

# Adaptive Space-Time Finite Element Approximations of Parabolic Optimal Control Problems

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# Abstract

We consider adaptive space-time finite element approximations of parabolic optimal control problems with distributed controls based on an approach where the optimality system is stated as a fourth order elliptic boundary value problem. The numerical solution relies on the formulation of the fourth order equation as a system of two second order ones which enables the discretization by P1 conforming finite elements with respect to simplicial triangulations of the space-time domain. The resulting algebraic saddle point problem is solved by preconditioned Richardson iterations featuring preconditioners constructed by means of appropriately chosen left and right transforms. The space-time adaptivity is realized by a reliable residual-type a posteriori error estimator which is derived by the evaluation of the two residuals associated with the underlying second order system. Numerical results are given that illustrate the performance of the adaptive space-time finite element approximation.

## 1. INTRODUCTION

In this contribution, we study adaptive space-time finite element approximations of unconstrained optimally controlled initial-boundary value problems for linear parabolic partial differential equations (PDE) with distributed controls based on simplicial triangulations of the space-time domain.

We note that the efficient numerical solution of boundary and initial-boundary value problems for PDE and systems thereof by adaptive finite element methods has reached some state of maturity as documented by the monographs [1, 5, 7, 25, 69, 82] and the references therein. Several error concepts have been developed over the past decades including residual-type estimators [1, 5, 82] that rely on the appropriate evaluation of the residual in a dual norm, hierarchical type estimators [5] where the error equation is solved locally using higher order elements, error estimators that are based on local averaging [19, 86], the so-called goal oriented dual weighted approach [7, 25] where information about the error is extracted from the solution of the dual problem, and functional type error majorants [69] that provide guaranteed sharp upper bounds for the error. A systematic comparison of the performance of these estimators for a basic linear second order elliptic PDE has been provided recently in [21]. While the majority of the contributions has been dealing with elliptic PDE, adaptive methods for parabolic PDE have been investigated, e.g., in [7, 13, 14, 16, 17, 18, 23, 26, 27, 52, 63, 67, 70]. These contributions typically consider a discretization in space by finite elements with respect to triangulations of the spatial domain in combination with finite difference methods for discretization in time with respect to a partitioning of the underlying time interval. A central issue is the separation of the error in time and the error in space to allow for an automatic time-stepping and spatial mesh adaption by refinement and coarsening.

The systematic mathematical treatment and numerical solution of optimally controlled PDE dates back to the late sixties of the last century (cf. the seminal monograph [56] and the more recent textbooks [29, 34, 41, 55, 81] as well as the references therein). However, it took roughly twenty more years until the a posteriori error analysis of adaptive finite element schemes for PDE constrained optimal control problems has been addressed. For optimally controlled elliptic problems, classical residual-based error estimators have been derived in [30, 31, 35, 39, 40, 42, 43, 44, 51, 53], whereas the goal-oriented dual weighted approach has been applied in [8, 11, 36, 37, 38, 83, 85]. Much less work has been devoted to other available techniques. In particular, hierarchical estimators have been considered in [12], those based on local averaging in [54], and those using functional type error majorants in [32, 33]. For further references, we refer to the recent monograph [60].

The numerical solution of optimal control problems for parabolic PDE has been dealt with in [9, 58, 59, 64, 65, 66, 71, 74]. One is faced with the problem that the optimality conditions give rise to a coupled system consisting of the forward-in-time state equation, the backward-in-time adjoint state equation, and an equation or variational inequality (in case of control constraints) which relates the adjoint state and the control at optimality. Finite element discretizations in space and implicit time integrators for discretization in time typically lead to very large algebraic systems whose efficient numerical solution represents a significant challenge with regard to computational complexity (see, e.g., [9]). Moreover, since the state and the the adjoint state may exhibit singularities at different space-time locations, individual time-stepping and mesh adaptivity for the state and the adjoint state equation would be advantageous which, however, would render the adaptive method computationally costly as well.

The latter difficulty can be circumvented by the simultaneous use of finite element discretizations based on triangulations of the space-time domain. For time-dependent PDE, such an approach has been initiated in [15, 28, 50, 61] and has been further dealt with in [3, 4, 10, 48, 49, 62]. The application to optimally controlled parabolic PDE has been considered in [68]. However, to our best knowledge, adaptive space-time finite element approximations for parabolic optimal control problems based on simplicial triangulations of the space-time domain have not yet been studied in the literature.

Adaptive finite element methods for optimal control problems associated with PDE consist of successive loops of the cycle

$$\text{SOLVE} \implies \text{ESTIMATE} \implies \text{MARK} \implies \text{REFINE} .$$

Here, SOLVE stands for the numerical solution of the discretized optimality system. The step ESTIMATE is devoted to the derivation of an a posteriori error estimator whose contributions are used for the realization of adaptivity in space (elliptic problems) or adaptivity in space and time (time-dependent problems). The subsequent step MARK deals with the selection of elements and faces (or edges) of the triangulation for refinement and/or coarsening based on the information provided by the local contributions of the a posteriori error estimator. Since in this contribution we use space-time finite elements, we will only be concerned with refinement for which we are going to use the bulk criterion from [24], meanwhile also known as Dörfler marking. The final step REFINE addresses the technical realization of the refinement/coarsening process. Here, refinement will be based on newest vertex bisection (cf., e.g., [6, 22, 75]).

The novelty of the adaptive approach in this contribution is that the optimality system for the optimally controlled parabolic PDE under consideration will be

stated as a fourth order elliptic boundary value problem. We note that for PDE constrained optimal control problems such an approach, merging the state and adjoint state equation, has been recently used for elliptic problems in [57] and for the parabolic case in [68]. The fourth order problem can be formulated equivalently as a boundary value problem for a system of two second order equations. This suggests the use of standard P1 conforming finite elements with respect to simplicial triangulations of the space-time domain. The P1 conforming space-time discretization leads to an algebraic saddle point problem which will be numerically solved by a preconditioned Richardson iteration. The adaptive space-time mesh refinement relies on a residual-type a posteriori error estimator which can be derived within the framework of unified a posteriori error control [20].

The thesis is organized as follows: In the following section 2, after providing basic functional analytic notations and preliminaries in subsection 2.1, we consider an unconstrained parabolic optimal control problem featuring a tracking type objective functional and distributed controls (subsection 2.2) and state the first order optimality conditions in terms of the forward-in-time state equation, the backward-in-time adjoint state equation, and an equation which relates the adjoint state and the control (subsection 2.3). In section 3, we show that the optimality system gives rise to a fourth order elliptic boundary value problem (Theorem 3). We rewrite the fourth order equation as a system of two second order equations and introduce a weak solution concept in an appropriate function space setting. In particular, we prove that the operator-theoretic formulation involves a linear continuous, bijective operator so that the solution depends continuously on the data (Theorem 4). Consequently, having an approximate solution at hand, the error can be estimated from above in terms of the associated residuals which have to be evaluated in the norms of the respective dual spaces (Corollary 5). Section 4 deals with P1 conforming finite element discretizations of the second order system with respect to simplicial triangulations of the space-time domain, whereas section 5 is concerned with the numerical solution of the resulting saddle point problem by a preconditioned Richardson iteration featuring preconditioners constructed by means of suitably chosen left and right transforms. After a brief introduction to the idea behind such transforms in subsection 5.1, the following subsections 5.2 and 5.3 are devoted to the construction of the preconditioners for the specific saddle point problem under consideration. The residual-type a posteriori error estimator is presented in section 6. Using Galerkin orthogonality, it can be derived by means of an appropriate evaluation of the two residuals from Corollary 5 which simultaneously proves reliability of the estimator (Theorem 8). For two representative examples, section 7 contains a documentation of



numerical results illustrating the performance of our adaptive approach. Some concluding remarks are given in the final section 8.

## 2. THE PARABOLIC OPTIMAL CONTROL PROBLEM

We consider an optimally controlled linear second order parabolic PDE with a quadratic tracking type objective functional and distributed controls. In this contribution, we only study the unconstrained case, i.e., constraints are neither imposed on the control nor on the state. Moreover, we restrict ourselves to the heat equation with homogeneous Dirichlet boundary conditions on the boundary of the spatial domain. We note, however, that the generalization to general linear second order parabolic PDE and other types of boundary conditions is straightforward and only requires some technical effort.

**2.1. Notations and preliminaries.** We use standard notation from Lebesgue and Sobolev space theory [78]. In particular, given a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$ , with boundary  $\Gamma := \partial\Omega$ , for  $D \subseteq \Omega$  we refer to  $L^p(D), 1 \leq p \leq \infty$  as the Banach spaces of  $p$ -th power integrable functions ( $p < \infty$ ) and essentially bounded functions ( $p = \infty$ ) on  $D$  with norm  $\|\cdot\|_{L^p(D)}$ . We denote by  $L^p(D)_+$  the positive cone in  $L^p(D)$ , i.e.,  $L^p(D)_+ := \{v \in L^p(D) \mid v \geq 0 \text{ a.e. in } D\}$ . In case  $p = 2$ , the space  $L^2(D)$  is a Hilbert space whose inner product and norm will be referred to as  $(\cdot, \cdot)_{L^2(D)}$  and  $\|\cdot\|_{L^2(D)}$ . For  $m \in \mathbb{N}_0$ , we denote by  $W^{m,p}(D)$  the Sobolev spaces with norms

$$\|v\|_{W^{m,p}(D)} := \begin{cases} \left( \sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(D)}^p \right)^{1/p}, & \text{if } p < \infty \\ \max_{|\alpha| \leq m} \|D^\alpha v\|_{L^\infty(D)}, & \text{if } p = \infty \end{cases},$$

where  $\alpha = (\alpha_1, \dots, \alpha_d)^T \in \mathbb{N}_0$  with  $|\alpha| := \sum_{i=1}^d \alpha_i$ , and refer to  $|\cdot|_{W^{m,p}(D)}$  as the associated seminorms. For  $p < \infty$  and  $s \in \mathbb{R}_+, s = m + \sigma, m \in \mathbb{N}_0, 0 < \sigma < 1$ , we denote by  $W^{s,p}(D)$  the Sobolev space with norm

$$\|v\|_{W^{s,p}(D)} := \left( \|v\|_{W^{m,p}(D)}^p + \sum_{|\alpha|=m} \int_{\Omega} \int_{\Omega} \frac{|D^\alpha v(\mathbf{x}) - D^\alpha v(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{d+\sigma p}} d\mathbf{x} d\mathbf{y} \right)^{1/p}.$$

