On the Löwner, Minus, and Star Partial Orderings of Nonnegative Definite Matrices and Their Squares

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ABSTRACT

We investigate how the ordering of two Hermitian nonnegative definite matrices $\mathbf{A}$ and $\mathbf{B}$ relates to the ordering of their squares $\mathbf{A}^2$ and $\mathbf{B}^2$, in the sense of the Löwner partial ordering, the minus partial ordering, and the star partial ordering. The condition that $\mathbf{A}$ and $\mathbf{B}$ commute appears essential in these investigations. We also give some comments on possible extensions of our results by replacing the squares $\mathbf{A}^2$ and $\mathbf{B}^2$ with the kth powers $\mathbf{A}^k$ and $\mathbf{B}^k$.

1. INTRODUCTION

For Hermitian nonnegative definite matrices $\mathbf{A}$ and $\mathbf{B}$, the Löwner partial ordering $\overset{\mathsf{L}}{\leqslant}$, the minus partial ordering $\overset{-}{\leqslant}$, and the star partial ordering $\overset{*}{\leqslant}$

are defined as follows:

$$\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{B} - \mathbf{A} = \mathbf{K}\mathbf{K}^* \quad \text{for some matrix } \mathbf{K}, \tag{1}$$

$$\mathbf{A} \overset{-}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^-\mathbf{A} = \mathbf{A}^-\mathbf{B} \quad \text{for some } \mathbf{A}^- \in \mathbf{A}\{1\}, \tag{2}$$

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^2 = \mathbf{A}\mathbf{B}, \tag{3}$$

where $\mathbf{K}^*$ in (1) is the conjugate transpose of $\mathbf{K}$, and $\mathbf{A}\{1\}$ in (2) denotes the set of generalized inverses of $\mathbf{A}$, i.e., $\mathbf{A}\{1\} = \{\mathbf{X} : \mathbf{AXA} = \mathbf{A}\}$. The ordering (1) dates back to Löwner (1934). The orderings (2) and (3) are restrictions to Hermitian matrices of the general definitions introduced by Hartwig (1980) and Drazin (1978), respectively. Hartwig (1980) showed in addition that the minus ordering is equivalent to *rank subtractivity*,

$$\mathbf{A} \overset{-}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathrm{r}(\mathbf{B} - \mathbf{A}) = \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{A}). \tag{4}$$

In view of Marsaglia and Styan (1974, p. 188) and Cline and Funderlic (1979, p. 195), an alternative form of (4) for Hermitian $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \overset{-}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B}) \text{ and } \mathbf{A}\mathbf{B}^+\mathbf{A} = \mathbf{A}, \tag{5}$$

where $\mathscr{R}(\cdot)$ stands for the range and $\mathbf{B}^+$ is the Moore-Penrose inverse of $\mathbf{B}$. It is known that for nonnegative definite matrices, the partial orderings (1), (2), and (3) follow the implications

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A} \overset{-}{\leqslant} \mathbf{B} \quad \Rightarrow \quad \mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}; \tag{6}$$

cd. Baksalary, Kala, and Kłaczyński (1983, p. 84) and Hartwig and Styan (1987, Theorem 2.1).

The purpose of this paper is to compare the relations $\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B}$, $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$, and $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ with the corresponding relations involving $\mathbf{A}^2$ and $\mathbf{B}^2$. Notice that on the cone of Hermitian nonnegative definite matrices, the relations $\overset{\mathrm{L}}{\preceq}$, $\overset{-}{\preceq}$, and $\overset{*}{\preceq}$, defined as

$$\mathbf{A} \overset{\mathsf{L}}{\leq} B \quad \Leftrightarrow \quad \mathbf{A}^2 \overset{\mathsf{L}}{\leqslant} \mathbf{B}^2,$$

$$\mathbf{A} \overset{-}{\leq} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^2 \overset{-}{\leqslant} \mathbf{B}^2,$$

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A}^2 \overset{*}{\leqslant} \mathbf{B}^2,$$

specify further partial orderings. This is due to the fact that if **A** and **B** are Hermitian nonnegative definite matrices, then the equality between $\mathbf{A}^2$ and $\mathbf{B}^2$ implies the equality between their unique square roots **A** and **B**. Stępniak (1987) studied the problem of how the ordering $\overset{\mathsf{L}}{\leq}$ behaves on *sets* of matrices; see also Horn (1988) for an example disproving Stępniak's conjecture. In contrast, we here concentrate on *pairs* of matrices **A** and **B**, without any further algebraic structure. Moreover, we consider the orderings $\overset{-}{\leq}$ and $\overset{*}{\leq}$ in addition to $\overset{\mathsf{L}}{\leq}$. In the final section, we give some comments on possible extensions of our results by replacing the squares $\mathbf{A}^2$ and $\mathbf{B}^2$ with the kth powers $\mathbf{A}^k$ and $\mathbf{B}^k$.

