Controlling the Dynamics of a Random System

WOLFGANG KLIEMANN AND FRITZ COLONIUS

Department of Mathematics
Iowa State University
U.S.A.

Institut für Mathematik
Universität Augsburg
Germany

ABSTRACT. Random systems, dynamical systems and control systems can all be described as flows on (finite or infinite dimensional) spaces, which allows for the use of unified concepts in the analysis of their qualitative long term behavior. In particular there is a close connection between the attractors of an undisturbed system, the stationary and ergodic solutions of the system under Markovian parameter noise, the invariant control sets of an associated control system, and the (chaotic) attractors of the corresponding control flow. This correspondence is used to analyze the possible changes in the ergodic behavior of controlled random nonlinear systems. Discussing four examples in some detail reveals a link to stochastic bifurcation theory.

1. Introduction.

Random vibrations occur in physical systems (often modelled by ordinary differential equations), which are subjected to noise. As long as the resulting system is Gaussian and Markov (in particular the solution of a linear stochastic differential equation with additive noise), there is a well-known range of techniques for the analysis of such systems. Parameter excited and/or nonlinear systems are in general non-Gaussian, and a wide range of approaches have been developed to deal with these situations analytically, approximately or numerically, among them in particular stochastic linearization, averaging for systems on different time scales, computation of solutions of the corresponding Kolmogorov (or Fokker-Planck) equations. The textbooks of Dimentberg [17] and Sobczyk [26], as well as many papers in this volume describe the current state of art in these areas. They all have in common that they deal with the statistical properties of random systems, i.e. in the case of solutions of stochastic differential equations these approaches are based on the generator of the diffusion process.

However, for certain properties of the trajectories, like their exponential convergence, it is not appropriate to consider the generator: Baxendale [5] has an example, where for processes with the same generator the Lyapunov exponents, indicating exponential convergence or divergence with probability one of the trajectories, can be negative or positive. These exponents depend on the dynamics, i.e. the vectorfields of the random system. In recent years, the dynamical analysis of stability and bifurcations in stochastic systems based on Lyapunov exponents has made considerable progress, we just mention

---

Research supported in part by NSF grant DMS 8813976 and by DFG grants Co 124/6-1 and Co 124/8-1 .

the papers of Arnold and Boxler [2], Boxler [7], and of Ariaratnam, Sri, and Wedig in these Proceedings.

In this paper we present another aspect of the dynamical analysis of random systems: we will gain insight by using concepts from the theory of dynamical systems and of control systems. We concentrate on the qualitative long term behavior, i.e. on stationarity and ergodicity, and analyze its change via deterministic control. (Note that this problem of controlling the dynamics of a random system is different from stochastic optimal control with adapted control processes.) After the introduction of some concepts and notations in Section 2, we will present an outline of the theory, linking stationarity to chaos and controllability. The examples in Section 4. show, how to apply this theory in some obvious and not so obvious situations. Besides for the control of random dynamics, these ideas also prove valuable for stability and stabilization of stochastic systems via Lyapunov exponents. We refer to [20], [9], and [10], where the last two papers also show some connections with stabilization theory for uncertain (deterministic) systems. Some of the common background material, which is needed for the following sections, can be found in the survey paper [20].

2. Random Vibrations, Stochastic Flows and Control Flows.

As a mathematical model of random vibrations we will use the following set-up: Given is a (deterministic) ordinary differential equation on a smooth manifold M, $\dim M = d$,

$$\dot{x} = X_0(x), \tag{2.1}$$

where X_0 is a smooth vectorfield, describing a physical system. The random system is

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} \xi_t^i X_i(x) \qquad \text{on } M, \tag{2.2}$$

where $(\xi_t^i)_{i=1,\dots,m} = \xi_t$ is an m-dimensional noise process, and the vectorfields $X_1 \dots X_m$ describe, how the noise enters into the system. This model covers parameter and/or additive noise (in the latter case the vectorfields $X_1 \dots X_m$ are constant). In this paper we consider the case, where ξ_t is bounded (in particular non-Gaussian), i.e. we assume that ξ_t has values in a compact, convex set $V \subset \mathbb{R}^m$, which contains 0 in its interior. Furthermore, since we are interested in stationary behavior, we assume that ξ_t is (strict sense) stationary, i.e. all finite dimensional distributions are invariant under time shift.

In order to link the random system (2.2) with dynamical and control systems, we describe it as a stochastic flow. This means first of all that we need an appropriate probability space for ξ_t: Let $\mathcal{V} = \{v\colon \mathbb{R} \to V \subset \mathbb{R}^m$, locally integrable$\}$ be the trajectory space of ξ_t. Then the standard Kolmogorov construction (see e.g. Rozanov [24]) yields a σ- algebra $\mathcal{B}$ and a probability measure P on $(\mathcal{V}, \mathcal{B})$ such that: If $\theta_t\colon \mathcal{V} \to \mathcal{V}$, defined by $\theta_t v(\cdot) = v(t + \cdot)$, denotes the time shift on $\mathcal{V}$, then $\theta_t P = P$, i.e. shift invariance of P on $\mathcal{V}$ corresponds to shift invariance of the finite dimensional distributions of ξ_t, indicating stationarity. Now take $(\mathcal{V}, \mathcal{B}, P)$ as the probability space, then $\xi_t(v) = v(t)$ for $v \in \mathcal{V}$ is the equivalent noise model, which we will use from now on.

