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Sensitivity Analysis for Optimization Problems in Hilbert Spaces with Bilateral Constraints*

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Abstract

A general framework for directional differentiability of the solutions and the associated Lagrange multipliers of optimization problems in Hilbert spaces involving bilateral constraints is developed. Applications are given to parameter estimation and to optimal control problems for ordinary and partial differential equations.

Key words: optimal control, sensitivity analysis, boundary control, parameter estimation

AMS Subject Classifications: 49B50, 34A55

1 Introduction

In this paper we develop a sensitivity analysis for optimization problems in Hilbert spaces involving bilateral constraints. The problems under investigation are of the type

$$\begin{aligned} &\min f(x,p) \\ &\text{subject to} \quad e(x,p) = 0 \\ &\qquad\qquad\quad g(x,p) \le 0 \\ &\qquad\qquad\quad z_1 \le b(x,p) \le z_2, \end{aligned} \tag{$\mathcal{P}_p$}$$

where x is an element of a Hilbert space X and p is a possibly infinite dimensional perturbation parameter. The image spaces of f, e, g and b are one–, infinite–, finite–, and infinite dimensional, respectively. In practical

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applications $g(x,p) \le 0$ can characterize a norm constraint of x or, if x has several components, of some of its components, and the last constraint in $(\mathcal{P}_p)$ can describe a pointwise constraint from above and below on x. In optimal control problems involving differential equations, the equality constraint could be used to describe the controlled differential equation, in which case x would consist of a pair (y,u) with y the state– and u the control variable. Alternatively one could consider the state variable y as a function of the control variable u; in this case $x = u$ and the differential equation is taken into account implicitly through the definition of f.

To facilitate the ensuing discussion let x_0 be a local solution of $(\mathcal{P}_{p_0})$ at a reference parameter p_0 and let $(\lambda_0, \mu_0, \eta_0)$ denote a Lagrange multiplier triple associated with the three constraints in $(\mathcal{P}_{p_0})$. Analogously we denote by $(x_p, \lambda_p, \mu_p, \eta_p)$ a solution and a multiplier triple for (P_p) for p in a neighborhood of p_0. Assuming progressively more stringent hypotheses we shall obtain Hölder continuous dependence of x_p on p, Lipschitz continuous dependence of $(x_p, \lambda_p, \mu_p, \eta_p)$ on p, Gateaux differentiability of the minimal value functions with respect to p, and finally directional differentiability of $(x_p, \lambda_p, \mu_p, \eta_p)$ at p_0 and an optimality system characterizing the directional derivative $(\dot{x}, \dot{\lambda}, \dot{\mu}, \dot{\eta})$ of $p \mapsto (x_p, \lambda_p, \mu_p, \eta_p)$ at p_0. The applicability of our general results will be illustrated by means of four examples involving an obstacle problem and problems in parameter estimation, optimal control of a hyperbolic partial differential equation and optimal control of a nonlinear ordinary differential equation.

The present paper uses ideas from [4] in an essential way. The results in [4], however, do not include bilateral constraints. The relationship of our results to those in [4] is discussed in detail in Remark 2.1, below. It is the main contribution of this paper to develop a general framework for a sensitivity analysis which includes bilateral constraints. Our work uses results from Robinson [15] on Lipschitz continuous dependence of solutions of "generalized equations" and of Haraux [3] and Mignot [11] on directional differentiability of the projection onto sets which are polyhedric at certain points. We point out that our results do not rely on any kind of strict complementarity assumption.

Several authors have studied differential stability properties for infinite dimensional optimization problems with constraints and we mention [9, 12, 13, 14, 16] in this respect. These contributions, however, are either focused on specific problems—optimal control of differential equations or parameter estimation—or they do not include the differential stability for the optimal solutions in the presence of bilateral constraints, which themselves depend on parameters.

In [10] Malanowski treats general optimization problems in Hilbert spaces and applies the results to optimal control problems. However, in this work only weak directional differentiability of the optimal solutions is obtained.

Let us also point out that results on the differentiability of the solutions to $(\mathcal{P}_p)$ and of the associated Lagrange multipliers are useful for numerical computations. This point is illustrated in [5], for example, where techniques on the optimal choice of the regularization parameter for nonlinear ill-posed inverse problems (e.g. parameter estimation problems) are proposed.

In Section 2 of this paper we develop the general theory and Section 3 is devoted to the applications.

2 General Theory

In this section we develop sensitivity results for the following parameter dependent optimization problem:

$$\begin{array}{lll} \min f(x,p) & \text{over } x & \\ \text{subject to} & e(x,p)=0 & (\mathcal{P}_p) \\ & g(x,p)\leq 0 & \\ & z_1 \leq b(x,p) \leq z_2, & \end{array}$$

where $f: X\times P\to\mathbb{R}$, $e: X\times P\to Y$, $g: X\times P\to\mathbb{R}^m$, $b: X\times P\to Z$. Here X, Y, Z are (real) Hilbert spaces, P is a normed linear space, $\mathbb{R}^m$ is considered with the natural negative cone $\mathbb{R}^m_-$ and K is a closed convex cone in Z with vertex at zero satisfying $K\cap(-K)=0$. The cone K induces a natural ordering on Z given by $z_1\leq z_2$ iff $z_1-z_2\in K$, similarly $\mathbb{R}^m_-$ induces a natural ordering on $\mathbb{R}^m$. For the parameter value $p_0\in P$, $(\mathcal{P}_{p_0})$ is considered as the unperturbed problem. Our general assumptions are that $z_1\neq z_2$, $x_0\in X$ is a local solution of $(\mathcal{P}_{p_0})$, the maps f, e, g and b are twice continuously Fréchet–differentiable with respect to x at (x_0,p_0) and the first and second derivatives are continuous in a neighbourhood of (x_0,p_0).

We refer to the constraints specified by e, b and g above as the equality constraint, the bilateral constraint and the (additional) finite dimensional constraint, respectively.

In a first step we transform the bilateral constraint into a unilateral one, which is of a special nature. We introduce

$$\begin{aligned} \tilde{Z} &:= \left\{\begin{pmatrix} z \\ -z\end{pmatrix}+\rho\begin{pmatrix} z_1 \\ -z_2\end{pmatrix} : z\in Z, \rho\in\mathbb{R}\right\} \\ &= \operatorname{diag}(Z\times -Z)\oplus\operatorname{span}\left\{\begin{pmatrix} z_1 \\ -z_2\end{pmatrix}\right\}. \end{aligned}$$

Then $\tilde{Z}$ is a closed linear subspace of $Z \times Z$, and hence it is a Hilbert space with respect to the induced scalar product. Define further

$$\tilde{b} : X \times P \to \tilde{Z} \quad \text{by}$$
$$\tilde{b}(x,p) := \begin{pmatrix} -b(x,p) + z_1 \\ b(x,p) - z_2 \end{pmatrix}$$

and

$$\tilde{K} := \tilde{Z} \cap (K \times K).$$

Note that $\tilde{K}$ is a closed and convex cone with vertex at zero in $\tilde{Z}$. Clearly $(\mathcal{P}_p)$ may be rewritten in the form

$$\begin{array}{rll} \min f(x,p) & \text{over } x & \\ \text{subject to} & e(x,p) = 0 & \qquad (\mathcal{P}'_p) \\ & g(x,p) \leq 0 & \\ & \tilde{b}(x,p) \leq 0, & \end{array}$$

where $\tilde{b}(x,p) \leq 0$ means $\tilde{b}(x,p) \in \tilde{K}$.

Next we introduce some notation. We define the Lagrange functional $\mathcal{L}$ for $(\mathcal{P}_p)$ by

$$\mathcal{L} : X \times P \times Y \times \mathbb{R}^m \times \tilde{Z} \to \mathbb{R}$$
$$\mathcal{L}(x,p,\lambda,\mu,\tilde{\eta}) = f(x,p) + \langle \lambda, e(x,p)\rangle_Y + \langle \mu, g(x,p)\rangle_{\mathbb{R}}^m + \langle \tilde{\eta}, \tilde{b}(x,p)\rangle_{\tilde{Z}}.$$

In the following, derivative with respect to x will be denoted by a prime. A triple $(\lambda_0, \mu_0, \tilde{\eta}_0) \in Y \times \mathbb{R}^m \times \tilde{Z}$ is called a *Lagrange multiplier* for $(\mathcal{P}_{p_0})$ at x_0 if

$$\begin{array}{ll} \mathcal{L}'(x_0,p_0,\lambda_0,\mu_0,\tilde{\eta}_0) & = 0 \\ e(x_0,p_0) & = 0 \\ \langle \mu_0, g(x_0,p_0)\rangle_{\mathbb{R}}^m & = 0, \ g(x_0,p_0) \leq 0, \ \mu_0 \in \mathbb{R}_+^m \\ \langle \tilde{\eta}_0, \tilde{b}(x_0,p_0)\rangle_{\tilde{Z}} & = 0, \ \tilde{b}(x_0,p_0) \leq 0, \ \tilde{\eta}_0 \in \tilde{K}_+, \end{array} \tag{2.1}$$

where $\tilde{K}_+$ is the dual cone of $\tilde{K}$, i.e.

$$\tilde{K}_+ := \left\{ z \in \tilde{Z} : \langle z, k\rangle \leq 0 \quad \text{for all} \quad k \in \tilde{K} \right\}.$$

We will identify $\mathcal{L}'(x_0,p_0,\lambda_0,\mu_0,\tilde{\eta}_0)$ with an element of X. In abbreviated notation, (2.1) can be rewritten as

$$0 \in \begin{cases} \mathcal{L}'(x_0,p_0,\lambda_0,\mu_0,\eta_0) \\ e(x_0,p_0) \\ -g(x_0,p_0) + \partial\Psi_{\mathbb{R}_+^m}(\mu_0) \\ -\tilde{b}(x_0,p_0) + \partial\Psi_{\tilde{K}_+}(\tilde{\eta}_0), \end{cases} \tag{2.2}$$

where for a closed convex subset C of a Hilbert space H, $\partial\Psi_C(x)$ denotes the subdifferential at x of the indicator function Ψ_C of C,

$$\Psi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{if } x \notin C, \end{cases}$$

which is given by

$$\partial\Psi_C(x) = \begin{cases} \{y \in H : \langle y, c - x\rangle \leq 0 \quad \text{for all } x \in C\} & \text{if } x \in C \\ \emptyset & \text{if } x \notin C. \end{cases}$$

Furthermore we define $A : X \to X$ as the operator representation of $\mathcal{L}''(x_0, p_0, \lambda_0, \mu_0, \eta_0)$ i.e.

$$\langle Ax, y\rangle = \mathcal{L}''(x_0, p_0, \lambda_0, \mu_0, \eta_0)(x, y) \quad \text{for all } x, y, \in X, \tag{2.3}$$

and

$$E := e'(x_0, p_0),\ G = g'(x_0, p_0),\ B = b'(x_0, p_0),\ \tilde{B} = \begin{pmatrix} -B \\ B \end{pmatrix}. \tag{2.4}$$

The Lagrange multiplier associated with the finite dimensional inequality constraint is decomposed with respect to the sign of the components of μ_0 and $g(x_0, p_0)$:

$$\begin{cases} \mu_0 = (\mu_0^+, \mu_0^0, \mu_0^-) \in \mathbb{R}^{m^+} \times \mathbb{R}^{m^0} \times \mathbb{R}^{m^-} \\ g = (g^+, g^0, g^-) : X \to \mathbb{R}^{m^+} \times \mathbb{R}^{m^0} \times \mathbb{R}^{m^-} \end{cases} \tag{2.5}$$

according to

$$\begin{aligned} g^+(x_0, p_0)6 &= 0, \quad \mu_0^+ > 0 \\ g^0(x_0, p_0) &= 0, \quad \mu_0^0 = 0 \\ g^-(x_0, p_0) &< 0, \quad \mu_0^- = 0, \end{aligned}$$

where $m = m^+ + m^0 + m^-$.

For our local analysis, g^- can be deleted and g^+ will be treated like an equality constraint.