We refer to  $W_0^{s,p}(D)$  as the closure of  $C_0^\infty(D)$  in  $W^{s,p}(D)$ . For  $s < 0$ , we denote by  $W^{s,p}(D)$  the dual space of  $W_0^{-s,q}(D), p^{-1} + q^{-1} = 1$ . In case  $p = 2$ , the spaces  $W^{s,2}(D)$  are Hilbert spaces. We will write  $H^s(D)$  instead of  $W^{s,2}(D)$  and refer to  $(\cdot, \cdot)_{H^s(D)}$  and  $\|\cdot\|_{H^s(D)}$  as the inner products and associated norms. Moreover, for  $T > 0$  we consider the space-time domain  $Q := \Omega \times (0, T)$ . For functions  $y$  on  $Q$ , derivatives with respect to time  $t \in [0, T]$  will be denoted by  $y_t, y_{tt}$  and so forth. Given a Banach space  $X$ , we denote by  $L^p((0, T), X), 1 \leq p \leq \infty$ , and  $C([0, T], X)$  the Banach spaces of functions  $v : [0, T] \rightarrow X$  with

norms

$$\|v\|_{L^p((0,T),X)} := \begin{cases} \left( \int_0^T \|v(t)\|_X^p dt \right)^{1/p}, & 1 \leq p < \infty \\ \text{ess sup}_{t \in [0,T]} \|v(t)\|_X, & p = \infty \end{cases},$$

$$\|v\|_{C([0,T],X)} := \max_{t \in [0,T]} \|v(t)\|_X.$$

The spaces  $W^{s,p}((0,T),X)$  and  $H^s((0,T),X)$ ,  $s \in \mathbb{R}_+$ , are defined likewise. In particular, for a subspace  $V \subset H^1(\Omega)$  with dual  $V^*$  we will consider the space

$$(2.1) \quad W(0,T) := H^1((0,T),V^*) \cap L^2((0,T),V),$$

and note that the following continuous embedding holds true (cf., e.g., [72])

$$(2.2) \quad W(0,T) \subset C([0,T],L^2(\Omega)).$$

**2.2. Parabolic optimal control problem with distributed controls.** Let  $\Omega$  be a bounded polyhedral domain in  $\mathbb{R}^d$ ,  $d \in \mathbb{N}$ , with boundary  $\Gamma := \partial\Omega$ . For  $T > 0$  we set  $Q := \Omega \times (0,T)$  and  $\Sigma := \partial Q$  with  $\Sigma_{lat} := \Gamma \times (0,T)$ ,  $\Sigma_{bot} := \Omega \times \{0\}$ ,  $\Sigma_{top} := \Omega \times \{T\}$ . We further set  $Q_p := \bar{Q} \setminus \Sigma_p$  where  $\Sigma_p$  stands for the parabolic boundary  $\Sigma_p := \Sigma_{bot} \cup \Sigma_{lat}$ . Given a desired state  $y^d \in L^2(Q)$ , a shift control  $u^d \in L^2(Q)$  with  $\text{supp } u^d \subset Q$ , an initial state  $y^0 \in L^2(\Omega)$  as well as a regularization parameter  $\alpha > 0$ , we consider the following unconstrained distributed parabolic optimal control problem

$$(2.3a) \quad \inf_{y,u} J(y,u),$$

where

$$(2.3b) \quad J(y,u) := \frac{1}{2} \int_0^T \int_{\Omega} |y - y^d|^2 dx dt + \frac{\alpha}{2} \int_0^T \int_{\Omega} |u - u^d|^2 dx dt,$$

subject to

$$(2.3c) \quad y_t - \Delta y = u \quad \text{in } Q,$$

$$(2.3d) \quad y = 0 \quad \text{on } \Sigma_{lat},$$

$$(2.3e) \quad y(\cdot, 0) = y^0 \quad \text{on } \Sigma_{bot}.$$

We note that (2.3c)-(2.3e) has to be understood in a weak sense. In particular, we are looking for  $(y,u) \in W(0,T) \times L^2(Q)$ , where  $V := H_0^1(\Omega)$  in (2.1), such that for all  $v \in W(0,T)$  the variational equation

$$(2.4) \quad (y_t, v)_{L^2(Q)} + (\nabla y, \nabla v)_{L^2(Q)} = (u, v)_{L^2(Q)}$$

and the initial condition  $y|_{\Sigma_{bot}} = y^0$  are satisfied.

**Theorem 1.** *Under the assumptions on the data of (2.3a)-(2.3e) there exists a unique solution  $(y, u) \in W(0, T) \times L^2(Q)$ .*

*Proof.* We refer to [81]. □

**2.3. Optimality conditions.** We derive the first order necessary optimality conditions which for the optimal control problem (2.3a)-(2.3e) under consideration are also sufficient due to the strict convexity of the objective functional.

**Theorem 2.** *Let  $(y, u) \in W(0, T) \times L^2(Q)$  be the optimal solution of (2.3a)-(2.3e). Then, there exists an adjoint state  $p \in W(0, T)$  such that the triple  $(y, p, u)$  satisfies the state equation*

$$(2.5a) \quad y_t - \Delta y = u \quad \text{in } Q,$$

$$(2.5b) \quad y = 0 \quad \text{on } \Sigma_{lat},$$

$$(2.5c) \quad y = y^0 \quad \text{on } \Sigma_{bot},$$

the adjoint state equation

$$(2.5d) \quad -p_t - \Delta p = y - y^d \quad \text{in } Q,$$

$$(2.5e) \quad p = 0 \quad \text{on } \Sigma_{lat},$$

$$(2.5f) \quad p = 0 \quad \text{on } \Sigma_{top},$$

and the relationship

$$(2.5g) \quad p + \alpha(u - u^d) = 0 \quad \text{in } Q.$$

*Proof.* We formally use the Lagrange multiplier approach to derive (2.5a)-(2.5g). Using Lagrange multipliers  $p \in H^1((0, T), H^{-1}(\Omega)) \cap L^2((0, T), H_0^1(\Omega))$  for (2.5a) and  $p^0 \in L^2(\Omega)$  for (2.5c), we introduce the Lagrangian

$$(2.6) \quad L(y, u, p, p^0) := J(y, u) + \int_0^T \langle y_t - \Delta y - u, p \rangle dt + (p^0, y - y^0)_{L^2(\Omega)},$$

where  $\langle \cdot, \cdot \rangle$  stands for the respective dual product. Critical points of the Lagrangian are characterized by

$$(2.7a) \quad L_y(y, u, p, p^0) = 0,$$

$$(2.7b) \quad L_u(y, u, p, p^0) = 0,$$

$$(2.7c) \quad L_p(y, u, p, p^0) = 0,$$

$$(2.7d) \quad L_{p^0}(y, u, p, p^0) = 0.$$

Obviously, (2.7c) and (2.7d) readily yield (2.5a) and (2.5c), whereas (2.5b) is a direct consequence of  $y \in W(0, T)$ . On the other hand, partial integration

with respect to  $t$  gives

$$\int_0^T \langle y_t, p \rangle dt = - \int_0^T \langle p_t, y \rangle dt + (y|_{\Sigma_{top}}, p|_{\Sigma_{top}})_{L^2(\Omega)} - ((y|_{\Sigma_{bot}}, p|_{\Sigma_{bot}})_{L^2(\Omega)}).$$

Moreover, observing  $y_{\Sigma_{lat}} = 0$ , applying Green's formula twice, we find

$$- \int_0^T \langle \Delta y, p \rangle dt = - \int_0^T \langle \Delta p, y \rangle dt + \int_0^T \langle \mathbf{n}_\Gamma \cdot \nabla y, p \rangle dt,$$

and hence,

(2.8)

$$\begin{aligned} L(y, u, p, p^0) &= J(y, u) + \int_0^T \langle -p_t - \Delta p, y \rangle dt + \int_0^T \langle \mathbf{n}_\Gamma \cdot \nabla y, p \rangle dt \\ &\quad + (p|_{\Sigma_{top}}, y|_{\Sigma_{top}})_{L^2(\Omega)} - (p|_{\Sigma_{bot}} - p^0, y|_{\Sigma_{bot}})_{L^2(\Omega)} - (p^0, y^0)_{L^2(\Omega)}. \end{aligned}$$

Taking  $J_y(y, u) = y - y^d$  into account and using (2.8) in (2.7a) gives rise to  $p^0 = p|_{\Sigma_{bot}}$  and (2.5d)-(2.5f). Finally, in view of  $J_u(y, u) = \alpha(u - u^d)$ , from (2.7b) we deduce (2.5g).

For a justification of the formal Lagrangian approach for the optimal control problem under consideration we refer to [29].  $\square$

### 3. OPTIMALITY SYSTEM AS A FOURTH ORDER ELLIPTIC EQUATION

In this section, we will show that for sufficiently smooth state  $y$  and adjoint state  $p$  the optimality system (2.5a)-(2.5g) can be formulated as a fourth order elliptic boundary value problem. Further, this fourth order problem will be equivalently stated as a boundary value problem for a system of two second order equations

**Theorem 3.** *Assume that the state  $y$  and the adjoint state  $p$  are sufficiently smooth. Then, the optimality system (2.5a)-(2.5g) is equivalent to the fourth order elliptic boundary value problem*

$$(3.1a) \quad -y_{tt} + \Delta^2 y + \alpha^{-1}y = f \quad \text{in } Q,$$

$$(3.1b) \quad y = 0 \quad \text{on } \Sigma_{lat},$$

$$(3.1c) \quad y_t - \Delta y = 0 \quad \text{on } \Sigma_{lat},$$

$$(3.1d) \quad y = y^0 \quad \text{on } \Sigma_{bot},$$

$$(3.1e) \quad y_t - \Delta y = 0 \quad \text{on } \Sigma_{top}.$$

where the right-hand side  $f$  in (3.1a) is given by

$$(3.2) \quad f := \alpha^{-1}y^d - \Delta u^d - u_t^d.$$

*Proof.* Substituting  $u$  in (2.5a) by means of (2.5g) yields

$$(3.3) \quad y_t - \Delta y = -\alpha^{-1}p + u^d.$$

Differentiating (3.3) with respect to time  $t$  results in

$$(3.4) \quad y_{tt} - \Delta y_t = -\alpha^{-1}p_t + u_t^d.$$

On the other hand, (2.5d) gives

$$(3.5) \quad -\alpha^{-1}p_t = \alpha^{-1}\Delta p + \alpha^{-1}(y - y^d).$$

Thus, inserting (3.5) into (3.4), we obtain

$$(3.6) \quad y_{tt} - \Delta y_t = \alpha^{-1}\Delta p + \alpha^{-1}(y - y^d) + u_t^d.$$

Now, we apply the Laplacian  $\Delta$  to (3.3):

$$\Delta y_t - \Delta^2 y = -\alpha^{-1}\Delta p + \Delta u^d,$$

which results in

$$(3.7) \quad \alpha^{-1}\Delta p = -\Delta y_t + \Delta^2 y + \Delta u^d.$$

Using (3.7) in (3.6) yields (3.1a). The boundary conditions (3.1b) and (3.1d) follow readily from (2.5b) and (2.5c). On the other hand, observing  $\text{supp } u^d \subset Q$  and (2.5e),(2.5f), from (2.5g) we deduce  $u|_{\Sigma_{lat}} = 0$  and  $u|_{\Sigma_{top}} = 0$ . Hence, (3.1c) and (3.1e) are a direct consequence of (2.5a).  $\square$

