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COHOMOLOGY OF BIQUOTIENTS

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Biquotients are non-homogeneous quotient spaces of Lie groups. Using the Serre spectral sequence and the method of Borel, we compute the cohomology algebra of these spaces in cases where the Lie group cohomology is not too complicated. Among these are the biquotients which are known to carry a metric of positive curvature.

Introduction

Biquotients are natural generalisations of homogeneous spaces. They are the source of many interesting examples in Riemannian geometry, among them compact spaces of positive curvature (cf.[E1, E2, E3]). Extending the method of A.Borel [B], we show how to compute the cohomology ring of a certain class of compact biquotients. In particular, we discuss a number of examples, e.g. the non-homogeneous analogues of the Wallach sapces. We are indebted to M.Kreck for many hints and discussion.

1. Biquotients

A *biquotient* is the base space of a homogeneous principal bundle. More precisely, let P be a compact homogeneous manifold and L a compact Lie group acting transitively on P. Let $U \subset L$ be a closed subgroup which acts freely, i.e. if $u \in U$ has a fixed point ($u \cdot p = p$ for some $p \in P$), then u is the unit element $1 \in U$. Consequently, the orbit space $M = P/U$ is a manifold and $P \to M$ a principal bundle with structure group U.

Consider the particular case where $P = G$ is a compact Lie group and U a closed freely acting subgroup of $L := G^2 = G \times G$ (which acts on $P = G$ by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$). This example is universal: We can always represent P as left coset space $P = L/K$ and get $M = U \backslash L/K$. Putting $\hat{G} = L$ and $\hat{U} = U \times K \subset (\hat{G})^2$, we obtain $M = \hat{G}/\hat{U}$ as in the special case.

In the following, we consider always this case $P = G$, $L = G^2$. The group $U \subset G^2$ acts freely if the components u_1, u_2 are not conjugate in G for any $(u_1, u_2) \in U \backslash \{1\}$. The simplest examples are of course the homogeneous spaces where $U \subset \{1\} \times G$. (In this case, we may also choose $L = G$ acting on $P = G$ by right translations.) We give some more interesting examples:

EXAMPLES

1. Let $G = Sp(2)$ and

$$U = \{(A(q), B(q));\ q \in S^3 \subset H\}$$

with

$$A(q) = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \quad B(q) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}.$$

U acts freely since $B(q)$ has a fixed vector while $A(q)$ has not; so they are non-conjugate. This example has been discovered by Gromoll and Meyer [GM]; they showed that $M = G/U$ is an exotic 7-sphere.

2. Let $G = U(3)$ and $U = U_1 \times U_2$ where

$$U_1 = \{D(a,a,\bar{a}) \; ; \; a \in S^1\},$$

$$U_2 = \{D(b,c,1) \; ; \; b,c \in S^1\},$$

where $D(a_1,\ldots,a_k)$ denotes the diagonal matrix with entries $a_1,\ldots,a_k$. This group acts freely since complex diagonal matrices are conjugate only if their entries agree up to permutation.

3. Let $G = U(3)$ and

$$U' = \{(D(a^{x_1},a^{x_2},a^{x_3}),(D(a^{y_1},a^{y_2},a^{y_3})); \; a \in S^1\}$$

where x_i, y_j are integers without a common divisor. If the determinant of the components is the same, i.e. if $\Sigma\, x_i = \Sigma\, y_i$, then $SU(3) \subset U(3)$ is invariant under U'. The action is free iff $x_i - y_{\tau(i)}$ have no common divisor for $i = 1,2,3$, for any permutation $\tau \in S_3$. This generalizes a construction in [E1]. However, $U' \not\subset SU(3)^2$. But note that the mapping $SU(3) \subset U(3) \to U(3)/U(1)$ is a diffeomorphism, where $U(1) = \{D(b,1,1); \; b \in S^1\}$, and the action of $U(1)$ and U commute. So $SU(3)/U' = U(3)/U$ where

$$U = U' \cdot (\{1\} \times U(1)).$$

4. Let $G = Sp(3)$ and $U = \{(A(q),B(r,s));\ q,r,s \in S^3\}$ with

$$A(q) = D(q,q,q)\ ,\quad B(r,s) = D(r,s,1)\ .$$

Since $B(r,s)$ does have a fixed vector while $A(q)$ for $q \neq 1$ does not, these matrices are non-conjugate and U acts freely.

