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HOMOTOPY OF EXCEPTIONAL SYMMETRIC SPACES

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Abstract. We apply a result due to Mitchell to determine explicitly homotopy groups of some exceptional symmetric spaces.

1. Introduction

Using cell decomposition Mitchell [Mit-88, Thm. 7.1] proved a generalized version of Bott's periodicity theorem [Bo-59] that also applies to some exceptional spaces, where no periodicity phenomenon occurs. Mitchell's theorem says that the low degree homotopy groups of the based loop space of an irreducible simply connected symmetric space P of compact type coincide with the homotopy groups of isotropy orbits (s-orbits) of P formed by extrinsically symmetric elements. These s-orbits are extrinsically symmetric submanifolds in a tangent space of P (see [Fe-80]) and can be identified with certain spaces of shortest geodesics in P (see [MQ-11]). Some of the arising spaces of shortest geodesics are examples of centrioles (see [CN-88, Nag-88] for the definition of centrioles and [Bu-85, Bu-92, Nag-88] for applications of centrioles and related geometric objects to homotopy).

In this note we apply Mitchell's result to exceptional symmetric spaces and determine explicitly some homotopy groups of certain exceptional symmetric spaces, in particular for $\mathrm{EVII} = \mathrm{E}_7/\left(S^1\mathrm{E}_6\right)$, see Table 7. As a starting point we use known homotopy groups of exceptional Lie groups and symmetric spaces, see Tables 3 and 4, that can be found in [BS-58, Co-65, Co-66, Bu-85, Bu-92, MT-91, Mim-95].

The determination of the homotopy groups presented here is a part of the author's habilitation thesis [Qu-10]. The author thanks J.-H. Eschenburg for helpful discussions and remarks, and A.-L. Mare for valuable hints.

2. Mitchell's theorem

Let P be a simply connected irreducible symmetric space of compact type and G the identity component of its isometry group. We choose a base point o in P and denote by s_o the geodesic symmetry of P at o and by K the stabilizer of o in G. Thus we can identify P with the coset space G/K. Since the sequence $\{0\} = \pi_1(\mathrm{G/K}) \to \pi_0(\mathrm{K}) \to \pi_0(\mathrm{G}) = \{0\}$ is exact, K is connected.

Conjugation by s_0 is an involutive Lie group automorphism of G. We denote the induced automorphism on its Lie algebra $\mathfrak{g}$ by σ. There is a well known identification of T_oP with the (-1)-eigenspace $\mathfrak{p}$ of σ (see e.g. [He-78, p. 208, Thm. 3.3]). Using this identification the linear isotropy representation of K on T_oP becomes the adjoint representation $\mathrm{Ad}_{\mathrm{G}}(\mathrm{K})$ restricted to $\mathfrak{p}$.

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An element $\xi \in \mathfrak{p}$ is called *extrinsically symmetric* (or a *minuscule coweight*) if $\mathrm{ad}(\xi)^3 = -\mathrm{ad}(\xi)$. Let ξ be such an element and $\mathfrak{a}^+$ a Weyl chamber in some maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ containing ξ in its closure, i.e. $\xi \in \overline{\mathfrak{a}^+}$. Denote by $\Sigma = \{\alpha_1, ..., \alpha_r\}$ be the system of positive simple roots, considered as real valued linear forms on $\mathfrak{a}$, that corresponds to $\mathfrak{a}^+$. Then $\xi = \alpha_j^*$ for some $\alpha_j \in \Sigma$ whose coefficient in the expansion of the highest root δ corresponding to Σ is 1 (see [MQ-11, Lem. 2.1] and [KN-64]). Here α_j^* is defined by the property that $\alpha_k(\alpha_j^*) = 0$ if $k \neq j$ and $\alpha_j(\alpha_j^*) = 1$. We refer to [He-78] for the non-explained terminology used here.

