot\nu_u^{(i)}| + |\mathbf{v}_k\cdot\nu_u^{(i)}|\right)d\mathcal{H}^1.$$

The factors in front occur since $\mathcal{H}^1(h_{\mathbf{v}_j}) = \frac{\varepsilon}{2}$ and, letting ν_j be a normal of $h_{\mathbf{v}_j}$, one has $|\nu_j\cdot\mathbf{v}_j| = 0$ and $|\nu_j\cdot\mathbf{v}_k| = \frac{\sqrt{3}}{2}$. Consequently, defining $\phi_i^{\rho}(r,\nu) = \psi^{\rho}(r)\left(|\mathbf{v}_j\cdot\nu| + |\mathbf{v}_k\cdot\nu|\right)$ and $\tilde{\phi}_i^{\rho}(r,\nu) = \tilde{\psi}^{\rho}(r)\left(|\mathbf{v}_j\cdot\nu| + |\mathbf{v}_k\cdot\nu|\right)$, respectively, we get

$$W_{\triangle,3}\left((\tilde{y}_\varepsilon)_{\triangle}\right) = \frac{1}{\sqrt{3}\varepsilon}\sum_{i=1}^{3}\int_{J_{u'_{\varepsilon,V_i}}\cap\mathrm{int}(\triangle)} \tilde{\phi}_i^{\rho}(\varepsilon^{-\frac{1}{4}}|[u'_{\varepsilon,V_i}]|,\nu_u^{(i)})\,d\mathcal{H}^1,$$

on every $\triangle \in \bar{\mathcal{C}}_{\varepsilon,3}$. For $\triangle \in \bar{\mathcal{C}}_{\varepsilon,2}$ we proceed analogously. Assuming $|(\tilde{y}_\varepsilon)_{\triangle}\,\mathbf{v}_i| \leq 2$ we compute for the springs in $\mathbf{v}_j,\mathbf{v}_k$ direction (abbreviated by $\mathbf{v}_{j,k}$) as in (39)

$$\varepsilon(\tilde{y}_\varepsilon)_{\triangle}\,\mathbf{v}_{j,k} = \varepsilon(y'_\varepsilon)_{\triangle}\,\mathbf{v}_{j,k} + \sqrt{\varepsilon}[u'_\varepsilon]_{h_{\mathbf{v}_i}}. \tag{42}$$

Note that in this case we do not have to take a special variant of u'_ε into account. Repeating the steps (40) and (41) we find

$$\frac{1}{2}\left(W(|(\tilde{y}_\varepsilon)_{\triangle}\,\mathbf{v}_j|) + W(|(\tilde{y}_\varepsilon)_{\triangle}\,\mathbf{v}_k|)\right) \geq \tilde{\psi}^{\rho}\left(\varepsilon^{-\frac{1}{4}}|[u'_\varepsilon]_{h_{\mathbf{v}_i}}|\right) =: W_{\triangle,2}\left((\tilde{y}_\varepsilon)_{\triangle}\right).$$

Noting that $|\mathbf{v}_j\cdot\nu_i| = |\mathbf{v}_k\cdot\nu_i| = \frac{\sqrt{3}}{2}$, $|\mathbf{v}_i\cdot\nu_i| = 0$ and that every of these terms occurs twice in the sum of the right hand side of the following formula, it is not hard to see that this energy satsifies the same integral representation formula as $W_{\triangle,3}$:

$$W_{\triangle,2}\left((\tilde{y}_\varepsilon)_{\triangle}\right) = \frac{1}{\sqrt{3}\varepsilon}\sum_{i=1}^{3}\int_{J_{u'_{\varepsilon,V_i}}\cap\mathrm{int}(\triangle)} \tilde{\phi}_i^{\rho}(\varepsilon^{-\frac{1}{4}}|[u'_{\varepsilon,V_i}]|,\nu_u^{(i)})\,d\mathcal{H}^1.$$

Let $\sigma > 0$. Then for ε sufficiently small the crack energy can be estimated by

$$\begin{aligned} \mathcal{E}_\varepsilon^{\mathrm{crack}}(u_\varepsilon) &\geq \frac{1}{\sqrt{3}}\sum_i\int_{J_{u'_{\varepsilon,V_i}}\cap\Omega_\varepsilon} \tilde{\phi}_i^{\rho}(\varepsilon^{-\frac{1}{4}}|[u'_{\varepsilon,V_i}]|,\nu_u^{(i)})\,d\mathcal{H}^1 - E^{\rho}_{\varepsilon,\cup\partial\triangle}(\tilde{y}_\varepsilon) \\ &\geq \frac{1}{\sqrt{3}}\sum_i\int_{J_{u'_{\varepsilon,V_i}}\cap\Omega_\varepsilon}\left(\phi_i^{\rho}(\sigma^{-1}|[u'_{\varepsilon,V_i}]|,\nu_u^{(i)}) - \rho\right)d\mathcal{H}^1 - E^{\rho}_{\varepsilon,\cup\partial\triangle}(\tilde{y}_\varepsilon), \end{aligned}$$

where $E^{\rho}_{\varepsilon,\cup\partial\triangle}(\tilde{y}_\varepsilon)$ compensates for the extra contribution provided by jumps lying on the boundary of some $\triangle \in \bar{\mathcal{C}}_\varepsilon$. We will show that this term vanishes in the limit.

