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# Inhomogeneous spaces of positive curvature

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## Introduction

In this work, we consider compact simply connected manifolds which admit a Riemannian metric with curvature  $K \geq 1$ , i.e. which are curved stronger than spheres. Besides the rank-one symmetric spaces, there are very few examples of such manifolds. The homogeneous ones have been classified by Berger, Wallach and Berard Bergery ([3, 14, 2], cf. also [1, 6, 11]). The lowest dimensional non-symmetric example among those was found by Wallach; it is the (6-dimensional) flag manifold  $F$  over  $\mathbb{C}P^2$ , and it is the base space of an infinite number of simply connected circle bundles  $M_{p,q}$  which also admit such metrics [1]. Some years ago, we found new examples which admit only an inhomogeneous metric of positive curvature ([7, 8]), in particular a six-dimensional manifold  $F'$  which is closely related to  $F$ . Like  $F$ , it is an  $S^2$ -bundle over  $\mathbb{C}P^2$  and there are also corresponding circle bundles  $M'_{p,q}$  with the same properties as  $M_{p,q}$ . It is the aim of the present paper to describe the geometry and topology of these examples in some detail.

## 1. How to construct spaces of positive curvature

All known examples of compact Riemannian manifolds with positive sectional curvature arise from two sources: The noncommutativity of compact Lie groups and the contraction property of Riemannian submersions:

**Fact 1** (cf. [13, 5]). *If  $G$  is a compact Lie group with biinvariant metric, then the curvature of any 2-plane  $\sigma \subset \mathfrak{g}$  with orthonormal basis  $X, Y$  is*

$$K(\sigma) = K(X, Y) = \frac{1}{4} \|[X, Y]\|^2 \geq 0. \quad (1)$$

**Fact 2** (cf. [13, 12, 5]). *Let  $E, B$  be Riemannian manifolds and  $\pi : E \rightarrow B$  a Riemannian submersion. For  $e \in E$  let  $\sigma^\wedge \subset T_e E$  be a horizontal 2-plane and  $\sigma = d\pi(\sigma^\wedge)$ . Then*

$$K(\sigma) \geq K(\sigma^\wedge). \quad (2)$$

Both facts have been observed first by H. Samelson [13]. Of course, Fact 2 follows also from O'Neill's formula, but Samelson's proof is easier: Let  $\gamma_1, \gamma_2$  be horizontal geodesics starting at  $e$  tangent to  $\sigma^\wedge$ . Then  $\pi \circ \gamma_j$  are geodesics in  $M$  starting tangent to  $\sigma$  with the same angle. Since  $\pi$  contracts distances, we have  $d(\pi \circ \gamma_1(t), \pi \circ \gamma_2(t)) \leq d(\gamma_1(t), \gamma_2(t))$  for all positive  $t$  and therefore  $K(\sigma) \geq K(\sigma^\wedge)$ .

Examples of Riemannian submersions are the so called *orbital submersions* which arise as follows. Let  $G$  be a group of isometries acting with closed orbits on a Riemannian manifold  $E$ . Then the orbits have constant distance from each other. Hence the orbit space  $E/G$  becomes a metric space. If the action of  $G$  is free, this metric is Riemannian, called *orbital metric*.

## 2. Normally homogeneous metrics on Lie groups

Let  $G_0$  be a compact Lie group with a biinvariant metric and  $K_0$  a closed subgroup. Then  $K_0$  acts isometrically and freely on  $G_0$  by right translation:  $(k, g) \rightarrow gk^{-1}$ . The orbit space is the homogeneous space  $G_0/K_0$  with a normally homogeneous metric. Thus, all normally homogeneous spaces have curvature  $K \geq 0$ , and the zero curvature planes are spanned by  $X, Y \in \mathfrak{p}$  with  $[X, Y] = 0$ , where  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is an  $\text{Ad}(K)$ -invariant decomposition of the Lie algebra. (In fact, O'Neill's defect term in (2) which we omitted shows that this condition is also sufficient [12, 5].)

