

THE SPECTRUM AND EIGENSPACE STRUCTURE OF LINEAR UNCERTAIN SYSTEMS

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Abstract

We present a Lyapunov exponents approach to output feedback stabilization of linear systems with time varying uncertainties. This allows the analysis of precise stability and stabilization radii. The results are compared to those obtained using quadratic Lyapunov functions.

1. Introduction

The stability behavior of linear, time constant systems is determined by the spectrum and the corresponding eigenspaces of the system matrix. In this paper we show that one can also use these concepts — in an appropriately generalized form — to describe the stability behavior of linear uncertain systems.

Typically, in linear systems uncertain parameters change independently or jointly within prescribed bounds, given e.g. by intervals or Euclidian norm restrictions. We use the following general model, which remains within the realm of linear systems theory

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^m v_i(t)A_i x(t) + B\dot{u}(t) \\ y(t) &= Cx(t), \end{aligned} \quad (1.1)$$

where $v = (v_i)$ represents the uncertainty with values in $V_\rho := \rho V_1$, $\rho \geq 0$, V_1 a given compact subset of \mathbb{R}^m with $0 \in V_1$.

For the uncontrolled system (i.e. $\dot{u} \equiv 0$) we will study the stability properties; of particular interest is their dependence on the size ρ of the uncertainty. For the controlled system we allow static output feedback $u = Fy$, where F has to satisfy given bounds. Again, the dependence of the stabilizability properties on the bounds for $v(\cdot)$ and F are of particular interest.

For time constant uncertainties stability properties of linear systems are mainly analyzed following the already classical work of Kharitonov [18], cp. e.g. the proceedings [13]. A measure for the robustness

of the stability of an equation $\dot{x} = Ax$ with respect to time constant uncertainties in V_ρ is the real stability radius [14]: Let $\mu_v := \max\{\mu; \mu \in \text{spec}(A + \Sigma v_i A_i)\}$, then

$$r_{\mathbb{R}}(A) = \inf\{\rho \geq 0; \sup_{v \in V_\rho} \mu_v \geq 0\}. \quad (1.2)$$

For time varying uncertainties $v \in V_\rho := \{v: \mathbb{R} \rightarrow \mathbb{R}^m; v(t) \in V_\rho \text{ a.e.}\}$ the use of quadratic Lyapunov functions is a well-established tool for stability analysis. They yield sufficient conditions for stability and lead to the following measure of robustness

$$\begin{aligned} r_{LJ}(A) &= \sup\{\rho \geq 0; \text{there exist a positive} \\ &\quad \text{definite matrix } P \in Gl(d, \mathbb{R}) \\ &\quad \text{and } \alpha > 0 \text{ such that for all} \\ &\quad (x, v) \in \mathbb{R}^d \times V_\rho \text{ we have} \\ &\quad x^T [P(A + v) + (A + v)^T P] x \\ &\quad \leq -\alpha |x|^2\}. \end{aligned} \quad (1.3)$$

This stability radius is implicit in much of the quadratic stability literature, see e.g. Rotea and Khargonekar [19]. Recall that for a large class of uncertainties, r_{LJ} coincides with the complex stability radius $r_{\mathbb{C}}$ of Hinrichsen and Pritchard [14–16]. Compare also Doyle et al. [8] for connections with H^∞ theory [9].

In this paper, we study the precise (exponential) stability radius given by

$$\begin{aligned} r(A) &= \inf\{\rho \geq 0; \text{there is } v \in V_\rho \\ &\quad \text{such that (1.1) with } \dot{u} = 0 \\ &\quad \text{is not exponentially stable}\}. \end{aligned} \quad (1.4)$$

This radius is characterized by (nonnegativity of) the natural generalization of eigenvalues for linear systems with time-varying coefficients, namely Lyapunov exponents. They yield precise criteria for stability and stabilization of linear uncertain systems. More importantly, it turns out that these Lyapunov exponents are actually eigenvalues of a certain family of matrices and the corresponding eigenspaces determine the controllability structure. The Lyapunov exponents are numerically computable using optimal

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control methods. This precise characterization of stability and stabilization can already be different from $r_{\mathbf{x}}$ and r_{L_f} for simple two-dimensional systems with output feedback.

The contents of this paper is as follows: In Section 2. we introduce Lyapunov exponents of linear uncertain systems, in Section 3. several results on perturbations of linear differential equations are given, and in Section 4. these results are applied to the output feedback stabilization of linear, uncertain systems, including several examples.