To capture the random dynamics of (2.2) we consider the family of differential equations

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} v^i(t) X_i(x) \tag{2.3}$$

with $(v^i)_{i=1,\dots,m} = v \in \mathcal{V}$. The solutions are denoted by $\varphi(t,x,v)$ for $(v,x) \in \mathcal{V} \times M$, with initial value $\varphi(0,x,v) = x$ and using $v \in \mathcal{V}$ on the r.h.s. of (2.3). Putting things together, we arrive at the random flow description of (2.2):

$$\begin{aligned} \phi\colon \mathbb{R} \times \mathcal{V} \times M &\to \mathcal{V} \times M \\ (t,v,x) &\mapsto (\theta_t v, \varphi(t,x,v)) \end{aligned} \tag{2.4}$$

where the first component (i.e. the shift θ with invariant measure P) describes the stationary noise, and the second component describes the dynamics of the system under the noise. (2.4) is called a flow on $\mathcal{V} \times M$, because it has the group property $\phi_t \circ \phi_s = \phi_{t+s}$ for all times $t, s \in \mathbb{R}$ (we have used the notation $\phi_t = \phi(t,\cdot,\cdot)$). The shift component θ_t is also a flow (on $\mathcal{V}$), but the second component by itself does not satisfy this property, because of the time varying noise. It is called the skew component of ϕ, and the notion 'skew product flow' is often used for a flow ϕ on a product space with skew component. (If ξ_t is not a bounded, stationary process but white noise, one can also construct a corresponding stochastic flow for the system, see e.g. Kunita [21], Arnold and Crauel [3], or the survey paper [20].)

For the topological analysis of (2.4) as a dynamical system or a control system we need a topology on the trajectory space $\mathcal{V}$ (and we disregard the measure P of the stochastic flow). For control theoretic purposes the appropriate one is the weak* topology of $L^\infty(\mathbb{R},\mathbb{R}^m) = \left(L^1(\mathbb{R},\mathbb{R}^m)\right)^*$ (see e.g. [12]), since convergence of $v_n \to v$ in $\mathcal{V}$ then implies uniform convergence of the solutions $\varphi(\cdot,x,v_n) \to \varphi(\cdot,x,v)$ on finite time intervals. With this topology ϕ is continuous in all its components.

The next section is devoted to the study of ϕ from stochastic, dynamical systems, and control systems points of view. We will do this under an assumption which guarantees that we have the "right" state space for (2.2), namely

$$\dim \mathcal{LA}\{X_0, X_1, \dots, X_m\}(x) = d \quad \text{for all } x \in M. \tag{H}$$

This Lie algebra rank condition, which is discussed in detail e.g. in [20], means that the noise dynamics is sufficiently rich, i.e. that the set of points reached by some solution of (2.2) up to time $t > 0$ has nonvoid interior, or, in the language of Markov processes, that the supports of the transition probabilities up to time $t > 0$ have nonvoid interior. In fact, it is this assumption, on which in the white noise case the famous support theorem of Stroock and Varadhan is based, saying intuitively that the trajectories of (2.2) as a random system "agree" with the trajectories of (2.2) as a control system, see Stroock and Varadhan [27], or the survey [20] for details. In what follows, we develop a theory for real (bounded) noise systems, which reflects some of the consequences of the support theorem.

3. Stationary Random Systems, Controllability, and Chaotic Attractors.

In this section we will analyze the problem, under which conditions the random systems (2.2) has a bounded stationary solution. This is the basis for the subsequent discussion about changing the dynamical behavior via control.

As indicated in Section 2, stationarity means finding a flow invariant probability measure, i.e. for the stationarity of (2.2) under stationary noise ξ_t: Find an invariant measure μ on $\mathcal{V} \times M$, such that $\phi_t\mu = \mu$ for all $t \in \mathbb{R}$. Furthermore, since ξ_t and hence P on $\mathcal{V}$ are given, we need that the marginal of μ on $\mathcal{V}$ is P. This amounts to the following construction: μ on $\mathcal{V} \times M$ can be desintegrated as $\mu = \mu_v P$, where for each $v \in \mathcal{V}$ μ_v is a measure on M with the invariance property $\varphi(t, \cdot, v)\mu_v = \mu_{\theta_t v}$, see e.g. [11]. In order to find these measures, one has to study the ergodic properties of the flow ϕ on $\mathcal{V} \times M$. This can be done using the limit structure of ϕ. Carrying out this program, however, is quite a technical task, if done in full generality, and we refer to [11] and the references therein for several results in this direction.

Fortunately, the solution is much simpler, if we work in the context of Markov processes, i.e. stochastic processes, for which past and future are independent, given the presence. Since physical systems, described by ordinary differential equations, do not foresee their future, this seems to be a natural class of noises to consider for random vibrations. In particular, it includes the following models, which are quite common in random mechanics (colored noise):

$$d\eta_t = Z_0(\eta_t)dt + \sum_{i=1}^{k} Z_i(\eta_t) \circ dw_t^i \tag{3.1}$$

is a (nondegenerate) stationary diffusion process on a compact manifold N and $f\colon N \to V$ a smooth map with $f[N] = V$, and with coordinate functions $f_1, \dots, f_m$. η is the 'background' noise, which enters the system $\dot{x} = X_0(x) + \sum_{i=1}^{m} f_i(\eta_t)X_i(x)$ via the transformation f. In this set-up we have: η_t is a Markov diffusion process, the pair (η_t, x_t) is a Markov diffusion process, and supp $P = \mathcal{N}$, where $\mathcal{N}$ is the trajectory space of η_t. (Strictly spaeking we also need that the initial values of η_t and of x_t are independent of the driving Wiener process W_t^i.) From now on we assume that these properties are satisfied.

In the Markov context stationarity of (2.2) can be described through the control structure of (2.3), which is quite surprising, because controllability is a finite time result, while stationarity and invariant measures refer to the long term behavior as $t \to \infty$. We need the following definitions and notions for the system (2.3):

Denote by $\mathcal{O}^+_{\leq T}(x) = \{y \in M;$ there is $v \in \mathcal{V}$ and $0 \leq t \leq T$ with $\varphi(t, x, v) = y\}$ the so called positive orbit of a point $x \in M$ up to time $T > 0$, and let $\mathcal{O}^+(x) = \bigcup_{T>0} \mathcal{O}^+_{\leq T}(x)$.

