Inhomogeneous spaces of positive curvature

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Introduction

In this work, we consider compact simply connected manifolds which admit a Riemannian metric with curvature $K \geqslant 1$, i.e. which are curved stronger than spheres. Besides the rank-one symmetric spaces, there are very few examples of such manifolds. The homogeneous ones have been classified by Berger, Wallach and Berard Bergery ([3, 14, 2], cf. also [1, 6, 11]). The lowest dimensional non-symmetric example among those was found by Wallach; it is the (6-dimensional) flag manifold F over $\mathbb{C}P^2$, and it is the base space of an infinite number of simply connected circle bundles $M_{p,q}$ which also admit such metrics [1]. Some years ago, we found new examples which admit only an inhomogeneous metric of positive curvature ([7, 8]), in particular a six-dimensional manifold F' which is closely related to F. Like F, it is an S^2-bundle over $\mathbb{C}P^2$ and there are also corresponding circle bundles $M'_{p,q}$ with the same properties as $M_{p,q}$. It is the aim of the present paper to describe the geometry and topology of these examples in some detail.

1. How to construct spaces of positive curvature

All known examples of compact Riemannian manifolds with positive sectional curvature arise from two sources: The noncommutativity of compact Lie groups and the contraction property of Riemannian submersions:

Fact 1 (cf. [13, 5]). *If G is a compact Lie group with biinvariant metric, then the curvature of any 2-plane $\sigma \subset \mathfrak{g}$ with orthonormal basis X, Y is*

$$K(\sigma) = K(X,Y) = \tfrac{1}{4}\|[X,Y]\|^2 \geqslant 0. \tag{1}$$

Fact 2 (cf. [13, 12, 5]). *Let E, B be Riemannian manifolds and $\pi : E \to B$ a Riemannian submersion. For $e \in E$ let $\sigma^\wedge \subset T_eE$ be a horizontal 2-plane and $\sigma = d\pi(\sigma^\wedge)$. Then*

$$K(\sigma) \geqslant K(\sigma^\wedge). \tag{2}$$

Both facts have been observed first by H. Samelson [13]. Of course, Fact 2 follows also from O'Neill's formula, but Samelson's proof is easier: Let γ_1, γ_2 be horizontal geodesics starting at e tangent to $\sigma^\wedge$. Then $\pi \circ \gamma_j$ are geodesics in M starting tangent to σ with the same angle. Since π contracts distances, we have $d(\pi \circ \gamma_1(t), \pi \circ \gamma_2(t)) \leqslant d(\gamma_1(t), \gamma_2(t))$ for all positive t and therefore $K(\sigma) \geqslant K(\sigma^\wedge)$.

Examples of Riemannian submersions are the so called *orbital submersions* which arise as follows. Let G be a group of isometries acting with closed orbits on a Riemannian manifold E. Then the orbits have constant distance from each other. Hence the orbit space E/G becomes a metric space. If the action of G is free, this metric is Riemannian, called *orbital metric.*

2. Normally homogeneous metrics on Lie groups

Let G_0 be a compact Lie group with a biinvariant metric and K_0 a closed subgroup. Then K_0 acts isometrically and freely on G_0 by right translation: $(k, g) \to gk^{-1}$. The orbit space is the homogeneous space G_0/K_0 with a normally homogeneous metric. Thus, all normally homogeneous spaces have curvature $K \geqslant 0$, and the zero curvature planes are spanned by $X, Y \in \mathfrak{p}$ with $[X,Y] = 0$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an $\mathrm{Ad}(K)$-invariant decomposition of the Lie algebra. (In fact, O'Neill's defect term in (2) which we omitted shows that this condition is also sufficient [12, 5].)