Let

$$\begin{cases} G_+ = g^+(x_0, p_0)',\ G_0 = g^0(x_0, p_0)' \\ E_+ = \begin{pmatrix} E \\ G_+ \end{pmatrix} : X \to Y \times \mathbb{R}^{m^+} \end{cases} \tag{2.6}$$

and define

$$\mathcal{E} : X \times \mathbb{R} \to (Y \times \mathbb{R}^{m^+}) \times \mathbb{R}^{m^0} \times \tilde{Z},\ \mathcal{E} := \begin{pmatrix} E_+ & 0 \\ G_0 & 0 \\ \tilde{B} & \tilde{b}(x_0, p_0) \end{pmatrix}. \tag{2.7}$$

The following hypotheses will be employed below:

(H1) (regular point condition at (x_0, p_0)):

$$0 \in \operatorname{int}\left\{\begin{pmatrix} E \\ G \\ \tilde{B} \end{pmatrix} X + \begin{pmatrix} 0 \\ \mathbb{R}^m_+ \\ -\tilde{K} \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 \\ g(x_0, p_0) \\ \tilde{b}(x_0, p_0) \end{pmatrix}\right\}.$$

(H2) There exists a neighborhood V of (x_0, p_0) and a constant $\nu > 0$ such that

$$\begin{aligned} |f(x,p) &- f(x,q)| + |e(x,p) - e(x,q)|_Y + |g(x,p) - g(x,q)|_{\mathbb{R}}^m \\ &+ |b(x,p) - b(x,q)|_Z \leq \nu|p - q| \end{aligned}$$

for all $(x,p), (x,q) \in V$.

(H3) There exists $\kappa > 0$ such that

$$\langle Ax, x\rangle_X \geq \kappa |x|_X^2$$

for all $x \in \ker E_+$.

(H4) $\mathcal{E}$ is surjective.

The following relationship holds between (H1) and (H4).

Lemma 2.1 *Hypothesis (H4) implies (H1).*

The proof of this and all the following lemmas will be given at the end to this section.

For $r > 0$ we define the local extremal value function μ_r by

$$\Sigma_r(p) = \{x \in X : e(x,p) = 0, g(x,0) \leq 0, \tilde{b}(x,p) \leq 0, |x - x_0| \leq r\}$$

and

$$\mu_r(p) = \inf\{f(x,p) : x \in \Sigma_r(p)\}.$$

With (H4) holding, $\mu_r(p)$ is well defined provided that p is sufficiently close to p_0 and that $r > 0$ is sufficiently small.

As a special case of [1, theorems 3 and 6], one obtains the following results on Lipschitz continuity of the local extremal value function and on Hölder continuity of the local extrema.

Theorem 2.1 *Let (H1) and (H2) hold. Then there exist a constant $r > 0$ and a neighbourhood $\hat{V}$ of p_0 such that μ_r is finite on $\hat{V}$ and Lipschitz continuous at p_0, i.e.*

$$|\mu_r(p) - \mu_r(p_0)| \leq L_r |p - p_0|$$

for some $L_r > 0$ independent of $p \in \hat{V}$. If, moreover, (H3) holds, then r, L_r and V can be chosen such that the following conclusion holds: If for $p \in P$, there exists $x_p \in \Sigma_r(p)$ with $f(x_p, p) = \mu_r(p)$, then $x_p \in \operatorname{int} \Sigma_r(p)$ (i.e. x_p is a local minimizer of $(\mathcal{P}_p)$) and

$$|x_p - x_0|_X \le L_r |p - p_0|^{1/2}.$$

We already observed in Lemma 2.1 that (H4) implies (H1). Next we turn to sufficient conditions on the original problem $(\mathcal{P}_p)$ which ensure (H1).

We shall make use of the following two hypotheses:

(A1) The map $\binom{E}{B} : X \to Y \times Z$ is surjective.

(A2) There exists $\omega \in X$ such that

$$z_1 \le B\omega \le z_2 \quad \text{and}$$

$$G\left[\begin{pmatrix} E \\ B \end{pmatrix}^* \begin{pmatrix} E \\ B \end{pmatrix}\Big|_{\operatorname{Im}\begin{pmatrix} E \\ B \end{pmatrix}^*}\right]^{-1} B^*[B\omega - b(x_0, p_0)] < 0 \text{ in } \mathbb{R}^m,$$

(i.e. every component is less than zero).

Note that (A1) implies the existence of the inverse appearing in (A2).

We refer to section 3.4 for an example in which (A2) and consequently (H1) hold, but where (H4) is not satisfied.

Proposition 2.1 *Hypotheses (A1) and (A2) imply (H1). Moreover, if g is not present in $(\mathcal{P}_p)$, then (A1) implies (H1) and (H4).*

Proof of Proposition 2.1: We have to verify that in $Y \times \mathbb{R}^m \times \tilde{Z}$

$$0 \in \operatorname{int} \left\{ \begin{pmatrix} E \\ G \\ \tilde{B} \end{pmatrix} X + \begin{pmatrix} 0 \\ \mathbb{R}^m_+ \\ -\tilde{K} \end{pmatrix} + \mathbb{R} \begin{pmatrix} 0 \\ g(x_0, p_0) \\ \tilde{b}(x_0, p_0) \end{pmatrix} \right\}.$$

Consider the equation

$$\begin{pmatrix} Eh \\ Gh \\ -Bh \\ Bh \end{pmatrix} + \begin{pmatrix} 0 \\ r_+ \\ -v - \rho z_1 \\ v + \rho z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(x_0, p_0) \\ -b(x_0, p_0) + z_1 \\ b(x_0, p_0) - z_2 \end{pmatrix} = \begin{pmatrix} y \\ r \\ z + \alpha z_1 \\ -z - \alpha z_2 \end{pmatrix}. \tag{2.8}$$

Condition (H1) will be verified if the existence of $\varepsilon > 0$ can be established such that for all $(y, r, z, \alpha) \in Y \times \mathbb{R}^m \times Z \times \mathbb{R}$ with

$$|y| + |r| + |z + \alpha z_1|_Z + |z + \alpha z_2|_Z < \varepsilon \tag{2.9}$$

there exists a solution $(h, r_+, v, \rho) \in X \times \mathbb{R} \times Z \times \mathbb{R}$ of (2.8) satisfying

$$r_+ \in \mathbb{R}^m_+, \quad v + \rho z_1 \in K, \quad v + \rho z_2 \in -K. \tag{2.10}$$

In order to show this, we first observe that (2.9) implies

$$|r| < \varepsilon, \quad |\alpha| \leq \frac{\varepsilon}{|z_1 - z_2|} < \infty, \quad |z| \leq \varepsilon \left(1 + \frac{|z_1|}{|z_1 - z_2|}\right). \tag{2.11}$$

The first inequality in (2.11) is obvious, the second one follows from

$$\alpha|z_1 - z_2| \leq |\alpha z_1 + z| + |z + \alpha z_2| < \varepsilon$$

and $z_1 \neq z_2$, and the last inequality is a consequence of

$$|z| \leq |z + \alpha z_1| + |\alpha z_1| \leq \varepsilon + \frac{\varepsilon}{|z_1 - z_2|}|z_1|.$$

Without loss of generality we assume $|\alpha| < 1$. Now choose ρ, v, and h such that

$$\begin{gathered} \rho = 1 - \alpha, \quad v = (\alpha - 1)B\omega, \\ Eh = y, \quad Bh = -v - z - b(x_0, p_0) \\ h \in \left[\ker \begin{pmatrix} E \\ B \end{pmatrix}\right]^{\perp}. \end{gathered} \tag{2.12}$$

Here we have used (A1) and (A2). Observe that by definition of ω and v and since $|\alpha| < 1$ we find

$$\begin{aligned} v + \rho z_1 &= (\alpha - 1)(B\omega - z_1) \leq 0 \quad \text{and} \\ v + \rho z_2 &= (\alpha - 1)(B\omega - z_2) \geq 0, \end{aligned}$$

as required in (2.10).

The first equation in (2.8) is satisfied by definition of h. The third equation in (2.8) is verified by

$$\begin{aligned} &-Bh - v - \rho z_1 - b(x_0, p_0) + z_1 \\ &= v + z + b(x_0, p_0) - v + (\alpha - 1)z_1 - b(x_0, p_0) + z_1 \\ &= z + \alpha z_1; \end{aligned}$$

and similarly, the fourth equation is verified:

$$\begin{aligned} &Bh + v + \rho z_2 + b(x_0, p_0) - z_2 \\ &= -v - z - b(x_0, p_0) + v + (1 - \alpha)z_2 + b(x_0, p_0) - z_2 \\ &= -z - \alpha z_2. \end{aligned}$$

The proof will be finished if we can solve the second equation of (2.8) for $r_+ \in \mathbb{R}^m_+$ by choosing $\varepsilon > 0$ sufficiently small.

Observe first that

$$\begin{pmatrix} E \\ B \end{pmatrix} h = \begin{pmatrix} y \\ (1-\alpha)B\omega - z - b(x_0,p_0) \end{pmatrix},$$

and hence

$$\begin{pmatrix} E \\ B \end{pmatrix}^* \begin{pmatrix} E \\ B \end{pmatrix} h = E^*y + (1-\alpha)B^*B\omega - B^*z - B^*b(x_0,p_0).$$

Since $\binom{E}{B}$ is surjective, $\binom{E}{B}^*\binom{E}{B}_{|\operatorname{Im}\binom{E}{B}^*}$ is an isomorphism on $\operatorname{Im}\binom{E}{B}^*$. By the choice of $h \in \left[\ker\binom{E}{B}\right]^\perp = \operatorname{Im}\binom{E}{B}^*$, we find, abbreviating $D = \left(\binom{E}{B}^*\binom{E}{B}_{|\operatorname{Im}\binom{E}{B}^*}\right)^{-1}$ that

$$h = D[E^*y + (1-\alpha)B^*B\omega - B^*z - B^*b(x_0,p_0)].$$

Note that $E^*y - B^*z$, $B^*B\omega$ and $B^*b(x_0,p_0)$ are elements of $\operatorname{Im}\binom{E}{B}^*$. This implies that

$$h = (1-\alpha)DB^*B\omega - DB^*b(x_0,p_0) + D[E^*y - B^*z].$$

This expression for h used in the second equation of (2.8) yields

$$\begin{aligned} r &= Gh + r_+ + g(x_0,p_0) \\ &= (1-\alpha)GDB^*B\omega - GDB^*b(x_0,p_0) \\ &\quad +GD[E^*y - B^*z] + r_+ + g(x_0,p_0). \end{aligned} \tag{2.13}$$

By (A2) we have

$$GDB^*B\omega - GDB^*b(x_0,p_0) < 0.$$

Hence one finds $\varepsilon > 0$ and $(\eta_j) \in \mathbb{R}^m$ with $\eta_j > 0$ for $j = 1,\dots,m$ such that

$$(1-\alpha)GDB^*B\omega - GDB^*b(x_0,p_0) \le -\eta$$

for any $|\alpha| < \frac{\varepsilon}{|z_1 - z_2|}$.

In view of (2.11) and $g(x_0,p_0) \le 0$ one can now decrease $\varepsilon > 0$ further such that (2.13) has a solution r_+ in $\mathbb{R}^m_+$ whenever (y,r,z,α) satisfy (2.9). This shows that (A1) and (A2) imply (H1).

Next let us assume that g is not present in $(\mathcal{P}_p)$. We shall show that (A1) implies (H4). By Lemma 2.1 this will conclude the proof. For any $(y,z,\alpha) \in Y \times Z \times \mathbb{R}$ one has to find a solution of

$$\begin{pmatrix} Eh \\ -Bh \\ Bh \end{pmatrix} + r \begin{pmatrix} 0 \\ -b(x_0,p_0) + z_1 \\ b(x_0,p_0) - z_2 \end{pmatrix} = \begin{pmatrix} y \\ z + \alpha z_1 \\ -z - \alpha z_2 \end{pmatrix}$$

for any $(y, z, \alpha) \in Y \times Z \times \mathbb{R}$. This is achieved by choosing $r = \alpha$ and using (A1) to find h such that

$$\begin{pmatrix} E \\ L \end{pmatrix} h = \begin{pmatrix} y \\ -z - \alpha b(x_0, p_0) \end{pmatrix}.$$

This ends the proof of Proposition 2.1. □

In the following theorem, Hölder continuity of the local minima is improved to Lipschitz continuity. Moreover, Lipschitz continuity of the corresponding Lagrange multipliers is obtained.

Theorem 2.2 *Assume that (H2) – (H4) hold at a local solution x_0 of $(\mathcal{P}_{p_0})$. Then there exist neighbourhoods $V(p_0)$ of p_0 and $V(x_0, \lambda_0, \mu_0, \tilde{\eta}_0)$ of $(x_0, \lambda_0, \mu_0, \tilde{\eta}_0)$ and a constant $\kappa > 0$ such that for all $p \in V(p_0)$ there exists a solution $\xi(p) = (x_p, \lambda_p, \mu_p, \tilde{\eta}_p) \in X \times Y \times \mathbb{R}^m \times \tilde{Z}$ of*

$$0 \in \begin{cases} \mathcal{L}'(x, p, \lambda, \mu, \tilde{\eta}) \\ e(x, p) \\ -g(x, p) + \partial \Psi_{\mathbb{R}^m_+}(\mu) \\ -\tilde{b}(x, p) + \partial \Psi_{\tilde{K}_+}(\tilde{\eta}). \end{cases} \tag{2.14}$$

This solution is unique in $V(x_0, \lambda_0, \mu_0, \eta_0)$ and satisfies

$$|(x_p, \lambda_p, \mu_p, \tilde{\eta}_p) - (x_q, \lambda_q, \mu_q, \tilde{\eta}_q)| \leq k|p - q|_P$$

for any p, q in $V(p_0)$ and, moreover, x_p is a solution of $(\mathcal{P}_p)$ if $p \in V(p_0)$.

