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A Note on the Boundedness of the Variables in Two-Sector Models of Optimal Economic Growth with Learning by Doing

von

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Abstract

It is shown that for two-sector models of optimal economic growth where learning by doing is present, the existence of an upper bound for the marginal products in the social production function is sufficient for the boundedness of per-capita variables.

The goal of this paper is to demonstrate that an upper bound for the marginal products of productive input factors in the production possibility frontier is sufficient to guarantee the boundedness of economic per-capita variables. This result holds for two-sector models of optimal economic growth with learning by doing [see Arrow (1962)].

We consider a closed economy with one consumption good, $c(t)$, and one investment good, $I(t)$. The stock of physical capital, $K(t)$, evolves according to $\dot{K}(t) = I(t) - \delta K(t)$, δ constant depreciation rate. The stock of human capital, $A(t)$, is formed as a by-product of past gross investment with exponentially declining weights put on investment flows further back in time and is given by $A(t) = \rho \int_{-\infty}^{t} e^{-\rho(t-s)} I(s) ds$, such that $\dot{A}(t) = \rho(I(t) - A(t))$. We make this assumption here because from the economic point of view it seems plausible to suppose that investment further back in time contributes less to the formation of workers' skill than more recent investment. Supposing efficient production, consumption is given by $c(t) = T(A(t), K(t), I(t))$, with $T(A(t), K(t), I(t))$ denoting the production possibility frontier (PPF). The amount of labour is assumed to be constant and is normalized to 1 such that the variables give per-capita values.

As to the PPF, we make the following assumptions.

(A1) $T(A, K, I)$ is increasing in A and K and decreasing in I. Furthermore, $T_{II}(.,.,.) < 0$, $T_{AA}(.,.,.) \leq 0$, $T_{KK}(.,.,.) \leq 0$, $T_{ij}(.,.,.) \geq\leq 0$, $i, j = A, K, I$, $i \neq j$.

(A2) There exists an upper bound for $T_A(.,.,.)$ and $T_K(.,.,.)$, i.e. $T_A(.,.,.) \leq M$, $T_K(.,.,.) \leq M$, $0 < M =$ constant.

The maximization problem for a social planner is then given by:

$$\max_{I} \int_0^{\infty} e^{-rt} T(A(t), K(t), I(t)) dt \tag{1}$$

subject to

$$\dot{K}(t) = I(t) - \delta K(t), \; K(0) = K_0 > 0, \tag{2}$$

$$\dot{A}(t) = \rho(I(t) - A(t)), \; A(0) = A_0 > 0, \tag{3}$$

$$I(t) \in \Omega(K(t), A(t)) = \{I | I \geq 0, \; T(A, K, I) \geq 0, \; I \in \mathbb{R}\}. \tag{4}$$

We now show that for our problem the domain of all possibly optimal values for $I(t)$ is bounded. To do this we adopt an idea from Sorger (1986), p. 73/74.

Theorem *Under assumptions (A1) and (A2) the domain of all possibly optimal values of the control I(t) for problem (1)-(4) is bounded.*

Proof: To prove that $I(t)$ is bounded from above we consider two controls $I_1(t)$, $I_2(t)$ and a constant $\bar{I}$ for which:

$$I_2(t) \begin{cases} = \bar{I} < I_1(t) \text{ for } t \in (\tau_1, \tau_3) \\ = I_1(t) \text{ for } t \notin (\tau_1, \tau_3), \end{cases}$$

with $\tau_3 = min(\tau_1 + 1, \tau_2)$ and $\int_0^{\tau_3} I_1(t)dt < \infty$.

