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TITS DISTANCE OF HADAMARD MANIFOLDS AND ISOPARAMETRIC HYPERSURFACES

ABSTRACT. We prove that the Tits distance of nonhomogeneous isoparametric hypersurfaces in spheres cannot occur as Tits distance (in the sense of Gromov) at the boundary of a Hadamard manifold.

It is an open problem in the theory of Hadamard manifolds (simply connected Riemannian manifolds of sectional curvature $K \leqslant 0$) whether there exist Hadamard manifolds of higher rank which are neither symmetric spaces nor Riemannian products. 'Higher rank' means that any geodesic is contained in a totally geodesic Euclidean subspace ($K = 0$) with dimension $\geqslant 2$, called *flat*. In a symmetric Hadamard manifold of rank 2, the structure of the set of flats can be read off from the isotropy orbits which form an isoparametric hypersurface family inside the unit tangent sphere. One may ask whether one can construct nonsymmetric Hadamard manifolds of rank 2 where this homogeneous isoparametric hypersurface family is replaced with a nonhomogeneous one. We will show in this paper that this is impossible.

This will follow from a result about the Tits distance Td on the sphere at infinity $X(\infty)$ of a Hadamard manifold X. The pseudometric Td was introduced by M. Gromov (cf. [BGS, lecture I.4]) as the interior metric of the angle distance

$$\measuredangle(a, b) = \sup\{\measuredangle_o(a, b); o \in X\}$$

for any $a, b \in X(\infty)$ where $\measuredangle_o(a, b)$ is the angle between the two geodesics from o to a and b. Gromov called this metric the *Tits distance* since, in the case where X is a symmetric space, it is closely related to the Tits Building of X (cf. [BGS, App. 5]). Since for any $o \in X$, we have a canonical homeomorphism from the unit tangent sphere $S_oX \subset T_oX$ onto $X(\infty)$ through the geodesic rays starting at o, we may consider Td as a distance on S_oX for arbitrary $o \in X$.

In the case of a rank-2 symmetric space $X = G/K$, this metric can be described as follows. Choose $o = 1 \cdot K \in X$. The orbits of K form an *isoparametric hypersurface family* ('i.h.f.') inside $S = S_oX$. This is a family of compact equidistant hypersurfaces of S which foliate the whole sphere up to a subset F, which is the disjoint union of two compact submanifolds of higher codimension, called the *focal manifolds* of the i.h.f. Then the Tits distance of two points $a, b \in S$ is the length of the shortest admissible great circle polygon connecting a and b, where 'admissible' means that the vertices lie on the focal manifolds and the great circle arcs intersect the isoparametric hypersurfaces orthogonally.

Besides the homogeneous i.h.f.'s, which are isotropy orbits of rank-2 symmetric spaces, there exist many nonhomogeneous ones, but they are not yet completely classified (cf. [FKM]). *Any* i.h.f., homogeneous or not, in a sphere S defines a distance on S in the same way as described above. Let us call this the *Tits distance of the i.h.f.* Can this also be isometric to the Tits distance of a Hadamard manifold if the i.h.f. is nonhomogeneous? The following theorem says that the answer is No:

THEOREM. *Suppose that X is an irreducible Hadamard manifold whose Tits distance is isometric to that of an isoparametric hypersurface family. Then X is a symmetric space.*

The proof combines results of P. Eberlein and G. Thorbergsson. We first recall Eberlein's theorem. Let X be a Hadamard manifold. For any $x \in X$ let s_x be the *geodesic reflection at* x which sends $\gamma(t)$ to $\gamma(-t)$ for any geodesic γ on X with $\gamma(0) = x$. This is a diffeomorphism of X and can be extended to a homeomorphism of $X(\infty)$. Let $G^* \subset \mathrm{Diff}(X)$ be the group generated by all geodesic reflections s_x, $x \in X$ and G_e^* the subgroup of index 2 which consists of all products of an even number of geodesic reflections. Then Eberlein ([E, Th. A]) has stated, based on previous work by Ballmann [B], that the closures of the orbits of G_e^* and of the holonomy group are the same, if we identify $X(\infty)$ with S_oX for some $o \in X$. Consequently, by the result of Berger [Be], if G_e^* leaves invariant a proper closed subset of $X(\infty)$, then X is either a Riemannian product (which is excluded by irreducibility) or a symmetric space ([E, Th. B).

