Invariant Measure of a Driven Nonlinear Oscillator with External Noise

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The effect of external white Gaussian noise on the invariant measure of a periodically driven damped nonlinear oscillator is studied by solving for the first time the full three-dimensional Fokker-Planck equation by numerical means. We critically discuss and interpret deterministic concepts and stochastic notions in the presence of noise and chaos.

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It is well known, that periodically driven nonlinear oscillators exhibit deterministic chaos.[1] For systems with dissipation—but without fluctuations—the driving amplitude has to exceed a certain homoclinic threshold[2] in order that the solution becomes chaotic and a strange attractor builds up. For Hamiltonian systems with more than one degree of freedom, local chaos shows up for every finite driving amplitude,[3] while the transition to global chaos occurs at a certain threshold value of the driving amplitude. In spite of considerable progress in understanding the chaotic behavior and the routes to chaos in isolated systems like those mentioned above, there is only little knowledge on the impact of those results for more realistic models which include *dissipation and fluctuations.* The influence of noise on a driven nonlinear oscillator may be described in terms of a Langevin equation, i.e., the differential equation for the driven oscillator supplemented by a stochastic force.[4] Assuming this stochastic force to be δ correlated and Gaussian distributed (Gaussian white noise), the Langevin equation is statistically equivalent to a two-dimensional Fokker-Planck equation (FPE) with a periodic drift coefficient.[5]

Fokker-Planck equations are linear partial differential equations of first order in time for the probability distribution regardless of whether the trajectories of the deterministic system show regular or chaotic behavior. A similar situation arises in the discussion of quantum-mechanical properties of classically chaotic systems.[6] There, the Schrödinger equation is a linear partial differential equation for the probability amplitude and thus the wave functions are periodic or at most quasiperiodic in time, although the classical trajectories may exhibit chaotic behavior. In our system the situation is even more complicated, since the generator of a Markovian stochastic process (the Fokker-Planck operator) is generally a non-Hermitian differential operator and thus the spectrum—or the Floquet spectrum for periodically driven systems—is generally complex valued. The goal of this paper is the discussion of properties of the solution of the FPE which are characteristic for deterministic chaotic behavior.

The influence of weak noise has been studied for discrete maps[7] as well as for Hamiltonian systems[8] by utilizing path-integral methods. The influence of periodic driving on a bistable Duffing oscillator has been studied recently within an inverse system-size expansion.[9] This method, however, is valid only for short time intervals compared to the inverse of the largest positive Lyapunov coefficient and for weak noise. Thus, on a chaotic attractor only transient properties of the stochastic systems are described correctly by the inverse system-size expansion. In this paper we consider the influence of generally nonweak noise on a driven nonlinear oscillator in regular as well as in chaotic regions. The particular system under investigation is a shunted Josephson junction exposed to coherent microwave radiation.[10] The equation of motion for the phase difference between the macroscopic wave functions of the superconductors is given within the resistively shunted junction model in dimensionless units by[10,11]

$$\ddot{\varphi}+\gamma\dot{\varphi}+\sin\varphi=F+A\sin(\Omega t)+\xi(t)\,. \tag{1}$$

Here F is proportional to the dc part of the bias current, Ω and A are the frequency and the amplitude of the microwave field, respectively, and $\xi(t)$ denotes white Gaussian noise with zero mean, i.e., $\langle\xi(t)\xi(t')\rangle=2D\delta(t-t')$, $\langle\xi(t)\rangle=0$.

