Minimal and Maximal Lyapunov Exponents of Bilinear Control Systems

FRITZ COLONIUS*

Institut für Mathematik, Universität Augsburg, 8900 Augsburg, Germany

AND

WOLFGANG KLIEMANN*

Department of Mathematics, Iowa State University, Ames, Iowa 50011

For a bilinear control system with bounded and unbounded controls

$$\dot{x} = A_0(u_0)\,x + \sum_{i=1}^{m} u_i A_i(u_0)\,x \qquad \text{in } \mathbb{R}^d,$$

where $u_0(t) \in \Omega \subset \mathbb{R}^l$ compact, $u_i(t) \in \mathbb{R}$ for $i = 1, \ldots, m$, the extremal exponential growth rates of the solutions $x(\cdot, x_0, u)$ are analyzed: If $\lambda(x_0, u) = \limsup_{t \to \infty} (1/t) \log |x(t, x_0, u)|$, then $\mathscr{K} = \sup_{u \in \mathscr{U}} \sup_{x_0 \neq 0} \lambda(x_0, u)$ and $\mathscr{K}^* = \inf_{u \in \mathscr{U}} \inf_{x_0 \neq 0} \lambda(x_0, u)$ are the maximal (and minimal, respectively) Lyapunov exponents of the system. This paper gives several characterizations of these rates, together with the corresponding uniform concepts (with respect to the initial value or the control). We describe the situations, in which $\mathscr{K} = +\infty$ and $\mathscr{K}^* = -\infty$, and characterize the sets of initial values, from which $\mathscr{K}$ and $\mathscr{K}^*$ can actually be realized. The techniques are applied to high gain stabilization, and the example of the linear oscillator with parameter controlled restoring force is treated in detail. Finally we indicate how the results can be used for feedback stabilization of linear systems, when the feedback is allowed to be time varying, but restricted to certain types of (bounded) gain matrices.

1. INTRODUCTION

The problem of destabilization and, in particular, of stabilization of bilinear (and more generally of nonlinear) systems has attracted a great deal of interest since the first systematic study by Mohler [Mo]. The

* Work partially supported by NSF Grant DMS-8813976 and DFG Grant Co124/6-1.

problem has been approached basically with two methods: via feedback (using only specific, a priori given classes of feedbacks), and via high gain techniques. For an outline of the first approach we refer, e.g., to Gauthier [Ga], Yong [Y], Rotea and Khargonekar [RK], Jacobson [J], Gutman [Gu], and Koditschek and Narenda [KN]. For problems without constraints in the controls, Isidori [I, Chapt. 7] describes a general approach.

High gain techniques have been developed, e.g., by Meerkov *et al.* [Me, BBM1, BBM2] using the idea of vibrational stabilization, or Knobloch [Kn], who uses a theorem on the global existence of center manifolds and the theory of singular perturbations. The recent monograph by Kokotovic *et al.* [KKO] contains some interesting ideas on this topic.

This paper introduces a different approach to the analysis of bilinear control systems with control constraints. It is based on Lyapunov exponents associated with open loop controls: The possibility of (exponential) stabilization (or destabilization) in a control system depends on the minimal (or maximal, resp.) exponential growth rate that can be realized in the system. For example, if the minimal growth rate is negative then there exists an (open loop) control such that the system response, using this control, decays exponentially. It seems therefore desirable, to characterize the extremal growth rates of a bilinear control system. These rates are the minimal (and maximal resp.) Lyapunov exponents, which generalize the real parts of the eigenvalues of the matrix A in the linear, autonomous equation $\dot{x} = Ax$.

More precisely, the following setup is used in this paper: Consider the bilinear control system with bounded and unbounded controls

$$\dot{x}(t) = A_0(u_0(t))\,x(t) + \sum_{i=1}^{m} u_i(t)\,A_i(u_0(t))\,x(t) \qquad \text{in } \mathbb{R}^d, \tag{1.1}$$

where

$$u_0\colon \mathbb{R} \to \Omega \subset \mathbb{R}^l, \qquad \Omega \text{ compact},$$

$$A_j\colon \Omega \to gl(d, \mathbb{R}) \text{ (the } d \times d \text{ matrices) is analytic for } j = 0, ..., m,$$

$$u_i\colon \mathbb{R} \to \mathbb{R} \qquad \text{for} \quad i = 1, ..., m.$$

Denote

$$U = \Omega \times \mathbb{R}^m, \text{ and choose as admissible controls}$$

$$\mathscr{U} = \{u\colon \mathbb{R} \to U, \text{ locally integrable}\},$$

such that for each $x_0 \in \mathbb{R}^d$ and each $u \in \mathscr{U}$ Eq. (1.1) has a unique solution, denoted by $x(t, x_0, u)$, for $t \in \mathbb{R}$.

Define the Lyapunov exponents of the solutions of (1.1) by

$$\lambda(x_0, u) = \limsup_{t \to \infty} \frac{1}{t} \log |x(t, x_0, u)|. \tag{1.2}$$

In this paper we analyze the maximal and the minimal Lyapunov exponents of the bilinear control system (1.1):

$$\mathscr{K} := \sup_{u \in \mathscr{U}} \sup_{x_0 \neq 0} \lambda(x_0, u), \qquad \mathscr{K}^* := \inf_{u \in \mathscr{U}} \inf_{x_0 \neq 0} \lambda(x_0, u), \tag{1.3}$$

the uniform concepts (with respect to the initial value)

$$\hat{\mathscr{K}} := \sup_{u \in \mathscr{U}} \inf_{x_0 \neq 0} \lambda(x_0, u), \qquad \hat{\mathscr{K}}^* := \inf_{u \in \mathscr{U}} \sup_{x_0 \neq 0} \lambda(x_0, u), \tag{1.4}$$

and the uniform growth rates (with respect to the controls)

$$\tilde{\mathscr{K}} := \sup_{x_0 \neq 0} \inf_{u \in \mathscr{U}} \lambda(x_0, u), \qquad \tilde{\mathscr{K}}^* := \inf_{x_0 \neq 0} \sup_{u \in \mathscr{U}} \lambda(x_0, u). \tag{1.5}$$

Since we allow unbounded controls in (1.1), $\mathscr{K} < +\infty$ and $\mathscr{K}^* > -\infty$ will occur only in certain degenerate situations (iff there exists a nonsingular matrix $T \in Gl(d, \mathbb{R})$ such that $T\{A_i(u_0), u_0 \in \Omega, i = 1, ..., m\} T^{-1} \subset so(d, \mathbb{R})$, i.e., the system matrices of the unbounded part can be made simultaneously skew symmetric; see Theorem 4.5). The main emphasis is then on bounded controls, and we show that the concepts in (1.3) and (1.4) are related to the exponential growth of the norm, the conorm, the spectral radius, and the spectral coradius of the systems semigroup (Theorems 4.1 and 4.10).

The extremal Lyapunov exponents, as defined above, involve suprema (and infima, resp.) over all $x_0 \in \mathbb{R}^d \setminus \{0\}$. It is therefore important to characterize the initial values $x_0 \neq 0$ for which these growth rates can be realized. Since $\lambda(\alpha x_0, u) = \lambda(x_0, u)$ for $\alpha \in \mathbb{R}$, $\alpha \neq 0$, it suffices to consider $s_0 \in \mathbb{P}$, the projective space obtained from $\mathbb{R}^d$, and it turns out that the controllability properties of the projected system of (1.1) onto $\mathbb{P}$ play a crucial role in determining the appropriate initial values: Under a weak nondegeneracy assumption (cf. (H) in Sect. 2), $\mathscr{K}$ can be realized for all $x_0 \neq 0$ (and hence $\mathscr{K} = \tilde{\mathscr{K}}^*$), while $\mathscr{K}^*$ is in general attainable only for a certain subset of initial values, which is characterized in Theorem 5.1. Of course, all quantities in (1.3)–(1.5) can be realized from all $x_0 \neq 0$ if the projected system is completely controllable on $\mathbb{P}$ (which is a weaker requirement than contrallability of (1.1) in $\mathbb{R}^d \setminus \{0\}$), and then $\mathscr{K}^* = \tilde{\mathscr{K}}$.

If $\mathscr{K}^* < 0$, then there exist $x_0 \in \mathbb{R}^d$ and $u \in \mathscr{U}$ such that $\lambda(x_0, u) < 0$, i.e., system (1.1) is exponentially stabilizable from x_0 via the (open loop) control u. The existence of feedback stabilization at the exponential rate $\mathscr{K}^*$ is not addressed in this paper, because the solution of that problem requires a complete characterization of the control properties of the projected system on $\mathbb{P}$, and of the complete Lyapunov spectrum $\{\lambda(x_0, u), x_0 \neq 0, u \in \mathscr{U}\} \subset \mathbb{R}$ of (1.1). The problem is currently under investigation and requires quite different techniques.

It may be interesting to point out, how the results of this paper can be used for feedback stabilization and robustness analysis of *linear* systems.

Consider the linear system

$$\dot{x} = Ax + Bu \qquad \text{in } \mathbb{R}^d,$$

for which we want to construct a time varying linear feedback $F(t)$ in a given class of (bounded) matrices, in order to stabilize (or destabilize) the system. Assume that the feedback matrix $F(t)$ has to satisfy constraints for its elements. Then the closed loop system

$$\dot{x} = (A + BF(t))\,x, \qquad F(t) \in V, \text{ a set of constriants,}$$

is of the form (1.1). For stabilization we have to distinguish two cases:

(a) The projected system is controllable on $\mathbb{P}$: Then $\mathscr{K}^* < 0$ means that for each initial value x_0 there exists $F(\cdot)$, such that $|x(t, x_0, F(\cdot))|$ decays exponentially, and $\tilde{\mathscr{K}}^* > 0$ means that there exists a (global) $F(\cdot)$ such that $\dot{x} = (A + BF(\cdot))\,x$ is stable for all $x_0 \in \mathbb{R}^d$.

(b) The projected system is not controllable on $\mathbb{P}$: Here $\mathscr{K}^* < 0$ indicates that for certain initial values x_0 (characterized in Theorem 5.1) a stabilizing $F(\cdot)$ can be found, while $\tilde{\mathscr{K}}^* < 0$ again means that the system is stabilizable with some admissible feedback $F(\cdot)$.

For destabilization we obtain: If $\mathscr{K} > 0$, then for each $x_0 \neq 0$ there exists a $F(\cdot)$ such that $|x(t, x_0, F(\cdot))|$ is exponentially increasing, and $\hat{\mathscr{K}} > 0$ implies the existence of a global destabilizing feedback $F(\cdot)$.

An example is the linear, additively controlled oscillator

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \qquad \text{in } \mathbb{R}^2.$$

For unbounded u this system is completely controllable in $\mathbb{R}^2$. With time varying (bounded) feedback of the type $f(t) = (f(t)\;0)$ the system reads

$$\dot{x} = \left[\begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} (f(t)\;0)\right] x, \qquad f(t) \in [A, B] \subset \mathbb{R}.$$

Depending on the constraints interval $[A, B]$, this system (or even its projection onto the projective space $\mathbb{P}$) may not be controllable. All extremal Lyapunov exponents for this example are computed in Section 6, and, by the above, the results can be interpreted as feedback stabilization (for $b<0$) or destabilization (for $b>0$) results in the context of the linear system.

For the robustness analysis of linear differential equations $\dot{x}=Ax$ the stability radius of a matrix plays an important role; see, e.g., [HP1, HP2, HIP] or [VL]. This concept, and also various types of instability radii, are closely related to the extremal Lyapunov exponents of bilinear systems: Let $\{U_\rho, \rho \geqslant 0\}$ be an increasing family of subsets of $gl(d, \mathbb{R})$, expressing the uncertainty (of size ρ) about the parameters in a matrix $A \in gl(d, \mathbb{R})$.

Consider for a stable matrix A the bilinear control system

$$\dot{x}=(A+u_\rho(t))\,x \qquad \text{in } \mathbb{R}^d,\ u_\rho(t)\in U_\rho,$$

and define $\mathscr{K}_\rho := \sup_{u_\rho \in \mathscr{U}_\rho} \sup_{x_0 \neq 0} \lambda(x_0, u_\rho)$. Then the (Lyapunov) stability radius of A with respect to time varying uncertainties is given by

$$r_L(A)=\inf\{\rho \geqslant 0,\ \mathscr{K}_\rho \geqslant 0\}.$$

For the linear oscillator, this stability radius is discussed in Section 6. For a general discussion of the relation between Lyapunov exponents and stability, as well as instability radii, using the extremal growth rates $\mathscr{K}$, $\mathscr{K}^*$, and $\tilde{\mathscr{K}}$, see [CK2].

In Section 2 we introduce some techniques and preliminary results. Growth concepts for linear semigroups in $Gl(d, \mathbb{R})$ are introduced in Section 3, and are used in Section 4 to characterize the extremal Lyapunov exponents. There also some results on the magnitude of the exponents are derived. Section 5 concentrates on the characterization of the points, from which system (1.1) can be (de-)stabilized in an optimal fashion, and the example of the linear parameter controlled oscillator is treated in great detail in Section 6. We conclude the paper with some applications of this theory to high gain stabilization.

2. Techniques and Preliminary Results

In this section we will briefly outline three major tools used in this paper: geometric control theory, time reversal, and reduction to periodic controls.

The individual Lyapunov exponents $\lambda(x_0, u)$, describing the radial behavior of the solutions of (1.1), can be expressed in terms of the angular behavior by projecting the system onto the projective space $\mathbb{P}$ in $\mathbb{R}^d$, an idea going back to Bogolyubov. A simple application of the chain rule shows that the projected system reads, with $s=x/|x| \in \mathbb{P}$

$$\dot{s}(t) = h(u(t), s(t)) = h_0(u, s) + \sum_{i=1}^{m} u_i h_i(u, s)$$

$$\begin{aligned} &\text{with } h_j(u, s) = (A_j(u_0) - q_j(u_0, s) \cdot \mathrm{Id})\, s, \qquad j = 0, ..., m, \\ &\quad q_j(u_0, s) = s^T A_j(u_0)\, s, \\ &\quad q(u, s) = q_0(u_0, s) + \sum_{i=1}^{m} u_i q_i(u_0, s), \end{aligned} \tag{2.1}$$

where ^{T} denotes transposition and Id is the $d \times d$ identity matrix.

We have

$$|x(t, x_0, u)| = |x_0| \exp\left(\int_0^t q(u(\tau), s(\tau, s_0, u))\, d\tau\right) \tag{2.2}$$

and therefore

$$\lambda(x_0, u) = \limsup_{t \to \infty} \frac{1}{t} \int_0^t q(u(\tau), s(\tau, s_0, u))\, d\tau. \tag{2.3}$$

Thus computing $\mathscr{K}$ and $\mathscr{K}^*$ means solving an optimal control problem for system (2.1) on $\mathbb{P}$ with the "average cost" functional (2.3).

The control structure of system (2.1) is crucial for the following analysis. Denote by $\mathscr{L}$ the Lie algebra generated by the systems vector fields, i.e.,

$$\mathscr{L} = \mathscr{L}\mathscr{A}\{h(u, \cdot), u = (u_0, u_1, ..., u_m) \in U\}.$$

By assumption on the A_j, $j = 0, ..., m$, the system is analytic; hence the distribution $\Delta_{\mathscr{L}}$ generated by $\mathscr{L}$ in $T\mathbb{P}$, the tangent bundle of $\mathbb{P}$, is completely integrable by Nagano's theorem (see, e.g., [I]). Chow's theorem implies that (2.1) "lives" on the maximal integral manifolds of $\Delta_{\mathscr{L}}$. In order to exclude degenerate situations, we will assume throughout this paper

$$\Delta_{\mathscr{L}}(p) = T_p\mathbb{P} \qquad \text{for all} \quad p \in \mathbb{P}, \tag{H}$$

i.e., the Lie algebra $\mathscr{L}$ has full rank on $\mathbb{P}$. Hypothesis (H) implies local accessibility, i.e.:

For $t > 0$, $p_0 \in \mathbb{P}$ let

$$\mathcal{O}^+_{\leqslant t}(p_0) := \{y \in \mathbb{P}, \text{there exists } 0 \leqslant \tau \leqslant t \text{ and } u \in \mathscr{U} \text{ with } s(\tau, p_0, u) = y\};$$

then $\operatorname{int} \mathcal{O}^+_{\leqslant t}(p_0) \neq \varnothing$ for all $p_0 \in \mathbb{P}$, $t > 0$ and similarly for $\mathcal{O}^-_{\leqslant t}(p_0)$ (see, e.g., [I]). $\mathcal{O}^+(p_0) = \bigcup_{t \geqslant 0} \mathcal{O}^+_{\leqslant t}(p_0)$ will denote the positive (i.e., forward in time) orbit of the point p_0.

For the investigation of the extremal Lyapunov exponents of (1.1), we will make use of the time reversed systems corresponding to (1.1) and (2.1): If we solve either equation from $t=0$ backwards in time, and set $x^*(t)=x(-t)$, $s^*(t)=s(-t)$, $u^*(t)=u(-t)$ we arrive at

$$\dot{x}^*(t) = -\left(A_0(u_0^*(t))\, x^*(t) + \sum_{i=1}^{m} u_i^*(t)\, A_i(u_0^*(t))\, x^*(t)\right) \tag{2.4}$$

$$\dot{s}^*(t) = -h(u^*(t), s^*(t)). \tag{2.5}$$

Again we consider (2.4) and (2.5) as control systems with $t \in \mathbb{R}$, and with controls $u^* \in \mathcal{U}^* = \mathcal{U}$. Then (2.5) is the projection of (2.4) and for the corresponding functions q^*, h^* one obtains $q^* = -q$, $h^* = -h$. If we denote the (forward) orbit of (2.5) from $p \in \mathbb{P}$ by $\mathcal{O}^{*+}_{\leqslant t}(p)$, then it is easy to see that $\mathcal{O}^{*+}_{\leqslant t}(p) = \mathcal{O}^-_{\leqslant t}(p)$. Furthermore, (H) holds for (2.1) iff it holds for (2.5).

2.1. *Remark.* Associated with (1.1) is also the adjoint system $\dot{y} = -(A_0^T(u_0)\, y + \sum u_i A_i^T(u_0)\, y)$, whose projection onto $\mathbb{P}$ is $\dot{s} = -h(u^T, s)$, with $h_j(u^T, s) = (A_j^T(u_0) - s^T A_j^T(u_0)\, s \cdot \mathrm{Id})\, s$. Now $s^T A s = s^T A^T s$, and hence by (2.3) the individual Lyapunov exponents of the adjoint system are the same as those of the time reversed system (2.4).

The Lie algebra condition (H) does in general not imply that (2.1) or (2.5) is completely controllable on $\mathbb{P}$. But there will always be subsets of $\mathbb{P}$, on which controllability holds.

2.2. DEFINITION. A set $D \subset \mathbb{P}$ is called a *control set* for (2.1), if

(i) for all $p \in D$ there exists $u \in \mathcal{U}$ such that $s(t, p, u) \in D$ for all $t \geqslant 0$,

(ii) for all $p \in D$ we have $\overline{\mathcal{O}^+(p)} \supset \bar{D}$,

and if D is maximal with respect to this property. If furthermore, $\overline{\mathcal{O}^+(p)} = \bar{D}$ for all $p \in D$, then D is an *invariant cotrol set.* (For the relation of this definition to other notions of control sets see Remark 3.2 in [CK3].)