2. Linear Uncertain Systems and their Lyapunov Spectrum

We will consider linear, time constant output feedbacks, satisfying an *a priori* bound, for linear uncertain systems of the following kind:

$$\begin{aligned} \dot{x} &= Ax + v(t)x + B\hat{u}, \\ y &= Cx, \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^l \end{aligned} \quad (2.1)$$

where $A \in gl(d, \mathbb{R})$, the real $d \times d$ matrices, B and C are real matrices of dimension $d \times k$ and $l \times d$, respectively. The uncertainties are denoted by v , and we assume: $V \subset gl(d, \mathbb{R})$ is a linear subspace, $V_1 \subset V$ a compact, connected subset with $0 \in \text{int } V_1$ and $\text{cl}(\text{int } V_1) = V_1$, where int and cl denote the interior and the closure, respectively, with respect to V . Define

$$\begin{aligned} \rho V_1 &:= V_\rho \subset gl(d, \mathbb{R}), \quad \rho \geq 0 \\ \mathcal{V}_\rho &:= \{v: \mathbb{R} \rightarrow V_\rho, \text{ measurable}\}. \end{aligned}$$

\mathcal{V}_ρ are the time varying uncertainties of size ρ . This model includes in particular norm bounded and interval type uncertainties.

As inputs \hat{u} we allow time invariant output feedbacks of the form $\hat{u} = FCx$ with: \hat{U} is a linear subspace of the real $k \times l$ matrices, $\hat{U}_1 \subset \hat{U}$ is a compact, connected subset with $\text{int}_{\hat{U}} \hat{U}_1 \neq \emptyset$. Denote $U := B\hat{U}C$, $U_1 := B\hat{U}_1C$ and

$$\sigma U_1 := U_\sigma \subset gl(d, \mathbb{R}), \quad \sigma \geq 0.$$

Then any output feedback gain matrix F corresponds to an element of U . With these notations, the system (2.1) can be written as

$$\begin{aligned} \dot{x} &= (A + v(t))x + ux, \\ x &\in \mathbb{R}^d, \quad v \in \mathcal{V}_\rho, \quad u \in U_\sigma, \end{aligned} \quad (2.2)$$

which represents a linear system with time varying uncertainties of size ρ and constant output feedback of size σ .

For a given initial point $0 \neq x \in \mathbb{R}^d$, given uncertainty $v \in \mathcal{V}_\rho$ and given feedback $u \in U_\sigma$, the exponential growth rate, or Lyapunov exponent, of the corresponding trajectory $\varphi(t, x, v, u)$, $t \geq 0$, of (2.2) is given by

$$\lambda(x, v, u) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log |\varphi(t, x, v, u)|; \quad (2.3)$$

here and for the rest of the paper $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^d . (Observe that by equivalence of norms in \mathbb{R}^d , (2.3) is independent of the norm employed.) Contrary to state space concepts, like Lyapunov functions, the Lyapunov exponents are defined on infinite dimensional spaces, i.e.

$$\lambda: \mathbb{R}^d \setminus \{0\} \times \mathcal{V}_\rho \times U_\sigma \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}.$$

The collection of all Lyapunov exponents is called the Lyapunov spectrum of (2.2). Note that for constant $v \in \mathcal{V}_\rho$, these exponents coincide with the real parts of the eigenvalues of $A + v + u$; and for T -periodic $v \in \mathcal{V}_\rho$, they are the logarithms of the Floquet multipliers, multiplied by $\frac{1}{T}$, cp. e.g. [10,11]. It is easily seen that the Lyapunov exponents depend only on the direction of the initial value, i.e. $\lambda(\alpha x, v, u) = \lambda(x, v, u)$ for $\alpha \neq 0$. Hence λ can also be defined on the space of directions in \mathbb{R}^d , i.e. on the $(d-1)$ -dimensional projective space \mathbb{P}^{d-1} , obtained by identifying opposite points $s = \frac{x}{|x|}$ on the sphere S^{d-1} , i.e. λ is a map $\lambda: \mathbb{P}^{d-1} \times \mathcal{V}_\rho \times U_\sigma \longrightarrow \overline{\mathbb{R}}$. A straightforward application of the chain rule yields for the projected trajectories $s(t) = \frac{x(t)}{|x(t)|}$ of (2.2) the system (on S^{d-1} or \mathbb{P}^{d-1})

$$\dot{s}(t) = h(s(t), v(t), u), \quad t \in \mathbb{R} \quad (2.4)$$

and for the Lyapunov exponents

$$\lambda(x, v, u) = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t q(s(\tau), v(\tau), u) d\tau \quad (2.5)$$

with

$$\begin{aligned} h(s, v, u) &:= [A + v + u - s^T(A + v + u)s \cdot Id] s, \\ q(s, v, u) &:= s^T(A + v + u)s. \end{aligned}$$

In Section 3. we will link the Lyapunov exponents to the dynamics of the nonlinear control system (2.4).

