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Resonantly driven Brownian motion: Basic concepts and exact results

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We consider the Brownian motion of a particle in a force field exposed to external periodic modulations. Discussed are ergodic properties, the long-time behavior of correlation functions, and spectral densities for general (nonlinear) force fields. For a parabolic potential we present analytical results for the quasispectrum (quasieigenvalues and quasieigenfunctions) at all values of the damping constant as well as spectral densities and cycle-averaged probability distributions. The theory is also applied to a Brownian oscillator with memory damping, yielding additional resonances between the external field and internal degrees of freedom.

I. INTRODUCTION

Periodically driven Brownian motion is often used for modeling activation processes in physical systems or devices, such as in lasers,[1,2] shunted Josephson junctions,[3,4] and earth climate models.[5–8] In spite of the large number of applications, the theory of periodically modulated stochastic processes is developed only in a fragmentary fashion. The activation out of a metastable state in the presence of periodic perturbations has been considered in the weak-damping regime by Larkin and Ovchinikov,[9] Ivlev and Melnikov,[10] Fonseca and Grigolini,[11] Carmeli and Nitzan,[12] and Munakata.[13,14] One of the main problems seems to be the treatment of purely coherent modulations, since an infinite sequence of resonance zones (resonance between the higher harmonics of the well oscillations and the external modulation) does not allow a description in terms of energy or action variables, where the other variables are eliminated adiabatically. For the high-damping regime (Smoluchowski limit) an approximate discrete state description has been proposed by Nicolis,[5] McNamara and Wiesenfeld,[15] and Gammaitoni *et al.*[16] This analysis, however, is based on adiabatic arguments and thus is only valid for small modulation frequencies (smaller than all typical system frequencies). An approximate continuous-state description has been presented recently by Fox[17] and Presilla, Marchesoni, and Gammaitoni.[18] The analysis of Fox and Marchesoni is based on an eigenfunction expansion of the Green's functions. For the overdamped-limit case we recently have developed a Floquet-type theory.[19,20] We have proven rigorously some generic ergodic characteristics such as δ spikes in the phase-averaged spectral power density and the sensitive dependence on the initial conditions of the large-time behavior of the system. Further, we have presented numerical results for the asymptotic (large-time) probability distributions, the dynamical susceptibilities[19] and the modulation-enhanced escape rates.[20] Analog simulations have been performed[15,16,21,22] to study the phenomenon of *stochastic resonance.* The goal of this paper is to present some general properties of periodically modulated stochastic differential equations for small-to-moderate-to-large values of the damping and to introduce a general language for the description of such systems.

This paper is a generalization of our recent papers,[19,20] where we have restricted ourselves to the overdamped regime, to the case with inertia, and *arbitrary friction.* In Sec. II we introduce the two-dimensional nonstationary description with quasieigenvalues and quasieigenfunctions as well as the three-dimensional stationary description with ordinary eigenvalues and eigenfunctions. Ergodic properties and their relations to the correlations functions and spectral densities are discussed. In Sec. III explicit analytical results for the resonantly driven Ornstein-Uhlenbeck process, such as the quasispectrum, asymptotic probabilities, correlation functions, and spectral densities as well as phase-averaged probabilities are given. In Sec. IV our formalism is applied to the problem of Brownian motion in an oscillator with memory friction and colored noise.

II. BASIC CONCEPTS

We consider the motion of a Brownian particle in a force field $U'(x)$ under the action of white Gaussian noise and an external periodic modulation. The corresponding Langevin equation in normalized (dimensionless) variables reads

$$\ddot{x}+\gamma\dot{x}+U'(x)=A\sin(\omega_0 t+\phi)+\xi(t)\ , \tag{2.1a}$$

where γ is the friction coefficient, ϕ is a random initial phase, and $\xi(t)$ denotes Gaussion white noise, i.e.,

$$\begin{aligned}&\langle\xi(t)\rangle=0\ ,\\&\langle\xi(t)\xi(t')\rangle=2\gamma T\delta(t-t')\ .\end{aligned} \tag{2.1b}$$

Without the modulation, i.e., for $A=0$, the system is in equilibrium with its heat bath, with T being the normalized temperature. With $A\neq 0$, Eqs. (2.1) describe a nonstationary Markovian process in the two-dimensional $x,v(=\dot{x})$ phase space.

A. Spectrum and quasispectrum

The corresponding Fokker-Planck equation for the probability density $P(x,v,t)$, i.e.,

$$\frac{\partial P}{\partial t}=\left[-v\frac{\partial}{\partial x}+\gamma\frac{\partial}{\partial v}v+\frac{\partial}{\partial v}U'(x)+\gamma T\frac{\partial^2}{\partial v^2}\right.$$
$$\left.-A\sin(\omega_0 t+\phi)\frac{\partial}{\partial v}\right]P(x,v,t)$$
$$\equiv\left[\underline{L}_0-A\sin(\omega_0 t+\phi)\frac{\partial}{\partial v}\right]P(x,v,t)\ , \tag{2.2}$$

has a drift coefficient with discrete time-translation symmetry. This guarantees Floquet-type solutions of the form[19,20,23,24]

$$P^{(\mu)}(x,v,t)=\exp(-\mu t)p^{(\mu)}(x,v,t)\ ,$$
$$p^{(\mu)}(x,v,t)=p^{(\mu)}(x,v,t+T)\ . \tag{2.3}$$

The dynamics of the periodically modulated stochastic systems is completely described by the quasieigenvalues μ and quasieigenfunctions $p^{(\mu)}(x,v,t)$. They are the natural extension to the eigenvalue-eigenfunction concept for stationary stochastic processes.

Equivalently, the stochastic process (2.1) can also be treated as a three-dimensional time-homogeneous Markovian process

$$\dot{x}=v\ ,$$
$$\dot{v}=-\gamma v-U'(x)+A\sin\theta+\xi(t)\ , \tag{2.4}$$
$$\dot{\theta}=\omega_0\ ,$$

with the corresponding three-dimensional Fokker-Planck equation (FPE) for the probability density W, obeying

$$\frac{\partial}{\partial t}W(x,v,\theta,t)=\underline{L}_{\rm FP}W\ , \tag{2.5}$$

$$\frac{\partial}{\partial t}W(x,v,\theta,t)\equiv\underline{L}_0W-A\sin\theta\frac{\partial}{\partial v}W-\omega_0\frac{\partial}{\partial\theta}W\ , \tag{2.6}$$

where θ is now a random variable due to random initial conditions. Proposing indistinguishable phases θ, $\theta+2\pi,\ldots$, we have to impose periodic boundary conditions in θ. The periodicity of W, i.e., $W(x,v,\theta+2\pi,t)=W(x,v,\theta,t)$, immediately implies that the quasieigenvalues μ are identical with the eigenvalues of $\underline{L}_{\rm FP}$ (2.6).[19,20]

B. Ergodic properties

Exactly in the same way as in the overdamped case[19,20] it can be shown that the Fokker-Planck operator $\underline{L}_{\rm FP}$ in (2.6) has a branch of purely imaginary eigenvalues

$$\mu_{\rm on}=\lambda_{\rm on}=in\omega_0,\quad n=-\infty,\ldots,0,\ldots,\infty \tag{2.7a}$$

where the spectrum of the Fokker-Planck operator (2.6) is defined by

$$\underline{L}_{\rm FP}\psi=-\lambda\psi\ . \tag{2.7b}$$

An initial probability which has a *nonvanishing* weight on the subspace S, spanned up by the left and right eigenfunctions

$$\{\phi_{\rm on}(x,v,\theta)=\exp(-in\theta)\}_{n=-\infty}^{\infty}\ ,$$
$$\{\psi_{\rm on}(x,v,\theta)\}_{n=-\infty}^{\infty}$$

of $\underline{L}_{\rm FP}$ in (2.6) thus does not decay to a steady state on S, but rather moves dispersion-free in S. Therefore, although the coefficients of the FPE (2.6) are time independent, a general initial probability does not approach a steady-state probability for large times. The conditions for zero weight on S is—just like in the overdamped case[19]—a uniformly distributed initial phase ϕ in Eq. (2.1) i.e.,

$$w(\phi)=\frac{1}{2\pi}\ . \tag{2.8}$$

C. Asymptotic probability density

For large times the solution of the two-dimensional Fokker-Planck equation (2.2) approaches an asymptotic periodic probability density $P_{\rm as}^{\phi}(x,v,t)$. On the other hand, the solution of the three-dimensional Fokker-Planck equation (2.8) converges in the case of uniformly distributed phases ϕ to a stationary probability for large times, i.e.,

$$\frac{\partial W(x,v,\theta,t)}{\partial t}=\left[\underline{L}_0-A\sin\theta\frac{\partial}{\partial v}-\omega_0\frac{\partial}{\partial\theta}\right]$$
$$\times W(x,v,\theta,t)\underset{t\to\infty}{\longrightarrow}0\ . \tag{2.9}$$

