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CONVEXITY OF REFLECTIVE SUBMANIFOLDS IN SYMMETRIC R-SPACES

PETER QUAST AND MAKIKO SUMI TANAKA

Abstract. We show that every reflective submanifold of a symmetric R-space is (geodesically) convex.

Introduction

The main result in this article is:

Theorem 1. *Reflective submanifolds of symmetric R-spaces are (geodesically) convex.*

We organized this article as follows: In Section 1 we define all notions used in Theorem 1. Reflective submanifolds in symmetric R-spaces are described in Section 2. The proof of Theorem 1 can be found in Section 3. In Section 4 we explain why the assumption "symmetric R-space" in Theorem 1 can not be easily generalized.

Symmetric R-spaces form a class of compact symmetric spaces that have very peculiar geometric properties and appear in various contexts. For example symmetric R-spaces arise as certain spaces of shortest geodesics, namely as those centrioles (see [CN88]) that are formed by midpoints of shortest geodesics arcs joining a base point to a pole (see e.g. [MQ11a]). Reflective submanifolds in symmetric spaces include among others polars and centrioles (see e.g. [CN88, Na88, Qu11]). An iterative construction involving such centrioles has been used by Bott [Bo59] in the first proof of his famous periodicity result for the homotopy groups of classical Lie groups (see also [Mi69, § 23,24] and [Mit-88, § 7]). For the construction described in [MQ11b, Sect. 1.2] it is important that the distance between a base point and a pole in a centriole of certain R-spaces measured in the centriole is the same as the distance measured in the ambient R-space. This follows directly from Theorem 1.

Theorem 1 also provides a conceptional proof of [NS91, Remark 3.2b] in the case where the ambient space is a symmetric R-space.

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1. Preliminaries

We first define the terminology used in Theorem 1.

Reflective submanifolds. A *reflective submanifold* M of a Riemannian manifold P is a connected component of the fixed point set of some involutive isometry τ of P, that is τ^2 equals the identity. Thus reflective submanifolds are totally geodesic (see e.g. [BCO-03, Prop. 8.3.4]). To contain many reflective submanifolds the ambient Riemannian manifold P must have a large isometry group. An interesting class of ambient manifolds are therefore symmetric spaces. In the series of papers [Le73, Le74, Le79a, Le79b] D.S.P. Leung studied and classified reflective submanifolds in simply connected irreducible symmetric spaces of compact type.

Convexity. We call a totally geodesic submanifold $M \subset P$ of a Riemannian manifold P *(geodesically) convex*, if the (Riemannian) distance $d_M(m_1, m_2)$ in M between any pair of points $m_1, m_2 \in M$ coincides with the Riemannian distance $d_P(m_1, m_2)$ in the abient space P. In other words, a submanifold $M \subset P$ is convex if any shortest geodesic arc in M joining two arbitrarily chosen points m_1 and m_2 in M is still shortest in P (see also [Sa96, p. 26, 84]).

Symmetric spaces. Before defining the terminology "symmetric R-space" we shortly introduce some useful notions about symmetric spaces. We refer to the Helgason's standard monograph [He78] for proofs and further details about symmetric spaces. Let S be a (Riemannian) *symmetric space,* that is a connected Riemannian manifold such that for point $p \in S$ there exists an isometry s_p of S that fixes p and whose differential at p is $-\mathrm{Id}$ on T_pS. One can show that symmetric spaces are geodesically complete and homogeneous.

We now fix an origin $o \in S$ and get an involutive Lie group automorphism σ of the isometry group $\mathrm{I}(S)$ of S defined by $\sigma(g) = s_o \circ g \circ s_o$ for any $g \in \mathrm{I}(S)$. Its differential σ_* at the identity is an involutive automorphism of the Lie algebra of $\mathrm{I}(S)$.

The (-1)-eigenspace $\mathfrak{s}$ of σ_* is called the *Lie triple* corresponding to (S, o). It is identified with T_oS by the differential at the identity of the principal bundle $\mathrm{I}(S) \to S,\ g \mapsto g.o$, where $g.o$ denotes the point in S obtained by applying g to the origin o. By this identification $\mathfrak{s}$ carries a scalar product denoted by $\langle . , . \rangle$ induced from the Riemannian metric on T_oS. The curvature tensor on T_oP is the expressed in $\mathfrak{s}$ by double Lie brackets and the geodesics of P emanating from o are of the form

$$t \mapsto \exp(tX).o \text{ with } X \in \mathfrak{s},$$

where exp is the Lie theoretic exponential map. The linear isotropy action on $\mathfrak{s}$ coincides with the adjoint action restricted to $\mathfrak{s}$.

