CONTROLLABILITY NEAR A HOPF BIFURCATION

Fritz Colonius
Gerhard Häckl
Institut für Mathematik
Universität Augsburg
89 Augsburg, Germany

Wolfgang Kliemann
Department of Mathematics
400 Carver Hall
Iowa State University
Ames, IA 50011, U.S.A.

Abstract

We study controllability properties of control affine systems depending on a parameter and with constrained control values. If the uncontrolled system is subject to a Hopf bifurcation, a continuum of periodic solutions bifurcates from an equilibrium. This, together with an accessibility condition, induces for small control range a 2-parameter bifurcation of the control sets (i.e. the regions of complete controllability) around the equilibria and the periodic solutions, respectively. The proofs are based on methods from dynamical systems theory applied to the associated control flow.

1. Introduction

The behavior of parameter-dependent control systems near Hopf bifurcation points has been studied for some time, starting with [AF]. In this paper we study controllability near Hopf bifurcation points, and show that the bifurcation behavior of an (uncontrolled) differential equation is reflected in the controllability behavior of the controlled system with 'small' control range.

More specifically, consider an ordinary differential equation

$$(1.1)^\alpha \qquad \dot{x} = f_0(x, \alpha)$$

with $f_0 : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ a smooth (C^∞) function, and $\alpha \in I$, an open interval.

Suppose that for $\alpha = \alpha^0$ a Hopf bifurcation occurs (cp. e.g.[Ru, MM]). Thus suppose that there is for α close to α^0 a continuous family of equilibria x^α of $(1.1)^\alpha$,

$$(1.2) \qquad 0 = f_0(x^\alpha, \alpha)$$

such that the Jacobian $f_{0,x}(x^\alpha, \alpha) := \frac{\partial}{\partial x} f_0(x^\alpha, \alpha)$ has a pair of complex conjugate eigenvalues $\lambda^\alpha, \overline{\lambda}^\alpha$ crossing the imaginary axis at $\alpha = \alpha^0$ with nonvanishing velocity. For $\alpha > \alpha_0$, a continuous family of periodic solutions of $(1.1)^\alpha$ occurs approaching x^{α_0} for $\alpha \searrow \alpha^0$. Now suppose that the differential equations $(1.1)^\alpha$ are embedded into a 2-parameter family of control systems with $\alpha \in I, \rho \geq 0$,

$$(1.3)^{\alpha,\rho} \qquad \dot{x} = f_0(x, \alpha) + \sum_{i=1}^{m} u_i(t) f_i(x, \alpha)$$

$$(u_i) \in \mathcal{U}^\rho := \{u : \mathbb{R} \to \mathbb{R}^m; u(t) \in U^\rho \text{ a.e., measurable}\}$$

where $f_i : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ are smooth, $U \subset \mathbb{R}^m$ is a fixed compact convex set with $0 \in intU$ and $U^\rho := \rho \cdot U, \rho \geq 0$. We will show that the control sets of this family of control systems are subject to a 2-parameter bifurcation. For (α, ρ) close to $(\alpha^0, 0)$ there exists a control set $D^{\alpha,\rho}$ with $x^\alpha \in intD^{\alpha,\rho}$; for (α, ρ) close to $(\alpha^0, 0)$ with $\alpha > \alpha^0$ another control set $\tilde{D}^{\alpha,\rho}$ occurs containing the periodic orbit H^α in its interior. This result is based on the theory of control flows associated with a control system. Hence in Section 2 we recall some relevant material on control flows from [CK^a]. Section 3 contains the main result, and Section 4 presents a numerical study of a continuous flow stirred tank reactor, where this scenario occurs.

Notation: cl and int denote topological closure and interior, respectively.

2. Control Flows and Control Sets

In this section we will recall the definition of the control flow associated with a control system and some properties of control sets. Then a result on the existence of control sets around chain recurrent sets will be cited from [CK^a].