Introducing the "periodic" time $\bar{t}=(\theta-\phi)/\omega_0$, then, with $\partial W_{\rm st}/\partial t=0$, Eq. (2.9) has again the form of Eq. (2.2). Thus, $W_{\rm st}(x,v,\omega_0\bar{t}+\phi)$ is the strictly periodic (in time) solution of the two-dimensional Fokker-Planck equation (2.2) and agrees with $P_{\rm as}^{\phi}(x,v,t)$ up to a normalization constant, i.e.,

$$W_{\rm st}((x,v,\omega_0\bar{t}+\phi))=\frac{1}{2\pi}P_{\rm as}^{\phi}(x,v,\bar{t})\ . \tag{2.10}$$

The integral of $W_{\rm st}(x,v,\theta)$ over x and v does not depend on θ, which is readily seen from (2.9). This is consistent with the uniform distribution of the initial phase ϕ, cf. Eq. (2.8), and with the interpretation of $\bar{t}=\theta/\omega_0-\phi/\omega_0$ as a time (since the normalization $\int dx\,dv\,P_{\rm as}^{\phi}(x,v,\bar{t})$ has to be invariant under time $\bar{t}$ translation).

Let us now discuss the case of a nonuniform distribution of the initial phases ϕ. In this case the two-dimensional asymptotic large-time probability

$$W_{\rm as}(x,v,\theta,t)=\lim_{t_0\to-\infty}W(x_0,v_0,\theta_0,t_0|x,y,\theta,t) \tag{2.11}$$

approaches a nonconstant periodic function in time t, i.e.,

$$W_{\rm as}(x,v,\theta,t)=W_{\rm as}(x,v,\theta,t+T)$$
$$=\sum_{n=-\infty}^{\infty}W_n(x,v,\theta)\exp(in\omega_0t)\ . \tag{2.12}$$

The Fourier coefficients $W_n(x,v,\theta)$ obey the equation

$$\left[\underline{L}_0 - A\sin\theta\frac{\partial}{\partial v} - \omega_0\frac{\partial}{\partial\theta} - in\omega_0\right] W_n(x,v,\theta)=0 \; , \qquad (2.13)$$

which is solved by

$$W_n(x,v,\theta)=\exp(-in\theta)\rho_n W_{st}(x,v,\theta) \; , \qquad (2.14)$$

where ρ_n is the weight of the initial probability $W(x,v,\theta,t=0)$ on the left eigenfunction $\phi_{on}(\theta)=\exp(-in\theta)$ corresponding to the eigenvalue $\lambda_{on}=in\omega_0$. Note that the weight of $W_{st}(x,v,\theta)$ on $\phi_{on}(\theta)=\exp(-in\theta)$ is nonzero only for $n=0$, i.e.,

$$\rho_n^{st}=\int_{-\infty}^{\infty}dv\int dx\int_0^{2\pi}d\theta\exp(-in\theta)W_{st}(x,v,\theta)$$
$$=\int_0^{2\pi}d\theta\exp(-in\theta)\frac{1}{2\pi}=\delta_{n0} \; . \qquad (2.15)$$

The general asymptotic two-dimensional solution is thus given by

$$W_{as}(x,v,\theta,t)=W_{st}(x,v,\theta)\sum_{n=-\infty}^{\infty}\rho_n\exp[-in(\theta-\omega_0 t)]$$
$$=\rho(\theta-\omega_0 t)W_{st}(x,v,\theta) \; , \qquad (2.16a)$$

where $\rho(\theta-\omega_0 t)$ is a positive periodic weight function. One can show by insertion that (2.16a) in fact is a solution of the time-dependent FPE (2.6). The case of a uniform initial-phase distribution is represented in (2.16) by $\rho(\theta-\omega_0 t)=1$. The relation between the three-dimensional asymptotic probability and the two-dimensional asymptotic probability, which is a generalization of (2.10), then reads

$$P_{as}^{\phi}(x,v,t)=2\pi\frac{W_{as}(x,v,\omega_0 t+\phi,t)}{\rho(\phi)} \; . \qquad (2.16b)$$

The angle-averaged asymptotic probability

$$\overline{W}_{as}(x,v,t)\equiv\int_0^{2\pi}d\theta\,\rho(\theta-\omega_0 t)W_{st}(x,v,\theta)$$
$$=\int_0^{2\pi}d\phi\rho(\phi)W_{st}(x,v,\omega_0 t+\phi) \qquad (2.17)$$

is time independent only for $\rho=$const. For general initial phase distributions the angle averaging *does not* provide time-independent two-dimensional probabilities. Inserting the general relation (2.10) into (2.17) we find the connection between the angle-averaged probability $\overline{W}_{as}(x,v,t)$ and the phase-averaged asymptotic probability $\overline{P}_{as}(x,v)$, i.e.,

$$\overline{W}_{as}(x,v,t)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\,\rho(\phi)P_{as}^{\phi}(x,v,t)$$
$$=\langle P_{as}^{\phi}(x,v,t)\rangle_{\rho(\phi)} \; . \qquad (2.18)$$

D. Correlation functions and susceptibilities

The correlation function $\langle\langle x(t)x(0)\rangle_{\xi}\rangle_{\phi}$, averaged over the process and the initial phase is written within our three-dimensional framework, see Eqs. (2.5) and (2.6), as

$$K(t,t')=\langle\langle x(t)x(t')\rangle_{\xi}\rangle_{\phi}$$
$$=\int dx\int dv\int d\theta\, x\exp[\underline{L}_{FP}(t-t')]$$
$$\times x W_{as}(x,v,\theta,t')$$
$$=\langle x\exp[\underline{L}_{FP}(t-t')]x\rangle_{as} \; , \qquad (2.19)$$

where $W_{as}(x,v,\theta,t)$ is the asymptotic probability, given by $\rho(\theta-\omega_0 t)W_{st}(x,v,\theta)$. Note, that $K(t,t')$ has the stationary form $K(t,t')=K(t-t')$ *only* for $\rho(\theta-\omega_0 t)=1$, i.e., only for uniformly distributed initial phases we have stationary (but still periodic) correlation functions.[19] Let us first discuss the case of uniformly distributed initial phases. The correlation function can then be expressed in terms of eigenfunctions and eigenvalues of the three-dimensional Fokker-Planck operator.[25] With extremely large times only the terms with the purely imaginary eigenvalues survive, i.e.,

$$K_{as}(t,t_0\to-\infty)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} g_n\exp(-in\omega_0 t)+\langle x\rangle^2$$
$$=2\sum_{n=1}^{\infty}g_n\cos(n\omega_0 t)+\langle x\rangle^2 \; , \qquad (2.20a)$$

where

$$g_n=\int dx\int d\theta\int dv\, x\psi_{on}(x,v,\theta)$$
$$\times\int dx\int d\theta\int dv\, x\exp(-in\theta)$$
$$\times W_{st}(x,v,\theta) \qquad (2.20b)$$

and $\psi_{on}(x,v,\theta)$ is the right eigenfunction of $\underline{L}_{FP}$ corresponding to $\lambda_{on}=in\omega_0$. (Note that $\exp(-in\theta)$ is the corresponding left eigenfunction.) The spectral power density $S(\omega)$ of the ϕ-averaged stochastic process is given via the Wiener-Khintchin theorem by the Fourier transform of the correlation function $K(t)$, i.e.,

$$S(\Omega)=\int_{-\infty}^{\infty}\exp(-i\Omega t)K(t)dt \; . \qquad (2.21)$$

The asymptotic contribution

$$S_{as}(\Omega)=\int_{-\infty}^{\infty}K_{as}(\tau)\exp(-i\Omega\tau)d\tau \qquad (2.22)$$

can now be expressed in terms of $\lambda_{on}=in\omega_0$ by the use of Cauchy's formula

$$\frac{1}{x+i\epsilon}=\mathrm{P}\left[\frac{1}{x}\right]-i\pi\delta(x) \; , \qquad (2.23)$$

where P denotes the principal value, yielding

$$S_{as}(\Omega)=2\pi\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}g_n\delta(\Omega-n\omega_0)+2\pi\langle x\rangle^2\delta(\Omega) \; . \qquad (2.24)$$

Since the spectral densities, i.e., the Fourier transforms of the diagonal elements of the time-dependent covariance matrix have to be positive for a real-valued stationary process in virtue of the Wiener-Khintchin theorem, g_n has to be real and positive. This has been used to derive (2.20a) and (2.24). Thus, with $g_{-n}=g_n^*=g_n$, we finally obtain

$$S_{as}(\Omega)=2\pi\sum_{n=1}^{\infty} g_n[\delta(\Omega-n\omega_0)+\delta(\Omega+n\omega_0)] +2\pi\langle x\rangle^2\delta(\Omega) . \tag{2.25}$$

As a result, the spectrum contains δ spikes at the multiples of the driving frequency.