The Langevin equation (1) is statistically equivalent to the two-dimensional ($x\equiv\varphi$, $v\equiv\dot{\varphi}$) *nonstationary* Markovian stochastic process generated by the Fokker-Planck operator

$$\mathbf{L}_2=-\frac{\partial}{\partial x}v+\frac{\partial}{\partial v}[\gamma v+\sin x+F-A\sin(\Omega t)]+D\frac{\partial^2}{\partial v^2}\,. \tag{2}$$

Equivalently, we may consider the *three-dimensional stationary* Markovian process, generated by the Fokker-Planck operator

$$\mathbf{L}_3=-\frac{\partial}{\partial x}v+\frac{\partial}{\partial v}(\gamma v+\sin x+F-A\sin\theta)-\Omega\frac{\partial}{\partial\theta}+D\frac{\partial^2}{\partial v^2}\,. \tag{3}$$

It follows from Floquet theory[12] that the asymptotic long-time solution $P_{\rm as}(x,v,t)$ of the Fokker-Planck equation, $\dot{P}(x,v,t)=\mathbf{L}_2P(x,v,t)$, is periodic with period $T=2\pi/\Omega$. The stationary solution $W_{\rm st}(x,v,\theta)$ of the

FPE, $\dot{W}(x,v,\theta,t) = \mathbf{L}_3 W(x,v,\theta,t)$, is identical to $(1/2\pi)P_{as}(x,v,t)$, if θ is substituted by Ωt (see Ref. 12). The θ-averaged stationary distribution function, $W_{av}(x,v) \equiv \int_0^{2\pi} W_{st}(x,v,\theta)d\theta$, is thus identical to the time-averaged probability density $P_{av}(x,v) \equiv (1/T)\int_0^T P_{as}(x,v,t)dt$. In the noiseless case $(D=0)$ this probability density is given by the invariant measure and can be obtained by solving the deterministic differential equation numerically. The *time-averaged* probability density $P_{av}(x,v)$ is thus the generalization of the deterministic invariant measure, i.e., it represents the noisy invariant measure.[13] Noisy invariant measures and their shapes as a function of the noise strength have been discussed thus far for discrete maps[14] only.

Before we discuss the numerical solution of the FPE's above, some general remarks on the time dependence of the invariant measure are necessary. In the limit of zero noise $(D \to 0)$, the FPE generated by $\mathbf{L}_2$ is reduced to Liouville's equation. For an initial δ function on phase space the solution remains a δ function following the deterministic trajectory forever; i.e., the dynamics of the solution may be chaotic. For an initial probability which has a *finite width* (but still without noise), the probability density changes its shape in the the course of time and approaches a *time-periodic* probability distribution for large times with the period of the driving frequency. This follows from the existence of an invariant measure on any Poincaré cross section. Therefore, possible period-doubling scenarios and irregular behavior for probability distributions are generically (with or without noise) only transient properties of distribution functions.

In contrast to other work,[13] where the Langevin equations have been solved via numerical simulations, we solve the corresponding three-dimensional FPE $\dot{W}(x,v,\theta,t) = \mathbf{L}_3 W(x,v,\theta,t)$ with the matrix-continued-fraction technique[15] in order to obtain the most precise results. The probability distribution $W(x,v,\theta,t)$ is expanded into complete sets with respect to x (trigonometric functions), v [Hermite functions $\psi_n(v)$], and θ (trigonometric functions), i.e.,

$$W(x,v,\theta,t) = \psi_0(v) \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_n^{k,m} \psi_n(v) \times \exp(ikx + in\theta) . \quad (4)$$

The coefficients are ordered into column matrices in such a way that we obtain a tridiagonal vector differential equation of first order in time. For the stationary distribution $W_{st}(x,v,\theta)$ the tridiagonal vector recurrence relation is solved in terms of matrix continued fractions. The asymptotic time-dependent probability in x, $P_{as}(x,t)$, is obtained by integrating $W_{st}(x,v,\theta)$ over v, while the time- (or equivalently phase-) averaged probability $P_{av}(x)$, i.e., the noisy invariant measure in x only, is obtained by performing an integration of $W_{st}(x,v,\theta)$ over v and θ.