The following properties of invariant control sets of (2.1) and (2.5) play a crucial role:

2.3. PROPOSITION. *Consider system* (2.1) *on* $\mathbb{P}$ (*under assumption* (H)). *Then*

(i) *there is exactly one invariant control set* $C = \bigcap_{p \in \mathbb{P}} \overline{\mathcal{O}^+(p)}$, *C is compact with* int $C \neq \varnothing$, *and* $\mathcal{O}^+(p) \supset \operatorname{int} C$ *for all* $p \in C$;

(ii) *there is exactly one open control set* C^-, $C^- = \operatorname{int} \bigcap_{p \in \mathbb{P}} \overline{\mathcal{O}^-(p)} = \operatorname{int} C^*$, *and* C^* *is the unique invariant control set of* (2.5);

(iii) *if* (2.1) *is completely controllable on* $\mathbb{P}$, *then* $C = C^- = \mathbb{P}$; *otherwise* $C \cap C^- = \varnothing$.

Note, however, that $C \cap \overline{C^-} \neq \varnothing$ *is possible, see Section 6.*

Proof. (i) This is Theorem 3.1 in [AKO] and Lemma 2.1 in [Kl].

(ii) Consider the time reversed system (2.5): By (i) there exists a unique invariant control set $C^* = \bigcap_{p \in \mathbb{P}} \overline{\mathcal{O}^{*+}(p)} = \bigcap_{p \in \mathbb{P}} \overline{\mathcal{O}^-(p)}$ for (2.5). Since $\mathcal{O}^{*+}(p) \supset \operatorname{int} C^*$ for all $p \in C^*$, we have $\mathcal{O}^-(p) \supset \operatorname{int} C^* = C^-$ for all $p \in C^-$. Thus $\mathcal{O}^+(p) \supset C^-$ for all $p \in C^-$. To complete the proof, we first show: If $p \in C^* \backslash \operatorname{int} C^*$, then $p \notin D$, the (maximal) control set containing $\operatorname{int} C^*$. Indeed, if $\mathcal{O}^+(p) \cap C^- \neq \varnothing$, then there is a control $u \in \mathcal{U}$ such that $y = s(t, p, u) \in \operatorname{int} C^*$ for some $t > 0$. Hence there exists a neighborhood $V(p)$ such that $s(t, q, u) \in \operatorname{int} C^*$ for all $q \in V(p)$. Therefore $\mathcal{O}^-(y) = \mathcal{O}^{*+}(y) \cap (C^*)^c \neq \varnothing$, and C^* is not invariant for (2.5).

We conclude that all points $p \in \partial C^*$ are not in the (maximal) control set containing C^-. Because control sets are path connected, we obtain that C^- is a control set.

It remains to show: C^- is the only open control set of (2.1). The argument above says that, if D is an open control set of (2.1), then $\bar{D}$ is an invariant control set of (2.5). But $\bar{D}$ is unique by part (i).

(iii) If (2.1) is completely controllable on $\mathbb{P}$, then $\mathbb{P}$ is the only control set for (2.1) and (2.5), i.e., $C = C^- = \mathbb{P}$. Vice versa $C = \mathbb{P}$ implies complete controllability of (2.1) and thus of (2.5). Furthermore, because control sets are pairwise disjoint by maximality, $C \neq \mathbb{P}$ implies $C^- \neq \mathbb{P}$, and therefore $C \cap C^- = \varnothing$. ∎

Note that the control sets in Definition 2.2 are defined via the closures of orbits, and not through the orbits themselves. Nevertheless, Proposition 2.3 ensures that in the interior of the control sets C and C^- we have (precise) controllability. Definition 2.2 allows us to obtain the same control sets also for restricted classes of admissible controls; compare [I] and [S2]. We will make use of this fact, when we consider piecewise constant controls in Section 3.

Finally we introduce two restricted sets of controls that will turn out to be sufficient for the characterization of the extremal growth rates:

$$\mathcal{U}_p = \{u \colon \mathbb{R}^+ \to U, \text{ measurable and } T\text{-periodic for some } T > 0\}$$

$$\mathcal{U}_{pp} = \{u \colon \mathbb{R}^+ \to U, \text{ measurable and } (u(\cdot), s(\cdot, s_0, u)) \text{ } T\text{-periodic for some } T > 0 \text{ and some } s_0 \in \mathbb{P}\}. \tag{2.6}$$

The growth concepts in (1.3) and (1.4) can now be defined with respect to $\mathscr{U}_p$ and $\mathscr{U}_{pp}$, e.g.

$$\mathscr{K}_{pp} = \sup_{u \in \mathscr{U}_{pp}} \limsup_{t \to \infty} \frac{1}{t} \int_0^t q(u(\tau), s(\tau, s(0), u))\, d\tau; \tag{2.7}$$

here the sup is taken over all $u \in \mathscr{U}_{pp}$ and all initial values s_0 appearing in (2.6). Similarly for $\mathscr{K}_p, \mathscr{K}_p^*, \mathscr{K}_{pp}^*, \ldots$. To distinguish, when necessary, between the growth rates of (2.1) and the time reversed system (2.5), we will write $\mathscr{K}_{pp}(N)\ldots$ for the former and $\mathscr{K}_{pp}(-N)\ldots$ for the latter.

3. Exponential Growth Rates of Linear Control Semigroups

The solutions of control system (1.1) take a particularly simple form, if we allow only piecewise constant controls. This leads to the definition of the systems group $\mathscr{G}$ and semigroup $\mathscr{S}$ as follows:

Denote by N the possible constant right-hand sides of (1.1), i.e.,

$$N := \left\{ A_0(u_0) + \sum_{i=1}^{m} u_i A_i(u_0),\ u = (u_0, u_1, \ldots, u_m) \in U \right\} \subset gl(d, \mathbb{R}), \tag{3.1}$$

and define the systems group $\mathscr{G}$ and semigroup $\mathscr{S}$ by

$$\mathscr{G} := \{\exp(t_n B_n) \cdot \ldots \cdot \exp(t_1 B_1),\ t_j \in \mathbb{R},\ B_j \in N,\ j = 1, \ldots, n \in \mathbb{N}\} \subset Gl(d, \mathbb{R})$$

$$\mathscr{S} := \{\exp(t_n B_n) \cdot \ldots \cdot \exp(t_1 B_1),\ t_j \geqslant 0,\ B_j \in N,\ j = 1, \ldots, n \in \mathbb{N}\} \subset Gl(d, \mathbb{R}).$$

The subsets of the group and the semigroup at time t, i.e., $\sum |t_j| = t$, are denoted by $\mathscr{G}_t$ and $\mathscr{S}_t$, respectively. Note the following relationships between the system and the time reversed equations: $N^* = -N$, $\mathscr{G}^* = \mathscr{G}$, $\mathscr{S}^* = \mathscr{S}^{-1}$.

Instead of considering the Lyapunov exponents of the individual solutions of (1.1), we can measure the exponential growth behavior by looking at the growth rates of $\mathscr{S}_t \subset \mathscr{S}$. This can be done by either using any norm in $Gl(d, \mathbb{R})$ or by looking at the spectral radius: Define the norm $\|\cdot\|$, the conorm $m(\cdot)$, the spectral radius $r(\cdot)$, and the spectral coradius $\mathrm{cor}(\cdot)$ on $Gl(d, \mathbb{R})$ by

$$\|g\| := \max\{|gx|;\ |x| = 1\}, \qquad m(g) := \min\{|gx|;\ |x| = 1\}$$

$$r(g) := \max\{|\lambda|;\ \lambda \in \sigma(g)\}, \qquad \mathrm{cor}(g) := \min\{|\lambda|;\ \lambda \in \sigma(g)\},$$

where $\sigma(g)$ denotes the spectrum of g.

Note that all these quantities are strictly positive for all $g \in Gl(d, \mathbb{R})$. We now obtain the following concepts for extremal exponential growth rates of $\mathcal{S}$:

$$\begin{aligned}
\beta &:= \limsup_{t\to\infty} \frac{1}{t} \sup_{g\in\mathcal{S}_t} \log r(g), &\quad \beta^* &:= \liminf_{t\to\infty} \frac{1}{t} \inf_{g\in\mathcal{S}_t} \log \operatorname{cor}(g) \\
\hat{\beta} &:= \limsup_{t\to\infty} \frac{1}{t} \sup_{g\in\mathcal{S}_t} \log \operatorname{cor}(g), &\quad \hat{\beta}^* &:= \liminf_{t\to\infty} \frac{1}{t} \inf_{g\in\mathcal{S}_t} \log r(g) \\
\delta &:= \limsup_{t\to\infty} \frac{1}{t} \sup_{g\in\mathcal{S}_t} \log \|g\|, &\quad \delta^* &:= \limsup_{t\to\infty} \frac{1}{t} \inf_{g\in\mathcal{S}_t} \log m(g) \\
\hat{\delta} &:= \limsup_{t\to\infty} \frac{1}{t} \sup_{g\in\mathcal{S}_t} \log m(g), &\quad \hat{\delta}^* &:= \liminf_{t\to\infty} \frac{1}{t} \inf_{g\in\mathcal{S}_t} \log \|g\|.
\end{aligned} \tag{3.2}$$

Here $\beta(\hat{\beta}, \delta, \hat{\delta})$ are understood to be $+\infty$, if for some $t < \infty$ we have that $\sup_{g\in\mathcal{S}_t} \log r(g) = \infty$ (or $\sup_{g\in\mathcal{S}_t} \log \operatorname{cor}(g) = \infty$, etc.); similarly for the quantities β^*, $\hat{\beta}^*$, δ^*, $\hat{\delta}^*$, and $-\infty$. This section is devoted to the analysis of the concepts defined in (3.2).

3.1. LEMMA (On Time Reversal). *Denote again by $\beta(N)$... the growth rates of the system* (1.1), *and by $\beta(-N)$... the corresponding quantities of the time reversed system* (2.4). *Then we have*

(i) $\delta^*(N) = -\delta(-N)$

(ii) $\beta^*(N) = -\beta(-N)$

(iii) $\hat{\delta}^*(N) = -\hat{\delta}(-N)$

(iv) $\hat{\beta}^*(N) = -\hat{\beta}(-N)$.

Proof. Note first that for all $g \in Gl(d, \mathbb{R})$ we have

$$\|g\| = \max_{x\neq 0} \frac{|gx|}{|x|} = \max_{y\neq 0} \frac{|y|}{|g^{-1}y|} = \frac{1}{\min_{y\neq 0}(|g^{-1}y|/|y|)} = \frac{1}{m(g^{-1})}.$$

Furthermore, $\lambda \in \sigma(g)$ is equivalent to $\lambda^{-1} \in \sigma(g^{-1})$; hence

$$\begin{aligned}
r(g) &= \max\{|\lambda|, \lambda \in \sigma(g)\} = \max\{|\lambda^{-1}|, \lambda \in \sigma(g^{-1})\} \\
&= \frac{1}{\min\{|\lambda|, \lambda \in \sigma(g^{-1})\}} = \frac{1}{\operatorname{cor}(g^{-1})}.
\end{aligned}$$

Proof of (i):

$$\frac{1}{t}\inf_{g\in\mathcal{S}_t}\log m(g)=\frac{1}{t}\inf_{g\in\mathcal{S}_t}\log\|g^{-1}\|^{-1}$$
$$=-\frac{1}{t}\sup_{g\in\mathcal{S}_t}\log\|g^{-1}\|$$
$$=-\frac{1}{t}\sup_{g\in\mathcal{S}_t^*}\log\|g\|;$$

hence $\delta^*(N)=\liminf_{t\to\infty}(1/t)\inf_{g\in\mathcal{S}_t}\log m(g)=-\limsup_{t\to\infty}(1/t)\sup_{g\in\mathcal{S}_t^*}\log\|g\|=-\delta(-N)$. All other equalities are proved in exactly the same way. ∎

3.2. LEMMA. *All quantities in (3.1) are actually limits. Furthermore*

$$\beta=\sup_{t>0}\frac{1}{t}\sup_{g\in\mathcal{S}_t}\log r(g),\qquad \beta^*=\inf_{t>0}\frac{1}{t}\inf_{g\in\mathcal{S}_t}\log\operatorname{cor}(g)$$

$$\hat\beta=\sup_{t>0}\frac{1}{t}\sup_{g\in\mathcal{S}_t}\log\operatorname{cor}(g),\qquad \hat\beta^*=\inf_{t>0}\frac{1}{t}\inf_{g\in\mathcal{S}_t}\log r(g)$$

$$\delta=\inf_{t>0}\frac{1}{t}\sup_{g\in\mathcal{S}_t}\log\|g\|,\qquad \delta^*=\sup_{t>0}\frac{1}{t}\inf_{g\in\mathcal{S}_t}\log m(g)$$

$$\hat\delta=\sup_{t>0}\frac{1}{t}\sup_{g\in\mathcal{S}_t}\log m(g),\qquad \hat\delta^*=\inf_{t>0}\frac{1}{t}\inf_{g\in\mathcal{S}_t}\log\|g\|.$$

Again, if one of the quantities above for some $t<\infty$ is $+\infty$ (or $-\infty$), we define the corresponding growth rate to be $+\infty$ (or $-\infty$, respectively).

Proof. For the β-quantities recall that $r(g^n)=r(g)^n$, and therefore

$$\beta(nt):=\frac{1}{nt}\sup_{g\in\mathcal{S}_{nt}}\log r(g)\geqslant\frac{1}{nt}\sup_{g\in\mathcal{S}_t}\log r(g)^n$$
$$=\frac{1}{t}\sup_{g\in\mathcal{S}_t}\log r(g)=:\beta(t)$$

for all $n\in\mathbb{N}$, $t>0$. Furthermore, $\beta(t)$ is obviously continuous (if it is finite for all $t>0$), and hence $\beta=\limsup_{t\to\infty}\beta(t)=\sup_{t>0}\beta(t)=\lim_{t\to\infty}\beta(t)$.

Similarly we obtain for $\hat\beta$

$$\begin{aligned}\hat{\beta}(nt) &:= \frac{1}{nt} \sup_{g \in \mathcal{S}_{nt}} \log \operatorname{cor}(g) = \frac{1}{nt} \sup_{g \in \mathcal{S}_{nt}} \log \frac{1}{r(g^{-1})} \\ &\geqslant \frac{1}{nt} \sup_{g \in \mathcal{S}_t} \log \left(\frac{1}{r(g^{-1})}\right)^n \\ &= \frac{1}{t} \sup_{g \in \mathcal{S}_t} \log \operatorname{cor}(g) =: \hat{\beta}(t),\end{aligned}$$

which implies $\hat{\beta} = \limsup_{t \to \infty} \hat{\beta}(t) = \sup_{t > 0} \hat{\beta}(t) = \lim_{t \to \infty} \hat{\beta}(t)$. The results for β^* and $\hat{\beta}^*$ follow via time reversal from Lemma 3.1.

For the δ-quantities recall that $\|g_1 \cdot g_2\| \leqslant \|g_1\| \, \|g_2\|$ for all $g_1, g_2 \in Gl(d, \mathbb{R})$. All $g \in \mathcal{S}_{s+t}$ are of the form $g = g_1 g_2$, with $g_1 \in \mathcal{S}_s$, $g_2 \in \mathcal{S}_t$. Therefore

$$\begin{aligned}(s+t) \cdot \delta(s+t) &:= \sup_{g \in \mathcal{S}_{s+t}} \log \|g\| = \sup_{g_1 \in \mathcal{S}_s, g_2 \in \mathcal{S}_t} \log \|g_1 \cdot g_2\| \\ &\leqslant \sup_{g_1 \in \mathcal{S}_s} \log \|g_1\| + \sup_{g_2 \in \mathcal{S}_t} \log \|g_2\| \\ &=: s\delta(s) + t\delta(t).\end{aligned}$$

Thus $t\delta(t)$ is subadditive, and again continuous (if finite for all $t > 0$). This implies

$$\delta = \limsup_{t \to \infty} \delta(t) = \inf_{t > 0} \delta(t) = \lim_{t \to \infty} \delta(t) \qquad \text{(see, e.g., [D, p. 14]).}$$

Similarly, the function $t\hat{\delta}^*(t) := \inf_{g \in \mathcal{S}_t} \log \|g\|$ is subadditive, which gives the result for $\hat{\delta}^*$. The equations for δ^* and $\hat{\delta}$ follow again via time reversal from Lemma 3.1. ∎

In Section 4 we will relate the extremal Lyapunov exponents defined in Sections 1 and 2 to the semigroup concepts (3.2). Here we discuss the finiteness of these concepts.

Denote the unbounded part of the right-hand sides of (1.1) by

$$N_0 := \left\{ \sum_{i=1}^{m} u_i A_i(u_0), \, u \in U \right\} \tag{3.3}$$

and let $\mathcal{H}$ be the corresponding systems group.

3.3. Proposition. (i) *$\beta, \hat{\beta}, \delta, \hat{\delta} > -\infty$, and $\beta^*, \hat{\beta}^*, \delta^*, \hat{\delta}^* < +\infty$.*

(ii) $\beta = +\infty$ iff $\delta = +\infty$ iff $\bar{\mathcal{H}}$ is not compact, and $\beta^* = -\infty$ iff $\delta^* = -\infty$ iff $\bar{\mathcal{H}}$ is not compact.

3.4. *Remark.* Recall that a closed subgroup $\mathcal{H}$ of $Gl(d, \mathbb{R})$ is compact if and only if there exists a basis transformation $T \in Gl(d, \mathbb{R})$ such that $T\bar{\mathscr{H}}T^{-1} \subset SO(d, \mathbb{R})$, the special orthogonal group, or, equivalently in our setup, iff $TN_0T^{-1} \subset so(d, \mathbb{R})$, the skew symmetric matrices.

Proof. (i) $\beta, \hat{\beta}, \hat{\delta} > -\infty$ follows directly from Lemma 3.2. $\beta \leqslant \delta$ can be seen as follows: Recall that $r(g) = \lim_{n \to \infty} \|g^n\|^{1/n}$ for $g \in Gl(d, \mathbb{R})$. Hence for $g \in \mathscr{S}_t$

$$\frac{1}{t} \log r(g) = \lim_{n \to \infty} \frac{1}{nt} \log \|g^n\| \leqslant \lim_{n \to \infty} \frac{1}{nt} \sup_{g' \in \mathscr{S}_{nt}} \log \|g'\|$$

$$= \lim_{n \to \infty} \delta(nt) = \delta.$$

This means that $\beta(t) \leqslant \delta$ for all $t > 0$ and again by Lemma 3.2 we obtain $\beta \leqslant \delta$.

The result for $\beta^*, \hat{\beta}^*, \delta^*, \hat{\delta}^*$ now follows from Lemma 3.1 via time reversal.

(ii) Again because of Lemma 3.1 we have to prove the result only for β and δ.

(a) $\beta = \infty$ implies $\delta = \infty$ was shown in (i).

(b) Proof of "$\bar{\mathscr{H}}$ not compact implies $\beta = \infty$." The goal is to construct a point $p_0 \in \mathbb{P}$ and a sequence $g_n \in \mathscr{S}$ such that p_0 is an eigenvector for all g_n, and the log of the corresponding eigenvalue can be made arbitrarily large. This part of the proof resembles that of Proposition 3.1 in [AOP], where constant matrices $A_0 \cdots A_m$ in $sl(d, \mathbb{R})$ are considered.