Another important result is that the correlation function $K_{as}(t)$ in (2.20) contains no additional phases in the terms $\cos(n\omega_0 t)$ since g_n is real. This vanishing phase, however, is only true for perfect periodic modulation. Switching on the modulation at a fixed time point gives rise to a small phase shift, as observed by Presilla, Marchesoni, and Gammaitoni.[18]

Systems where the correlation functions $K(t)$ do not decay to zero, i.e.,

$$K(t)=\langle\langle x(t)x(0)\rangle\rangle-\langle\langle x(t)\rangle\rangle^2\neq 0 \quad \text{for } t\to\infty , \tag{2.26}$$

are called not strongly mixing.[25] This nonmixing property of our system on a functional subspace means that the system *does not* lose information in the course of time on that very subspace.

In the case of nonuniformly distributed initial phases ϕ the probability $W_{as}(x,v,\theta,t')$ can be expanded into a Fourier series with respect to the time t'. Using (2.16a) we find for the general asymptotic correlation function

$$K(t,t')=\sum_{n=-\infty}^{\infty}\rho_n\langle \exp[-in(\theta-\omega_0 t')]x \times\exp[\underline{L}_{FP}^{+}(t-t')]x\rangle_{st} \tag{2.27}$$

where the average has to be taken over the stationary distribution $W_{st}(x,v,\theta)$. Since the underlying three-dimensional process is not stationary, the Wiener-Khintchin theorem is not valid, and the spectral power density is not related directly to the Fourier transform of the correlation function.

In this section we have presented general characteristics for periodically modulated stochastic systems valid for all values of the damping constant γ and for general (nonlinear) force fields. We have extended the concepts introduced in Refs. 19 and 20 for the overdamped case to the more general system with inertia. In Sec. III we consider the analytically solvable case of the periodically modulated Ornstein-Uhlenbeck process. The N-dimensional Ornstein-Uhlenbeck process has been solved formally in Ref. 25. From this formal solution it seems, however, difficult to extract explicit expressions for the Floquet parameters and the Floquet functions. With our method we have direct access to all those quantities. The solutions for a nonlinear potential will be the subject of future publications.

III. PERIODICALLY MODULATED ORNSTEIN-UHLENBECK PROCESS

Within our three-dimensional framework, Eqs. (2.4)–(2.6), the Langevin equations for a parabolic potential $U(x)=\frac{1}{2}\omega^2x^2$ read with a unit mass

$$\dot{x}=v , \quad \dot{v}=-\gamma v-\omega^2x+A\sin\theta+\xi , \quad \dot{\theta}=\omega_0 . \tag{3.1}$$

Note that this stationary process is not a three-dimensional Ornstein-Uhlenbeck process. It is, however, possible to transform it to an Ornstein-Uhlenbeck process by introducing a set of shifted variables

$$\bar{x}=x-f(\theta) , \quad \bar{v}=v-g(\theta) , \tag{3.2}$$

with the still undetermined periodic functions $f(\theta)=f(\theta+2\pi)$ and $g(\theta)=g(\theta+2\pi)$. The Fokker-Planck operator (2.6) corresponding to (3.1) is, with $\underline{L}_0$ defined in (2.2), transformed to

$$\underline{\bar{L}}(\bar{x},\bar{v},\theta)=\underline{L}_0(\bar{x},\bar{v})-\omega_0\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\bar{x}}[\omega_0 f'(\theta)-g(\theta)] +\frac{\partial}{\partial\bar{v}}[\omega^2 f(\theta)-A\sin\theta+\omega_0 g'(\theta)+\gamma g(\theta)] . \tag{3.3}$$

Choosing the functions $f(\theta)$ and $g(\theta)$ in such a way that the square brackets in Eq. (3.3) vanish identically, i.e., with

$$\begin{aligned} f(\theta)&=\frac{A}{[(\omega^2-\omega_0^2)^2+\gamma^2\omega_0^2]^{1/2}}\sin(\theta+\varphi) \\ &\equiv A(\omega_0)\sin(\theta+\varphi) , \\ g(\theta)&=\frac{A\omega_0}{[(\omega^2-\omega_0^2)^2+\gamma^2\omega_0^2]^{1/2}}\cos(\theta+\varphi) \\ &=\omega_0 A(\omega_0)\cos(\theta+\varphi) , \\ \tan\varphi&=-\frac{\gamma\omega_0}{\omega^2-\omega_0^2} , \end{aligned} \tag{3.4}$$

the Fokker-Planck operator separates in $(\bar{x},\bar{v})$ and θ, i.e.,

$$\begin{aligned} \underline{\bar{L}}(\bar{x},\bar{v},\theta)&=\underline{L}_0(\bar{x},\bar{v})-\omega_0\frac{\partial}{\partial\theta} \\ &=-\bar{v}\frac{\partial}{\partial\bar{x}}+\gamma\frac{\partial}{\partial\bar{v}}\bar{v}+\omega^2\frac{\partial}{\partial\bar{v}}\bar{x}+\gamma T\frac{\partial^2}{\partial v^2}-\omega_0\frac{\partial}{\partial\theta} , \end{aligned} \tag{3.5}$$

which is of Ornstein-Uhlenbeck form.

A. Spectrum and quasispectrum

The eigenfunctions $\psi_{lmn}(\bar{x},\bar{v},\theta)$ and eigenvalues λ_{lmn} of $\underline{\bar{L}}$, i.e.,

$$\underline{\bar{L}}\psi_{lmn}=-\lambda_{lmn}\psi_{lmn} , \tag{3.6a}$$

are given in terms of the eigenfunctions $K_{mn}(\bar{x},\bar{v})$ and eigenvalues Λ_{mn} of $\underline{L}_0$ by

$$\psi_{lmn}(\bar{x},\bar{v},\theta)=\exp(il\theta)K_{mn}(\bar{x},\bar{v})\ , \quad (3.6b)$$

$$\lambda_{lmn}=il\omega_0+\Lambda_{mn},\quad l=-\infty,\ldots,0,\ldots,\infty\ , \quad (3.6c)$$

where the eigenvalues Λ_{mn} of the Kramers operator $\underline{L}_0$ are given for the parabolic potential by[25,26]

$$\Lambda_{mm}=\frac{\gamma}{2}\left[1+\left[1-\frac{4\omega^2}{\gamma^2}\right]^{1/2}\right]m +\frac{\gamma}{2}\left[1-\left[\frac{4\omega^2}{\gamma^2}\right]^{1/2}\right]n,\quad n,m=0,\ldots,\infty\ . \quad (3.7)$$

The eigenfunctions $K_{mn}(x,v)$ are given explicitly in Ref. 27. The particular eigenfunction $K_{00}(x,v)$ reads

$$K_{00}(x,v)=\frac{\omega}{2\pi T}\exp\left[-\frac{\omega^2}{2T}x^2-\frac{1}{2T}v^2\right]\ . \quad (3.8)$$