We consider a control system of the type $(1.3)^{\alpha,\rho}$ and suppress the respective index, whe α or ρ are kept fixed. Denote by $\varphi(t,x,u)$ the solution of (1.3) (assumed to exist uniquely for all $t \in \mathbb{R}$) using the control $u \in \mathcal{U}$ with $\varphi(0,x,u) = x \in \mathbb{R}^d$. For time- varying control functions $u \in \mathcal{U}$, the solutions of (1.3) do not define a flow on $\mathbb{R}^d$. However, we can add as an infinite dimensional component the time shift in $\mathcal{U}$ to obtain the control flow

(2.1) $$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d, \Phi(t,u,x) = (u(t+\cdot), \varphi(t,x,u)),$$

where $u(t+\cdot)(s) := u(t+s), s \in \mathbb{R}$. Using the weak*-topology on $\mathcal{U} \subset L^\infty(\mathbb{R},\mathbb{R}^m) = (L^1(\mathbb{R},\mathbb{R}^m))^*$, the map Φ is continuous with $\Phi_0 = id$, $\Phi_{t+s} = \Phi_t \circ \Phi_s, t,s \in \mathbb{R}$, where $\Phi_t := \Phi(t,\cdot,\cdot)$. In other words, Φ is a continuous flow on $\mathcal{U} \times \mathbb{R}^d$. It describes the time evolution of the control system (1.3).

The employed topology on the set $\mathcal{U}$ of control functions is appropriate, because convergence in $\mathcal{U}$ implies uniform convergence on bounded time intervals for the trajectories. Furthermore $\mathcal{U}$ is a compact complete metrizable space. The control flow Φ contains all possible trajectories of the control system (1.3) including those of the uncontrolled system (1.1)

In the rest of this paper the following standard integrability condition on the Lie algebra generated by the systems vector fields will be imposed:

(2.2)
$$\text{rank}\, \mathcal{LA}\{f_0 + \sum_{i=1}^m u_i f_i, (u_i) \in U\}(x) = d$$
for all $x \in \mathbb{R}^d$.

This implies (cp. e.g. [Is]) the following local accessibility property:

(2.3)
$$int\mathcal{O}^{\pm}_{\leq T}(x) \neq \emptyset \text{ for all } x \in \mathbb{R}^d \text{ and all } T > 0,$$

where $\mathcal{O}^+_{\leq T}(x)$ is the reachable set from x up to time $T, \mathcal{O}^+_{\leq T}(x) := \{y \in \mathbb{R}^d;$ there are $u \in \mathcal{U}$ and $0 < t \leq T$ with $\varphi(t,x,u) = y\}$; analogously, $\mathcal{O}^-_{\leq T}(x) := \{y \in \mathbb{R}^d;$ there are $u \in \mathcal{U}$ and $0 < t \leq T$ with $\varphi(t,y,u) = x\}$. We also write $\mathcal{O}^+(x) := \bigcup_{T>0} \mathcal{O}^+_{\leq T}(x)$.

Next we define the maximal regions of complete controllability.

Definition. A set $D \subset \mathbb{R}^d$ is a control set, if

(i) $D \subset cl\mathcal{O}^+(x)$ for all $x \in D$;

(ii) for all $x \in D$ there exists $u \in \mathcal{U}$ with $\varphi(t,x,u) \in D$ for all $t \in \mathbb{R}$; .

(iii D is maximal (with respect to set inclusion) with the properties (i) and (ii).

In this paper we consider only control sets with nonvoid interior. Then (ii) follows from (i), and all control sets are connected, pairwise disjoint and, by (2.3), $intD \subset \mathcal{O}^+(x)$ for all $x \in D$, i.e. we have exact controllability in the interior of D.

The order between control sets is given by the reachability properties of (1.3):

(2.4) $$D_1 \prec D_2 \text{ if there exists } x \in D_1 \text{ with } \mathcal{O}^+(x) \cap D_2 \neq \emptyset.$$

Closed control sets are maximal elements w.r.t. $\prec$, open control sets are minimal. Closed control sets C are also invariant, i.e. $C = cl\mathcal{O}^+(x)$ for all $x \in C$. If $N \subset \mathbb{R}^d$ is compact and invariant for (1.3), then there exists at least one (invariant) control set C in N, and $int\, C \neq \emptyset$.

Recall the following concept from the theory of dynamical systems:

Consider the differential equation (1.1) and abbreviate $\varphi(t,x) := \varphi(t,x,0)$.

Then for $\varepsilon, T > 0$ an (ε,T)-chain for (1.1) from $y \in \mathbb{R}^d$ to $z \in \mathbb{R}^d$ is given by $n \in \mathbb{N}$, $y = y_0, y_1, \ldots y_n = z \in \mathbb{R}^d$, $t_0, \ldots t_{n-1} > T$ such that $|\varphi(t_i, y_i) - y_{i+1}| < \varepsilon$ for $i = 0, 1, \ldots n-1$.