All these manifolds $M = G/U$ are compact, simply connected and carry a metric of nonnegative curvature; in 1. and 4. the curvature is positive on an open dense subset (cf. [GM], [E2]), while 2. and most of the examples in 3. carry a metric of positive curvature [E1,E2,E3].

2. How to compute the cohomology

Consider P,L,U,M as above. Let E be an acyclic space on which L acts freely. So $B_L = E/L$ and $B_U = E/U$ are the classifying spaces of L and U. Following Borel [B], to compute the cohomology of M we use the the Serre spectral sequence (e.g. cf. [H], p.37) of the bundle

$$M' = (P\times E)/U \ \rightarrow\ E/U = B_U$$

with fibre P ; note that M' and M are homotopic. Let $K \subset L$ be the stabilizer subgroup of some point $p \in P$. Then we have the bundle maps

$$\begin{array}{ccccc} M' = (P\times E)/U & \longrightarrow & (P\times E)/L & \stackrel{\cong}{\longleftarrow} & B_K = E/K \\ \downarrow & & \downarrow & & \downarrow \\ B_U & \longrightarrow & B_L & = & B_L \end{array}$$

where the isomorphism to the right is given by the mapping

$$\psi : B_K = E/K \to (P\times E)/L\ , \quad \psi(K\cdot e) = L\cdot(p,e)$$

which is well defined, keeps the points on the base space fixed, and has inverse

$$\psi^{-1}(L\cdot(q,e)) = K\cdot g^{-1}\cdot e$$

for $q = g\cdot p \in P$, $g \in L$. The bundle $B_K \to B_L$ will be called the *reference bundle*. In fact, we will compute first the spectral sequence of this bundle and then use naturality. This has been done by Borel in the case of homogeneous spaces where $P = G$ is a compact Lie group and $L = G$ acts by right translations. We are interested in the case $P = G$, $L = G^2$. The stabilizer of the unit element $p = 1 \in G$ is the subgroup $\{(g,g);\ g \in G\} \subset G^2$ (which we identify with G .) Thus it is sufficient to study the reference bundle $B_G \to B_{G^2}$ which is pulled back to the bundle $M' \to B_U$ by the map $B_\phi : B_U \to B_{G^2}$ induced by the inclusion $\phi : U \to G^2$. We have to compute the cohomology of this map B_ϕ .

From now on, we make the following assumption on the cohomology algebra of our group G (the coefficient ring R is $\mathbb{Z}$ or a field):

(A) $H^*(G)$ is a free exterior algebra $\Lambda[a_1,\dots,a_m]$ with generators $a_1,\dots,a_m$ of degree $\deg a_j = r_j-1$, r_j even, with $2 \le r_1 \le r_2 \le \dots \le r_m$.

Then (cf. [B], Th. 19.1)

$$H^*(B_G) = R[\bar a_1,\dots \bar a_m]$$

with $\deg \bar a_j = r_j$, and the generators a_j of $H^*(G)$ can be chosen so that $\bar{a}_j$ corresponds to a_j under transgression τ in the spectral sequence of the universal bundle $E_G \to B_G$. In particular, if $T \subset G$ is the maximal torus and $t_1,\ldots,t_k$ a basis of $H^1(T)$, then

$$H^*(B_T) = R[\bar{t}_1,\ldots,\bar{t}_k]$$

where $\deg(\bar{t}_j) = 2$ and $\bar{t}_j = \tau(t_j)$. The inclusion $T \subset G$ makes $H^*(B_G)$ into the subalgebra of $H^*(B_T)$ which contains precisely the Weyl group invariant polynomials over $\bar{t}_1,\ldots,\bar{t}_k$ (cf. [B], Prop. 27.1). In other words, the generators $\bar{a}_j$ of $H^*(B_G)$ can be identified with certain Weyl-invariant polynomials over $\bar{t}_1,\ldots,\bar{t}_k$. E.g., $H^*(B_{U(n)})$ is generated by the elementary symmetric polynomials in the variables $\bar{t}_j$,