In particular, let us consider normally homogeneous metrics on a Lie group  $G$  itself: If  $K$  is any subgroup of  $G$ , then  $G_0 := G \times K$  acts transitively on  $G$  by

$$((g, k), g') \rightarrow gg'k^{-1},$$

and the isotropy group of the identity element  $1 \in G$  is

$$\Delta K := \{(k, k) \mid k \in K\} \subset G \times K.$$

Consider a biinvariant metric  $\langle \cdot, \cdot \rangle_0$  on  $G$  and the induced metric on  $K$ . This gives us a biinvariant metric on  $G \times K$  and an induced normally homogeneous metric  $\langle \cdot, \cdot \rangle$  on  $G = (G \times K)/\Delta K$ , so that the map

$$\Phi : G \times K \rightarrow G, \quad \Phi(g, k) = gk^{-1}$$

becomes a Riemannian submersion. A vector  $(X, Y) \in \mathfrak{g} \oplus \mathfrak{k}$  is horizontal with respect to  $\Phi$  iff it is perpendicular to  $(Z, Z)$  for all  $Z \in \mathfrak{k}$ , i.e.

$$0 = \langle (X, Y), (Z, Z) \rangle = \langle X + Y, Z \rangle_0.$$

Thus  $Y = -X_k$ , where  $X_k$  denotes the projection of  $X$  into  $\mathfrak{k}$ . Let  $\mathfrak{p}$  be the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$ . So the horizontal subspace is

$$\mathfrak{H} = \{(X_1 + X_2, -X_2) \mid X_1 \in \mathfrak{p}, X_2 \in \mathfrak{k}\}.$$

On the other hand,

$$\Phi_*(X_1 + X_2, -X_2) = X_1 + X_2 + X_2 = X_1 + 2X_2 =: X$$

and the new metric on  $\mathfrak{g}$  is

$$\|X\|^2 = \|X_1 + X_2\|_0^2 + \|X_2\|_0^2 = \|X_1\|_0^2 + 2\|X_2\|_0^2.$$

Since the components of  $X$  in  $\mathfrak{p}$  and  $\mathfrak{k}$  are

$$X_p = X_1, \quad X_k = 2X_2,$$

we get

$$\|X\|^2 = \|X_p\|_0^2 + \frac{1}{2}\|X_k\|_0^2.$$

Hence in the new metric, all vectors in  $\mathfrak{k}$  are shortened by the factor  $2^{-1/2}$ .

Now a horizontal plane  $\sigma^\wedge \subset \mathfrak{g} \oplus \mathfrak{k}$  spanned by  $(X_1 + X_2, -X_2)$  and  $(Y_1 + Y_2, -Y_2)$  has curvature zero in  $G \times K$  iff

$$[X_2, Y_2] = 0, \quad [X_1 + X_2, Y_1 + Y_2] = 0.$$

In general, this is not a pleasant condition, but if  $G/K$  is a symmetric space, i.e. if  $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ , then  $[X_1, Y_1] \in \mathfrak{k}$  and  $[X_1, Y_2] + [X_2, Y_1] \in \mathfrak{p}$ . Hence, both terms must vanish. Thus,  $\sigma^\wedge$  has zero curvature if and only if

$$[X_k, Y_k] = 0, \quad [X, Y] = 0, \tag{N}$$

where  $X = \Phi_*(X_1 + X_2, X_2)$  and  $Y = \Phi_*(Y_1 + Y_2, Y_2)$ . Thus by Fact 2, (N) is a necessary (in fact also sufficient) condition for zero curvature on a 2-plane  $\sigma \subset \mathfrak{g}$  spanned by  $X$  and  $Y$ .