For each feedback $u \in U$ denote the maximal realizable exponent by

$$\mathcal{K}(\rho, u) = \sup_{v \in \mathcal{V}_\rho} \sup_{x \neq 0} \lambda(x, v, u), \quad (2.6)$$

and the minimal exponent that can be achieved by a feedback $u \in U_\sigma$ is

$$\mathcal{K}(\rho, \sigma) = \inf_{u \in U_\sigma} \mathcal{K}(\rho, u). \quad (2.7)$$

Observe that the system (1.1) can be written as a special case of (2.2) and that the stability radius $r(A)$ defined in (1.4) is given by $r(A) = \inf\{\rho \geq 0; \mathcal{K}(\rho, 0) \geq 0\}$.

The discussion above shows that uncertainty may be viewed as a bounded, time-varying structured perturbation of $\dot{x} = Ax$. The maximal spectral value under these perturbations is $\mathcal{K}(\rho, u)$. Time invariant feedback can be viewed as a time-constant structured perturbation $\dot{x} = (A + u)x + v(t)x$, and $\mathcal{K}(\rho, \sigma)$ is the minimal spectral value for $u \in U_\sigma$, given all time varying perturbations $v \in \mathcal{V}_\rho$. Therefore, we will discuss in the next section perturbation theory of linear ordinary differential equations.

In a natural way, $\mathcal{K}(\rho, u)$ and $\mathcal{K}(\rho, \sigma)$ can be used to define stability and stabilization radii. This will be studied in Section 4., together with the continuity properties of the functions \mathcal{K} .

3. Perturbations of Linear Differential Equations

Consider the system (2.2) with $u = 0$, i.e.

$$\dot{x} = Ax + v(t)x, \quad x \in \mathbb{R}^d, \quad v \in \mathcal{V}_\rho. \quad (3.1)_\rho$$

We are interested in the Lyapunov exponents $\lambda(x, v)$, $x \neq 0$, $v \in \mathcal{V}_\rho$ of this family of differential equations and in the corresponding directions. Hence we study the projected system

$$\dot{s} = h(s, v(t)), \quad s \in \mathbb{P}^{d-1}, \quad v \in \mathcal{V}_\rho \quad (3.2)_\rho$$

on the space \mathbb{P}^{d-1} of directions, and its exponents described via (2.5).

Only the subspace, in which the perturbations act, is of interest; in the other directions the stability (and the stabilizability) properties are described by the well-known linear theory. In order to keep the notation simple, we will assume that the perturbations affect every direction. This is accomplished by postulating that the Lie algebra generated by the vectorfields $h(\cdot, v)$, $v \in \mathcal{V}_\rho$, on \mathbb{P}^{d-1} has full rank for all $s \in \mathbb{P}^{d-1}$, i.e. we assume from now on that the following hypothesis (H), well known from geometric control theory, is satisfied, cp. e.g. [17]:

$$\dim \mathcal{LA}\{h(\cdot, v); v \in \mathcal{V}_\rho\}(s) = d - 1 \quad (H)$$

for all $s \in \mathbb{P}^{d-1}$.

We will describe the Lyapunov exponents of (3.1) $_\rho$ by analyzing specific perturbations, namely piecewise

constant periodic elements of \mathcal{V}_ρ , and their corresponding fundamental matrices. Define the positive semigroup \mathcal{S}^ρ of (3.1) $_\rho$ by

$$\mathcal{S}^\rho := \left\{ e^{t_n(A+v_n)} \dots e^{t_1(A+v_1)}; v_i \in \mathcal{V}_\rho, \right. \\ \left. t_i \geq 0, i = 1 \dots n \in \mathbb{N} \right\}$$

and the associated group $\mathcal{G}^\rho \subset Gl(d, \mathbb{R})$ as

$$\mathcal{G}^\rho := \left\{ e^{t_n(A+v_n)} \dots e^{t_1(A+v_1)}; v_i \in \mathcal{V}_\rho, \right. \\ \left. t_i \in \mathbb{R}, i = 1 \dots n \in \mathbb{N} \right\}.$$