$$\alpha_1 = \sum_j \bar{t}_j \ , \quad \alpha_2 = \sum_{j<k} \bar{t}_j\bar{t}_k \ , \ \ldots \ , \ \alpha_n = \bar{t}_1\bar{t}_2\ldots\bar{t}_n \ ,$$

while $H^*(B_{Sp(n)})$ is generated by the elementary symmetric polynomials in $\bar{t}_j^2$.

The second desciption allows us to compute the cohomology of the map $B_\rho : B_G \to B_{G'}$ for any Lie group embedding $\rho : G \to G'$. In fact, choose maximal tori T of G and T' of G' so that $\rho(T) \subset T'$. Call $\rho_T : T \to T'$ the restriction. Let $\underline{t}$ and $\underline{t}'$ denote the corresponding Lie algebras and $\rho_{T*} : \underline{t} \to \underline{t}'$ the differential of ρ_T . Then $\rho_T^* : H^1(T') \to H^1(T)$ is given by the adjoint map of ρ_{T*} . (We may identify $H_1(T;\mathbb{Z})$ with the unit lattice of T in $\underline{t}$, and similar for T' .) Then $B_\rho^* : H^*(B_{G'}) \to H^*(B_G)$ is the extension of ρ_T^* to the Weyl-invariant polynomials over $H^1(T')$ and $H^1(T)$. The following lemma is a simple application:

LEMMA. *Let* G *be any compact Lie group satisfying (A) with* $H^*(B_G) = R[\alpha_1,\dots,\alpha_m]$. *Let* $\Delta : G \to G^2$, $g \to (g,g)$ *be the diagonal embedding. Then* $H^*(B_{G^2}) = H^*(B_G) \otimes H^*(B_G)$, *and* $B_\Delta{}^*(\alpha_j \otimes 1) = B_\Delta{}^*(1 \otimes \alpha_j) = \alpha_j$. *In particular,* $\ker(B_\Delta{}^*)$ *is the ideal generated by* $\delta_1,\dots,\delta_m$ *where*

$$\delta_j = \alpha_j \otimes 1 - 1 \otimes \alpha_j \ .$$

PROOF. If E is an acyclic G-space, then $E^2 = E \times E$ is an acyclic G^2-space. Thus $B_{G^2} = B_G \times B_G$ is a classifying space for G^2 and $H^*(B_{G^2}) = H^*(B_G) \otimes H^*(B_G)$. We have

$$\Delta_{T*} = \binom{I}{I} : \underline{t} \to \underline{t} \oplus \underline{t}$$

where I denotes the identity matrix, and the same holds for Δ_{T*} on $H_1(T)$. Thus

$$\rho_T{}^* = (I,I) : H^1(T^2) = H^1(T) \oplus H^1(T) \to H^1(T) \ .$$

Since α_j can be considered as a polynomial over $H^1(T)$, this proves the lemma.

3. The reference bundle

In this section, we compute the spectral sequence E_r of the bundle $B_\Delta : B_G \to B_{G^2}$ where G satisfies (A). Recall that the Serre spectral sequence of a bundle $\pi : E \to B$ with fibre F starts with $E_2{}^{**} = H^*(B;H^*(F))$, and we have a projection $k_r : H^*(B) \to E_r{}^{*0}$ with $k_\infty = \pi^*$. Let δ_j be as in the lemma above.