(ba) Denote $M = \{p \in \mathbb{P}; \sup_{g \in \mathscr{H}} |gp| < \infty\}$. M does not contain a basis of $\mathbb{R}^d$: If $\{x_1, ..., x_d\} \subset \mathbb{P}$ were such a basis, then $\sup_{g \in \mathscr{H}} |gx_i| = \alpha_i < \infty$, and thus $\sup_{g \in \mathscr{H}} \|g\| \leqslant \sum \alpha_i < \infty$. But this means that $\bar{\mathscr{H}}$ would be compact.

(bb) Choose $p_1 \in \operatorname{int} C \setminus M$, where C is the invariant control set from Proposition 2.3. Then for all $n \in \mathbb{N}$ there exist $h_n \in \mathscr{H}$ of the form

$$h_n = \exp(t_{n,i_n} B_{n,i_n}(u_{i_n})) \cdot ... \cdot \exp(t_{n,1} B_{n,1}(u_1)),$$

$$t_{n,j} \in \mathbb{R},\ B_{n,j} \in N_0,\ j = 1, ..., i_n \in \mathbb{N}$$

such that $|h_n p_1| \geqslant n + 1$. For $\varepsilon > 0$ and a sequence $\varepsilon_n > 0$ with $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$, define

$$k_n = \exp \varepsilon_{i_n} \left(A_0(u_{i_n}) + \frac{t_{n,i_n}}{\varepsilon_{i_n}} B_{n,i_n}(u_{i_n}) \right)$$

$$\cdot ... \cdot \exp \varepsilon_1 \left(A_0(u_1) + \frac{t_{n,1}}{\varepsilon_1} B_{n,1}(u_1) \right).$$

Then ε_n can be chosen such that $|k_n p_1| \geqslant n$ for all $n \in \mathbb{N}$, because Ω is compact, and $k_n \in \mathcal{S}_{\leqslant \varepsilon}$.

Compactness of $C \subset \mathbb{P}$ implies that $\pi(k_n p_1)$, the projection of $k_n p_1$ onto $\mathbb{P}$, has an accumulation point in C, say w.l.o.g. $\lim_{n \to \infty} \pi(k_n p_1) = p_2 \in C$. By Proposition 2.3(i) together with local accessibility, there exists for all $t > 0$ a $k \in \mathcal{S}_t$ with $p_0 := \pi(kp_2) \in \text{int } C$, and a $\hat{k} \in \mathcal{S}$ with $\pi(\hat{k} p_0) = p_1 \in \text{int } C \backslash M$. Thus $\lim_{n \to \infty} \pi(k_n \circ \hat{k} \circ k(p_2)) = p_2 \in C$.

(bc) Now let $\varepsilon > 0$ be arbitrary and k_n, p_1, p_2, p_0 be chosen as above. Note that we can pick $k \in \text{int } \mathcal{S}_{\leqslant \varepsilon}$, such that there is a neighborhood W of $\text{Id} \in Gl(d, \mathbb{R})$ with $kW \subset \text{int } \mathcal{S}_{\leqslant 2\varepsilon}$. Let $V \subset \mathbb{P}$ be a neighborhood of p_2 with $p_0 \in kV \subset \text{int } C$ and $V \subset \{g^{-1} p_0; g \in kW\}$. Then there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $\pi(k_n p_1) \in V$. Therefore, by definition of V, we can find $l_n \in \mathcal{S}_{\leqslant 2\varepsilon}$ with $\pi(l_n \circ k_n(p_1)) = p_0$, and hence $\pi(g_n(p_0)) := \pi(l_n \circ k_n \circ \hat{k}(p_0)) = p_0$ for all $n \geqslant N$, i.e., p_0 is an eigenvector. Note that for all $p \in V$ and all $g \in kW \subset \text{int } \mathcal{S}_{\leqslant 2\varepsilon}$ we have $|gp| > 0$, uniformly in g, i.e., there is a constant $\alpha > 0$ with $|gp| \geqslant \alpha\, |p| > 0$. Therefore

$$|g_n p_0| \geqslant \alpha\, |k_n \circ \hat{k}(p_0)| = \alpha \cdot c_1\, |k_n p_1| \geqslant \alpha c_1 n,$$

where $\hat{k}(p_0) = c_1 p_1$. Thus we obtain for the spectral radius $r(g_n) \geqslant \alpha_1 n$ for all $n \geqslant N$ with $\alpha_1 = \alpha c_1 > 0$. Furthermore, if $\hat{k} \in \mathcal{S}_{\leqslant \hat{t}}$, then $g_n \in \mathcal{S}_{\leqslant 3\varepsilon + \hat{t}}$, and hence

$$\beta(2\varepsilon + \hat{t}) \geqslant \frac{\log \alpha_1 + \log n}{3\varepsilon + \hat{t}} \qquad \text{for all} \quad n \geqslant N.$$

Since $\varepsilon > 0$ was arbitrary, $\beta = \sup_{t > 0} \beta(t) = +\infty$.

(c) It remains to show that $\delta = +\infty$ implies that $\bar{\mathscr{H}}$ is not compact. If $\bar{\mathscr{H}}$ is compact, then there exists a basis transformation $T \in Gl(d, \mathbb{R})$ such that $TN_0 T^{-1} \subset so(d, \mathbb{R})$. We will assume that such a choice of basis is made, and will continue to use the notation $A_j(u_0)$, $j = 0, ..., m$, for the representation with respect to the new basis. Using (2.2) and Lemma 3.2 we can write

$$\delta = \lim_{t \to \infty} \sup_{u \in \mathscr{U}} \sup_{p \in \mathbb{P}} \frac{1}{t} \int_0^t q(u(\tau), s(\tau, p, u))\, d\tau. \tag{3.4}$$

Now $q(u, s) = q_0(u_0, s) + \sum_{i=1}^m u_i q_i(u_0, s)$, and for $B \in so(d, \mathbb{R})$, $s^T B s = 0$. Therefore $q(u, s) = q_0(u_0, s)$ and because Ω is compact, $\delta < \infty$. ∎

A characterization similar to Proposition 3.3(ii) is not possible for the uniform concepts $\hat{\beta}$, $\hat{\delta}$, $\hat{\beta}^*$, and $\hat{\delta}^*$, as the following examples show:

3.5. EXAMPLES. (i) If $\mathscr{H} = Gl(d, \mathbb{R})$, then $\hat{\beta} = \hat{\delta} = +\infty$, and $\hat{\beta}^* = \hat{\delta}^* = -\infty$.

(ii) For the parameter controlled linear oscillator

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} x + u \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x, \qquad u(t) \in \mathbb{R}$$

one obtains $\hat{\beta} = \hat{\delta} = \hat{\beta}^* = \hat{\delta}^* = -b$ (compare Sect. 6).

In both cases the group $\mathscr{H}$ is not compact.

3.6. *Remark.* If the group $\mathscr{H}$ is compact, then $\hat{\beta}$, $\hat{\delta}$, $\hat{\beta}^*$, and $\hat{\delta}^*$ are of course finite, but each of the cases $\hat{\beta} < \hat{\beta}^*$, $\hat{\beta} = \hat{\beta}^*$, and $\hat{\beta} > \hat{\beta}^*$ is possible. For $\hat{\beta} < \hat{\beta}^*$ and $\hat{\beta} = \hat{\beta}^*$ see Section 6. For $\hat{\beta} > \hat{\beta}^*$ consider the system

$$\dot{x} = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} x, \qquad u_1(t) \in [2, 4], u_2(t) \in [1, 3].$$

Using the constant control $(u_1, u_2) \equiv (3, 3)$, one obtains $\hat{\beta} \geqslant 3$, while for $(u_1, u_2) \equiv 2$ we see that $\hat{\beta}^* \leqslant 2$. Theorem 4.4 will show that the same cases can occur for $\hat{\delta}$ and $\hat{\delta}^*$.

4. MAXIMAL AND MINIMAL LYAPUNOV EXPONENTS

In this section we study the relations between the extremal Lyapunov exponents defined in Section 1 and the semigroup concepts from Section 3. The first main result is:

4.1. THEOREM. (i) $\hat{\beta} = \hat{\delta} = \mathscr{K} = \mathscr{K}_p = \mathscr{K}_{pp}$,

(ii) $\hat{\beta}^* = \hat{\delta}^* = \mathscr{K}^* = \mathscr{K}_p^* = \mathscr{K}_{pp}^*$.

The proof will be accomplished in two lemmas.

4.2. LEMMA. *$\mathscr{H}$ is not compact iff $\mathscr{K} = \infty$ iff $\mathscr{K}_p = \infty$ iff $\mathscr{K}_{pp} = \infty$.*

Proof. According to Proposition 3.3(ii), all we have to show is (a) $\hat{\beta} = \infty$ implies $\mathscr{K}_{pp} = \infty$, and (b) $\mathscr{K}_{pp} \leqslant \mathscr{K}_p \leqslant \mathscr{K} \leqslant \hat{\delta}$.

(a) Assume $\hat{\beta} = \infty$. By Lemma 3.2, for each $k > 0$ we can find $g = \exp(t_n B_n) \cdot \ldots \cdot \exp(t_1 B_1)$, $B_j \in N$ (as defined in (3.1)) and $g \in \mathscr{S}_T$, $T = \sum_{i=1}^n t_i$, such that $(1/T) \log r(g) > k$. Each B_j corresponds to a constant control $u_j \in U$ for the time interval $I_j = (\sum_{i=1}^{j-1} t_i, \sum_{i=1}^{j} t_i]$, with $t_0 = 0$.

Define a control u^p on $(0, T]$ by $u^p(t) = u_j$ for $t \in I_j$, and continue T-periodically. Then the system (1.1) corresponding to this control has g as its fundamental matrix at time T. If g has an eigenvector $s_0 \in \mathbb{P}$ corresponding to $r(g)$ then $|x(t, s_0, u^p)| \geqslant \exp tk$ for all t large enough, and $\pi(x(nT, s_0, u^p)) = s_0$ for all $n \in \mathbb{N}$, where $\pi: \mathbb{R}^d \to \mathbb{P}$ denotes again the natural projection. Therefore $\lim_{t \to \infty}(1/t)(\log |x(t, s_0, u^p)|) \geqslant k$, and, since $k > 0$ was arbitrary, $\mathscr{K}_{pp} = \infty$.

If $r(g)$ is only attained for a complex eigenvalue λ of g, we first approximate $(1/t)\, r(g)$ by $(1/t')\, r(g')$ with $g' \in \operatorname{int} \mathscr{S}_{\leqslant t} \cap \mathscr{S}_{t'}$ for some $t' < t$. This is possible, since by (H) we have $\operatorname{cl} \mathscr{S}_{\leqslant t} = \operatorname{cl} \operatorname{int} \mathscr{S}_{\leqslant t}$. Now let $\lambda' \in \sigma(g')$ satisfy $|\lambda'| = r'(g')$. Then for s_0' in the corresponding real eigenspace we have $\pi(s_0') \in C$, and hence $\pi(g' s_0') \in C$. In fact, even $\pi(g' s_0') \in \operatorname{int} C$ holds: By (H), $\mathscr{G}$ acts transitively on $\mathbb{P}$, and therefore the map $h \to h s_0'$ from $\mathscr{G}$ into $\mathbb{P}$ is open. Thus $g' \in \operatorname{int} \mathscr{S}$ implies $g' s_0' \in \operatorname{int} C$. Now Proposition 2.3 in [CK1] shows that for arbitrarily large n, $\pi(g'^n s_0') \in C$ can be steered in uniformly bounded time to $g' s_0' \in \operatorname{int} C$. Since $r(g'^n) = r(g')^n$, this shows that there is $g'' \in \mathscr{S}_{t''}$ with $(1/t'') \log |\lambda''| \geqslant (1/t') \log r(g') - \varepsilon$ for a real eigenvalue λ'' of g''. Now we can conclude the proof as above.

(b) $\mathscr{K}_{pp} \leqslant \mathscr{K}_p \leqslant \mathscr{K}$ follows directly from the definitions. To show $\mathscr{K} \leqslant \delta$, simply recall the representation of $\lambda(x_0, u)$ in (2.3), and of δ in (3.4). ∎

4.3. Lemma. *Assume that $\bar{\mathscr{K}}$ is compact. Then*

(i) $\beta = \mathscr{K}_p = \mathscr{K}_{pp}$,

(ii) $\delta = \mathscr{K} = \mathscr{K}_p = \mathscr{K}_{pp}$.

Proof. As in the proof of Proposition 3.3(ii)(c) we can assume w.l.o.g. that we work with a basis in $\mathbb{R}^d$, such that $N_0 \subset so(d, \mathbb{R})$, N_0 defined in (3.3).

(i) $\beta \leqslant \mathscr{K}_{pp}$ follows exactly along the same lines as the proof of Lemma 4.2(a), where we replace "for each $k > 0$" by "for each $\varepsilon > 0$," and we choose $g \in \mathscr{S}_T$ such that $(1/T) \log r(g) > \beta - \varepsilon$.

$\mathscr{K}_p \leqslant \beta$ was shown in [CK1], and a proof similar to that of Lemma 5.3(ii) for systems without unbounded controls, but for $\beta < \infty$, which is guaranteed here by Proposition 3.3, goes through for systems of type (1.1). Since $\mathscr{K}_{pp} \leqslant \mathscr{K}_p$, we conclude that $\beta = \mathscr{K}_p = \mathscr{K}_{pp}$.

(ii) $\mathscr{K} \leqslant \delta$ follows as in the proof of Lemma 4.2(b), and it remains to show $\delta \leqslant \mathscr{K}$.

(a) Denote by $C \subset \mathbb{P}$ the invariant control set of (2.1) and fix an open set $V \subset \operatorname{int} C$ with $\bar{V} \subset \operatorname{int} C$. Then V contains a basis of $\mathbb{R}^d$, say $\{e_1, ..., e_d\}$. Pick $g \in \mathcal{S}_t$ and denote $k_i = |ge_i|$, and $k = \max k_i$. Then

$$\begin{aligned}\frac{1}{t} \log \|g\| &\leqslant \frac{1}{t} \log \max_{|\alpha_i| \leqslant 1} \left| g\left(\sum_{i=1}^{d} \alpha_i e_i \right) \right| \\ &\leqslant \frac{1}{t} \log \sum_{i=1}^{d} k_i \leqslant \frac{1}{t} \log dk \\ &= \frac{1}{t} \log d + \frac{1}{t} \log k.\end{aligned}$$

(b) For $V \subset \operatorname{int} C$ as in (a), define

$$T := \sup_{x \in C, y \in \bar{V}} \inf_{u \in \mathcal{U}} \{t > 0; s(t, x, u) = y\}.$$

By Proposition 2.3 in [CK1], $T < \infty$.

(c) Recall that for $N_0 \subset so(d, \mathbb{R})$, $q(u, s) = q_0(u_0, s)$. Therefore by the definition of δ and using (a) and (b), for every $\varepsilon > 0$ there exist $s_0 \in V$, $\tilde{u} \in \mathcal{U}$, and $t_0 > 0$ with

$$\frac{1}{t_0 + t_1} \left\{ \int_0^{t_0} q_0(\tilde{u}_0(\tau), s(\tau, s_0, \tilde{u}))\, d\tau + t_1 \cdot \min_{(u_0, s) \in \Omega \times C} q_0(u_0, s) \right\} > \delta - \varepsilon$$

for all $t_1 \in (0, T]$. Since C is invariant, $s(t_0, s_0, \tilde{u}) =: y \in C$, and by (b) there exists a control $\hat{u} \in \mathcal{U}$ such that $s(t_1, y, \hat{u}) = s_0$ for some $t_1 \leqslant T$. Define

$$u(t) = \begin{cases} \tilde{u}(t), & t \in (0, t_0], \\ \hat{u}(t - t_0), & t \in (t_0, t_0 + t_1] \end{cases}$$

and continue $(t_0 + t_1)$-periodically to obtain the control u^p. We then have

$$\frac{1}{t_0 + t_1} \int_0^{t_0 + t_1} q_0(u_0^p(\tau), s(\tau, s_0, u^p))\, d\tau > \delta - \varepsilon,$$

and since $(u^p, s(\cdot, s_0, u^p))$ is $(t_0 + t_1)$-periodic, we see that $\mathcal{K}_{pp} \geqslant \delta$. ∎

Proof of Theorem 4.1. (i) If $\bar{\mathcal{H}}$ is not compact, then $\beta = \delta = \mathcal{K} = \mathcal{K}_p = \mathcal{K}_{pp} = \infty$ by Proposition 3.3(ii) and Lemma 4.2. If $\bar{\mathcal{H}}$ is compact, the result is Lemma 4.3.

(ii) It follows from (i) and time reversal (Lemma 3.1) that $\beta^* = \delta^* = \mathcal{K}^*_{pp}$. $\mathcal{K}^* \leqslant \mathcal{K}^*_{pp}$ is an immediate consequence of the definitions. All we have to show is therefore $\delta^* \leqslant \mathcal{K}^*$: For an admissible control $u \in \mathcal{U}$ denote by $\Phi_u(t)$ the fundamental matrix of (1.1) with $\Phi_u(0) = \mathrm{Id}$. Then

$$\begin{aligned}
\delta^* &= \lim_{t\to\infty} \frac{1}{t} \inf_{g\in\mathscr{S}_t} \log m(g) \\
&= \lim_{t\to\infty} \frac{1}{t} \inf_{u\in\mathscr{U}} \log \inf_{p\in\mathbb{P}} |\Phi_u(t)\, p| \\
&= \lim_{t\to\infty} \inf_{u\in\mathscr{U}} \inf_{p\in\mathbb{P}} \frac{1}{t} \log |\Phi_u(t)\, p| \\
&= \lim_{t\to\infty} \inf_{u\in\mathscr{U}} \inf_{p\in\mathbb{P}} \frac{1}{t} \int_0^t q(u, s)\, d\tau \\
&\leqslant \inf_{u\in\mathscr{U}} \inf_{p\in\mathbb{P}} \limsup_{t\to\infty} \frac{1}{t} \int_0^t q(u, s)\, d\tau \\
&= \mathscr{K}^*. \quad \blacksquare
\end{aligned}$$

4.4. Corollary. *$\mathscr{K}^*(N) = -\mathscr{K}(-N)$, $\mathscr{K}_p^*(N) = -\mathscr{K}_p(-N)$, $\mathscr{K}_{pp}^*(N) = -\mathscr{K}_{pp}(-N)$, where $(-N)$ refers again to the time reversed system.*

Proof. For $\mathscr{K}_{pp}$ the equality can be proved directly by noting that for $u\in\mathscr{U}_{pp}$ the lim sup in $\lambda(x_0, u)$ is actually a limit. Therefore for $u\in\mathscr{U}_{pp}$ one has $\lambda(x_0, u(\cdot), N) = -\lambda(x_0, u(-\cdot), -N)$, and hence

$$\begin{aligned}
K_{pp}^*(N) &= \inf_{u\in\mathscr{U}_{pp}} \lambda(x_0, u, N) = \inf_{u\in\mathscr{U}_{pp}} -\lambda(x_0, u, -N) \\
&= -\sup_{u\in\mathscr{U}_{pp}} \lambda(x_0, u, -N) = -\mathscr{K}_{pp}(-N).
\end{aligned}$$

For $\mathscr{K}$ the result follows from Theorem 4.1 and Lemma 3.1,

$$\mathscr{K}^*(N) = \delta^*(N) = -\delta(-N) = -\mathscr{K}(-N),$$

and similarly for $\mathscr{K}_p$. $\blacksquare$

The second main result of this section is a characterization of the extremal Lyapunov exponents $\mathscr{K}$ and $\mathscr{K}^*$ in terms of associated systems groups. Recall the definition of N in (3.1), and of N_0 in (3.3) with the corresponding groups $\mathscr{G}$ for N, and $\mathscr{H}$ for N_0. For a matrix $B\in gl(d, \mathbb{R})$ define $B^0 := B - (1/d)\operatorname{trace} B \cdot \mathrm{Id}$, and denote

$$N^0 := \left\{ A_0^0(u_0) + \sum_{i=1}^m u_i A_i^0(u_0),\ u\in U \right\}$$

$$N_0^0 := \left\{ \sum_{i=1}^m u_i A_i^0(u_0),\ u\in U \right\}.$$

The corresponding groups will be denoted by $\mathscr{G}^0$ for N^0, and $\mathscr{H}^0$ for N_0^0. Note that N^0, $N_0^0 \subset sl(d, \mathbb{R})$ and $\mathscr{G}^0$, $\mathscr{H}^0 \subset Sl(d, \mathbb{R})$, and that (H) holds for N iff it holds for N^0.