In the original variables (x,v,θ) the eigenfunctions have the same form but $\bar{x}$ and $\bar{v}$ have to be substituted by $x-f(\theta)$ and $v-g(\theta)$, respectively, (note that the Jacobian is equal to the unity). The results for the quasieigenvalues (Floquet coefficients) and the quasieigenfunctions (Floquet functions) then read

$$\mu_{lmn}=il\omega_0+\frac{\gamma}{2}(n+m)+(m-n)\left[\frac{\gamma^2}{4}-4\omega^2\right]^{1/2}\ , \quad (3.9a)$$

$$p_\mu(x,v,t)=K_{mn}(x-f(\omega_0t+\phi),v-g(\omega_0t+\phi)) \times\exp(il\omega_0t)\ , \quad (3.9b)$$

with

$$l=-\infty,\ldots,0,\ldots,\infty,\quad m,n=0,\ldots,\infty\ .$$

Note that in the complete Floquet solutions [Eq. (2.3)]

$$P^{(\mu)}(x,v,t)=\exp(-\mu t)p^{(\mu)}(x,v,t) =\exp(-\Lambda_{mn}t) \times K_{mn}(x-f(\omega_0t+\phi),v-g(\omega_0t+\phi)) \quad (3.10)$$

the $\exp(-il\omega_0t)$ have canceled. The results in Eqs. (3.9) and (3.10) are the natural extension of the well-known results for Brownian motion in a parabolic potential. For vanishing modulation ($A=0$), f and g both are zero, and the familiar results are recovered.[25-27] The Floquet solutions (3.10) "decay" with the same relaxation times $t_r=\Lambda_{nm}^{-1}$ as the probability distributions in the modulation-free system. The shapes of the "relaxation modes," however, are periodically modulated. This result is intuitively clear, since in linear systems the response to a modulation and to noise can be linear superimposed,[3] and thus one expects a "decoupled" action of modulation and noise.

In the overdamped limit ($\gamma\to\infty$) the modulated Ornstein-Uhlenbeck process reads (after a time-scale transformation $\bar{\omega}_0=\omega_0\gamma$, $\bar{t}=t/\gamma$)

$$\dot{x}=-\omega^2x+A\sin(\bar{\omega}_0\bar{t}+\phi)+\sqrt{T}\,\xi(t)\ . \quad (3.11)$$

The quasieigenfunctions are obtained in the same way as for the case with inertia, or may be derived from the results above to read

$$P^{(n)}(x,\bar{t})=\left[\frac{\omega^2}{2\pi T}\right]^{1/2}\frac{1}{(2^nn!)^{1/2}}\exp\left[-\frac{\omega^2}{2T}\left[x-\frac{A}{(\omega^4+\bar{\omega}_0^2)^{1/2}}\sin(\bar{\omega}_0\bar{t}+\bar{\varphi})\right]^2\right] \times H_n\left[\left[\frac{\omega^2}{2T}\right]^{1/2}\left[x-\frac{A}{(\omega^4+\bar{\omega}_0^2)^{1/2}}\sin(\bar{\omega}_0\bar{t}+\bar{\varphi})\right]\right]\exp(-n\omega^2\bar{t})\ , \quad (3.12a)$$

$$\mu_{nl}=n\omega^2+li\omega_0\ , \quad (3.12b)$$

where

$$n=0,\ldots,\infty,\quad m=-\infty,\ldots,0,\ldots,\infty\ ,$$

$$\bar{\varphi}=\phi-\arctan\frac{\bar{\omega}_0}{\omega^2}\ .$$

Here $H_n(x)$ denote Hermite polynomials and ϕ denotes an arbitrary initial phase of the modulation.

B. Asymptotic probability

The stationary solution of the three-dimensional FPE (2.6) is identical with the eigenfunction corresponding to the zero eigenvalues. Thus from Eqs. (3.2), (3.6a), and (3.8) we find

$$W_{\rm st}(x,v,\theta)=\frac{1}{2\pi}K_{00}(x-f(\theta),v-g(\theta)) =\frac{\omega}{4\pi^2T}\exp\left[-\frac{1}{2T}\omega^2[x-f(\theta)]^2 -\frac{1}{2T}[v-g(\theta)]^2\right]\ . \quad (3.13)$$

The asymptotic probability is thus given with (2.10) by

$$P_{\rm as}^{\phi}(x,v,t)=\frac{\omega}{2\pi T}\exp\left[-\frac{\omega^2}{2T}[x-f(\omega_0 t+\phi)]^2-\frac{1}{2T}[v-g(\omega_0 t+\phi)]^2\right] \quad (3.14)$$

which is, as expected, time periodic with a period of $2\pi/\omega_0$ and contains higher harmonics of ω_0.

In the overdamped case $\gamma\to\infty$ we find with the time-scale transformation [cf. (3.11)]

$$f(\theta)\to\frac{A}{(\bar\omega_0^2+\omega^2)^{1/2}}\sin(\theta+\varphi)\ ,\quad g(\theta)\to 0\ ,\quad \tan\varphi\to-\frac{\bar\omega_0}{\omega^2}\ , \quad (3.15)$$

and thus

$$P_{\rm as}^{\phi}(x,t)=\frac{\omega}{\sqrt{2\pi T}}\exp\left[-\frac{\omega^2}{2T}\left[x-\frac{A}{(\bar\omega_0^2+\omega^4)^{1/2}}\sin(\bar\omega_0\bar t+\varphi+\phi)\right]^2\right] . \quad (3.16)$$

C. Time-averaged asymptotic probabilities

We now consider the time-averaged asymptotic probabilities. Since the time t and phase ϕ enter only in the combination $\omega_0 t+\phi$, time averaging is equivalent to averaging over the initial phase ϕ for uniformly distributed initial phases. The resulting probabilities are time independent and important for comparison with experiments since the initial phase in such system is averaged out by taking the average over many experimentally obtained time series. The averaged probability in x is obtained from Eq. (3.13), i.e.,

$$p(x)=\int_{-\infty}^{\infty}dv\int_0^{2\pi}d\theta\, W_{\rm st}(x,v,\theta)=\frac{1}{2\pi}\frac{\omega}{\sqrt{2\pi T}}\int_0^{2\pi}\exp\left[-\frac{\omega^2}{2T}[x-A(\omega_0)\sin\theta]^2\right]d\theta\ . \quad (3.17)$$

Using the fact that $\exp[iz\sin(\theta)]$ and $\exp[iz\cos(\theta)]$ are generating functions of the modified Bessel functions $I_\nu(z)$,[28(a)] one finds

$$\bar p(x)=\frac{\omega}{\sqrt{2\pi T}}\exp\left[-\frac{1}{2T}\omega^2x^2\right]\exp\left[-\frac{1}{4}\frac{\omega^2}{T}A^2(\omega_0)\right]\left[I_0\left[\frac{\omega^2}{4T}A^2(\omega_0)\right]I_0\left[\frac{\omega^2}{T}A(\omega_0)x\right]+2\sum_{k=1}^{\infty}(-1)^kI_k\left[\frac{\omega^2A^2(\omega_0)}{4T}\right]I_{2k}\left[\frac{\omega^2}{T}A(\omega_0)x\right]\right] . \quad (3.18)$$

In Fig. 1 $\bar p(x)$ is plotted for various of ω_0 with $\omega=1$, $T=0.1$, $\gamma=0.1$, and $A=0.5$. Most important the probability $\bar p(x)$ becomes bimodal at a certain value of ω_0, or equivalently at a certain value of the amplitude $A(\omega_0)$. This can be understood as a consequence of the linear superposition of the response of the system to noise and to modulation.