We reformulate (3.1a)-(3.1e) as a boundary value problem for a system of two second order equations. Setting  $w = -\Delta y$ , the fourth order boundary value problem reads as follows

$$(3.8a) \quad -y_{tt} - \Delta w + \alpha^{-1}y = f \quad \text{in } Q,$$

$$(3.8b) \quad \Delta y + w = 0 \quad \text{in } Q,$$

$$(3.8c) \quad y = y^0 \quad \text{on } \Sigma_{bot},$$

$$(3.8d) \quad y = 0 \quad \text{on } \Sigma_{lat},$$

$$(3.8e) \quad \begin{aligned} y_t + w &= 0 \\ \Delta y + w &= 0 \end{aligned} \quad \text{on } \Sigma.$$

Multiplying (3.8a) by a smooth test function  $v$  on  $Q$  satisfying  $v|_{\Sigma_p} = 0$  and integrating over  $Q$  yields

$$(3.9) \quad -(y_{tt}, v)_{L^2(Q)} - (\Delta w, v)_{L^2(Q)} + \alpha^{-1}(y, v)_{L^2(Q)} = (f, v)_{L^2(Q)}.$$

In view of  $v(\cdot, 0) = 0$  and  $y_t(\cdot, T) = -w(\cdot, T)$ , by partial integration we find

$$(3.10) \quad \begin{aligned} -(y_{tt}, v)_{L^2(Q)} &= (y_t, v_t)_{L^2(Q)} - (y_t(\cdot, T), v(\cdot, T))_{L^2(\Omega)} \\ &= (y_t, v_t)_{L^2(Q)} + (w(\cdot, T), v(\cdot, T))_{L^2(\Omega)}. \end{aligned}$$

Moreover, observing  $v|_{\Sigma_{lat}} = 0$  as well as (3.2), Green's formula gives

$$(3.11) \quad -(\Delta w, v)_{L^2(Q)} = (\nabla w, \nabla v)_{L^2(Q)} - (\mathbf{n} \cdot \nabla w, v)_{L^2(\Sigma_{lat})} = (\nabla w, \nabla v)_{L^2(Q)}.$$

On the other hand, multiplying (3.8b) by a smooth test function  $z$  on  $Q$  and integrating over  $Q$  we obtain

$$(3.12) \quad (\Delta y, z)_{L^2(Q)} = -(\nabla y, \nabla z)_{L^2(Q)} + (\mathbf{n} \cdot \nabla y, z)_{L^2(\Sigma_{lat})}.$$

Taking (3.9)-(3.12) into account, for the weak formulation of the second order system (3.8a)-(3.8e) we introduce the function spaces

$$(3.13a) \quad W := L^2((0, T); H^1(\Omega)) \cap C([0, T]; L^2(\Omega)),$$

$$(3.13b) \quad Y := \{y \in H^1(Q) \cap C([0, T]; L^2(\Omega)) \mid y|_{\Sigma_{bot}} = y^0, y|_{\Sigma_{lat}} = 0\},$$

$$(3.13c) \quad Y_0 := \{y \in Y \mid y|_{\Sigma_{bot}} = 0\},$$

equipped with the norms

$$(3.13d) \quad \|w\|_W := \left( \int_0^T (\|w\|_{H^1(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2) dt + \|w(\cdot, T)\|_{L^2(\Omega)}^2 \right)^{1/2},$$

$$(3.13e) \quad \|y\|_Y := \left( \int_0^T (\|y\|_{H^1(\Omega)}^2 + \|y\|_{L^2(\Omega)}^2) dt \right)^{1/2},$$

as well as

$$(3.14a) \quad Z := \{z = (w, y)^T \in W \times Y \mid z \text{ satisfies (3.8e)}\},$$

$$(3.14b) \quad Z_0 := \{z \in Z \mid z|_{\Sigma_p} = 0\}.$$

Then, the weak formulation of (3.8a)-(3.8e) amounts to the computation of  $(w, y) \in Z$  such that for all  $v_1 \in Y_0$  and  $v_2 \in W$  there holds

$$(3.15a) \quad a_{11}(w, v_1) + a_{12}(y, v_1) = \ell_1(v_1),$$

$$(3.15b) \quad -a_{21}(w, v_2) + a_{22}(y, v_2) = \ell_2(v_2).$$

Here, the bilinear forms  $a_{ij}(\cdot, \cdot)$ ,  $1 \leq i, j \leq 2$ , are given by

$$(3.16a) \quad a_{11}(w, v_1) := (\nabla w, \nabla v_1)_{L^2(Q)} + (w(\cdot, T), v_1(\cdot, T))_{L^2(\Omega)},$$

$$(3.16b) \quad a_{12}(y, v_1) := (y_t, (v_1)_t)_{L^2(Q)} + \alpha^{-1}(y, v_1)_{L^2(Q)},$$

$$(3.16c) \quad a_{21}(w, v_2) := (w, v_2)_{L^2(Q)},$$

$$(3.16d) \quad a_{22}(y, v_2) := (\nabla y, \nabla v_2)_{L^2(Q)} - \int_0^T \langle \mathbf{n} \cdot \nabla y, v_2 \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} dt,$$

where  $\langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$  stands for the dual pairing between  $H^{-1/2}(\Gamma)$  and  $H^{1/2}(\Gamma)$ . In view of (3.2), the functionals  $\ell_i(\cdot)$ ,  $1 \leq i \leq 2$ , in (3.15a),(3.15b) read as follows

$$(3.17a) \quad \ell_1(v_1) := (f, v_1)_{L^2(Q)},$$

$$(3.17b) \quad \ell_2(v_2) := 0.$$

$$(3.17c)$$

The operator theoretic version of (3.15a),(3.15b) reads as follows

$$(3.18) \quad \mathcal{L}(w, y) = (\ell_1, \ell_2)^T,$$

where the operator  $\mathcal{L} : Z \rightarrow Z^*$  is given by

$$(3.19) \quad \langle \mathcal{L}(w, y), (v_1, v_2) \rangle_{Z^*, Z} := a_{11}(y, v_1) - a_{12}(w, v_1) + a_{21}(y, v_2) + a_{22}(w, v_2).$$

**Theorem 4.** *The operator  $\mathcal{L}$  is a continuous, bijective linear operator. Hence, for any  $(\ell_1, \ell_2)^T \in Z^*$ , the system (3.15a),(3.15b) admits a unique solution  $(y, w) \in Z$ . The solution depends on the data according to*

$$(3.20) \quad \|(w, y)\|_{W \times Y} \lesssim \|(\ell_1, \ell_2)\|_{W^* \times Y^*}.$$

*Proof.* Without loss of generality we may assume  $y^0 = 0$ , since otherwise we define  $\hat{y}$  as the solution of the state equation (2.5a)-(2.5c) with zero right-hand side and replace  $y$  with  $y - \hat{y}$ . Then, the corresponding pair  $(y, w)$  satisfies



(3.15a),(3.15b) with homogeneous Dirichlet data on  $\Sigma_{bot}$ , but modified right-hand sides.

The linearity and continuity of  $\mathcal{L}$  are straightforward. Moreover, we may resort to smooth trial and test functions  $y, v_1 \in Y_0 \cap C_{0, \Sigma_p}^\infty(Q)$  and  $w, v_2 \in W \cap C^\infty(Q)$ . Then,  $(y_t + w)|_{\Sigma_p} = 0$  implies  $w|_{\Sigma_p} = 0$  such that  $w$  is an admissible test function in (3.15a). For such  $z = (w, y)^T$  we have

$$\begin{aligned} \langle \mathcal{L}(w, y), (y + \alpha w, y - w) \rangle &= (y_t, y_t)_{L^2(Q)} + \alpha (y_t, w_t)_{L^2(Q)} \\ &+ \alpha a(w, w) + \alpha^{-1} (y, y)_{L^2(Q)} + \alpha (w(\cdot, T), w(\cdot, T))_{L^2(\Omega)} \\ &+ (w(\cdot, T), y(\cdot, T))_{L^2(\Omega)} + a(y, y) + (w, w)_{L^2(Q)}. \end{aligned}$$

Observing  $w = -\Delta y$  and  $w_t = -\Delta y_t$ , we find

$$\begin{aligned} (y_t, w_t)_{L^2(Q)} &= -(y_t, \Delta y_t)_{L^2(Q)} = (\nabla y_t, \nabla y_t)_{L^2(Q)} \geq 0, \\ (w(\cdot, T), y(\cdot, T))_{L^2(\Omega)} &= -(\Delta y(\cdot, T), y(\cdot, T))_{L^2(\Omega)} = (\nabla y(\cdot, T), \nabla y(\cdot, T))_{L^2(\Omega)} \geq 0, \end{aligned}$$

and hence, it follows that

(3.21)

$$\begin{aligned} \langle \mathcal{L}(w, y), (y + \alpha w, y - w) \rangle &\geq \\ \alpha \int_0^T |w|_{H^1(\Omega)}^2 dt + \|w\|_{L^2(Q)}^2 + \alpha \|w(\cdot, T)\|_{L^2(\Omega)}^2 + |y|_{H^1(Q)}^2 + \alpha^{-1} \|y\|_{L^2(Q)}^2. \end{aligned}$$

A density argument shows that (3.21) holds true for all  $(w, y) \in Z_0$ . This allows to deduce the inf-sup condition which implies bijectivity of  $\mathcal{L}$ .  $\square$

**Corollary 5.** *Let  $(y_h, w_h) \in Y_h \times W_h, Y_h \subset Y, W_h \subset W$ , be an approximate solution of (3.15a),(3.15b). Then, there holds*

$$(3.22) \quad \|(y - y_h, w - w_h)\|_{Y \times W} \lesssim \|(\text{Res}_1, \text{Res}_2)\|_{Y^* \times W^*},$$

where the residuals  $\text{Res}_1 \in Y^*, \text{Res}_2 \in W^*$  are given by

$$(3.23a) \quad \begin{aligned} \text{Res}_1(v) &:= \ell_1(v) - ((y_h)_t, v_t)_{0,Q} - a(w_h, v) \\ &\quad - \alpha^{-1} (y_h, v)_{0,Q} - (w_h(\cdot, T), v)_{0,\Omega}, \quad v \in V, \end{aligned}$$

$$(3.23b) \quad \text{Res}_2(z) := \ell_2(z) - a(y_h, z) + (w_h, z)_{0,Q}. \quad z \in W.$$