2. RESULTS

Let **A** and **B** be two Hermitian nonnegative definite matrices. For the Löwner ordering it is well known that

$$\mathbf{A}^2 \overset{\mathsf{L}}{\leqslant} \mathbf{B}^2 \quad \Rightarrow \quad \mathbf{A} \overset{\mathsf{L}}{\leqslant} \mathbf{B};$$

cf. Davis (1963, p. 199) and Marshall and Olkin (1979, p. 464). The converse implication fails to hold in general, as can be seen by taking

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}; \tag{7}$$

see also a more general counterexample in Marshall and Olkin (1979, p. 465). For the minus partial ordering, neither of the relations $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$ and $\mathbf{A}^2 \overset{-}{\leqslant} \mathbf{B}^2$ implies the other. The matrices in (7) form an example that $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$ does not imply $\mathbf{A}^2 \overset{-}{\leqslant} \mathbf{B}^2$, whereas

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \sqrt{15} & 0 \\ 0 & \sqrt{30} \end{pmatrix}$$

illustrate that $\mathbf{A}^2 \overset{-}{\leqslant} \mathbf{B}^2$ does not entail $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$. However, when **A** and **B** commute, then they more nearly behave like real (or complex) numbers, and relationships between orderings of matrices and orderings of their squares become feasible. For the three partial orderings (1), (2), and (3), the results are as follows.

THEOREM 1 (Löwner partial ordering). *For Hermitian nonnegative definite matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *consider the following*:

(a$_1$) $\mathbf{A} \overset{\mathsf{L}}{\leqslant} \mathbf{B}$,
(b$_1$) $\mathbf{A}^2 \overset{\mathsf{L}}{\leqslant} \mathbf{B}^2$,
(c) $\mathbf{AB} = \mathbf{BA}$.

Then (a$_1$), (c) $\Rightarrow$ (b$_1$), *and* (b$_1$) $\Rightarrow$ (a$_1$).

THEOREM 2 (Minus partial ordering). *For Hermitian nonnegative definite matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *consider the following*:

(a$_2$) $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$,
(b$_2$) $\mathbf{A}^2 \overset{-}{\leqslant} \mathbf{B}^2$,
(c) $\mathbf{AB} = \mathbf{BA}$.

Then any two of these statements imply the third statement.

THEOREM 3 (Star partial ordering). *For Hermitian nonnegative definite matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *consider the following*:

(a$_3$) $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$,
(b$_3$) $\mathbf{A}^2 \overset{*}{\leqslant} \mathbf{B}^2$,
(c) $\mathbf{AB} = \mathbf{BA}$.

Then (a$_3$) $\Leftrightarrow$ (b$_3$) $\Rightarrow$ (c).

We may summarize (6) and the results above by the following schematic diagram in which $\Rightarrow$ denotes the usual implication and $\rightarrow$ denotes the implication which is valid under the commutativity condition $\mathbf{AB} = \mathbf{BA}$:

$$\begin{array}{ccccc}
\mathbf{A} \overset{*}{\leqslant} \mathbf{B} & \Rightarrow & \mathbf{A} \overset{-}{\leqslant} \mathbf{B} & \Rightarrow & \mathbf{A} \overset{\mathsf{L}}{\leqslant} \mathbf{B} \\
\Uparrow \Downarrow & & \downarrow\uparrow & & \Uparrow \downarrow \\
\mathbf{A}^2 \overset{*}{\leqslant} \mathbf{B}^2 & \Rightarrow & \mathbf{A}^2 \overset{-}{\leqslant} \mathbf{B}^2 & \Rightarrow & \mathbf{A}^2 \overset{\mathsf{L}}{\leqslant} \mathbf{B}^2
\end{array}$$