Now by construction the $\phi_i^\rho(r,\nu)$, $i = 1,2,3$, are products of a positive, increasing and concave in function in r and a norm in ν. Moreover, u'_ε and its variants converge to u in L^1 with $\nabla u'_\varepsilon$ bounded in L^2 and thus equiintegrable. By Ambrosio's lower semicontinuity Theorem [2, Theorem 3.7] we obtain

$$\liminf_{\varepsilon\to 0} \mathcal{E}^{\mathrm{crack}}_\varepsilon(u_\varepsilon) \geq \frac{1}{\sqrt{3}} \int_{J_u} \sum_i \phi_i^\rho(\sigma^{-1}|[u]|, \nu_u)\, d\mathcal{H}^1 - CM\rho - \limsup_{\varepsilon\to 0} E^{\rho}_{\varepsilon,\cup\partial\triangle}(\tilde{y}_\varepsilon)\,,$$

where we used that $\sup_\varepsilon \mathcal{H}^1(J_{u'_\varepsilon}) \leq CM$ for a constant $C > 0$ by (37). We recall that $\psi^\rho(r) \to \beta$ for $r \to \infty$. Passing to the limit $\sigma \to 0$ this yields

$$\liminf_{\varepsilon\to 0} \mathcal{E}^{\mathrm{crack}}_\varepsilon(u_\varepsilon) \geq \frac{1}{\sqrt{3}} \int_{J_u} 2\beta \sum_{\mathbf{v}\in\mathcal{V}} |\mathbf{v}\cdot\nu_u|\, d\mathcal{H}^1 - CM\rho - \limsup_{\varepsilon\to 0} E^{\rho}_{\varepsilon,\cup\partial\triangle}(\tilde{y}_\varepsilon)\,. \quad (43)$$

Taking (36) and (37) into account we compute

$$\begin{aligned}\limsup_{\varepsilon\to 0} \sum_{\triangle\in\bar{\mathcal{C}}_\varepsilon} \int_{\partial\triangle} |\tilde{\psi}^\rho\left(\varepsilon^{-\frac{1}{4}}|[u'_\varepsilon]|\right)| &\leq \lim_{\varepsilon\to 0} CM \sup\left\{|\psi^\rho(r) - \rho| : \, r \leq \varepsilon^{-\frac{1}{4}}\cdot c\varepsilon^{\frac{1}{2}}\right\} \\ &= CM\rho.\end{aligned}$$

This proves $\limsup_\varepsilon |E^{\rho}_{\varepsilon,\cup\partial\triangle}(\tilde{y}_\varepsilon)| \leq \tilde{C}M\rho$ for some $\tilde{C} > 0$. We let $\rho \to 0$ in (43). This finishes the proof of (i).

(ii) The basic tool for the proof of the Γ-limsup-inequality is a density result for SBV functions due to Cortesani and Toader [17]. We suppose $\mathcal{W}(\Omega,\mathbb{R}^2)$ is the space of all SBV functions $u \in SBV(\Omega,\mathbb{R}^2)$ such that J_u is a finite, disjoint union of segments and $u \in W^{k,\infty}(\Omega\setminus J_u,\mathbb{R}^2)$ for all k. Then $\mathcal{W}(\Omega,\mathbb{R}^2)$ is dense in $SBV^2(\Omega,\mathbb{R}^2)\cap L^\infty(\Omega,\mathbb{R}^2)$ in the following way: For every $u \in SBV^2(\Omega,\mathbb{R}^2)\cap L^\infty(\Omega,\mathbb{R}^2)$ there exists a sequence $u_\varepsilon \in \mathcal{W}(\Omega,\mathbb{R}^2)$ such that $\|u_\varepsilon\|_\infty \leq \|u\|_\infty$ and

(i) $u_\varepsilon \to u$ strongly in $L^1(\Omega,\mathbb{R}^2)$, $\nabla u_\varepsilon \to \nabla u$ strongly in $L^2(\Omega,\mathbb{R}^2)$,

(ii) $\limsup_{\varepsilon\to 0}\int_{J_{u_\varepsilon}} \phi(\nu_{u_\varepsilon})d\mathcal{H}^1 \leq \int_{J_u}\phi(\nu_u)d\mathcal{H}^1$ for every upper semicontinuous function $\phi : S^1 \to [0,\infty)$ satisfying $\phi(\nu) = \phi(-\nu)$ for every $\nu \in S^1$.

In the following a problem arises as we cannot control the boundary values of such an approximating sequence. To overcome this difficulty, similar to the implementation of boundary values for the discrete configurations, we consider functions attaining the boundary data in small stripes at the boundary.