Orthogonal unit lattices. Choose a maximal abelian subspace $\mathfrak{t} \subset \mathfrak{s}$ in $\mathfrak{s}$, then $T := \exp(\mathfrak{t}).o$ is a maximal complete connected totally geodesic flat submanifold of S, a maximal flat torus.

For a compact symmetric space S, the *unit lattice*

$$\Gamma(S, \mathfrak{t}) := \{X \in \mathfrak{t} : \ \exp(X).o = o\}$$

of S is called *orthogonal*, if there exists a basis $\{b_1, ..., b_r\}$ of $\mathfrak{t}$ with the following properties:

(i) $\langle b_j, b_k \rangle = 0$, if $j \neq k$.

(ii) $\Gamma(S, \mathfrak{t}) = \mathrm{span}_{\mathbb{Z}}(b_1, ..., b_r) = \left\{ \sum_{j=1}^{r} x_j b_j \ : \ x_j \in \mathbb{Z} \right\}.$

Symmetric R-spaces. Symmetric R-spaces form a distinguished subclass of compact Riemannian symmetric spaces. They arise as particular orbits of s-representations, i.e. linear isotropy representations of symmetric spaces of compact type:

Let S be a symmetric space of compact type, that is the universal Riemannian cover of S is still compact, and o an origin in S. Using the notation introduced above we take a nonzero element $\xi \in \mathfrak{s}$ that satisfies

$$\mathrm{ad}(\xi)^3 = -\mathrm{ad}(\xi).$$

Then the connected isotropy orbit $P := \mathrm{Ad}_{\mathrm{I}(S)}(H)\xi \subset \mathfrak{s}$ is a *symmetric R-space.* Here $H \subset \mathrm{I}(S)$ denotes the identity component of the isotropy group of $o \in S$, which is a compact Lie group. Thus symmetric R-spaces are always compact.

The orbit $P \subset \mathfrak{s}$ is extrinsically symmetric in the Euclidean space $\mathfrak{s}$, that is, P is invariant under the reflections at all its affine normal spaces (see [Fe80]). In particular symmetric R-spaces are (Riemannian) symmetric spaces (w.r.t. the submanifold metric induced by the scalar product on $\mathfrak{s}$). Ferus [Fe74] (see also [Fe80, EH95]) has shown that the converse also holds, namely every full compact extrinsically symmetric submanifold in a Euclidean space is a symmetric R-space.

Irreducible symmetric R-spaces have been first classified by Kobayashi and Nagano in [KN64]. A list of them can also be found in [BCO-03, p. 311]. Takeuchi [Tak-84] has shown that irreducible symmetric R-spaces are either irreducible hermitian symmetric spaces of compact type or compact connected real forms of them and vice-versa.

Theorem 2 (Satz 3 in [Lo85])**.** *The unit lattice of a R-symmetric space P is orthogonal.*

Following Loos [Lo85] this property is actually an intrinsic characterization of symmetric R-spaces among compact symmetric spaces.

2. Reflective submanifolds of symmetric R-spaces

Let now $M \subset P$ be a reflective submanifold of a symmetric R-space P and $o \in M$ a chosen origin. Since P is compact and M a closed subset of P, M is also compact. Let G be the transvection group of P, that is the identity component of $\mathrm{I}(P)$. The topology underlying the Lie group structure of G is the compact-open topology (see e.g. [He78]). Thus the identity component L of $\{g \in G : \ g(M) \subset M\}$ is a closed subgroup of the compact Lie group G and therefore a compact Lie group, too. Since M is a totally geodesic submanifold of P, L contains all transvections of P along geodesics of M. Thus L acts transitively (but maybe highly non effectively) on M.

The involution σ of G given by $\sigma(g) = s_o \circ g \circ s_o$ for all $g \in G$ leaves $\{g \in G : \ g(M) \subset M\}$ and therefore also L invariant and induces an involutive automorphism of L which we also denote by σ. We set

$$H := \{l \in L : \ l.o = o\}.$$

Since H is a closed subgroup of the compact Lie group L, H is a compact Lie subgroup of L.

Observation 3. *(L, H) is a compact Riemannian symmetric pair (in the sense defined in* [Sa77, p. 137]*).*

Proof. We are left to show that $L^\sigma_e \subset H \subset L^\sigma$, where $L^\sigma \subset L$ is the fixed point set of σ in L and L^σ_e its identity component.