In particular, let us consider normally homogeneous metrics on a Lie group G itself: If K is any subgroup of G, then $G_0 := G \times K$ acts transitively on G by

$$((g,k), g') \to gg'k^{-1},$$

and the isotropy group of the identity element $1 \in G$ is

$$\Delta K := \{(k,k) \mid k \in K\} \subset G \times K.$$

Consider a biinvariant metric $\langle \cdot, \cdot \rangle_0$ on G and the induced metric on K. This gives us a biinvariant metric on $G \times K$ and an induced normally homogeneous metric $\langle \cdot, \cdot \rangle$ on $G = (G \times K)/\Delta K$, so that the map

$$\Phi : G \times K \to G, \quad \Phi(g,k) = gk^{-1}$$

becomes a Riemannian submersion. A vector $(X,Y) \in \mathfrak{g} \oplus \mathfrak{k}$ is horizontal with respect to Φ iff it is perpendicular to (Z,Z) for all $Z \in \mathfrak{k}$, i.e.

$$0 = \langle (X,Y),(Z,Z)\rangle = \langle X+Y, Z\rangle_0.$$

Thus $Y = -X_k$, where X_k denotes the projection of X into $\mathfrak{k}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. So the horizontal subspace is

$$\mathfrak{H} = \{(X_1+X_2,-X_2) \mid X_1 \in \mathfrak{p}, X_2 \in \mathfrak{k}\}.$$

On the other hand,

$$\Phi_*(X_1+X_2,-X_2) = X_1+X_2+X_2 = X_1+2X_2 =: X$$

and the new metric on $\mathfrak{g}$ is

$$\|X\|^2 = \|X_1+X_2\|_0^2 + \|X_2\|_0^2 = \|X_1\|_0^2 + 2\|X_2\|_0^2.$$

Since the components of X in $\mathfrak{p}$ and $\mathfrak{k}$ are

$$X_p = X_1, \quad X_k = 2X_2,$$

we get

$$\|X\|^2 = \|X_p\|_0^2 + \tfrac{1}{2}\|X_k\|_0^2.$$

Hence in the new metric, all vectors in $\mathfrak{k}$ are shortened by the factor $2^{-1/2}$.

Now a horizontal plane $\sigma^\wedge \subset \mathfrak{g}\oplus\mathfrak{k}$ spanned by $(X_1+X_2,-X_2)$ and $(Y_1+Y_2,-Y_2)$ has curvature zero in $G\times K$ iff

$$[X_2,Y_2]=0, \quad [X_1+X_2,Y_1+Y_2]=0.$$

In general, this is not a pleasant condition, but if G/K is a symmetric space, i.e. if $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, then $[X_1,Y_1] \in \mathfrak{k}$ and $[X_1,Y_2]+[X_2,Y_1] \in \mathfrak{p}$. Hence, both terms must vanish. Thus, $\sigma^\wedge$ has zero curvature if and only if

$$[X_k,Y_k]=0, \quad [X,Y]=0, \tag{N}$$

where $X = \Phi_*(X_1+X_2,X_2)$ and $Y = \Phi_*(Y_1+Y_2,Y_2)$. Thus by Fact 2, (N) is a necessary (in fact also sufficient) condition for zero curvature on a 2-plane $\sigma \subset \mathfrak{g}$ spanned by X and Y.

Example. We consider the groups $G = U(3)$ and $K = U(2)\times U(1) \subset U(3)$. Then (G,K) is a symmetric pair since G/K is the symmetric space $\mathbb{C}P^2$. We start with the biinvariant metric

$$\langle X,Y\rangle_0 = \operatorname{Re}\operatorname{trace} XY^*$$

on G and pass over to the normally homogeneous metric $\langle\cdot,\cdot\rangle$ as described above. Let $\sigma^\wedge = \operatorname{Span}(X,Y) \subset \mathfrak{g}$ be a 2-plane with zero curvature for the metric $\langle\cdot,\cdot\rangle$. Then by (N), $[X,Y]=0$ and $[X_k,Y_k]=0$. Consequently, $[X_p,Y_p]=0$. Since $\mathfrak{p}$ does not contain linearly independent commuting vectors ($\mathbb{C}P^2$ has positive curvature), the projection of $\sigma^\wedge$ onto $\mathfrak{p}$ is 1-dimensional. Hence we may assume that $Y \in \mathfrak{k}$, and