Let $\eta > 0$. We first construct a recovery sequence for $u \in \mathcal{W}(\Omega)$ with $u_1(x_1,\cdot) = 0$ for $0 < x_1 < \eta$ and $u_1(x_1,\cdot) = al$ for $l - \eta < x_1 < l$. We define $u_\varepsilon(x) = u(x)$ for $x \in \mathcal{L}_\varepsilon \cap \Omega$ and let $y_\varepsilon(x) = \mathbf{id} + \sqrt{\varepsilon}u_\varepsilon(x)$. Then $y_\varepsilon \in \mathcal{A}(a_\varepsilon)$

for all ε (here in the sense of (4)). By $\tilde{u}_\varepsilon, u'_\varepsilon$ we again denote the interpolations on Ω_ε.

J_u is the union of disjoint segments. Up to considering the translated lattice $\mathcal{L}_\varepsilon + (\xi_\varepsilon, 0)$ for appropriate $\xi_\varepsilon \to 0$ and passing to a slightly smaller η, we may assume that $J_u \cap \mathcal{L}_\varepsilon = \emptyset$. Let $\delta > 0$ and define $J_u^\delta = \{x \in J_u, |[u](x)| \geq \delta\}$. Let $\mathcal{D}_\varepsilon$ and $\mathcal{D}_\varepsilon^\delta$ be the sets of triangles where J_u and J_u^δ, respectively, cross at least one side of the triangle (i.e., J_u (J_u^δ respectively) and the side have nonempty intersection). Then

$$\#\mathcal{D}_\varepsilon^\delta \leq \#\mathcal{D}_\varepsilon \leq \frac{C\mathcal{H}^1(J_u)}{\varepsilon} + C\#(J_u) \tag{44}$$

for a constant $C > 0$ independent of $u \in \mathcal{W}(\Omega, \mathbb{R}^2)$ and ε, where $\#(J_u)$ denotes the (smallest) number of disjoint segments whose union gives J_u. From now on for the local nature of the arguments we may assume that J_u consists of one segment only. We show

$$\mathcal{D}_\varepsilon^\delta \subset \bar{\mathcal{C}}_\varepsilon \subset \mathcal{D}_\varepsilon. \tag{45}$$

for ε small enough. Let $\triangle \in \mathcal{D}_\varepsilon^\delta$. We see that, if J_u^δ crosses a spring $\mathbf{v}$ at point x_*, say, then a computation similar as in (42) together with $\nabla u \in L^\infty$ shows

$$|(\tilde{y}_\varepsilon)_\triangle \, \mathbf{v}| = \left| \frac{1}{\sqrt{\varepsilon}}[u(x_*)] + O(1) \right| \geq \frac{\delta}{\sqrt{\varepsilon}} + O(1) \tag{46}$$

Thus, $\triangle \in \bar{\mathcal{C}}_\varepsilon$ for ε small enough. On the other hand, if we assume $\triangle \notin \mathcal{D}_\varepsilon$, then for at least two springs $\mathbf{v} \in \mathcal{V}$ we have $|(\tilde{y}_\varepsilon)_\triangle \, \mathbf{v}| \leq \|\mathbf{Id} + \sqrt{\varepsilon}\nabla u\|_\infty < 2$ for ε small enough leading to $\triangle \notin \bar{\mathcal{C}}_\varepsilon$. In particular, it is not hard to see that $\bar{\mathcal{C}}_\varepsilon = \bar{\mathcal{C}}_{\varepsilon,2}$.

We claim that

$$|(u'_\varepsilon)_\triangle| = O(1) \quad \text{ for } \triangle \notin \mathcal{D}_\varepsilon \setminus \bar{\mathcal{C}}_\varepsilon. \tag{47}$$

This is clear for $\triangle \notin \mathcal{D}_\varepsilon$ as $\nabla u \in L^\infty$. For $\triangle \in \bar{\mathcal{C}}_\varepsilon = \bar{\mathcal{C}}_{\varepsilon,2}$ there is a $\mathbf{v} \in \mathcal{V}$ such that $(y'_\varepsilon)_\triangle \, \mathbf{v} = (\tilde{y}_\varepsilon)_\triangle \, \mathbf{v} = \mathbf{v} + O(\sqrt{\varepsilon})$. By Lemma 3.5(i) and (35) we get a rotation $R_\varepsilon \in SO(2)$ such that

$$|R_\varepsilon - (y'_\varepsilon)_\triangle|^2 = \mathrm{dist}^2((y'_\varepsilon)_\triangle, SO(2)) = \mathrm{dist}^2((y'_\varepsilon)_\triangle, O(2)) \leq CW_\triangle((y'_\varepsilon)_\triangle) = O(\varepsilon).$$

This yields $|(y'_\varepsilon)_\triangle - \mathbf{Id}| = O(\sqrt{\varepsilon})$ and thus $|(u'_\varepsilon)_\triangle| = O(1)$.