The chain recurrent set $\mathcal{CR}$ of (1.1) consists of all $y \in \mathbb{R}^d$ which satisfy: For all $\varepsilon, T > 0$ there is an (ε, T)-chain from y to y. In particular, all limit points of bounded trajectories are in the chain recurrent set; trivially, this is true for all equilibria and all periodic points.

Next we consider the family of control systems $(1.3)^\rho$ (where α is kept fixed) with $0 \leq \rho < \infty$. The following result is an immediate consequence of [CK^a, Corollary 5.3].

2.1 Theorem. *Let M be a bounded connected component of the chain recurrent set $\mathcal{CR}$ of (1.1) which is isolated, i.e. there exists a neighborhood N of M such that $\mathcal{CR} \cap N = M$. Assume that for all $\rho > 0$ and all $x \in M$ there are $T, s > 0$ such that*

$$\varphi(T, x, 0) \in int\mathcal{O}^{\rho,+}_{\leq T+s}(x),$$

where $\mathcal{O}^{\rho,+}_{\leq T+s}(x)$ denotes the reachable set from x up to time $T + x$ for the system $(1.3)^\rho$.

Then there is an increasing sequence of control sets D^ρ of $(1.3)^\rho$ with $M \subset intD^\rho$ such that $M = \bigcap_{\rho > 0} D^\rho$.

Vice versa, if for a sequence $\rho_k \to 0$ and control sets D^{ρ_k} of $(1.3)^{\rho_k}$ the set of limit points $L = \{y \in \mathbb{R}^d;$ there are $x^k \in D^k$ with $x^k \to y\}$ is nonvoid, then L is a component of the chain recurrent set of (1.1).

3. Bifurcation of Control Sets at a Hopf Bifurcation

The results in this section do not use linearization of the considered control system. Accordingly, we do not actually use that a Hopf bifurcation occurs in the uncontrolled system. Instead, only the following weaker properties (which are a consequence of a Hopf bifurcation) of the uncontrolled system are used.

(3.1) There exist an open interval $I \ni \alpha^0$ and a continuous map $\alpha \mapsto x^\alpha : I \to \mathbb{R}^d$ such that for all $\alpha \in I$ the point x^α is an equilibrium of the uncontrolled system $(1.1)^\alpha$, i.e. $0 = f(x^\alpha, \alpha)$. For $\alpha > \alpha^0$ there exists a nontrivial periodic orbit H^α of $(1.1)^\alpha$ such that

$$\{(\alpha, x) : \alpha > \alpha^0, x \in H^\alpha\}$$

is connected, with $H^\alpha \to \{x^{\alpha^0}\}$ for $\alpha \to \alpha^0$. For all $\rho > 0$ and all $x \in \{x^\alpha\} \cup H^\alpha$ there are $T, s > 0$ such that $\phi^\alpha(t, x, 0) \in int\mathcal{O}^{\alpha,\rho,+}_{\leq T+s}(x)$ (the reachable set up to time $T+s$ from x for the system $(1.3)^{\alpha,\rho}$). Furthermore there exists a neighborhood N of x^0 such that for all $\alpha \in I$ the intersection of N with the chain recurrent set of $(1.1)^\alpha$ coincides with $\{x^\alpha\} \cup H^\alpha$ and $\{x^\alpha\}$, respectively.

(3.2) In addition to (3.1), for all $\alpha < \alpha^0$ the equilibrium x^α is asymptotically stable for $(1.1)^\alpha$ and unstable for $\alpha > \alpha^0$; the periodic orbits H^α are asymptotically stable for all $\alpha > \alpha^0$.

The next theorem is the main result of this paper. It shows that in the situation above also the control sets are subject to a 'bifurcation'.