$$[X_p, Y] = 0, \quad [X_k, Y] = 0.$$

Case 1. $\sigma^\wedge \subset \mathfrak{k}$. Let $\mathfrak{su}(2)$ be the subalgebra of trace-0 matrices in $\mathfrak{u}(2)$. Then the map $\mathfrak{su}(2) \oplus \mathbb{R} \to \mathfrak{u}(2)$, $(X, t) \mapsto X + \mathrm{i} \cdot tI$ is an isomorphism of Lie algebras, and $\mathfrak{su}(2)$ contains no linearly independent commuting vectors. Thus the projection of $\sigma^\wedge$ in $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathbb{R} \oplus \mathfrak{u}(1)$ onto the first factor is 0- or 1-dimensional. Hence $\sigma^\wedge$ contains an element

$$H_1 := \mathrm{i} \cdot \mathrm{diag}(t, t, s)$$

for some $s, t \in \mathbb{R}$ with $(s, t) \neq (0, 0)$. (By $\mathrm{diag}(a_1, \ldots, a_n)$ we denote the diagonal matrix with entries $a_1, \ldots, a_n$.) This condition is also sufficient since H_1 is in the center of $\mathfrak{k}$.

Case 2. $\sigma^\wedge \not\subset \mathfrak{k}$. Then $\sigma^\wedge = \mathrm{Span}(X, Y)$ with $Y \in \mathfrak{k}$ and $X_p \neq 0$. We have

$$Y = \mathrm{i} \begin{pmatrix} A & 0 \\ 0 & t \end{pmatrix}, \qquad X_p = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}$$

where A is some hermitian 2×2-matrix, $t \in \mathbb{R}$ and $x \in \mathbb{C}^2 \backslash 0$. Now $[X_p, Y] = 0$ iff $Ax = tx$. Thus Y is conjugate in K to

$$H_2 = \mathrm{i}\, \mathrm{diag}(s, t, t)$$

for some $s, t \in \mathbb{R}$ with $(s, t) \neq (0, 0)$.

3. Biquotients with $K > 0$

Let $G, K, \langle \cdot, \cdot \rangle$ be as above and U a closed subgroup of $G \times K$. Then U acts isometrically on G by

$$((u_1, u_2), g) \to u_1 g u_2^{-1}.$$

An element $g \in G$ is a fixed point of $(u_1, u_2) \in U$ iff

$$u_2 = g^{-1} u_1 g.$$

Therefore U acts without fixed points if u_1 is not conjugate to u_2 for all $(u_1, u_2) \in U \backslash \{1\}$, and then the orbit space G/U is again a Riemannian manifold of nonnegative curvature, called a *biquotient*. This construction was used first by Gromoll and Meyer [10] to construct a metric of nonnegative curvature on an exotic 7-sphere.

The vertical subspace of the submersion $G \to G/U$ at $g \in G$ is

$$T_g\{u_1 g u_2^{-1} \mid (u_1, u_2) \in U\} = \{(R_g)_* X_1 - (L_g)_* X_2 \mid (X_1, X_2) \in \mathfrak{u}\}.$$

Using left translation back to T_1G, we get

$$\mathfrak{V}_g = (L_{g^{-1}})_*(T_g(Ug)) = \{\mathrm{Ad}(g^{-1})X_1 - X_2 \mid (X_1, X_2) \in \mathfrak{u}\}.$$

Thus G/U has positive curvature if no two linear independent vectors X, Y satisfying (N) lie in the orthogonal complement $\mathfrak{H}_g$ of $\mathfrak{V}_g$ in $\mathfrak{g}$, for any $g \in G$.