**Example.** We consider the groups  $G = U(3)$  and  $K = U(2) \times U(1) \subset U(3)$ . Then  $(G, K)$  is a symmetric pair since  $G/K$  is the symmetric space  $\mathbb{C}P^2$ . We start with the biinvariant metric

$$\langle X, Y \rangle_0 = \text{Re trace } XY^*$$

on  $G$  and pass over to the normally homogeneous metric  $\langle \cdot, \cdot \rangle$  as described above. Let  $\sigma^\wedge = \text{Span}(X, Y) \subset \mathfrak{g}$  be a 2-plane with zero curvature for the metric  $\langle \cdot, \cdot \rangle$ . Then by (N),  $[X, Y] = 0$  and  $[X_k, Y_k] = 0$ . Consequently,  $[X_p, Y_p] = 0$ . Since  $\mathfrak{p}$  does not contain

linearly independent commuting vectors ( $\mathbb{C}P^2$  has positive curvature), the projection of  $\sigma^\wedge$  onto  $\mathfrak{p}$  is 1-dimensional. Hence we may assume that  $Y \in \mathfrak{k}$ , and

$$[X_p, Y] = 0, \quad [X_k, Y] = 0.$$

**Case 1.**  $\sigma^\wedge \subset \mathfrak{k}$ . Let  $\mathfrak{su}(2)$  be the subalgebra of trace-0 matrices in  $\mathfrak{u}(2)$ . Then the map  $\mathfrak{su}(2) \oplus \mathbb{R} \rightarrow \mathfrak{u}(2)$ ,  $(X, t) \mapsto X + i \cdot tI$  is an isomorphism of Lie algebras, and  $\mathfrak{su}(2)$  contains no linearly independent commuting vectors. Thus the projection of  $\sigma^\wedge$  in  $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathbb{R} \oplus \mathfrak{u}(1)$  onto the first factor is 0- or 1-dimensional. Hence  $\sigma^\wedge$  contains an element

$$H_1 := i \cdot \text{diag}(t, t, s)$$

for some  $s, t \in \mathbb{R}$  with  $(s, t) \neq (0, 0)$ . (By  $\text{diag}(a_1, \dots, a_n)$  we denote the diagonal matrix with entries  $a_1, \dots, a_n$ .) This condition is also sufficient since  $H_1$  is in the center of  $\mathfrak{k}$ .

**Case 2.**  $\sigma^\wedge \not\subset \mathfrak{k}$ . Then  $\sigma^\wedge = \text{Span}(X, Y)$  with  $Y \in \mathfrak{k}$  and  $X_p \neq 0$ . We have

$$Y = i \begin{pmatrix} A & 0 \\ 0 & t \end{pmatrix}, \quad X_p = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}$$

where  $A$  is some hermitian  $2 \times 2$ -matrix,  $t \in \mathbb{R}$  and  $x \in \mathbb{C}^2 \setminus \{0\}$ . Now  $[X_p, Y] = 0$  iff  $Ax = tx$ . Thus  $Y$  is conjugate in  $K$  to

$$H_2 = i \text{diag}(s, t, t)$$

for some  $s, t \in \mathbb{R}$  with  $(s, t) \neq (0, 0)$ .

### 3. Biquotients with $K > 0$

Let  $G, K, \langle \cdot, \cdot \rangle$  be as above and  $U$  a closed subgroup of  $G \times K$ . Then  $U$  acts isometrically on  $G$  by

$$((u_1, u_2), g) \rightarrow u_1 g u_2^{-1}.$$

An element  $g \in G$  is a fixed point of  $(u_1, u_2) \in U$  iff

$$u_2 = g^{-1} u_1 g.$$

Therefore  $U$  acts without fixed points if  $u_1$  is not conjugate to  $u_2$  for all  $(u_1, u_2) \in U \setminus \{1\}$ , and then the orbit space  $G/U$  is again a Riemannian manifold of nonnegative curvature, called a *biquotient*. This construction was used first by Gromoll and Meyer [10] to construct a metric of nonnegative curvature on an exotic 7-sphere.

The vertical subspace of the submersion  $G \rightarrow G/U$  at  $g \in G$  is

$$T_g \{u_1 g u_2^{-1} \mid (u_1, u_2) \in U\} = \{(R_g)_* X_1 - (L_g)_* X_2 \mid (X_1, X_2) \in \mathfrak{u}\}.$$

Using left translation back to  $T_1G$ , we get

$$\mathfrak{V}_g = (L_{g^{-1}})_*(T_g(Ug)) = \{\text{Ad}(g^{-1})X_1 - X_2 \mid (X_1, X_2) \in \mathfrak{u}\}.$$

Thus  $G/U$  has positive curvature if no two linear independent vectors  $X, Y$  satisfying (N) lie in the orthogonal complement  $\mathfrak{H}_g$  of  $\mathfrak{V}_g$  in  $\mathfrak{g}$ , for any  $g \in G$ .