4.5. THEOREM. *$\bar{\mathscr{H}}$ is not compact (in $Gl(d, \mathbb{R})$) iff $\mathscr{K} = +\infty$ iff $\mathscr{K}^* = -\infty$.*

(ii) *If $\bar{\mathscr{H}}$ is compact and*

(a) *$\mathscr{G}$ and $\mathscr{G}^0$ are not compact, then $-\infty < \mathscr{K}^* < \mathscr{K} < +\infty$;*

(b) *$\mathscr{G}$ is not compact, but $\mathscr{G}^0$ is compact, then*

$$\text{if } \frac{1}{d} \text{trace } A_0(u_0) \equiv c\text{:} \qquad \mathscr{K}^* = \mathscr{K} = c$$

$$\text{if } \frac{1}{d} \text{trace } A_0(u_0) \not\equiv c\text{:} \qquad \mathscr{K}^* = \frac{1}{d} \min_{u_0 \in \Omega} \text{trace } A_0(u_0)$$

$$< \frac{1}{d} \max_{u_0 \in \Omega} \text{trace } A_0(u) = \mathscr{K};$$

(c) *$\mathscr{G}$ is compact iff $\mathscr{K}^* = \mathscr{K} = 0$.*

A version of this theorem was proved in [AK] in a stochastic setup under stronger assumptions. We will give a proof that uses our previous results from Sections 3 and 4 directly. First we will show that under assumption (H), $\mathscr{G}$ and $\mathscr{G}^0$ are closed (see [O]).

4.6. LEMMA. *If $\mathscr{G} \subset Gl(d, \mathbb{R})$ is a connected, real Lie group, acting transitively on $\mathbb{P}$, then $\mathscr{G}$ is closed in $Gl(d, \mathbb{R})$.*

Proof. If $\mathscr{G}$ acts transitively, thus irreducibly on $\mathbb{P}$, its Lie algebra g is reductive, i.e., $g = h \oplus z$, where h is semisimple and the center z consists only of semisimple elements (see [Bo, Chap. I.6.5]). Then the complexification $g_{\mathbb{C}} = g \oplus ig \subset gl(d, \mathbb{C})$ is also reductive. Let $\mathscr{G}_{\mathbb{C}} \subset Gl(d, \mathbb{C})$ be the complex (reductive) Lie group of $g_{\mathbb{C}}$. Then $\mathscr{G}_{\mathbb{C}}$ is closed in $Gl(d, \mathbb{C})$ (see, e.g., [HM, p. 94]). Denote by $\tilde{\mathscr{G}}$ the component of $Gl(d, \mathbb{R}) \cap \mathscr{G}_{\mathbb{C}}$ that contains the identity matrix Id, and note that $\mathscr{G} \subset \tilde{\mathscr{G}}$, and $\tilde{\mathscr{G}}$ is closed in $Gl(d, \mathbb{R})$. Now $\dim_{\mathbb{R}} \mathscr{G} = \dim_{\mathbb{R}} g = \dim_{\mathbb{R}}(g_{\mathbb{C}} \cap gl(d, \mathbb{R})) = \dim_{\mathbb{R}} \tilde{\mathscr{G}}$, where $g_{\mathbb{C}} \cap gl(d, \mathbb{R})$ is the Lie algebra of $\tilde{g}$. Since $\tilde{\mathscr{G}}$ is connected, we obtain $\mathscr{G} = \tilde{\mathscr{G}}$. ∎

4.7. COROLLARY. *Assume* (H). *Then $\mathscr{G}$* and *$\mathscr{G}^0$ are closed in $Gl(d, \mathbb{R})$.*

Proof. Under (H), the maximal integral manifold through any $p \in \mathbb{P}$ is $\mathbb{P}$ itself, and therefore $\mathscr{G}$ acts transitively on $\mathbb{P}$. Obviously, $\mathscr{G}$ is connected and a Lie group contained in $Gl(d \mathbb{R})$, and the result for $\mathscr{G}$ follows from

Lemma 4.6. Furthermore, (H) holds for N iff it holds for N^0, which implies that also $\mathcal{G}^0$ is closed. ▮

Before we can prove Theorem 4.5, we need the following result from [AKO, Proposition 6.2]:

4.8. LEMMA. *If $\bar{\mathcal{K}}$ is compact, then the following statements are equivalent*

(i) $\beta = 0$ *and* $N \subset sl(d, \mathbb{R})$,

(ii) $\mathcal{G}$ *is compact.*

Proof of Theorem 4.5. (i) By Proposition 3.3(ii), $\bar{\mathcal{K}}$ is not compact iff $\beta = +\infty$ iff $\beta^* = -\infty$. By Theorem 4.1 the same is true for $\mathcal{K}$ and $\mathcal{K}^*$.

(ii) If $\bar{\mathcal{K}}$ is compact, we can assume again w.l.o.g. that we work with a basis in $\mathbb{R}^d$ such that $N_0 \subset so(d, \mathbb{R})$.

First of all, we will show that $\mathcal{K}^* = \mathcal{K}$ implies trace $A_0(u_0)$ is constant: If $\mathcal{K}^* = \mathcal{K}$, then by Theorem 4.1(i) $\beta^* = \beta$, i.e., for all constant $u \in U$ and all $t > 0$ we have $(1/t) \log \operatorname{cor}(g_u) = (1/t) \log r(g_u) = \beta^* = \beta$, where $g_u \in \mathcal{S}_t$ is the element in the systems semigroup at time t corresponding to u. (Note that $\operatorname{cor}(g^n) = \operatorname{cor}(g)^n$ and $r(g^n) = r(g)^n$ for $g \in \mathcal{S}$.) Thus for all $u_0 \in \Omega$, all $u_i \in \mathbb{R}$, $i = 1, ..., m$, $\operatorname{trace}(A_0(u_0) + \sum u_i A_i(u_0))$ is constant (see, e.g., [LT, Theorem 4.11.2]), and hence $(1/d)$ trace $A_0(u_0) \equiv c$ for all $u_0 \in \Omega$.

(a) If $\bar{\mathcal{K}}$ is compact, we know from Proposition 3.3(ii) that $\mathcal{K}^* > -\infty$ and $\mathcal{K} < +\infty$. It remains to show that $\mathcal{K}^* < \mathcal{K}$. Consider the system

$$\dot{x} = A_0^0(u_0)\, x + \sum_{i=1}^{m} u_i A_i^0(u_0)\, x \tag{4.1}$$

with semigroup $\mathcal{S}^0$ and systems group $\mathcal{G}^0$. Denote by β^0 the exponential growth rate of $\mathcal{S}^0$ with respect to the spectral radius. Since $\mathcal{G}^0 \subset Sl(d, \mathbb{R})$, we know that $\beta^0 \geqslant 0$ (see, e.g., [LT, Theorem 4.11.2]). If $\mathcal{G}^0$ is not compact, then by Lemma 4.8 $\beta^0 > 0$, i.e., $\mathcal{K}^0 > 0$, where $\mathcal{K}^0$ is the maximal Lyapunov exponent of (4.1). Similarly, using the time reversed system, we obtain $\mathcal{K}^{0*} < 0$.

Now assume that $\mathcal{K}^* = \mathcal{K}$ and consider the individual Lyapunov exponents of (1.1) for $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$

$$\lambda(x_0, u) = \limsup_{t \to \infty} \frac{1}{t} \int_0^t q_0(u_0(\tau), s(\tau, s_0, u))\, d\tau$$

with $s_0 = x_0/|x_0|$. (Recall that in the chosen basis $q_i = 0$ for $i = 1, ..., m$). If we denote by q_0^0 and s^0 the corresponding functions for (4.1), we obtain

$$\lambda(x_0, u) = \limsup_{t \to \infty} \left[\frac{1}{t} \int_0^t q_0^0(u_0(\tau), s^0(\tau, s_0, u))\, d\tau + \frac{1}{t} \int_0^t \frac{1}{d} \operatorname{trace} A_0(u_0(\tau))\, d\tau \right]. \tag{4.2}$$

Since $\mathscr{K}^* = \mathscr{K}$ implies $(1/d)$ trace $A_0(u_0) \equiv c$ by the argument above, we have

$$\mathscr{K} = \mathscr{K}^0 + c \qquad \text{and} \qquad \mathscr{K}^* = \mathscr{K}^{0*} + c,$$

which contradicts the fact that $\mathscr{K}^{0*} < \mathscr{K}^0$.

(b) If $\mathscr{G}^0$ is compact, then Lemma 4.8 yields $\mathscr{K}^{0*} = \mathscr{K}^0 = 0$. Therefore the first assertion follows from (4.2). To see the second assertion, let $u_0^1, u_0^2 \in \Omega$ be such that

$$\min_{u_0 \in \Omega} \operatorname{trace} A_0(u_0) = \operatorname{trace} A_0(u_0^1),$$

$$\max_{u_0 \in \Omega} \operatorname{trace} A_0(u_0) = \operatorname{trace} A_0(u_0^2).$$

For the constant controls u_0^i, $i = 1, 2$, there are $x_0^i \in \mathbb{R}^d$, $x_0^i \neq 0$, with

$$\lambda(x_0^i, u_0^i) = \limsup_{t \to \infty} \frac{1}{t} \int_0^t q_0^0(u_0^i, s^0(\tau, s_0^i, u_0^i))\, d\tau + \frac{1}{d} \operatorname{trace} A_0(u_0^i).$$

The first summand must vanish, because $\mathscr{K}^{0*} = \mathscr{K}^0 = 0$. Hence we obtain

$$\mathscr{K}^* \leqslant \frac{1}{d} \min_{u_0 \in \Omega} \operatorname{trace} A_0(u_0)$$

$$\mathscr{K} \geqslant \frac{1}{d} \max_{u_0 \in \Omega} \operatorname{trace} A_0(u_0).$$

The converse inequalities are obvious.

(c) If $\mathscr{G}$ is compact, then $\mathscr{G} = \mathscr{G}^0$ and hence, by Lemma 4.8, $\mathscr{K}^* = \mathscr{K} = 0$. Vice versa, if $\mathscr{K}^* = \mathscr{K} = 0$, then by (i) and (ii)(a), (b) above, $\bar{\mathscr{H}}$ and $\mathscr{G}^0$ are compact, and trace $A_0(u_0) \equiv 0$. Since in the chosen basis $N_0 \subset so(d, \mathbb{R})$ and $A_0(u_0) \in sl(d, \mathbb{R})$ for all $u_0 \in \Omega$, we have that $N \subset sl(d, \mathbb{R})$. Hence by Lemma 4.8, $\mathscr{G}$ is compact. ▮

4.9. *Remark.* Assume that $\bar{\mathscr{H}}$ is compact. If $\bar{\mathscr{S}}$ is compact, where $\mathscr{S}$ is again the systems semigroup of (1.1), then $\mathscr{G} = \bar{\mathscr{S}}$, and system (2.1) is completely (exactly) controllable on $\mathbb{P}$ (see, e.g., [JS, Theorem 6.5] or [AKO, Corollary 3.2]). Likewise, if $\mathscr{G}^0$ is compact, then the projection of (4.1) onto $\mathbb{P}$, $\dot{s} = h^0(u, s)$, is completely controllable on $\mathbb{P}$. Hence, because $h = h^0$, system (2.1) is also completely controllable. The cases (ii)(b) and (c) in Theorem 4.5, therefore, correspond to controllable situations on $\mathbb{P}$.

We now turn to the uniform growth rates $\hat{\mathscr{K}}$ and $\hat{\mathscr{K}}^*$. The analogue of Theorem 4.1 for these quantities is

4.10. THEOREM. (i) $\hat{\beta}^* = \hat{\delta}^* = \hat{\mathscr{K}}^* = \hat{\mathscr{K}}_p^* = \hat{\mathscr{K}}_{pp}^*$,

(ii) $\hat{\beta} = \hat{\delta} = \hat{\mathscr{K}} = \hat{\mathscr{K}}_p = \hat{\mathscr{K}}_{pp}$.

Proof. Recall from Proposition 3.3(i) that $\hat{\beta}, \hat{\delta} > -\infty$, and $\hat{\beta}^*, \hat{\delta}^* < +\infty$.

(i) $\hat{\delta}^* \leqslant \hat{\mathscr{K}}^*$:

$$\begin{aligned} \hat{\delta}^* &= \lim_{t\to\infty} \frac{1}{t} \inf_{g\in\mathscr{S}_t} \log \|g\| \\ &= \lim_{t\to\infty} \frac{1}{t} \inf_{u\in\mathscr{U}} \log \sup_{p\in\mathbb{P}} |x(t, p, u)| \\ &\leqslant \inf_{u\in\mathscr{U}} \sup_{p\in\mathbb{P}} \limsup_{t\to\infty} \frac{1}{t} \log |x(t, p, u)| \\ &= \hat{\mathscr{K}}^*, \end{aligned}$$

where the inequality can be shown as in Lemma 4.3(ii).

$\hat{\mathscr{K}}_{pp}^* \leqslant \hat{\beta}^*$: Assume first that $\hat{\beta}^* > -\infty$.

Fix $\varepsilon > 0$. Then there exist $T > 0$ and $g = \Phi_u(T) \in \mathscr{S}_T$ with $(1/T) \log r(\Phi_u(T)) \leqslant \hat{\beta}^* + \varepsilon$, where $\Phi_u(T)$ is the fundamental matrix of (1.1), for the control function $u \in \mathscr{U}$ that is associated with $g \in \mathscr{S}_T$. Extend u, defined on $[0, T)$, T-periodically to u_p: $\mathbb{R}^+ \to U$. The periodic differential equation $\dot{x} = A(u_p)\, x$ has largest Floquet exponent $(1/T) \log r(\Phi_u(T))$. Hence for all $p_0 \in \mathbb{P}$ and all $t > T$ large enough $(1/t) \log |x(t, p_0, u_p)| \leqslant \hat{\beta}^* + 2\varepsilon$, i.e., $\hat{\mathscr{K}}_{pp}^* = \inf_{u\in\mathscr{U}_p} \sup_{p_0\in\mathbb{P}} \lambda(p_0, u) \leqslant \hat{\beta}^* + 2\varepsilon$, which implies the result, because $\varepsilon > 0$ was arbitrary. The infinite case follows similarly.

$\hat{\beta}^* \leqslant \hat{\delta}^*$: Recall that $r(g) \leqslant \|g\|$ for all $g \in Gl(d, \mathbb{R})$.

Hence $\hat{\beta}^* \leqslant \hat{\delta}^*$ follows from the definitions.

We now have the following chain of inequalities

$$\hat{\delta}^* \leqslant \hat{\mathscr{K}}^* \leqslant \hat{\mathscr{K}}_p^* \leqslant \hat{\mathscr{K}}_{pp}^* \leqslant \hat{\beta}^* \leqslant \hat{\delta}^*,$$

where the second and third inequality follow directly from the definitions.

(ii) $\hat{\beta} = \hat{\delta}$ follows via time reversal (Lemma 3.1) from (i), and $\hat{\mathscr{K}}_{pp} \leqslant \hat{\mathscr{K}}_p \leqslant \hat{\mathscr{K}}$ is obvious.

$\hat{\mathscr{K}} \leqslant \hat{\delta}$:

Fix $\varepsilon > 0$, then there exist $T > 0$ and $g = \Phi_u(T) \in \mathscr{S}_T$ such that $\hat{\mathscr{K}} \leqslant \inf_{p \in \mathbb{P}} (1/T) \log |\Phi_u(T)\, p| + \varepsilon$, i.e., $\hat{\mathscr{K}} \leqslant (1/T) \log m(\Phi_u(T)) + \varepsilon$. Extend again u T-periodically to u_p on $\mathbb{R}^+$, and we have

$$\hat{\mathscr{K}} \leqslant \limsup_{t \to \infty} \frac{1}{t} \log m(\Phi_{u_p}(t)) + \varepsilon \leqslant \limsup_{t \to \infty} \frac{1}{t} \sup_{g \in \mathscr{S}_t} \log m(g) + \varepsilon \leqslant \hat{\delta} + \varepsilon.$$

Finally $\hat{\beta} \leqslant \hat{\mathscr{K}}_{pp}$ is proved in complete analogy to $\hat{\mathscr{K}}^*_{pp} \leqslant \hat{\beta}^*$ in (i). ∎

4.11. Corollary. $\hat{\mathscr{K}}^*(N) = -\hat{\mathscr{K}}(-N)$, $\hat{\mathscr{K}}^*_p(N) = -\hat{\mathscr{K}}_p(-N)$, $\hat{\mathscr{K}}^*_{pp}(N) = -\hat{\mathscr{K}}_{pp}(-N)$.

Proof. Use time reversal (Lemma 3.1) and Theorem 4.10. ∎

4.12. *Remark.* From Example 3.5 we have seen that a group characterization similar to Theorem 4.5 is not possible for the uniform growth rates $\hat{\beta}$ and $\hat{\beta}^*$, and therefore neither for $\hat{\mathscr{K}}$ and $\hat{\mathscr{K}}^*$.

Finally we would like to point out that the growth rate $\tilde{\mathscr{K}}$ from (1.5), which is a uniform growth rate with respect to $u \in \mathscr{U}$, does not correspond to a (globally defined) growth rate of the systems semigroup $\mathscr{S}$. Indeed, $\tilde{\mathscr{K}}$ is (locally) defined "over the control set C," as we will see in the next section.

5. The Points from which the Extremal Growth Rates Can Be Realized

Let us briefly recall the situation for a linear differential equation $\dot{x} = Ax$ with constant matrix $A \in gl(d, \mathbb{R})$: Denote by $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ the real parts of the eigenvalues of A, and by $E_1, ..., E_k$, the corresponding sums of (generalized) eigenspaces. Then the maximal Lyapunov exponent of this system is λ_k, the minimal one is λ_1. λ_1 can be attained only for initial values $x_0 \in E_1$, while λ_k is realized for all $x_0 \in \mathbb{R}^d \setminus \bigoplus_{i=1}^{k-1} E_i$.