Denote furthermore for $t \geq 0$ by $\mathcal{S}_{\leq t}^\rho$ the subset of \mathcal{S}^ρ with $\sum t_i \leq t$. Note that each piecewise constant, T -periodic perturbation $v \in \mathcal{V}_\rho$ gives rise to an element of $\mathcal{S}_{\leq T}^\rho$ via: Let $v = v_i$ for $t \in \left[\sum_{j=1}^{i-1} t_j, \sum_{j=1}^i t_j \right)$,

$i = 1 \dots n$ with $\sum_{j=1}^n t_j = T$ and $v_i \in \mathcal{V}_\rho$. (We have set $t_0 = 0$.) Then $g_v := e^{t_n(A+v_n)} \dots e^{t_1(A+v_1)} \in \mathcal{S}_{\leq T}^\rho$. Similarly every element g of \mathcal{S}^ρ gives rise to a (not necessarily unique) piecewise constant periodic element v_g of \mathcal{V}_ρ .

The hypothesis (H) implies that the system semigroup satisfies for all $T > 0$

$$cl \mathcal{S}_{\leq T}^\rho = cl \text{int } \mathcal{S}_{\leq T}^\rho, \\ \text{in particular } \text{int } \mathcal{S}_{\leq T}^\rho \neq \emptyset, \quad (3.3)$$

where cl and int denote closure and interior with respect to $\mathcal{G}^\rho \subset Gl(d, \mathbb{R})$. Before we can characterize perturbations and their eigenvalues and eigenspaces, we have to introduce one concept describing maximal sets of complete controllability [1,2]:

3.1. Definition. A set $D^\rho \subset \mathbb{P}^{d-1}$ is called a *control set* of (3.2) $_\rho$, if for all $s \in D^\rho$ one has $cl \mathcal{O}^\rho(s) \supset D^\rho$, and if D^ρ is maximal with this property. Here $\mathcal{O}^\rho(s) := \{y \in \mathbb{P}^{d-1}; \text{there are } t \geq 0 \text{ and } v \in \mathcal{V}_\rho \text{ with } s(t, s, v) = y\}$ is the set of points reachable from s with some perturbation $v \in \mathcal{V}_\rho$.

3.2. Remark. Hypothesis (H) implies for control sets D^ρ with nonvoid interior that $\text{int } D^\rho \subset \mathcal{O}^\rho(s)$ for all $s \in D^\rho$, and that any such control set is closed iff it is invariant, i.e. $\mathcal{O}^\rho(s) \subset D^\rho$ for all $s \in D^\rho$.

The following theorem classifies the control sets of the projected system (3.2) $_\rho$:

3.3 Theorem. Let Hypothesis (H) be satisfied. Then the following holds:

(i) The system (3.2) $_\rho$ has $1 \leq k_\rho \leq d$ control sets D_i^ρ , $i = 1 \dots k_\rho$, with nonvoid interior. For a direction $s \in \mathbb{P}^{d-1}$ we have $s \in \text{int } D_i^\rho$ for some control set D_i^ρ

iff s is an eigenvector for some real eigenvalue of some element $g \in \text{int } S^\rho$.

(ii) Let $M_i \subset \mathbb{P}^{d-1}$, $i = 1 \dots k_0$, be the sum of the generalized eigenspaces of the unperturbed system matrix A , corresponding to the eigenvalues with equal real part. If $g_0 := e^{At} \in \text{int } S^\rho$ for some $t > 0$ and all $\rho > 0$ (small), then there exists $\hat{\rho} > 0$ such that $k_\rho = k_0$ for all $0 < \rho < \hat{\rho}$, and $M_i \subset \text{int } D_i^\rho$, $i = 1 \dots k_0$ — i.e. the number of control sets D_i^ρ coincides with the number of different M_i of A .

Furthermore, for $0 < \rho < \rho'$ and every control set D^ρ there is a control set $D^{\rho'}$ with $D^\rho \subset D^{\rho'}$, and vice versa, the map $\rho \mapsto k_\rho$ is nonincreasing.

(iii) For fixed $\rho > 0$, the control sets with nonvoid interior are linearly ordered by $D^\rho \prec \tilde{D}^\rho$ iff there are $x \in D^\rho$, $y \in \tilde{D}^\rho$ and $v \in \mathcal{V}_\rho$, $t > 0$ with $s(t, x, v) = y$. We enumerate these sets such that $i < j$ iff $D_i^\rho \prec D_j^\rho$. The maximal element $C^+ := D_{k_\rho}^\rho$ is closed and invariant, the minimal element $C_- := D_1^\rho$ is open.