Let (A_i, K_i) denote the trajectory belonging to I_i according to (2) and (3). $A(t)$ and $K(t)$ are then given by

$$A(t) = A(0)e^{-\rho t} + \rho \int_0^t I(s)e^{-\rho(t-s)}ds,$$

$$K(t) = K(0)e^{-\delta t} + \int_0^t I(s)e^{-\delta(t-s)}ds,$$

such that the following holds for all $t \in [0, \infty)$,

$$0 \leq A_1(t) - A_2(t) \leq \int_{\tau_1}^{\tau_3} \rho(I_1(\tau) - I_2(\tau))d\tau, \tag{5}$$

$$0 \leq K_1(t) - K_2(t) \leq \int_{\tau_1}^{\tau_3} (I_1(\tau) - I_2(\tau))d\tau. \tag{6}$$

We may write:

$T(A_2, K_2, I_2) - T(A_1, K_1, I_1) = T_1(A_2 + \kappa(A_1 - A_2), K_2 + \kappa(K_1 - K_2), I_2 + \kappa(I_1 - I_2))(A_2 - A_1) + T_2(A_2 + \kappa(A_1 - A_2), K_2 + \kappa(K_1 - K_2), I_2 + \kappa(I_1 - I_2))(K_2 - K_1) + T_3(A_2 + \kappa(A_1 - A_2), K_2 + \kappa(K_1 - K_2), I_2 + \kappa(I_1 - I_2))(I_2 - I_1)$, with $\kappa(t)$ function taking values in $(0, 1)$ and thus $\int_0^\infty e^{-rt}[T(A_2, K_2, I_2) -$

$T(A_1, K_1, I_1)]dt = \int_{\tau_1}^{\infty} e^{-rt}[T_1(.,.,.)(A_2 - A_1) + T_2(.,.,.)(K_2 - K_1)]dt + \int_{\tau_1}^{\tau_3} e^{-rt} \cdot [T_3(.,.,.)(I_2 - I_1)]dt$.

As $I_2 \leq I_1$, $\bar{I} < I_1$ and $T_I(.,.,.) < 0$, $T_{II}(.,.,.) < 0$, we know that $\int_{\tau_1}^{\tau_3} e^{-rt}[T_3(.,.,.)(I_2 - I_1)]dt > \int_{\tau_1}^{\tau_3} e^{-rt}[T_3(.,.,\bar{I})(I_2 - I_1)]dt \geq \int_{\tau_1}^{\tau_3} e^{-rt}[T_3(A_i, K_j, \bar{I}) \cdot (I_2 - I_1)]dt > T_3(C_1, C_1, \bar{I}) \int_{\tau_1}^{\tau_3} e^{-rt}(I_2 - I_1)dt.$ (7)

Note that $i = 1$ if $T_{IA}(.,.,.) \geq 0$, $i = 2$ if $T_{IA}(.,.,.) \leq 0$ and $j = 1$ if $T_{IK}(.,.,.) \geq 0$, $j = 2$ if $T_{IK}(.,.,.) \leq 0$. Since $A(t)$ and $K(t)$ are bounded for $t \in [0, \tau_3]$ we can find a non-negative constant C_1 such that the last inequality in (7) holds.

Knowing that there exists a constant M with $T_A(.,.,.) \leq M$, $T_K(.,.,.) \leq M$ we can write $|\int_{\tau_1}^{\infty} e^{-rt}[T_1(.,.,.)(A_2 - A_1) + T_2(.,.,.)(K_2 - K_1)]dt| \leq \int_{\tau_1}^{\infty} e^{-rt}M[(A_1 - A_2) + (K_1 - K_2)]dt \leq \int_{\tau_1}^{\infty}\int_{\tau_1}^{\tau_3} e^{-rt}M\rho(I_1(\tau) - I_2(\tau))d\tau dt + \int_{\tau_1}^{\infty}\int_{\tau_1}^{\tau_3} e^{-rt}M(I_1(\tau) - I_2(\tau))d\tau dt = \int_{\tau_1}^{\infty}\int_{\tau_1}^{\tau_3} e^{-rt}M_1(I_1(\tau) - I_2(\tau))d\tau dt = M_1 \cdot (e^{-r\tau_1}/r) \int_{\tau_1}^{\tau_3}(I_1 - I_2)dt \leq (M_1 e^r/r) \int_{\tau_1}^{\tau_3} e^{-rt}(I_1 - I_2)dt.$ (8)

Here we used $|\tau_3 - \tau_1| \leq 1$ and defined $(1 + \rho)M \equiv M_1$.