*Proof.* The assertion is an immediate consequence of Theorem 4.  $\square$

#### 4. SPACE-TIME FINITE ELEMENT DISCRETIZATION

We consider a shape regular family  $(\mathcal{T}_h(Q))_{h \in \mathbb{H}}$  of simplicial triangulations of the space-time domain  $Q$  where  $\mathbb{H}$  is a null sequence of positive real numbers. We refer to  $\mathcal{N}_h(D)$  as the set of vertices  $a_D^{(i)}, 1 \leq i \leq \text{card } \mathcal{N}_h(D)$ , and to  $\mathcal{F}_h(D)$  as the set of faces in  $D \subseteq \bar{Q}$ . For  $K \in \mathcal{T}_h(Q)$ , we denote by  $h_K$  the diameter of  $K$  and set  $h := \max\{h_K \mid K \in \mathcal{T}_h(Q)\}$ . We further refer to  $h_F$  as the diameter of  $F \in \mathcal{F}_h(\bar{Q})$ . We set  $Q_P := \bar{Q} \setminus \Sigma_P$ , where  $\Sigma_P$  stands for the parabolic boundary  $\Sigma_P := \Sigma_{bot} \cup \Sigma_{lat}$ , and define

$$\begin{aligned}
 (4.1a) \quad & N_{Q_P} := \text{card } \mathcal{N}_h(Q_P), \\
 (4.1b) \quad & N_{\Sigma_{bot}} := \text{card } \mathcal{N}_h(\Sigma_{bot}), \\
 (4.1c) \quad & N_{\Sigma_{lat}} := \text{card } \mathcal{N}_h(\Sigma_{lat}), \\
 (4.1d) \quad & N_{\Sigma_P} := N_{\Sigma_{bot}} + N_{\Sigma_{lat}}.
 \end{aligned}$$

For space-time discretization, we use P1 conforming finite elements with respect to the triangulation  $\mathcal{T}_h(Q)$ . Denoting by  $\varphi_{Q_P}^{(i)}, 1 \leq i \leq N_{Q_P}$ , and by  $\varphi_{\Sigma_{bot}}^{(i)}, 1 \leq i \leq N_{\Sigma_{bot}}$ , as well as  $\varphi_{\Sigma_P}^{(i)}, 1 \leq i \leq N_{\Sigma_P}$ , the nodal basis functions associated with the nodal points in  $\mathcal{N}_h(Q_P)$  and  $\mathcal{N}_h(\Sigma_{bot}), \mathcal{N}_h(\Sigma_P)$ , respectively, we introduce the finite element spaces

$$\begin{aligned}
 (4.2a) \quad & Y_{h, \Sigma_{bot}} := \text{span}(\varphi_{\Sigma_{bot}}^{(1)}, \dots, \varphi_{\Sigma_{bot}}^{(N_{\Sigma_{bot}})}), \\
 (4.2b) \quad & Y_{h, 0} := \text{span}(\varphi_{Q_P}^{(1)}, \dots, \varphi_{Q_P}^{(N_{Q_P})}), \\
 (4.2c) \quad & W_h := Y_{h, 0} \oplus W_h^{\Sigma_P}, \quad W_h^{\Sigma_P} := \text{span}(\varphi_{\Sigma_P}^{(1)}, \dots, \varphi_{\Sigma_P}^{(N_{\Sigma_P})}), \\
 (4.2d) \quad & Y_h := \{y_h \in W_h \mid y_h|_{\Sigma_{bot}} = y_{h,0}, y_h|_{\Sigma_{lat}} = 0\},
 \end{aligned}$$

where  $y_{h,0} \in Y_{h, \Sigma_{bot}}$  is a suitable approximation of  $y_0$ .

The space-time finite element approximation of the solution  $(w, y) \in W \times Y$  amounts to the computation of  $(w_h, y_h) \in W_h \times Y_h$  such that for all  $v_{h,1} \in Y_{h,0}$  and  $v_{h,2} \in W_h$  there holds

$$\begin{aligned}
 (4.3a) \quad & a_{11}(w_h, v_{h,1}) + a_{12}(y_h, v_{h,1}) = \ell_1(v_{h,1}), \\
 (4.3b) \quad & -a_{21}(w_h, v_{h,2}) + a_{22}(y_h, v_{h,2}) = \ell_2(v_{h,2}).
 \end{aligned}$$

The algebraic formulation of (4.3a),(4.3b) leads to a block-structured linear algebraic system. We set

$$(4.4a) \quad \mathbf{y}_0 := (y_0^{(1)}, \dots, y_0^{(N_{\Sigma_{bot}})})^T, \quad y_0^{(i)} := y_{h,0}(a_{\Sigma_{bot}}^{(i)}), \quad 1 \leq i \leq N_{\Sigma_{bot}},$$

$$(4.4b) \quad \mathbf{y}_P := (y_P^{(1)}, \dots, y_P^{(N_{Q_P})})^T, \quad y_P^{(i)} := y_h(a_{Q_P}^{(i)}), \quad 1 \leq i \leq N_{Q_P},$$

$$(4.4c) \quad \mathbf{w}_P := (w_P^{(1)}, \dots, w_P^{(N_{Q_P})})^T, \quad w_P^{(i)} := w_h(a_{Q_P}^{(i)}), \quad 1 \leq i \leq N_{Q_P},$$

$$(4.4d) \quad \mathbf{w}_\Sigma := (w_\Sigma^{(1)}, \dots, w_\Sigma^{(N_{\Sigma_P})})^T, \quad w_\Sigma^{(i)} := w_h(a_{\Sigma_P}^{(i)}), \quad 1 \leq i \leq N_{\Sigma_P},$$

$$(4.4e) \quad \mathbf{x} := (\mathbf{w}_P, \mathbf{y}_P, \mathbf{w}_\Sigma)^T.$$

The linear algebraic system is given by

$$(4.5) \quad \mathbf{K}\mathbf{x} = \mathbf{b},$$

where the system matrix  $\mathbf{K}$  and the right-hand side read as follows

$$(4.6a) \quad \mathbf{K} := \begin{pmatrix} \mathbf{A}_{PP} & \mathbf{A}_{P\Sigma} \\ \mathbf{A}_{\Sigma P} & -\mathbf{M}_{\Sigma\Sigma} \end{pmatrix},$$

$$(4.6b) \quad \mathbf{b} := (\mathbf{b}_P, \mathbf{0})^T.$$

The first diagonal block  $\mathbf{A}_{PP}$  of  $\mathbf{K}$  is the  $2 \times 2$  block matrix

$$(4.7) \quad \mathbf{A}_{PP} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{T} + \alpha^{-1}\mathbf{M} \\ -\mathbf{M} & \mathbf{A}_2 \end{pmatrix},$$

where the matrices  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{T}, \mathbf{M} \in \mathbb{R}^{N_{Q_P} \times N_{Q_P}}$  are given by

$$\begin{aligned} (\mathbf{A}_1)_{ij} &:= a_{11}(\varphi_{Q_P}^{(i)}, \varphi_{Q_P}^{(j)}), & (\mathbf{A}_2)_{ij} &:= a_{22}(\varphi_{Q_P}^{(i)}, \varphi_{Q_P}^{(j)}), \\ (\mathbf{T})_{ij} &:= ((\varphi_{Q_P}^{(i)})_t, (\varphi_{Q_P}^{(j)})_t)_{L^2(Q)}, & (\mathbf{M})_{ij} &:= (\varphi_{Q_P}^{(i)}, \varphi_{Q_P}^{(j)})_{L^2(Q)}. \end{aligned}$$

The off-diagonal matrices  $\mathbf{A}_{P\Sigma} \in \mathbb{R}^{N_{Q_P} \times N_{\Sigma_P}}$  and  $\mathbf{A}_{\Sigma P} \in \mathbb{R}^{N_{\Sigma_P} \times N_{Q_P}}$  are of the form

$$(4.8) \quad \mathbf{A}_{P\Sigma} = \begin{pmatrix} \mathbf{A}_{P\Sigma}^{(1)} \\ \mathbf{A}_{P\Sigma}^{(2)} \end{pmatrix}, \quad \mathbf{A}_{\Sigma P} = \begin{pmatrix} \mathbf{A}_{\Sigma P}^{(1)} & \mathbf{A}_{\Sigma P}^{(2)} \end{pmatrix}$$

with the matrices  $\mathbf{A}_{P\Sigma}^{(i)}, \mathbf{A}_{\Sigma P}^{(i)}, 1 \leq i \leq 2$ , being given by

$$\begin{aligned} (\mathbf{A}_{P\Sigma}^{(1)})_{ij} &:= a_{11}(\varphi_{\Sigma_P}^{(j)}, \varphi_{Q_P}^{(i)}), \\ (\mathbf{A}_{P\Sigma}^{(2)})_{ij} &:= -a_{21}(\varphi_{\Sigma_P}^{(j)}, \varphi_{Q_P}^{(i)}), \\ (\mathbf{A}_{\Sigma P}^{(1)})_{ij} &:= -a_{21}(\varphi_{Q_P}^{(j)}, \varphi_{\Sigma_P}^{(i)}), \\ (\mathbf{A}_{\Sigma P}^{(2)})_{ij} &:= a_{22}(\varphi_{Q_P}^{(j)}, \varphi_{\Sigma_P}^{(i)}). \end{aligned}$$

## 5. NUMERICAL SOLUTION OF THE SPACE-TIME DISCRETIZED PROBLEM

We will solve the linear algebraic system (4.5) by the preconditioned Richardson iteration [2]

$$(5.1) \quad \mathbf{x}^{(\nu+1)} = \mathbf{x}^{(\nu)} - \hat{\mathbf{K}}^{-1}(\mathbf{K}\mathbf{x}^{(\nu)} - \mathbf{b}), \quad \nu \in \mathbb{N}_0,$$

where  $\hat{\mathbf{K}}$  is an appropriate preconditioner for  $\mathbf{K}$  and  $\mathbf{x}^{(0)}$  is a given initial iterate. The preconditioner  $\hat{\mathbf{K}}$  will be constructed by means of left and right transforms.

**5.1. Left and right transforms.** Let  $\mathbf{K}_L, \mathbf{K}_R$  be regular matrices. Then, (4.5) can be equivalently written as

$$(5.2) \quad \mathbf{K}_L \mathbf{K} \mathbf{K}_R \mathbf{K}_R^{-1} \mathbf{x} = \mathbf{K}_L \mathbf{b}.$$

Assuming  $\tilde{\mathbf{K}}$  to be a suitable preconditioner for  $\mathbf{K}_L \mathbf{K} \mathbf{K}_R$ , we consider the transforming iteration

$$(5.3) \quad \mathbf{K}_R^{-1} \mathbf{x}^{(\nu+1)} = \mathbf{K}_R^{-1} \mathbf{x}^{(\nu)} - \tilde{\mathbf{K}}^{-1}(\mathbf{K}_L \mathbf{K} \mathbf{x}^{(\nu)} - \mathbf{K}_L \mathbf{b}).$$

Backtransformation yields

$$(5.4) \quad \mathbf{x}^{(\nu+1)} = \mathbf{x}^{(\nu)} - \mathbf{K}_R \tilde{\mathbf{K}}^{-1} \mathbf{K}_L (\mathbf{K} \mathbf{x}^{(\nu)} - \mathbf{b})$$

$$(5.5) \quad = \mathbf{x}^{(\nu)} - (\mathbf{K}_L^{-1} \tilde{\mathbf{K}} \mathbf{K}_R^{-1})^{-1} (\mathbf{K} \mathbf{x}^{(\nu)} - \mathbf{b}).$$

Consequently,

$$(5.6) \quad \hat{\mathbf{K}} := \mathbf{K}_L^{-1} \tilde{\mathbf{K}} \mathbf{K}_R^{-1}$$

is an appropriate preconditioner for  $\mathbf{K}$ .