The proof of this result is given in [4, Theorem 2.1], where, additionally, it is assumed that $\tilde{b}$ ($= \ell$ in the notation of [4]) is affine in x; this assumption, however, is not necessary, as an inspection of the proof shows.

Once Lipschitz continuity of x with respect to p is established, Gateaux–differentiability of the minimal value function μ_r follows from [7, Section 3]:

Theorem 2.3 *Let (H2) – (H4) hold at a local solution x_0 of $(\mathcal{P}_{p_0})$ and assume that f is differentiable and e, g, b are continuously differentiable in the sense of Fréchet at (x_0, p_0). Then for all sufficiently small $r > 0$, the Gateaux derivative of μ_r at p_0 exists and is given by*

$$D\mu_r(p_0, p) = \mathcal{L}_p(x_0, p_0, \lambda_0, \mu_0, \tilde{\eta}_0)p$$

for any $p \in P$.

Next we turn to directional differentiability of the local optima and of the corresponding Lagrange multipliers with respect to the parameter.

Before we introduce some additional hypotheses, we recall the following definitions.

Definition: *A closed convex set C in a Hilbert space H is called polyhedric with respect to $x \in H$ if*

$$\overline{\bigcup_{\lambda>0} \lambda(C - Px) \cap [x - Px]^{\perp}} = \overline{\bigcup_{\lambda>0} \lambda(C - Px)} \cap [x - Px]^{\perp},$$

where Px denotes the metric projection of x onto C.

Let $\mathcal{C}$ be the closed convex set in X given by

$$\mathcal{C} = \{x \in \ker E : z_1 \leq Bx \leq z_2\}$$

and observe that

$$\mathcal{C} = \left\{ x \in \ker E : \tilde{B}x + \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} \in \tilde{K} \right\}.$$

Definition: *A function H between normed linear spaces P and Q is said to be directionally differentiable at $p_0 \in P$ in direction $p \in P$ if $\lim_{t \to 0^+} t^{-1}(H(p_0 + tp) - H(p_0))$ exists.*

(H5) The functions $e(x_0, \cdot)$, $g(x_0, \cdot)$, $b(x_0, \cdot)$, $f'(x_0, \cdot)$, $g'(x_0, \cdot)$, $e'(x_0, \cdot)$ and $b'(x_0, \cdot)$ are directionally differentiable at p_0 in every direction q.

(H6) The set $\mathcal{C}$ is polyhedric at every $x \in \mathcal{C}$.

(H7) There exists $\nu > 0$ such that

$$\langle Ax, x \rangle \geq \nu |x|^2 \quad \text{for all} \quad x \in \ker E.$$

Note that (H7) implies (H3). Below, we will see that polyhedricity of $\mathcal{C}$ is needed only at some specific point $x \in \mathcal{C}$.

Recall the decomposition of the finite dimensional inequality constraint and the notation introduced above. Due to the complementarity condition and continuous dependence of ξ on p one can always assume that

$$g^+(x_p, p) = 0,\ g^-(x_p, p) < 0,\ \mu_p^+ > 0,\ \mu_p^- = 0$$

for all p sufficiently close to p_0. We shall put $\tilde{\lambda}_p = (\lambda_p, \mu_p^+)$ and we shall not distinguish between $\mathcal{L}'_p(x_0, p_0, \lambda_0, \mu_0, \tilde{\eta}_0)q$, the directional derivative of $\mathcal{L}'(x_0, p_0, \lambda_0, \mu_0, \tilde{\eta}_0)$ at p_0 in direction q, and its Riesz representation in X.

We note the following result:

Proposition 2.2 *Let (H2) - (H5) hold at a local solution x_0 of $(\mathcal{P}_{p_0})$, and let $(\dot{x}, \dot{\lambda}, \dot{\mu}, \dot{\tilde{\eta}})$ denote a weak cluster point of $t^{-1}[\xi(p_0+tq)-\xi(p_0)]$ for $t \to 0^+$, with $q \in P$. Then $(\dot{x}, \dot{\lambda}, \dot{\mu}, \dot{\tilde{\eta}})$ satisfies*

$$0 \in \begin{cases} \mathcal{L}'_p(x_0, p_0, \lambda_0, \mu_0, \tilde{\eta}_0)q + A\dot{x} + E_+^*\dot{\lambda} + G_0^*\dot{\mu}^0 + \tilde{B}^*\dot{\tilde{\eta}} \\ -e_p^+(x_0, p_0)q - E_+\dot{x} \\ \dot{\mu}^- \\ -g_p^0(x_0, p_0) - G_0\dot{x} + \partial\Psi_{\mathbb{R}_+^{m^0}}(\dot{\mu}_0) \\ \langle \dot{\tilde{\eta}}, \tilde{b}(x_0, p_0)\rangle + \langle \tilde{\eta}_0, \tilde{B}\dot{x} + \tilde{b}_p(x_0, p_0)q\rangle. \end{cases}$$

This is proved in [4, Theorem 3.2]. The proof there considers the case of $\tilde{b}(=\ell)$ affine in x; however, it remains almost literally the same for general $\tilde{b}$. □

The following theorem shows directional differentiability of the local minimum with respect to the perturbation parameter. It presents the main result of this paper.

Theorem 2.4 *Assume that (A1), (H2) and (H4) - (H7) hold and let $(\dot{x}, \dot{\lambda}, \dot{\mu}, \dot{\tilde{\eta}})$ denote a weak cluster point of $t^{-1}[\xi(p_0+tq)-\xi(p_0)]$ for $t \to 0^+$ with $q \in P$, where $\xi(p) = (x_p, \lambda_p, \mu_p, \tilde{\eta}_p)$. Then $(\dot{x}, \dot{\lambda}, \dot{\mu}, \dot{\tilde{\eta}})$ is the directional derivative of $\xi(p)$ at $p = p_0$ in direction q and*

$$0 \in \begin{cases} \mathcal{L}'_p(x_0, p_0, \lambda_0, \mu_0, \tilde{\eta}_0)q + A\dot{x} + E_+^*\dot{\lambda} + G_0^*\dot{\mu}^0 + \tilde{B}^*\dot{\tilde{\eta}} \\ -e_p^+(x_0, p_0)q - E_+\dot{x} \\ \dot{\mu}^- \\ -g_p^0(x_0, p_0)q - G_0\dot{x} + \partial\Psi_{\mathbb{R}_+^{m^0}}(\dot{\mu}_0) \\ -\tilde{b}(x_0, p_0)q - \tilde{B}\dot{x} + \partial\Psi_{\hat{K}_+}(\dot{\tilde{\eta}}) \end{cases} \tag{2.15}$$

where $\hat{K}_+$ is the dual cone in $\tilde{Z}^*$ of $\hat{K} := \overline{\bigcup_{\lambda>0}\lambda(\tilde{K} - \tilde{b}(x_0, p_0))} \cap [\tilde{\eta}_0]^\perp$.

Since in view of Theorem 2.2, $t^{-1}[\xi(p_0+tq)-\xi(p_0)]$ has a weak cluster point as $t \to 0^+$ for every $q \in P$, Theorem 2.4 implies in particular the directional differentiability of ξ at p_0, in every direction $q \in P$.

Remark 2.1: The proof of Theorem 2.4 will be constructed in a similar manner as that of Theorem 3.2 in [4]. Our hypotheses (H2) – (H4) for $(\mathcal{P}_p)'$ coincide with (H2) - (H4) in [4] and hence all results from the proof in [4] relying only on these properties remain valid for the bilateral problem in rewritten from $(\mathcal{P}_p)'$. In [4] additional hypotheses denoted by (H5) – (H9) are used in order to derive differential stability properties. It turns out that (H5) coincides with our hypothesis (H5). We cannot guarantee the polyhedricity requirement (H6) of [4] for our cone $\tilde{K}$. This motivates us to reconstruct the proof requiring polyhedricity of the original cone K only, (H6). Hypothesis (H7) in [4] corresponds to our hypothesis (H7), while (H8) in [4] requiring surjectivity of

$$\begin{pmatrix} E \\ \tilde{B} \end{pmatrix} : X \to Y \times \tilde{Z}$$

for the $(\mathcal{P}_p)'$ problem is *never* satisfied. In our analysis it is replaced by (A1).

Hypotheses (H5) and (H6) are somewhat stronger than necessary. We shall discuss this issue in Remark 2.3 after the proof.

Proof of Theorem 2.4: Let $\{t_n\}$ be a sequence of positive numbers with $\lim_{n\to\infty} t_n = 0$ and with $w-\lim_{n\to\infty} t_n^{-1}[\xi(p_0+t_nq)-\xi(p_0)] = (\dot{x}, \dot{\lambda}, \dot{\mu}, \dot{\tilde{\eta}})$. Together with (2.14) we also consider the following linearized optimality system

$$0 \in \begin{cases} \mathcal{L}'(x_0, p, \lambda_0, \mu_0, \tilde{\eta}_0) + A(x-x_0) + E^*(\lambda-\lambda_0) \\ \quad +G^*(\mu-\mu_0) + \tilde{B}^*(\tilde{\eta}-\tilde{\eta}_0) \\ -e(x_0,p) - E(x-x_0) \\ -g(x_0,p) - G(x-x_0) + \partial\Psi_{\mathbb{R}^m_+}(\mu) \\ -\tilde{b}(x_0,p) - \tilde{B}(x-x_0) + \partial\Psi_{\tilde{K}_+}(\tilde{\eta}). \end{cases} \tag{2.16}$$

Using a theorem of Robinson one shows (cf. [4]) that there exists a neighborhood V_1 of $(x_0, \lambda_0, \mu_0, \tilde{\eta}_0)$ and a real valued function α defined on $V(p_0)$ with $\lim_{p\to p_0} \alpha(p) = 0$, such that for each $p \in V(p_0)$ there exists a solution $\hat{\xi}(p) = (\hat{x}(p), \hat{\lambda}(p), \hat{\mu}(p), \hat{\tilde{\eta}}(p))$ of (2.16) that is unique in V_1 such that

$$|\xi(p) - \hat{\xi}(p)|_{X\times Y\times\mathbb{R}^m\times\tilde{Z}} \le \alpha(p)|p-p_0|_P,$$

where $\xi(p) = (x_p, \lambda_p, \mu_p, \tilde{\eta}_p)$ is the unique solution in $V(x_0, \lambda_0, \mu_0, \tilde{\eta}_0)$ of (2.14). In particular, this implies that $p \to \hat{\xi}(p)$ is Lipschitz–continuous at p_0, that a weak cluster point of $t^{-1}[\hat{\xi}(p_0+tq) - \hat{\xi}(p_0)]$ as $t \to 0^+$ exists and that weak and strong limits of $t_n^{-1}[\hat{\xi}(p_0+t_nq) - \hat{\xi}(p_0)]$ and $t_n^{-1}[\xi(p_0 + t_nq) - \xi(p_0)]$ coincide. Our analysis of $\lim_{t\to 0^+} t^{-1}[\xi(p_0+tq) - \xi(p_0)]$ can therefore concentrate on that of $\lim_{t\to 0^+} t^{-1}[\hat{\xi}(p_0+tq) - \hat{\xi}(p_0)]$. Let us also observe that $\hat{\xi}(p_0) = \xi(p_0)$. The proof will now be given in several steps.

(i) We show that the strong limit of $t_n^{-1}(x_{p_0+t_nq} - x_0)$ exists. Put $p_n = p_0 + t_nq$ and $(x(t_n), \lambda(t_n), \mu(t_n), \tilde{\eta}(t_n)) := \hat{\xi}(p_n)$. We have to show that

$$\lim_{n\to\infty} t_n^{-1}[x(t_n) - x_0] \tag{2.17}$$

exists.