The proof of (i) and (iii) was given in [1, Theorem 3.10]; there the control sets were constructed around the eigenspaces of elements $g \in \text{int } S_{\leq T}$. Assertion (ii) is a consequence of a much more general result in [2]: The sets M_i are the Morse sets of the flow associated with $\dot{x} = Ax$ on the projective space \mathbb{P}^{d-1} . The assumption on $g_0 = e^{At}$ guarantees that every pair $(v_{g_0}, s) \in \mathcal{V}_\rho \times \mathbb{P}^{d-1}$ is an inner pair for all $\rho > 0$, [2, Remark 3.3]. Hence the first claim in (ii) follows from [2, Theorem 4.12]. The second one is an obvious consequence of the definitions.

3.4. Remark. The structure of the eigenspaces of elements in $\text{int } S^\rho$, $\rho \geq 0$ is therefore the following: For $\rho = 0$ we have the Morse sets M_i of A on \mathbb{P}^{d-1} . As ρ grows, the M_i extend to sets of eigenspaces D_i^ρ , which are characterized by controllability properties of $(3.2)_\rho$, with $M_i \subset \text{int } D_i^\rho$. Depending on the structure of A and V , certain D_i^ρ may unite for increasing ρ . In particular, $k_0 = d$ iff all eigenvalues of A have different real parts, and $k_0 = 1 = k_\rho$ for all $\rho > 0$ iff all real parts of eigenvalues of A are equal.

Interesting enough, the eigenvalues of the elements in $\text{int } S^\rho$ determine the stability of the uncertain system $(3.1)_\rho$ via the Lyapunov spectrum:

3.5. Theorem. Let Hypothesis (H) be satisfied. Then the following holds:

(i) To each control set D_i^ρ corresponds an interval $[a_i^\rho, b_i^\rho] = \text{cl } I_i^\rho$, $i = 1 \dots k_\rho$, via $I_i^\rho := \{\lambda(x, v_g); x \in D_i^\rho \text{ and } x \text{ is an eigenvector of some } g \in \text{int } S^\rho\}$. Hence each value in any I_i^ρ corresponds to an eigenvalue of some $g \in \text{int } S^\rho$, i.e. to a Floquet multiplier of the fundamental matrix g_v of some periodic $v \in \mathcal{V}_\rho$.

(ii) The intervals I_i^ρ depend continuously on ρ on the intervals, where k_ρ is constant, and all real parts

of eigenvalues of A are in one of these intervals. Furthermore, the largest and the smallest eigenvalues, $\sup I_{k_\rho}^\rho$ and $\inf I_1^\rho$ respectively, depend continuously on $\rho \in [0, \infty)$.

(iii) For each $\rho \geq 0$, the intervals I_i^ρ are ordered in the sense that $i < j$ implies $a_i^\rho \leq a_j^\rho$ and $b_i^\rho \leq b_j^\rho$.

For a proof of this result see [6]. The complete stability behavior of $(3.1)_\rho$ is given by these intervals of eigenvalues together with the eigenspace structure D_i^ρ [6]. Here we are only interested in the stability properties with respect to all $x \in \mathbb{R}^d \setminus \{0\}$, hence only the largest spectral value plays a role and one obtains the following result:

3.6. Theorem. Let Hypothesis (H) be satisfied. Denote for each $u \in U$ by $I_i^\rho(u)$ the corresponding spectral intervals of $\dot{x} = (A + u)x + v(t)x$, $v \in \mathcal{V}_\rho$. Then $\mathcal{K}(\rho, u) = \sup I_{k_\rho}^\rho(u)$ and $\mathcal{K}(\rho, u)$ depends continuously on $\rho \in [0, \infty)$, for $u \in U$ fixed.

For a proof see [4], where also further characterizations of \mathcal{K} are given.

3.7. Remark. It is well known that in general Lyapunov exponents — in contrast to eigenvalues for time invariant matrices — do not depend continuously on the right hand side of a linear differential equation, cp. e.g. Hahn [11]. Therefore, the continuity statement with respect to ρ given above appears remarkable, cp. also the discussion in Hinrichsen and Pritchard [16] about perturbations of eigenvalues.

4. Stabilization of Linear Uncertain Systems

The perturbation results of the preceding section will be applied now to the stabilization of linear, uncertain systems described by (2.2). Note that all results of Section 3. remain valid for the feedback system (2.2) for fixed feedback $u \in U_\sigma$.