We note that $\chi_{\Omega_\varepsilon}\tilde{u}_\varepsilon \to u$ in L^1 as u and thus every $\tilde{u}_\varepsilon$ is bounded uniformly in L^∞ and, u being smooth away from J_u, $\tilde{u}_\varepsilon \to u$ uniformly on $\Omega_\varepsilon \setminus \bigcup_{\triangle \in \mathcal{D}_\varepsilon} \triangle$, where $|\bigcup_{\triangle \in \mathcal{D}_\varepsilon} \triangle| \leq C\varepsilon$. Letting $\mathcal{C}_\varepsilon^* = \mathcal{D}_\varepsilon$ this shows that $u_\varepsilon \to u$ in the sense of Definition 2.4 recalling (44) and the fact that $|(\tilde{u}_\varepsilon)_\triangle| = O(1)$ for $\triangle \notin \mathcal{D}_\varepsilon$. We next establish an even stronger convergence of the derivatives. Consider $\nabla \tilde{u}_\varepsilon$ on triangles in $\mathcal{C}_\varepsilon \setminus \mathcal{D}_\varepsilon$. As u is smooth there, the oscillation on such a triangle, $\mathrm{osc}_\varepsilon^\triangle(\nabla u) := \sup\{\|\nabla u(x) - \nabla u(x')\|_\infty, x, x' \in \triangle\}$, tends uniformly to zero (i.e., not depending on the choice of the triangle). We thus obtain

$$\int_{\Omega_\varepsilon \setminus \cup_{\triangle \in \mathcal{D}_\varepsilon} \triangle} \|\nabla \tilde{u}_\varepsilon - \nabla u\|_\infty^2 \leq \int_{\Omega_\varepsilon \setminus \cup_{\triangle \in \mathcal{D}_\varepsilon} \triangle} (\mathrm{osc}_\varepsilon^\triangle(\nabla u))^2 \to 0$$

for $\varepsilon \to 0$, so that even $\chi_{\Omega_\varepsilon \setminus \bigcup_{\triangle \in \mathcal{D}_\varepsilon} \triangle} \nabla \tilde{u}_\varepsilon \to \nabla u$ strongly in $L^2(\Omega)$. Note that in fact $\chi_{\Omega_\varepsilon} \nabla u'_\varepsilon \to \nabla u$ in $L^2(\Omega)$. Indeed, on the set of broken triangles we get

$$\begin{aligned}\int_{\bigcup_{\triangle \in \mathcal{D}_\varepsilon} \triangle} |\nabla u'_\varepsilon - \nabla u|^2 &\le \int_{\bigcup_{\triangle \in \bar{\mathcal{C}}_\varepsilon} \triangle} |\nabla u'_\varepsilon - \nabla u|^2 + \int_{\bigcup_{\triangle \in \mathcal{D}_\varepsilon \setminus \bar{\mathcal{C}}_\varepsilon} \triangle} |\nabla u'_\varepsilon - \nabla u|^2 \\ &\le C\#\bar{\mathcal{C}}_\varepsilon \varepsilon^2 + C\#(\mathcal{D}_\varepsilon \setminus \bar{\mathcal{C}}_\varepsilon)\varepsilon^2 \cdot (\frac{C}{\sqrt{\varepsilon}})^2\end{aligned}$$

Using (47), (45) and $\#\mathcal{D}_\varepsilon \le C\varepsilon^{-1}$ this leads to

$$\limsup_{\varepsilon\to 0} \int_{\bigcup_{\triangle \in \mathcal{D}_\varepsilon} \triangle} |\nabla u'_\varepsilon - \nabla u|^2 \le \limsup_{\varepsilon \to 0} C\#(\mathcal{D}_\varepsilon \setminus \mathcal{D}^\delta_\varepsilon)\varepsilon \le C\mathcal{H}^1(J_u \setminus J^\delta_u)$$

and letting $\delta \to 0$ yields the claim.

We now split up the energy in bulk and crack parts

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}^{\text{elastic}}_\varepsilon(u_\varepsilon) + \mathcal{E}^{\text{crack}}_\varepsilon(u_\varepsilon) + O(\varepsilon)$$

as defined in (38) (Note that the contribution $\varepsilon E^{\text{boundary}}_\varepsilon$ is of order $O(\varepsilon)$). Repeating the steps in the elastic energy estimate in (i), applying $\chi_{\Omega_\varepsilon} \nabla u'_\varepsilon \to \nabla u$ strongly in $L^2(\Omega)$ and $Q(F) \le C|F|^2$ for a constant $C > 0$ we conclude that

$$\limsup_{\varepsilon\to 0} \mathcal{E}^{\text{elastic}}_\varepsilon(u_\varepsilon) = \frac{4}{\sqrt{3}} \int_\Omega \frac{1}{2} Q(e(u(x)))\, dx. \tag{48}$$

It is elementary to see that J_u crosses

$$\mathcal{H}^1(J_u) \frac{2|\nu_u \cdot \mathbf{v}|}{\sqrt{3}\varepsilon} + O(1) \tag{49}$$

springs in $\mathbf{v}$-direction for $\mathbf{v} \in \mathcal{V}$. It is not restrictive to suppose that J^δ_u consists of only one segment. Indeed, by the continuity of $[u]$ in J_u we see that $J^\delta(u)$ is the union of open intervals. Thus, up to a set of arbitrarily small size, $J^\delta(u)$ consists of a finite number of open intervals and then it suffices to consider one of these intervals. Consequently, (49) also holds for the subset J^δ_u replacing $\mathcal{H}^1(J_u)$ by $\mathcal{H}^1(J^\delta_u)$. Recalling (46), the crack energy may be estimated by