Example. We start with the flag manifold over $\mathbb{C}P^2$, namely

$$F := U(3)/U(1)^3 = U(3)/U(1)^2 Z$$

where

$$Z = \{\mathrm{diag}(z,z,z) \mid z \in \mathbb{C}, |z| = 1\}.$$

Since Z is the center of $U(3)$, right and left translations agree, so we can write equally well

$$F = Z \backslash U(3)/U(1)^2$$

which denotes the orbit space of $Z \times U(1)^2$, where Z acts by left and $U(1)^2$ by right translation. Now we replace Z with

$$Z' := \{\mathrm{diag}(z,z,\bar{z}) \mid z \in \mathbb{C}, |z| = 1\}$$

and put

$$F' = Z' \backslash U(3)/U(1)^2,$$

i.e. F' is the orbit space of $U = Z' \times U(1)^2$ acting on $G = U(3)$ by $(u_1, u_2) \to L_{u_1} R_{u_2^{-1}}$. This action is free: We have $u_1 = \mathrm{diag}(z, z, \bar{z})$ and $u_2 = \mathrm{diag}(z_1, z_2, 1)$. These are conjugate if both matrices have the same eigenvalues. This implies $z = 1$ (or $\bar{z} = 1$) and hence $z_1 = z_2 = 1$, so $u_1 = u_2 = I$.

The flag manifold F is base space of infinitely many simply connected S^1-bundles, namely

$$M_{p,q} = U(3)/U_{p,q}Z'$$

where

$$U_{p,q} = \{\mathrm{diag}(z^p, z^q, 1) \mid |z| = 1\}$$

for relative prime integers p, q; note that $F = U(3)/U(1)^2 Z'$ and that $U(1)^2/U_{p,q}$ is a circle. These spaces have been described by Aloff and Wallach [1]; they have positive curvature iff $p \cdot q > 0$. Likewise, F' is the base space of simply connected circle bundles $M'_{p,q}$, namely

$$M'_{p,q} = Z' \backslash U(3)/U_{p,q}.$$

Now we show that F' and $M'_{p,q}$ for $p, q > 0$ carry a metric of positive curvature. However, we have to interchange the factors, i.e. we rather put

$$F' = U(1)^2 \backslash U(3)/Z', \qquad M'_{p,q} = U_{p,q} \backslash U(3)/Z'.$$

On $G := U(3)$ we take the normally homogeneous metric with $K = U(2) \times U(1)$ which was introduced in Section 2; note that $U(1)^2 \times Z' \subset G \times K$ acts by isometries with respect to this metric.

Theorem 1. *The induced metrics on F' and $M'_{p,q}$ for $p, q > 0$ have positive curvature.*

Proof. Let $G = U(3)$. All we have to show is that no zero curvature plane $\sigma^\wedge \subset \mathfrak{g}$ is perpendicular to the subspace

$$\mathfrak{V}_g = \mathrm{Ad}(g^{-1})\mathfrak{u}_{p,q} + \mathfrak{z}',$$

i.e. perpendicular to B and $\mathrm{Ad}(g^{-1})A$, for any $g \in G$ where

$$A = \mathrm{i} \cdot \mathrm{diag}(p, q, 0), \quad B = \mathrm{i} \cdot \mathrm{diag}(1, 1, -1).$$

So, let us consider a zero curvature plane $\sigma^\wedge \subset \mathfrak{g}$ which is already perpendicular to B. By Section 2, two cases are possible:

Case 1. $H_1 = i \cdot \mathrm{diag}(t, t, s) \in \sigma^\wedge$ for some $(s, t) \neq (0, 0)$. For all $X \in \mathfrak{g}, Y \in \mathfrak{k}$ we have $\langle X, Y\rangle = \frac{1}{2}\langle X, Y\rangle_0$. Thus

$$0 = \langle H_1, B\rangle = \tfrac{1}{2}\langle H_1, B\rangle_0 = \tfrac{1}{2}(2t - s),$$

and so we assume

$$H_1 = \mathrm{i} \cdot \mathrm{diag}(1, 1, 2).$$

It is well known that the extremal values of the height function

$$f : \mathrm{Ad}(G)A \to \mathbb{R}, \quad f(X) = \langle X, H_1\rangle_0$$

are attained on the set $\mathrm{Ad}(G)A \cap \{\text{diagonal matrices}\}$ (see Lemma below) which consists of the matrices $\mathrm{i} \cdot \mathrm{diag}(a, b, c)$ where (a, b, c) is a permutation of $(p, q, 0)$. So the extremal values of f are in the set $\{p + q, p + 2q, 2p + q\} \subset (0, \infty)$ which shows $\langle \mathrm{Ad}(g)A, H_1\rangle > 0$ for all $g \in G$.