First of all we note the following continuity properties of the functions \mathcal{K} as defined in Section 2.

4.1. Theorem. Assume that for all $u \in U_\sigma$ and all $\sigma \geq 0$, $\rho > 0$, the system $(2.4)_\rho$ satisfies Hypothesis (H). Then the following holds:

- (i) The function $\mathcal{K}: [0, \infty) \times U_\sigma \rightarrow \overline{\mathbb{R}}$, $(\rho, u) \mapsto \mathcal{K}(\rho, u)$ is continuous, and increasing in ρ for u fixed.
- (ii) The function $\mathcal{K}: [0, \infty) \times [0, \infty) \rightarrow \overline{\mathbb{R}}$, $(\rho, \sigma) \mapsto \mathcal{K}(\rho, \sigma)$ is continuous, and increasing in ρ , decreasing in σ .

Proof. The monotonicity statements follow directly from the definitions. Continuity in (i) is a consequence of Theorem 3.6, and (ii) follows from a standard perturbation argument, see [7].

4.2. Corollary. The zero level sets of the functions \mathcal{K}

$$\Gamma(u) := \{(\rho, u) \in [0, \infty) \times U; \mathcal{K}(\rho, u) = 0\}$$

$$\Gamma := \{(\rho, \sigma) \in [0, \infty) \times [0, \infty); \mathcal{K}(\rho, \sigma) = 0\}$$

are closed and connected, possibly empty.

We will use these results to characterize the following stabilization radii:

4.3. Definition. Given a matrix $A \in gl(d, \mathbb{R})$, a family of uncertainties $\{\mathcal{V}_\rho, \rho \geq 0\}$ and of output feedbacks $\{U_\delta, \delta \geq 0\}$, define the stabilization radius for constant $u \in U$

$$r(u) := \inf \{ \rho \geq 0; \text{there is } v \in \mathcal{V}_\rho \text{ such that (2.2) is not exponentially stable} \}$$

and for feedbacks u of a given size σ

$$r(\sigma) := \inf \{ \rho \geq 0; \text{for every } u \in U_\sigma \text{ there is } v \in \mathcal{V}_\rho \text{ such that (2.2) is not exponentially stable} \}.$$

If $A + u + v(\cdot)$ is exponentially stable for all $v \in \mathcal{V}_\rho$, $\rho \geq 0$, then we set $r(u) := \infty$, and similarly for $r(\sigma)$.

4.4. Corollary. The stabilization radii defined above are characterized by

$$r(u) = \begin{cases} \min\{\rho \geq 0; \mathcal{K}(\rho, u) = 0\} & \text{if } \Gamma(u) \cap \{(\rho, u); \rho \geq 0\} \neq \emptyset \\ 0 & \text{if } \mathcal{K}(0, u) > 0, \text{ i.e. if } A + u \text{ is unstable} \\ \infty & \text{otherwise} \end{cases}$$

$$r(\sigma) = \begin{cases} \min\{\rho \geq 0; \mathcal{K}(\rho, \sigma) = 0\} & \text{if } \Gamma \cup \{(\rho, \sigma); \rho \geq 0\} \neq \emptyset \\ 0 & \text{if } \mathcal{K}(0, \sigma) > 0 \\ \infty & \text{otherwise.} \end{cases}$$

Proof. The expressions above with min replaced by inf are clear from the definitions. Since the functions \mathcal{K} are continuous, the assertions follow.

The continuity and monotonicity properties of \mathcal{K} imply also certain continuity and monotonicity results for the stabilization radii:

4.5. Corollary. Denote by $\text{dom}_u := \{u \in U; r(u) < \infty\}$ and by $\text{dom}_\sigma := \{\sigma \geq 0; r(\sigma) < \infty\}$ the effective domains of $r(u)$, and of $r(\sigma)$ respectively. Then

(i) $r(\sigma)$ is right continuous with left hand limits, and increasing in dom_σ , in particular $r(\sigma)$ is lower semi continuous;

(ii) $r(u)$ is semi continuous in dom_u , i.e. if $u_0 \in \text{dom}_u$ is a point of discontinuity, then there is a sequence $\{u_n, n \geq 1\}$ in dom_u such that $u_n \rightarrow u_0$ and $r(u_n) \rightarrow r(u_0)$ as $n \rightarrow \infty$.

Proof. Both parts follow from Theorem 4.1 and Corollary 4.2 via standard analysis arguments.