PROPOSITION *For* $r_{j-1} < r \le r_j$, $j \ge 1$ *(with* $r_0 := 0$*) we have*

(1) $$E_r{}^{*0} = H^*(B_{G^2})/I_{j-1}$$

where $I_k \subset H^*(B_{G^{\cong}})$ *is the ideal generated by* $\delta_1,\ldots,\delta_k$ *, and*

$$(2) \qquad d_r(1\otimes a_j) = \begin{cases} 0 & \textit{for all} \quad r < r_j \\ k_r(\delta_j) & \textit{for} \quad r = r_j \end{cases}$$

PROOF. We prove (1) for all $r_j \le t$ and (2) for all $r_j \le t-1$ by induction over t . For $t = 2$ the statement is true. Suppose that it holds for some $t \ge 2$. First, we show (2) for all j with $r_j = t$. Let

$$A = \text{Span}\,\{a_j\ ;\ r_j = t\} \subset H^{t-1}(G)\ ,$$

$$D = \text{Span}\,\{\delta_j\ ;\ r_j = t\} \subset H^t(B_{G^{\cong}})\ .$$

We have $D \subset \ker\,(B_\Delta{}^*\colon\ H^*(B_G) \to H^*(B_{G^{\cong}}))$. Hence, for any $\delta \in D$ there exist some $r \ge t$ such that $0 \ne k_r(\delta) \in (\text{im}\ d_r)^{t,0}$ (note that $k_t(\delta) \ne 0$ by induction hypothesis). But $(\text{im}\ d_r)^{t,0} = 0$ for $r > t$ ("the arrow of d_r is too long"), thus

$$k_t(D) \subset (\text{im}\ d_t)^{t,0} = d_t(1\otimes A)\ .$$

Since A and $k_t(D)$ (by induction hypothesis) are free modules with the same dimensions, d_t is an isomorphism between these modules, and we get (2) for $r = t$ up to a change of the basis of A or D .

Now suppose that $r_{k-1} < t \le r_k$. Since $k_t(D) \subset \text{im}\ d_t$, it follows that $I_k \subset \ker k_{t+1}$. We have to show equality. Thus assume that $k_{t+1}(\gamma) = 0$ for some $\gamma \in H^*(B_{G^{\cong}})\backslash I_k$. Then

$$\gamma = d_t([u]_t)$$

where $u = \Sigma\, c_\mu\otimes\gamma_\mu$ with $c_\mu \in H^{t-1}(G)$, and $[\ ]_t$ denotes the projection onto E_t . Let $C \subset H^{t-1}(G)$ be the submodule generated by products with factors a_j of lower degree

$r_j < t$. Since the A-component of c_μ gives rise to a term of u which lies in I_k , we may assume (mod I_k) that $c_\mu \in C$. Now we show that no such element $u \in C \otimes H^*(B_{G^2})$ can survive up to E_t . In fact, let r be the lowest degree of any factor in any of the c_μ . Then we have a decomposition

$$u = \sum_{i \in J(r)} a_i u_i + u'$$

where $J(r) = \{i ; r_i = r\}$, such that u_i and u' contain only factors a_j with $r_j > r$. (We have written a_i for $1 \otimes a_i$.) By induction hypothesis, the elements a_i, u_i and u' survive up to E_r , and

$$d_r(u) = \sum_{i \in J(r)} \delta_j u_j \neq 0 .$$

Thus u does not survive in E_{r+1} , and we have finished the proof.

4. The spectral sequence of a biquotient

THEOREM 1. *Let* G *be a compact Lie group with (A),* $\phi : U \to G^2$ *a Lie group embedding such that* U *acts freely on* G *, and* E *an acyclic space with a free* G^2*-action. Let* E_r *denote the spectral sequence of the bundle*

$$M' = (G \times E)/U \to E/U = B_U .$$

Let a_j , $j = 1,\dots,m$ *be the generators of* H^*G *. Then all* $1 \otimes a_j \in E_2^{0*}$ *are transgressive, and*

$$d_{r_j}(1 \otimes a_j) = k_{r_j}(B_\phi^* \delta_j)$$

where δ_j *is as above.*

REMARK. Since $B_\phi^* : H^*(B_{G^2}) \to H^*(B_U)$ can be computed from the adjoint of the map $\phi_{T*} = \phi_*|\underline{t}' : \underline{t}' \to \underline{t}\oplus\underline{t}$, where $\underline{t}'$ and $\underline{t}$ denote the Lie algebras of the maximal tori $T' \subset U$ and $T \subset G$, Theorem 1 allows us to compute $H^*(G/U)$ if we know ϕ_{T*}. We have an important special case:

THEOREM 2. *Suppose that* $B_\phi^*\delta_1,\ldots B_\phi^*\delta_m$ *generate a direct factor of the free module* $H^*(B_U)$ *and that* $B_\phi^*\delta_j$ *does not lie in the ideal* $(B_\phi^*\delta_1,\ldots,B_\phi^*\delta_{j-1})$ *for* $j = 2,\ldots,m$. *Then the cohomology algebra of* $M = G/U$ *is*

$$H^*(G/U) = H^*(B_U)/(B_\phi^*\delta_1,\ldots,B_\phi^*\delta_m)$$

PROOF. The proof of Theorem 1 is clear by the previous proposition and the naturality of spectral sequences. To prove Theorem 2, we observe first by induction over r that all E_r in the spectral sequence of $M' \to B_U$ are free R-modules since $d_{r_j}(1\otimes a_j) = k_{r_j}(B_\phi^*\delta_j)$ is indivisible. Moreover, no products with $1\otimes a_j$ can survive in E_∞ which means $E_\infty^{**} = E_\infty^{*0} = H^*(B_U)/(\phi^*\delta_1,\ldots,\phi^*\delta_m)$. Since E_∞^{*0} is a subalgebra of $H^*(M')$ and the spectral sequence (E_r) converges to $H^*(M')$, the theorem is proved.

Now we compute the integral cohomology $(R = \mathbb{Z})$ of the examples given in Ch.1:

1. $G = Sp(2)$,

   $U = \{(D(q,q),D(q,1));\ q \in S^3\}$.

Then $\phi_T^* = (1,1,1,0)$. Let v,w denote the basis of $H^1(T)$ and t a generator of $H^1(T')$ (where T, T' are maximal tori in G and U). Thus ϕ_T^* maps $v\otimes 1$, $w\otimes 1$, $1\otimes v$ onto t and $1\otimes w$ onto 0. We have $H^*(Sp(2)) = \Lambda[a_1,a_2]$ with $r_1 = 4$, $r_2 = 8$, and $H^*(B_{Sp(2)}) = \mathbb{Z}[\alpha_1,\alpha_2]$ where

$$\alpha_1 = v^2 + w^2 \in H^4 , \quad \alpha_2 = v^2\cdot w^2 \in H^8$$

(we omit the distinction between v,w and $\bar{v},\bar{w}$), while $H^*(B_U) = \mathbb{Z}[t^2]$ with $t^2 \in H^4$. Since $\delta_j = \alpha_j\otimes 1 - 1\otimes\alpha_j$, we get $B_\phi^*(\delta_1) = t^2$, $B_\phi^*(\delta_2) = t^4$. Thus $d_4(1\otimes a_1) = k_4(t^2)$ which implies $k_j(t^2) = 0$ for $j \geq 5$, and $d_8(1\otimes a_2) = k_8(t^4) = 0$. So $1\otimes a_2$ survives in E_∞ and generates $H^7(G/U) = \mathbb{Z}$, while any other $H^p(G/U)$ for $p \neq 0$ and $p \neq 7$ vanishes. This shows $H^*(G/U) = H^*(S^7)$ which of course follows also from the result of [GM].

<u>2.</u> $G = U(3)$,

$U = \{(D(a,a,\bar{a}),D(b,c,1));\ a,b,c \in S^1\}$.

We have

$$\phi_T^* = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

Let u_1,u_2,u_3 and v_1,v_2,v_3,w_1,w_2,w_3 be bases of $H^1(U)$ and $H^1(T^2)$. Then $H^*(B_{G^2})$ is generated by

$$\sum_j v_j \ , \ \sum_j w_j \ , \ \sum_{j<k} v_jv_k \ , \ \sum_{j<k} w_jw_k \ , \ v_1v_2v_3 \ , \ w_1w_2w_3$$

which is mapped by B_ϕ^* onto the following elements of $H^*(B_U)$:

$$u_1 \ , \ u_2+u_3 \ , \ -u_1^2 \ , \ u_2u_3 \ , \ -u_1^3 \ , \ 0 \ .$$

Thus

$$B_\phi^*(\delta_1) = u_1 - u_2 - u_3 ,$$
$$B_\phi^*(\delta_2) = -u_1^2 - u_2u_3 ,$$
$$B_\phi^*(\delta_3) = -u_1^3 .$$