The main tool in this note is Mitchell's generalized version of Bott's periodicity theorem [Bo-59]:

Theorem 1 (Thm. 7.1 in [Mit-88] and Prop. 2.6 in[Mit-87])**.** *Let $P = \mathrm{G}/\mathrm{K}$ be an irreducible simply connected symmetric space of compact type and o a point in P. Assume that $\xi \in \mathfrak{p}$ is extrinsically symmetric. Then*

$$\pi_{i+1}(P) \cong \pi_i(\mathrm{Ad}_{\mathrm{G}}(\mathrm{K})\xi) \qquad \textit{for} \quad 0 \leq i \leq d_\xi - 2. \tag{1}$$

The number d_ξ is obtained as follows: Denote $\xi = \alpha_j^*$ as above. Let γ be a path in the extended Dynkin diagram of P that joins α_j to $-\delta$. The sum of the multiplicities of all vertices (representing simple roots in the Dynkin diagram) along γ including the end points α_j and $-\delta$ is denoted d_γ . The minimum of d_γ over all such paths is d_ξ.

Besides the explicit validity bound for Equation 1, another advantage of Mitchell's above version of Bott's periodicity theorem is that it also applies to exceptional symmetric spaces, where no periodicity can occur.

3. Determination of homotopy groups

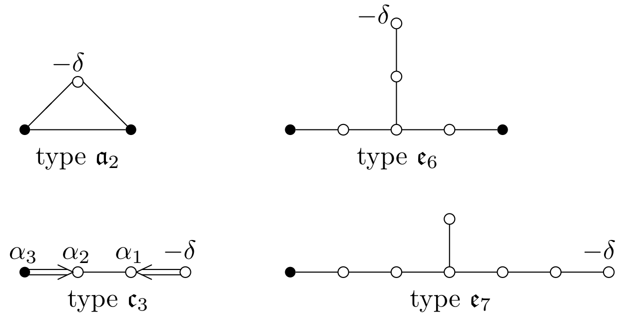
To apply Mitchell's result, Theorem 1, to exceptional symmetric spaces, we summarize all information we need about irreducible exceptional symmetric spaces of compact type that admit extrinsically symmetric elements in the following table:

Table 1: Extrinsically symmetric s-orbits

$P = \mathrm{G}/\mathrm{K}$	type of P	Multiplicities	$\mathrm{Ad}_{\mathrm{G}}(\mathrm{K})\xi$	d_ξ
E_6	$\mathfrak{e}_6$	always 2	$\mathrm{E}_6/\left(S^1\mathrm{Spin}_{10}\right)$	10
E_7	$\mathfrak{e}_7$	always 2	$\mathrm{E}_7/\left(S^1\mathrm{E}_6\right)$	14
$\mathrm{E}_6/\mathrm{Sp}_4$	$\mathfrak{e}_6$	always 1	$G_2(\mathbb{H}^4)/\mathbb{Z}_2$	5
$\mathrm{E}_6/\mathrm{F}_4$	$\mathfrak{a}_2$	always 8	$\mathbb{O}P^2 = \mathrm{F}_4/\mathrm{Spin}_9$	16
$\mathrm{E}_7/\mathrm{SU}_8$	$\mathfrak{e}_7$	always 1	$(\mathrm{SU}_8/\mathrm{Sp}_4)/\mathbb{Z}_2$	7
$\mathrm{E}_7/\left(S^1\mathrm{E}_6\right)$	$\mathfrak{c}_3$	$m_{\alpha_1} = m_{\alpha_2} = 8$ $m_{\alpha_3} = m_{-\delta} = 1$	$\left(S^1\mathrm{E}_6\right)/\mathrm{F}_4$	18

By E_7, E_6 and F_4 we denote the connected and simply connected real compact Lie group whose Dynkin diagram is of type $\mathfrak{e}_7$, $\mathfrak{e}_6$ and $\mathfrak{f}_4$ respectively (see [He-78]). The description of $\mathrm{Ad}_{\mathrm{G}}(\mathrm{K})\xi$ is taken from [BCO-03, p. 311] (see also [KN-64]), and the Dynkin diagram type of P with multiplicities can be found e.g. in [He-78, p. 534]. The value of d_ξ can now be read off from the extended Dynkin diagrams in Table 2 below that are taken from [He-78, p. 503]. We marked every simple root with coefficient 1 in the highest root, i.e. those giving rise to extrinsically symmetric elements, by a black node (see [He-78, p. 476 - 478] for the coefficients of simple roots in the highest root). Observe that the diagrams of type $\mathfrak{a}_2$ and $\mathfrak{e}_6$ admit two such roots. Since they can be identified by a diagram automorphism, the corresponding extrinsically symmetric s-orbits are isomorphic.