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \mathcal{E}^{\text{crack}}_\varepsilon(u_\varepsilon) \\ &\le \limsup_{\varepsilon\to 0} \Big(\mathcal{H}^1(J^\delta_u)\ \sup\Big\{W(r) : r \ge \delta\varepsilon^{-\frac{1}{2}} + O(1)\Big\} \\ &\qquad\qquad + \mathcal{H}^1(J_u \setminus J^\delta_u)\ \max\{W(r) : r \ge 1\}\Big) \frac{2}{\sqrt{3}} \sum_{\mathbf{v}\in\mathcal{V}} |\nu_u \cdot \mathbf{v}| + O(\varepsilon) \\ &= \Big(\mathcal{H}^1(J^\delta_u)\ \beta + \mathcal{H}^1(J_u \setminus J^\delta_u)\ \max_{r\ge 1} W(r) \Big)\ \frac{2}{\sqrt{3}} \sum_{\mathbf{v}\in\mathcal{V}} |\nu_u \cdot \mathbf{v}|.\end{aligned}$$

We finally pass to the limit $\delta \to 0$ to obtain

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon^{\mathrm{crack}}(u_\varepsilon) \le \int_{J_u} \sum_{\mathbf{v}\in\mathcal{V}} \frac{2\beta}{\sqrt{3}} |\nu_u \cdot \mathbf{v}| \, d\mathcal{H}^1. \tag{50}$$

Applying (48) and (50) then shows that u_ε is a recovery sequence for $u \in \mathcal{W}(\Omega)$ with $u_1(x_1, \cdot) = 0$ for $0 < x_1 < \eta$ and $u_1(x_1, \cdot) = al$ for $l - \eta < x_1 < l$. Now let $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ satisfying the boundary conditions $u_1(0, \cdot) = 0$ and $u_1(l, \cdot) = la$. Define $u^\eta \in SBV^2(\Omega)$ by

$$u^\eta(x) = \begin{cases} (0, u_2(2\eta - x_1, x_2)) & \text{for } 0 < x_1 < 2\eta, \\ u(\frac{l}{l-4\eta} x_1 - \frac{2l\eta}{l-4\eta}, x_2) & \text{for } 2\eta \le x_1 \le l - 2\eta, \\ (la, u_2(2l - 2\eta - x_1, x_2)) & \text{for } l - 2\eta < x_1 < l. \end{cases}$$

We approximate u^η with a sequence $u^{\eta,j} \in \mathcal{W}(\Omega)$ such that $\limsup_{j\to\infty} \mathcal{E}(u^{\eta,j}) \le \mathcal{E}(u^\eta)$. Due to the proof of the density result in [17] the approximating sequence can be chosen in such a way that $u_1^{\eta,j}$ equals u_1^η in $(0,\eta)\times(0,1)$ and $(l-\eta,l)\times(0,1)$ as u_1^η is constant in $(0,2\eta)\times(0,1)$ and $(l-2\eta,l)\times(0,1)$. Consequently, we may assume that $u_1^{\eta,j}(x_1,\cdot) = 0$ for $0 < x_1 < \eta$ and $u_1^{\eta,j}(x_1,\cdot) = al$ for $l - \eta < x_1 < l$ for j large enough.

Let $(\varepsilon_i)_i$ be an arbitrary null sequence. According to the above computation for j large enough let $(u_{\varepsilon_i}^{\eta,j})_i$ be a recovery sequence for $u^{\eta,j}$ satisfying $u_{\varepsilon_i}^{\eta,j} \in \mathcal{A}(a_{\varepsilon_i})$ for $\varepsilon_i \ll \eta$. In particular, we have $\chi_{\Omega_{\varepsilon_i}} (\tilde{u})_{\varepsilon_i}^{\eta,j} \to u^{\eta,j}$ in L^1 and $\chi_{\Omega_{\varepsilon_i}} \nabla (u')_{\varepsilon_i}^{\eta,j} \to \nabla u^{\eta,j}$ in L^2 for $i \to \infty$ (as before, $(\tilde{u})_{\varepsilon_i}^{\eta,j}$ and $(u')_{\varepsilon_i}^{\eta,j}$ denote the different interpolations of $u_{\varepsilon_i}^{\eta,j}$). Choose $j(i)$ for $i \in \mathbb{N}$ such that $u_i^* := u_{\varepsilon_i}^{\eta,j(i)} \in \mathcal{A}(a_{\varepsilon_i})$ satisfies $\chi_{\Omega_{\varepsilon_i}} (\tilde{u})_i^* \to u^\eta$ in L^1, $\chi_{\Omega_{\varepsilon_i}} \nabla (u')_i^* \to \nabla u^\eta$ in L^2 for $i \to \infty$ and in addition $\mathcal{H}^1(J_{u^{\eta,j(i)}}) \le C\mathcal{H}^1(J_{u^\eta})$ and $\#(J_{u^{\eta,j(i)}}) \le \frac{C}{\varepsilon_i}$. Observing that $(\tilde{u})_i^* \neq (u')_i^*$ only on a set of triangles $\mathcal{D}_i^*$ with