3.1. Definition. A set $D \subset M$ is a *control set* of (2.3), if $\overline{\mathcal{O}^+(x)} \supset D$ for all $x \in D$, and D is maximal with this property. $C \subset M$ is an *invariant control set*, if furthermore $\overline{\mathcal{O}^+(x)} = \overline{C}$ for all $x \in C$. (For a set A we denote by $\bar{A}$ its closure.)

In the context of this paper we are interested only in those control sets that have nonvoid interior, denoted by int D. Under our standing assumption (H), this is always true for invariant control sets, see e.g. [20].

In order to obtain concepts for the flow ϕ in (2.4), we have to lift the control sets D of (2.3), which are subsets of M, to the product space $\mathcal{V} \times M$. We define

$$\mathcal{D} = c\ell\{(v,x) \in \mathcal{V} \times M;\ \varphi(t,x,v) \in \text{int } D \text{ for all } t \in \mathbb{R}\}, \tag{3.2}$$

where "$c\ell$" is the closure in $\mathcal{V} \times M$ with respect to the described topologies. The interplay between the control structure of (2.2) and the dynamical systems structure of (2.3) yields the following result:

3.2. Theorem. (control and chaos)

(i) *Let $D \subset M$ be a control set of (2.3) with* int $D \neq \phi$, *and $\mathcal{D}$ its lift to $\mathcal{V} \times M$. Then the flow $(\mathcal{D}, \phi)$ is chaotic, i.e. (a) there is a dense set of periodic points, (b) $(\mathcal{D}, \phi)$ is maximal topologically transitive (and mixing), (c) $(\mathcal{D}, \phi)$ has sensitive dependence on initial values.*

(ii) *Vice versa, if $\mathcal{D} \subset \mathcal{V} \times M$ is maximal topologically transitive (or mixing) with* int $\Pi_M \mathcal{D} \neq \phi$, *then there is a control set $D \subset M$ of (2.3) with* int $D =$ int $\Pi_M \mathcal{D}$ *and $c\ell D = c\ell \Pi_M \mathcal{D}$. Here $\Pi_M \colon \mathcal{V} \times M \to M$ denotes the projection.*

Theorem 3.2 says that the sets, in which the system (2.3) can be controlled, correspond to the chaotic components $\mathcal{D}$ of the flow ϕ. We will briefly recall the concepts used in Theorem 3.2: Let $\psi \colon \mathbb{R} \times S \to S$ be a continuous flow on a complete metric space. For a subset $A \subset S$ the limit set $\omega(A)$ is defined as $\omega(A) = \bigcap_{t \geq 0} c\ell\{\psi(\tau, A);\ \tau \in [t, \infty)\}$ and similarly for $\omega^*(A)$ using $t \leq 0$.

(S, ψ) is called topologically transitive, if there is $s \in S$ with $\omega(s) = S$, and topologically mixing, if for any two open sets $V_1, V_2 \subset S$ there are times $T_0 \in \mathbb{R}$, $T_1 > 0$ such that $\psi(-nT_1 + T_0, V_1) \cap V_2 \neq \phi$ for all $n \in \mathbb{N}$. (S, ψ) has sensitive dependence on initial values, if there is a uniform $\delta > 0$ such that for all $s \in S$ and all neighborhoods N of s there are $p \in N$ and $t > 0$ with $d(\psi(t,s), \psi(t,p)) \geq \delta$. Here $d(\cdot,\cdot)$ is the metric (distance) on S. Finally $s \in S$ is a periodic point of ψ, if there is a time T with $\psi(t,s) = \psi(T+t, s)$ for all $t \in \mathbb{R}$, and denseness means that the closure of the set of periodic points is all of S. (See e.g. Mañé [22] or Devaney [16] for these definitions.) Setting $S = \mathcal{V} \times M$ and $\psi = \phi$ one obtains the corresponding statements in Theorem 3.2, where "maximal" of course means a maximal (with respect to set inclusion) set with these properties.

A flow with the properties (a)–(c) from Theorem 3.2(i) is 'chaotic' (compare e.g. Devaney [16]), because it has the three ingredients: a dense set of regular (periodic) points, mixing, and nearby trajectories drift apart everywhere. (In differentiable dynamics, which we do not have because of the shift component θ, a concept similar to (c) can be based on hyperbolicity and Lyapunov exponents, see e.g. Mañé [22].)

Because of the maximal mixing in Theorem 3.2(ib), we expect the following 'meta theorem' to hold: Control sets D of (2.3) correspond to maximal mixing components

$\mathcal{D}$ of ϕ, which correspond to the recurrent components of the stochastic flow ϕ, since recurrence is a mixing property, namely that the solution process visits every open set infinitely often. For general stationary noises this meta theorem is 'almost' true, compare Section 4. in [11]. In the Markov context we arrive at an even stronger result, for which we need the following concepts.

Consider the control system (2.3) and its control sets D_α, $\alpha \in I$, where I is some index set. We define a (partial) *order* on the control sets through

$$D_\alpha \prec D_\beta \text{ if there exists } x \in D_\alpha,\ v \in \mathcal{V} \text{ and } t > 0 \text{ such that } \varphi(t,x,v) \in D_\beta \tag{3.3}$$

in other words, if D_β can be reached from D_α (but not vice versa, since control sets are maximal sets with the reachability condition in Definition 3.1). A control set D_γ is said to be maximal with respect to the order '$\prec$', if, whenever $D_\gamma \prec D_\delta$, then $D_\gamma = D_\delta$. It is easy to see that invariant control sets are maximal w.r.t. $\prec$.