Table 2: Extended Dynkin diagrams



As a consequence of Mitchell's theorem (Theorem 1) we get:

$$\pi_{i+1}(\mathrm{E}_6) \cong \pi_i\left(\mathrm{E}_6/(S^1\mathrm{Spin}_{10})\right), \qquad 0 \le i \le 8; \tag{2}$$

$$\pi_{i+1}(\mathrm{E}_7) \cong \pi_i\left(\mathrm{E}_7/(S^1\mathrm{E}_6)\right), \qquad 0 \le i \le 12; \tag{3}$$

$$\pi_{i+1}(\mathrm{E}_6/\mathrm{Sp}_4) \cong \pi_i\left(G_2(\mathbb{H}^4)/\mathbb{Z}_2\right), \qquad 0 \le i \le 3; \tag{4}$$

$$\pi_{i+1}(\mathrm{E}_6/\mathrm{F}_4) \cong \pi_i(\mathbb{O}P^2), \qquad 0 \le i \le 14; \tag{5}$$

$$\pi_{i+1}(\mathrm{E}_7/\mathrm{SU}_8) \cong \pi_i((\mathrm{SU}_8/\mathrm{Sp}_4)/\mathbb{Z}_2), \qquad 0 \le i \le 5; \tag{6}$$

$$\pi_{i+1}\left(\mathrm{E}_7/(S^1\mathrm{E}_6)\right) \cong \pi_i\left((S^1\mathrm{E}_6)/\mathrm{F}_4\right), \qquad 0 \le i \le 16; \tag{7}$$

Equation 6 coincides with [Bu-92, Prop. 2.4] and Equation 7 is also stated in [Nag-88, p. 74] with reference to Burns' thesis [Bu-85].

Since the homotopy group $\pi_i(M)$ of a space M is the same as the homotopy group $\pi_i(\tilde{M})$ of its universal cover $\tilde{M}$ for $i \ge 2$ and as $\pi_i(M_1 \times M_2) \cong \pi_i(M_1) \times \pi_i(M_2)$ we get with $\pi_i(\mathbb{R}) \cong 0$

$$\pi_{i+1}(\mathrm{E}_6/\mathrm{Sp}_4) \cong \pi_i\left(\mathrm{G}_2(\mathbb{H}^4)\right), \qquad i = 2, 3; \tag{8}$$

$$\pi_{i+1}(\mathrm{E}_7/\mathrm{SU}_8) \cong \pi_i(\mathrm{SU}_8/\mathrm{Sp}_4), \qquad 2 \le i \le 5; \tag{9}$$

$$\pi_{i+1}\left(\mathrm{E}_7/(S^1\mathrm{E}_6)\right) \cong \pi_i(\mathrm{E}_6/\mathrm{F}_4), \qquad 2 \le i \le 16. \tag{10}$$

In particular there is the equation chain:

$$\pi_{i+3}(\mathrm{E}_7) \cong \pi_{i+2}\left(\mathrm{E}_7/(S^1\mathrm{E}_6)\right) \cong \pi_{i+1}(\mathrm{E}_6/\mathrm{F}_4) \cong \pi_i\left(\mathbb{O}P^2\right), \qquad 1 \le i \le 10.$$

Known homotopy groups. Using these equations we want to describe the above mentioned homotopy groups explicitly. To start we need to know some homotopy groups explicitly.

The fundamental groups of irreducible symmetric spaces of compact type that do not nontrivially cover another symmetric space (called adjoint spaces in [He-78]) have been determined by É. Cartan [Ca-27] (see also [Ta-64]).

The second homotopy groups of irreducible compact symmetric spaces that are not Lie groups can be found in [Ta-64, p. 122].

For a compact connected simple non-abelian real Lie group É. Cartan showed that the second homotopy group vanishes. By Bott [Bo-56, p. 253] the third homotopy group of a compact simple non-abelian Lie group is isomorphic to $\mathbb{Z}$.