The following result characterizes the extreme cases, where either $r(u) = 0$ or $r(u) = \infty$.

4.6. Proposition. (i) $r(u) > 0$ iff $A + u$ is stable.

(ii) $r(u) = \infty$ iff $A + u$ is stable and there exists a transformation matrix $T \in Gl(d, \mathbb{R})$ such that TVT^{-1} consists only of skew symmetric matrices.

Proof. (i) If $A + u$ is stable, then $\mathcal{K}(0, u) = \max \{ \text{Re } \mu; \mu \in \text{spec}(A + u) \} < 0$, where $\text{spec}(A + u)$ denotes the set of eigenvalues of $A + u$. Hence $r(u) > 0$ by continuity of $\mathcal{K}(\cdot, u)$. Vice versa, if $r(u) > 0$, then there exists $\rho > 0$ with $\mathcal{K}(\rho, u) < 0$. But $A + u \in A + u + \mathcal{V}_\rho$ for all $\rho > 0$. For a proof of (ii) see [7].

4.7. Remark. Our definition of the stabilization radius for time varying uncertainties uses exponential stability, i.e. it is based on Lyapunov exponents. An alternative definition in Hinrichsen et al. [12] uses uniform exponential stability, i.e. is based on Bohl exponents. Since by the results of Section 3., $\mathcal{K}(\rho, u)$ is, for each $u \in U$, the supremum over eigenvalues of periodic matrix functions, the two concepts agree in the situation considered here, see also Theorem 5. in [3].

4.8. Remark. The model (2.2) allows for measurable uncertainties with values in \mathcal{V}_ρ . If more information concerning the stochastic nature of the uncertainties is available, assertions holding with probability one are of interest. In [3] results of this type and their relation to the deterministic radii defined above are discussed.

Finally we examine more closely the relation between the stability radii $r_{\mathbb{N}}(A)$, see (1.3), $r(A)$ as in Definition 4.3, and $r_{L_f}(A)$ from (1.4). It is an immediate consequence of the definitions that for fixed feedback gain $u \in U$ these radii satisfy

$$r_{\mathbb{N}}(u) \geq r(u) \geq r_{L_f}(u); \quad (4.1)$$

hence, taking suprema over $u \in U_\sigma$, we also have

$$r_{\mathbb{N}}(\sigma) \geq r(\sigma) \geq r_{L_f}(\sigma). \quad (4.2)$$

Townley and Ryan [20] give conditions, which ensure that for σ large $r_{\mathbb{N}}(\sigma) = r_{L_f}(\sigma)$, and then all these radii agree. This is in particular the case for $d = 2$, A stable, (A, B) controllable, and static linear state feedback \hat{u} . The following two examples illustrate further the relation between the three radii. The first

one shows that even if $r_{\mathbb{R}}(\sigma) = r_{L_f}(\sigma)$ for σ large, it is possible that $r_{\mathbb{R}}(\sigma) > r(\sigma) > r_{L_f}(\sigma)$ for σ small. The second example demonstrates that with output feedback the strict inequalities $r_{\mathbb{R}}(\sigma) > r(\sigma) > r_{L_f}(\sigma)$ for all $\sigma > 0$ can occur even in 2-dimensional systems.

4.9. Example. (linear oscillator with uncertain restoring force) Consider the linear oscillator $\ddot{y} + 2u\dot{y} + (1+v(t))y = 0$, or in the form of (2.1) with $x = (x_1, x_2)^T = (y, \dot{y})^T$

$$\dot{x} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -v(t) & 0 \end{pmatrix} \right] x + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \dot{u},$$

$$y = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x.$$

Using $F = \begin{pmatrix} a & b \\ c & u \end{pmatrix}$, the resulting closed loop system is

$$\dot{x} = \left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} + v(t) \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right] x + u \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} x. \quad (4.3)$$

We choose $v(t) \in V_\rho := [-\rho, \rho]$ as the uncertainty range, and $u \in U_\sigma := [0, \sigma]$ as the feedback gain range for $\rho, \sigma \geq 0$. Then we have (cp. Figure 1):

$$\begin{aligned} r_{\mathbb{R}}(\sigma) &> r(\sigma) > r_{L_f}(\sigma) \\ &\text{for } 0 < \sigma < \sigma_0, \text{ with } \sigma_0 \sim 0.405 \\ r_{\mathbb{R}}(\sigma) &= r(\sigma) > r_{L_f}(\sigma) \\ &\text{for } \sigma_0 \leq \sigma < 1/\sqrt{2} \\ r_{\mathbb{R}}(\sigma) &= r(\sigma) = r_{L_f}(\sigma) \\ &\text{for } 1/\sqrt{2} \leq \sigma. \end{aligned}$$