**Example.** We start with the flag manifold over  $\mathbb{C}P^2$ , namely

$$F := U(3)/U(1)^3 = U(3)/U(1)^2 Z$$

where

$$Z = \{\text{diag}(z, z, z) \mid z \in \mathbb{C}, |z| = 1\}.$$

Since  $Z$  is the center of  $U(3)$ , right and left translations agree, so we can write equally well

$$F = Z \backslash U(3)/U(1)^2$$

which denotes the orbit space of  $Z \times U(1)^2$ , where  $Z$  acts by left and  $U(1)^2$  by right translation. Now we replace  $Z$  with

$$Z' := \{\text{diag}(z, z, \bar{z}) \mid z \in \mathbb{C}, |z| = 1\}$$

and put

$$F' = Z' \backslash U(3)/U(1)^2,$$

i.e.  $F'$  is the orbit space of  $U = Z' \times U(1)^2$  acting on  $G = U(3)$  by  $(u_1, u_2) \rightarrow L_{u_1} R_{u_2^{-1}}$ . This action is free: We have  $u_1 = \text{diag}(z, z, \bar{z})$  and  $u_2 = \text{diag}(z_1, z_2, 1)$ . These are conjugate if both matrices have the same eigenvalues. This implies  $z = 1$  (or  $\bar{z} = 1$ ) and hence  $z_1 = z_2 = 1$ , so  $u_1 = u_2 = I$ .

The flag manifold  $F$  is base space of infinitely many simply connected  $S^1$ -bundles, namely

$$M_{p,q} = U(3)/U_{p,q} Z'$$

where

$$U_{p,q} = \{\text{diag}(z^p, z^q, 1) \mid |z| = 1\}$$

for relative prime integers  $p, q$ ; note that  $F = U(3)/U(1)^2 Z'$  and that  $U(1)^2/U_{p,q}$  is a circle. These spaces have been described by Aloff and Wallach [1]; they have positive curvature iff  $p \cdot q > 0$ . Likewise,  $F'$  is the base space of simply connected circle bundles  $M'_{p,q}$ , namely

$$M'_{p,q} = Z' \backslash U(3)/U_{p,q}.$$

Now we show that  $F'$  and  $M'_{p,q}$  for  $p, q > 0$  carry a metric of positive curvature. However, we have to interchange the factors, i.e. we rather put

$$F' = U(1)^2 \backslash U(3)/Z', \quad M'_{p,q} = U_{p,q} \backslash U(3)/Z'.$$

On  $G := U(3)$  we take the normally homogeneous metric with  $K = U(2) \times U(1)$  which was introduced in Section 2; note that  $U(1)^2 \times Z' \subset G \times K$  acts by isometries with respect to this metric.

**Theorem 1.** *The induced metrics on  $F'$  and  $M'_{p,q}$  for  $p, q > 0$  have positive curvature.*

**Proof.** Let  $G = U(3)$ . All we have to show is that no zero curvature plane  $\sigma^\wedge \subset \mathfrak{g}$  is perpendicular to the subspace

$$\mathfrak{V}_g = \text{Ad}(g^{-1})\mathfrak{u}_{p,q} + \mathfrak{J}',$$

i.e. perpendicular to  $B$  and  $\text{Ad}(g^{-1})A$ , for any  $g \in G$  where

$$A = i \cdot \text{diag}(p, q, 0), \quad B = i \cdot \text{diag}(1, 1, -1).$$

So, let us consider a zero curvature plane  $\sigma^\wedge \subset \mathfrak{g}$  which is already perpendicular to  $B$ . By Section 2, two cases are possible:

*Case 1.*  $H_1 = i \cdot \text{diag}(t, t, s) \in \sigma^\wedge$  for some  $(s, t) \neq (0, 0)$ . For all  $X \in \mathfrak{g}, Y \in \mathfrak{k}$  we have  $\langle X, Y \rangle = \frac{1}{2}\langle X, Y \rangle_0$ . Thus

$$0 = \langle H_1, B \rangle = \frac{1}{2}\langle H_1, B \rangle_0 = \frac{1}{2}(2t - s),$$

and so we assume

$$H_1 = i \cdot \text{diag}(1, 1, 2).$$

It is well known that the extremal values of the height function

$$f : \text{Ad}(G)A \rightarrow \mathbb{R}, \quad f(X) = \langle X, H_1 \rangle_0$$

are attained on the set  $\text{Ad}(G)A \cap \{\text{diagonal matrices}\}$  (see Lemma below) which consists of the matrices  $i \cdot \text{diag}(a, b, c)$  where  $(a, b, c)$  is a permutation of  $(p, q, 0)$ . So the extremal values of  $f$  are in the set  $\{p + q, p + 2q, 2p + q\} \subset (0, \infty)$  which shows  $\langle \text{Ad}(g)A, H_1 \rangle > 0$  for all  $g \in G$ .

*Case 2.*  $\text{Ad}(k)H_2 \in \sigma^\wedge$  for some  $k \in K$ , where  $H_2 = i \cdot \text{diag}(s, t, t)$ . Then

$$0 = \langle \text{Ad}(k)H_2, B \rangle = \langle H_2, \text{Ad}(k^{-1})B \rangle = \langle H_2, B \rangle = \frac{1}{2}s,$$

since  $B$  commutes with all elements of  $K = U(2) \times U(1)$ . Thus we may assume  $H_2 = i \cdot \text{diag}(0, 1, 1)$ . Then

$$2\langle \text{Ad}(g)A, \text{Ad}(k)H_2 \rangle = \langle \text{Ad}(g)A, \text{Ad}(k)H_2 \rangle_0 = \langle \text{Ad}(k^{-1}g)A, H_2 \rangle_0,$$

and as above we see that extremal values of  $\langle X, H_2 \rangle_0$  for  $X \in \text{Ad}(G)A$  are within the set  $\{p, q, p + q\} \subset (0, \infty)$ , and so  $\text{Ad}(k)H_2$  is never perpendicular to  $\text{Ad}(g)A$ . Thus no curvature-zero plane  $\sigma^\wedge \subset \mathfrak{g}$  can be perpendicular to  $\mathfrak{V}_g$  for any  $g \in G$ , and the theorem is proved.  $\square$

**Lemma.** Let  $G$  be a compact Lie group with biinvariant metric  $\langle \cdot, \cdot \rangle_0$ . Let  $\mathfrak{t} \subset \mathfrak{g}$  be a maximal abelian subalgebra and  $H \in \mathfrak{t}$ . Let  $M = \text{Ad}(G)H \subset \mathfrak{g}$ . Then the extremal values of the height function

$$f_H : M \rightarrow \mathbb{R}, \quad f_H(X) = \langle H, X \rangle_0$$

are attained on  $M \cap \mathfrak{t}$ .

**Proof.** Suppose first that  $H \in \mathfrak{t}$  is regular, i.e. that it lies in no other maximal abelian subalgebra. A point  $X \in M$  is a critical point for  $f_H$  iff  $H \perp T_X M = \text{Ad}(\mathfrak{g})X$ , i.e. iff  $0 = \langle [Z, X], H \rangle_0 = \langle Z, [X, H] \rangle_0$  for any  $Z \in \mathfrak{g}$ , which means  $[X, H] = 0$ . Since  $H$  is regular,  $X \in \mathfrak{t}$ . In particular, the extremal values of  $f_H$  are taken on  $M \cap \mathfrak{t}$ . Since the regular vectors are dense in  $\mathfrak{t}$ , the same fact holds for any  $H \in \mathfrak{t}$ .  $\square$

**Remarks.** (1) Note that  $\mathfrak{W}_1 = \mathfrak{u}_{p,q} + \mathfrak{z}'$  is the vertical space corresponding to the homogeneous spaces  $M_{p,q}$ . So, we have also proved that  $M_{p,q}$  and  $F$  have positive curvature, where  $p, q > 0$ .