3.1 Theorem. *Assume that condition (3.1) is satisfied. Then there exists an open subinterval J of I containing α^0 such that for all $\alpha \in J$ there is $\rho(\alpha) > 0$ with the following properties:*

(i) *For all $\alpha \in J$, $0 < \rho < \rho(\alpha)$ there is a control set $D^{\alpha,\rho}$ of $(1.3)^{\alpha,\rho}$ with $x^\alpha \in intD^{\alpha,\rho}$.*

(ii) *For all $\alpha \in J$, $\alpha > \alpha^0$, $0 < \rho \leq \rho(\alpha)$ there is a control set $\tilde{D}^{\alpha,\rho}$ of $(1.3)^{\alpha,\rho}$ having void intersection with $D^{\alpha,\rho}$, and $H^\alpha \subset int\tilde{D}^{\alpha,\rho}$ holds.*

(iii) *There is $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$ there is a maximal $\alpha^0 < \alpha(\rho) \leq \infty$ with the following property: For all $\alpha^0 < \alpha < \alpha(\rho)$ one has $\{x^\alpha\}, H^\alpha \subset intD^{\alpha,\rho}$.*

Remark. The number $\alpha(\rho)$ may be viewed as the bifurcation value for the control system $(1.3)^{\alpha,\rho}$. In the next section, we will numerically determine $\alpha(\rho)$ in an example.

Proof. As a direct consequence of assumption (3.1) and Theorem 2.1 we obtain $J \subset I$ such that: For all $\alpha \in J$ and all $\rho > 0$ there are control sets $D^{\alpha,\rho}$ of $(1.3)^{\alpha,\rho}$ with $x^\alpha \in intD^{\alpha,\rho}$ and

$$\{x^\alpha\} = \bigcap_{\rho>0} D^{\alpha,\rho};$$

furthermore, for all $\alpha > \alpha^0, \alpha \in J$ and all $\rho > 0$ there are control sets $\tilde{D}^{\alpha,\rho}$ of $(1.3)^{\alpha,\rho}$ with $H^\alpha \subset int\tilde{D}^{\alpha,\rho}$ and

$$H^\alpha = \bigcap_{\rho>0} \tilde{D}^{\alpha,\rho}.$$

Thus there is for every $\alpha \in J$, $\alpha > \alpha^0$, a number $\rho = \rho(\alpha) > 0$ with $D^{\alpha,\rho} \cap \tilde{D}^{\alpha,\rho} = \emptyset$. Since, obviously, $D^{\alpha,\rho_1} \subset D^{\alpha,\rho_2}$ and $\tilde{D}^{\alpha,\rho_1} \subset \tilde{D}^{\alpha,\rho_2}$ for $0 < \rho_1 < \rho_2$, the assertions (i) and (ii) follow.

Assertion (iii) follows immediately from (3.1) and the next lemma (cp.[Wi]) showing continuous dependence of the control sets $D^{\alpha,\rho}$ on the parameter α.

3.2 Lemma. *Let $K \subset intD^{\alpha^0}$ be compact. Then for all α in a neighborhood of α^0 it follows that $K \subset intD^\alpha$.*

Proof (of Lemma 3.2). Let $x, y \in K$. By (2.2) there are $T > 0$ and $\varepsilon > 0$ such that a point $z \in \bigcap_{|\alpha-\alpha^0|<\varepsilon} \mathcal{O}^{\alpha-}_{\leq T}(y) \cap K$ exists, where $\mathcal{O}^{\alpha-}_{\leq T}(y)$ is the set of points which can in system $(1.3)^\alpha$ be steered to y in time less or equal to T. Since $z \in K$ can be reached in finite time from x in system $(1.3)^{\alpha^0}$, it follows, by continuous dependence of trajectories on α, that for all α with $|\alpha - \alpha^0| < \varepsilon_0$, $\varepsilon_0 > 0$, the point y can be reached from x in system $(1.3)^\alpha$. Now ε^0 can be chosen uniformly for all y' in a neighborhood of y. An analogous argument for $\mathcal{O}^{\alpha+}_{\leq T}(x)$ combined with compactness of K shows that ε_0 can be chosen uniformly for all $x, y \in K$. This proves the lemma and hence Theorem 3.1 follows. □

The next result shows that also the stability properties carry over from the uncontrolled system to the control system.

3.3 Theorem. *Assume that condition (3.2) is satisfied. Then there exists an open subinterval $J_0 \subset J \subset I$ containing α^0 such that for all $\alpha \in J_0$ there is $\rho_0(\alpha) > 0$ with the following properties:*

(i) *For all $\alpha \in J_0$ with $\alpha < \alpha^0$, and all $0 < \rho < \rho_0(\alpha)$ the control sets $D^{\alpha,\rho}$ are invariant.*

(ii) *For all $\alpha \in J_0$ with $\alpha > \alpha^0$ and all $0 < \rho < \rho_0(\alpha)$ the control sets $D^{\alpha,\rho}$ are not invariant, and the control sets $\tilde{D}^{\alpha,\rho}$ are invariant; furthermore $D^{\alpha,\rho} \prec \tilde{D}^{\alpha,\rho}$.*