Case 2. $\mathrm{Ad}(k)H_2 \in \sigma^\wedge$ for some $k \in K$, where $H_2 = \mathrm{i} \cdot \mathrm{diag}(s, t, t)$. Then

$$0 = \langle \mathrm{Ad}(k)H_2, B\rangle = \langle H_2, \mathrm{Ad}(k^{-1})B\rangle = \langle H_2, B\rangle = \tfrac{1}{2}s,$$

since B commutes with all elements of $K = U(2) \times U(1)$. Thus we may assume $H_2 = \mathrm{i} \cdot \mathrm{diag}(0, 1, 1)$. Then

$$2\langle \mathrm{Ad}(g)A,\ \mathrm{Ad}(k)H_2\rangle = \langle \mathrm{Ad}(g)A,\ \mathrm{Ad}(k)H_2\rangle_0 = \langle \mathrm{Ad}(k^{-1}g)A,\ H_2\rangle_0,$$

and as above we see that extremal values of $\langle X, H_2\rangle_0$ for $X \in \mathrm{Ad}(G)A$ are within the set $\{p, q, p + q\} \subset (0, \infty)$, and so $\mathrm{Ad}(k)H_2$ is never perpendicular to $\mathrm{Ad}(g)A$. Thus no curvature-zero plane $\sigma^\wedge \subset \mathfrak{g}$ can be perpendicular to $\mathfrak{V}_g$ for any $g \in G$, and the theorem is proved. □

Lemma. *Let G be a compact Lie group with biinvariant metric $\langle\cdot,\cdot\rangle_0$. Let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian subalgebra and $H \in \mathfrak{t}$. Let $M = \mathrm{Ad}(G)H \subset \mathfrak{g}$. Then the extremal values of the height function*

$$f_H : M \to \mathbb{R}, \quad f_H(X) = \langle H, X\rangle_0$$

are attained on $M \cap \mathfrak{t}$.

Proof. Suppose first that $H \in \mathfrak{t}$ is regular, i.e. that it lies in no other maximal abelian subalgebra. A point $X \in M$ is a critical point for f_H iff $H \perp T_X M = \mathrm{Ad}(\mathfrak{g})X$, i.e. iff $0 = \langle [Z,X], H\rangle_0 = \langle Z, [X,H]\rangle_0$ for any $Z \in \mathfrak{g}$, which means $[X,H] = 0$. Since H is regular, $X \in \mathfrak{t}$. In particular, the extremal values of f_H are taken on $M \cap \mathfrak{t}$. Since the regular vectors are dense in $\mathfrak{t}$, the same fact holds for any $H \in \mathfrak{t}$. □

Remarks. (1) Note that $\mathfrak{V}_1 = \mathfrak{u}_{p,q} + \mathfrak{z}'$ is the vertical space corresponding to the homogeneous spaces $M_{p,q}$. So, we have also proved that $M_{p,q}$ and F have positive curvature, where $p, q > 0$.

(2) The projections

$$M'_{p,q} = U_{p,q}\backslash U(3)/Z' \to U(1)^2\backslash U(3)/Z' = F',$$
$$F' = U(1)^2\backslash U(3)/Z' \to U(2)\backslash U(3)/Z' = S^5/Z' = \mathbb{C}P^2$$

are Riemannian submersions. However, note that the fibres are not totally geodesic and that the metric on $\mathbb{C}P^2$ is not homogeneous (it has cohomogeneity one).