We note that transforming iterations have been used as smoothers within multigrid methods [84] as well as for the iterative solution of KKT systems in PDE constrained optimization [45, 46, 47, 76, 77].

**5.2. Construction of a preconditioner  $\mathbf{I}$ .** We assume  $\hat{\mathbf{A}}_{PP}$  to be an appropriate preconditioner for  $\mathbf{A}_{PP}$  (for its construction see subsection 5.3 below) and choose the left transform  $\mathbf{K}_L$  and the right transform  $\mathbf{K}_R$  according to

$$(5.7) \quad \mathbf{K}_L = \mathbf{I}, \quad \mathbf{K}_R = \begin{pmatrix} \mathbf{I} & -\hat{\mathbf{A}}_{PP}^{-1} \mathbf{A}_{P\Sigma} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

Recalling the definition of  $\mathbf{K}$  (cf. (4.6)), it follows that

$$\mathbf{K}_L \mathbf{K} \mathbf{K}_R = \begin{pmatrix} \mathbf{A}_{PP} & (\mathbf{I} - \mathbf{A}_{PP} \hat{\mathbf{A}}_{PP}^{-1}) \mathbf{A}_{P\Sigma} \\ \mathbf{A}_{\Sigma P} & -(\mathbf{M}_{\Sigma\Sigma} + \mathbf{A}_{\Sigma P} \hat{\mathbf{A}}_{PP}^{-1} \mathbf{A}_{P\Sigma}) \end{pmatrix}.$$

The matrix  $\mathbf{K}_L \mathbf{K} \mathbf{K}_R$  admits the regular splitting

$$\mathbf{K}_L \mathbf{K} \mathbf{K}_R = \mathbf{M}_1 + \mathbf{M}_2,$$

where the matrices  $\mathbf{M}_1$  and  $\mathbf{M}_2$  are given by

$$\mathbf{M}_1 := \begin{pmatrix} \hat{\mathbf{A}}_{PP} & \mathbf{0} \\ \mathbf{A}_{\Sigma P} & -(\mathbf{M}_{\Sigma\Sigma} + \mathbf{A}_{\Sigma P} \hat{\mathbf{A}}_{PP}^{-1} \mathbf{A}_{P\Sigma}) \end{pmatrix},$$

$$\mathbf{M}_2 := \begin{pmatrix} \mathbf{A}_{PP} - \hat{\mathbf{A}}_{PP} & (\mathbf{I} - \mathbf{A}_{PP} \hat{\mathbf{A}}_{PP}^{-1}) \mathbf{A}_{P\Sigma} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

We note that  $\mathbf{M}_2 \approx \mathbf{0}$ , if  $\hat{\mathbf{A}}_{PP} \approx \mathbf{A}_{PP}$ . Hence,  $\tilde{\mathbf{K}} = \mathbf{M}_1$  is a suitable preconditioner for  $\mathbf{K}_L \mathbf{K} \mathbf{K}_R$ . In view of (5.6), we thus obtain

$$(5.8) \quad \hat{\mathbf{K}} = \mathbf{K}_L^{-1} \mathbf{M}_1 \mathbf{K}_R^{-1}$$

as an appropriate preconditioner for the system matrix  $\mathbf{K}$ . We thus arrive at the following preconditioned Richardson iteration:

**Algorithm (Preconditioned Richardson Iteration)**

**Step 1 (Initialization)**

Choose an initial iterate  $\mathbf{x}^{(0)} = (\mathbf{w}_P^{(0)}, \mathbf{y}_P^{(0)}, \mathbf{w}_\Sigma^{(0)})^T$ , prescribe some tolerance  $TOL > 0$ , and set  $\nu = 0$ .

**Step 2 (Iteration loop)**

**Step 2.1 (Computation of the residual)**

Compute the residual with respect to  $\mathbf{x}^{(\nu)}$ :

$$\mathbf{d}^{(\nu)} = \mathbf{K} \mathbf{x}^{(\nu)} - \mathbf{b}.$$

**Step 2.2 (Implementation of the preconditioner)**

Solve the staggered linear algebraic system

$$\mathbf{M}_1 \mathbf{y}^{(\nu)} = \mathbf{d}^{(\nu)}$$

by forward substitution.

**Step 2.3 (Computation of the new iterate)**

Compute

$$\mathbf{x}^{(\nu+1)} = \mathbf{x}^{(\nu)} - \mathbf{K}_R \mathbf{y}^{(\nu)}.$$

**Step 2.4 (Termination criterion)**

If

$$\frac{\|\mathbf{x}^{(\nu+1)} - \mathbf{x}^{(\nu)}\|}{\|\mathbf{x}^{(\nu+1)}\|} < TOL,$$

stop the algorithm. Otherwise, set  $\nu := \nu + 1$  and go to Step 2.1

5.3. **Construction of a preconditioner II.** As far as the construction of a preconditioner for  $\mathbf{A}_{PP}$  is concerned, we choose a left transform  $\mathbf{A}_{PP}^L$  and a right transform  $\mathbf{A}_{PP}^R$  as the following block-diagonal matrices

$$(5.9a) \quad \mathbf{A}_{PP}^L = \begin{pmatrix} \alpha^{1/2}(\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{-1/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix},$$

$$(5.9b) \quad \mathbf{A}_{PP}^R = \begin{pmatrix} \alpha^{-1/2}(\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

We thus obtain the symmetric block matrix

$$\mathbf{A}_{PP}^L \mathbf{A}_{PP} \mathbf{A}_{PP}^R = \begin{pmatrix} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{-1/2} \mathbf{A}_1 (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} & \alpha^{-1/2} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} \mathbf{M} \\ \alpha^{-1/2} \mathbf{M} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} & -\mathbf{A}_2 \end{pmatrix}.$$

The Schur complement associated with  $\mathbf{A}_{PP}^L \mathbf{A}_{PP} \mathbf{A}_{PP}^R$  is given by

$$\mathbf{S} = \mathbf{A}_2 + \alpha^{-1} \mathbf{M} \mathbf{A}_1^{-1} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1}) \mathbf{M} = \mathbf{A}_2 + \mathbf{M} \mathbf{A}_1^{-1} \mathbf{T} + \alpha^{-1} \mathbf{M} \mathbf{A}_1^{-1} \mathbf{M}.$$

Consequently, we have

$$\mathbf{A}_{PP}^L \mathbf{A}_{PP} \mathbf{A}_{PP}^R = \begin{pmatrix} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{-1/2} \mathbf{A}_1 (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} & \alpha^{-1/2} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} \mathbf{M} \\ \alpha^{-1/2} \mathbf{M} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} & -\mathbf{S} + \alpha^{-1} \mathbf{M} \mathbf{A}_1^{-1} \mathbf{M} + \mathbf{M} \mathbf{A}_1^{-1} \mathbf{T} \end{pmatrix}.$$

With  $\hat{\mathbf{A}}_1$  as a preconditioner for  $\mathbf{A}_1$  and

$$(5.10) \quad \hat{\mathbf{S}} := \tau^{-1} \text{diag}(\mathbf{A}_2 + \alpha^{-1} \mathbf{M} \hat{\mathbf{A}}_1^{-1} \mathbf{M} + \mathbf{M} \hat{\mathbf{A}}_1^{-1} \mathbf{T}), \quad \tau > 0,$$

as a symmetric Uzawa preconditioner for  $\mathbf{A}_{PP}^L \mathbf{A}_{PP} \mathbf{A}_{PP}^R$  we choose

$$\tilde{\mathbf{A}}_{PP} = \begin{pmatrix} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{-1/2} \hat{\mathbf{A}}_1 (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} & \alpha^{-1/2} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} \mathbf{M} \\ \alpha^{-1/2} \mathbf{M} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1})^{1/2} & -\hat{\mathbf{S}} + \alpha^{-1} \mathbf{M} \hat{\mathbf{A}}_1^{-1} \mathbf{M} + \mathbf{M} \hat{\mathbf{A}}_1^{-1} \mathbf{T} \end{pmatrix}.$$

Backtransformation yields

$$(5.11) \quad \hat{\mathbf{A}}_{PP} = (\mathbf{A}_{PP}^L)^{-1} \tilde{\mathbf{A}}_{PP} (\mathbf{A}_{PP}^R)^{-1} = \begin{pmatrix} \hat{\mathbf{A}}_1 & \alpha^{-1} (\mathbf{I} + \alpha\mathbf{T}\mathbf{M}^{-1}) \mathbf{M} \\ -\mathbf{M} & \hat{\mathbf{S}} - \alpha^{-1} \mathbf{M} \hat{\mathbf{A}}_1^{-1} \mathbf{M} - \mathbf{M} \hat{\mathbf{A}}_1^{-1} \mathbf{T} \end{pmatrix}.$$

**Remark 6.** Step 2.2 of the preconditioned Richardson iteration requires the solution of linear algebraic systems of the form

$$(5.12) \quad \hat{\mathbf{A}}_{PP} \begin{pmatrix} \mathbf{w}_P \\ \mathbf{y}_P \end{pmatrix} = \begin{pmatrix} \mathbf{b}_P^1 \\ \mathbf{b}_P^2 \end{pmatrix}.$$

In view of (5.11), this can be done by the successive solution of the two linear subsystems

$$(5.13a) \quad \hat{\mathbf{S}}\mathbf{y}_P = \mathbf{b}_P^2 + \mathbf{M}\hat{\mathbf{A}}_1^{-1}\mathbf{b}_P^1,$$

$$(5.13b) \quad \hat{\mathbf{A}}_1\mathbf{w}_P = \mathbf{b}_P^1 - (\alpha^{-1}\mathbf{M} + \mathbf{T})\mathbf{y}_P.$$

An appropriate preconditioner  $\hat{\mathbf{A}}_1$  for  $\mathbf{A}_1$  is

$$(5.14) \quad \hat{\mathbf{A}}_1 = \sigma^{-1} \text{diag}(\mathbf{A}_1), \quad \sigma > 0,$$

which facilitates the solution of (5.13a),(5.13b).