Concerning the existence of invariant control sets for a control system (2.3) we mention that in each bounded, invariant subset of the state space M (in particular, if M itself is compact) there exists at least one (and at most finitely many) invariant control sets C; C is closed and has nonvoid interior. Furthermore, C is said to be ***isolated*** if there is an open neighborhood N of C such that $\Pi_M\omega(v,N) \subset C$ for all $v \in \mathcal{V}$. With these preparations we can state the following results on the generic behavior of the trajectories of (2.3);

3.3. Theorem. (genericity and attractors)

(i) *The set* $\{(v,x) \in \mathcal{V} \times M;$ *there is* $t_0 \geq 0$ *such that* $\varphi(t,x,v) \in \operatorname{int} C$, *some invariant control set for all* $t \geq t_0$, *or* $\varphi(t,x,v) \to \infty$ *for* $t \to \infty\}$ *is an open and dense set in* $\mathcal{V} \times M$.

(ii) *If* $\mathcal{C} \subset \mathcal{V} \times M$ *is the lift of an isolated invariant control set* $C \subset M$, *then* $\mathcal{C}$ *is a chaotic attractor of the flow* $(\mathcal{V} \times M, \phi)$.

Part (i), whose proof can be found in [13] , says that, except for a thin set of control functions $v \in \mathcal{V}$ and initial values $x \in M$, all trajectories will end up in an invariant control set or go towards the boundary of M. In particular, for all initial values in a bounded invariant subset of M, these trajectories will enter some invariant control set and stay there. Concerning part (ii), we first have to make precise the concept of an attractor: We are interested in the structure of the union of all ω-limit set $\bigcup\{\omega(v,x);\ (v,x) \in \mathcal{V} \times M\}$. Since this set is difficult to describe, it is replaced in dynamical systems theory by a (possibly somewhat larger) set CR of *chain recurrent points*, see e.g. Conley [15]: For a dynamical system (S,ψ) we define $x \in CR$ if for all $\varepsilon > 0$, $T > 0$ there are $n \in \mathbb{N}$, times $t_0,\dots,t_{n-1} \geq T$, and points $x = x_0,\dots,x_n = x$ in S with $d(\psi(t_i,x_i),x_{i+1}) < \varepsilon$ for $i = 0,\dots,n-1$. The connected components $\mathcal{M}_\alpha$, $\alpha \in \mathcal{J}$ ($\mathcal{J}$ is again some index set) of CR are the *Morse sets* of (S,ψ). Between the Morse sets there is a (partial) order defined by

$$\mathcal{M}_\alpha \prec \mathcal{M}_\beta \quad \text{if there is } s \in S \text{ with } \omega^*(s) \subset \mathcal{M}_\alpha \text{ and } \omega(s) \subset \mathcal{M}_\beta. \tag{3.4}$$

An *attractor* is a maximal Morse set w.r.t. the order $\prec$, see e.g. Ruelle [25]. Using these concepts for the flow $(\mathcal{V} \times M, \phi)$ and for the lifts $\mathcal{C}$ of isolated invariant control sets C one can now easily prove statement (ii).

In the Markov set-up described above, we obtain the following consequence:

3.4. Corollary. *$P\{v \in \mathcal{V};$ there is $t_0 \geq 0$ such that $\varphi(t,x,v) \in C$, some invariant control set, for all $t \geq t_0$, or $\varphi(t,x,v) \to \infty\} = 1$ for all $x \in M$.*

It is clear from this corollary that solutions of (2.2), such that (ξ_t, x_t) is a Markov process, can be stationary only in the invariant control sets C of (2.3). In order to show that there actually is a stationary solution in each C, one needs a tightness condition on the transition probabilities, see e.g. Hasminskii [19]. Since we are only interested in *bounded* stationary solutions, the situation is simple:

3.5. Corollary. *Let ξ_t be given by a background noise η_t as in (3.1). Then we have:*

(i) *In each bounded invariant control set C of (2.3) there exists a stationary solution (η_t^0, x_t^0) of (2.2). The corresponding invariant probability measure μ of (η_t^0, x_t^0) is a solution of the (stationary) Fokker-Planck equation for (η_t, x_t), whose marginal on the state space N of η_t is the invariant measure of η_t. μ has a smooth density with support $N \times C$, in particular the stationary Markov solution on C is unique.*

(ii) *All bounded stationary and ergodic Markov solutions of (2.2) are concentrated on $N \times C$ for some bounded invariant control set C of (2.3), i.e. they are of the form (i). All bounded stationary Markov solutions are finite convex combinations of these ergodic processes.*

(iii) *Any Markovian solution of (2.2), which enters some C with probability 1, will converge (in distribution) to the unique ergodic process in C, according to the strong law of large numbers.*

For remarks on the proofs of the Corollaries 3.4 and 3.5 see [20] and [11].

The results 3.3–3.5 allow the precise formulation of our meta theorem above in the Markovian context:

3.6. Theorem. *Let M be a bounded, invariant set and assume that the invariant control sets $C \subset M$ are isolated. Then the invariant control sets $C \subset M$ of (2.3) correspond exactly to the (chaotic) attractors $\mathcal{C}$ of $(\mathcal{V} \times M, \phi)$, which in turn correspond exactly to the stationary, ergodic Markov solutions (η_t^0, x_t^0) of (2.2) on $N \times M$.*

If the invariant control sets are not isolated, the statement above remains true with 'attractor' replaced by 'maximal topologically mixing component', where maximality is understood w.r.t. the order defined in (3.3).

Theorem 3.6 connects several interpretations of (2.4) as a control, a dynamical, and a random system. It does, however, not relate these interpretations to the behavior of the undisturbed physical system (2.1). We will now explain such a relationship for small, Markovian noise.