Let K be the subgroup of G formed by all transvections of P that leave o fix. It is well-known that $G^\sigma_e \subset K \subset G^\sigma$ (here G^σ is the fixed point set of σ in G and G^σ_e its identity component), see e.g. [He78]. Since $H = L \cap K$, $L^\sigma = L \cap G^\sigma$ and L^σ_e is the identity component of $L \cap G^\sigma_e$ the claims follows, because

$$L^\sigma_e \subset L \cap G^\sigma_e \subset H = L \cap K \subset L \cap G^\sigma = L^\sigma.$$

□

Let $\mathfrak{p}$ be the Lie triple corresponding to (P, o) and $\mathfrak{m} \subset \mathfrak{p}$ the subtriple of $\mathfrak{p}$ corresponding to (M, o) (see [He78, p. 224 f.] for further explications). If τ denotes the involutive isometry of P such that M is a connected component of the fixed point set of τ and τ_* the involution on $\mathfrak{p}$ induced by the differential of τ at o, then $\mathfrak{m}$ is the fix point set and its orthogonal complement $\mathfrak{m}^\perp$ in $\mathfrak{p}$ is the (-1)-eigenspace of τ_*. Notice that s_o and τ commute (see [Le73, p. 156]). We get an involutive Lie group automorphism

$$\tilde{\tau} : G \to G, \ g \mapsto \tau \circ g \circ \tau.$$

Since the curves $t \mapsto (\tau \circ \exp(tX) \circ \tau).o$ and $t \mapsto (\tau \circ \exp(tX)).o$ in P coincide, we see that on $\mathfrak{p} \cong T_oP$ the differential $\tilde{\tau}_*$ of $\tilde{\tau}$ at the identity coincides with the differential τ_* of τ at o and therefore leaves $\mathfrak{p}$ invariant.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}$ and $\mathfrak{t}$ a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$.

Observation 4 (see Lemma 3.1 in [TT11]). *$\mathfrak{t}$ is invariant under τ_*.*

Proof. The arguments given here are similar to the proof of [Lo-69-II, Prop. 3.2, p. 125]. Take $T \in \mathfrak{t}$, then $T + \tau_*(T)$ lies in $\mathfrak{m}$. Since

$$\begin{array}{rclcl} [A, T + \tau_*(T)] & = & [A, T] + [A, \tau_*(T)] & = & [\tau_*(A), \tau_*(T)] \\ & = & [\tilde{\tau}_*(A), \tilde{\tau}_*(T)] & = & \tilde{\tau}_*([A, T]) \\ & = & 0 & & \end{array}$$

for all $A \in \mathfrak{a}$ and $\mathfrak{a}$ is a maximal abelian subset of $\mathfrak{m}$, we see that $T + \tau_*(T) \in \mathfrak{a}$ and hence $\tau_*(T) = (T + \tau_*(T)) - T \in \mathfrak{t}$. □

Observation 4 implies that $\mathfrak{t}$ splits as an orthogonal direct sum

$$\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{a}^\perp$$

with $\mathfrak{a}^\perp = \mathfrak{t} \cap \mathfrak{m}^\perp$.

Since τ_* is the differential of an involutive isometry of P that leaves $\mathfrak{t}$ invariant, $\tau_*|_\mathfrak{t}$ is an orthogonal transformation of $\mathfrak{t}$ that squares to the identity and preserves the unit lattice $\Gamma(P, \mathfrak{t}) \subset \mathfrak{t}$. As the unit lattice of the symmetric R-space P is orthogonal (see Theorem 2 due to Loos [Lo85]), there exists an orthogonal basis $\{b_1, ..., b_r\}$ of $\mathfrak{t}$ that generates $\Gamma(P, \mathfrak{t})$ over $\mathbb{Z}$.

Proposition 5 (Prop. 3.3 in [TT11]). *There exists an orthogonal basis $\{e_1, ..., e_r\}$ of $\mathfrak{t}$ with the following properties:*

(i) $\Gamma(P, \mathfrak{t}) = \mathrm{span}_\mathbb{Z}(e_1, ..., e_r) = \left\{ \sum_{j=1}^{r} x_j e_j : \ x_j \in \mathbb{Z} \right\}$,

(ii) There exist integer numbers p, q with $0 \leq 2p \leq q \leq r$ such that
- $\tau_*(e_{2j}) = e_{2j-1}$, *for* $1 \leq j \leq p$,
- $\tau_*(e_j) = e_j$, *for* $2p + 1 \leq j \leq q$,
- $\tau_*(e_j) = -e_j$, *for* $q + 1 \leq j \leq r$.