By definition of $\hat{\xi}$ one has

$$0 \in \left\{ \begin{array}{l} \mathcal{L}'(x_0, p_n, \lambda_0, \mu_0, \tilde{\eta}_0) + f'(x_0, p_0) - Ax_0 + Ax(t_n) + E^*\lambda(t_n) \\ \quad +G^*\mu(t_n) + \tilde{B}^*\tilde{\eta}(t_n) \\ -e(x_0, p_n) - E[x(t_n) - x_0] \\ -g(x_0, p_n) - G[x(t_n) - x_0] + \partial\Psi_{\mathbb{R}^m_+}(\mu(t_n)) \\ -\tilde{b}(x_0, p_n) - \tilde{B}[x(t_n) - x_0] + \partial\Psi_{\tilde{K}_+}(\tilde{\eta}(t_n)). \end{array} \right. \tag{2.18}$$

For every $n \in \mathbb{N}$ we introduce the closed convex set

$$\mathcal{K}_n = \left\{ c \in X : \begin{array}{l} \tilde{B}c - \tilde{B}x_0 + \tilde{b}(x_0, p_n) \in \tilde{K} \\ Ec - Ex_0 + e(x_0, p_n) = 0 \end{array} \right\}.$$

Observe that

$$\mathcal{K}_n = \left\{ c \in X : \begin{array}{l} -Bc + Bx_0 - b(x_0, p_n) + z_1 \leq 0 \\ B.c. - Bx_0 + b(x_0, p_n) - z_2 \leq 0 \\ Ec - Ex_0 + e(x_0, p_n) = 0 \end{array} \right\},$$

and define

$$\varphi(t_n) = \mathcal{L}'(x_0, p_n, \lambda_0, \mu_0, \tilde{\eta}_0) + f'(x_0, p_0) - Ax_0.$$

By (A1) there exists a unique element $w(t_n) \in [\ker \binom{E}{B}]^\perp$ with

$$\begin{pmatrix} E \\ B \end{pmatrix} w(t_n) = \begin{pmatrix} -e(x_0, p_0) + Ex_0 \\ -b(x_0, p_n) + Bx_0 \end{pmatrix}. \tag{2.19}$$

We put $y(t_n) := x(t_n) - w(t_n)$. By (A1) the operator $\binom{E}{B}$ is invertible on $\operatorname{Im} \binom{E}{B}^* = \ker \binom{E}{B}^\perp$. Hence by (H5) and (A1) there exists $\dot{w} \in X$ such that

$$\lim t_n^{-1}[w(t_n) - w(0)] = \dot{w}.$$

Therefore in order to prove (2.16) is suffices to show the existence of

$$\lim_{n \to \infty} t_n^{-1}[y(t_n) - y(0)]. \tag{2.20}$$

We will accomplish this by identifying $y(t_n)$ with the metric projection on a certain closed convex set, which is polyhedric at an appropriately specified point.

Observe that $By(0) = B(x(0) - w(0))$, and that for each $c \in \mathcal{K}_n$ we have

$$\begin{aligned} &\langle E^*\lambda(t_n) + \tilde{B}^*\tilde{\eta}(t_n), c - x(t_n)\rangle \\ &\quad = \langle \lambda(t_n), Ec - Ex(t_n)\rangle + \langle \tilde{\eta}(t_n), \tilde{B}c - \tilde{B}x(t_n)\rangle \\ &\quad = \langle \lambda(t_n), E[x_0 - x(t_n)] - e(x_0, p_n)\rangle \\ &\qquad + \langle \tilde{\eta}(t_n), \tilde{B}[x_0 - x(t_n)] - \tilde{b}(x_0, p_n)\rangle \leq 0, \end{aligned}$$

where for the last two inequalities we used the choice of $c \in \mathcal{K}_n$ and (2.18). The last estimate and (2.18) further imply for each $c \in \mathcal{K}_n$

$$\begin{aligned} &\langle \varphi(t_n) + Ax(t_n) + G^*\mu(t_n), c - x(t_n)\rangle \\ &\quad = \langle \mathcal{L}'(x_0, p_n, \lambda_0, \mu_0, \tilde{\eta}_0) \\ &\qquad + f'(x_0, p_0) - Ax_0 + Ax(t_n) + G^*\mu(t_n), c - x(t_n)\rangle \\ &\quad \geq 0. \end{aligned} \tag{2.21}$$

Next we claim that for $c \in \mathcal{C}$

$$c + w(t_n) \in \mathcal{K}_n. \tag{2.22}$$

Due to the choice of c we find that $Ec = 0$, $-B.c.+z_1 \leq 0$ and $B.c.-z_2 \leq 0$. Hence (2.19) implies

$$\begin{aligned} &-B[c + w(t_n)] + Bx_0 - b(x_0, p_n) + z_1 \\ &\quad = -B.c. + b(x_0, p_n) - Bx_0 + Bx_0 - b(x_0, p_n) + z_1 \\ &\quad = -B.c. + z_1 \leq 0 \end{aligned}$$

and

$$\begin{aligned} &B[c + w(t_n)] - Bx_0 + b(x_0, p_n) - z_2 \\ &\quad = B.c. - b(x_0, p_n) + Bx_0 - Bx_0 + b(x_0, p_n) - z_2 \\ &\quad = B.c. - z_2 \leq 0. \end{aligned}$$

Together with the first equality in (2.19) this proves (2.22). Recalling that $x(t_n) = y(t_n) + w(t_n)$ and applying (2.21), (2.22) we find that for $c \in \mathcal{C}$

$$\begin{aligned} &\langle Ay(t_n) + \varphi(t_n) + Aw(t_n) + G^*\mu(t_n), c - y(t_n)\rangle \\ &\quad = \langle Ax(t_n) + \varphi(t_n) + G^*\mu(t_n), c + w(t_n) - x(t_n)\rangle \\ &\quad \geq 0. \end{aligned}$$

Furthermore $y(t_n) \in \mathcal{C}$, since by (2.19) and (2.18)

$$\begin{aligned} Ey(t_n) &= Ex(t_n) - Ew(t_n) \\ &= Ex(t_n) + e(x_0, p_n) - Ex_0 = 0, \end{aligned}$$

and

$$\begin{aligned} \tilde{B}y(t_n) &= \tilde{B}x(t_n) - \tilde{B}w(t_n) \\ &= \tilde{B}x(t_n) + \tilde{b}(x_0, p_n) - \tilde{B}x_0 - \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix}, \end{aligned}$$

and, again by (2.18),

$$\tilde{B}y(t_n) + \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} \in \tilde{K}.$$

We have therefore shown that

$$\begin{cases} \langle Ay(t_n) + \varphi(t_n) + Aw(t_n) + G^*\mu(t_n), c - y(t_n)\rangle \\ \geq 0 \quad \text{for all } x \in C, \\ y(t_n) \in C. \end{cases} \tag{2.23}$$

Now define

$$\psi(t_n) := P_{\ker E}(\varphi(t_n) + Aw(t_n) + G^*\mu(t_n)),$$

where $P_{\ker E}$ denotes the orthogonal projection of X onto $\ker E$.

Observe that

$$\lim t_n^{-1}[\psi(t_n) - \psi(0)] \text{ exists,}$$

since $\lim t_n^{-1}[\varphi(t_n) - \varphi(0)]$ exists by (H5), $\lim t_n^{-1}[w(t_n) - w(0)]$ exists by construction, and $\lim t_n^{-1}[G^*\mu(t_n) - G^*\mu(0)]$ exists due to finite dimensionality of $\mu(t_n)$, and since $P_{\ker E}$ is a bounded linear operator.

Due to (H7),

$$\langle\langle x, y\rangle\rangle := \langle A_P x, y\rangle$$

with $A_P := P_{\ker E}A$, defines a positive definite inner product on $\ker E$, and (2.23) is equivalent to

$$\begin{cases} \langle\langle y(t_n), c - y(t_n)\rangle\rangle + \langle\langle A_p^{-1}\psi(t_n), c - y(t_n)\rangle\rangle \geq 0 \quad \text{for all } c \in C, \\ y(t_n) \in C. \end{cases}$$

This variational inequality shows that

$$y(t_n) = \hat{P}_C[-A_P^{-1}\psi(t_n)], \tag{2.24}$$

where $\hat{P}_C$ is the metric projection in $\ker E$ onto C with respect to $\langle\langle\cdot,\cdot\rangle\rangle$. In order to verify the existence of $\lim t_n^{-1}(y(t_n) - y(0))$, we require the following lemma.

Lemma 2.2 *The closed convex set C considered as subset of* $\ker E$ *with respect to $\langle\langle\cdot,\cdot\rangle\rangle$ as inner product is polyhedric at the point $\hat{y} = -A_P^{-1}\psi(0)$.*

We are now prepared to finish the proof of step (i). Due to Lemma 2.1, (2.24) and the existence of $\lim_{n\to\infty} t_n^{-1}(x(t_n) - x_0)$, a variant of Haraux's theorem on the differentiability of the metric projection [4, Proposition 3.3] is applicable and implies the existence of the limit in (2.20) and consequently the existence of $\lim_{n\to\infty} t_n^{-1}[x(t_n) - x_0]$.

(ii) We verify the inclusions in (2.15). The first four of them were already established in Proposition 2.2, so that it remains to prove the last one which is equivalent to

$$\langle \dot{\tilde{\eta}}, u\rangle \leq 0 \text{ for all } u \in \hat{K} \tag{2.25}$$

$$\langle \dot{\tilde{\eta}}, \tilde{b}_p(x_0,p_0)q + \tilde{B}\dot{x}\rangle = 0, \tag{2.26}$$

$$\tilde{b}_p(x_0,p_0)q + \tilde{B}\dot{x} \in \hat{K}. \tag{2.27}$$

We first verify that

$$\langle \tilde{\eta}_0, \tilde{b}_p(x_0,p_0)q + \tilde{B}\dot{x}\rangle = 0. \tag{2.28}$$

We define

$$\gamma(t_n) := t_n^{-1}[x(t_n) - x_0],$$

and

$$\hat{\nu}(t_n) := t_n^{-1}[\tilde{\eta}(t_n) - \tilde{\eta}_0].$$

Since $\tilde{\eta}(t_n)$ solves the linearized optimality system (2.16), it is known that

$$\langle \tilde{\eta}(t_n), \tilde{w} - \tilde{b}(x_0,p_n) - \tilde{B}[x(t_n) - x_0]\rangle \leq 0 \quad \text{for all } \tilde{w} \in \tilde{K}.$$

Thus choosing $\tilde{w} = \tilde{b}(x_0,p_0) \in \tilde{K}$ one obtains

$$\begin{aligned}
&\langle \tilde{\eta}(t_n), \tilde{b}(x_0,p_0) - \tilde{b}(x_0,p_n) - \tilde{B}[x(t_n) - x_0]\rangle \\
&= \langle \tilde{\eta}_0 + t_n\nu(t_n), \frac{t_n[\tilde{b}(x_0,p_0) - \tilde{b}(x_0,p_n)]}{t_n} - t_n\tilde{B}\gamma(t_n)\rangle \\
&\leq 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&-t_n^2\langle \nu(t_n), \frac{\tilde{b}(x_0,p_n) - \tilde{b}(x_0,p_0)}{t_n} + \tilde{B}\gamma(t_n)\rangle \\
&\quad \leq \langle \tilde{\eta}_0, \tilde{b}(x_0,p_n) - \tilde{b}(x_0,p_0) + t_n\tilde{B}\gamma(t_n)\rangle \\
&\quad \leq \langle \tilde{\eta}_0, \tilde{b}(x_0,p_n) - \tilde{b}(x_0,p_0) + \tilde{B}[x(t_n) - x_0]\rangle \\
&\quad \leq 0.
\end{aligned}$$

Therefore the following inequalities hold:

$$\begin{aligned}
&-t_n\langle \nu(t_n), \frac{\tilde{b}(x_0,p_n) - \tilde{b}(x_0,p_0)}{t_n} + \tilde{B}\gamma(t_n)\rangle \\
&\quad \leq \langle \tilde{\eta}_0, t_n^{-1}[\tilde{b}(x_0,p_n) - \tilde{b}(x_0,p_0) + \tilde{B}[x(t_n) - x_0]\rangle \\
&\quad \leq 0.
\end{aligned}$$

For $n \to \infty$ this implies that $\langle \tilde{\eta}_0, \tilde{b}_p(x_0, p_0)q + \tilde{B}\dot{x} \rangle = 0$ which proves (2.28).

Next we prove (2.27). By (2.28) we have

$$\tilde{b}_p(x_0, p_0)q + \tilde{B}\dot{x} \in [\tilde{\eta}_0]^{\perp}.$$

Furthermore

$$\begin{aligned} t_n^{-1}[\tilde{b}(x_0, p_n) - \tilde{b}(x_0, p_0)] + \tilde{B}t_n^{-1}[x(t_n) - x_0] \\ = t_n^{-1} \left\{ \tilde{b}(x_0, p_n) + \tilde{B}[x(t_n) - x_0] - \tilde{b}(x_0, p_0) \right\}, \end{aligned}$$

and by (2.18)

$$\tilde{b}(x_0, p_n) - \tilde{B}[x(t_n) + x_0] \in \tilde{K}.$$

Hence the difference quotients above are in

$$\bigcup_{\lambda > 0} \lambda(\tilde{K} - \tilde{b}(x_0, p_0))$$

and therefore $\tilde{b}_p(x_0, p_0)q + \tilde{B}\dot{x}$ is in the closure of this cone. Thus (2.27) holds.