The averaged probability $\bar p(x)$ close to the origin ($x=0$) is obtained by using the following expansions for modified Bessel functions:[28(b)]

$$I_0(z)=1+\tfrac{1}{4}z^2+O(z^4)\ ,\quad I_k(z)=\frac{1}{2^k}\frac{1}{\Gamma(k+1)}z^k+O(z^{k+1})\ , \quad (3.19)$$

to be

$$\bar p(x)=\frac{\omega}{\sqrt{2\pi T}}\exp\left[-\frac{1}{4}\frac{\omega^2}{T}A^2(\omega_0)\right]\times I_0\left[\frac{\omega^2}{4T}A^2(\omega_0)\right]-\kappa x^2\ , \quad (3.20a)$$

where

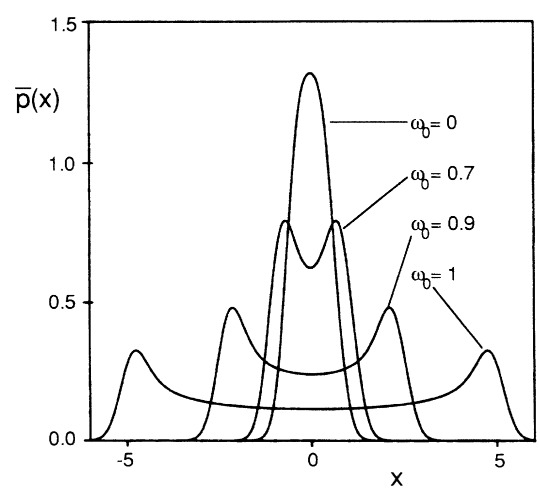

FIG. 1. The phase-averaged asymptotic probability $\bar p(x)$ (3.18) is shown at $D=0.1$, $\omega=1$, and $\gamma=0.1$ for $\omega_0=0$, 0.7, 0.9, and 1.

$$\kappa=\frac{\exp\left[-\frac{1}{4}\frac{\omega^2}{T}A^2(\omega_0)\right]}{\sqrt{2\pi T}}\left\{\frac{1}{2}\frac{\omega^2}{T}I_0\left[\frac{1}{4}\frac{\omega^2A^2(\omega_0)}{T}\right]+\frac{1}{4}\frac{\omega^4A^2(\omega_0)}{T^2}\left[I_1\left[\frac{1}{4}\frac{\omega^2A^2(\omega_0)}{T}\right]-I_0\left[\frac{1}{4}\frac{\omega^2A^2(\omega_0)}{T}\right]\right]\right\}. \tag{3.20b}$$

The sign of κ is positive for an unimodal averaged probability, and negative for a bimodal averaged probability. In Fig. 2 we have plotted the numerically evaluated κ (3.20b) as a function of $\overline{A}$ at $\omega=T=1$.

For large values of the damping γ the curvature quickly approaches for increasing ω_0 a positive-valued limit, being identical with the curvature of the stationary probability without modulation. For decreasing γ this limit is approached more slowly. At $\gamma=1$, the curvature decreases for small modulation frequencies, but increases to the limit above resonance $\omega_0=\omega=1$. In the underdamped limit (see also Fig. 1) the curvature has two minima, one below and one above resonance, and a relative maximum between at resonance. This relative maximum at resonance corresponds to an extremely flat probability close to the origin $x=0$.

The averaged two-dimensional probability

$$\overline{p}(x,v)=\int_0^{2\pi}d\theta W_{\text{st}}(x,v,\theta) \tag{3.21}$$

can also be obtained by using the generating functions for modified Bessel functions,

$$\begin{aligned}\overline{p}(x,v)=&\frac{\omega}{2\pi T}\exp\left[-\frac{\omega^2}{2T}x^2-\frac{1}{2T}v^2\right]\exp\left[-\frac{1}{4T}A^2(\omega_0)(\omega^2+\omega_0^2)\right]\\&\times\left[I_0\left[\frac{\omega^2-\omega_0^2}{4T}A^2(\omega_0)\right]I_0\left[\frac{\omega_0A(\omega_0)}{T}v\right]I_0\left[\frac{\omega^2A(\omega_0)}{T}x\right]\right.\\&+2\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}I_k\left[\frac{\omega^2-\omega_0^2}{4T}A^2(\omega_0)\right]I_{2l}\left[\frac{\omega^2A(\omega_0)}{T}x\right]I_{2l+2k}\left[\frac{\omega_0A(\omega_0)}{T}v\right](-1)^l\\&\left.+2\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}I_k\left[\frac{\omega^2-\omega_0^2}{4T}A^2(\omega_0)\right]I_{2l}\left[\frac{\omega^2A(\omega_0)}{T}x\right]I_{|2l-2k|}\left[\frac{\omega_0A(\omega_0)}{T}v\right](-1)^l\right].\end{aligned} \tag{3.22}$$

In Fig. 3 $\overline{p}(x,v)$ is plotted for $A=0.5$, $T=0.1$, $\gamma=0.1$, and $\omega=1$ for modulation frequencies below and above resonance; i.e., $\omega_0=0.2,\ldots,1.8$ in Figs. 3(a)–3(f).

For frequencies far from resonance $\omega_0=\omega$, $\overline{p}(x,v)$ is unimodal with the peak centered at the origin $x=v=0$ [Fig. 3(a)]. For some frequency $\omega_0^{(1)}(A,T,\gamma,\omega)$ the averaged probability becomes bimodel with the two peaks centered on the x axis at finite values of x. The origin has become a saddle point [Fig. 3(b)]. A second topological bifurcation takes place at some larger value of $\omega_0=\omega_0^{(2)}(A,T,\gamma,\omega)$. The averaged probability now has two saddle points lying on the v axis, two peaks lying on the x axis, and a minimum at the origin [Figs. 3(c) and 3(d)]. For $\omega_0=\omega=1$ [Fig. 3(e)] the off-axis saddle points have vanished and the probability is peaked on a circle. For $\omega_0>\omega$ [Fig. 3(f)] the off-axis saddle points occur again, but now on the x axis and the peaks are now lying on the v axis. For further increasing modulation frequencies, an inverse cascade of topological bifurcations takes place [cf. Figs. 3(e)–3(g)].

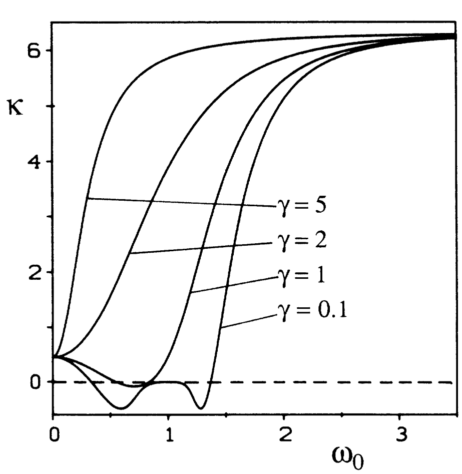

FIG. 2. The curvature κ (3.20) is plotted at $A=0.5$, $D=0.1$ and $\omega=1$ for $\gamma=0.1$, 1, 2, and 5 as a function of the modulation frequency ω_0.

The observed rich topological behavior is quite surprising in view of the linearity of the underlying stochastic process. The phase-averaging procedure, however, produces an averaged probability density corresponding to a *non-Markovian stochastic process*.[20] The underlying non-Markovian (and generally non-Gaussian) process is responsible for the complex topology of $\overline{p}(x,v)$. Bifurcations of the second type (i.e., $\omega_0^{(2)}$) mentioned above have also been observed for joint probability densities of colored-noise-driven nonlinear dynamical systems (for an overview, see Ref. 29). The joint probability density