Proof. Tanaka and Tasaki presented a differential geometric proof of this result (see [TT11, proof of Prop. 3.3]). In this paper we are inclined to give an elementary linear algebraic construction of the orthogonal basis $\{e_1, ..., e_r\}$.

Without loss of generality we may assume that the orthogonal basis $B = \{b_1, ..., b_r\}$ that generates the unit lattice $\Gamma(P, \mathfrak{t})$ over $\mathbb{Z}$ is ordered by length, that is $\|b_1\| \leq \|b_2\| \leq ... \leq \|b_r\|$. Let $s \in \{1, ..., r\}$ be the integer number such that $\|b_1\| = \|b_j\|$ for $j = 1, ..., s$ and $\|b_1\| < \|b_{s+1}\|$. If $0 \neq x = \sum_{j=1}^r x_j b_j \in \Gamma(P, \mathfrak{t})$, that is $x_j \in \mathbb{Z}$, then $\|x\|^2 = \sum_{j=1}^r x_j^2 \|b_j\|^2 \geq \|b_1\|$, and $\|x\| = \|b_1\|$ holds true if and only if $x \in \{\pm b_1, ..., \pm b_s\}$. Since τ_* is an orthogonal map that preserves $\Gamma(P, \mathfrak{t})$, we conclude that

$$\tau_*(b_j) \in \{\pm b_1, ..., \pm b_s\} \quad \text{for all} \quad j \in \{1, ..., s\}.$$

Let $V := \mathrm{span}_{\mathbb{R}}\{b_1, ..., b_s\}$, then $V^{\perp} = \mathrm{span}_{\mathbb{R}}\{b_{s+1}, ..., b_r\}$. Since the orthogonal endomorphism τ_* leaves V invariant, the same holds true for $V^{\perp}$. By applying the above arguments to $\tau_*|_{V^{\perp}}$ and by iterating this scheme we get

$$\tau_*(b_j) \in \{\pm b_1, ..., \pm b_r\} \quad \text{for all} \quad j \in \{1, ..., r\}.$$

Since τ_* is involutive $\tau_*(b_j) = b_k$ implies $\tau_*(b_k) = b_j$ and $\tau_*(b_j) = -b_k$ implies $\tau_*(b_k) = -b_j$.

After renumbering $\{b_1, ..., b_r\}$ suitably, we can therefore assume that

- $\tau_*(b_{2j}) = \pm b_{2j-1}$, for $1 \leq j \leq p$,
- $\tau_*(b_j) = b_j$, for $2p + 1 \leq j \leq q$,
- $\tau_*(b_j) = -b_j$, for $q + 1 \leq j \leq r$.

for some integers p, q with $0 \leq 2p \leq q \leq r$. We now choose the desired basis $\{e_1, ..., e_r\}$ as follows:

- $e_{2j-1} = b_{2j-1}$, for $1 \leq j \leq p$,
- $e_{2j} = \begin{cases} b_{2j}, & \text{if} \quad \tau_*(b_{2j}) = b_{2j-1} \\ -b_{2j}, & \text{if} \quad \tau_*(b_{2j}) = -b_{2j-1} \end{cases}$, for $1 \leq j \leq p$,
- $e_j = b_j$, for $2p + 1 \leq j \leq r$.

□

Since $\mathfrak{a}$ is the fixed point set of τ_* in $\mathfrak{t}$, Proposition 5 implies:

Corollary 6 (Prop. 3.3 in [TT11])**.**

$$\mathfrak{a} = \left\{ \sum_{j=1}^{r} x_j e_j : \ x_{2j-1} = x_{2j} \textit{ for } 1 \leq j \leq p, \ \textit{and } x_{q+1} = ... = x_r = 0 \right\}.$$

3. Proof of the main result, Theorem 1

A reflective submanifold $M \subset P$ in a compact symmetric R-space is itself a compact connected symmetric space and hence complete. The classical theorem of Hopf and Rinow (see e.g. [Sa96, p. 84]) tells us that any two points $m_1, m_2 \in M$ can be joined by a geodesic in M that is shortest within M. If such a shortest geodesic in M is still shortest within P, then M is geodesically convex.