For the bilinear control system (1.1) the situation is somewhat similar, but assumption (H) guarantees that the systems semigroup $\mathscr{S}$ leaves no proper subspace of $\mathbb{R}^d$ invariant, although it does not imply complete controllability of the projected system on $\mathbb{P}$. It turns out that under (H) the maximal Lyapunov exponent $\mathscr{K}$ can be realized from all $x_0 \in \mathbb{R}^d \setminus \{0\}$ and even the minimal growth rate $\mathscr{K}^*$ is attained for an open set of initial values in $\mathbb{R}^d$. This set is described in the next theorem.

5.1. THEOREM. (i) *$\mathcal{K} = \sup_{u\in\mathcal{U}} \lambda(x_0, u)$ for all $x_0 \neq 0$; in particular $\mathcal{K} = \inf_{x_0\neq 0} \sup_{u\in\mathcal{U}} \lambda(x_0, u) = \tilde{\mathcal{K}}^*$ as defined in* (1.5).

(ii) *$\mathcal{K}^* = \inf_{u\in\mathcal{U}} \lambda(x_0, u)$ for all $x_0 \neq 0$ with $x_0/|x_0| \in C^-$.*

In general, $\mathcal{K}^*$ cannot be realized from $p_0 \notin C^-$.

Proof. (i) By Proposition 2.3, int $C \neq \emptyset$ in $\mathbb{P}$; hence each open set $V \subset \text{int } C$ with $\bar{V} \subset \text{int } C$ contains a basis of $\mathbb{R}^d$. Therefore, for all $u \in \mathcal{U}$ there exists $p_u \in V$ such that $\lambda(p_u, u) = \sup_{x_0\neq 0} \lambda(x_0, u)$ (see, e.g., [C] for standard properties of Lyapunov exponents).

Assume $\mathcal{K} < +\infty$. Pick $\varepsilon > 0$. Then by the above there exist $v \in \mathcal{U}$ and $p_v \in V$ such that $\lambda(p_v, v) > \mathcal{K} - \varepsilon$. Fix $p_0 \in \mathbb{P}$. Then there exist, by Proposition 2.3(i) a time $t(p_0)$ and a control $u(p_0) \in \mathcal{U}$ such that $s(t_0, p_0, u) = p_v$. Define the composite control by

$$w(t) = \begin{cases} u(p_0)(t) & \text{for } t \in [0, t_0) \\ v(t - t_0) & \text{for } t \in [t_0, \infty). \end{cases}$$

We obtain $\lambda(p_0, w) = \lambda(p_v, v) > \mathcal{K} - \varepsilon$, and, since $\varepsilon > 0$ was arbitrary, $\mathcal{K} = \sup_{u\in\mathcal{U}} \lambda(x_0, u)$ for all $x_0 \in \mathbb{R}^d \setminus \{0\}$. Note that by Theorem 4.1(i), for each $p_0 \in \mathbb{P}$ the maximal growth rate $\mathcal{K}$ can be realized as the supremum over all controls that are in $\mathcal{U}_{pp}$ after time $t_0(p_0)$.

(ii) Using the time reversed system (2.5) and Corollary 4.4, the proof is exactly the same as in (i) for all $p_0 \in C^- = \text{int } C^*$. Note, however, that for $p_0 \in \mathbb{P} \setminus C^-$ we have $\overline{\mathcal{O}^+(p_0)} \cap C^- = \emptyset$ according to the proof of Proposition 2.3(ii). Hence, we do not expect that $\mathcal{K}^*$ can be realized from initial values outside C^-; the example in Section 6 shows indeed that $\mathcal{K}^* < \inf_{u\in\mathcal{U}} \inf_{p_0\in\mathbb{P}\setminus C^-} \lambda(p_0, u)$ is possible. ∎

Theorem 5.1 not only characterizes exactly the points from which the extremal growth rates $\mathcal{K}$ and $\mathcal{K}^*$ can be realized, but also shows that the uniform (in $\mathcal{U}$) growth rate $\inf_{x_0\neq 0} \sup_{u\in\mathcal{U}} \lambda(x_0, u)$ is equal to $\mathcal{K}$. For $\tilde{\mathcal{K}} = \sup_{x_0\neq 0} \inf_{u\in\mathcal{U}} \lambda(x_0, u)$ and $\mathcal{K}^*$ we cannot expect a similar result, because the points $p_0 \in \mathbb{P} \setminus C^-$ cannot be steered into C^-. We have, however, the following theorem:

5.2. THEOREM. (i) $\tilde{\mathcal{K}} < +\infty$.

(ii) *$\tilde{\mathcal{K}} = \inf_{u\in\mathcal{U}} \lambda(x_0, u)$ for all $x_0 \neq 0$ with $x_0/|x_0| \in \text{int } C$.*

(iii) *$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_p = \tilde{\mathcal{K}}_{pp}$; in particular $\tilde{\mathcal{K}}_{pp} = \{\lambda(p_0, u); s(\cdot, p_0, u) \subset \text{int } C$ and periodic$\}$.*

(iv) *$\mathcal{K}^* < \mathcal{K}$ iff $\tilde{\mathcal{K}} < \mathcal{K}$. Recall that the cases in which $\mathcal{K}^* < \mathcal{K}$ occurs are characterized in Theorem* 4.5.

Proof. (i) Consider the constant control $u(t) \equiv c \in U$. Then system (1.1) with this control is periodic with maximal characteristic exponent λ_1, say. Hence $\tilde{\mathscr{K}} < \lambda_1 < \infty$.

(ii) We first show that $\tilde{\mathscr{K}} \leqslant \sup_{p_0 \in \operatorname{int} C} \inf_{u \in \mathscr{U}} \lambda(p_0, u) := \tilde{\mathscr{K}}(\operatorname{int} C)$. Take again, as in the proof of Theorem 5.1(i), an open set $V \subset \operatorname{int} C$ with $\bar{V} \subset \operatorname{int} C$, and fix a basis $\{e_1, ..., e_d\}$ of $\mathbb{R}^d$ in V. For each $x_0 = \sum_{i=1}^d \alpha_i e_i$ with $\alpha_i \neq 0$ for all $i = 1, ..., d$, we have that $\inf_{u \in \mathscr{U}} \lambda(x_0, u)$ is attained for one of the e_i's, and thus $\tilde{\mathscr{K}} \leqslant \max_{e_i} \inf_{u \in \mathscr{U}} \lambda(e_i, u)$. Together with the obvious inequality $\tilde{\mathscr{K}} \geqslant \tilde{\mathscr{K}}(\operatorname{int} C)$, we obtain $\tilde{\mathscr{K}} = \tilde{\mathscr{K}}(\operatorname{int} C)$. Next, pick $p_0 \in \operatorname{int} C$. Then for all $p \in \operatorname{int} C$ there exist $t(p) \geqslant 0$ and $u(p) \in \mathscr{U}$ such that $p_0 = s(t(p), p, u(p))$. Define for $u \in \mathscr{U}$

$$w_u(t) = \begin{cases} u(p)(t) & \text{for} \quad t \in [0, t(p)) \\ u(t - t(p)) & \text{for} \quad t \geqslant t(p). \end{cases}$$

Then $\lambda(p_0, u) = \lambda(p, w_u)$. Hence $\inf_{u \in \mathscr{U}} \lambda(p_0, u) \geqslant \inf_{u \in \mathscr{U}} \lambda(p, u)$. Exchanging the roles of p_0 and p we obtain

$\inf_{u \in \mathscr{U}} \lambda(p, u)$ is constant, independent of $p \in \operatorname{int} C$, and therefore

$\inf_{u \in \mathscr{U}} \lambda(p, u) = \tilde{\mathscr{K}}(\operatorname{int} C) = \tilde{\mathscr{K}}$ for all $p \in \operatorname{int} C$.

(iii) $\tilde{\mathscr{K}} \leqslant \tilde{\mathscr{K}}_p \leqslant \tilde{\mathscr{K}}_{pp} < \infty$ follows from the definitions. We thus have to show by (ii) that for $p_0 \in \operatorname{int} C$, all $\varepsilon > 0$, and all $u \in \mathscr{U}$ there exists a periodic trajectory $(u^p(\cdot), s(\cdot, p_0, u^p))$ in $\mathscr{U} \times \operatorname{int} C$ such that $\lambda(p_0, u^p) \leqslant \lambda(p_0, u) + \varepsilon$.
Denote

$$\hat{T} = \sup_{x \in C} \inf_{u \in \mathscr{U}} \{t > 0; s(t, x, u) = p_0\}.$$

By Proposition 2.3 in [CK1], $T < \infty$, and since C and $\{p_0\}$ are compact, there even exists a compact set $U^0 \subset U$ such that

$$T := \sup_{x \in C} \inf_{u \in \mathscr{U}^0} \{t > 0; s(t, x, u) = p_0\} < \infty,$$

where $\mathscr{U}^0$ denotes the set of admissible controls with values in U^0. Define

$$K := \max_{x \in C} \max_{u \in U^0} q(u, s).$$

For $p_0 \in \operatorname{int} C$ and $u \in \mathscr{U}$ there is a sequence of times $\{t_n, n \geqslant 0\}$, $t_n \uparrow \infty$ as $n \to \infty$ such that $(1/t_n) \int_0^{t_n} q(u(\tau), s(\tau, p_0, u))\, d\tau \leqslant \lambda(p_0, u) + \varepsilon/2$. Choose $n_0 \in \mathbb{N}$ large enough such that

$$\frac{1}{t_n + \hat{t} + 1} \left\{ \int_0^{t_n} q(u(\tau), s(\tau, p_0, u))\, d\tau + (\hat{t} + 1) K \right\}$$
$$\leqslant \lambda(p_0, u) + \varepsilon, \qquad \text{for all} \quad n \geqslant n_0,$$

and for all $\hat{t} \in [0, T]$. Since int C is invariant, $s(t_{n_0}, p_0, u) := y \in \operatorname{int} C$, and let $v \in \mathscr{U}^0$ be a control such that $s(t_v, y, v) = p_0$ with $t_v < T + 1$. Such a control exists by the argument above. Now define $u^p \in \mathscr{U}_{pp}$ by

$$u^p(t) = \begin{cases} u(t) & \text{for} \quad t \in [0, t_{n_0}) \\ v(t - t_{n_0}) & \text{for} \quad t \in [t_{n_0}, t_{n_0} + t_v) \end{cases}$$

and continue $(t_{n_0} + t_v)$-periodically. Then $(u^p(\cdot), s(\cdot, p_0, u^p))$ is periodic and $\lambda(p_0, u^p) \leqslant \lambda(p_0, u) + \varepsilon$.

(iv) If $\mathscr{K}$ is not compact, the result follows from Theorem 4.5 and the fact that $\tilde{\mathscr{K}} < \infty$. If $\mathscr{G}^0$ is compact (and in particular, if $\mathscr{G}$ is compact), then by Remark 4.9 system (2.1) is completely controllable, and hence by (ii) $\mathscr{K}^* = \tilde{\mathscr{K}}$. Thus the results hold in this case.

It remains to consider the case where $\mathscr{G}$ and $\mathscr{G}^0$ are not compact. Assume that $\mathscr{K}$ is compact. It follows from (ii) above and Theorem 5.5(ii) below that $\tilde{\mathscr{K}} \leqslant \hat{\mathscr{K}}^* \leqslant \mathscr{K}$. Thus the result is a consequence of the stronger statement in Theorem 5.5(iii). ∎

A "probabilistic proof" of Theorem 5.2(iv) can be found in [AK, proof of Theorem 2.3]. It can be shown that $\gamma_- = \mathscr{K}^*, \gamma_+ = \mathscr{K}$, and $\lambda \in [\tilde{\mathscr{K}}, \mathscr{K}]$, where γ_-, γ_+, and λ are stochastic exponential growth rates analyzed in [AK].

5.3. Corollary. *If system (2.1) is completely controllable on $\mathbb{P}$, then $\mathscr{K}$, $\tilde{\mathscr{K}}$, and $\mathscr{K}^*$ can be realized from all $x_0 \neq 0$, and $\tilde{\mathscr{K}} = \mathscr{K}^*$.*

5.4. *Remark.* We have seen in the proofs of Theorems 5.1 and 5.2 that

$$\mathscr{K} = \sup_{u \in \mathscr{U}} \lambda(x_0, u) \qquad \text{for all} \quad x_0 \neq 0,$$

$$\text{i.e., } \mathscr{K} = \sup_{x_0 \neq 0} \sup_{u \in \mathscr{U}} \lambda(x_0, u)$$
$$= \inf_{x_0 \neq 0} \sup_{u \in \mathscr{U}} \lambda(x_0, u) = \sup_{p_0 \in \operatorname{int} C} \sup_{u \in \mathscr{U}} \lambda(p_0, u),$$

$$\mathcal{K}^* = \inf_{u \in \mathcal{U}} \lambda(x_0, u) \qquad \text{for all} \quad x_0 \text{ with } \frac{x_0}{|x_0|} \in C^-,$$

$$\text{i.e., } \mathcal{K}^* = \inf_{x_0 \neq 0} \inf_{u \in \mathcal{U}} \lambda(x_0, u) = \inf_{p_0 \in C^-} \inf_{u \in \mathcal{U}} \lambda(p_0, u),$$

$$\tilde{\mathcal{K}} = \inf_{u \in \mathcal{U}} \lambda(x_0, u) \qquad \text{for all} \quad x_0 \text{ with } \frac{x_0}{|x_0|} \in \operatorname{int} C,$$

$$\text{i.e., } \tilde{\mathcal{K}} = \inf_{p_0 \in \operatorname{int} C} \inf_{u \in \mathcal{U}} \lambda(p_0, u).$$

It remains an open question at this moment, how to characterize $\inf_{u \in \mathcal{U}} \lambda(x_0, u)$ for $x_0 \neq 0$, when $x_0/|x_0|$ is not in int C or C^-. The answer to this question requires a complete characterization of the control properties of system (2.1) on $\mathbb{P}$, which will be addressed elsewhere. Here we would like to point out that the above characterizations allow us to use $\mathcal{K}$ for the definition of a stability radius of the linear system $\dot{x} = Ax$, when A is a stable matrix, and to use $\tilde{\mathcal{K}}$ and $\mathcal{K}^*$ for the definition of instability radii of $\dot{x} = Ax$ when A is unstable. For details on these concepts and corresponding robust design results compare [CK2].

Finally we consider the quantities $\hat{\mathcal{K}}$ and $\hat{\mathcal{K}}^*$, which are the extremal growth rates, uniformly with respect to the initial value.

5.5. THEOREM. (i) $\hat{\mathcal{K}} > -\infty, \hat{\mathcal{K}}^* < +\infty$.

(ii)
$$\hat{\mathcal{K}} := \sup_{u \in \mathcal{U}} \inf_{x_0 \neq 0} \lambda(x_0, u)$$

$$= \sup\{ \inf_{p_0 \in C^-} \lambda(p_0, u); u \in \mathcal{U} \textit{ with } s(\cdot, p_0, u) \subset C^- \};$$

$$\hat{\mathcal{K}}^* := \inf_{u \in \mathcal{U}} \sup_{x_0 \neq 0} \lambda(x_0, u)$$

$$= \inf\{ \sup_{p_0 \in \operatorname{int} C} \lambda(p_0, u); u \in \mathcal{U} \textit{ with } s(\cdot, p_0, u) \subset \operatorname{int} C \}$$

$$= \inf_{u \in \mathcal{U}} \sup_{p_0 \in \operatorname{int} C} \lambda(p_0, u).$$

Proof. (i) By Proposition 3.3(i) $\hat{\beta} > -\infty$ and $\hat{\beta}^* < +\infty$, and by Theorem 4.10 $\hat{\mathcal{K}} = \hat{\beta}$ and $\hat{\mathcal{K}}^* = \hat{\beta}^*$.

(ii) For each $u \in \mathcal{U}$, $\sup_{x_0 \neq 0} \lambda(x_0, u)$ is attained on a basis in int C.

Furthermore, int C is invariant and therefore $\hat{\mathcal{K}}^* = \inf_{u \in \mathcal{U}} \sup_{p_0 \in \operatorname{int} C} \lambda(x_0, u)$. The result for $\hat{\mathcal{K}}$ follows this via time reversal. But note that C is not an invariant set of (2.1); hence the sup has to be taken over all $u \in \mathcal{U}$ with $s(\cdot, p_0, u) \subset C^-$. ∎

6. An Example: The Parameter Controlled Linear Oscillator

In this section we will discuss a typical two dimensional bilinear control system, the parameter controlled linear oscillator. Computing the exponential growth rates (1.3)–(1.5) means in general, solving the associated infinite time optimal control problem based on formula (2.3) for the individual Lyapunov exponents. By Theorems 4.1 and 4.10 it is sufficient to consider periodic, piecewise constant controls that lead to periodic solutions of the projected system (2.1). For this example, the optimal control problems can be reduced to two dimensional optimization problems, which can be solved numerically, and some growth rates can be computed directly.

Consider the linear oscillator with bounded controls

$$\ddot{y} + 2b\dot{y} + (1+u)\, y = 0, \tag{6.1}$$

or with $x = (x_1, x_2) = (y, \dot{y})$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} x + u \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x =: A(u)\, x. \tag{6.2}$$

The uncontrolled part of (6.2) is exponentially stable iff $b > 0$, marginally stable iff $b = 0$, and exponentially unstable iff $b < 0$. We want to investigate the following problem: If $b \geqslant 0$, does there exist a (bounded) control function, such that (6.2) becomes exponentially unstable, and if $b < 0$, can the system be stabilized. In particular, we want to compute for each damping b and a given range of control values the exponential growth rates (1.3)–(1.5). Choose

$$U = \sigma\Omega, \qquad \Omega = [A, B] \subset \mathbb{R} \tag{6.3}$$

to study this problem depending on the control range via $\sigma \geqslant 0$.

In order to obtain an algorithm for the computation of all exponential growth rates, we will use the methods developed in this paper and study the projected system. To make the notation simpler, we first use the transformation $\bar{y} = y \exp(bt)$ and obtain

$$\dot{\bar{x}} = \begin{pmatrix} 0 & 1 \\ -1+b^2 & 0 \end{pmatrix} \bar{x} + u \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \bar{x} =: \bar{A}(u)\, \bar{x}. \tag{6.4}$$

Note that $\lambda(x_0, u) = -b + \bar{\lambda}(x_0, u)$.

Projection of (6.4) onto the projective space $\mathbb{P}$ in $\mathbb{R}^2$ yields with $p = (\cos\varphi, \sin\varphi)$, $\varphi \in [0, \pi)$

$$\begin{aligned} \dot{\varphi} &= -\sin^2 \varphi(t) + (b^2 - 1 - u(t)) \cos^2 \varphi(t) =: f(\varphi, u) \\ q(\varphi, u) &= (b^2 - u) \sin\varphi \cos\varphi \end{aligned} \tag{6.5}$$

with q as defined in (2.1), here expressed in terms of the angle φ.

The Control Sets of (6.5) *on* $\mathbb{P}$

Depending on the parameters b and σ, the control sets can be of the following type (compare Fig. 1):

(i) $(b^2-1)/\sigma < A$: U is in the "rotation regime" of (6.6) and $C = C^- = \mathbb{P}$.