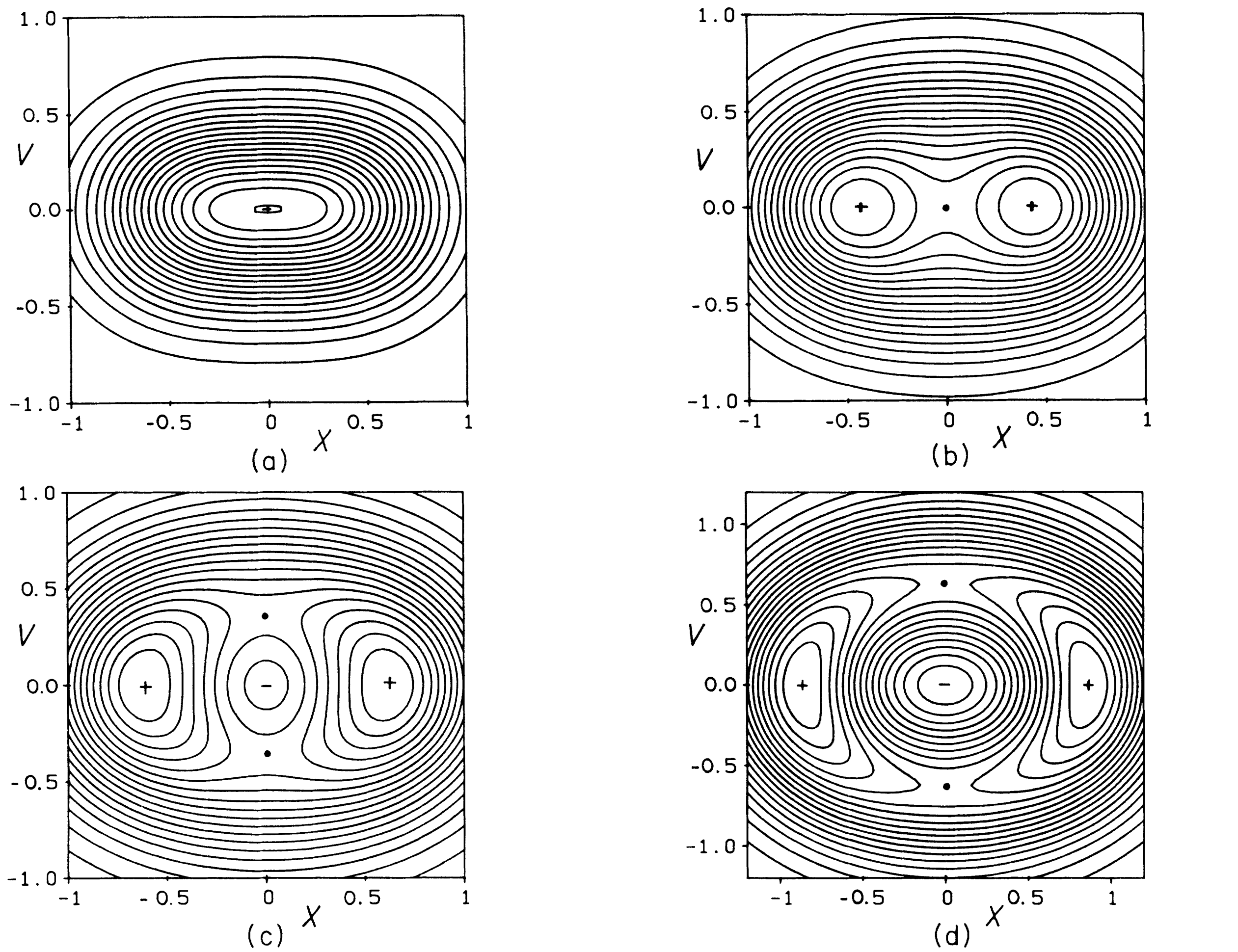

FIG. 3. The altitude charts of the phase-averaged two-dimensional asymptotic probability $\bar{p}(x,v)$ (3.22) are plotted at $D=0.1$, $\gamma=0.1$, $A=0.5$, and $\omega=1$ for $\omega_0=0.2$ (a), 0.5 (b), 0.6 (c), 0.7 (d), 1 (e), 1.3 (f), and 1.5 (g). The lines are equidistant with respect to the probability in each figure. Dots denote hyperbolic points while maxima are marked by plus signs and minima by minus signs.

$P_{st}(x,\epsilon)$ of $\dot{x}=-x^3+\epsilon$, $\dot{\epsilon}=-(1/\tau)\epsilon+(1/\tau)\sqrt{D}\,\Gamma(t)$ has for $\tau<\tau_c(D)$ one saddle point at $(x=\epsilon=0)$ and has for $\tau<\tau_c(D)$ two off-axis saddle points, and a minimum at $x=\epsilon=0$.[29–31] Important for the bifurcation of the second type is the existence of an unstable deterministic point $(x=\epsilon=0)$. In our system $\dot{x}=v$, $\dot{v}=-\gamma v-\omega^2 x+A\sin\omega_0 t$, the origin is also an unstable point yielding in connection with the non-Markovian process the mentioned bifurcation. It is also worthwhile to remark about the nonspherical symmetry of $\bar{p}(x,v)$ in the x and v direction, cf. Fig. 3. The reason for this is the ω_0 prefactor in the deterministic solution of $\dot{x}=v$, $\dot{v}=-\gamma v-\omega^2 x+A\sin\omega_0 t$. This asymmetry, however, may also be responsible for the observed asymmetry in the resonance curve of the rate enhancement in resonance activation experiments.[3]

D. Correlation functions

The position-position correlation function is defined by

$$K_{xx}(t,t')=\langle\langle x(t)x(t')\rangle_{\xi}\rangle_{\phi}\,, \tag{3.23}$$

where the first average ξ has to be taken over the realizations of the noise, and the second average ϕ has to be taken over the initial phases of the oscillator. In the framework of our three-dimensional description, the correlation function (3.22) reads

$$K_{xx}(t,t')=\langle x\exp[\underline{L}^{+}_{\mathrm{FP}}(t-t')]x\rangle_{st}\,, \tag{3.24}$$

where the average has to be taken over the three-dimensional stationary probability. The equation of motion for the correlation function reads

$$\ddot{K}_{xx}+\gamma\dot{K}_{xx}+\omega^2 K_{xx}=\langle x\sin\theta\rangle_{st}\cos\omega_0 t+\langle x\cos\theta\rangle_{st}\sin\omega_0 t\,. \tag{3.25}$$

The solution is with $\gamma<4\omega$ given by

$$K_{xx}(\tau=t-t')=\frac{\gamma}{2\bar{\omega}}\frac{T}{\omega^2}\exp\left[-\frac{\gamma}{2}\tau\right]\sin\bar{\omega}\tau+\frac{T}{\omega^2}\exp\left[-\frac{\gamma}{2}\tau\right]\cos\bar{\omega}\tau+\tfrac{1}{2}A^2(\omega_0)\cos(\omega_0\tau)\,, \tag{3.26}$$

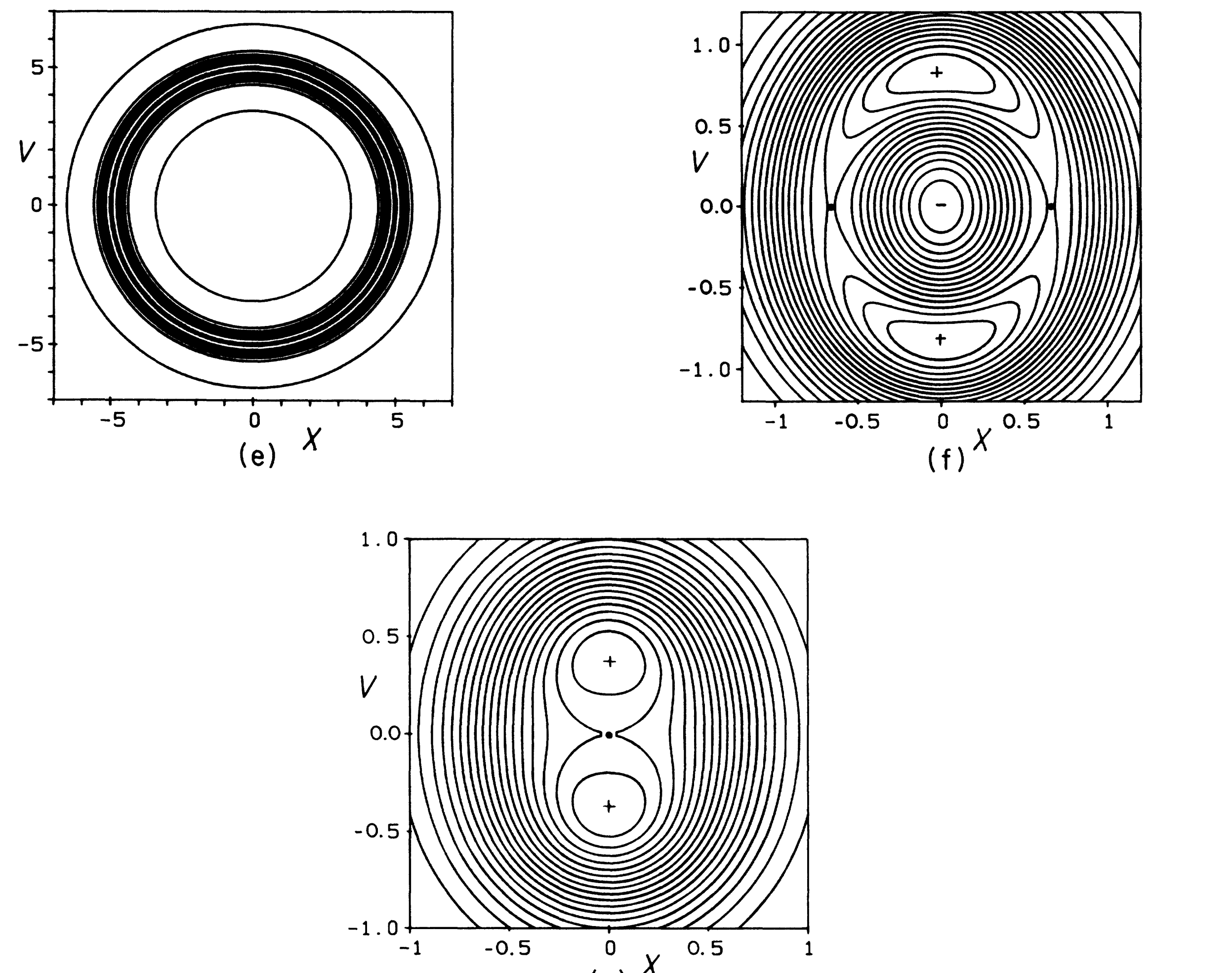

FIG. 3. (*Continued*).

while for $\gamma > 4\omega$, the trigonometric functions are substituted by the corresponding hyperbolic functions. The frequency $\bar{\omega} = |\omega^2 - \gamma^2/4|^{1/2}$, and the resonant amplitude $A(\omega_0)$ is given in Eq. (3.4). The correlation function is the linear superposition of the deterministic correlation function (noise strength $T=0$) and the correlation function without modulation ($A=0$).