3. PROOFS

In the course of the proofs we will use the following lemma on commutativity of functions defined on the linear space $\mathbb{H}_n$ of all $n \times n$ Hermitian matrices. If $\mathbf{A} \in \mathbb{H}_n$ has the spectral decomposition $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^*$, where $\mathbf{U}$ is a unitary matrix and $\boldsymbol{\Lambda}$ is the diagonal matrix with the diagonal elements $\lambda_1, \cdots, \lambda_n$ equal to the eigenvalues of $\mathbf{A}$, then a function $f : \mathbb{R} \to \mathbb{R}$ gives rise to the function $f_n : \mathbb{H}_n \to \mathbb{H}_n$ by way of $f_n(\mathbf{A}) = \mathbf{U} f_n(\boldsymbol{\Lambda})\mathbf{U}^*$, where $f_n(\boldsymbol{\Lambda})$ is the diagonal matrix with the diagonal elements equal to $f(\lambda_1), \cdots, f(\lambda_n)$.

LEMMA. *Let $\mathbf{A} \in \mathbb{H}_n$ and $\mathbf{B} \in \mathbb{H}_n$, let $\lambda_1, \cdots, \lambda_a$ and $\mu_1, \cdots, \mu_b$ be the distinct eigenvalues of* $\mathbf{A}$ *and* $\mathbf{B}$, *and let the functions $f : \{\lambda_1, \cdots, \lambda_a\} \to \mathbb{R}$ and $g : \{\mu_1, \cdots, \mu_b\} \to \mathbb{R}$ be one-to-one. Then $\mathbf{AB} = \mathbf{BA}$ if and only if $f_n(\mathbf{A}) g_n(\mathbf{B}) = g_n(\mathbf{B}) f_n(\mathbf{A})$.*

The proof of this lemma follows by using the fact that the commutativity is a necessary and sufficient condition for two Hermitian matrices to admit spectral decompositions with the same unitary matrix.

Proof of Theorem 1. For three alternative proofs that (b_1) implies (a_1) see Davis (1983, p. 199) and Marshall and Olkin (1979, p. 464).

If $\mathbf{A}$ and $\mathbf{B}$ commute, then

$$\mathbf{A} = \mathbf{U}\mathbf{D}_{\mathbf{A}}\mathbf{U}^* \quad \text{and} \quad \mathbf{B} = \mathbf{U}\mathbf{D}_{\mathbf{B}}\mathbf{U}^* \tag{8}$$

for some unitary matrix $\mathbf{U}$ and nonnegative diagonal matrices $\mathbf{D}_{\mathbf{A}}$ and $\mathbf{D}_{\mathbf{B}}$. Consequently,

$$\mathbf{A} \overset{\mathrm{L}}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{D}_{\mathbf{A}} \overset{\mathrm{L}}{\leqslant} \mathbf{D}_{\mathbf{B}} \quad \Leftrightarrow \quad \mathbf{D}_{\mathbf{A}}^2 \overset{\mathrm{L}}{\leqslant} \mathbf{D}_{\mathbf{B}}^2 \quad \Leftrightarrow \quad \mathbf{A}^2 \overset{\mathrm{L}}{\leqslant} \mathbf{B}^2. \qquad \blacksquare$$

Proof of Theorem 2. For the proof that (a_2) and (b_2) imply (c) first notice that, in view of (5), condition (b_2) is equivalent to

$$\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B}) \quad \text{and} \quad \mathbf{A}^2(\mathbf{B}^2)^+\mathbf{A}^2 = \mathbf{A}^2. \tag{9}$$

The equality $\mathbf{AB}^+\mathbf{A} = \mathbf{A}$ in (5) entails

$$(\mathbf{A}^2\mathbf{B}^+ - \mathbf{A})(\mathbf{A}^2\mathbf{B}^+ - \mathbf{A})^* = \mathbf{A}^2(\mathbf{B}^+)^2\mathbf{A}^2 - \mathbf{A}^2,$$

and then the equality in (9) yields

$$\mathbf{A}^2\mathbf{B}^+ = \mathbf{A} = \mathbf{B}^+\mathbf{A}^2, \tag{10}$$

since $(\mathbf{B}^+)^2 = (\mathbf{B}^2)^+$. Applying the Lemma with f and g defined as $f(x) = x^2$ and $g(x) = 1/x$ when $x \neq 0$ and $g(0) = 0$ to (10) leads to the commutativity of $\mathbf{A}$ and $\mathbf{B}$.