(ii) $A \leqslant (b^2-1)/\sigma < B$: U is in the rotation and in the "switching curve" regime of (6.6), where the switching curves are the zeros of $f(\varphi, b, u)$, here $C = C^- = \mathbb{P}$.

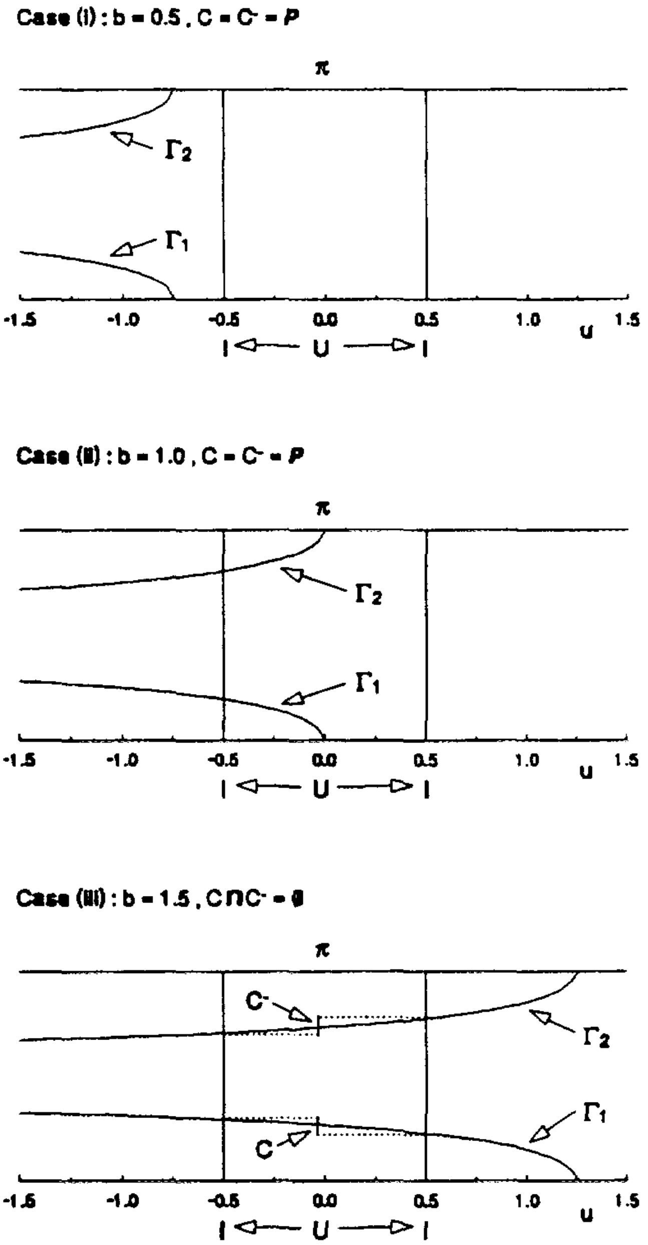

FIG. 1. Control sets for various values of b; $N = [-1, 1]$, $\sigma = 0.5$.

(iii) $B \leqslant (b^2-1)/\sigma$: U is in the switching curve area, and there are two disjoint control sets C and C^-. Note, however, that for $(b^2-1)/\sigma = B$ we have $C \cap C^* \neq \phi$ (compare Proposition 2.3).

As in Fig. 1 we will denote the switching curves by

$$\Gamma_1(u) = \arctan \sqrt{b^2-1-u}, \qquad u \leqslant b^2-1$$

$$\Gamma_2(u) = \pi - \Gamma_1(u), \qquad u \leqslant b^2-1.$$

The Lyapunov Exponents for Constant Controls

For $u(t) \equiv u \in U$ we obtain

$$\text{if } u \geqslant b^2-1\text{: } \bar{\lambda}(x_0, u) = 0 \qquad \text{for all} \quad x_0 \neq 0,$$

$$\text{if } u < b^2-1\text{: } \bar{\lambda}(x_0, u) = q(\Gamma_1(u), u) = \sqrt{b^2-1-u}$$

$$\text{for all} \quad x_0 \neq 0 \text{ with } \varphi_0 \notin \Gamma_2(u)$$

$$\bar{\lambda}(x_0, u) = q(\Gamma_2(u), u) = -\sqrt{b^2-1-u}$$

$$\text{for} \quad x_0 \neq 0 \text{ with } \varphi_0 \in \Gamma_2(u).$$

The Individual Lyapunov Exponents $\lambda(x_0, u)$

Using the results from Section 4, we have to compute $\bar{\lambda}(x_0, u)$ only for periodic, piecewise constant $u \in \mathscr{U}$. According to formula (2.3), for each constant piece u: $[0, t_u] \to U$, $u(t) \equiv u$ we need $\int_0^{t_u} q(u(\tau), s(\tau, s_0, u))\, d\tau$. This integral and the time t_u can be expressed in terms of the coordinates of the solution on $\mathbb{P}$:

$$\begin{aligned} t(u, \varphi) &= \int \frac{1}{|f(\varphi, u)|}\, d\varphi = \int \frac{1}{1+(u-b^2)\cos^2\varphi}\, d\varphi \\ &= \frac{1}{\sqrt{1+u-b^2}} \arctan \frac{\tan\varphi}{\sqrt{1+u-b^2}} \qquad \text{for} \quad u > b^2-1 \\ &= -\cot\varphi \qquad \text{for} \quad u = b^2-1 \\ &= -\frac{1}{2\sqrt{-1-u+b^2}} \ln \frac{\tan\varphi - \sqrt{-1-u+b^2}}{\tan\varphi + \sqrt{-1-u+b^2}} \qquad \text{for} \quad u < b^2-1 \end{aligned} \tag{6.6}$$

and

$$\begin{aligned}
\bar{\lambda}(u, \varphi) &:= \int q(u(\tau), s(\tau, s_0, u))\, d\tau = \int \frac{q(\varphi, u)}{|f(\varphi, u)|}\, d\varphi \\
&= -\frac{1}{2}(b^2 - u) \int \frac{dz}{1 + (u - b^2)\, z} \qquad \text{with} \quad z = \cos^2 \varphi \\
&= \frac{1}{2} \ln(1 + (u - b^2)\, z) \qquad \text{for} \quad u \neq b^2 \text{ and } 1 + (u - b^2)\, z > 0 \\
&= 0 \qquad \text{for} \quad u = b^2 \\
&= -\frac{1}{2} \ln(1 + (u - b^2)\, z) \qquad \text{for} \quad u \neq b^2 \text{ and } 1 + (u - b^2)\, z < 0. \qquad (6.7)
\end{aligned}$$

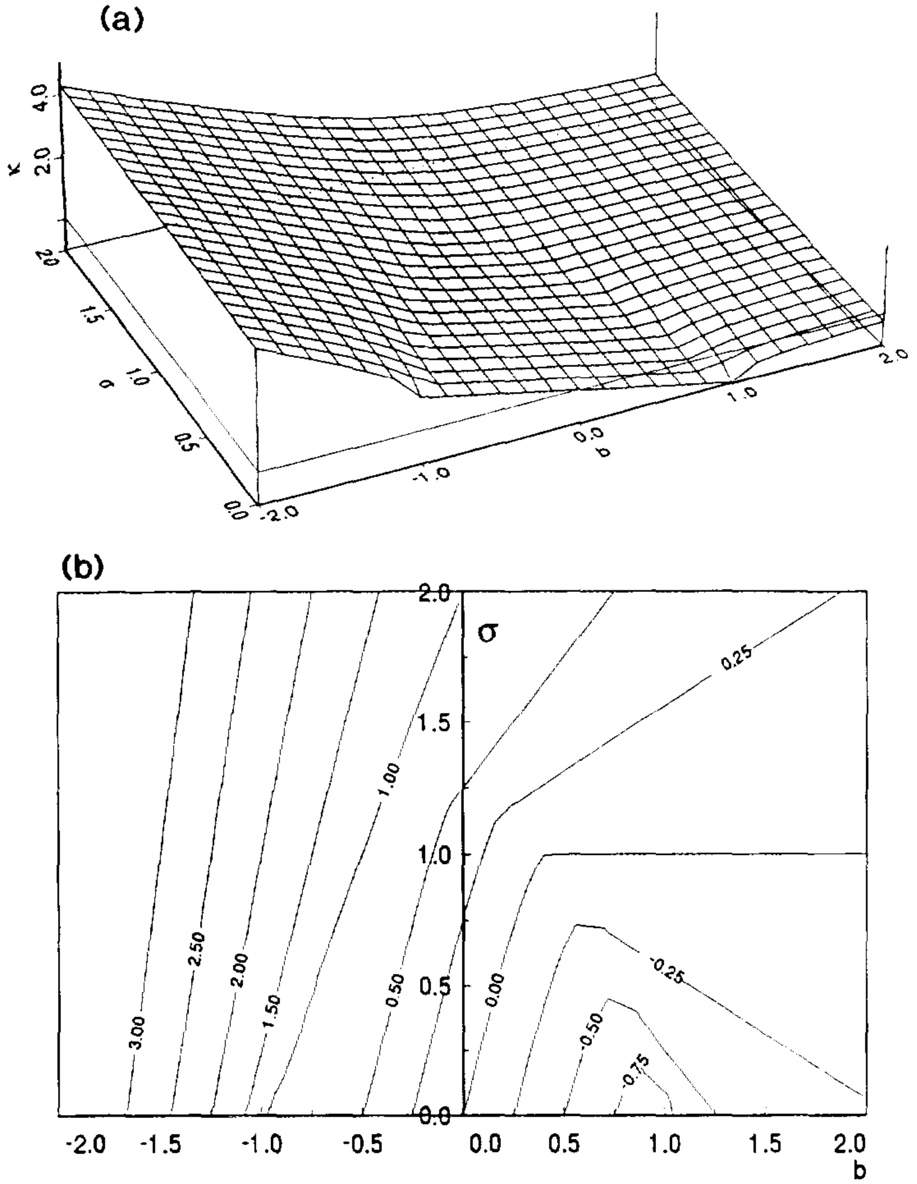

FIG. 2. (a) Maximal Lyapunov exponent $\mathscr{K}(b, \sigma)$, $\Omega = [-1, 1]$; (b) level curves of (a) with control range $\Omega = [-1, 1]$; (c) maximal Lyapunov exponent $\mathscr{K}(b, 0.5)$, $\Omega = [-1, 1]$.

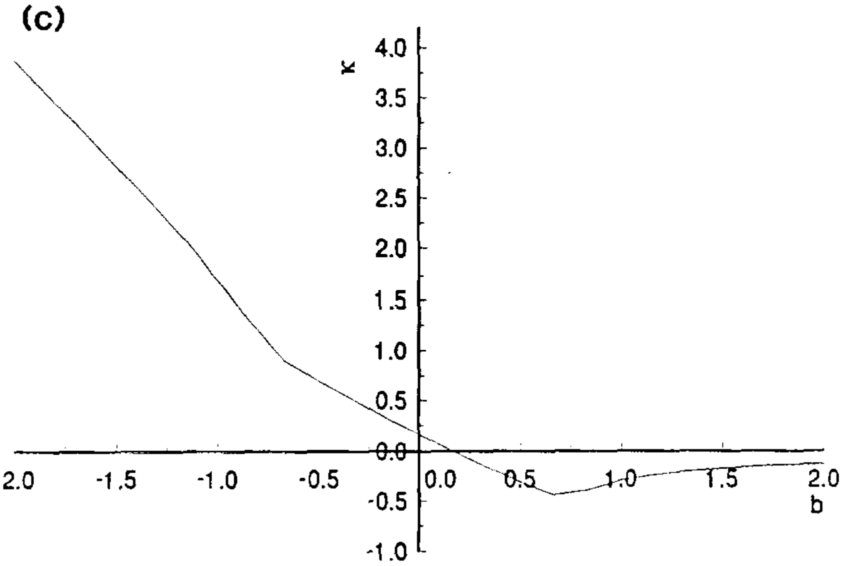

FIG. 2—*Continued*

From the monotonicity of f and q it is clear that solutions in the area between the switcing curves Γ_1 and Γ_2, where f is positive, do not contribute to the extremal Lyapunov exponents.

The Maximal and the Minimal Lyapunov Exponent $\mathscr{K}$ and $\mathscr{K}^*$

Using Eqs. (6.6) and (6.7), one can reformulate the optimal control problem based on formula (2.3) as a 2-dimensional optimization problem: Note that for b fixed the function q, defined in (6.5), is monotone increasing in u for all $\varphi \in [\pi/2, \pi)$, and monotone decreasing for $\varphi \in [0, \pi/2)$, while $f(\varphi, \cdot)$ is decreasing for all $\varphi \in [0, \pi)$. Therefore it suffices to consider piecewise constant controls with at most one switch in $(0, \pi/2)$, one switch in $(\pi/2, \pi)$, and switches at $\varphi = 0$ and $\varphi = \pi/2$, where q changes its sign. This leads to the following 2-dimensional optimization problem for $\mathscr{K}$:

Given $b \in \mathbb{R}$, $[A, B] \subset \mathbb{R}$, and $\sigma \geqslant 0$, consider

$$\alpha_1 = \max_{u \in [\sigma A, \sigma B]} \max_{\psi \in [0, \pi/2]} J_1(u, \psi)$$

$$\alpha_2 = \max_{u \in [\sigma A, \sigma B]} \max_{\psi \in [\pi/2, \pi)} J_2(u, \psi);$$

then

$$\mathscr{K}(b, \sigma) = -b + \alpha_1 + \alpha_2, \tag{6.8}$$

where

$$J_1(u,\psi)\begin{cases} =\sqrt{b^2-1-\sigma A} & \text{if } \sigma B\leqslant b^2-1 \\ =\sqrt{b^2-1-u} & \\ \qquad \text{if } \psi\leqslant \arctan\sqrt{b^2-1-u} \text{ and } u\leqslant b^2-1 & \\ =\left(t(\sigma B,\psi)+t\left(u,\dfrac{\pi}{2}\right)-t(u,\psi)\right)^{-1} & \\ \quad \times(\bar{\lambda}(\sigma B,\psi)-\bar{\lambda}(\sigma B,0)-\bar{\lambda}(u,\psi)) & \\ \qquad \text{if } \psi>\arctan\sqrt{b^2-1-u} \text{ or } u>b^2-1, & \end{cases}$$

with $t(u,\varphi)$ and $\bar{\lambda}(u,\varphi)$ defined as in (6.6) and (6.7), and

$$J_2(u,\psi)\begin{cases} =-\sqrt{b^2-1-\sigma B} & \text{if } \sigma B\leqslant b^2-1 \\ =-\sqrt{b^2-1-u} & \\ \qquad \text{if } \psi\geqslant \pi-\arctan\sqrt{b^2-1-u} \text{ and } u\leqslant b^2-1 & \\ =-J_1(u,\pi-\psi) & \\ \qquad \text{if } \psi<\pi-\arctan\sqrt{b^2-1-u} \text{ or } u>b^2-1, & \end{cases}$$

where the last equality holds because $f(\psi,u)=f(\pi-\psi,u)$ and $q(\psi,u)=-q(\pi-\psi,u)$ for $\psi\in(\pi/2,\pi)$.

Recall that by Theorem 5.1 the maximal exponent $\mathscr{K}$ can be realized for all $x_0\neq 0$.

Figure 2(a) shows a 3-dimensional graph of $\mathscr{K}(b,\sigma)$, and Figure 2(b) the corresponding level curves. These figures demonstrate an interesting feature of the system: For $\sigma_0>1$ the exponent $\mathscr{K}(b,\sigma_0)$ is monotone decreasing with the damping parameter b. For $\sigma_0=1$ the system hits a threshold at $b_0\sim 0.405$ (as computed from Eq. (6.8)): $\mathscr{K}(b,1)=0$ for $b\geqslant b_0$, i.e., increasing the damping above b_0 does not decrease the maximal destabilization rate further. For $\sigma_0<1$ this trend is even reversed: For each $\sigma_0\in(0,1)$ there exists $b(\sigma_0)\in(b_0,1)$ such that $\mathscr{K}(b,\sigma_0)$ is decreasing for $b<b(\sigma_0)$, and increasing (but negative) for $b>b(\sigma_0)$, i.e., further increase in damping above $b(\sigma_0)$ diminishes the stability reserve of the system (overdamping); see Fig. 2(c).

The minimal Lyapunov exponent $\mathscr{K}^*$ can be computed via the minimization problem corresponding to (6.8). However, because of the symmetries in f and q, one obtains immediately:

$$\mathscr{K}^*(b,\sigma)=-b-\alpha_1-\alpha_2. \tag{6.9}$$

According to Theorem 5.1, $\mathscr{K}^*$ can only be realized from C^{-}, i.e., for the control sets in Cases (i) and (ii), $C^{-}=\mathbb{P}$, and in Case (iii), C^{-} is a proper subset of $\mathbb{P}$. Figure 3 shows the open control set C^{-} for $\sigma=0.5$, depending on b. Figures 4(a) and 4(b) present the 3-dimensional graph and the level curves of $\mathscr{K}^*(b,\sigma)$. Here a similar effect as for the maximal Lyapunov

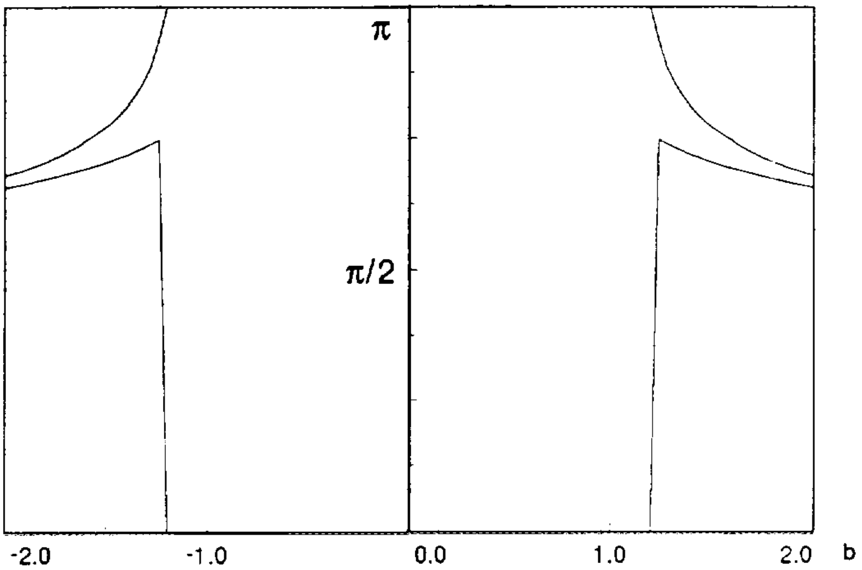

FIG. 3. Minimal control set for $b \in [-2, 2]$, $\Omega = [-1, 1]$, $\sigma = 0.5$.

exponent occurs, but now for negative damping: Again the system hits a threshold for $\sigma_0 = 1$ at $b_0 \sim -0.405$, and for all $\sigma_0 \in (0, 1)$ there exists a $b(\sigma_0) \in (-1, b_0)$ such that the minimal exponent decreases toward 0 as $b \to -\infty$ (see Fig. 4(c)).

As pointed out in the Introduction and in Remark 5.4, the maximal (and minimal) Lyapunov exponents can be used to define stability radii (and instability raddii, respectively) for linear equations $\dot{x} = Ax$. The level curves $\mathscr{K} = 0$ (and $\mathscr{K}^* = 0$) characterize these radii (compare [CK2] for details).