$$\#\mathcal{D}_i^* \le \frac{C\mathcal{H}^1(J_{u^{\eta,j(i)}})}{\varepsilon_i} + C\#(J_{u^{\eta,j(i)}}) \le \frac{C\mathcal{H}^1(J_{u^\eta})}{\varepsilon_i} + \frac{C}{\varepsilon_i} \le \frac{C}{\varepsilon_i} \tag{51}$$

by (44) we conclude that $\chi_{\Omega_{\varepsilon_i}} (\tilde{u})_i^* \to u^\eta$ in the sense of Definition 2.4. Since

$$\limsup_{i\to\infty} \mathcal{E}_{\varepsilon_i}(u_i^*) \le \mathcal{E}(u^\eta)$$

u_i^* is a recovery sequence for u^η. Repeating the above arguments we can find a sequence $\eta_i \to 0$ with $\varepsilon_i \ll \eta_i$ such that $u_i^{**} := u_{\varepsilon_i}^{\eta_i,j(i)} \in \mathcal{A}(a_{\varepsilon_i})$ and $u_{\varepsilon_i}^{**} \to u$ in the sense of Definition 2.4 (in particular, we choose η_i such that $\mathcal{H}^1(J_{u^{\eta_i,j(i)}}) \le C\mathcal{H}^1(J_{u^{\eta_i}}) \le C\mathcal{H}^1(J_u)$ holds and therefore the argument in (51) can be applied). Moreover, it is not hard to see that $\mathcal{E}(u^\eta) \to \mathcal{E}(u)$ for $\eta \to 0$ as $\mathcal{H}^1(J_{u^\eta} \cap (0,4\eta) \times (0,1)) + \mathcal{H}^1(J_{u^\eta} \cap (l-4\eta,l) \times (0,1)) \to 0$ for $\eta \to 0$ due to the construction of u^η. Then

$$\limsup_{i\to\infty} \mathcal{E}_i(u_i^{**}) \le \mathcal{E}(u).$$

This together with the arbitrariness of $(\varepsilon_i)_i$ shows the existence of a recovery sequence for $u \in SBV^2(\Omega)\cap L^\infty(\Omega)$. Finally, we may drop the hypothesis $u \in L^\infty$ by applying a truncation argument and taking $Q(F) \le C|F|^2$ into account. $\square$

Analysis of the limiting variational problem

We finally give the proof of Theorem 2.6, i.e., we determine the minimizers of the limiting functional $\mathcal{E}$ directly. An analogous result for isotropic energy functionals has been obtained in [27]. We thus do not repeat all the steps of the proof provided in [27] but rather concentrate on the additional arguments necessary to handle anisotropic surface contributions.

Proof of Theorem 2.6. Let $\mathbf{v}_\gamma$ be unique and thus $P(\gamma) \neq 0$, $P(\gamma)$ as in (7). We first establish a lower bound for the energy $\mathcal{E}$. An elementary computation yields

$$\begin{aligned}\sum_{\mathbf{v}\in\mathcal{V}} |\mathbf{v}\cdot\nu| &\ge |\mathbf{v}_\gamma\cdot\nu| + \sqrt{3}|\mathbf{v}_\gamma^\perp\cdot\nu| = |\mathbf{v}_\gamma\cdot\nu| + \sqrt{3}\left|\pm\frac{1}{\gamma}\mathbf{e}_1\cdot\nu \pm \frac{\sqrt{1-\gamma^2}}{\gamma}\mathbf{v}_\gamma\cdot\nu\right| \\ &\ge \frac{\sqrt{3}}{\gamma}|\mathbf{e}_1\cdot\nu| + 2P(\gamma)|\mathbf{v}_\gamma\cdot\nu|\end{aligned}$$

for $\nu \in S^1$. In the first step we used that $\sum_{\mathbf{v}\in\mathcal{V}\setminus\{\mathbf{v}_\gamma\}}\mathbf{v} = \pm\sqrt{3}\mathbf{v}_\gamma^\perp$. Thus, we get

$$\mathcal{E}(u) \ge \frac{4}{\sqrt{3}}\int_\Omega \frac{1}{2}Q(e(u(x)))\,dx + \int_{J_u}\frac{2\beta}{\gamma}|\mathbf{e}_1\cdot\nu_u| + \frac{4\beta}{\sqrt{3}}P(\gamma)|\mathbf{v}_\gamma\cdot\nu_u|\,d\mathcal{H}^1.$$