(3) Note that the map $U(3)/Z' \to SU(3)$ sending the coset $A\cdot Z'$ onto $A\cdot\mathrm{diag}(\bar\alpha,\bar\alpha,\alpha)$ for any $A \in U(3)$ with $\alpha := \det A$, is a diffeomorphism. This shows that $M_{p,q}$ and $M'_{p,q}$ can be considered as quotient spaces of $SU(3)$ (cf. [1,8]) and in particular, they are simply connected. More precisely, we have $M_{p,q} = SU(3)/U'_{p,q}$ and $M'_{p,q} = SU(3)/U''_{p,q}$ with

$$U'_{p,q} = \{\mathrm{diag}(z^{-p}, z^{-q}, z^{p+q}) \mid z \in S^1\},$$
$$U''_{p,q} = \{(\mathrm{diag}(z^p, z^q, 1), \mathrm{diag}(z^{p+q}, z^{p+q}, z^{-p-q})) \mid z \in S^1\}.$$

(4) There are many more positively curved 7-dimensional biquotients of $SU(3)$ (cf. [8]). In fact, if $a, b \in \mathbb{Z}^3$, then

$$U_{a,b} := \left\{(\mathrm{diag}(z^{a_1}, z^{a_2}, z^{a_3}), \mathrm{diag}(z^{b_1}, z^{b_2}, z^{b_3})) \;\middle|\; |z| = 1\right\}$$

acts on $SU(3)$ iff $\Sigma a_i = \Sigma b_i$, and it acts freely if any integer n dividing all components of $a - A_\sigma b$ for some permutation matrix A_σ also divides $a_i - b_j$ for all $i, j \in \{1,2,3\}$. Moreover, $SU(3)/U_{a,b}$ has positive curvature if and only if

$$b_i \notin [a_{\min}, a_{\max}]$$

for $i \in \{1,2,3\}$, where $a_{\min}(a_{\max})$ is the minimum (maximum) of a_1, a_2, a_3 (cf. [8, p. 45]). In particular, the condition $p \cdot q > 0$ is also necessary for $M'_{p,q}$ to have positive curvature.

(5) It has been shown in [8] that among the even dimensional biquotients of any simple compact Lie group with a left invariant metric which is right invariant with

respect to some maximal torus, F' is the only one which is positively curved and not diffeomorphic to a homogeneous space of positive curvature.

4. Topology of F and F'

We start by considering the homogeneous space

$$F^\wedge = \{(x,[y]) \in S^5 \times \mathbb{C}P^2 \mid x \perp [y]\}.$$

Clearly, $U(3)$ acts transitively on $F^\wedge$, and the stabilizer subgroup of $(e_3,[e_1])$ is $U(1)^2 = \{\mathrm{diag}(\alpha,\beta,1) \mid \alpha,\beta \in S^1\}$, hence $F^\wedge = U(3)/U(1)^2$. The projection onto S^5 makes $F^\wedge$ into a $\mathbb{C}P^1$-bundle over S^5, namely the projectivization of the $\mathbb{C}^2$-bundle

$$E^\wedge = \{(x,y) \in S^5 \times \mathbb{C}^3 \mid y \perp \mathbb{C}x\}.$$

The complement of $E^\wedge$ in the trivial bundle $S^5 \times \mathbb{C}^3$ is the trivial bundle

$$T^\wedge = \{(x,\lambda x) \mid x \in S^5, \lambda \in \mathbb{C}\} \cong S^5 \times \mathbb{C}.$$

Now we let $S^1 = S \subset U(3)$ be a subgroup which acts freely on S^5. It is easy to see that up to conjugation there are only two such subgroups, namely Z and Z'. The group S acts on $E^\wedge$ by bundle isomorphisms. Hence, $E = E^\wedge/S$ is a $\mathbb{C}^2$-bundle over S^5/S, and $M := F^\wedge/S = S\backslash U(3)/U(1)^2$ is the projectivization of E. Moreover, $T = T^\wedge/S$ is the trivial bundle $(S^5/S) \times \mathbb{C}$ since $s(x,\lambda x) = (sx, \lambda sx)$ for any $s \in S$, i.e. the S-action fixes the $\mathbb{C}$-factor. Therefore, E is stable equivalent to the $\mathbb{C}^3$-bundle $E_0 = (S^5 \times \mathbb{C}^3)/S$. If $S = Z$, this is 3-times the inverse Hopf bundle over $S^5/Z = \mathbb{C}P^2 : E_0 = \bar{H} \oplus \bar{H} \oplus \bar{H}$ where $\bar{H} = (S^5 \times \mathbb{C})/S^1$ with the S^1-action $z(v,\lambda) = (zv, z\lambda)$ on $S^5 \times \mathbb{C}$. If $S = Z'$, we note that the map