The result for $\gamma=0.5$, $A=1$, and $\Omega=1$ is shown for decreasing noise strength D in Fig. 1. The multipeaked structure for small D, which turns out to be typical for deterministic chaotic behavior, is washed out for increasing noise strength. In Fig. 2, the noisy invariant measure

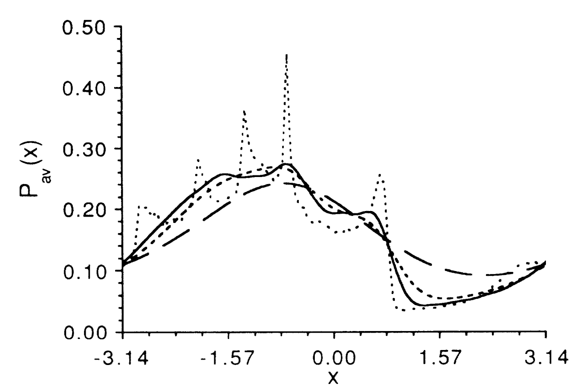

FIG. 1. The noise-invariant measure in x for $\gamma=0.5$, $F=-0.5$, $\Omega=1$, and $A=1$ for decreasing noise strengths, i.e., $D=1$ (long-dashed curve), $D=0.1$ (short-dashed curve), $D=0.03$ (solid curve), and the deterministic invariant measure (dotted curve).

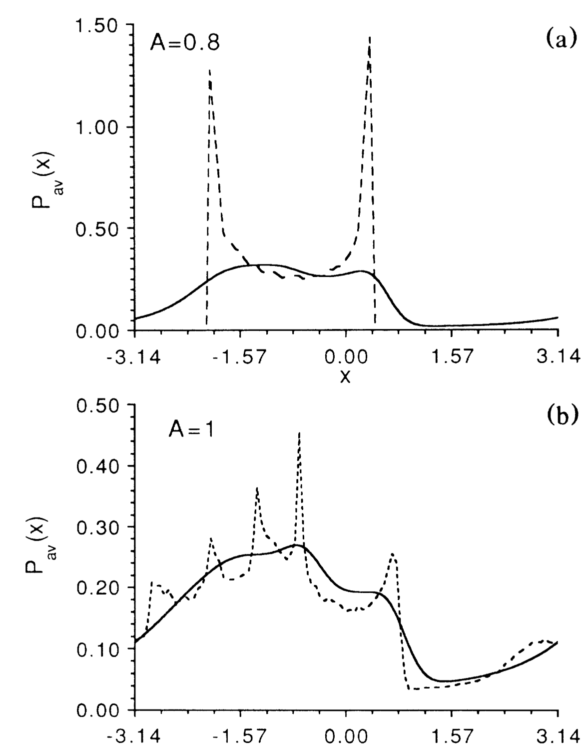

FIG. 2. The noisy invariant measure for $D=0.05$ (solid lines) is compared against the deterministic invariant measure (dashed lines) for $F=-0.5$, $\gamma=0.5$, and $\Omega=1$ (a) below homoclinic threshold, i.e., $A=0.8$, and (b) above homoclinic threshold, i.e., $A=1$.

for $D=0.05$ is compared with the invariant measure of the deterministic system below [2(a)] and above [2(b)] the homoclinic threshold. The invariant measure of the deterministic system as well as the noisy invariant measure develop multipeaked structure above the homoclinic threshold. The smoothing of the multipeak structure in the presence of noise, however, is not uniform in x. Some peaks at $x<0$, for instance, are not visible with noise, while even smaller details are clearly visible elsewhere. This is an indication that the sensitivity to noise is different in different regions of phase space. The transition from a smooth to a multipeaked invariant measure for noisy systems is, however, not as sharp as compared with the deterministic case. The transition to a multipeaked structure starts already for smaller values of A as compared to the deterministic homoclinic threshold value.