**Remark 7.** Denoting by  $\mathbf{A}_1^{sym}$  and  $\mathbf{S}^{sym}$  the symmetric part of  $\mathbf{A}_1$  and  $\mathbf{S}$ , respectively, an appropriate choice of the parameters  $\tau$  and  $\sigma$  (cf. (5.10) and (5.14)) is as follows

$$(5.15) \quad \tau \leq \lambda_{max} \left( (\text{diag}(\mathbf{S}^{sym}))^{-1} \mathbf{S}^{sym} \right)^{-1}, \quad \sigma \leq \lambda_{max} \left( (\text{diag}(\mathbf{A}_1^{sym}))^{-1} \mathbf{A}_1^{sym} \right)^{-1},$$

where  $\lambda_{max}(\cdot)$  denotes the maximum eigenvalue of the respective matrix.

## 6. RESIDUAL-TYPE A POSTERIORI ERROR ESTIMATION

The residual a posteriori error estimator

$$(6.1) \quad \eta_h := \left( \sum_{K \in \mathcal{T}_h(Q)} (\eta_{K,1}^2 + \eta_{K,2}^2) + \sum_{F \in \mathcal{F}_h(Q)} (\eta_{F,1}^2 + \eta_{F,2}^2) + \sum_{F \in \mathcal{F}_h(\Sigma_{top})} \eta_{F,3}^2 \right)^{1/2}$$

consists of element residuals  $\eta_{K,i}, K \in \mathcal{T}_h(Q), 1 \leq i \leq 2$ , and face residuals  $\eta_{F,i}, F \in \mathcal{F}_h(Q), 1 \leq i \leq 2, \eta_{F,3}, F \in \mathcal{F}_h(\Sigma_{top})$ . In particular, the element residuals  $\eta_{K,i}, 1 \leq i \leq 2$ , are given by

$$(6.2a) \quad \eta_{K,1} := h_K \|f - \alpha^{-1}y_h\|_{L^2(K)}, \quad K \in \mathcal{T}_h(Q),$$

$$(6.2b) \quad \eta_{K,2} := h_K \|w_h\|_{L^2(K)}, \quad K \in \mathcal{T}_h(Q).$$

The face residuals  $\eta_{F,i}, 1 \leq i \leq 3$ , read as follows

$$(6.3a) \quad \eta_{F,1} := h_F^{1/2} \|\mathbf{n}_F \cdot [\nabla w_h]_F\|_{L^2(F)}, \quad F \in \mathcal{F}_h(Q),$$

$$(6.3b) \quad \eta_{F,2} := h_F^{1/2} \|\mathbf{n}_F \cdot [\nabla y_h]_F\|_{L^2(F)}, \quad F \in \mathcal{F}_h(Q),$$

$$(6.3c) \quad \eta_{F,3} := h_F^{1/2} \|(y_h)_t + w_h\|_{L^2(F)}, \quad F \in \mathcal{F}_h(\Sigma_{top}),$$

where  $[\nabla w_h]_F$  stands for the jump of  $\nabla w_h$  across  $F = K_+ \cap K_-, K_\pm \in \mathcal{T}_h(Q)$ , according to

$$[\nabla w_h]_F := (\nabla w_h)|_{K_+} - (\nabla w_h)|_{K_-}.$$

We note that  $[\nabla y_h]_F$  is defined analogously.

**Theorem 8.** *Let  $(w, y) \in W \times Y$  and  $(w_h, y_h) \in W_h \times V_h$  be the solution of (3.15a),(3.15b) and the space-time finite element approximation (4.3a),(4.3b), respectively. Let further  $\eta_h$  be the residual a posteriori error estimator as given by (6.1). Then, there holds*

$$(6.4) \quad \|(w - w_h, y - y_h)\|_{W \times Y} \lesssim \eta_h.$$

*Proof.* We first recall the definition of Clément's quasi-interpolation operator and state its stability and local approximation properties (cf., e.g., [82]).

For  $a \in \mathcal{N}_h(\bar{Q})$  we denote by  $\varphi_a$  the nodal basis function with supporting point  $a$ , and we refer to  $D_a$  as the patch

$$D_a := \bigcup \{ K \in \mathcal{T}_h(Q) \mid a \in \mathcal{N}_h(K) \}.$$

We refer to  $\pi_a$  as the  $L^2$ -projection onto  $P_1(D_a)$ , i.e.,  $\pi_a(w), w \in W$  is given by

$$(\pi_a(w), z)_{L^2(D_a)} = (w, z)_{L^2(D_a)}, \quad z \in P_1(D_a).$$



Then, Clément's interpolation operator  $P_C$  is defined as follows

$$P_C w := \sum_{a \in \mathcal{N}_h(\bar{Q})} \pi_a(w) \varphi_a.$$

For  $K \in \mathcal{T}_h(Q)$  and  $F \in \mathcal{F}_h(\bar{Q})$  we denote by  $D_K$  and  $D_F$  the patches

$$\begin{aligned} D_K &:= \bigcup \{K' \in \mathcal{T}_h(Q) \mid \mathcal{N}_h(K') \cap \mathcal{N}_h(K) \neq \emptyset\}, \\ D_F &:= \bigcup \{K' \in \mathcal{T}_h(Q) \mid \mathcal{N}_h(K') \cap \mathcal{N}_h(F) \neq \emptyset\}. \end{aligned}$$

Then, for  $v \in Y$  and  $K \in \mathcal{T}_h(Q), F \in \mathcal{F}_h(\bar{Q})$  there holds

$$(6.5a) \quad \|P_C v\|_{L^2(K)} \leq C \|v\|_{L^2(D_K)},$$

$$(6.5b) \quad \|P_C v\|_{L^2(F)} \leq C \|v\|_{L^2(D_F)},$$

$$(6.5c) \quad \|\nabla P_C v\|_{L^2(K)} \leq C \|\nabla v\|_{L^2(D_K)},$$

$$(6.5d) \quad \|v - P_C v\|_{L^2(K)} \leq C h_K \|v\|_{H^1(D_K)},$$

$$(6.5e) \quad \|v - P_C v\|_{L^2(F)} \leq C h_F^{1/2} \|v\|_{H^1(D_F)}.$$

Further, due to the finite overlap of the patches  $D_K$  and  $D_F$  we have

$$(6.6a) \quad \left( \sum_{K \in \mathcal{T}_h(Q)} \|v\|_{H^\mu(D_K)}^2 \right)^{1/2} \leq C \|v\|_{H^\mu(Q)}, \quad 0 \leq \mu \leq 1,$$

$$(6.6b) \quad \left( \sum_{F \in \mathcal{F}_h(\bar{Q})} \|v\|_{H^\mu(D_F)}^2 \right)^{1/2} \leq C \|v\|_{H^\mu(Q)}, \quad 0 \leq \mu \leq 1.$$

We note that (6.5a)-(6.5e) and (6.6a),(6.6b) hold true as well for  $w \in W$ .

The evaluation of the residuals  $\text{Res}_1$  and  $\text{Res}_2$  (cf. (3.23a),(3.23b)) in the dual norms relies on the Galerkin orthogonality

$$(6.7a) \quad \text{Res}_1(v_h) = 0, \quad v_h \in Y_h,$$

$$(6.7b) \quad \text{Res}_2(z_h) = 0, \quad z_h \in W_h.$$

For  $v \in Y$ , we choose  $v_h = P_C v$ . Then, due to (3.23a) and (6.7a) we have

$$\begin{aligned} (6.8) \quad \text{Res}_1(v) &= \text{Res}_1(v - P_C v) = \sum_{K \in \mathcal{T}_h(Q)} (f, v - P_C v)_{L^2(K)} \\ &\quad - \sum_{K \in \mathcal{T}_h(Q)} ((y_h)_t, (v - P_C v)_t)_{L^2(K)} - \sum_{K \in \mathcal{T}_h(Q)} (\nabla w_h, \nabla(v - P_C v))_{L^2(K)} \\ &\quad - \alpha^{-1} \sum_{K \in \mathcal{T}_h(Q)} (y_h, v - P_C v)_{L^2(K)} - \sum_{K \in \mathcal{T}_h(Q)} (w_h, v - P_C v)_{L^2(\partial K \cap \Sigma_{top})}. \end{aligned}$$

Observing  $(y_h)_{tt}|_K = 0$ ,  $K \in \mathcal{T}_h(Q)$ , as well as  $(v - P_C v)|_{\Sigma_{bot}} = 0$ , for the second term on the right-hand side of (6.8) partial integration yields

$$(6.9) \quad \sum_{K \in \mathcal{T}_h(Q)} ((y_h)_t, (v - P_C v)_t)_{L^2(K)} = \sum_{K \in \mathcal{T}_h(Q)} ((y_h)_t, v - P_C v)_{L^2(\partial K \cap \Sigma_{top})}.$$

On the other hand, taking  $(\Delta w_h)|_K = 0$ ,  $K \in \mathcal{T}_h(Q)$ , and  $(v - P_C v)|_{\Sigma_{lat}} = 0$  into account, for the third term on the right-hand side of (6.8) an application of Green's formula gives

$$(6.10) \quad \begin{aligned} \sum_{K \in \mathcal{T}_h(Q)} (\nabla w_h, \nabla(v - P_C v))_{L^2(K)} &= \sum_{K \in \mathcal{T}_h(Q)} (\mathbf{n}_{\partial K} \cdot \nabla w_h, v - P_C v)_{L^2(\partial K)} \\ &= \sum_{F \in \mathcal{F}_h(Q)} (\mathbf{n}_F \cdot [\nabla w_h]_F, v - P_C v)_{L^2(F)}. \end{aligned}$$

Using (6.9) and (6.10) in (6.8) yields

$$\begin{aligned} \text{Res}_1(v) &= \sum_{K \in \mathcal{T}_h(Q)} (f - \alpha^{-1} y_h, v - P_C v)_{L^2(K)} \\ &\quad - \sum_{F \in \mathcal{F}_h(Q)} (\mathbf{n}_F \cdot [\nabla w_h]_F, v - P_C v)_{L^2(F)} - \sum_{F \in \mathcal{F}_h(\Sigma_{top})} ((y_h)_t + w_h, v - P_C v)_{L^2(F)}. \end{aligned}$$