Scale the noise range V through $V^\varepsilon = \varepsilon \cdot V \subset \mathbb{R}^m$ for $\varepsilon > 0$, then $\bigcap_{\varepsilon>0} V^\varepsilon = \{0\}$, which corresponds to the undisturbed system (2.1). Denote by M_α, $\alpha \in \mathcal{J}$ the Morse sets of (2.1), defined as in the explanations after Theorem 3.3, but now for the flow $\chi: \mathbb{R} \times M \to M$ induced by (2.1) on the state space M. Assume that χ has finitely many attractors $A_1 \dots A_n$ in M. A point x in one of these attractors is called in inner point, if for all $\varepsilon > 0$ there exist times $T, S > 0$ such that $\varphi(T, x, 0) \in \operatorname{int} \mathcal{O}^{\varepsilon+}_{\leq S}(x)$. Here $\varphi(t, x, 0)$ is the trajectory of the undisturbed system, and $\mathcal{O}^{\varepsilon+}$ is the positive orbit of the control system (2.3) with range V^ε, compare the notation introduced before Definition 3.1. The inner point property is discussed in [13] (see also [20]) and can be verified through a Lie algebraic criterion similar to (H).

3.7. Theorem. *Consider the control system (2.3^ε) with control range V^ε. Assume that for some $\hat{\varepsilon} > 0$ M is a bounded, invariant set of ($2.3^{\hat{\varepsilon}}$) with finitely many attractors $A_1 \dots A_k$ of the undisturbed system (2.1), such that all points $x \in \bigcup_{i=1}^{k} A_i$ are inner points. Then there exists $\varepsilon^0 > 0$ such that*

(i) *for all $0 < \varepsilon < \varepsilon^0$ there exist invariant control sets C_i^ε of (2.3^ε) such that $A_i \subset \operatorname{int} C_i^\varepsilon$ for all $i \in \{1, \dots, k\}$;*
(ii) *vice versa, if for a family C^ε, $0 < \varepsilon < \varepsilon^0$, of invariant control sets we have $\bigcap_{\varepsilon > 0} C^\varepsilon \neq \emptyset$, then $\bigcap_{\varepsilon > 0} C^\varepsilon$ is one of the attractors $A_1, \dots, A_k$.*

This theorem says that for all ε the attractors of (2.1) "blow up" to corresponding invariant control sets of (2.3^ε). The result is a special case of Theorem 4.12 in [13]. Thus, we obtain under the assumptions of Theorems 3.6 and 3.7 an extension of our meta theorem:

3.8. Corollary. *For $\varepsilon > 0$ small enough, the following objects are in one-to-one correspondence:*

(a) *attractors of the undisturbed system (2.1) on M,*
(b) *invariant control sets of the control system (2.3^ε) on M,*
(c) *(chaotic) attractors of the flow ϕ in (2.4) on $\mathcal{V} \times M$,*

(or maximal topologically mixing components of ϕ, if the invariant control sets are not isolated),

(d) *stationary, ergodic solutions (η_t^0, x_t^0) of (2.2) on $N \times M$.*

Note that this correspondence also holds, if the attractors of (2.1) are not chaotic, even for linear systems with additive noise, if the origin is the unique attractor of $\dot{x} = Ax$. The chaotic nature of the attractor of the flow ϕ on $\mathcal{V} \times M$ comes from its infinite dimensional component, described by the shift on the space $\mathcal{V}$ of bounded trajectories. Corollary 3.8 applies in particular to stable fixed points and stable limit cycles of $\dot{x} = X_0(x)$, around which one finds, for small noises, unique stationary, ergodic solutions of the random system (2.2).

4. Control of Long Term Random Dynamics.

Section 3. was devoted to the study of the long term behavior of the random system (2.2). Now we introduce an additional control component and analyze the possible changes of the qualitative behavior via deterministic control functions. Let us consider

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} \xi_t^i X_i(x) + \sum_{j=1}^{n} u_j(t) Y_j(x), \tag{4.1}$$

where the $Y_1 \dots Y_n$ are again smooth vectorfields on the state space M, and the admissible control functions $(u_j(t))_{j=1,\dots,n} = u(t)$ have values in a bounded set $U \subset \mathbb{R}^n$, again with $0 \in \text{int } U$, so that for $u \equiv 0$ the uncontrolled system corresponds to (2.2). We address the following problem: Find a control function $u(t)$ such that (4.1) has a stationary solution (or converges towards a stationary solution) with prescribed behavior. By prescribed behavior we mean here: possesses an invariant measure in a certain, given area. (We are not interested in the shape of the resulting invariant density, only in its support, since in this very general set-up analytical statements about the shape are not possible.) Recall that by the law of large numbers almost surely the trajectories of x_t will visit any open set in the support of the invariant measure infinitely often. Our results will therefore describe the area, in which the trajectories of the controlled system fluctuate.

We will continue to consider the Markov set up with noise processes described by (3.1), and we will use the correspondence between invariant control sets and ergodic solutions from Section 3. If the controlled equation is to have stationary solutions for t sufficiently large, then the vector fields in (4.1) have to be time independent for $t \geq t_0$ (see Hasminskii [19]), i.e. the controls $u(t)$ have to be constant for $t \geq t_0$. As we will see, this relates the problem studied here closely to the Markovian bifurcation theory of random systems.

Four examples will be studied according to the following scheme: Given a random system of the form (2.2) and control vectorfields Y_j with control range $U \subset \mathbb{R}^n$, and given certain required properties concerning the stationary behavior of the controlled system, is it possible to achieve this behavior. All examples will be in dimension 1 or 2, although the results work for any finite dimension, because the principles can be explained in an intuitive way for low dimensional state spaces. A software package is under development to compute the necessary invariant control sets ("CS" by Gerhard Häckl, University of Augsburg), which was used for Example 3.

4.1. Example. Consider the Verhulst equation in $\mathbb{R}^1$

$$\dot{x} = X_0(x) + \xi_t X_1(x) + u(t) Y_1(x) = \alpha x - x^2 + \xi_t x + ux, \tag{4.2}$$

$\alpha \in \mathbb{R}$, $\xi_t \in V = [-\delta, \delta] \subset \mathbb{R}$, $u(t) \in \mathbb{R}$. The goal is to obtain a stationary solution with support bounded away from 0.