(2) The projections

$$\begin{aligned} M'_{p,q} &= U_{p,q} \backslash U(3)/Z' \rightarrow U(1)^2 \backslash U(3)/Z' = F', \\ F' &= U(1)^2 \backslash U(3)/Z' \rightarrow U(2) \backslash U(3)/Z' = S^5/Z' = \mathbb{C}P^2 \end{aligned}$$

are Riemannian submersions. However, note that the fibres are not totally geodesic and that the metric on  $\mathbb{C}P^2$  is not homogeneous (it has cohomogeneity one).

(3) Note that the map  $U(3)/Z' \rightarrow SU(3)$  sending the coset  $A \cdot Z'$  onto  $A \cdot \text{diag}(\bar{\alpha}, \bar{\alpha}, \alpha)$  for any  $A \in U(3)$  with  $\alpha := \det A$ , is a diffeomorphism. This shows that  $M_{p,q}$  and  $M'_{p,q}$  can be considered as quotient spaces of  $SU(3)$  (cf. [1, 8]) and in particular, they are simply connected. More precisely, we have  $M_{p,q} = SU(3)/U'_{p,q}$  and  $M'_{p,q} = SU(3)/U''_{p,q}$  with

$$\begin{aligned} U'_{p,q} &= \{\text{diag}(z^{-p}, z^{-q}, z^{p+q}) \mid z \in S^1\}, \\ U''_{p,q} &= \{(\text{diag}(z^p, z^q, 1), \text{diag}(z^{p+q}, z^{p+q}, z^{-p-q})) \mid z \in S^1\}. \end{aligned}$$

(4) There are many more positively curved 7-dimensional biquotients of  $SU(3)$  (cf. [8]). In fact, if  $a, b \in \mathbb{Z}^3$ , then

$$U_{a,b} := \{(\text{diag}(z^{a_1}, z^{a_2}, z^{a_3}), \text{diag}(z^{b_1}, z^{b_2}, z^{b_3})) \mid |z| = 1\}$$

acts on  $SU(3)$  iff  $\Sigma a_i = \Sigma b_i$ , and it acts freely if any integer  $n$  dividing all components of  $a - A_\sigma b$  for some permutation matrix  $A_\sigma$  also divides  $a_i - b_j$  for all  $i, j \in \{1, 2, 3\}$ . Moreover,  $SU(3)/U_{a,b}$  has positive curvature if and only if

$$b_i \notin [a_{\min}, a_{\max}]$$

for  $i \in \{1, 2, 3\}$ , where  $a_{\min}(a_{\max})$  is the minimum (maximum) of  $a_1, a_2, a_3$  (cf. [8, p. 45]). In particular, the condition  $p \cdot q > 0$  is also necessary for  $M'_{p,q}$  to have positive curvature.

(5) It has been shown in [8] that among the even dimensional biquotients of any simple compact Lie group with a left invariant metric which is right invariant with



respect to some maximal torus,  $F'$  is the only one which is positively curved and not diffeomorphic to a homogeneous space of positive curvature.

#### 4. Topology of $F$ and $F'$

We start by considering the homogeneous space

$$F^\wedge = \{(x, [y]) \in S^5 \times \mathbb{C}P^2 \mid x \perp [y]\}.$$

Clearly,  $U(3)$  acts transitively on  $F^\wedge$ , and the stabilizer subgroup of  $(e_3, [e_1])$  is  $U(1)^2 = \{\text{diag}(\alpha, \beta, 1) \mid \alpha, \beta \in S^1\}$ , hence  $F^\wedge = U(3)/U(1)^2$ . The projection onto  $S^5$  makes  $F^\wedge$  into a  $\mathbb{C}P^1$ -bundle over  $S^5$ , namely the projectivization of the  $\mathbb{C}^2$ -bundle

$$E^\wedge = \{(x, y) \in S^5 \times \mathbb{C}^3 \mid y \perp \mathbb{C}x\}.$$

The complement of  $E^\wedge$  in the trivial bundle  $S^5 \times \mathbb{C}^3$  is the trivial bundle

$$T^\wedge = \{(x, \lambda x) \mid x \in S^5, \lambda \in \mathbb{C}\} \cong S^5 \times \mathbb{C}.$$