Taking (7) and (8), we see that $\int_0^{\infty} e^{-rt}[T(A_2, K_2, I_2) - T(A_1, K_1, I_1)]dt > (-T_3(C_1, C_1, \bar{I}) - M_1 e^r/r) \int_{\tau_1}^{\tau_3} e^{-rt}(I_1 - I_2)dt$.

Putting $\bar{I} = -T_3^{-1}(.,., M_1 e^r/r)$, it is immediately seen that $\int_0^{\infty} e^{-rt}(T(A_2, K_2, I_2) - T(A_1, K_1, I_1))dt > 0$ and it is shown that the domain of all possibly optimal controls is bounded. □

Remarks:

1. The assumption of an upper bound for the marginal products of human capital and physical capital was crucial for condition (8). This demonstrates that in this sort of models the variables remain bounded as long as there is an upper bound for the marginal products in the production possibility frontier $T(A, K, I)$.

The boundedness of $A(t)$ and $K(t)$ follows immediately from the existence of an upper bound for investment, i.e. from $I(t) < \bar{I}$.

The boundedness of investment is also assured if no weight is put on investment contributing to the formation of human capital, i.e. if $A(t) = \int_{-\infty}^{t} I(s)ds$. But then, for $t \to \infty$, $A(t) = A(0) + \int_0^t I(s)ds$ may become unbounded and thus $c(t) = T(A(t), K(t), I(t))$, too. This demonstrates, on the other hand, that increasing returns are not necessary to generate persistent growth of per-capita consumption if the input factors (here $A(t)$) increase without bound. A fact used by Lucas (1988) in his paper.

2. It can also be shown that the domain of all possibly optimal controls is bounded from below. The proof that a lower bound for $I(t)$ exists proceeds as follows.

Let us denote the lower bound with $\underline{I}$ and assume,

$$I_2(t) \begin{cases} = \underline{I} < I_1(t) < 2\underline{I} \text{ for } t \in (\tau_1, \tau_3) \\ = I_1(t) \text{ for } t \notin (\tau_1, \tau_3). \end{cases}$$

Now we have to show $\int_0^\infty e^{-rt}[T(A_2, K_2, I_2) - T(A_1, K_1, I_1)]dt < 0$.

As the lower bound for the partial derivatives of A and K is 0, it follows that $\int_{\tau_1}^\infty e^{-rt}[T_1(.,.,.)(A_2 - A_1) + T_2(.,.,.)(K_2 - K_1)]dt \leq 0.$ (8′)
Moreover, in analogy to (7) it can be shown that $\int_{\tau_1}^{\tau_3} e^{-rt}[T_3(.,.,.)(I_2 - I_1)]dt < -T_3(C_1, C_1, 2\underline{I}) \int_{\tau_1}^{\tau_3} e^{-rt}(I_1 - I_2)dt.$ (7′)
Here we used the fact $I(t) < 2\underline{I}$.

Combining (7′) and (8′) the existence of a lower bound is immediately established if the opportunity costs of investment tend to zero for a finite value of investment. For demonstrating that $I(t)$ is bounded from below the assumption of an upper bound for the partial derivatives of A and K is of course not needed.

The restriction $I(t) \geq 0$, i.e. the assumption of the irreversibility of investment (condition (4)), often encountered in economic literature, may be neglected by assuming that the set $I(t) \in \Omega(K(t), A(t)) = \{I | I > 0,\ T(A,K,I) > 0,\ I \in [0,\bar{I}]\}$ is non-empty and by imposing the conditions $\lim_{I \to 0} T_I(.,.,.) = 0$ and $\lim_{I \to \bar{I}} T_I(.,.,.) = -\infty$.

If $\lim_{I \to \check{I}} T_I(.,.,.) = 0$, with $\check{I} < 0$, negative gross investment cannot be excluded.

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