One finds some low degree homotopy groups of exceptional Lie groups in [BS-58, Thm. V, p. 995], [MT-91, p. 363] and [Mim-95, p. 970]. For our purpose we use:

Table 3: Homotopy groups of E_6 and E_7

	$\pi_4 - \pi_8$	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
E_6	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_{12}$	0	0	$\mathbb{Z}$
E_7	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Conlon [Co-65, Co-66] described the homotopy of some exceptional symmetric spaces, among them $\mathrm{E}_6/(S^1\mathrm{Spin}_{10})$ and $\mathrm{E}_6/\mathrm{F}_4$, in terms of the homotopy groups of spheres. In this note we use:

$$\pi_{i+1}\left(\mathrm{E}_6/(S^1\mathrm{Spin}_{10})\right) \cong \pi_i\left(S^7\right), \quad 2 \le i \le 14 \qquad \text{[Co-65, Cor. 1];} \tag{11}$$

$$\pi_i(\mathrm{E}_6/\mathrm{F}_4) \cong \pi_i\left(S^9\right), \quad 1 \le i \le 15 \qquad \text{[Co-66, p. 411];} \tag{12}$$

$$\pi_{16}(\mathrm{E}_6/\mathrm{F}_4) \cong 0. \qquad \text{[Co-66, p. 411].} \tag{13}$$

Using geometrical methods that have been introduced by Chen and Nagano [CN-78] to study totally geodesic submanifolds of symmetric spaces, Burns [Bu-92] (see also [Bu-85]) found various relations among homotopy groups of symmetric spaces. We are interested in:

$$\pi_{i+1}\left(\mathrm{E}_6/(S^1\mathrm{Spin}_{10})\right) \cong \pi_i\left(S^1\right), \quad 1 \le i \le 5 \qquad \text{[Bu-92, Prop. 2.3];} \tag{14}$$

$$\pi_{i+1}(\mathrm{E}_6/\mathrm{F}_4) \cong \pi_i\left(S^8\right), \quad 1 \le i \le 14 \qquad \text{[Bu-92, Prop. 2.2];} \tag{15}$$

$$\pi_{i+1}\left(\mathrm{E}_7/(S^1\mathrm{E}_6)\right) \cong \pi_i\left(S^1\right), \quad 1 \le i \le 7 \qquad \text{[Bu-92, Prop. 2.5];} \tag{16}$$

$$\pi_{i+1}\left(\mathbb{O}P^2\right) \cong \pi_i\left(S^7\right), \quad 1 \le i \le 13 \qquad \text{[Bu-92, Prop. 2.1].} \tag{17}$$

Looking up the well-known homotopy groups of the spheres mentioned above (see e.g. [To-62]) we get from the equations 11 to 17:

Table 4: Known homotopy groups for some exceptional symmetric spaces

	π_2	$\pi_3 - \pi_7$	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}	π_{16}
$\mathrm{E}_6/\left(S^1\mathrm{Spin}_{10}\right)$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	
$\mathrm{E}_6/\mathrm{F}_4$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	0
$\mathrm{E}_7/\left(S^1\mathrm{E}_6\right)$	$\mathbb{Z}$	0	0								
$\mathbb{O}P^2$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$		

Notice that Equation 5 is consistent with Table 4, but does not provide any further information. Some homotopy groups in the above table follow directly from the homotopy groups of Lie groups given in [Mim-95, p. 969 - 970] and the long exact homotopy sequence for coset spaces

$$\ldots \to \pi_{n+1}(\mathrm{K}) \to \pi_{n+1}(\mathrm{G}) \to \pi_{n+1}(\mathrm{G}/\mathrm{K}) \to \pi_n(\mathrm{K}) \to \ldots \quad . \tag{18}$$

Our goal is to determine some further homotopy groups of $\mathrm{E}_6/\mathrm{Sp}_4$, $\mathrm{E}_7/\mathrm{SU}_8$ and $\mathrm{E}_7/\left(S^1\mathrm{E}_6\right)$ using the equations 2 to 10. The most interesting results certainly concern $\mathrm{E}_7/\left(S^1\mathrm{E}_6\right)$.