We turn to the verification of (2.25). Observe that

$$\langle \tilde{\eta}(t_n) - \tilde{\eta}_0, \tilde{v} \rangle \leq 0 \quad \text{for all } \tilde{v} \in \tilde{K} \text{ with } \langle \tilde{\eta}_0, \tilde{v} \rangle = 0.$$

Therefore, for all such $\tilde{v}$

$$\langle \dot{\tilde{\eta}}, \tilde{v} - \tilde{b}(x_0, p_0) \rangle \leq 0,$$

where we have used the fact that (2.28) and the last equation in the differential inclusion of Proposition 2.2 imply $\langle \dot{\tilde{\eta}}, \tilde{b}(x_0, p_0) \rangle = 0$. We have thus established that

$$\langle \dot{\tilde{\eta}}, \tilde{u} \rangle \leq 0 \quad \text{for all} \quad \tilde{u} \in \overline{\bigcup_{\lambda > 0} \lambda(\tilde{K} - \tilde{b}(x_0, p_0)) \cap [\tilde{\eta}_0]^{\perp}}.$$

The next lemma shows that this cone coincides with $\hat{K}$ and hence it establishes (2.25).

Lemma 2.3 *m:2.3 If (H6) holds, then*

$$\overline{\bigcup_{\lambda > 0} \lambda(\tilde{K} - \tilde{b}(x_0, p_0)) \cap [\tilde{\eta}_0]^{\perp}} = \overline{\bigcup_{\lambda > 0} \lambda(\tilde{K} - \tilde{b}(x_0, p_0))} \cap [\tilde{\eta}_0]^{\perp}.$$

Since $P_{\tilde{K}}(\tilde{\eta}_0 + \tilde{b}(x_0, p_0)) = \tilde{b}(x_0, p_0)$, Lemma 2.3 asserts polyhedricity of $\tilde{K}$ at $\tilde{\eta}_0 + \tilde{b}(x_0, p_0)$.

It remains to establish (2.26). By (2.25) and (2.27) we find

$$\langle \dot{\tilde{\eta}}, \tilde{b}_p(x_0,p_0)q + \tilde{B}\dot{x}\rangle \le 0. \tag{2.29}$$

For each n we find

$$\langle \tilde{\eta}(t_n) - \tilde{\eta}_0, \tilde{b}(x_0,p_n) + \tilde{B}(x(t_n) - x_0) - \tilde{b}(x_0,p_0)\rangle \ge 0$$

by (2.18). Using (H5) this inequality implies

$$\langle \dot{\tilde{\eta}}, \tilde{b}_p(x_0,p_0)q + \tilde{B}\dot{x}\rangle \ge 0. \tag{2.30}$$

From (2.29) and (2.30) we conclude (2.26).

(iii) Finally, we show uniqueness of the weak limits of $t^{-1}[\xi(t) - \xi(0)]$ as $t \to 0^+$ and the fact that this limit is indeed a strong one. Let $(\dot{x}_i, \dot{\lambda}_i, \dot{\mu}_i, \dot{\tilde{\eta}}_i)$, $i = 1,2$, be two weak limit points. Then by (2.15) with $\dot{\bar{\lambda}}_i = (\dot{\lambda}_i, \dot{\mu}_i^+)$ one finds that

$$\begin{aligned}
0 &= \langle A(\dot{x}_1 - \dot{x}_2) + E_+^*(\dot{\bar{\lambda}}_1 - \dot{\bar{\lambda}}_2) + G_0^*(\dot{\mu}_1^0 - \dot{\mu}_2^0) + \tilde{B}^*(\dot{\tilde{\eta}}_1 - \dot{\tilde{\eta}}_2), \dot{x}_1 - \dot{x}_2\rangle \\
&= \langle A(\dot{x}_1 - \dot{x}_2), \dot{x}_1 - \dot{x}_2\rangle + \langle \dot{\mu}_1^0 - \dot{\mu}_2^0, G_0(\dot{x}_1 - \dot{x}_2)\rangle \\
&\quad + \langle \dot{\tilde{\eta}}_1 - \dot{\tilde{\eta}}_2, \tilde{B}(\dot{x}_1 - \dot{x}_2)\rangle \\
&= \langle A(\dot{x}_1 - \dot{x}_2), \dot{x}_1 - \dot{x}_2\rangle + \langle \dot{\mu}_1^0, -g_p^0(x_0,p_0) - G_0\dot{x}_2\rangle \\
&\quad - \langle \dot{\mu}_2^0, g_p^0(x_0,p_0)q + G_0\dot{x}_1\rangle + \langle \dot{\tilde{\eta}}_1, -\tilde{b}_p(x_0,p_0)q - \tilde{B}\dot{x}_2\rangle \\
&\quad - \langle \dot{\tilde{\eta}}_2, \tilde{b}_p(x_0,p_0)q + \tilde{B}\dot{x}_1\rangle \\
&\quad \ge \langle A(\dot{x}_1 - \dot{x}_2), \dot{x}_1 - \dot{x}_2\rangle.
\end{aligned}$$

Since A is positive definite on $\ker E$, this implies $\dot{x}_1 = \dot{x}_2$. That is, we have established uniqueness of the x–coordinate. Together with (i) this implies that $\dot{x}$ is the directional derivative of the x–coordinate at p_0 in direction q.

By the first equality in (2.15) we find

$$0 = E_+^*(\dot{\bar{\lambda}}_1 - \dot{\bar{\lambda}}_2) + G_0^*(\dot{\mu}_1 - \dot{\mu}_2) + \tilde{B}^*(\dot{\tilde{\eta}}_1 - \dot{\tilde{\eta}}_2).$$

For $i = 1,2$ the last inclusion in Proposition 2.2 yields

$$\langle \dot{\tilde{\eta}}_i, \tilde{b}(x_0,p)\rangle + \langle \tilde{\eta}_0, \tilde{B}\dot{x}_i + \tilde{b}_p(x_0,p_0)q\rangle = 0$$

and therefore

$$\langle \dot{\tilde{\eta}}_1 - \dot{\eta}_2, \tilde{b}(x_0,p_0)\rangle = 0.$$

We thus obtain

$$0 = \mathcal{E}^* \begin{pmatrix} \dot{\bar{\lambda}}_1 - \dot{\bar{\lambda}}_2 \\ \dot{\mu}_1 - \dot{\mu}_2 \\ \dot{\tilde{\eta}}_1 - \dot{\tilde{\eta}}_2 \end{pmatrix}.$$

Now (H3) implies that $\dot{\tilde{\lambda}}_1 = \dot{\tilde{\lambda}}_2$, $\dot{\mu}_1 = \dot{\mu}_2$, $\dot{\tilde{\eta}}_1 = \dot{\tilde{\eta}}_2$. It remains to show that these unique weak limits are also strong limits. This will establish the claim of the theorem.

Again it suffices to consider the solutions of the linearized generalized equation (2.18). We use the notation

$$(x(t), \lambda(t), \mu(t), \tilde{\eta}(t)) = \hat{\xi}(p_0 + tq) \quad \text{and} \quad \tilde{\lambda}(t) = (\lambda(t), \mu^+(t)).$$

Then we find

$$\begin{aligned} E_+^*[\tilde{\lambda}(t) - \tilde{\lambda}_0] + G_0^*[\mu^0(t) - \mu_0^0] + \tilde{B}^*[\tilde{\eta}(t) - \tilde{\eta}_0] \\ = -\mathcal{L}'(x_0, p_0 + tq, \mu_0, \tilde{\eta}_0) - A[x(t) - x_0] \\ + \mathcal{L}'(x_0, p_0, \mu_0, \tilde{\eta}_0) \end{aligned}$$

and further

$$\mathcal{E}^* \begin{pmatrix} \tilde{\lambda}(t) - \tilde{\lambda}_0 \\ \mu^0(t) - \mu_0^0 \\ \tilde{\eta}(t) - \tilde{\eta}_0 \end{pmatrix} = \tag{2.31}$$

$$\begin{pmatrix} -\mathcal{L}'(x_0, p_0 + tq, \lambda_0, \mu_0, \tilde{\eta}_0) + \mathcal{L}'(x_0, p_0, \lambda_0, \mu_0, \tilde{\eta}_0) - A[x(t) - x_0] \\ \langle \tilde{b}(x_0, p_0), \tilde{\eta}(t) \rangle \end{pmatrix}.$$

Since by (2.18) we have $\langle \tilde{\eta}(t), \tilde{b}(x_0, p_0 + tq) + \tilde{B}[x(t) - x_0] = 0$, and $\langle \tilde{\eta}_0, \tilde{b}(x_0, p_0) \rangle = 0$, the following equality holds:

$$\langle \tilde{b}(x_0, p_0), \tilde{\eta}(t) - \tilde{\eta}_0 \rangle = -\langle \tilde{b}(x_0, p_0 + tq) - \tilde{b}(x_0, p_0) + \tilde{B}[x(t) - x_0], \tilde{\eta}(t) \rangle.$$

Inserting this into (2.31), dividing by t and taking the limit for $t \to 0^+$, the right hand side of (2.31) tends to

$$\begin{pmatrix} \mathcal{L}_p'(x_0, p_0, \lambda_0, \mu_0, \tilde{\eta}_0)q - A\dot{x} \\ -\langle \tilde{b}_p(x_0, p_0)q + \tilde{B}\dot{x}, \tilde{\eta}_0 \rangle \end{pmatrix}.$$

In view of (H3) this implies that

$$\lim_{t \to 0+} t^{-1}[(\lambda(t), \mu(t), \tilde{\eta}(t)) - (\lambda_0, \mu_0, \tilde{\eta}_0)] = (\dot{\lambda}, \dot{\mu}, \dot{\tilde{\eta}}).$$

This concludes the proof of Theorem 2.4.

Proof of Lemma 2.1: Let $(y, \rho, \tilde{z})$ be an arbitrary element in $Y \times \mathbb{R}^m \times \tilde{Z}$ with $\rho = (\rho^+, \rho^0, \rho^-) \in \mathbb{R}^{m^+} \times \mathbb{R}^{m^0} \times \mathbb{R}^{m^-}$. As a consequence of (H1) there exists $(x, \alpha) \in X \times \mathbb{R}$ such that

$$Ex = y, \; G_+x = \rho^+, \; G_0x = \rho^0, \; \tilde{B}x + \alpha\tilde{b} = \tilde{z}.$$

We choose $r^- \in \mathbb{R}_+^{m^-}$ and $r \geq 0$ such that

$$r^- + rg^-(x_0, p_0) = \rho^- - G_- x.$$

Next we consider the cases $\alpha < 0$ and $\alpha \geq 0$. For $\alpha < 0$ we find

$$\begin{pmatrix} E \\ G_+ \\ G_0 \\ G_- \\ \tilde{B} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ -rg^0(x_0,p_0) \\ r^- \\ (\alpha - r)\tilde{b}(x_0,p_0) \end{pmatrix} + r \begin{pmatrix} 0 \\ g^+(x_0,p_0) \\ g^0(x_0,p_0) \\ g^-(x_0,p_0) \\ \tilde{b}(x_0,p_0) \end{pmatrix} = \begin{pmatrix} y \\ \rho^+ \\ \rho^0 \\ \rho^- \\ \tilde{z} \end{pmatrix}$$

which is of the form required for (H1). Similarly, if $\alpha \geq 0$ then

$$\begin{pmatrix} E \\ G_+ \\ G_0 \\ G_- \\ \tilde{B} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ -(\alpha + r)g^0(x_0,p_0) \\ r^- - \alpha g^-(x_0,p_0) \\ -r\tilde{b} \end{pmatrix} + (r+\alpha) \begin{pmatrix} 0 \\ g^+(x_0,p_0) \\ g^0(x_0,p_0) \\ g^-(x_0,p_0) \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} y \\ \rho^+ \\ \rho^0 \\ \rho^- \\ \tilde{z} \end{pmatrix}$$

which is again of the necessary form. This concludes the proof of Lemma 2.1.