The *tangent cut locus* $\tilde{C}(T_pP)$ of a compact Riemannian manifold P at a point $p \in M$ is the set of all tangent vectors $X \in T_pP$ such that

- $d_P(p, \gamma_X(t)) = t\|X\|$ for $t \in [0, 1]$ and
- $d_P(p, \gamma_X(t)) < t\|X\|$ for $t > 1$,

where d_P denotes the Riemannian distance in P (see e.g. [Sa96, p. 26] for the definition) and γ_X is the geodesic in P that emanates from p in the direction X. We refer to [Sa96, p. 104] for further explication concerning the cut locus.

Thus $M \subset P$ is convex, if

$$\tilde{C}(T_mM) = T_mM \cap \tilde{C}(T_mP) \tag{1}$$

holds for any point $m \in M$. Since M is homogeneous, it suffices to verify Equation (1) at just one point $o \in M$.

Sakai [Sa77, Thm. 2.5] has shown that the tangent cut locus of a compact symmetric space is determined up to the isotropy action by the tangent cut locus of a maximal flat totally geodesic torus. Tasaki [Tas10, Lemma 2.2] adapted Sakai's result to totally geodesic submanifolds. We now state Tasaki's result in a version that is specialized to fit best our needs and set up. We use again the notions established in Sections 1 and 2. Tasaki's assumptions in [Tas10, Lemma 2.2] concerning the symmetric pairs hold true by Observation 3.

Lemma 7 (see Lemma 2.2 in [Tas10])**.** *Let M be a reflective submanifold of a symmetric R-space P, o a point in M, $\mathfrak{a}$ an arbitrarily chosen maximal abelian linear subspace of $\mathfrak{m} \cong T_oM$ and $\mathfrak{t}$ a maximal abelian linear subspace of $\mathfrak{p} \cong T_oP$ that contains $\mathfrak{a}$. Let A be the maximal flat torus of M corresponding to $\mathfrak{a}$ and T the maximal flat torus of P corresponding to $\mathfrak{t}$, that is $\mathfrak{a} \cong T_oA$ and $\mathfrak{t} \cong T_oT$. If*

$$\tilde{C}(\mathfrak{a}) = \mathfrak{a} \cap \tilde{C}(\mathfrak{t}) \tag{2}$$

then

$$\tilde{C}(\mathfrak{m}) = \mathfrak{m} \cap \tilde{C}(\mathfrak{p})$$

and M is a (geodesically) convex submanifold of P.

Thus, to prove Theorem 1, we just need to show that Equation (2) is satisfied. We do this by showing

Claim 8. *For all points $a \in A$ we have*

$$d_A(o,a) = d_T(o,a).$$

Proof. Both maps

$$\mathfrak{a} \to A,\ X \mapsto \exp(X).o \quad \text{and} \quad \mathfrak{t} \to T,\ Y \mapsto \exp(Y).o$$

are Riemannian coverings between flat spaces. Thus they map straight lines in $\mathfrak{a}$ and $\mathfrak{t}$ onto geodesics of A and T, and very geodesic arises in this way.

Let $a \in A$ be an arbitrary chosen point in A, then $a = \exp(X).o$ for some $X \in \mathfrak{a}$. In view of Corollary 6 one can write $X = \sum_{j=1}^{r} x_j e_j$ with

- $x_{2j} = x_{2j-1}$ for $1 \leq j \leq p$,
- $x_{q+1} = ... = x_r = 0$,

where $\{e_1, ..., e_r\}$ is the orthogonal basis of $\mathfrak{t}$ mentioned in Proposition 5. Using Theorem 2 we get

$$\begin{aligned} d_T^2(o,a) &= \min\{\|X+Y\|^2 : \ Y \in \Gamma(P,\mathfrak{t})\} \\ &= \min\left\{\left\|\sum_{j=1}^{r}(x_j+y_j)e_j\right\|^2 : \ y_1,...,y_r \in \mathbb{Z}\right\} \\ &= \min\left\{\sum_{j=1}^{r}(x_j+y_j)^2 : \ y_1,...,y_r \in \mathbb{Z}\right\}. \end{aligned}$$

We now choose integer numbers $z_1, ..., z_r \in \mathbb{Z}$ as follows:

- for $1 \leq j \leq p$ we choose z_{2j} such that
$$(x_{2j}+z_{2j})^2 = \min\{(x_{2j}+y_{2j})^2 : \ y_{2j} \in \mathbb{Z}\}$$
and set $z_{2j-1} := z_{2j}$. Since $x_{2j} = x_{2j-1}$, we also get
$$(x_{2j-1}+z_{2j-1})^2 = \min\{(x_{2j-1}+y_{2j-1})^2 : \ y_{2j-1} \in \mathbb{Z}\},$$
- for $2p+1 \leq j \leq q$ we choose $z_j \in \mathbb{Z}$ such that
$$(x_j+z_j)^2 = \min\{(x_j+y_j)^2 : \ y_j \in \mathbb{Z}\},$$
- $z_{q+1} = ... = z_r = 0$.