The Gonzalez Criterion for Exponential Stability

An algorithm to study the destabilization problem for general two-dimensional linear systems with bounded intervals as input range was given by Gonzalez [Go]. For our example his result says:

The system (6.2) is exponentially stable for all $u \in \mathscr{U}$ iff

(i) $\max_{u \in U}$ trace $A(u) < 0$ and $\min_{u \in U} \det A > 0$, and

(ii) (a) $(\text{trace } A(u))^2 - 4 \det A(u) \geq 0$ for all $u \in U$, or

(b) the solution $\hat{y}(\hat{T})$ of the following optimal control problem satisfies $\hat{y}(\hat{T}) < 1$.

Here $\hat{y}(\hat{T})$ is the solution of

$$\text{maximize } y(T) \text{ with } T > 0, \text{ where } y(t) \text{ solves (6.1)}$$
$$\text{with } y(0) = -1,\ \dot{y}(0) = 0,\ \dot{y}(t) \neq 0 \text{ for } t \in [0, T]. \qquad (6.10)$$

Gonzalez shows that the following control function solves problem (6.10),

$$\hat{u}(t) = \begin{cases} B & \text{for } y(t, \hat{u}) < 0 \\ A & \text{for } y(t, \hat{u}) \geq 0, \end{cases}$$

and $\hat{T} = \inf\{t > 0, \dot{y}(t) = 0\}$.

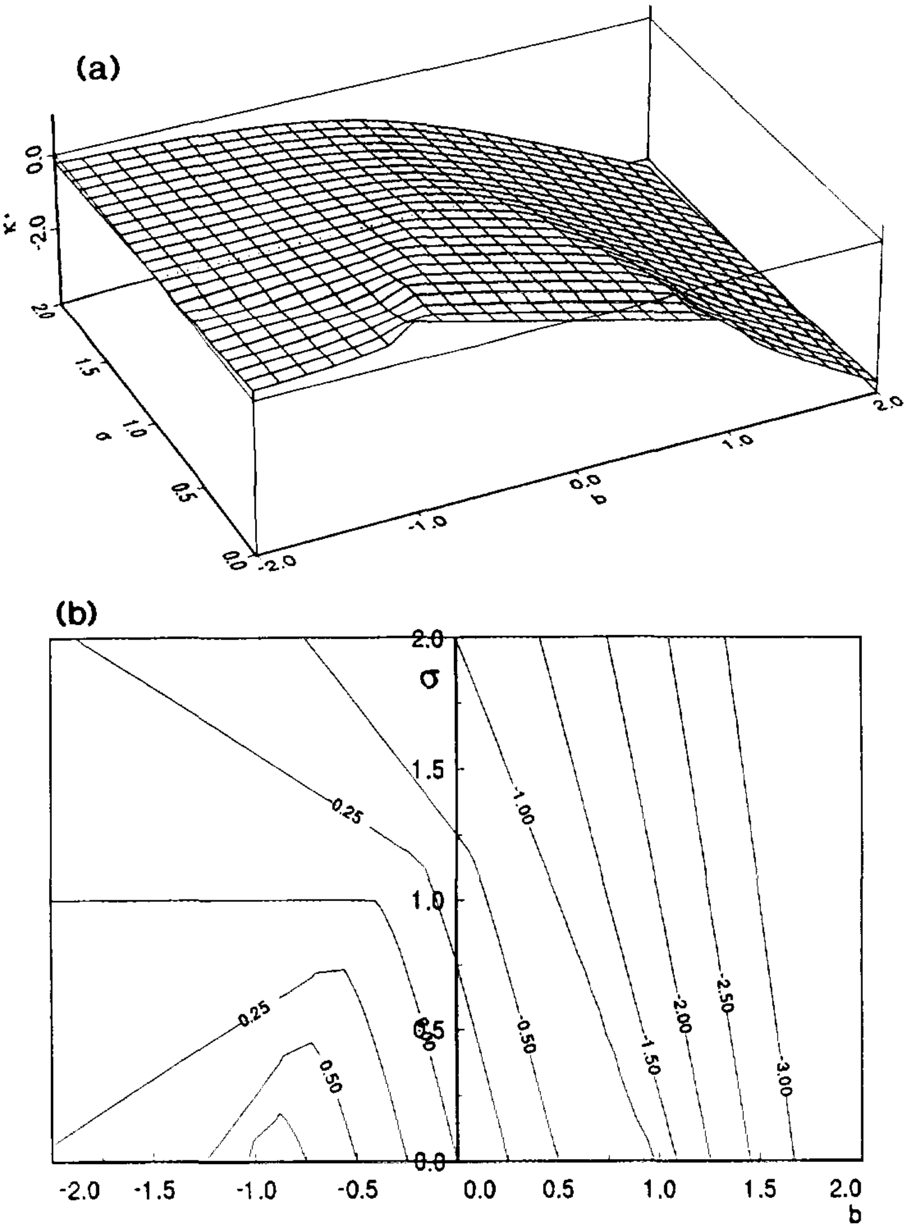

FIG. 4. (a) Minimal Lyapunov exponent $\mathscr{K}^*(b, \sigma)$, $\Omega = [-1, 1]$; (b) level curves of (a) with control range $\Omega = [-1, 1]$; (c) minimal Lyapunov exponent $\mathscr{K}^*(b, 0.5)$, $\Omega = [-1, 1]$.

Conditions (i) and (ii)(a) above are obvious, because they concern the real parts of the eigenvalues of $A(u)$ for constant $u \in U$. Condition (ii)(b) is related to a periodic, piecewise constant control, which solves the optimal control problem (1.3) for $\mathscr{K}$. We have $\hat{y}(\hat{T}) < 1$ iff the Lyapunov exponent corresponding to $\hat{y}$ is negative. Therefore (6.10) reads in our setup:

$$\begin{aligned}&\text{maximize } \lambda(\varphi_0, u), \text{ where}\\ &\varphi(t, \varphi_0, u) \text{ solves (6.5) with } \varphi_0 = \arctan b.\end{aligned} \tag{6.11}$$

(Note that $y(0) = 0$ implies $\dot{y} = b\bar{y}(0)$, and hence in terms of the angle $\varphi_0 = \arctan b$.)

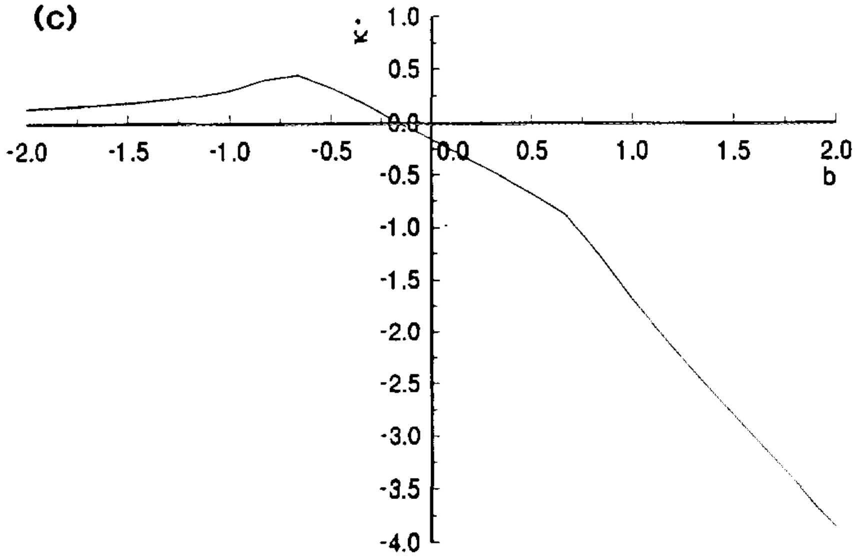

FIG. 4—*Continued*

The following control function solves the optimal control problem (6.11) for $b>0$ and $\sigma B > b^2-1$:

$$u^0(t) = \begin{cases} u_1^0(t) \equiv B & \text{for} \quad \varphi(t, \varphi_0, u^0) \in [\pi/2, \pi) \\ u_2^0(t) \equiv A & \text{for} \quad \varphi(t, \varphi_0, u^0) \in [\arctan b, \pi/2) \\ u_3^0(t) \equiv B & \text{for} \quad \varphi(t, \varphi_0, u^0) \in [0, \arctan b). \end{cases} \tag{6.12}$$

(Note that $y=0$ means $\bar{y}=0$; hence the switching point in terms of the angle is $\varphi_1 = \pi/2$.)

For the corresponding Lyapunov exponent of this periodic, piecewise constant control we obtain using (6.6) and (6.7),

$$\bar{\lambda}(\varphi_0, u^0) = \frac{1}{t(u_1^0) + t(u_2^0) + t(u_3^0)} (\bar{\lambda}(u_1^0) + \bar{\lambda}(u_2^0) + \bar{\lambda}(u_3^0)), \tag{6.13}$$

where

$$t(u_1^0) = \frac{\pi}{2\sqrt{1+\sigma B - b^2}}$$

$$t(u_2^0) = \begin{cases} \dfrac{1}{\sqrt{1+\sigma A - b^2}} \left(\dfrac{\pi}{2} - \arctan \dfrac{b}{\sqrt{1+\sigma A - b^2}} \right) & \text{if} \quad \sigma A > b^2 - 1 \\ \dfrac{1}{b} & \text{if} \quad \sigma A = b^2 - 1 \\ \dfrac{-1}{2\sqrt{b^2-1-\sigma A}} \ln \dfrac{b - \sqrt{b^2-1-\sigma A}}{b + \sqrt{b^2-1-\sigma A}} & \text{if} \quad \sigma A < b^2 - 1 \end{cases}$$

$$t(u_3^0) = \frac{1}{\sqrt{1+\sigma B - b^2}} \arctan \frac{b}{\sqrt{1+\sigma B - b^2}}$$

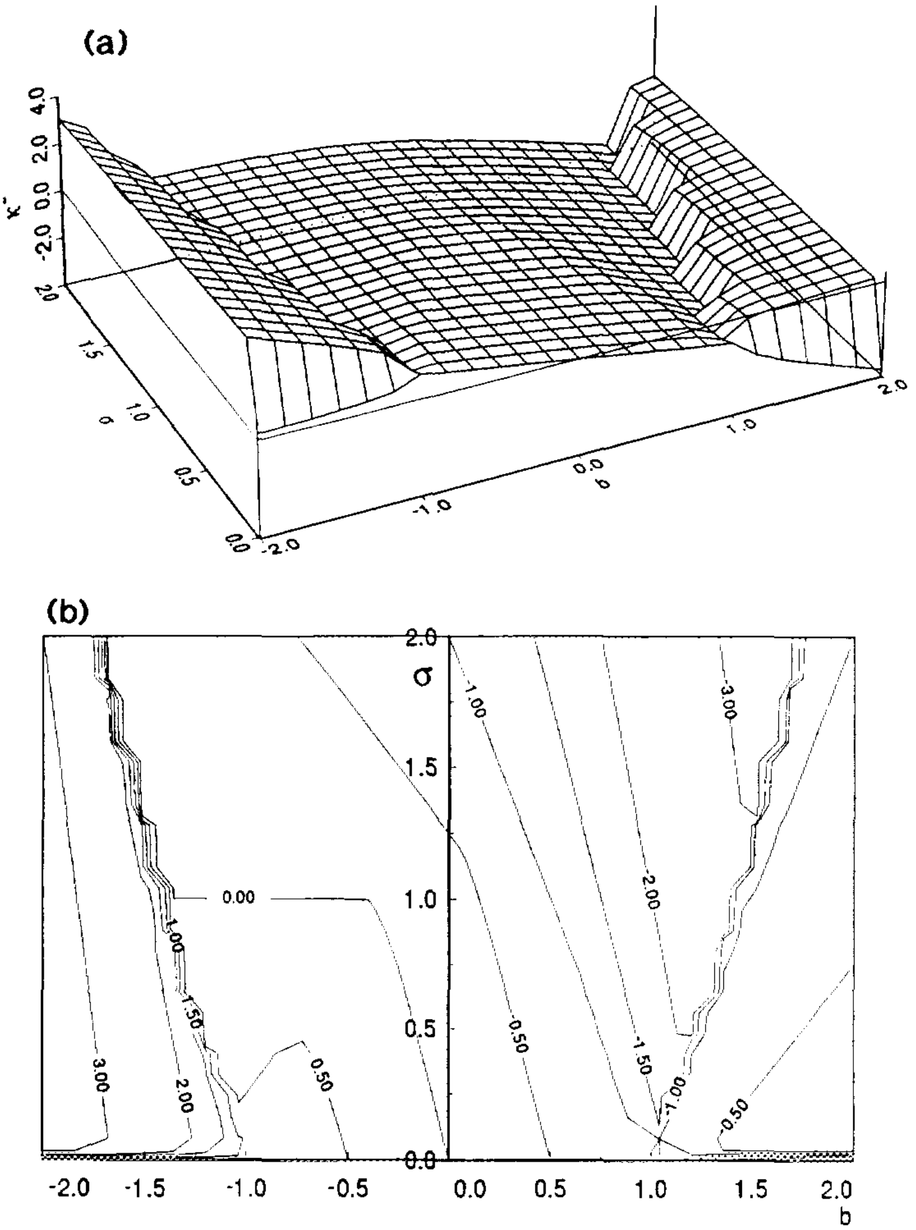

FIG. 5. (a) Minimal Lyapunov exponent $\mathcal{K}^-(b, \sigma)$, $\Omega = [-1, 1]$; (b) level curves of (a) with control range $\Omega = [-1, 1]$; (c) minimal Lyapunov exponent $\mathcal{K}^-(b, 0.5)$, $\Omega = [-1, 1]$.

and

$$\bar{\lambda}(u_1^0) = \frac{1}{2} \ln(1 + \sigma B - b^2)$$

$$\bar{\lambda}(u_2^0) = -\frac{1}{2} \ln \frac{\sigma A + 1}{b^2 + 1}$$

$$\bar{\lambda}(u_3^0) = \frac{1}{2} \ln \frac{\sigma B + 1}{(b^2 + 1)(1 + \sigma B - b^2)},$$

i.e., $\bar{\lambda}(u_1^0) + \bar{\lambda}(u_2^0) + \bar{\lambda}(u_3^0) = (1/2) \ln(1 + \sigma B)/(1 + \sigma A)$.

(c)

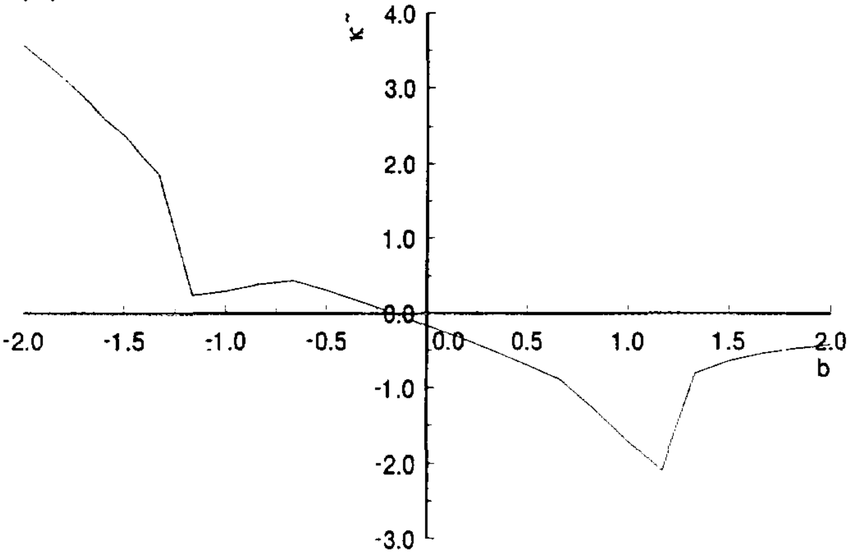

FIG. 5—*Continued*

Note that the terms in Eq. (6.13) depend on b, A, B, and σ.

For a given damping $b > 0$ and control range Ω with $A < 0 < B$, define

$$r_L(b) = \inf\{\sigma \geqslant 0;\ \lambda(\varphi_0, u^0) \geqslant 0\},$$

where $\lambda(\varphi_0, u^0) = -b + \bar{\lambda}(\varphi_0, u^0)$. $r_L(b)$ gives the minimal size of U, such that system (6.1) can be destabilized. This, of course, corresponds to the level 0 of $\mathscr{K}(b, \sigma)$, the maximal Lyapunov exponent of (6.1) depending on b and σ (compare Figure 2(b)).

The Growth Rate $\tilde{\mathscr{K}}$

According to Corollary 5.3 we have $\mathscr{K}^* = \tilde{\mathscr{K}}$ if $C^- = \mathbb{P}$, i.e., for Cases (i) and (ii) above. In Case (iii), U is in the switching curve area, and hence

$$\tilde{\mathscr{K}}(b, \sigma) = -b + \sqrt{b^2 - 1 - \sigma B} \qquad \text{if} \quad \sigma B \leqslant b^2 - 1. \tag{6.14}$$

Figures 5(a) and 5(b) present the 3-dimensional graph and the level curves of $\tilde{\mathscr{K}}(b, \sigma)$. Figure 5(c) shows the behavior of $\tilde{\mathscr{K}}(b, \sigma_0)$ for $\sigma_0 = 0.5$. The $\tilde{\mathscr{K}} = 0$ curve agrees with $\mathscr{K}^* = 0$ for $b \in (-\sqrt{2}, 0]$; for $b \leqslant -\sqrt{2}$ the curves disagree according to the bifurcation of the control sets.

The Uniform Growth Rates $\hat{\mathscr{K}}$ *and* $\hat{\mathscr{K}}^*$

Using Theorem 5.5(ii) and the symmetries of the functions f and q it is easy to see that one obtains for the uniform growth rates:

$$\begin{aligned} &\hat{\mathscr{K}} = \hat{\mathscr{K}}^* = -b && \text{if} \quad C = C^- = \mathbb{P}, \\ &\left.\begin{aligned} \hat{\mathscr{K}} &= -b - \sqrt{b^2 - 1 - \sigma B} \\ \hat{\mathscr{K}}^* &= -b + \sqrt{b^2 - 1 - \sigma A} \end{aligned}\right\} && \text{if} \quad C \cap C^- = \varnothing. \end{aligned} \tag{6.15}$$

In all cases we have

$$-\infty < \mathscr{K}^* < \hat{\mathscr{K}} \leqslant \hat{\mathscr{K}}^* < \mathscr{K} < +\infty,$$

and $\hat{\mathscr{K}} < \hat{\mathscr{K}}^*$ holds, if $C \cap C^{-} = \varnothing$. In our example all uniform growth rates can be realized using constant controls.

7. Maximal Lyapunov Exponents and High Gain Stability

In this section we use the techniques developed so far to obtain a criterion for exponential high gain stability of bilinear systems. Similar tools for linear, stochastic systems were developed in [ACW, A], but their approach uses rotation matrices, which may not be appropriate for the special structure of a given system.