Proof of Lemma 2.2: We have to show that

$$\overline{\bigcup_{\lambda>0} \lambda(\mathcal{C} - \hat{P}_{\mathcal{C}}\hat{y}) \cap [\hat{y} - \hat{P}_{\mathcal{C}}\hat{y}]^{\perp\!\!\!\perp}} = \overline{\bigcup_{\lambda>0} \lambda(\mathcal{C} - \hat{P}_{\mathcal{C}}\hat{y})} \cap [\hat{y} - \hat{P}_{\mathcal{C}}\hat{y}]^{\perp\!\!\!\perp} \tag{2.32}$$

where $\perp\!\!\!\perp$ denotes orthogonal complement w.r.t $\langle\langle\cdot,\cdot\rangle\rangle$. First we claim that

$$[\hat{y} - \hat{P}_{\mathcal{C}}\hat{y}]^{\perp\!\!\!\perp} = [P_{\ker E}\tilde{B}^*\eta_0]^{\perp}, \tag{2.33}$$

where the orthogonal complement on the right hand side is taken w.r.t. $\langle\cdot,\cdot\rangle$, restricted to $\ker E$, and the whole identity is interpreted in $\ker E$. For any $h \in \ker E$

$$\begin{aligned}
\langle\langle h, \hat{y} - \hat{P}_{\mathcal{C}}\hat{y}\rangle\rangle &= \langle A_P h, \hat{y} - \hat{P}_{\mathcal{C}}\hat{y}\rangle \\
&= \langle h, A_P\hat{y} - A_P\hat{P}_{\mathcal{C}}\hat{y}\rangle \quad \text{(since } A \text{ is selfadjoint)} \\
&= \langle h, -\psi(0) - A_P y(0)\rangle \quad \text{(since } A_P\hat{y} = -\psi(0) \text{ and } \hat{P}_{\mathcal{C}}\hat{y} = y(0)) \\
&= \langle h, -\psi(0) - Ay(0)\rangle \quad \text{(since } h \in \ker E) \\
&= \langle h, -P_{\ker E}(\varphi(0) + Aw(0) + G^*\tilde{\mu}(0)) - Ay(0)\rangle \\
&= \langle h, -\mathcal{L}'(x_0, p_0, \lambda_0, \mu_0, \tilde{\eta}_0) - f'(x_0,p_0) - Ax(0) \\
&\quad -Aw(0) - Ay(0) - G^*\tilde{\mu}(0)\rangle \\
&= \langle h, -\mathcal{L}'(x_0, p_0, \lambda_0, \mu_0, \tilde{\eta}_0\rangle - f'(x_0,p_0) - G^*\tilde{\mu}(0) - E^*\lambda_0\rangle \\
&\quad \text{(since } \langle h, E^*\lambda_0\rangle = 0) \\
&= \langle h, \tilde{B}^*\eta_0\rangle \quad \text{(by (2.18))}.
\end{aligned}$$

This proves (2.32).

Next we recall that

$$y(0) = \hat{P}_C[-A_P^{-1}\psi(0)] = \hat{P}_C\hat{y},$$

and hence (2.32) is equivalent to the following equality in $\ker E$

$$\overline{\bigcup_{\lambda>0} \lambda(C - y(0)) \cap [P_{\ker E}\tilde{B}^*\tilde{\eta}_0]^\perp} = \overline{\bigcup_{\lambda>0} \lambda(C - y(0))} \cap [P_{\ker E}\tilde{B}^*\eta_0]^\perp. \tag{2.34}$$

Since $C \subset \ker E$ and $y(0) \in \ker E$ it is simple to argue that (2.34) is equivalent to

$$\overline{\bigcup_{\lambda>0} \lambda(C - y(0)) \cap [\tilde{B}^*\tilde{\eta}_0]^\perp} = \overline{\bigcup_{\lambda>0} \lambda(C - y(0))} \cap [\tilde{B}^*\tilde{\eta}_0]^\perp, \tag{2.35}$$

where the orthogonal complement is interpreted in X. We claim that (2.35) is equivalent to polyhedricity of C in X at the point $\tilde{B}^*\tilde{\eta}(0) + y(0)$. Once this is shown the assertion of the lemma follows from (H6). Clearly, it suffices to prove that

$$P_C[\tilde{B}^*\tilde{\eta}(0) + y(0)] = y(0),$$

since then also $[I - P_C][\tilde{B}^*\tilde{\eta}(0) + y(0)] = \tilde{B}^*\tilde{\eta}(0)$, where P_C is the metric projection in X onto C.

Let us put $x = \tilde{B}^*\tilde{\eta}_0 + y(0)$ and observe that $P_C x$ is characterized by $P_C x \in C$ and the variational inequality

$$\langle x - P_C x, c - P_C x\rangle \leq 0 \quad \text{for all } c \in C.$$

We have seen in (2.23) that $y(0) \in C$. Furthermore, for all $c \in C$,

$$\begin{aligned}
\langle x - y(0), c - y(0)\rangle &= \langle \tilde{B}^*\tilde{\eta}(0), c - y(0)\rangle \\
&= \langle \tilde{\eta}(0), \tilde{B}c - \tilde{B}y(0)\rangle \\
&= \langle \tilde{\eta}(0), \tilde{B}c\rangle - \langle \tilde{\eta}(0), \tilde{B}y(0)\rangle \\
&= \langle \tilde{\eta}(0), \tilde{B}c\rangle - \langle \tilde{\eta}(0), \tilde{b}(x_0, p_0) - \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix}\rangle \\
&= \langle \tilde{\eta}(0), \tilde{B}c + \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix}\rangle \leq 0,
\end{aligned}$$

where we have used the definition of $\tilde{b}(x_0, p_0)$, the fact that

$$\tilde{B}c + \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} = \begin{pmatrix} -B.c. & +z_1 \\ B.c. & -z_2 \end{pmatrix} \in \tilde{K}$$

and $\tilde{\eta}(0) \in \tilde{K}_+$, by (2.18). We have thus shown that $P_C x = y(0)$, which concludes the proof of the lemma.

Proof of Lemma 2.3: From the proof of Lemma 2.1 we recall equation (2.35)

$$\overline{\bigcup_{\lambda>0} \lambda(C - y(0)) \cap [\tilde{B}^* \tilde{\eta}_0]^\perp} = \overline{\bigcup_{\lambda>0} \lambda(C - y(0))} \cap [\tilde{B}^* \tilde{\eta}_0]^\perp.$$

This is a consequence of the polyhedricity of C (at the point $\tilde{B}^* \tilde{\eta}(0) + y(0)$).

Obviously the set appearing on the left hand side of the equality in the statement of Lemma 2.3 is contained in the set appearing on the right hand side of this equality, and it suffices to prove the converse inclusion. Every element in the set

$$\overline{\bigcup_{\lambda>0} \lambda(\tilde{K} - \tilde{b}(x_0, p_0))} \cap [\tilde{\eta}_0]^\perp$$

can be expressed in the form

$$\begin{aligned} \begin{pmatrix} -w + \rho z_1 \\ w - \rho z_2 \end{pmatrix} &= \lim \lambda_n \left\{ \begin{pmatrix} -k_n + \rho_n z_1 \\ k_n - \rho_n z_2 \end{pmatrix} \right. \\ &\qquad \left. - \begin{pmatrix} -b(x_0, p_0) + z_1 \\ b(x_0, p_0) - z_2 \end{pmatrix} \right\} \end{aligned} \tag{2.36}$$

where $\lambda_n > 0$, $\rho_n \in \mathbb{R}$, $k_n \in K$, and it satisfies

$$\left\langle \begin{pmatrix} -w + \rho z_1 \\ w - \rho z_2 \end{pmatrix}, \tilde{\eta}_0 \right\rangle = 0, \quad \begin{matrix} -k_n + \rho_n z_1 \le 0 \\ k_n + \rho_n z_2 \le 0. \end{matrix} \tag{2.37}$$

Clearly $\rho_n z_1 \le k_n \le \rho_n z_2$, and since $K \cap (-K) = \Phi$, we conclude that $\rho_n \ge 0$. Henceforth we assume that $\rho_n > 0$ for all n. The cases that finitely or infinitely many ρ_n equal zero can than be treated in a trivial manner. For future reference we record that

$$\rho_n > 0 \quad \text{and} \quad z_1 \le \frac{k_n}{\rho_n} \le z_2 \quad \text{for all } n. \tag{2.38}$$

We also observe that by adding the two components in (2.36), one obtains

$$\begin{aligned} \rho(z_1 - z_2) &= \lim\{\lambda_n[\rho_n(z_1 - z_2) + z_2 - z_1]\} \\ &= \lim\{\lambda_n(\rho_n - 1)(z_1 - z_2)\}. \end{aligned}$$

Since $z_1 \ne z_2$, this implies

$$\rho = \lim \lambda_n(\rho_n - 1). \tag{2.39}$$

From (2.36) we obtain

$$\begin{aligned}\begin{pmatrix} -w+\rho z_1 \\ w-\rho z_1 \end{pmatrix} &= \lim\left\{\lambda_n\left[\begin{pmatrix} -k_n \\ k_n \end{pmatrix} + \rho_n \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} + \rho_n \begin{pmatrix} -b(x_0,p_0) \\ b(x_0,p_0) \end{pmatrix}\right.\right. \\ &\quad \left.\left. -\rho_n \begin{pmatrix} -b(x_0,p_0) \\ b(x_0,p_0) \end{pmatrix} - \tilde{b}(x_0,p_0)\right]\right\} \\ &= \lim\left\{\lambda_n\left[\begin{pmatrix} -k_n \\ k_n \end{pmatrix} + (\rho_n - 1)\tilde{b}(x_0,p_0) - \rho_n \begin{pmatrix} -b(x_0,p_0) \\ b(x_0,p_0) \end{pmatrix}\right]\right\}.\end{aligned} \tag{2.40}$$

As a consequence of (2.39) and (2.40) one derives that

$$\left\{\begin{array}{l} \lim\{\lambda_n(\rho_n-1)\tilde{b}(x_0,p_0)\} \text{ and} \\ \lim\left\{\lambda_n\left[\begin{pmatrix} -k_n \\ k_n \end{pmatrix} - \rho_n \begin{pmatrix} -b(x_0,p_0) \\ b(x_0,p_0) \end{pmatrix}\right]\right\} \text{ exist.} \end{array}\right. \tag{2.41}$$

By (A1) there exits unique $c_n \in \left(\ker \binom{E}{B}\right)^{\perp}$ such that

$$\begin{pmatrix} E \\ B \end{pmatrix} c_n = \begin{pmatrix} 0 \\ k_n \end{pmatrix}.$$

Since $Ec_n = 0$, $B(\rho_n^{-1}c_n) = \rho_n^{-1}k_n$ and $z_1 \leq \rho_n^{-1}k_n \leq z_2$ by (2.38) it follows that

$$\rho_n^{-1}c_n \in \mathcal{C}. \tag{2.42}$$

Using (2.19) and (2.41) one finds that the following limits exist:

$$\begin{aligned}\lim \begin{pmatrix} 0 \\ \lambda_n(k_n - \rho_n b(x_0,p_0)) \end{pmatrix} &= \lim\left\{\lambda_n\left[\begin{pmatrix} Ec_n \\ Bc_n \end{pmatrix} - \rho_n \begin{pmatrix} Ey(0) \\ By(0) \end{pmatrix}\right]\right\} \\ &= \lim\left\{\begin{pmatrix} E \\ B \end{pmatrix} \lambda_n(c_n - \rho_n y(0))\right\}.\end{aligned}$$

Since $\lambda_n(c_n - \rho_n y(0)) \in [\ker \binom{E}{B}]^{\perp}$, (A1) implies the existence of the strong limit of $\lambda_n c_n - \rho_n y(0)$. We define

$$a = \lim \lambda_n(c_n - \rho_n y(0)),$$

and observe that

$$\tilde{B}a = \begin{pmatrix} B \\ -B \end{pmatrix} a = \lim\left\{\lambda_n\left[\begin{pmatrix} -k_n \\ k_n \end{pmatrix} - \rho_n \begin{pmatrix} -b(x_0,p_0) \\ b(x_0,p_0) \end{pmatrix}\right]\right\}. \tag{2.43}$$

The element can be expressed as $a = \lim\left\{\lambda_n\rho_n\left(\frac{c_n}{\rho_n} - y\right)\right\}$. Since $\rho_n^{-1}c_n \in \mathcal{C}$ by (2.42) and $\lambda_n\rho_n > 0$ by (2.38) we conclude that

$$a \in \overline{\bigcup_{\lambda>0} \lambda(\mathcal{C} - y(0))}. \tag{2.44}$$

Furthermore, by (2.43), (2.40), (2.37) and since $\langle\tilde{b}(x_0,p_0),\tilde{\eta}_0\rangle = 0$ we find

$$\begin{aligned}
\langle a, \tilde{B}^*\tilde{\eta}_0\rangle &= \langle\tilde{B}a,\tilde{\eta}_0\rangle \\
&= \Big\langle \lim\left\{\lambda_n\left[\begin{pmatrix} -k_n \\ k_n \end{pmatrix} - \rho_n\begin{pmatrix} -b(x_0,p_0) \\ b(x_0,p_0)\end{pmatrix}\right]\right\},\tilde{\eta}_0\Big\rangle \\
&= \Big\langle \lim\Big\{\lambda_n\Big[\begin{pmatrix} -k_n \\ k_n \end{pmatrix} - \rho_n\begin{pmatrix} -b(x_0,p_0) \\ b(x_0,p_0)\end{pmatrix} \\
&\quad +(\rho_n - 1)\tilde{b}(x_0,p_0)\Big]\Big\},\tilde{\eta}_0\Big\rangle \\
&= \Big\langle \begin{pmatrix} -w + \rho z_1 \\ w - \rho z_2 \end{pmatrix},\tilde{\eta}_0\Big\rangle \\
&= 0.
\end{aligned}$$