Proof. Assumption (3.2) implies that for $\alpha < \alpha^0$ the equilibrium x^α is an attractor for $(1.1)^\alpha$ and for $\alpha > \alpha^0$ the periodic solution H^α is an attractor for $(1.1)^\alpha$ (cp. [Ru]). Hence by Theorem 9(iii) in [CK^b] it follows that for $\rho > 0$, small enough, $D^{\alpha,\rho}$ and $\tilde{D}^{\alpha,\rho}$ are invariant control sets of system $(1.3)^{\alpha,\rho}$ for $\alpha < \alpha^0$ and $\alpha > \alpha^0$, respectively. By construction $D^{\alpha,\rho} \prec \tilde{D}^{\alpha,\rho}$ and hence $D^{\alpha,\rho}$ is not invariant for $\alpha > \alpha^0$ and $\rho > 0$, small. □

Remark. One cannot (at least in the nonanalytic case) exclude that in every neighborhood of x^{α^0} further control sets of $(1.3)^{\alpha,\rho}$ occur, compare Example 5.5 in [CK^a].

4. A Numerical Case Study

In this section we present numerical results on bifurcating control sets for a continuous flow stirred tank reactor (CSTR).

Consider the following simple model of a controlled CSTR (cp. [OM], [URP], [GS]):

$(4.1)^{\alpha,\rho}$

$$\dot{x}_1 = -x_1 - [u_0 + u(t)][x_1 - x_c] + B\alpha[1 - x_2]exp\left(\frac{\gamma x_1}{\gamma + x_1}\right)$$

$$\dot{x}_2 = -x_2 + \alpha[1 - x_2]exp\left(\frac{\gamma x_1}{\gamma + x_1}\right)$$

$$u(t) \in U^\rho = \rho[-1, 1].$$

Here x_1 and x_2 are the (normalized) temperature and product concentration, respectively, while γ, B, and x_c (corresponding to the coolant temperature) are constants. The parameter α (representing the Damkoehler number) is taken as a bifurcation parameter, and $u_0 + u(t)$ (the coolant flow rate which is proportional to the heat transfer coefficient) is taken as the control variable. In the numerical study the following parameter values were used (without regard to physical significance):

$$B = 12,\ x_c = 0,\ \gamma = 40,\ u_0 = 3,\ \rho = 0.02.$$

For $\alpha \in [0.25, 0.286]$ a Hopf bifurcation occurs in the uncontrolled system (where $u(t) \equiv 0$). For different values of α, the control sets of this system were computed, using the program 'cs' from [Ha]. Figures 1-4 present the obtained control sets (the shaded regions) and the phase portrait of the uncontrolled system.

In Figure 1, with $\alpha = 0.2500$, there is a unique (invariant) control set $D^{\alpha,\rho}$ around the asymptotically stable equilibrium of the uncontrolled system. In Figure 2, with $\alpha = 0.2824$, a stable periodic solution H^α has bifurcated from the equilibrium x^α, which now is unstable, both are contained in a single (invariant) control set $D^{\alpha,\rho}$. In Figure 3, with $\alpha = 0.2828$ there are two control sets $D^{\alpha,\rho}$ and $\tilde{D}^{\alpha,\rho}$ containing the (unstable) equilibrium x^α and the (stable) periodic solution H^α, respectively.

The control set $\tilde{D}^{\alpha,\rho}$ is invariant, while $D^{\alpha,\rho}$ is not. In Figure 4, with $\alpha = 0.2860$ the control sets have moved further apart. Hence the numerical result is that the bifurcation value $\alpha(\rho)$, $\rho = 0.2$, from Theorem 3.1(iii) lies in the interval $[0.2824, 0.2828]$.

The computations used a grid of 1200×1200 meshes on the state space; the calculation time (on an IBM 6000) varied between 17 minutes 13 seconds (for Figure 4) and 37 minutes 45 seconds (for Figure 3).