The dynamical susceptibility, defined by

$$\chi_{xx}(\Omega) = \int_0^\infty d\tau \exp(-i\Omega\tau) K_{xx}(\tau) , \tag{3.27a}$$

reads

$$\begin{aligned}\chi_{xx}(\Omega) = {} & \frac{\gamma T}{(\omega^2-\Omega^2)^2+\gamma^2\Omega^2} \\ & + \tfrac{1}{4}\pi A^2(\omega_0)[\delta(\Omega-\omega_0)+\delta(\Omega+\omega_0)] \\ & + \frac{i\Omega T}{\omega^2}\frac{\omega^2-\Omega^2-\gamma^2}{(\omega^2-\Omega^2)^2+\gamma^2\Omega^2} - i\tfrac{1}{2}A^2(\omega_0)\frac{\Omega}{\Omega^2-\omega_0^2}\end{aligned} \tag{3.27b}$$

for $\gamma \neq 0$, and

$$\begin{aligned}\chi_{xx}(\Omega) = {} & \tfrac{1}{4}\pi A^2(\omega_0)[\delta(\Omega-\omega_0)+\delta(\Omega+\omega_0)] \\ & + \frac{\pi T}{2\omega^2}[\delta(\Omega-\omega)+\delta(\Omega+\omega)] \\ & - \tfrac{1}{2}iA^2(\omega_0)\frac{\Omega}{\Omega^2-\omega_0^2} - i\frac{T}{\omega^2}\frac{\Omega}{\Omega^2-\omega^2}\end{aligned} \tag{3.27c}$$

for $\gamma = 0$. The spectral power density (being twice the real part of the susceptibility) shows, as predicted in Sec. II D, a δ spike at the driving frequency $\Omega = \pm\omega_0$. In the overdamped limit $\gamma \to \infty$ the correlation function and the susceptibility reduced to

$$K_{xx}(\tau) = \frac{T}{\omega^2}\exp(-\omega^2\tau) + \frac{1}{2}\frac{A^2}{(\omega^2+\omega_0^2)}\cos\omega_0\tau , \tag{3.28}$$

$$\begin{aligned}\chi_{xx}(\Omega) = {} & \frac{T}{\omega^2+\Omega^2} + \frac{1}{4}\frac{A^2\pi}{\omega_0^2+\omega^2}[\delta(\omega_0-\Omega)+\delta(\omega_0+\Omega)] \\ & + i\left[-\frac{\Omega}{\omega}\frac{T}{\Omega^2+\omega^2} - \frac{1}{2}\omega\frac{A^2}{(\omega_0^2+\omega^2)(\Omega^2-\omega_0^2)}\right] .\end{aligned} \tag{3.29}$$

Like in the inertia case, cf. Eq. (3.27b), the real part of the susceptibility has δ spikes at the driving frequency.

IV. APPLICATION TO NON-MARKOVIAN EQUILIBRIUM SYSTEMS

In many situations[32,33] the memory time of the non-Markovian system is not small in comparison to the typical system time scale, and the white-noise assumption is not valid. In this section we consider a linear non-Markovian system, which is perturbed by periodic modulations. The Langevin equations reads in normalized (dimensionless) variables

$$\ddot{x}+\beta\int_0^t \phi(t-t')\dot{x}(t')dt'+\omega^2 x=\beta^{1/2}\xi(t)+A\sin(\omega_0 t+\phi)$$

with the Gaussian noise $\xi(t)$ obeying the fluctuation-dissipation theorem of the second kind

$$\langle \xi(t)\xi(t')\rangle = T\phi(t-t') , \quad \langle \xi(t)\rangle = 0 . \tag{4.1}$$

For the memory kernel we choose an exponential, i.e.,

$$\phi(\tau)=\gamma\exp(-\gamma|\tau|) .$$

Introducing the variable z, i.e.,

$$z=\int_0^t \phi(t-t')\dot{x}(t')dt'-\beta^{-1/2}\xi(t) \tag{4.2}$$

the equivalent four-dimensional Markovian system reads

$$\begin{aligned} \dot{x}&=v , \\ \dot{v}&=-\beta z-\omega^2 x+A\sin\theta , \\ \dot{z}&=-\gamma z+\gamma v-\gamma\sqrt{T/\beta}\,\Gamma(t) , \\ \dot{\theta}&=\omega_0 , \end{aligned} \tag{4.3a}$$

with the Gaussian white noise $\Gamma(t)$

$$\langle \Gamma(t)\Gamma(t)\rangle = 2\delta(t-t), \quad \langle \Gamma(t)\rangle = 0 . \tag{4.3b}$$

A. Spectrum and quasispectrum

The Fokker-Planck equation corresponding to Eqs. (4.3a) and (4.3b) is transformed to the simple form

$$\frac{\partial}{\partial t}W(\bar{x},\bar{v},\bar{z},\theta,t)=\left[\underline{L}_0(\bar{x},\bar{v},\bar{z})-\omega_0\frac{\partial}{\partial\theta}\right]W(\bar{x},\bar{v},\bar{z},\theta,t) \tag{4.4a}$$

with

$$\underline{L}_0(\bar{x},\bar{v},\bar{z})=-\bar{v}\frac{\partial}{\partial\bar{x}}+\beta\bar{z}\frac{\partial}{\partial\bar{v}}+\omega_0^2\bar{x}\frac{\partial}{\partial\bar{v}} +\gamma\frac{\partial}{\partial\bar{z}}\bar{z}-\gamma v\frac{\partial}{\partial z}+\frac{\gamma^2 T}{\beta}\frac{\partial^2}{\partial\bar{z}^2} . \tag{4.4b}$$

The variables $\bar{x},\bar{v},\bar{z}$ are defined by

$$\bar{x}=x-f(\theta),\quad \bar{v}=v-g(\theta),\quad \bar{z}=z-h(\theta) , \tag{4.5a}$$

where the functions $f(\theta)$, $g(\theta)$, $h(\theta)$ obey the linear system of differential equations

$$\begin{aligned} \omega_0 f'(\theta)&=g(\theta) , \\ \omega_0 g'(\theta)&=-\beta h(\theta)-\omega_0^2 f(\theta)+A\sin\theta , \\ \omega_0 h'(\theta)&=\gamma g(\theta)-\gamma h(\theta) . \end{aligned} \tag{4.5b}$$

The periodic solutions of (4.5b) are found to be

$$\begin{aligned} f(\theta)&=B(\omega_0)\cos(\theta+\varphi) , \\ g(\theta)&=-B(\omega_0)\omega_0\sin(\theta+\varphi) , \\ h(\theta)&=Z(\omega_0)\cos(\theta+\varphi_z) , \end{aligned} \tag{4.6a}$$

where

$$B(\omega_0)=A\left[\left[\omega^2-\omega_0^2+\frac{\omega_0^2\beta\gamma}{\gamma^2+\omega_0^2}\right]^2+\frac{\omega_0^2\beta^2\gamma^4}{(\gamma^2+\omega_0^2)^2}\right]^{-1/2} \tag{4.6b}$$

$$Z(\omega_0)=\frac{A\,\omega_0\gamma}{(\omega_0^2+\gamma^2)^{1/2}}\left[\left[\omega^2-\omega_0^2+\frac{\omega_0^2\beta\gamma}{\gamma^2+\omega_0^2}\right]^2+\frac{\omega_0^2\beta^2\gamma^4}{(\gamma^2+\omega_0^2)^2}\right]^{-1/2} \tag{4.6c}$$

and

$$\tan\varphi=\frac{\omega_0}{\gamma}-\frac{(\omega^2-\omega_0^2)(\omega_0^2+\delta^2)}{\beta\gamma^2\omega_0} , \tag{4.6d}$$

$$\tan\varphi_z=-\frac{\omega_0}{\gamma}+\beta\frac{\omega_0}{\omega^2-\omega_0^2} . \tag{4.6e}$$