We will formulate our results in the context of

$$\dot{x}_\varepsilon(t) = \left(A_0 + \frac{1}{\varepsilon}\, u\left(\frac{t}{\varepsilon}\right)\right) x_\varepsilon(t) \qquad \text{in } \mathbb{R}^d, \tag{7.1}$$

where A_0 is a given $d \times d$ matrix and $u \in \mathscr{U} := \{u\colon \mathbb{R}^+ \to \Omega,$ locally integrable$\}$, with $\Omega \subset gl(d, \mathbb{R})$ bounded and $0 \in \operatorname{co} \Omega$, the convex hull of Ω. (Note that $0 \in \operatorname{co} \Omega$ is not really a restriction, because one can choose $B_0 \in \Omega$ and consider the system $\dot{x}_\varepsilon = (A_0 + B_0 + (1/\varepsilon)\, v(t/\varepsilon))\, x_\varepsilon$ with $\Omega' = \{B - B_0,\ B \in \Omega\}$.) $\varepsilon > 0$ is a small parameter.

The semigroup of (7.1) is given by

$$\mathscr{S}_\varepsilon := \left\{\exp\left(t_n \varepsilon\left(A_0 + \frac{1}{\varepsilon} B_n\right)\right) \cdot \ldots \cdot \exp\left(t_1 \varepsilon\left(A_0 + \frac{1}{\varepsilon} B_1\right)\right)\right.$$

$$\left. t_j \geqslant 0,\ B_j \in \Omega,\ j = 1, ..., n \in \mathbb{N}\right\},$$

which coincides with the semigroup $\bar{\mathscr{S}}$ of the system

$$\dot{\bar{x}}_\varepsilon(t) = (\varepsilon A_0 + u(t))\, \bar{x}_\varepsilon(t). \tag{7.2}$$

The projected system of (7.1) reads

$$\dot{s}_\varepsilon(t) = h_0(s_\varepsilon(t)) + \frac{1}{\varepsilon} h_1\left(u\left(\frac{t}{\varepsilon}\right), s_\varepsilon(t)\right) =: h_\varepsilon(u, s), \tag{7.3}$$

where $h_0(s) = (A_0 - q_0(s)\,\mathrm{Id})\, s$, $h_1(u, s) = (u - q_1(u, s)\,\mathrm{Id})\, s$, and $q_0(s) = s^T A_0 s$, $q_1(u, s) = s^T u s$. Denote $q_\varepsilon(u, s) = q_0(s) + (1/\varepsilon)\, s^T u(t/\varepsilon)\, s$. We assume throughout this section

$$\dot{s} = h_1(u, s), \qquad u \in \mathscr{U} \text{ is completeley controllable on } \mathbb{P}. \tag{A}$$

Condition (A) means that the control structure of the system is rich enough to ensure controllability on $\mathbb{P}$ without using the systems matrix A_0. This is, e.g., not true for the linear oscillator treated in Section 6.

7.1. LEMMA. *Let α be any of the exponential growth rates defined in Sections 1–3 for system (7.1), and let $\bar{\alpha}$ be the corresponding quantity for (7.2). Then $\alpha = (1/\varepsilon)\,\bar{\alpha}$.*

Proof. We have $\mathscr{S}_{\varepsilon,t} = \bar{\mathscr{S}}_t$, and the result for the rates defined in Section 3 follows from this. Theorems 4.1 and 4.10 then yield the claim for all $\mathscr{K}$-quantities, except for $\tilde{\mathscr{K}}$. For $\tilde{\mathscr{K}}$ note that $\lambda(s_0, u) = \limsup_{t\to\infty} (1/t) \int_0^t q_\varepsilon(u, s)\, d\tau = (1/\varepsilon) \limsup_{t\to\infty} (1/t) \int_0^t \bar{q}(u, s)\, d\tau$, where $\bar{q}(u, s) = \varepsilon s^T \underset{\sim}{A}_0 s + s^T u s$ corresponds to system (7.2), which implies $\tilde{\mathscr{K}} = (1/\varepsilon)\,\tilde{\bar{\mathscr{K}}}$. Furthermore, it is interesting to observe that for $\varepsilon > 0$ small enough, Assumption (A) implies that system (7.3) is completely controllable on $\mathbb{P}$ (see [S1]). Therefore, for these ε's we have that $\tilde{\mathscr{K}} = \mathscr{K}^*$ for system (7.1). ∎

7.2. LEMMA. *Fix $p_0 \in \mathbb{P}$ and let μ be a probability measure on $\mathbb{P}$. Then there is a sequence $\{u^k, k \geq 1\}$ in $\mathscr{U}$ such that*

$$\lim_{k\to\infty} \lim_{t\to\infty} \frac{1}{t} \int_0^t q(u^k(\tau), s(\tau, p_0, u^k(\tau)))\, d\tau = \int_{\mathbb{P}} s^T A_0 s \mu(ds),$$

where $s(t, p_0, u^k(t))$ solves the equation $\dot{s} = [u^k - s^T u^k s\, \mathrm{Id}]\, s$, and $q(u, s) = s^T(A_0 + u)\, s$.

Proof. Construct the sequence $\{u^k, k \geq 1\}$ in the following way: for each $k > 1$ decompose $\mathbb{P}$ into subsets $R_1^k, ..., R_k^k$ with $\mu(R_i^k) = 1/k$. (If μ has atoms, one may choose the corresponding point as multiple R_i^k's.) Pick a $p_i^k \in R_i^k$ and without loss of generality $p_1^k = p_0$ for all k. Now by (A) and the compactness of $\mathbb{P}$, the first hitting time map $h: \mathbb{P} \times \mathbb{P} \to \mathbb{R}$, defined by $h(x, y) = \inf_{u \in \mathscr{U}} \{t \geq 0; s(t, x, u) = y\}$ is uniformly bounded by say T (compare Proposition 2.3 in [CK1]). Define u^k such that the trajectory $s(\cdot, p_0, u^k)$ goes from p_0 to p_2^k in time $\leq T$, stays there for time k (which is possible since $0 \in \mathrm{co}\, \Omega$), then goes to p_3^k in time $\leq T$, stays for time k, etc.; from p_k^k it returns to p_0, stays for time k, etc.

Consider

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t q(u(\tau), s(\tau, p_0, u^k(\tau)))\, d\tau$$

$$= \frac{1}{T_k} \int_0^{T_k} q(u(\tau), s(\tau, p_0, u^k(\tau)))\, d\tau,$$

where T_k is the period length of $(s(\cdot, p_0, u^k), u^k)$. Since $T < \infty$, we have $T_k/k^2 \to 1$. And because Ω is bounded, say by c, we also have

$$\frac{1}{T_k}\int_0^{T_k} s(\tau, p_0, u^k)^T u^k(\tau)\, s(\tau, p_0, u^k)\, d\tau$$

$$\leqslant c\,\frac{kT}{T_k} \leqslant c\,\frac{kT}{k^2}\,\frac{k^2}{T_k} \to 0 \qquad \text{for} \quad k \to \infty.$$

Hence for all $n \in \mathbb{N}$ we find k large enough with

$$\frac{1}{T_k}\int_0^{T_k} q(u(\tau), s(\tau, p_0, u^k))\, d\tau$$

$$\leqslant \sum_{i=1}^{k} (p_i^k)^T A_0 p_i^k \frac{k}{T_k} + \frac{1}{n}$$

$$= \sum_{i=1}^{k} (p_i^k)^T A_0 p_i^k \frac{k^2}{T_k} \cdot \frac{1}{k} + \frac{1}{n}$$

$$\leqslant \sum_{i=1}^{k} (p_i^k)^T A_0 p_i^k \mu(R_i^k) + \frac{2}{n}$$

$$\to \int_{\mathbb{P}} s^T A_0 s \mu(ds) + \frac{2}{n} \qquad \text{as} \quad k \to \infty.$$

The lower bound can be proved exactly in the same manner, which implies the result. ∎

We now apply this lemma to obtain a high gain stabilization result for (7.1):

7.3. PROPOSITION. *Let μ be a probability measure on $\mathbb{P}$ and $p_0 \in \mathbb{P}$. Then for $\varepsilon > 0$ small enough there exists a sequence $\{u^k, k \geqslant 1\}$ in $\mathscr{U}$ such that*

$$\lim_{k \to \infty} \lim_{t \to \infty} \frac{1}{t}\int_0^t q_\varepsilon(u(\tau), s_\varepsilon(\tau, p_0, u^k))\, d\tau = \int_{\mathbb{P}} s^T A_0 s \mu(ds),$$

where s_ε denotes the solution of (7.3).

Proof. Assumption (A) implies that for ε small enough the system

$$\dot{\bar{s}}_\varepsilon = [\varepsilon A_0 + u - \bar{s}_\varepsilon^T(\varepsilon A_0 + u)\, \bar{s}_\varepsilon \operatorname{Id}]\, \bar{s}_\varepsilon, \qquad u \in \mathscr{U}$$

is controllable on $\mathbb{P}$ (see Sussmann [S1]). Fix $\varepsilon > 0$ and apply Lemma 7.2 to this system, in order to obtain $\{u^k, k \geqslant 1\} \subset \mathscr{U}$ with

$$\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{t}\int_0^t \bar{s}_\varepsilon(\tau, p_0, u^k)^T\left[A_0+\frac{1}{\varepsilon}u^k(\tau)\right]$$
$$\times\bar{s}_\varepsilon(\tau, p_0, u^k)\,d\tau=\int_{\mathbb{P}} s^T A_0 s\mu(ds).$$

Recall that $s_\varepsilon(t, p_0, u^k) := \bar{s}_\varepsilon(t/\varepsilon, p_0, u^k)$ solves (7.3). Hence

$$\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{t}\int_0^t s_\varepsilon(\tau, p_0, u^k)^T\left[A_0+\frac{1}{\varepsilon}u^k\left(\frac{\tau}{\varepsilon}\right)\right]s_\varepsilon(\tau, p_0, u^k)\,d\tau$$
$$=\lim_{k\to\infty}\lim_{t\to\infty}\frac{\varepsilon}{t}\int_0^{t/\varepsilon} s_\varepsilon(\varepsilon\tau, p_0, u^k)^T\left[A_0+\frac{1}{\varepsilon}u^k(\tau)\right]s_\varepsilon(\varepsilon\tau, p_0, u^k)\,d\tau$$
$$=\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} \bar{s}_\varepsilon(\tau, p_0, u^k)^T\left[A_0+\frac{1}{\varepsilon}u^k(\tau)\right]\bar{s}_\varepsilon(\tau, p_0, u^k)\,d\tau$$
$$=\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{t}\int_0^t \bar{s}_\varepsilon(\tau, p_0, u^k)^T\left[\frac{A_0+1}{\varepsilon}u^k(\tau)\right]\bar{s}_\varepsilon(\tau, p_0, u^k)\,d\tau$$
$$=\int_{\mathbb{P}} s^T A_0 s\mu(ds).\quad\blacksquare$$

7.4. Corollary. *$\limsup_{\varepsilon\to 0^+}\mathcal{K}_\varepsilon^* \leqslant \min_{p\in\mathbb{P}} p^T A_0 p$ and the minimal exponential growth rates $\mathcal{K}_\varepsilon^*$ of* (7.1) *can be realized for all $x_0\neq 0$, for ε small enough.*

Proof. The inequality follows from Proposition 7.3 by choosing $\mu=\delta_{p_0}$, the Dirac measure at p_0, where $p^T A_0 p$ attains it minimum. Furthermore, by Assumption (A) and Sussmann's theorem on controllability under small perturbations (see [S1]), we see that the projected system (7.3) is controllable for ε small enough. Hence for these ε's $\mathcal{K}_\varepsilon^*$ can be realized for all $x_0\neq 0$ by Corollary 5.3. ▮

We thus obtain the following necessary and sufficient criterion for high gain exponential stabilization of (7.1):

7.5. Theorem. *Suppose that system* (7.1) *satisfies Assumptions* (H) *and* (A). *Then it is exponentially stabilizable as $\varepsilon\downarrow 0$ for all $x_0\neq 0$ iff there exists $B\in \operatorname{co}\Omega$ with A_0+B not positive semidefinite.*

Proof. (i) If all matrices in $A_0+\Omega$ are positive semidefinite, then $p^T(A_0+B)\,p\geqslant 0$ for all $B\in\Omega$ all $p\in\mathbb{P}$, and the system cannot be stabilized exponentially.

(ii) If there is $B \in \text{co}\,\Omega$, such that $A_0 + B$ is not positive semidefinite, then the system

$$\dot{x} = \left[A_0 + B + \frac{1}{\varepsilon} u\left(\frac{t}{\varepsilon}\right) \right] x, \qquad u \in \Omega - B = \Omega' \tag{7.4}$$

can be stabilized by Corollary 7.4, and hence system (7.1) is stabilizable via high gain. All we have to show is that (7.4) satisfies our assumptions. Now Ω' is bounded if Ω is, and $0 \in \text{co}\,\Omega'$ by construction. It remains to show that Assumption (A) implies that

$$\dot{s}_\varepsilon = \frac{1}{\varepsilon} h_1 \left(u\left(\frac{t}{\varepsilon}\right), s_\varepsilon(t) \right), \qquad u \in \Omega',$$

is completely controllable on $\mathbb{P}$ for all ε small enough.

If $B \in \text{rel int co}\,\Omega$, where rel int denotes the relative interior with respect to the span of Ω, then this is obvious in the high gain context, i.e., $(1/\varepsilon) \to \infty$. Otherwise we can choose $B' \in \Omega$ such that $A_0 + B'$ is not positive semidefinite, but $B' \in \text{rel int co}\,\Omega$. ∎

References

[A] L. Arnold, "Stabilization by Noise Revisited," Report No. 208, Institute for Dynamical Systems, University of Bremen, 1989.

[ACW] L. Arnold, H. Crauel, and V. Wihstutz, Stabilization of linear systems by noise, *SIAM J. Control Optim.* **21** (1983), 451–461.

[AK] L. Arnold and W. Kliemann, "Large Deviations of Linear Stochastic Differential Equations," pp. 117–151, Lecture Notes in Control and Information Sciences, Vol. 96, Springer-Verlag, New York, 1987.

[AKO] L. Arnold, W. Kliemann, and E. Oeljeklaus, Lyapunov exponents of linear stochastic systems, *in* "Lyapunov Exponents" (L. Arnold and V. Wihstutz, Eds.), Lecture Notes in Mathematics, Vol. 1186, Springer-Verlag, New York/Berlin, 1986.

[AOP] L. Arnold, E. Oeljeklaus, and E. Pardoux, Almost sure and moment stability for linear Ito equations, *in* "Lyapunov Exponents" (L. Arnold and V. Wihstutz, Eds.), Lecture Notes in Mathematics, Vol. 1186, Springer-Verlag, New York/Berlin, 1986.

[AW] L. Arnold and V. Wihstutz, Eds., "Lyapunov Exponents," Lecture Notes in Mathematics, Vol. 1186, Springer-Verlag, New York, 1986.

[Bo] N. Bourbaki, "Elements de mathématique: Groupes et algèbre de Lie," Hermann, Paris, 1960–1972.

[BBM1] R. Bellman, J. Bentsman, and S. M. Meerkov, On the vibrational stabilizability of nonlinear systems, *J. Optim. Theory Appl.* **46** (1985), 421–430.

[BBM2] R. Bellman, J. Bentsman, and S. M. Meerkov, Vibrational control of nonlinear systems: Vibrational stabilizability, *IEEE Trans. Aut. Control* **AC-31** (1986), 710–716.

[C] L. Cesari, "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations," 3rd ed., Springer-Verlag, Berlin, 1971.

[CK1] F. Colonius and W. Kliemann, Infinite time optimal control and periodicity, *Appl. Math. Optim.* **20** (1989), 113–130.

[CK2] F. Colonius and W. Kliemann, Stability radii and Lyapunov exponents, *in* "Control of Uncertain Systems—Proceedings of the Bremen Conference, 1989" (D. Hinrichsen and B. Martensson, Eds.), pp. 19–55, Birkhäuser, Boston, 1990, in press.

[CK3] F. Colonius and W. Kliemann, Some aspects of control systems as dynamical systems, submitted for publication.

[D] E. B. Davies, "One-Parameter Semigroups," Academic Press, New York, 1980.

[Ga] J. P. Gauthier, "Structure des Systèmes Nonlinéaires," CNRS, Paris, 1984.

[Go] H. Gonzales, Estabilidad absoluta de sistemas de segundo orden, Proc. I. Simposium acerca del Desarrollo de la Matematica, Habana, in press.

[Gu] P.-O. Gutman, Stabilizing controllers for bilinear systems, *IEEE Trans. Automat. Control* **AC-26** (1981), 917–922.

[HIP] D. Hinrichsen, A. Ilchmann, and A. J. Pritchard, Robustness of time-varying linear systems, *J. Differential Equations* **82** (1989), 219–250.

[HM] G. Hochschildt and G. D. Mostow, Representations and representation functions of Lie groups, III, *Ann. Math.* **70** (1959), 85–100.

[HP1] D. Hinrichsen and A. J. Pritchard, Stability radius of linear systems, *Systems Control Lett.* **7** (1986), 1–10.

[HP2] D. Hinrichsen and A. J. Pritchard, Stability radius for structured perturbations and the algebraic Riccati equations, *Systems Control Lett.* **8** (1986), 105–113.

[I] A. Isidori, "Nonlinear Control Systems: An Introduction," 2nd ed., Springer-Verlag, New York, 1989.

[J] D. H. Jacobson, "Extensions of Linear Quadratic Control Theory," Academic Press, London, 1977.

[JS] V. Jurdjevic and H. Sussmann, Control Systems on Lie groups, *J. Differential Equations* **12** (1972), 313–329.

[Kl] W. Kliemann, Recurrence and invariant measures for degenerate diffusions, *Ann. Probab.* **15** (1987), 690–707.

[Kn] H. W. Knobloch, Stabilization of control systems by means of high-gain feedback, *in* "Optimal Control Theory and Economic Analysis 3" (G. Feichtinger, Ed.), pp. 153–173, North-Holland, New York, 1988.

[KN] E. E. Koditschek and K. S. Narenda, Stability of second-order bilinear systems, *IEEE Trans. Automat. Control* **AC-28** (1983), 987–989.

[KKO] P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, "Singular Perturbation Methods in Control: Analysis and Design," Academic Press, New York, 1987.

[LT] P. Lancaster and M. Tismenetsky, "The Theory of Matrices," 2nd ed., Academic Press, New York, 1985.

[Me] S. M. Meerkov, Principle of vibrational control: Theory and applications, *IEEE Trans. Automat. Control* **AC-25** (1980), 755–762.

[Mo] R. R. Mohler, "Bilinear Control Processes," Academic Press, New York, 1973.

[O] E. Oeljeklaus, private communication.

[RK] M. A. Rotea and P. P. Khargonekar, Stabilization of uncertain systems with norm bounded uncertainty-A control Lyapunov function approach, *SIAM J. Control Optim.* **27** (1989), 1462–1476.

[S1] H. Sussmann, Some properties of vector fields that are not altered by small perturbations, *J. Differential Equations* **20** (1976), 292–315.

[S2] H. Sussmann, Subanalytic sets and feedback control, *J. Differential Equations* **31** (1979), 31–52.

[VL] C. Van Loan, How near is a matrix to an unstable matrix, *Contemp. Math.* **47** (1985), 465–478.

[Y] J. Yong, Feedback stabilization of nonlinear uncertain dynamical systems, *J. Math. Anal. Appl.* **129** (1988), 153–165.