Now we let  $S^1 = S \subset U(3)$  be a subgroup which acts freely on  $S^5$ . It is easy to see that up to conjugation there are only two such subgroups, namely  $Z$  and  $Z'$ . The group  $S$  acts on  $E^\wedge$  by bundle isomorphisms. Hence,  $E = E^\wedge/S$  is a  $\mathbb{C}^2$ -bundle over  $S^5/S$ , and  $M := F^\wedge/S = S \backslash U(3)/U(1)^2$  is the projectivization of  $E$ . Moreover,  $T = T^\wedge/S$  is the trivial bundle  $(S^5/S) \times \mathbb{C}$  since  $s(x, \lambda x) = (sx, \lambda sx)$  for any  $s \in S$ , i.e. the  $S$ -action fixes the  $\mathbb{C}$ -factor. Therefore,  $E$  is stable equivalent to the  $\mathbb{C}^3$ -bundle  $E_0 = (S^5 \times \mathbb{C}^3)/S$ . If  $S = Z$ , this is 3-times the inverse Hopf bundle over  $S^5/Z = \mathbb{C}P^2$ :  $E_0 = \bar{H} \oplus \bar{H} \oplus \bar{H}$  where  $\bar{H} = (S^5 \times \mathbb{C})/S^1$  with the  $S^1$ -action  $z(v, \lambda) = (zv, \lambda v)$  on  $S^5 \times \mathbb{C}$ . If  $S = Z'$ , we note that the map

$$\alpha : S^5 \rightarrow S^5, \quad \alpha(x_1, x_2, x_3) = (x_1, x_2, \bar{x}_3)$$

conjugates the actions of  $Z'$  and  $Z$  on  $S^5$ . Thus the map

$$\alpha \times \text{id} : S^5 \times \mathbb{C}^3 \rightarrow S^5 \times \mathbb{C}^3$$

descends to a bundle isomorphism between  $E_0$  over  $S^5/Z'$  and  $\bar{H} \oplus \bar{H} \oplus H$  over  $S^5/Z = \mathbb{C}P^2$ , where  $H = (S^5 \times \mathbb{C})/S^1$  with the action  $z(v, \lambda) = (zv, \bar{\lambda}v)$  on  $S^5 \times \mathbb{C}$ . Thus we have shown:

**Theorem 2.** *The manifolds  $F$  and  $F'$  are diffeomorphic to the projectivizations of  $\mathbb{C}^2$ -bundles over  $\mathbb{C}P^2$  which are stably equivalent to  $\bar{H} \oplus \bar{H} \oplus \bar{H}$  and  $\bar{H} \oplus \bar{H} \oplus H$ .*

Now it is easy to compute the cohomology of these manifolds. In fact, by the the Leray-Hirsch theorem, the (integer) cohomology ring of the projectivization  $PE$  of a  $\mathbb{C}^2$ -bundle  $E$  over  $\mathbb{C}P^2$  is a truncated polynomial algebra over  $H^*(\mathbb{C}P^2)$ , namely

$$H^*(PE) = H^*(\mathbb{C}P^2)[y]/(y^2 + c_1(E)y + c_2(E))$$

where  $c_i(E) \in H^{2i}(\mathbb{C}P^2)$  are the Chern classes of  $E$  and  $y \in H^2(PE)$  is the first Chern class of the inverse Hopf bundle over  $PE$  (cf. [4, p. 270]). Since  $c_1(\bar{H}) = x$  where  $x$  denotes the canonical generator of  $H^*(\mathbb{C}P^2)$ , we get