By straightforward estimation and the local approximation properties (6.5d),(6.5e) of Clément's quasi-interpolation operator we obtain

$$\begin{aligned} |\text{Res}_1(v)| &\leq \sum_{K \in \mathcal{T}_h(Q)} h_K \|f - \alpha^{-1} y_h\|_{L^2(K)} h_K^{-1} \|v - P_C v\|_{L^2(K)} \\ &\quad + \sum_{F \in \mathcal{F}_h(Q)} h_F^{1/2} \|\mathbf{n}_F \cdot [\nabla w_h]_F\|_{L^2(F)} h_F^{-1/2} \|v - P_C v\|_{L^2(F)} \\ &\quad + \sum_{F \in \mathcal{F}_h(\Sigma_{top})} h_F^{1/2} \|(y_h)_t + w_h\|_{L^2(F)} h_F^{-1/2} \|v - P_C v\|_{L^2(F)} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{K \in \mathcal{T}_h(Q)} h_K \|f - \alpha^{-1} y_h\|_{L^2(K)} \|v\|_{H^1(D_K)} \\
&\quad + \sum_{F \in \mathcal{F}_h(Q)} h_F^{1/2} \|\mathbf{n}_F \cdot [\nabla w_h]_F\|_{L^2(F)} \|v\|_{H^1(D_F)} \\
&\quad + \sum_{F \in \mathcal{F}_h(\Sigma_{top})} h_F^{1/2} \|(y_h)_t + w_h\|_{L^2(F)} \|v\|_{H^1(D_F)} \\
&\leq \left( \sum_{K \in \mathcal{T}_h(Q)} h_K^2 \|f - \alpha^{-1} y_h\|_{L^2(K)}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h(Q)} \|v\|_{H^1(D_K)}^2 \right)^{1/2} \\
&\quad + \left( \sum_{F \in \mathcal{F}_h(Q)} h_F \|\mathbf{n}_F \cdot [\nabla w_h]_F\|_{L^2(F)}^2 \right)^{1/2} \left( \sum_{F \in \mathcal{F}_h(Q)} \|v\|_{H^1(D_F)}^2 \right)^{1/2} \\
&\quad + \left( \sum_{F \in \mathcal{F}_h(Q)} h_F \|\mathbf{n}_F \cdot [\nabla w_h]_F\|_{L^2(F)}^2 \right)^{1/2} \left( \sum_{F \in \mathcal{F}_h(\Sigma_{top})} \|v\|_{H^1(D_F)}^2 \right)^{1/2}.
\end{aligned}$$

Observing (6.2a),(6.3a),(6.3c) and (6.6a),(6.6b) results in

(6.11)

$$\|\text{Res}_1\|_{Y^*} \lesssim \left( \sum_{K \in \mathcal{T}_h(Q)} \eta_{K,1}^2 \right)^{1/2} + \left( \sum_{F \in \mathcal{F}_h(Q)} \eta_{F,1}^2 \right)^{1/2} + \left( \sum_{F \in \mathcal{F}_h(\Sigma_{top})} \eta_{F,3}^2 \right)^{1/2}.$$

As far as the evaluation of  $\text{Res}_2$  is concerned, for  $z \in W$  we choose  $z_h = P_C z$ . Taking (3.23b) and (6.7b) into account, we find

(6.12)

$$\begin{aligned}
\text{Res}_2(z) &= \text{Res}_2(z - P_C z) = \sum_{K \in \mathcal{T}_h(Q)} (w_h, z - P_C z)_{L^2(K)} \\
&\quad - \sum_{K \in \mathcal{T}_h(Q)} (\nabla y_h, \nabla(z - P_C z))_{L^2(K)} + \sum_{F \in \mathcal{F}_h(\Sigma_{lat})} (\mathbf{n}_F \cdot \nabla y_h, z - P_C z)_{L^2(F)}.
\end{aligned}$$

Applying Green's formula elementwise and taking  $(\Delta y_h)|_K = 0$ ,  $K \in \mathcal{T}_h(Q)$ , into account, for the second term on the right-hand side in (6.12) it follows

that

$$\begin{aligned}
(6.13) \quad & \sum_{K \in \mathcal{T}_h(Q)} (\nabla y_h, \nabla(z - P_C z))_{L^2(K)} \\
&= \sum_{K \in \mathcal{T}_h(Q)} (\mathbf{n}_{\partial K} \cdot \nabla y_h, z - P_C z)_{L^2(\partial K \cap (Q \cup \Sigma_{\text{int}}))} \\
&= \sum_{F \in \mathcal{F}_h(Q)} (\mathbf{n}_F \cdot [\nabla y_h]_F, z - P_C z)_{L^2(F)} \\
&\quad + \sum_{F \in \mathcal{F}_h(\Sigma_{\text{int}})} (\mathbf{n}_F \cdot \nabla y_h, z - P_C z)_{L^2(F)}.
\end{aligned}$$

Using (6.13) in (6.12), we obtain

$$\text{Res}_2(z) = \sum_{K \in \mathcal{T}_h(Q)} (w_h, z - P_C z)_{L^2(K)} - \sum_{F \in \mathcal{F}_h(Q)} (\mathbf{n}_F \cdot [\nabla y_h]_F, z - P_C z)_{L^2(F)}.$$

Similar arguments as for the estimation of  $\text{Res}_1(v)$  give

$$\begin{aligned}
|\text{Res}_2(z)| &\lesssim \left( \sum_{K \in \mathcal{T}_h(Q)} h_K^2 \|w_h\|_{L^2(K)}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h(Q)} \|z\|_{H^1(D_K)}^2 \right)^{1/2} \\
&\quad + \left( \sum_{F \in \mathcal{F}_h(Q)} h_F \|\mathbf{n}_F \cdot [\nabla y_h]_F\|_{L^2(F)}^2 \right)^{1/2} \left( \sum_{F \in \mathcal{F}_h(Q)} \|z\|_{D_F}^2 \right)^{1/2}.
\end{aligned}$$

Due to (6.6a),(6.6b) and (6.2b),(6.3b) this results in

$$(6.14) \quad \|\text{Res}_2\|_{W^*} \lesssim \left( \sum_{K \in \mathcal{T}_h(Q)} \eta_{K,2}^2 \right)^{1/2} + \left( \sum_{F \in \mathcal{F}_h(Q)} \eta_{F,2}^2 \right)^{1/2}.$$

Combining (6.11) and (6.14) gives the assertion.  $\square$

In the step MARK of the adaptive cycle we use Dörfler marking [24]. In particular, given a universal constant  $0 < \theta < 1$ , we determine a set of elements  $\mathcal{M}_K$  and a set of faces  $\mathcal{M}_F$  such that

$$(6.15) \quad \theta \eta_h^2 \leq \sum_{K \in \mathcal{M}_K} (\eta_{K,1}^2 + \eta_{K,2}^2) + \sum_{F \in \mathcal{M}_F} (\eta_{F,1}^2 + \eta_{F,2}^2 + \eta_{F,3}^2)$$

The Dörfler marking can be realized by a greedy algorithm (cf., e.g., [39]).

## 7. NUMERICAL RESULTS

This section is devoted to a detailed documentation of numerical results for two examples illustrating the performance of the AFEM. The first example is set up in  $Q := (0, 1) \times (0, 1)$  and features a state  $y$  which rapidly changes in a vicinity of  $(x, t) = (0.25, 0.50)$  and  $(0.75, 0.50)$ . The second example is the adaptation of a benchmark from [23, 52] where  $Q := (-1, +1) \times (0, 1)$  and the state  $y$  decays exponentially around  $t = 0.50$ .

**Example 1.:** We choose  $\Omega = (0, 1)$ ,  $T = 1$ , and  $y^d = g - 0.1(g_{tt} - g_{xxxx})$ ,  $u^d = 0.9(g_t - g_{xx})$ ,  $y^0 = g(x, 0)$ ,  $x \in \Omega$ , as well as  $\alpha = 0.1$  where  $g(x, t) = r(x)s(t)$ ,  $(x, t) \in Q := \Omega \times (0, 1)$ , with

$$r(x) := \frac{10000x^4(1-x)^4}{1+1000(x-0.5)^2},$$

$$s(t) := \frac{1000t^2(1-t)^2}{1+100(t-0.25)^2} - \frac{1000t^2(1-t)^2}{1+100(t-0.75)^2}.$$

The solution  $(y, u, p)$  of the optimal control problem (2.3a)-(2.3e) is given by

$$y = g, \quad u = g_t - g_{xx}, \quad p = -\alpha (g_t - g_{xx}).$$

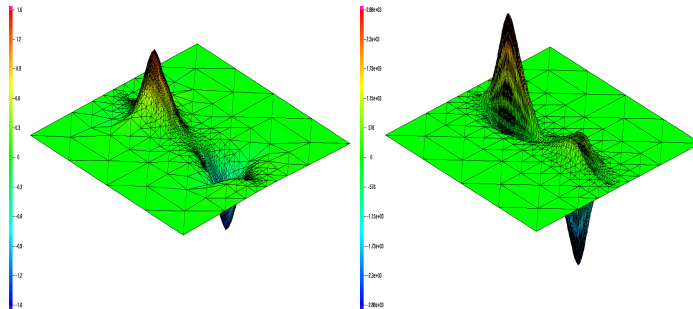


FIGURE 1. Example 1: Optimal state (left) and optimal control (right)

Figure 1 contains visualizations of the optimal state  $y$  (left) and the optimal control  $u$  (right), whereas Figure 2 displays the adaptively generated triangulations after 4 (left) and 8 (right) refinement steps.

Table 1 reflects the convergence history of the adaptive finite element method (AFEM). In particular, it contains the total number of degrees of freedom (DOF) and the discretization errors in  $y$  and  $w$  per refinement step  $\ell$ .

Figures 3, 4, and 5 provide a comparison between adaptive and uniform refinement. On a logarithmic scale, the decrease in the errors  $\|y - y_h\|_{L^2(Q)}$ ,  $\|y - y_h\|_Y$ ,  $\|w - w_h\|_{L^2(Q)}$ ,  $\|w - w_h\|_W$ , and  $|J(y, u) - J_h(y_h, u_h)|$  is shown as a function of the degrees of freedom (DOF) ( $\theta = 0.5$  in the Dörfler marking).

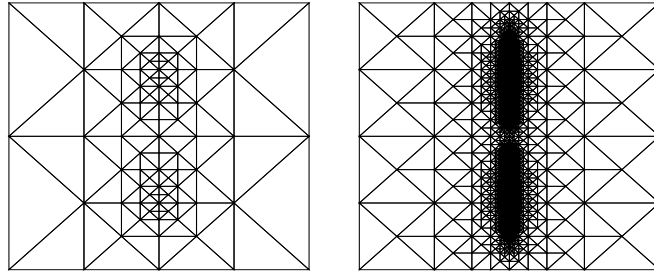


FIGURE 2. Example 1: Adaptively refined triangulations after 4 (left) and 8 (right) cycles of the adaptive algorithm

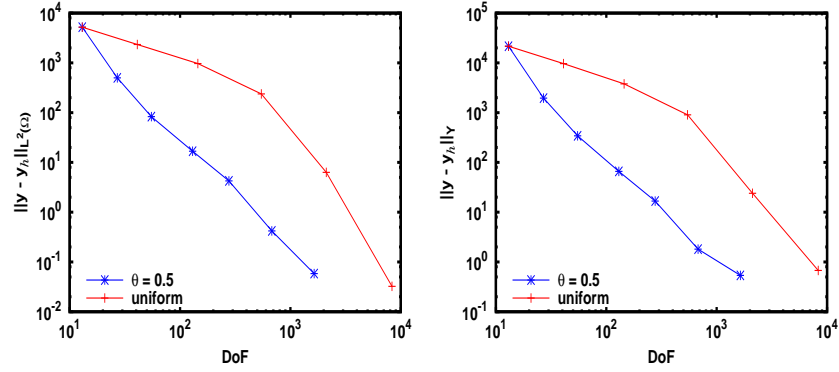


FIGURE 3. Example 1: Adaptive versus uniform refinement: Error in  $y$  ( $L^2$ -norm (left) and  $Y$ -norm (right)).