Now let us assume that $X(\infty)$ is Tits isometric to a sphere S with an i.h.f. We identify $X(\infty)$ and S by this isometry. We claim that the set $F \subset S$, the union of the focal manifolds, is invariant under G^*. In fact, if $a \in F$ and $x \in X$, then $\mathrm{Td}(a, s_x(a)) \geqslant \pi$ since a and $s_x(a)$ are joined by a geodesic ([BGS, I.4.10(ii)]). On the other hand, Thorbergsson ([T1], cf. Appendix) has shown that the Tits distance of an i.h.f. has diameter π. Thus a and $s_x(a)$ are joined by an admissible polygon of length π. Since the spherical distance between the focal manifolds is constant, it is π/g where $2g$ is the (even) number of times an admissible great circle intersects F. (In fact, Münzner [M] has shown that $g \in \{1, 2, 3, 4, 6\}$.) Thus the end point of an admissible polygon of length π starting at $a \in F$ must lie in F. Hence $s_x(a) \in F$ which shows that F is G^*-

invariant and hence (by Eberlein) invariant under the holonomy group at o.

To finish the proof, it remains to show that F is closed when $S = X(\infty)$ is identified with $S_o X$ for some $o \in X$. (Note that a Tits isometry between $X(\infty)$ and S may not preserve the usual topologies.) The following argument is due to Jens Heber. Let $a \in S \backslash F$. Suppose that $\mathrm{Td}(a, F) \geqslant \varepsilon$. Let

$$B = \{b \in S;\ \measuredangle_o(a, b) < \varepsilon\}.$$

We have to show that $B \subset S \backslash F$. Let $-a \in S$ be the point opposite to a with respect to o, i.e. $-a = \gamma(-\infty)$ if γ is the geodesic with $\gamma(0) = o$ and $\gamma(\infty) = a$. Then

$$(*) \qquad \mathrm{Td}(b, -a) \geqslant \measuredangle_o(b, -a) > \pi - \varepsilon$$

for any $b \in B$. Choose a shortest Tits geodesic λ from $-a$ to b and let c be the first point of $\lambda \cap F$. Let λ_1 be the segment between $-a$ and c and λ_2 the one between c and b. Since $\mathrm{Td}(a, F) \geqslant \varepsilon$ and the Tits distance between the components of F is π/g, we have $L(\lambda_1) \leqslant \pi/g - \varepsilon$. Now assume that $b \in F$. Then $L(\lambda_2) = k \cdot \pi/g$ for some $k \in \{0, \dots, g-1\}$ and therefore,

$$\mathrm{Td}(b, -a) \leqslant (k+1)\pi/g - \varepsilon \leqslant \pi - \varepsilon$$

which is a contradiction to $(*)$. Thus $b \in S \backslash F$ for all $b \in B$ and so F is a closed subset of S.

REMARK. Recently, Thorbergsson [T2] has shown that all *known* i.h.f.'s (including the nonhomogeneous ones) give rise to compact metric locally connected topological Tits buildings,of rank 2 (cf. [BS]). This is not known for arbitrary i.h.f.'s. However, our result applies to *all* i.h.f.'s.

APPENDIX

Thorbergsson's result on the Tits diameter of i.h.f.'s was not published in the final version of [T1]. In [T2], there is a simpler proof, but only for the known examples. Therefore, we give here a proof of the general result.