$$\alpha : S^5 \to S^5, \quad \alpha(x_1,x_2,x_3) = (x_1,x_2,\bar{x}_3)$$

conjugates the actions of Z' and Z on S^5. Thus the map

$$\alpha \times \mathrm{id} : S^5 \times \mathbb{C}^3 \to S^5 \times \mathbb{C}^3$$

descends to a bundle isomorphism between E_0 over S^5/Z' and $\bar{H} \oplus \bar{H} \oplus H$ over $S^5/Z = \mathbb{C}P^2$, where $H = (S^5 \times \mathbb{C})/S^1$ with the action $z(v,\lambda) = (zv, \bar{z}\lambda)$ on $S^5 \times \mathbb{C}$. Thus we have shown:

Theorem 2. *The manifolds F and F' are diffeomorphic to the projectivizations of $\mathbb{C}^2$-bundles over $\mathbb{C}P^2$ which are stably equivalent to $\bar{H} \oplus \bar{H} \oplus \bar{H}$ and $\bar{H} \oplus \bar{H} \oplus H$.*

Now it is easy to compute the cohomology of these manifolds. In fact, by the the Leray-Hirsch theorem, the (integer) cohomology ring of the projectivization PE of a $\mathbb{C}^2$-bundle E over $\mathbb{C}P^2$ is a truncated polynomial algebra over $H^*(\mathbb{C}P^2)$, namely

$$H^*(PE) = H^*(\mathbb{C}P^2)\,[y]\,/(y^2 + c_1(E)y + c_2(E))$$

where $c_i(E) \in H^{2i}(\mathbb{C}P^2)$ are the Chern classes of E and $y \in H^2(PE)$ is the first Chern class of the inverse Hopf bundle over PE (cf. [4, p. 270]). Since $c_1(\bar{H}) = x$ where x denotes the canonical generator of $H^*(\mathbb{C}P^2)$, we get

$$c(\bar{H} \oplus \bar{H} \oplus H) = (1+x)(1+x)(1-x) = 1 + x - x^2$$

so in this case, $c_1(E) = x,\ c_2(E) = -x^2$ and therefore

$$H^*(F') = \mathbb{Z}[x,y]/(y^2 + xy - x^2, x^3).$$

Using the substitution $x = u,\ y = u + v$, we receive

$$H^*(F') = \mathbb{Z}[u,v]/(u^2 + 3uv + v^2, u^3).$$

In the other case we get $c_1(E) = 3x,\ c_2(E) = 3x^2$ which shows

$$H^*(F) = \mathbb{Z}[x,y]/(y^2 + 3xy + 3x^2, x^3).$$

With the substitution $x = u,\ y = v - u$ this becomes

$$H^*(F) = \mathbb{Z}[u,v]/(u^2 + uv + v^2, u^3).$$

Note that the quadratic form $u^2 + uv + v^2$ is positive definite while $x^2 + xy - y^2$ is indefinite, so F and F' have nonisomorphic cohomology rings already over the reals. This shows that F' is not homotopic to any other known space which carries a metric of positive curvature.

Remark 1. Alternatively, the bundle PE can also be considered as the unit sphere bundle of an $\mathbb{R}^3$-bundle V over $\mathbb{C}P^2$. These bundles are classified by the first Pontrjagin class $p_1(V)$. It is not difficult to compute that

$$p_1(V) = c_1(E)^2 - 4c_2(E).$$

In particular, the Pontrjagin numbers of the $\mathbb{R}^3$-bundles corresponding to F' and F are 5 and -3.

Remark 2. The cohomology of $SU(3)/U_{a,b}$ has been computed in [8,9]. The invariant which distinguishes these spaces is the order $r = \frac{1}{2}|(\|a\|^2 - \|b\|^2)|$ of the torsion group H^5. So by Remark 3 in Section 3 we have

$$r(M'_{p,q}) = p^2 + q^2 + 3pq,$$
$$r(M_{p,q}) = p^2 + q^2 + pq.$$

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