Using the slicing method (see, e.g., [3, Section 3.11]) and applying properties of the reduced energy proved in Lemma 3.3 we get

$$\mathcal{E}(u) \ge \int_0^1\left(\int_0^l \frac{\alpha}{\sqrt{3}}\left(\mathbf{e}_1^T\nabla u(x_1,x_2)\mathbf{e}_1 \vee 0\right)^2 dx_1 + \frac{2\beta}{\gamma}\#S^{x_2}(u)\right)dx_2 + \mathcal{E}^\gamma(u),$$

where $\#S^{x_2}$ denotes the number of jumps on a slice $(0,l)\times\{x_2\}$ and

$$\mathcal{E}^\gamma(u) = \int_{J_u}\frac{4\beta}{\sqrt{3}}P(\gamma)|\mathbf{v}_\gamma\cdot\nu|\,d\mathcal{H}^1.$$

It is easy to see that the energy of a slice is larger or equal to $\min\left\{\alpha l a^2/\sqrt{3}, 2\beta/\gamma\right\}$ leading to $\inf\mathcal{E} \ge \min\left\{\alpha l a^2/\sqrt{3}, 2\beta/\gamma\right\}$. Testing with u^{el} and u^{cr} depending on a, we see that this bound is attained and that u^{el} and u^{cr}, respectively, is a minimizer. It remains to prove uniqueness:

(i) Let $a < a_{\mathrm{crit}}$ and u be a minimizer of $\mathcal{E}$. Then u has no jump on a.e. slice $(0,l)\times\{x_2\}$ and satisfies a.e. $\mathbf{e}_1^T\nabla u\,\mathbf{e}_1 = a$ by the strict convexity of the mapping $t\mapsto(t\vee 0)^2$ on $(0,\infty)$. Thus, if $J_u\neq\emptyset$, the crack normal must satisfy a.e. $\nu_u = \pm\mathbf{e}_2$. Taking $\mathcal{E}^\gamma(u)$ into account, we then may assume $J_u=\emptyset$ up to an $\mathcal{H}^1$ negligible set, i.e., $u \in H^1(\Omega)$. We find $u_1(x_1, x_2) = ax_1 + f(x_2)$ a.e. for a suitable function f, and the boundary condition $u_1(0, x_2) = 0$ yields $f = 0$ a.e. In particular, $\mathbf{e}_1^T \nabla u \, \mathbf{e}_2 = 0$ a.e. Applying strict convexity of Q on symmetric matrices (Lemma 3.2) we now observe $\mathbf{e}_2^T \nabla u \, \mathbf{e}_2 = -\frac{a}{3}$ and $\mathbf{e}_1^T \nabla u \, \mathbf{e}_2 + \mathbf{e}_2^T \nabla u \, \mathbf{e}_1 = 0$ a.e. So the derivative has the form

$$\nabla u(x) = \begin{pmatrix} a & 0 \\ 0 & -\frac{a}{3} \end{pmatrix} \text{ for a.e. } x.$$

Since Ω is connected, we conclude $u(x) = (0, s) + F^a x = u^{\mathrm{el}}(x)$ a.e.

(ii) Let $a > a_{\mathrm{crit}}$ and u be a minimizer of $\mathcal{E}$. We again consider the lower bound for the energy $\mathcal{E}$ and now obtain that on a.e. slice $(0, l) \times \{x_2\}$ a minimizer u has one jump and a.e. $\mathbf{e}_1^T \nabla u \, \mathbf{e}_1 = 0$. The arguments in (i) show that ∇u is antisymmetric a.e. Now the linearized rigidity estimate for SBD functions of Chambolle, Giacomini and Ponsiglione [16] yields that there is a Caccioppoli partition (E_i) of Ω such that

$$u(x) = \sum_i (A_i x + b_i) \chi_{E_i} \quad \text{and} \quad J_u = \bigcup_i \partial^* E_i,$$

where $A_i^T = -A_i \in \mathbb{R}^{2 \times 2}$ and $b_i \in \mathbb{R}^2$. (See [3] for the definition and basic properties of Caccioppoli partitions.) As $\mathcal{E}^\gamma(u) = 0$, we also note that $\nu_u \perp \mathbf{v}_\gamma$ a.e. on J_u. Following the arguments in [27], in particular using regularity results for boundary curves of sets of finite perimeter and exhausting the sets $\partial^* E_i$ with Jordan curves, we find that

$$J_u = \bigcup_i \partial^* E_i \subset (p, 0) + \mathbb{R}\mathbf{v}_\gamma$$

for some p such that $(p, 0) + \mathbb{R}\mathbf{v}_\gamma$ intersects both segments $(0, l) \times \{0\}$ and $(0, l) \times \{l\}$. We thus obtain that (E_i) consists of only two sets: E_1 to the left and E_2 to the right of $(p, 0) + \mathbb{R}\mathbf{v}_\gamma$, say. Due to the boundary conditions we conclude that $A_1 = A_2 = 0$ and $b_1 = (0, s)$, $b_2 = (al, t)$ for suitable $s, t \in \mathbb{R}$. $\square$

References

[1] R. Alicandro, M. Focardi, M. S. Gelli. *Finite-difference approximation of energies in fracture mechanics.* Ann. Scuola Norm. Sup. **29** (2000), 671–709.

[2] L. Ambrosio. *Existence theory for a new class of variational problems.* Arch. Ration. Mech. Anal. **111** (1990), 291–322.