The proof of the remaining two parts follows straightforwardly from the fact that, in view of (5) and (8),

$$\mathbf{A} \overset{-}{\leqslant} \mathbf{B} \quad \Leftrightarrow \quad \mathscr{R}(\mathbf{D}_\mathbf{A}) \subseteq \mathscr{R}(\mathbf{D}_\mathbf{B}) \text{ and } \mathbf{D}_\mathbf{A}\mathbf{D}_\mathbf{B}^{+}\mathbf{D}_\mathbf{A} = \mathbf{D}_\mathbf{A},$$

and

$$\mathbf{A}^2 \overset{-}{\leqslant} \mathbf{B}^2 \quad \Leftrightarrow \quad \mathscr{R}(\mathbf{D}_\mathbf{A}) \subseteq \mathscr{R}(\mathbf{D}_\mathbf{B}) \text{ and } \mathbf{D}_\mathbf{A}^2\left(\mathbf{D}_\mathbf{B}^2\right)^{+}\mathbf{D}_\mathbf{A}^2 = \mathbf{D}_\mathbf{A}^2.$$

This shows that if (c) holds, then conditions (a_2) and (b_2) are equivalent: each of them is satisfied if and only if every diagonal element of $\mathbf{D}_\mathbf{A}$ is equal either to zero or to the corresponding element of $\mathbf{D}_\mathbf{B}$. ■

Proof of Theorem 3. The statement "(a_3) $\Rightarrow$ (c)" is a direct consequence of the definition (3), and the statement "(a_3) $\Leftrightarrow$ (b_3)" follows from the part "(a_2), (c) $\Leftrightarrow$ (b_2), (c)" of Theorem 2, noting that (a_3) $\Leftrightarrow$ (a_2), (c) [cf. Hartwig and Styan (1986, Theorem 2)] and, similarly, (b_3) $\Leftrightarrow$ (b_2), (c). ■

4. COMMENTS ON POSSIBLE EXTENSIONS

Professor Ingram Olkin asked about possible extensions of Theorems 1, 2, and 3 by replacing the squares of $\mathbf{A}$ and $\mathbf{B}$ with the values $\varphi(\mathbf{A})$ and $\varphi(\mathbf{B})$ of some more general function $\varphi : \mathbb{H}_n \to \mathbb{H}_n$. We comment here on this question in the particular situation when φ is the kth power function, that is, when conditions (b_1), (b_2), and (b_3) are replaced by ($b_1^{\#}$) $\mathbf{A}^k \overset{L}{\leqslant} \mathbf{B}^k$, ($b_2^{\#}$) $\mathbf{A}^k \overset{-}{\leqslant} \mathbf{B}^k$, and ($b_3^{\#}$) $\mathbf{A}^k \overset{*}{\leqslant} \mathbf{B}^k$, respectively.

Our first observation is that the modified version of Theorem 1 remains true for any $k > 1$. The part "($b_1^{\#}$) $\Rightarrow$ (a_1)" is due to Loŵner (1934) [cf. Marshall and Olkin (1979, p. 464)], and the part "(a_1), (c) $\Rightarrow$ ($b_1^{\#}$)" can be established by similar arguments to those in the proof of Theorem 1; see also a related result of Man (1970, Corollary 2).

The second observation is that, using the same argument of simultaneous diagonalization of $\mathbf{A}$ and $\mathbf{B}$, the parts "(a_2), (c) $\Rightarrow$ (b_2)" and "(b_2), (c) $\Rightarrow$ (a_2)" can be extended to "(a_2), (c) $\Rightarrow$ ($b_2^{\#}$)" and "($b_2^{\#}$), (c) $\Rightarrow$ (a_2)." We were unable, however, to prove or disprove that $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$ and $\mathbf{A}^k \overset{-}{\leqslant} \mathbf{B}^k$ together imply the commutativity of $\mathbf{A}$ and $\mathbf{B}$ when $k \neq 2$.

Our final observation is again positive, viz. that the statement in Theorem 3 can be generalized to the form (a_3) $\Leftrightarrow$ (b_3) $\Rightarrow$ (c).

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