All these elements are indivisible, so they span a direct factor (i.e. the quotient module has no torsion). We put

$$x = u_1 , \quad y = u_2 ,$$

and the relation $B_\phi^*(\delta_1) = 0$ gives

$$u_3 = x - y .$$

Then from $B_\phi^*(\delta_2) = 0$ we get the new relation

$$xy = y^2 - x^2 .$$

Thus $B_\phi^*(\delta_2)$ does not lie in the ideal generated by $B_\phi^*(\delta_1)$. Likewise, $B_\phi^*(\delta_3) = 0$ gives the new relation $x^3 = 0$, so $B_\phi^*(\delta_3)$ does not lie in the ideal generated by $B_\phi^*(\delta_2)$ and $B_\phi^*(\delta_3)$. Hence Theorem 2 is applicable:

$$H^*(G/U) = \mathbb{Z}[x,y]/(xy-y^2+x^2, x^3) .$$

In other words, the cohomology is without torsion and generated by x, y, x^2, y^2 and x^2y with the relations

$$xy = y^2 - x^2 , \quad x^3 = 0 , \quad xy^2 = x^2y , \quad y^3 = 2xy^2 .$$

There is a homogeneous space which is very similar to G/U , namely the flag manifold G/T where $T = U(1)^3$ is the maximal torus in $G = U(3)$. It is well known (cf. [B], Prop. 29.2) that

$$H^*(G/T) = \mathbb{Z}[x,y,z]/I$$

where I is the ideal generated by the elementary symmetric polynomials $x+y+z$, $xy + yz + zx$, xyz . Replacing z with $-(x+y)$, we get the same cohomology module, but the relations are

$$xy = -y^2 - x^2 , \quad x^3 = 0 , \quad x^2y = -xy^2 , \quad y^3 = 0 .$$

Since the two quadratic forms $xy - y^2 + x^2$ and $xy + y^2 + x^2$ are inequivalent over $\mathbb{R}$ (the first is indefinite, the second positive definite), the real cohomology algebras of G/U and G/T are not isomorphic (cf. [E3] for a different proof) while these spaces have the same integral homology and the same cohomology algebra mod 2.

3. $G = U(3)$,

$$U = \{(D(a^{x_1},a^{x_2},a^{x_3}),D(a^{y_1}b,a^{y_2},a^{y_3}));\quad a,b \in S^1\}$$
$$=: U_{x,y}$$

where x_i, y_j are integers such that $x_1+x_2+x_3 = y_1+y_2+y_3$ and $x_1-y_{\tau(1)}$, $x_2-y_{\tau(2)}$, $x_3-y_{\tau(3)}$ are relatively prime for any permutation $\tau \in S_3$. Then

$$\phi_T{}^* = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

If we let s,t be the generators of $H^1(U) = H^2(B_U)$, then

$$B_\phi{}^*(\delta_1) = -s\ ,\ B_\phi{}^*(\delta_2) = m\cdot t^2\ ,\ B_\phi{}^*(\delta_3) = n\cdot t^3\ ,$$