This equality together with (2.44) and (2.35) implies

$$a \in \overline{\bigcup_{\lambda>0}\lambda(\mathcal{C} - y(0))} \cap [\tilde{B}^*\tilde{\eta}_0]^{\perp} = \overline{\bigcup_{\lambda>0}\lambda(\mathcal{C} - y(0)) \cap [\tilde{B}^*\tilde{\eta}_0]^{\perp}}. \tag{2.45}$$

From (2.40), (2.41), (2.43) and (2.39) we obtain

$$\begin{aligned}
\begin{pmatrix} -w + \rho z_1 \\ w - \rho z_2 \end{pmatrix} &= \lim\left\{\lambda_n\left[\begin{pmatrix} -k_n \\ k_n \end{pmatrix} - \rho_n\begin{pmatrix} -b(x_0,p_0) \\ b(x_0,p_0)\end{pmatrix}\right]\right\} \\
&\quad + \lim\{\lambda_n(\rho_n - 1)\tilde{b}(x_0,p_0)\} \\
&= \tilde{B}a + \rho\tilde{b}(x_0,p_0).
\end{aligned}$$

By (2.42) there exist $\mu_n > 0$ and $\xi_n \in \mathcal{C}$ with

$$a = \lim\{\mu_n(\xi_n - y(0)\} \text{ and } \langle\xi_n - y(0), \tilde{B}^*\tilde{\eta}_0\rangle = 0, \tag{2.46}$$

and therefore

$$\begin{pmatrix} -w + \rho z_1 \\ w - \rho z_2 \end{pmatrix} = \lim\{\tilde{B}[\mu_n(\xi_n - y(0))] + \rho\tilde{b}(x_0,p_0)\}.$$

In order to establish Lemma 2.3, it remains to prove that for every n the expression in { } lies in $\cup_{\lambda>0}\lambda[\tilde{K} - \tilde{b}(x_0,p_0)] \cap [\tilde{\eta}_0]^{\perp}$. By (2.46) we find

$$\langle\tilde{B}[\mu_n(\xi_n - y(0))] + \rho\tilde{b}(x_0,p_0),\tilde{\eta}_0\rangle = \mu_n\langle(\xi_n - y(0)),\tilde{B}^*\tilde{\eta}_0\rangle = 0,$$

and hence the expression in { } lies in $[\tilde{\eta}_0]^\perp$.

On the other hand, this expression equals

$$\begin{aligned}
&\mu_n\{[\tilde{B}(\xi_n - y(0))] + \frac{\rho}{\mu_n}\tilde{b}(x_0,p_0)\} \\
&= \mu_n\left[\begin{pmatrix} -B\xi_n \\ B\xi_n \end{pmatrix} - \begin{pmatrix} -b(x_0,p_0) \\ b(x_0,p_0) \end{pmatrix} + \frac{\rho}{\mu_n}\tilde{b}(x_0,p_0)\right] \\
&= \mu_n\left[\begin{pmatrix} -B\xi_n \\ B\xi_n \end{pmatrix} + \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} + (\frac{\rho}{\mu_n} - 1)\tilde{b}(x_0,p_0)\right].
\end{aligned}$$

Now we consider two cases. If $\rho \le 0$, then $(1 - \mu_n^{-1}\rho) > 0$ and the above expression equals

$$\mu_n(1-\mu_n^{-1}\rho)\left[(1-\mu_n^{-1}\rho)^{-1}\begin{pmatrix} -B\xi_n + z_1 \\ B\xi_n - z_2 \end{pmatrix} - \tilde{b}(x_0,p_0)\right].$$

Since $\xi_n \in \mathcal{C}$ we have $(1-\mu_n^{-1}\rho)^{-1}\begin{pmatrix} -B\xi_n + z_1 \\ B\xi_n - z_2 \end{pmatrix} \in \tilde{K}$, and the proof is finished in this case. If $\rho > 0$, the expression in { } can be expressed as

$$\mu_n\left[\begin{pmatrix} -B\xi_n + z_1 \\ B\xi_n - z_2 \end{pmatrix} + \mu_n^{-1}\rho\begin{pmatrix} -b(x_0,p_0) + z_1 \\ b(x_0,p_0) - z_2 \end{pmatrix} - \tilde{b}(x_0,p_0)\right]$$

where

$$\begin{pmatrix} -B\xi_n + z_1 \\ B\xi_n - z_2 \end{pmatrix} + \mu_n^{-1}\rho\begin{pmatrix} -b(x_0,p_0) + z_1 \\ b(x_0,p_0) - z_2 \end{pmatrix} \in \tilde{K},$$

since $z_1 \le b(x_0,p_0) \le z_2$. This concludes the proof of the lemma.

Remark 2.2 An inspection of the proof of Proposition 2.2 and Theorem 2.4 shows that (H5) can be weakened to

$$e(x_0,\cdot), g(x_0,\cdot), \mathcal{L}'(x_0,\cdot,\lambda_0,\mu_0,\eta_0) \text{ and } \langle\tilde{\eta}_0, \tilde{b}(x_0,\cdot)\rangle$$

are directionally differentiable at p_0 in every direction $q \in P$.

Moreover, from the proof of Lemma 2.1 and 2.2 it can be seen (compare (2.35)) that (H6) can be replaced by the requirement that $\mathcal{C}$ is polyhedric at $y(0) + \tilde{B}^*\tilde{\eta}_0$ only.

3 Applications

In this section we present problems for which the theory developed in Section 2 is applicable. Our aim is to illustrate that the hypotheses of Section 2 are satisfied for a variety of different problems. We do not reach for the greatest generality.

3.1 A bilateral obstacle problem

We consider the bilateral obstacle problem in the form

$$\min |a\nabla u|^2_{L^2} + \langle k, u\rangle_{L^2} \text{ over } Q_{ad}(p). \tag{3.1}$$

where $Q_{ad}(p) = \{v \in H^1_0 : z_1 \leq p^1 v + p^2 \leq z_2 \text{ a.e.}\}$, $k \in H^{-1}$, $a \in L^\infty$, $p^1 \in H^2$, $p^2 \in H^1_0$, $z_1 \in H^1_0$, $z_2 \in H^1_0$, with $z_1 \leq z_2$, $z_1 \not\equiv z_2$. All function spaces are considered over a bounded domain Ω in $\mathbb{R}^n$, with $n = 1, 2$ or 3, with sufficiently smooth (Lipschitzian-) boundary Γ. Observe that due to the requirement $n \leq 3$, $p^1 v \in H^1_0$ for $p^1 \in H^2$ and $v \in H^1_0$. To relate the present problem to the general theory of Section 2 one puts

$$X = Z = H^1_0, \quad P = H^2 \times H^1_0 \times L^\infty \times H^{-1},$$

with a generic element $p \in P$ of the form $p = (p^1, p^2, a, k)$, and

$$\begin{aligned} f(u,p) &= \frac{1}{2}|a\nabla u|^2 + \langle k, u\rangle, \\ b(u,p) &= p^1 u + p^2. \end{aligned}$$

We fix a reference parameter $p_0 = (p^1_0, p^2_0, a_0, k_0)$, which is required to satisfy $a_0 \geq \alpha$ and $p^1_0 \geq \alpha$ a.e. on Ω for some $\alpha > 0$. It is simple to argue that there exists a unique solution u_0 of (3.1) with $p = p_0$. We proceed to argue the applicability of the results of Section 2. Clearly, f and b are twice continuously Fréchet–differentiable w.r.t. u at (a_0, p_0). Hypotheses (A1) and (A2) hold since $B = p^1_0$ is surjective due to $p^1_0 \geq \alpha > 0$ and since $g \equiv 0$. By Proposition 2.1 therefore (H1) and (H4) hold as well. Local Lipschitz continuity as required in (H2) is obvious for $f(u,p)$. Concerning b we observe that for p, $q \in P$ and $u \in H^1_0$

$$\begin{aligned} |b(u,p) - b(u,q)|_{H^1_0} &= |(p^1 - p^2)u + q^1 - q^2|_{H^1_0} \\ &\leq K\left(|p^1 - p^2|_{H^2}|u|_{H^1_0} + |q^1 - q^2|_{H^1_0}\right) \end{aligned}$$

where K depends only on embedding constants of H^2 into L^∞ and into $W^{1,4}$. Hence (H2) follows. Conditions (H3), (H5) and (H7) are obviously satisfied. Finally, due to [11, Theorem 3.2], $Q_{ad}(p_0)$ is polyhedric at every point of H^1_0 and hence (H6) holds as well. Thus all results of Section 2 are applicable.

3.2 A parameter estimation problem

A regularized least squares formulation of estimating the potential c in

$$\begin{aligned} -\Delta u + cu &= k \text{ in } \Omega \\ u|\Gamma &= 0 \end{aligned} \tag{3.2}$$

from measurements $z \in H_0^1$ is given by

$$\min \frac{1}{2}|u(c) - z|^2_{H_0^1} + \frac{\beta}{2}|c|^2_{L^2} \text{ over}$$

$$c \in \{c \in L^2 : z_1 \leq p^1 c + p^2 \leq z_2\}. \tag{3.3}$$

In (3.3) the state variable u is considered as a function of the unknown coefficient c. It has recently been observed [5] that for numerical purposes it is advantageous to consider both c and u as independent variables and to impose the state equation (3.2) as an explicit constraint. The problem of estimating c in (3.2) from data $z \in H_0^1$ can then be formulated as

$$\min \frac{1}{2}|u - z|^2_{H_0^1} + \frac{\beta}{2}|c|^2_{L^2} \quad \text{over } Q_{ad}, \tag{3.4}$$

where $Q_{ad} = \{(c,u) \in L^2 \times H_0^1 : z_1 \leq p^1 c + p^2 \leq z_2, (-\Delta)^{-1}(\Delta u - cu + k) = 0\}$, $k \in H^{-1}$, $z_1 \in L^2$, $z_2 \in L^2$, with $z_1 \leq z_2$, $z_1 \neq z_2$, Δ denotes the Laplacian from H_0^1 to H^{-1} and Ω is a bounded domain in $\mathbb{R}^n$ with $n \leq 4$ and Lipschitz continuous boundary Γ. Observe that (3.3) and (3.4) are equivalent in the sense that c_0 is a solution of (3.3) if and only if $(c_0, u(c_0))$ is a solution of (3.4). This problem is related to the general theory of Section 2 by choosing $X = L^2 \times H_0^1$, $Y = H_0^1$, $Z = L^2$, $P = L^\infty \times L^2 \times L^2 \times H^{-1}$ with a generic element $p \in P$ of the form $p = (p^1, p^2, z, k)$. Further we put

$$\begin{aligned} f(c,u,p) &= \frac{1}{2}|u - z|^2_{H_0^1} + \frac{\beta}{2}|c|^2_{L^2}, \\ e(c,u,p) &= (-\Delta)^{-1}(\Delta u - cu + k), \\ b(c,p) &= p_1 c + c_2. \end{aligned}$$