For one dimensional systems it is convenient to represent the dynamics in the (u, x)-plane as a 'bifurcation diagram', see Figure 1. The arrows indicate the sign of the right hand side of (4.2), and the lines the zeros, i.e. the fixed points. Using the techniques described in [4] or [11], it is easy to find the control sets for each u and a given noise range V.

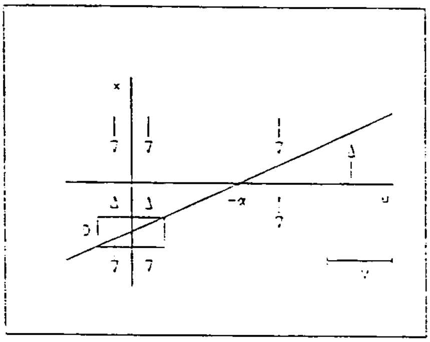

Figure 1: Uncontrolled system

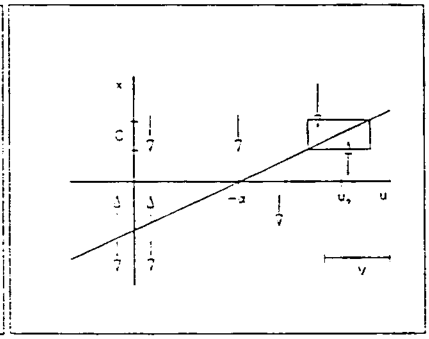

Figure 2: Controlled system

For the uncontrolled system, i.e. $u \equiv 0$, we obtain (see Figure 1): $x = 0$ is a fixed point (independent of the noise trajectory), there is one variant control set D, and no invariant control set (other than $\{x = 0\}$). Hence there is no stationary solution, bounded away from 0. Actually we obtain for the limit behavior of this system, with $D = (a, b)$: $x_t(y) \to 0$ almost surely for $t \to \infty$ for all initial values $y \geq b$, $x_t(y) \to -\infty$ almost surely for $t \to \infty$ for $y \leq a$, and for $y \in D$ there are positive probabilities p^0, p^∞ such that $P\{x_t(y) \to 0\} = p^0$ and $P\{x_t(y) \to -\infty\} = p^\infty$.

For the controlled system we have: (4.2) has a stationary solution, bounded away from 0, iff $u_0 > -\alpha + \delta$, see Figure 2. Here C is a unique invariant control set in $\mathbb{R}_+$. Increasing the control u means moving C, and hence the stationary solution, further away from 0. Note that for this system $x_t(y)$ converges towards the stationary solution iff the initial value $y > 0$. For nonpositive initial values the goal cannot be accomplished. Looking at Figures 1 and 2, these results are not surprising, but rather intuitive. We next consider a 2-dimensional system, which is also basically determined by its deterministic structure.

4.2. Example. Consider the Lotka-Volterra (or predator-prey) system in $\mathbb{R}^2_+$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = X_0(x) + \xi_t X_1(x) + u(t) Y_1(x) = \begin{pmatrix} (\alpha - \beta x_2) x_1 \\ -\gamma x_2 \end{pmatrix} + \xi_t \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix} + u(t) \begin{pmatrix} -x_1^2 \\ 0 \end{pmatrix}, \tag{4.3}$$

where α, β, γ are positive constants, $\xi_t \in [a, b] \subset \mathbb{R}_+$ and $u(t) \in U \subset \mathbb{R}_+$. The goals are to (a) obtain a stationary solution in $\mathbb{R}_+$, bounded away from the x_1-and the x_2-axis, (b) obtain a stationary solution with $x_2 = 0$. Note that $x_1 = 0$ or $x_2 = 0$ indicates the extinction of the corresponding species.

The uncontrolled system with $u \equiv 0$: For $\xi_t \equiv v \in [a, b]$, the solutions are closed curves in $\mathbb{R}^2_+$, covering all of $\mathbb{R}^2_+$. Thus, even with the smallest noise in $[a, b]$, $a \neq b$, the system cannot have a bounded, stationary solution. Hence for all thresholds $L > 1$ and all initial values $(y_1, y_2) \in \mathbb{R}^2_+$ there exists a positive probability that the solution $x_t(y_1, y_2)$ will grow above L, and fall below $\frac{1}{L}$.

Controlled system with u small, i.e. $0 < u < \frac{\alpha a}{\gamma}$: For $\xi_t \equiv v \in [a, b]$, the solutions have a unique stable fixed point $\left(\frac{\gamma}{v}, \frac{\alpha v - u\gamma}{\beta v}\right)$ in $\mathbb{R}^2_+$. Thus, by Proposition 2.4 in [8], for each $u \in \left(0, \frac{\alpha a}{\gamma}\right)$ there exists a unique compact invariant control set $C_u \subset \mathbb{R}^2_+$, on which the system (4.3) has a unique stationary ergodic solution. If u increases within this interval, the control set C_u moves closer to the x_1-axis, i.e. the x_2-population becomes smaller. For all initial values in $\mathbb{R}^2_+$, the solutions of (4.3) convergence towards the stationary one.

Controlled system with u large, i.e. $u > \frac{\alpha b}{\gamma}$: There exists a unique stationary solution with support $\left[\frac{\alpha}{b}, \frac{\alpha}{a}\right] \times \{0\}$ on the positive x_1-axis, and all solutions of (4.3) with initial value in $\mathbb{R}^2_+$ converge towards this solution, i.e. the x_2-population becomes extinct w.p.1.

For a deterministic analysis of this system see e.g. Amann [1], for some aspects of a white noise analysis see e.g. Dimentberg [17].