where $m = \sigma_2(x) - \sigma_2(y)$ and $n = \sigma_3(x) - \sigma_3(y)$ and σ_j denote the elementary symmetric polynomials. By assumption for x_i and y_j , the numbers m and n are relatively prime. By Theorem 1, $E_3 = E_4 = \mathbb{Z}[t] \otimes \Lambda[a_2,a_3]$. Moreover, $1\otimes a_2$ does not survive in $E_5 = E_6$, and $E_5{}^{4 0} = \mathbb{Z}t^2/m\mathbb{Z}t^2$. Thus $(\ker d_6)^{05} = \mathbb{Z}m(1\otimes a_3)$ since $(m,n) = 1$, and $m(1\otimes a_3)$ survives in E_7 . So the surviving elements in E_∞ are $t = t\otimes 1$, t^2 (with $mt^2 = 0$), $\bar{u} := m(1\otimes a_3)$ and $t\bar{u}$. Let $(F_p)_{p\geq 0}$ be the filtration of $H^*(G/U)$ which corresponds to the Serre spectral sequence (cf. [H], p.37). We have $E_\infty{}^{pq} = (F_p/F_{p+1})^{p+q}$. Thus $F_1 = F_2$, $F_3 = F_4$, $F_5 = 0$. Moreover, $t\bar{u} \in E_\infty{}^{25} = F_2{}^7 \subset H^7(G/U)$ since in $F_3 = F_4 = E_\infty{}^{4*}$ there are no elements of total degree 7. Let $u \in H^5(G/U)$ be a preimage of $\bar{u} \in (F_0/F_2)^5$, i.e. $\bar{u} = u + F_2{}^5$. Then $tu = t\bar{u}$ since $tu - t\bar{u} \in t\cdot F_2{}^5 \subset F_4{}^7 = 0$. So the cohomology

$H^*(G/U)$ is generated by $t \in H^2$, $t^2 \in H^4$ with torsion $m \cdot t^4 = 0$, $u \in H^5$ and $tu \in H^7$. The number

$$m = x_1x_2 + x_2x_3 + x_3x_1 - (y_1y_2 + y_2y_3 + y_3y_1)$$
$$= \tfrac{1}{2}(\|y\|^2 - \|x\|^2)$$

(remember $(\Sigma x_i)^2 = (\Sigma y_i)^2$) is the only cohomology invariant which distinguishes the cohomologies of $G/U_{x,y}$ for different pairs of integral vectors (x,y).

In the special case $x = 0$ we recover the the homogeneous spaces which have been considered by Aloff and Wallach [AW]. In this case we have $y_3 = -(y_1+y_2)$, and the corresponding number $m = y_1^2 + y_2^2 + y_1y_2$ satisfies certain restrictions, e.g. m is *not* congruent to 0 mod 2, 2 mod 3, 0 mod 5, 4 mod 5, 2 mod 7, 4 mod 7 etc. So one easily finds x,y so that $G/U_{x,y}$ are not Wallach spaces (cf. [E1]). It would be interesting to know whether the classification of the Wallach spaces up to diffeomorphisms by Kreck and Stolz [KS] extends to these inhomogeneous spaces as well.

<u>4.</u> $G = Sp(3)$,

$U = \{(D(q,q,q),D(r,s,1));\ q,r,s \in S^3\}$.

Let u_1,u_2,u_3 be the standard basis of $H^1(T) = H^2(B_T)$. The generators α_j of $H^*(B_G)$ are the elementary symmetric polynomials in the variables u_1^2, u_2^2, u_3^2. We have

$$\phi_T{}^* = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

hence, if t_1,t_2,t_3 denotes the basis of $H^1(T_U) = H^2(B_{T_U})$

(where T_U denotes the maximal torus of U), then

$$B_\phi^*(\delta_1) = 3t_1^2 - t_2^2 - t_3^2 ,$$

$$B_\phi^*(\delta_2) = 3t_1^4 - t_2^2 t_3^2 ,$$

$$B_\phi^*(\delta_3) = t_1^6 .$$

Note that $H^*(B_U)$ is generated by $x = t_1^2$, $y = t_2^2$, $z = t_3^2$. By Theorem 2 we get (cf. example 2)

$$H^*(G/U) = \mathbb{Z}[x,y]/(3x^2-3xy+y^2, x^3) .$$

Using the substitution $x = v$, $y = v - w$, we get

$$H^*(G/U) = \mathbb{Z}[v,w]/(v^2+w^2+vw, v^3) .$$

Thus G/U has the same cohomology algebra as $G/Sp(1)^3$, the manifold of flags in HP^3 . Recently, S.Stolz has shown that G/U has nontrivial Pontrjagin classes; therefore, G/U and $G/Sp(1)^3$ are not diffeomorphic.

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