These choices ensure that
$$\sum_{j=1}^{r}(x_j+z_j)^2 = d_T^2(o,a).$$
Moreover the vector $Z = \sum_{j=1}^{r} z_j e_j \in \Gamma(P,\mathfrak{t})$ satisfies

- $z_{2j} = z_{2j-1}$ for $1 \leq j \leq p$,
- $z_{q+1} = ... = z_r = 0$,

that is $Z \in \mathfrak{a}$.

Notice that $\exp(X+Z).o = \exp(X)\exp(Z).o = \exp(X).o = a$ and $d_T^2(o,a) = \|X+Z\|^2$. Since $X+Z \in \mathfrak{a}$ and $d_T^2(o,a) \leq d_A^2(o,a)$, we get $d_T^2(o,a) = d_A^2(o,a)$, and Claim 8 follows. □

4. Counterexamples

Though our proof of Theorem 1 relies on Loos' characterization of symmetric R-spaces in terms of orthogonal unit lattices, one may ask if the statement of Theorem 1 holds true for reflective submanifolds in arbitrary compact symmetric spaces. In this section we present two counterexamples for such a statement, that arose in discussion with Jost-Hinrich Eschenburg.

Example 9. Take a flat 2-torus P with a non-rectangular rhombic lattice. Then the long diagonal in the rhombic lattice gives a reflective submanifold M of P. The shortest geodesic in P joining the midpoint of a rhombic fundamental domain to a vertex of it follows the short diagonal and does therefore not lie in the reflective submanifold M. Thus M is not convex.

With this picture in mind for a maximal flat torus in a symmetric space, one gets a first counterexample of the statement Theorem 1, if one replaces the assumption "symmetric R-space" by symmetric space of compact type.

Example 10. Consider SU_3 equipped with the bi-invariant metric induced by

$$\langle X, Y\rangle = \mathrm{trace}(XY),\ X, Y \in \mathfrak{su}_3.$$

The complex conjugation is an involutive isometry of SU_3, whose fixed point set is SO_3. Since the complex conjugation leaves the center

$$C = \left\{I_3,\ e^{\frac{2\pi}{3}i} I_3,\ e^{\frac{4\pi}{3}i} I_3\right\}$$

of SU_3 invariant, it descends to an involutive isometry σ of the irreducible symmetric spaces $P = \mathrm{SU}_3/C \cong \mathrm{Ad}(\mathrm{SU}_3)$. Since SO_3 does not meet the center of SU_3, the restriction of the Riemannian covering

$$\pi : \mathrm{SU}_3 \to \mathrm{SU}_3/C,\ x \mapsto [x]$$

to SO_3 is an injective map and $M = \pi(\mathrm{SO}_3)$ is the fixed point set of σ and therefore a reflective submanifold of P.

A shortest geodesic arc within M joining $[I_3]$ to $\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} e^{\frac{\pi}{3}i} & 0 & 0 \\ 0 & e^{\frac{\pi}{3}i} & 0 \\ 0 & 0 & e^{-\frac{2\pi}{3}i} \end{pmatrix}\right]$ is given by

$$\gamma_1 : [0, \pi] \to M \subset P,\ t \mapsto \left[e^{tX_1}\right] \text{ with } X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

But there is a considerably shorter geodesic arc in P with the given endpoints, namely

$$\gamma_2 : [0, \pi] \to M \subset P,\ t \mapsto \left[e^{tX_2}\right] \text{ with } X_2 = \begin{pmatrix} \frac{1}{3}i & 0 & 0 \\ 0 & \frac{1}{3}i & 0 \\ 0 & 0 & -\frac{2}{3}i \end{pmatrix}.$$

Notice that $\|X_2\|^2 = \frac{2}{3} < 2 = \|X_1\|^2$. This shows that the reflective submanifold M of P is not geodesically convex.

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