The next example shows a 2-dimensional system, whose deterministic part can be controlled to stay in a certain area of the state space, while even small noise will drive it into a different region.

4.3. Example. Consider the model of a well-stirred chemical reactor in $M = (0, \infty) \times (0, 1)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = X_0(x) + \xi_t X_1(x) + u(t) Y_1(x)$$
$$= \begin{pmatrix} -x_1 - 0.15(x_1 - x_c) + 0.35(1 - x_2)e^{x_1} \\ -x_2 + 0.05(1 - x_2)e^{x_1} \end{pmatrix} + \xi_t \begin{pmatrix} x_c - x_1 \\ 0 \end{pmatrix} + u(t) \begin{pmatrix} x_c - x_1 \\ 0 \end{pmatrix}, \tag{4.4}$$

where x_1 denotes temperature, x_2 product concentration and noise and control affect the heat transfer coefficient. Noise and control affecting the same parameter means that one cannot steer the system precisely, but the control input into the reactor is disturbed by a (small) Markovian noise. For a discussion of the deterministic, uncontrolled system see e.g. Golubitsky and Schaeffer [18] or Poore [23]. We choose $u(t) \in U = [-0.15, 0.15]$ and $\xi_t \in (-\varepsilon, \varepsilon)$. The goal is to control the system such that the product concentration x_2 is as high as possible, under technological constraints.

The deterministic, controlled system (i.e. $\xi_t \equiv 0$) was analyzed in [14], and the results are shown in Figure 3. and 4. The control system has 3 control sets, 2 are invariant (the lower one, C_1, and the upper one, C_2), and one is variant (the middle one, D). Figure 3. shows these control sets, together with the phase portrait of the uncontrolled system. The problem is that the upper control set C_2, which is the most desirable region of operation because of the high product concentration, is technically not feasible, compare Bellman et al. [6]. However, the reactor with undisturbed controls can be steered from the region of attraction of the variant control set D, shown in Figure 4., into D and it can be kept in this set using for each initial value an appropriate control function $u(t)$, compare [14].

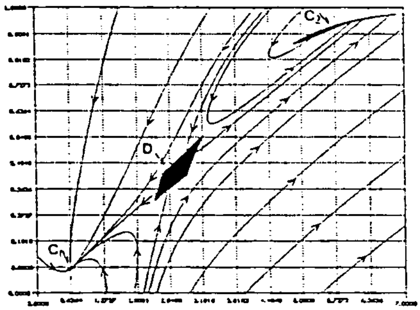

Figure 3: Control set of the undisturbed reactor model (4.4)

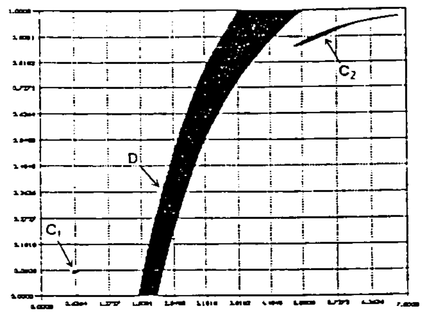

Figure 4: Region of attraction of the variant control set in (4.4)

Now, if the chosen control input $u(t)$ is disturbed by noise $\xi_t \in [-\varepsilon, \varepsilon]$, then by Corollary 3.4, for all $\varepsilon > 0$ however small, the system will tend towards one of the invariant control sets, which means lower product concentration in case of C_1, and destruction in case of C_2. Each of these possibilities will result with positive probability, if the initial value is in the region of attraction of D. If one starts to the left of this region, one converges to C_1 w.p.1, and similarly to C_2 for starting values to the right. This example shows that in order to stabilize input-disturbed systems in variant control sets, one has to design the controls depending on the noise.

4.4. Example. This example shows that even in one dimensional systems some surprising effects can occur. We will start with a 'bifurcation diagram' as in Figure 5. The noise and the control have the same dynamics (see e.g. Example 4.1 or 4.3), although the same effects can also occur, if this is not the case — the simple graphical representation, however, would not be possible. The size V of the noise is indicated in Figure 5. The goal is to obtain a stationary solution x_t with values above the point x_2. Figure 6 shows, for each control value u, the invariant control sets, corresponding to the noise range V, as areas around parts of the stable bifurcation curves.

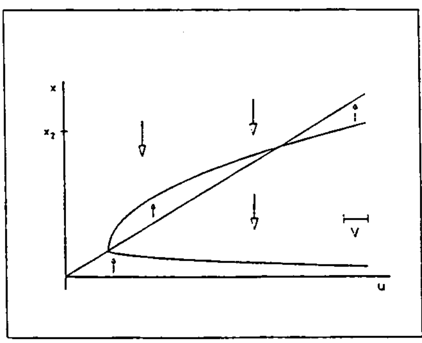

Figure 5: Bifurcation diagram of Example 4.4.

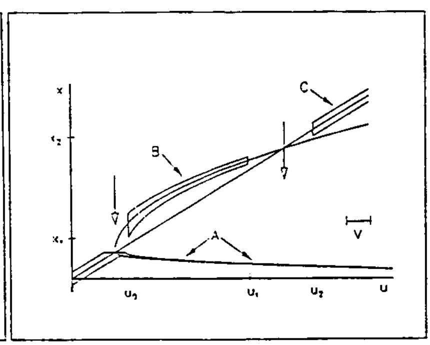

Figure 6: Invariant controls set of Example 4.4

For $u \equiv 0$ there is a unique invariant control set with low x-values. Increasing the control u leads for initial values $< x_1$ to unique invariant control sets in the area A around the lower bifurcation curve, i.e. the goal cannot be reached. For initial values in $[x_1, x_2)$ a control $\geq u_0$ leads to stationary solutions in the area B. However, increasing u above u_1 will cause the system to converge towards a stationary solution in area A. Finally, if the initial value is $\geq x_2$, a control $\geq u_2$ will result in a unique stationary solution in area C. Hence we see that, although the points 0 and x_2 are linked by a continuous line of stable bifurcation branches, it is not possible to steer the system to a stationary solution around x_2 for initial values $< x_2$. This is prohibited by the random dynamics of the Markov system. Note that this effect is present even for small noises; increasing the range V will lead to smaller areas B, and C will be shifted to larger x-values. For V large enough, the area B will disappear completely.