Having computed the noisy invariant measure one is led to the question of how the system approaches the invariant measure in the course of time. This dynamical information is contained, for instance, in the functional

$$H(W(x,v,\theta,t))=\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{2\pi}P(x,v,\theta,t)\ln P(x,v,\theta,t)\,dx\,dv\,d\theta\,. \qquad (5)$$

For short to intermediate times (which is the range of validity of the calculations in Ref. 9) the functional H is mainly determined by the specific choice of the initial probability distribution and is thus not characteristic for the system. For large times, the time dependence of the functional H is determined by the smallest nonvanishing eigenvalue of the Fokker-Planck operator (3). Our numerical calculations for those eigenvalues at intermediate noise strength, however, do not indicate a significant transition at a certain value of the field strength A.

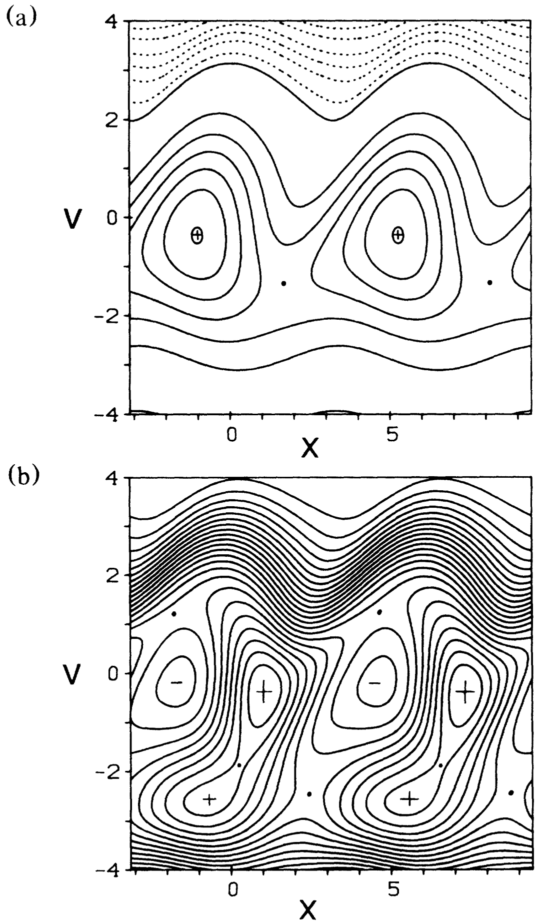

FIG. 3. The altitude charts of the two-dimensional invariant measure $P_{av}(x,v)$ are shown for $F=0.5$, $\gamma=0.5$, and $\Omega=1$ at (a) $A=0.5$ and (b) $A=2.75$. The solid lines are probabilistic equidistant, while the dotted lines, corresponding to smaller probabilities, are not equidistant. The plus signs denote maxima of the probability distribution, the minus signs minima, and the solid circles denote saddle points.

Finally, we consider the two-dimensional invariant measure in velocity-position phase space $P_{av}(x,v)$, which is defined above. In Fig. 3, the two-dimensional invariant measure is plotted in the slightly underdamped regime, $\gamma=0.5$, $\Omega=1$, $D=1$, and $F=0.5$, for $A=0.5$ and $A=2.75$. Again a transition from a smooth invariant measure to a more structured invariant measure for $D=1$ (characterizing a large noise strength) takes place at a certain value of the modulation strength.

With the numerical results above, we can address again the question of how chaos enters into the statistical properties of a periodically driven noisy nonlinear oscillator. The temporal evolution of a distribution function is in contrast to that of a single trajectory, not chaotic but strictly periodic at large times. In the presence of noise, the dynamics of single trajectories clearly does not represent an appropriate characterization of the system. The notion of invariant measure, however, can be discussed in noisy systems within a Fokker-Planck description. The noisy invariant measure shows a significant topological transition, when the deterministic trajectories become chaotic. A similar behavior has been observed in quantum-mechanical systems which are chaotic in the classical limit: The Husimi representation[16] of a wave function undergoes in the limit $h\rightarrow 0$ a complicated topological change if the classical limit is chaotic.[17]

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