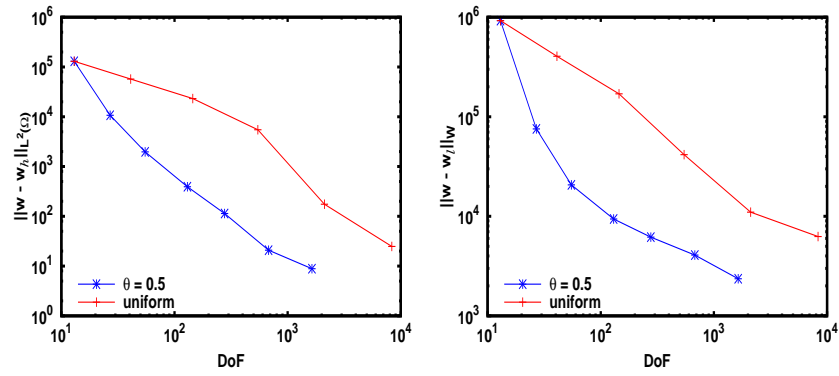


FIGURE 4. Example 1: Adaptive versus uniform refinement: Error in  $w$  ( $L^2$ -norm (left) and  $W$ -norm (right)).

TABLE 1. Example 1: Convergence history of the AFEM. Discretization errors in  $y$  and  $w$

l	DOF	$\ y - y_h\ _{L^2(Q)}$	$\ y - y_h\ _Y$	$\ w - w_h\ _{L^2(Q)}$	$\ w - w_h\ _W$
1	13	5.21e+03	2.16e+04	1.30e+05	9.24e+05
2	27	4.99e+02	1.96e+03	1.07e+04	7.55e+04
3	55	8.32e+01	3.42e+02	1.97e+03	2.07e+04
4	130	1.68e+01	6.61e+01	3.93e+02	9.40e+03
5	277	4.25e+00	1.67e+01	1.14e+02	6.18e+03
6	678	4.20e-01	1.81e+00	2.09e+01	4.08e+03
7	1639	5.84e-02	5.38e-01	8.87e+00	2.36e+03
8	4317	5.64e-02	4.18e-01	4.31e+00	1.42e+03
9	11391	4.16e-02	2.74e-01	1.94e+00	8.57e+02

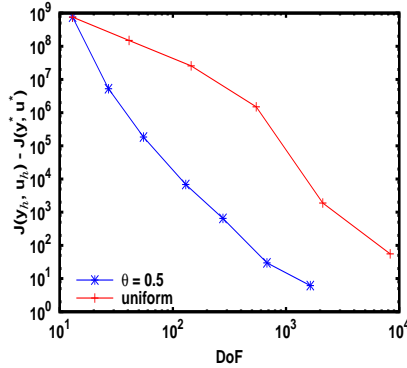


FIGURE 5. Example 1: Adaptive versus uniform refinement: Error in the objective functional.

**Example 2:** We choose  $\Omega = (0, 1)$ ,  $T = 1$ , and  $y^d = g - \alpha(g_{tt} - g_{xxxx})$ ,  $u^d = 0$ ,  $y^0 = g(x, 0)$ ,  $x \in \Omega$ , as well as  $\alpha = 1.0$  where  $g(x, t)$ ,  $(x, t) \in Q := \Omega \times (0, 1)$ , is given by

$$g(x, t) = x^3(1-x)^3t^2(1-t)^2 \arctan(60(r-1)),$$

$$r^2 := (x - 5/4)^2 + (t + 1/4)^2.$$

The solution  $(y, u, p)$  of the optimal control problem (2.3a)-(2.3e) reads as follows

$$y = g, \quad u = g_t - g_{xx}, \quad p = -\alpha(g_t - g_{xx}).$$

Figure 6 displays the optimal state  $y$  (left) and the optimal control  $u$  (right). Figure 7 shows the adaptively generated triangulations after 4 (left) and 8 (right) refinement steps.

Table 2 documents the convergence history of the AFEM for Example 2 with the same legends as for Example 1.

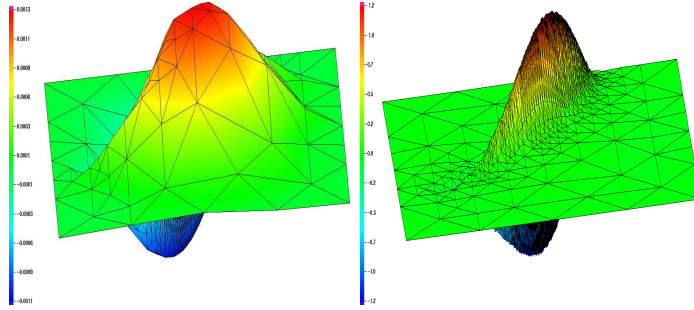


FIGURE 6. Example 2: Optimal state (left) and optimal control (right)

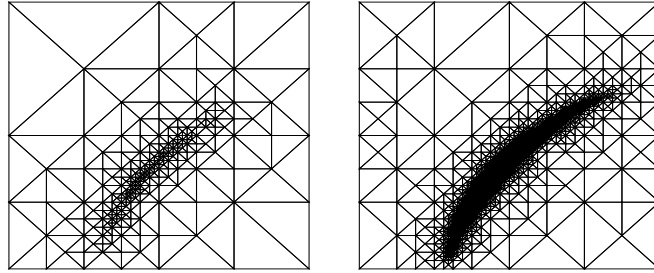


FIGURE 7. Example 2: Adaptively refined triangulations after 4 (left) and 8 (right) cycles of the adaptive algorithm

TABLE 2. Example 2: Convergence history of the AFEM. Discretization errors in  $y$  and  $w$

1	DOF	$\ y - y_h\ _{L^2(Q)}$	$\ y - y_h\ _Y$	$\ w - w_h\ _{L^2(Q)}$	$\ w - w_h\ _W$
1	18	3.19e-01	1.26e+00	7.41e+00	5.23e+01
2	42	1.22e-01	4.60e-01	2.93e+00	2.38e+01
3	93	4.53e-02	1.83e-01	1.11e+00	1.24e+01
4	190	3.31e-03	1.59e-02	1.28e-01	7.13e+00
5	388	4.73e-04	2.65e-03	3.27e-02	4.40e+00
6	935	1.89e-04	1.19e-03	1.40e-02	2.61e+00
7	2570	9.64e-05	8.08e-04	6.44e-03	1.48e+00
8	6963	5.53e-05	5.60e-04	3.20e-03	8.71e-01
9	18936	3.87e-05	3.67e-04	1.43e-03	4.43e-01

Finally, Figures 8, 9, and 10 display the performance of the AFEM in comparison to uniform refinement.



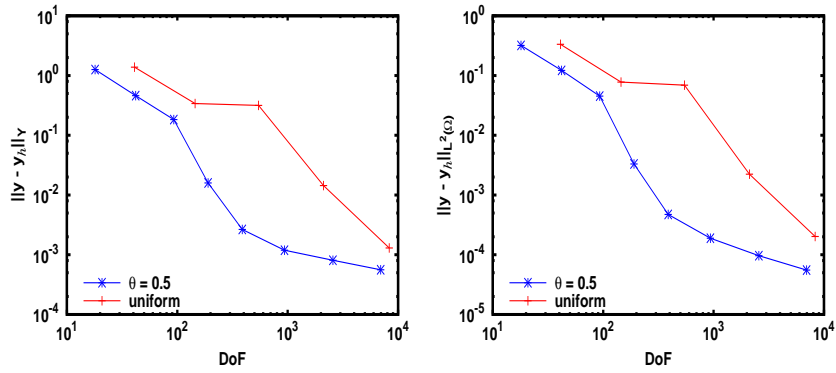


FIGURE 8. Example 2: Adaptive versus uniform refinement: Error in  $y$  ( $L^2$ -norm (left) and  $Y$ -norm (right)).

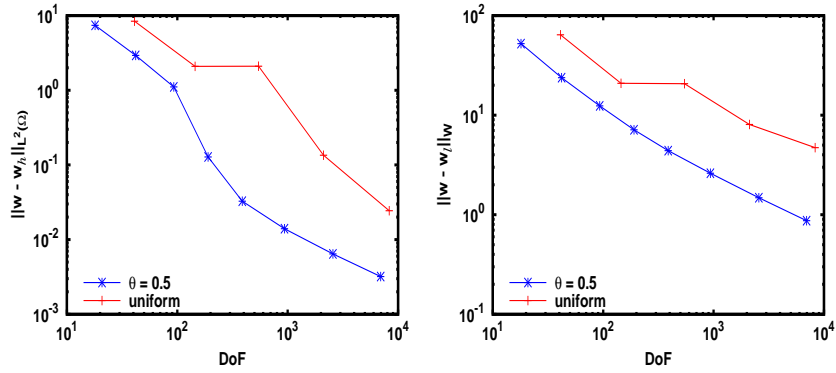


FIGURE 9. Example 2: Adaptive versus uniform refinement: Error in  $w$  ( $L^2$ -norm (left) and  $W$ -norm (right)).

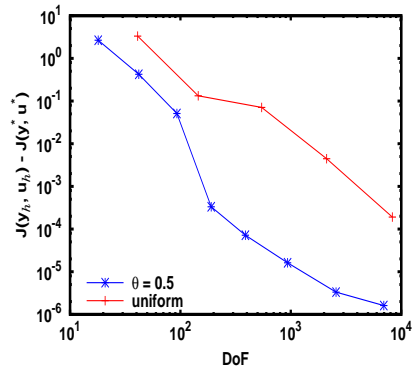


FIGURE 10. Example 2: Adaptive versus uniform refinement: Error in the objective functional.

## 8. CONCLUSIONS

For the numerical solution of optimal control problems with distributed controls for linear second order parabolic initial-boundary value problems we have developed, analyzed, and implemented an adaptive finite element method based on the formulation of the optimality system as a fourth order elliptic boundary value problem which can be equivalently stated as a second order system. This enables the use of P1 conforming finite elements with respect to simplicial triangulations of the space-time domain. We have put emphasis on

- the iterative solution of the resulting algebraic saddle point problem by a preconditioned Richardson-type iterative scheme featuring preconditioners constructed by means of appropriately chosen left and right transforms,
- the derivation of a reliable residual-type a posteriori error estimator with the framework of a unified a posteriori error control.

Numerical results have confirmed the theoretical findings and thus documented the feasibility of this novel adaptive approach.

So far we have only considered the unconstrained case, i.e., we have imposed neither constraints on the control nor on the state. Future work will be devoted to the application of the adaptive approach to control constrained as well as to state constrained optimally controlled parabolic problems.

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