The separation between areas A and B is an example of 'noise induced symmetry breaking', while the separation between B and C (and the subsequent convergence towards A) has not been discussed in the literature. These effects, and many others, can be analyzed in the context of stochastic bifurcation theory for systems with bounded Markov noise.

Conclusions. We summarize our findings in the examples above in a few rules:

(a) Control to stationary Markov solutions in prescribed areas of the state space needs invariant control sets in these areas.

(b) The control to a stationary solution consisting of one point s_0 (i.e. the invariant measure is the Dirac measure at x_0) requires the existence of a control $u^0 \in U$ such that x_0 is a fixed point of $\dot{x} = X_0(x) + \sum \xi_t^i X_i(x) + \sum u_j^0 Y_j(x)$ for all $X_1 \ldots X_m$.

(c) The bifurcation behavior of the random systems with u as a (multidimensional) bifurcation parameter, and its control structure determine the the possible supports of stationary, ergodic solutions.

(d) A deterministically controlled system may drastically change its behavior in the presence of noise, if the control is designed independent of the noise.

References.

[1] Amann, H. (1983), 'Gewöhnliche Differentialgleichungen,' De Gruyter.

[2] Arnold, L., P. Boxler (1991),'Stochastic bifurcation: Instructive examples in dimension one,' to appear in: 'Stochastic Flows,' Pinsky, M., V. Wihstutz (eds.), Birkhäuser.

[3] Arnold, L., H. Crauel (1991), Introduction to 'Lyapunov Exponents, Proceedings of the Oberwolfach Conference 1990,' Arnold, L., H. Crauel, J.-P. Eckmann (eds.), Springer.

[4] Arnold, L., W. Kliemann (1983), 'Qualitative theory of stochastic systems,' In: Probabilistic Analysis and Related Topics Vol. 3,' Bharucha-Reid, A. T. (ed.), Academic Press, 1–79.

[5] Baxendale, P. (1986), 'Asymptotic behavior of stochastic flows of diffeomorphisms.' In: Proceedings of the 15th SPA, Nagoya 1985. LN Mathematics 1203, Springer, 1–19.

[6] Bellman, R., J. Bentsman, S. M. Meerkov (1983), 'Vibrational control of systems with arrhenius dynamics,' J. Math. Anal. Appl. **91**, 152–191.

[7] Boxler, P. (1989), 'A stochastic version of center manifold theory,' Probab. Th. Related Fields **83**, 509–545.

[8] Colonius, F., W. Kliemann (1989), 'Infinite time optimal control and periodicity,' Appl. Math. Opt. **20**, 113–130.

[9] ______________ (1990), 'Stability radii and Lyapunov exponents.' In: Control of Uncertain Systems, Hinrichsen D., B. Mårtensson (eds.) Birkhäuser, 19–55.

[10] ______________ (1991), 'Stabilization of uncertain linear systems,' to appear in: Modeling and Control of Uncertain Systems, DiMasi, G., A. Gombani, A. Kurzhanski (eds.), Birkhäuser.

[11] ______________ (1991), 'Remarks on ergodic theory of stochastic flows and control flows,' to appear in: 'Stochastic Flows,' Pinsky, M., V. Wihstutz (eds.), Birkhäuser.

[12] ______________ (1991), 'Some aspects of control systems as dynamical systems,' submitted.

[13] ______________ (1991), 'Limit behavior and genericity for nonlinear control systems,' submitted.

[14] ______________ (1991), 'Kontrolltheorie and Dynamische Systeme,' submitted.

[15] Conley, C. (1978), 'Isolated Invariant Sets and the Morse Index,' Regional Conference Series in Mathematics no. 38, American Mathematical Society.

[16] Devaney, R. L. (1986), 'An Introduction to Chaotic Dynamical Systems,' Benjamin-Cummings.

[17] Dimentberg, M. F. (1988), 'Statistical Dynamics of Nonlinear and Time-Varying Systems,' Research Studies Press, Wiley.

[18] Golubitsky, M., D. G. Schaeffer (1985), 'Singularities and Groups in Bifurcation Theory,' Springer.

[19] Hasminskii, R. Z. (1980), 'Stochastic Stability of Differential Equations,' Sijthoff and Nordhoff. (Russian edition 1969).

[20] Kliemann, W. (1988), 'Analysis of nonlinear stochastic systems,' in: Analysis and Estimation of Stochastic Mechanical Systems, W. Schiehlen, W. Wedig (eds.), Springer, 43–102.

[21] Kunita, H. (1990), 'Stochastic Flows and Stochastic Differential Equations,' Cambridge University Press.

[22] Mañé, R. (1987), 'Ergodic Theory and Differentiable Dynamics,' Springer.

[23] Poore, A. B. (1974), 'A model equation arising from chemical reactor theory,' Arch. Rational Mech. Anal. **52**, 358–388.

[24] Rozanov, Y. A. (1967), 'Stationary Random Processes,' Holden Day.

[25] Ruelle, D. (1989), 'Elements of Differentiable Dynamics and Bifurcation Theory,' Academic Press.

[26] Sobczyk, K. (1991), 'Stochastic Differential Equations,' Kluwer.

[27] Stroock, D. W., S. R. S. Varadhan (1972), 'On the support of diffusion processes with applications to the strong maximum principle,' Proc. 6th Berkeley Symp. Math. Stat. Probab., Vol. 3, 333–359.