$$c(\bar{H} \oplus \bar{H} \oplus H) = (1+x)(1+x)(1-x) = 1+x-x^2$$

so in this case,  $c_1(E) = x$ ,  $c_2(E) = -x^2$  and therefore

$$H^*(F') = \mathbb{Z}[x, y]/(y^2 + xy - x^2, x^3).$$

Using the substitution  $x = u$ ,  $y = u + v$ , we receive

$$H^*(F') = \mathbb{Z}[u, v]/(u^2 + 3uv + v^2, u^3).$$

In the other case we get  $c_1(E) = 3x$ ,  $c_2(E) = 3x^2$  which shows

$$H^*(F) = \mathbb{Z}[x, y]/(y^2 + 3xy + 3x^2, x^3).$$

With the substitution  $x = u$ ,  $y = v - u$  this becomes

$$H^*(F) = \mathbb{Z}[u, v]/(u^2 + uv + v^2, u^3).$$

Note that the quadratic form  $u^2 + uv + v^2$  is positive definite while  $x^2 + xy - y^2$  is indefinite, so  $F$  and  $F'$  have nonisomorphic cohomology rings already over the reals. This shows that  $F'$  is not homotopic to any other known space which carries a metric of positive curvature.

**Remark 1.** Alternatively, the bundle  $PE$  can also be considered as the unit sphere bundle of an  $\mathbb{R}^3$ -bundle  $V$  over  $\mathbb{C}P^2$ . These bundles are classified by the first Pontrjagin class  $p_1(V)$ . It is not difficult to compute that

$$p_1(V) = c_1(E)^2 - 4c_2(E).$$

In particular, the Pontrjagin numbers of the  $\mathbb{R}^3$ -bundles corresponding to  $F'$  and  $F$  are 5 and  $-3$ .

**Remark 2.** The cohomology of  $SU(3)/U_{a,b}$  has been computed in [8, 9]. The invariant which distinguishes these spaces is the order  $r = \frac{1}{2}(|\|a\|^2 - \|b\|^2|)$  of the torsion group  $H^5$ . So by Remark 3 in Section 3 we have

$$r(M'_{p,q}) = p^2 + q^2 + 3pq,$$

$$r(M_{p,q}) = p^2 + q^2 + pq.$$

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## References

- [1] S. Aloff and N.L. Walach, An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures, *Bull. Amer. Math. Soc.* **81** (1975) 93–97.
- [2] L. Berard Bergery, Les variétés Riemanniennes homogènes simplement connexes de dimension impair à courbure strictement positive, *J. Math. Pures Appl.* **55** (1976) 47–68.
- [3] M. Berger, Les variétés Riemanniennes homogènes normales simplement connexes à courbure strictement positive, *Ann. Scuola Norm. Sup. Pisa* **15** (1961) 179–246.
- [4] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology* (Springer, Berlin, 1982).
- [5] J. Cheeger and D.G. Ebin, *Comparison Theorems in Riemannian Geometry* (North Holland, Amsterdam, 1975).
- [6] H.I. Eliasson, Die Krümmung des Raumes  $Sp(2)/SU(2)$  von Berger, *Math. Ann.* **164** (1966) 317–323.
- [7] J.H. Eschenburg, New examples of manifolds with strictly positive curvature, *Invent. Math.* **66** (1982) 469–480.
- [8] J.H. Eschenburg, Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen, *Schr. Math. Inst. Univ. Münster* **32** (2) (1984).
- [9] J.H. Eschenburg, Cohomology of biquotients, *Manuscripta Math.*, to appear.
- [10] D. Gromoll and W.T. Meyer, An exotic sphere with nonnegative sectional curvature, *Ann. of Math.* **100** (1974) 401–406.
- [11] E. Heintze, The curvature of  $SU(5)/(Sp(2) \times S^1)$ , *Invent. Math.* **13** 205–212.
- [12] B. O'Neill, The fundamental equations of a submersion, *Michigan Math. J.* **23** (1966) 459–469.
- [13] H. Samelson, On curvature and characteristic of homogeneous spaces, *Michigan Math. J.* **5** (1958) 13–18.
- [14] N.L. Walach, Compact homogeneous Riemannian manifolds with strictly positive curvature, *Ann. of Math.* **96** (1972) 277–295.