THEOREM (Thorbergsson). *The Tits distance of any isoparametric hypersurface family in a sphere S has diameter π.*

Proof. Let $a, b \in S$. We may assume that $a, b \in S \backslash F$. Let $a \in M$ where M is one of the isoparametric hypersurfaces. We will show that for any other point $c \in M$ there exists a closed admissible polygon p of length $\leqslant 2\pi$ passing through a and c. In particular, if c is the point of M which is closest to b, then p has to pass through b, so $\mathrm{Td}(a, b) \leqslant 2\pi/2 = \pi$ and we are finished.

To construct p, recall that M can be considered as a tube around either focal manifold $F_j, j = 1, 2$. Thus M is the total space of two sphere bundles

$\pi_j \colon M \to F_j$, and S is the union of the corresponding disc bundles B_1 and B_2 with $M = B_1 \cap B_2 = \partial B_1 = \partial B_2$. Now we consider homology with $\mathbb{Z}_2$-coefficients in order to avoid problems with nonorientability. The Mayer–Vietoris sequence

$$H_{k+1}(S) \to H_k(M) \xrightarrow[\pi_{1*} \oplus \pi_{2*}]{} H_k(F_1) \oplus H_k(F_2) \to H_k(S)$$

shows that π_{1*} and π_{2*} are surjective for $k < n + 1 := \dim(S)$, and for $k + 1 < n + 1$, the map $\pi_{1*} \oplus \pi_{2*}$ is injective which means

$$(*) \qquad \ker \pi_{1*} \cap \ker \pi_{2*} = 0.$$

On the other hand, we have the Gysin sequences for the sphere bundles $\pi_j \colon M \to F_j$, namely

$$\xrightarrow[\pi_{j*}]{} H_{k+m_j+1}(F_j) \to H_k(F_j) \xrightarrow[\tau_j]{} H_{k+m_j}(M) \xrightarrow[\pi_{j*}]{} H_{k+m_j(F_j)}$$

where m_j is the fiber dimension and τ_j the transfer map which assigns to each k-cycle z in F_j its full preimage $\pi_j^{-1}(z)$ which is a $(k + m_j)$-cycle in M. Since π_{j*} is surjective, the last sequence shows that τ_j is injective, i.e. it maps a nontrivial cycle onto a nontrivial cycle. Thus $S_1 := \pi_1^{-1}(\pi_1(a))$ carries the nontrivial m_1-cycle $\tau_1([\pi_1(a)])$. Since $\pi_{1*} \circ \tau_1 = 0$, we have $[S_1] \in \ker \pi_{1*}$ and hence $[\pi_2(S_1)] \neq 0$, by $(*)$. Thus $S_2 := \pi_2^{-1}(\pi_2(S_1))$ carries the nontrivial $(m_1 + m_2)$-cycle $\tau_2(\pi_{1*}[S_1])$. Next, $S_3 := \pi_1^{-1}(\pi_1(S_2))$ carries a nontrivial $(m_1 + m_2 + m_1)$-cycle. Note that the hypersurface $M \subset S$ has g curvature distributions which focalize onto F_1 or F_2, and so the dimensions of the curvature distributions (which add up to $n = \dim(M)$) are m_1 or m_2 in turns. Hence, after g steps, we arrive at a subset $S_g \subset M$ which carries a nontrivial cycle of the top dimension n. This shows that $S_g = M$.

Thus, by construction of S_g, the point $a \in M$ can be joined to any other point $c \in M$ by an admissible great circle polygon p_1 with at most g vertices, where the first vertex lies on F_1. In the same way, we can join a to c by an admissible polygon p_2 with at most g vertices, where the first vertex lies in F_2. These two polygons together form a closed admissible polygon $p = p_1 \cup p_2$ with at most $2g$ vertices. Since any of the great circle arcs between two consecutive vertices has length π/g (which is the distance between F_1 and F_2), we see that p has length $\leqslant 2\pi$.

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