[3] L. Ambrosio, N. Fusco, D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford University Press, Oxford 2000.

[4] L. Ambrosio, V. M. Tortorelli. *On the approximation of free discontinuity problems.* Boll. Un. Mat. Ital. B **7** (1992), 105–123.

[5] X. Blanc, C. Le Bris, P.-L. Lions. *From molecular models to continuum mechanics.* Arch. Ration. Mech. Anal. **164** (2002), 341–381.

[6] B. Bourdin, G. A. Francfort, J. J. Marigo. *Numerical experiments in revisited brittle fracture.* J. Mech. Phys. Solids **48** (2000), 797–826.

[7] A. Braides. *Γ-convergence for Beginners.* Oxford University Press, Oxford 2002.

[8] A. Braides. *Non-local variational limits of discrete systems.* Commun. Contemp. Math. **2** (2000), 285–297.

[9] A. Braides, M. Cicalese. *Surface energies in nonconvex discrete systems.* Math. Models Methods Appl. Sci. **17** (2007), 985–1037.

[10] A. Braides, G. Dal Maso, A. Garroni. *Variational formulation of softening phenomena in fracture mechanics. The one-dimensional case.* Arch. Ration. Mech. Anal. **146** (1999), 23–58.

[11] A. Braides, M. S. Gelli. *Limits of discrete systems without convexity hypotheses.* Math. Mech. Solids **7** (2002), 41–66.

[12] A. Braides, M. S. Gelli. *Limits of discrete systems with long-range interactions.* J. Convex Anal. **9** (2002), 363–399.

[13] A. Braides, A. Lew, M. Ortiz. *Effective cohesive behavior of layers of interatomic planes.* Arch. Ration. Mech. Anal. **180** (2006), 151–182.

[14] A. Braides, M. Solci, E. Vitali. *A derivation of linear elastic energies from pair-interaction atomistic systems.* Netw. Heterog. Media **2** (2007), 551–567.

[15] G. Buttazzo. *Energies on BV and variational models in fracture mechanics.* Proceedings of "'Curvautre Flows and Related Topics"', Levico 27 June-2 July 1994, Gakkotosho, Tokyo 1995.

[16] A. Chambolle, A. Giacomini, M. Ponsiglione. *Piecewise rigidity.* J. Funct. Anal. Solids **244** (2007), 134–153.

[17] G. Cortesani, R. Toader. *A density result in SBV with respect to non-isotropic energies.* Nonlinear Analysis **38** (1999), 585–604.

[18] G. Dal Maso. *An introduction to Γ-convergence.* Birkhäuser, Boston · Basel · Berlin 1993.

[19] G. Dal Maso, G. A. Francfort, R. Toader. *Quasistatic crack growth in nonlinear elasticity.* Arch. Ration. Mech. Anal. **176** (2005), 165–225.

[20] G. Dal Maso, R. Toader. *A model for quasi-static growth of brittle materials: Existence and approximation results.* Arch. Ration. Mech. Anal. **162** (2002), 101–135.

[21] E. De Giorgi, L. Ambrosio. *Un nuovo funzionale del calcolo delle variazioni.* Acc. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. **82** (1988), 199–210.

[22] G. A. Francfort, C, J. Larsen. *Existence and convergence for quasi-static evolution in brittle fracture.* Comm. Pure Appl. Math. **56** (2003), 1465–1500.

[23] G. A. Francfort, J, J. Marigo. *Revisiting brittle fracture as an energy minimization problem.* J. Mech. Phys. Solids **46** (1998), 1319–1342.

[24] G. Friesecke, R. D. James, S. Müller. *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity.* Comm. Pure Appl. Math. **55** (2002), 1461–1506.

[25] G. Friesecke, R. D. James, S. Müller. *A hierarchy of plate models derived from nonlinear elasticity by* Γ*-convergence.* Arch. Ration. Mech. Anal. **180** (2006), 183–236.

[26] A. A. Griffith. *The phenomena of rupture and flow in solids.* Philos. Trans. R. Soc. London **221** (1921), 163–198.

[27] C. Mora-Corral. *Explicit energy-minimizers of incompressible elastic brittle bars under uniaxial extension.* C. R. Acad. Sci. Paris **348** (2010), 1045–1048.

[28] M. Negri. *Finite element approximation of the Griffith's model in fracture mechanics.* Numer. Math. **95** (2003), 653–687.

[29] M. Negri. *A discontinuous finite element approximation of free discontinuity probems.* Adv. Math. Sci. Appl. **15** (2005), 283–306.

[30] B. Schmidt. *Linear* Γ*-limits of multiwell energies in nonlinear elasticity theory.* Continuum Mech. Thermodyn. **20** (2008) 375–396.

[31] B. Schmidt. *On the derivation of linear elasticity from atomistic models.* Netw. Heterog. Media **4** (2009), 789–812.

[32] B. Schmidt, F. Fraternali, M. Ortiz. *Eigenfracture: an eigendeformation approach to variational fracture.* SIAM Mult. Model. Simul. **7** (2009), 1237 – 1266.