Homotopy groups of $\mathrm{E}_6/\mathrm{Sp}_4$**.** To explore Equation 8 we need to know some low degree homotopy groups of $G_2(\mathbb{H}^4)$. This space appears in the geometric inclusion chain leading to Bott's periodicity theorem for the orthogonal group (see [Bo-59], [Mil-69, §24] or [Mit-88, Cor. 7.2]) and we obtain:

$$\pi_i\left(G_2(\mathbb{H}^4)\right) \cong \pi_{i+1}(\mathrm{U}_8/\mathrm{Sp}_4) \cong \pi_{i+2}(\mathrm{SO}_{16}/\mathrm{U}_8) \cong \pi_{i+3}(\mathrm{SO}_{16}),\ 2 \le i \le 5. \tag{19}$$

As $\pi_5(\mathrm{SO}_{16}) \cong \pi_6(\mathrm{SO}_{16}) \cong 0$ (see e.g. [Mim-95, p. 970]) we obtain $\pi_2\left(G_2(\mathbb{H}^4)\right) \cong \pi_3\left(G_2(\mathbb{H}^4)\right) \cong 0$. Since $G_2(\mathbb{H}^4)$ is simply connected we can deduce the following table from Equations 4 and 8:

Table 5: Homotopy groups of $\mathrm{E}_6/\mathrm{Sp}_4$

	π_2	$\pi_3 - \pi_4$
$\mathrm{E}_6/\mathrm{Sp}_4$	$\mathbb{Z}_2$	0

The homotopy groups thus obtained can also be calculated using the long exact homotopy sequence, Equation 18. The second homotopy group of $\mathrm{E}_6/\mathrm{Sp}_4$ can be found in [Ta-64, p. 122].

Homotopy groups of $\mathrm{E}_7/\mathrm{SU}_8$**.** To explore Equations 9 we need to know some low degree homotopy groups of $\mathrm{SU}_8/\mathrm{Sp}_4$. The space $\mathrm{U}_8/\mathrm{Sp}_4$ appears in Equation 19. As $\pi_4(\mathrm{SO}_{16}) \cong \pi_5(\mathrm{SO}_{16}) \cong \pi_6(\mathrm{SO}_{16}) \cong 0$ and $\pi_7(\mathrm{SO}_{16}) \cong \mathbb{Z}$ (see e.g. [Mim-95, p. 969 - 970]) we obtain with $\pi_i(\mathrm{SU}_8/\mathrm{Sp}_4) \cong \pi_i(\mathrm{U}_8/\mathrm{Sp}_4)$, $i \ge 2$, the homotopy groups $\pi_2(\mathrm{SU}_8/\mathrm{Sp}_4) \cong \pi_3(\mathrm{SU}_8/\mathrm{Sp}_4) \cong \pi_4(\mathrm{SU}_8/\mathrm{Sp}_4) \cong 0$ and $\pi_5(\mathrm{SU}_8/\mathrm{Sp}_4) \cong \mathbb{Z}$. Since $\mathrm{SU}_8/\mathrm{Sp}_4$ is simply connected, the equations 6 and 9 yield:

Table 6: Homotopy groups of $\mathrm{E}_7/\mathrm{SU}_8$

	π_2	$\pi_3 - \pi_5$	π_6
$\mathrm{E}_7/\mathrm{SU}_8$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Observe that this could also be obtained from the long exact homotopy sequence, Equation 18, since $\pi_4(\mathrm{SU}_8) \cong 0$ and $\pi_5(\mathrm{SU}_8) \cong \mathbb{Z}$ (see e.g. [Mim-95, p. 969 - 970]). The second homotopy group of $\mathrm{E}_7/\mathrm{SU}_8$ can again be found in [Ta-64, p. 122].

Higher homotopy groups of $\mathrm{E}_7/(S^1\mathrm{E}_6)$**.** From Equation 10 and the second line of Table 4 we obtain:

Table 7: Further homotopy groups of $\mathrm{E}_7/(S^1\mathrm{E}_6)$

	π_9	π_{10}	$\pi_{11} - \pi_{12}$	π_{13}	$\pi_{14} - \pi_{15}$	π_{16}	π_{17}
$\mathrm{E}_7/\left(S^1\mathrm{E}_6\right)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}_2$	0

Notice that the above table is consistent with Equation 3. Some of the homotopy groups in Table 7 , e.g. $\pi_9\left(\mathrm{E}_7/(S^1\mathrm{E}_6)\right)$, $\pi_{10}\left(\mathrm{E}_7/(S^1\mathrm{E}_6)\right)$ and $\pi_{14}\left(\mathrm{E}_7/(S^1\mathrm{E}_6)\right)$, can also be read off from the long exact homotopy sequence in Equation 18.

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