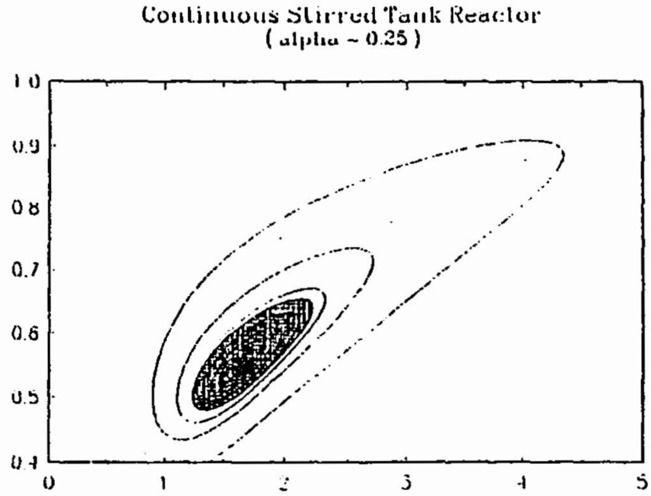

Figure 1. Control set and phase portrait for $\alpha = 0.25$.

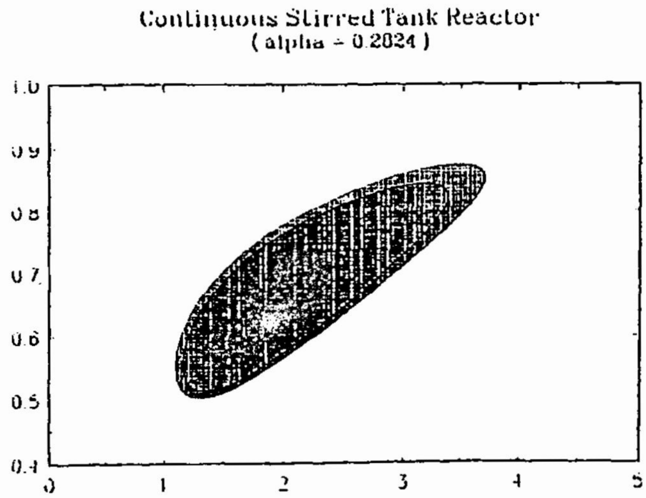

Figure 2. Control set and phase portrait for $\alpha = 0.2824$.

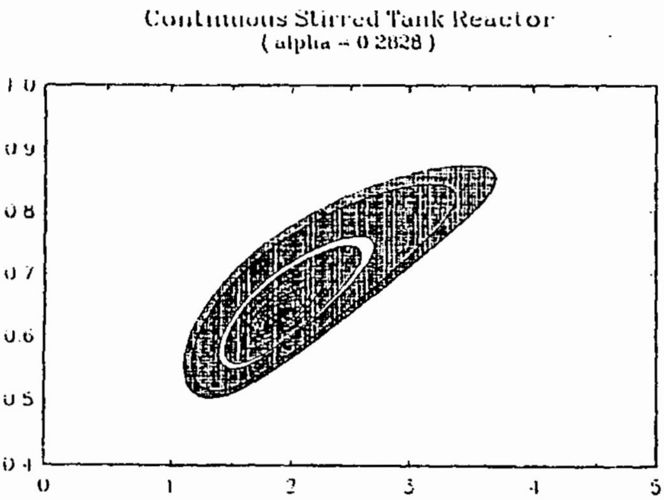

Figure 3. Control sets and phase portraits for $\alpha = 0.2828$.

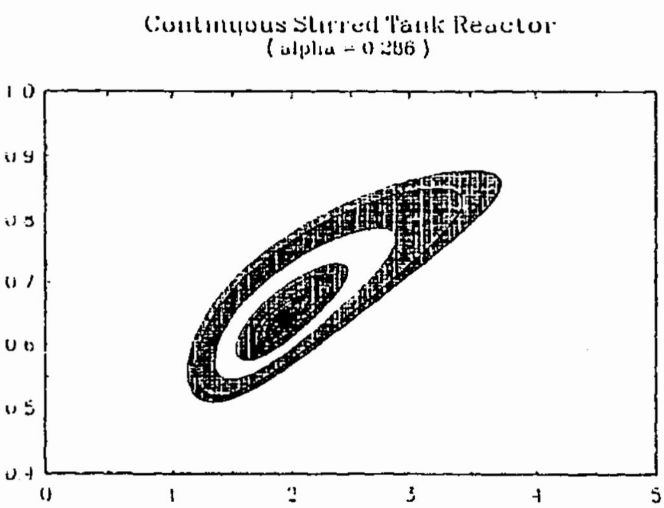

Figure 4. Control sets and phase portraits for $\alpha = 0.286$.

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