The eigenvalues of the FPE (4.4a), being identical with the Floquet coefficient of the equivalent three-dimensional nonstationary FPE, are

$$\begin{aligned} \lambda_{klmn}&=ki\omega_0-l\Lambda_1-m\Lambda_2-n\Lambda_3 , \\ k&=-\infty,\dots,0,\dots,\infty , \\ l,m,n&=0,1,\dots,\infty , \end{aligned} \tag{4.7}$$

where $\Lambda_{1,2,3}$ are the complex roots of the polynomial

$$(x^2+\omega^2)(x+\gamma)+\gamma\beta x=0 . \tag{4.8}$$

For a derivation of the eigenvalues of an N-dimensional Ornstein-Uhlenbeck process, see our recent paper.[34] In the Markovian limit $\gamma\to\infty$ the roots are given by $x_{1,2}^{\infty}=-\frac{1}{2}\beta\pm\frac{1}{2}(\beta^2-4\omega^2)^{1/2}$. At $\beta=2\omega$ two real branches ($\beta>2\omega_0$) approach each other to create a complex pair of eigenvalues ($\beta<2\omega_0$). In the non-Markovian case we have one additional complex root. For small correlation times γ^{-1}, there is another bifurcation at large values of the damping β ($\gamma=100$ in Fig. 4). For increasing correlations times the two bifurcation points come closer ($\gamma=10$ in Fig. 4) and finally cancel each other at the same value for the correlation time (cf. Fig. 4 with $\gamma=5$). For $\tau>t_c\equiv\gamma_c^{-1}$ ($\gamma=1$ in Fig. 4) one has a real and a complex root without a bifurcation.

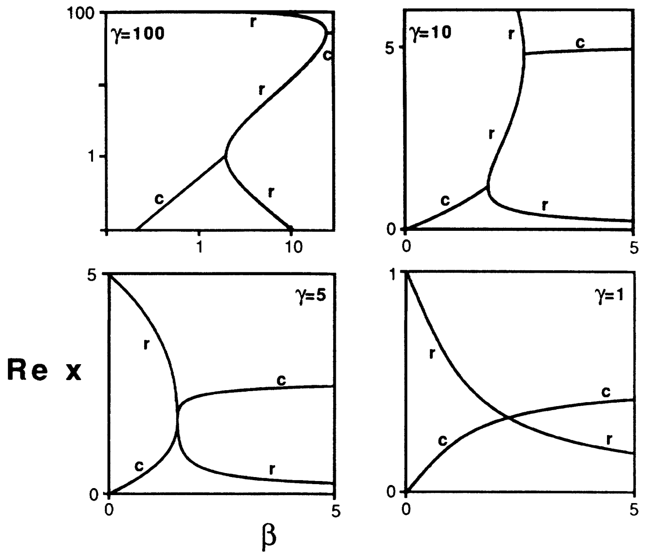

FIG. 4. The real parts of the complex roots of Eq. (4.8) are shown at $\omega=1$ for $\gamma=100$, 10, 5, and 1. Real branches are marked by r and complex branches by c.

B. Asymptotic probability

The asymptotic (periodic) probability is the Floquet function corresponding to the vanishing Floquet coefficients and thus is given by

$$\underline{\bar{L}}(\bar{x},\bar{v},\bar{z})W_{st}(\bar{x},\bar{v},\bar{z})=0 \;, \tag{4.9}$$

where $\underline{\bar{L}}(\bar{x},\bar{v},\bar{z})$ is given in Eq. (4.4b). The solution of Eq. (4.9) is

$$W_{st}=\left[\frac{\beta}{2\pi\gamma T}\right]^{1/2}\frac{\omega_0}{4\pi^2 T}\times\exp\left[-\frac{\omega_0^2}{2T}\bar{x}^2-\frac{1}{2T}\bar{v}^2-\frac{\beta}{2\gamma T}\bar{z}^2\right] . \tag{4.10}$$

Integrating out the auxiliary variable $\bar{z}$ the asymptotic probability in x,v reads

$$\begin{aligned}P_{as}^{\phi}(x,v,t)&=2\pi\int_{-\infty}^{\infty}dz\; W_{st}(x-f(\omega_0 t+\phi),v-g(\omega_0 t+\phi),z-h(\omega_0 t+\phi))\\&=\frac{\omega_0}{2T\pi}\exp\left[-\frac{\omega_0^2}{2T}[x-f(\omega_0 t+\phi)]^2-\frac{1}{2T}[v-g(\omega_0 t+\phi)]\right] .\end{aligned} \tag{4.11}$$

The time-averaged probability

$$\bar{P}_{as}(x,v)=\frac{1}{2\pi}\int_0^{2\pi}d\phi P_{as}^{\phi}(x,v,t) \tag{4.12}$$

may also be expressed analytically in a series of Bessel functions. Since Eqs. (3.14) and (4.11) have the same structure, the phase-averaged probabilities of the Markovian system ($\gamma=\infty$) (see Fig. 3) and the non-Markovian system ($\gamma=\infty$) are very similar. The differences are due to the different amplitude functions $B(\omega_0)$ (4.6b) and $A(\omega_0)$ (3.4). The non-Markovian resonant amplitude function $B(\omega_0)$, being identical with the frequency response of the unperturbed system is plotted in Fig. 5 for $A=0.5$, $\omega=1$ at $\beta=0.5$ (a) and $\beta=10$ (b). In both cases we observe that the amplitude function behaves for increasing correlation times more underdamped, i.e., the resonance curve becomes more sharp. Note, however, that in the overdamped case the Lorenzian peak at $\omega_0=0$ does not vanish for decreasing γ and we thus have a double resonance. It is also interesting that the position of the resonance peak at $\omega_0\neq0$ crucially depends on the correlation time γ^{-1}. For very large correlation times

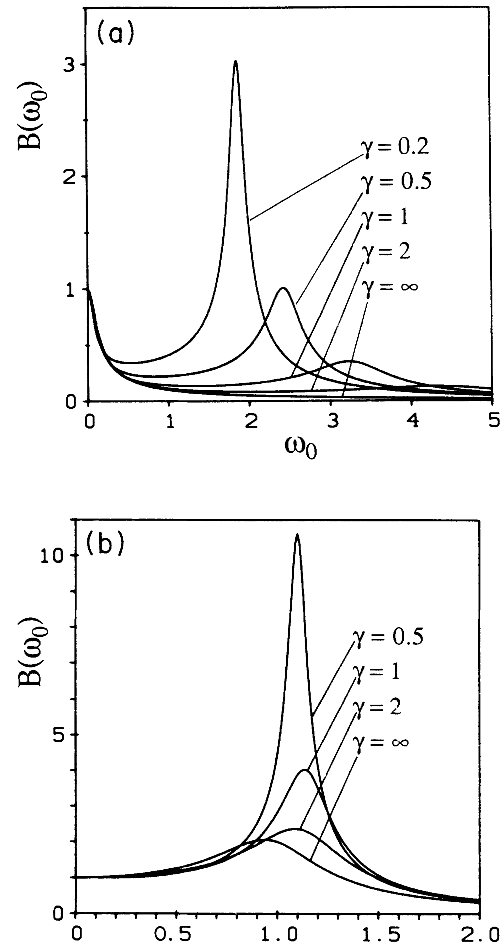

FIG. 5. The resonance amplitude $B(\omega_0)$ (4.6b) is depicted for (a) high-damping $\beta=10$ and (b) low-damping $\beta=0.5$ for various values of the noise color $1/\gamma$.

the resonant peak at $\omega_0 \neq 0$ converges to $\omega_0 = \omega = 1$. The physical meaning of the amplitude function becomes evident in the discussion of escape rates out of a metastable state: for larger $B(\omega_0)$, the averaged probability becomes more widespread. The activation process out of a metastable state (modeled by a cut-off oscillator) is thus strongly enhanced. In the overdamped case we therefore find an enhancement at a resonance frequency $\omega_0 \approx 0$.

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