(Note that all three radii satisfy $r_0(u) = r_0(\sigma)$, if $u = \sigma$.) If one is interested in time varying uncertainties, the stabilization criterion based on quadratic Lyapunov functions is too conservative for $\sigma < 1/\sqrt{2}$, and the one based on constant uncertainties is too optimistic for $\sigma < 0.405$. In this example, the pair (A, B) is controllable and A can be shifted to $\begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}$, with $0 < \alpha < \sigma_0$, to produce a system that satisfies the assumption of Townley and Ryan [20], which shows that for large σ the radii can agree also for nontrivial output feedback, see [5] for more details on this example.

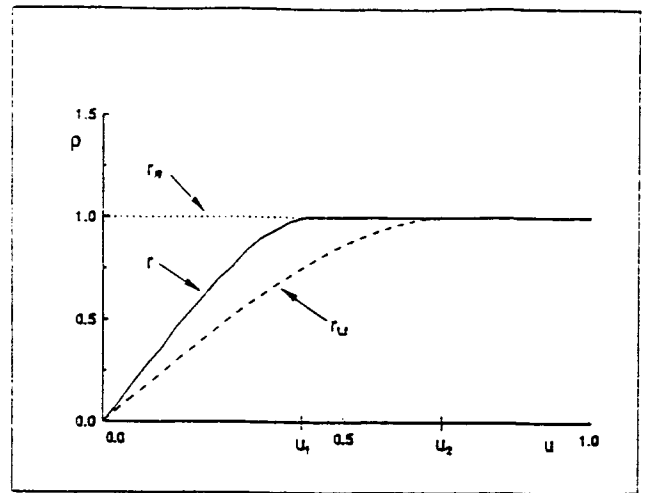


Figure 1: The stabilization radii for the linear oscillator (4.3)

4.10. Example. Consider the system

$$\dot{x} = \left[\begin{pmatrix} 1/4 & 1 \\ -1 & 1/4 \end{pmatrix} + v(t) \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right] x + u \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} x, \quad (4.4)$$

which again can be interpreted as an output feedback system as above. The stabilization radii are shown in Figure 2. We see that for $u = 1/4$ and $u = 17/8$ the radii depending on u are 0, and each of them has unique maximum, namely $r_{\mathbb{R}}(1/4) = 0.9375$, $r(0.59) = 0.77$, and $r_{L_f}(0.77) = 0.67$. Therefore we obtain that

$$r_{\mathbb{R}}(\sigma) > r(\sigma) > r_{L_f}(\sigma) \text{ for all } \sigma > 1/4.$$

Again, the pair (A, B) is controllable and a shift of A to $\begin{pmatrix} 1/4 & 1 \\ -1 & 1/4 + \alpha \end{pmatrix}$, with $0 < \alpha < 15/8$, produces a stable systems matrix. However, the stabilization radii depending on σ , and for $\sigma \rightarrow \infty$ are different. Note that in this example, as well as in Example 4.9 the values of the radii in a bounded u -interval determine the possibilities of stabilization, i.e. the stabilization via bounded output feedback for linear systems with time-varying uncertainties is not a high gain problem, but should be formulated realistically with bounds on the size of the uncertainty and on the size of the feedback gain. For more details see [7].

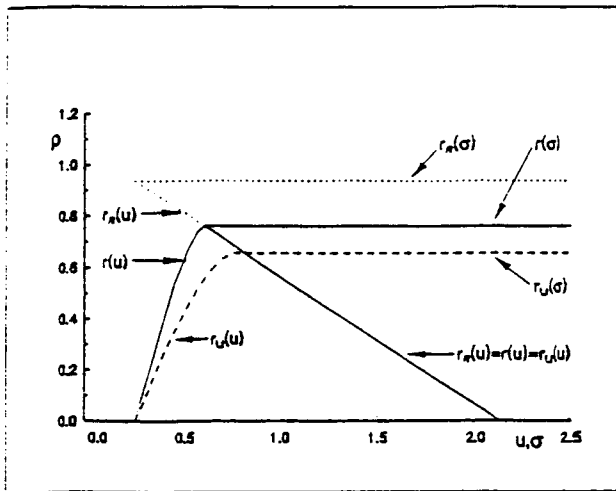


Figure 2: The stabilization radii for the system (4.4)

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