Throughout a reference parameter $p_0 = (p_0^1, p_0^2, z_0^2, k_0) \in P$ with $p_1^0 \geq \alpha$ a.e. for some $\alpha > 0$ and $(z_1 - p_0^2)/p_0^1 \geq 0$ is fixed. Obviously there exists at least one solution (c^0, u^0) for p^0. Clearly f, b and e are twice continuously Fréchet differentiable at (c^0, u^0). The operators E and B are found to satisfy

$$E(h,v) = e'(c_0, u_0, p_0) = (-\Delta)^{-1}(\Delta v - c_0 v) - (-\Delta)^{-1}(h u_0),$$

and

$$Bh = p_0^1 h.$$

A simple calculation shows that $\binom{E}{B} : X \to Y \times Z$ is surjective and hence by Proposition 2.1, (H1) and (H4) are satisfied for the present example. Hypotheses (H2) and (H5) can easily be checked and (H6) follows again from [11, Theorem 3.2]. Hence it remains to consider (H3) and (H7). The arguments for these coercivity estimates are quite similar to calculations that were required to obtain a coercivity estimate for the diffusion coefficient in an elliptic equation [2, 5] and we therefore only outline them here. Since (H1) holds there exists a Lagrange multiplier $(\lambda_0^\beta, \eta_0^\beta)$ for the solution (c_0, u_0) such that the Lagrangian

$$\mathcal{L}(c,u,p,\lambda_0^\beta,\eta_0^\beta) = \frac{1}{2}|u-z|^2_{H_0^1} + \frac{\beta}{2}|c|^2 + \langle \lambda_0^\beta, e(c,u)\rangle_{H_0^1} + \langle \eta_0^\beta, \tilde{b}(c,p)\rangle_{\tilde{Z}}$$

satisfies $\mathcal{L}'(c_0,u_0,p_0,\lambda_0^\beta,\eta_0^\beta) = 0$. Here the prime denotes differentiation with respect to (c,u). In our notation for the Lagrange multipliers we indicate the dependence on β. Henceforth we also use (c_0^β, u_0^β) to stress the dependence of (c_0,u_0) on β. Evaluating the first Fréchet derivative of the Lagrangian for $(0,v)$ one finds that λ_0^β is the unique solution in H_0^1 of

$$B(c_0^\beta)\lambda = (-\Delta)(u_0^\beta - z_0), \tag{3.5}$$

where $B(c_0^\beta) : H_0^1 \to H^{-1}$ is given by $B(c_0^\beta)\lambda = -\Delta\lambda + c_0^\beta\lambda$. For $\mathcal{L}''(\beta) = \mathcal{L}''(c_0^\beta, u_0^\beta, p_0, \lambda_0^\beta, \eta_0^\beta)$ we find

$$\begin{aligned}\mathcal{L}''(\beta)(h,v) &= |v|^2_{H_0^1} + \beta|h|^2_{L^2} - 2\langle\lambda_0^\beta, (-\Delta)^{-1}(hv)\rangle_{H_0^1} \\ &= |v|^2_{H_0^1} + \beta|h|^2_{L^2} - 2\langle\lambda_0^\beta, hv\rangle_{L^2}, \text{ for } (h,v) \in L^2 \times H_0^1.\end{aligned}$$

Due to (3.5) the following estimate can be obtained

$$\begin{aligned}\mathcal{L}''(\beta)(h,v) &= |v|^2_{H_0^1} + \beta|h|^2_{L^2} - 2|B(c_0^\beta)^{-1}\Delta(u_0^\beta - z_0)|_{L^4}|h|_{L^2}|v|_{L^4} \\ &\geq |v|^2_{H_0^1} + \beta|h|^2_{L^2} - 2K^2|B(c_0^\beta)^{-1}\Delta(u_0^\beta - z_0)|_{H_0^1}|h|_{L^2}|v|_{H_0^1},\end{aligned}$$

where K is the embedding constant of H_0^1 into L^4. Here we used the assumption that $n \leq 4$. Since $|B(c)^{-1}\Delta\varphi|_{H_0^1} \leq |\varphi|_{H_0^1}$ for every $\varphi \in H_0^1$ and every $c \geq 0$ we obtain

$$\begin{aligned}\mathcal{L}''(\beta)(h,v) &\geq |v|^2_{H_0^1} + \beta|h|^2_{L^2} - 2K^2|u(c_0^\beta) - z_0|_{H_0^1}|h|_{L^2}|v|_{H_0^1} \\ &\geq \frac{1}{2}|v|^2_{H_0^1} + \left(\beta - 4K^4|u(c_0^\beta) - z_0|^2_{H_0^1}\right)|h|^2_{L^2}. \qquad (3.6)\end{aligned}$$

Henceforth let (c_0^0, u_0^0) denote a solution of (3.4) with $\beta = 0$. Every solution (c_0^β, u_0^β) of (3.4) with $\beta \geq 0$ satisfies $u(c_0^\beta) = u_0^\beta$ and, we have

$$-|u(c_0^\beta) - z_0|^2_{H_0^1} \geq \beta\left(|c_0^\beta|^2_{L^2} - |c_0^0|^2_{L^2}\right) - \text{dist}\,(z_0, \mathcal{V})^2,$$

where $\mathcal{V} = \{u(c) : c \in L^2, z_1 \leq p_0^1 c + p_0^2 \leq z_2\}$. From (3.6) it follows that

$$\begin{aligned}&\mathcal{L}''(\beta)(h,v) \qquad (3.7)\\ &\geq \frac{1}{2}|v|^2_{H_0^1} + \left[\beta\left(1 + 4K^4(|c_0^\beta|^2_{L^2} - |c_0^0|^2_{L^2})\right) - \text{dist}\,(z_0,\mathcal{V})^2\right]|h|^2_{L^2}.\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Then from the results in [2] there exists $\bar{\beta}$ such that $1 > 4K^4(|c_0^0|_{L^2}^2 - |c_0^\beta|_{L^2}^2) + \varepsilon$ for any c_0^β, where (c_0^β, c_0^β) is a solution of (3.4) with $\beta \leq \bar{\beta}$ and c_0^0 is a minimum norm solution of (3.3). From (3.7) we deduce that for $\beta \in (0, \bar{\beta}]$

$$\mathcal{L}''(\beta)(h,v) \geq \frac{1}{2}|v|_{H_0^1}^2 + \left[\beta\varepsilon - \operatorname{dist}(z_0, \mathcal{V})^2\right]|h|_{L^2}^2, \text{ for all } (h,v) \in L^2 \times H_0^1. \tag{3.8}$$

Finally let us assume that $\operatorname{dist}(z_0, \mathcal{V})^2 < \varepsilon\bar{\beta}$ and choose $\underline{\beta}$ such that

$$\frac{\operatorname{dist}(z_0, \mathcal{V})^2}{\varepsilon} < \underline{\beta} < \overline{\beta}. \tag{3.9}$$

We thus conclude from (3.8) that (H3) and (H7) hold for all $\beta \in [\underline{\beta}, \overline{\beta}]$ provided that (3.9) is satisfied.

One can argue in an analogous manner the applicability of our results to the problem of estimating the diffusion coefficient a in

$$\begin{aligned} -\operatorname{div}(a \operatorname{grad} u) &= k \text{ in } \Omega \\ u|\Gamma &= 0 \end{aligned} \tag{3.10}$$

from observation $z \in H_0^1$ by means of the formulation

$$\min \frac{1}{2}|u - z|_{H_0^1}^2 + \frac{\beta}{2}|\nabla a|_{L^2}^2 + \frac{\beta}{2} \sum_{i,j=1,\dots,n} |a_{x_i,x_j}|^2$$

over $\{(a,u) \in H^2 \times H_0^1 : (-\Delta)^{-1}(\operatorname{div}(a \operatorname{grad} u) + k) = 0,\ z_1 \leq p^1 a + p^2 \leq z^2$ a.e.$\}$, where it is required that $n \leq 3$ and that $p_0^1 \geq \alpha > 0$ and $\frac{z_1 - p_0^2}{p_0^1} \geq \alpha$ a.e. on Ω. Coercivity can be argued similarly as in [5] and polyhedricity is proved in [14]. Directional differentiability of the solution of the least squares formulation of estimating a in (3.10) with respect to perturbations in z was proved previously in [14].

3.3 A boundary control problem

Here we consider

$$\begin{aligned} &\min |By(T,u)|_{L^2(\Omega)}^2 + |u|_{L^2(\Sigma)}^2 \\ &\text{over } Q_{ad} = \{u \in L^2(\Sigma) : z_1 \leq p^1 u + p^2 z_2, \\ &\qquad \text{a.e. for } (t,x) \in \Sigma\} \end{aligned} \tag{3.11}$$

where $z_1 \leq z_2$ a.e., $z_1 \neq z_2$ and $y = y(t,x;u)$ is the solution of

$$\begin{cases} y_{tt} = \Delta y & \text{in } \Omega \text{ for } t > 0, \\ \frac{\partial y}{\partial \kappa} = u & \text{on } \Gamma \text{ for } t > 0, \\ y(0,\cdot) = y_0,\ y_t(0,\cdot) = y_1. \end{cases} \tag{3.12}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ, $\Sigma = \{(t,x) : 0 < t \leq T, x \in \Gamma\}$, with $T > 0$ and κ denotes the unit outer normal to Ω on Γ. Moreover we choose $B \in \mathcal{L}(L^2(\Omega))$ and $y_0, y_1 \in L^2(\Omega)$. It is well known that the Laplace operator Δ in $L^2(\Omega)$ with $\operatorname{dom}\Delta = H^2(\Omega) \cap H_0^1$ generates a strongly continuous cosine family $C(t)$ on $L^2(\Omega)$ with associated sine family $S(t)y = \int_0^t C(s)y\,ds$ for $y \in L^2(\Omega)$. The operator theoretic solution to (3.12) is given by

$$y(t) = C(t)y_0 + S(t)y_1 + A_N \int_0^t S(t-\tau)Nu(\tau)d\tau \tag{3.13}$$

where $N : L_2(\Gamma) \to L_2(\Omega)$ is the Neumann boundary operator defined by $Nu = v$ with v the solution of

$$\begin{aligned} -\Delta v + v &= 0 \quad \text{in } \Omega, \\ \frac{\partial v}{\partial \kappa} &= u, \end{aligned}$$

and the operator A_N in $L^2(\Omega)$ is given by $\operatorname{dom}(A_N) = \{\varphi \in H^2(\Omega) : \frac{\partial \varphi}{\partial \kappa}|\Gamma = 0\}$ and $A_N\varphi = (\Delta - I)\varphi$, [6]. Throughout we use the identification $L^2(\Sigma) = L^2(0,T;L^2(\Gamma))$. From (3.13) it follows that

$$y(T) = C(T)y_0 + S(T)y_1 + L(T)u,$$

where $L(t)u = A_N \int_0^t S(t-\tau)Nu(\tau)d\tau$. It is known that $L(\cdot) \in \mathcal{L}(L^2(\Sigma), C(0,T;L^2(\Omega))$ [8, 6], and hence $Y(T)$ is a continuous affine mapping from $L^2(\Sigma)$ into $L^2(\Omega)$. With these preliminaries (3.11) can be expressed as

$$\begin{aligned} &\min |By(T)|^2_{L^2(\Omega)} + |u|^2_{L^2(\Sigma)} \quad \text{over } Q_{ad} \\ &\text{with } y(T) = C(T)y_0 + S(T)y_1 + L(T)u. \end{aligned} \tag{3.14}$$

The applicability of the results of Section 2 is obtained with $X = Z = L^2(\Sigma)$, $P = L^\infty(\Sigma) \times L^2(\Sigma)$ with a generic element $p \in P$ of the form $p = (p^1, p^2)$, and with

$$\begin{aligned} f(u) &= |By(T)|^2_{L^2(\Omega)} + |u|^2_{L^2(\Sigma)}, \\ b(u,p) &= p^1 u + p^2. \end{aligned}$$

For every fixed reference parameter $p_0 = (p_0^1, p_0^2) \in P$ there exists a unique solution u_0 of (3.14). In view of Proposition 2.1 and the special form of f and b, conditions (H1) – (H5) and (H7) are clearly satisfied. (H6) again follows from the results of [11].

3.4 Optimal control of ordinary differential systems

The purpose of this example is to demonstrate the applicability of the general results of Section 2 to optimal control problems in the presence of two sided pointwise constraints as well as a norm bound on the control energy and in particular to illustrate (A2) and (H4). A good reference for related results on the differential stability in nonlinear optimal control problems for more general systems is [10].

We consider the optimal control problem:

$$\min \int_0^T \tilde{h}(y(t), u(t))dt$$

such that

$$\begin{aligned} \dot{y}(t) &= Ay(t) + B_0 u(t) \quad \text{a.e. on } (0,T], \\ y(0) &= y_0, \\ u &\in Q_{ad}, \end{aligned} \tag{3.15}$$

where $T > 0$ and

$$\begin{aligned} Q_{ad} = \{& u \in L^2(0,T;\mathbb{R}^m)\ 0 \le p_i(t)u_i(t) \le z_i(t), \\ & \text{a.e. on } (0,T), \text{for } i = 1, \ldots, m\}, \end{aligned}$$

with $m \ge 2$.

The following specifications are made: $\tilde{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times m}$, $y_0 \in \mathbb{R}^n$, $z \in L^2(0,T;\mathbb{R}^m)$ with $z_i(t) \ge 0$ a.e. on $(0,T)$ for $i = 1, \ldots, m$, $z \ne 0$, and the perturbation vector p is in $L^\infty(0,T;\mathbb{R}^m)$ with unperturbed reference vector $p_0 = \text{col}(1,\ldots,1) \in L^\infty(0,T;\mathbb{R}^m)$. For a vector $v \in \mathbb{R}^m$, v_i denotes its i–th coordinate.

We put $x = (y,u)$ and $X = H^1(0,T;\mathbb{R}^n) \times L^2(0,T;\mathbb{R}^m)$, $Y = L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n$, $P = L^\infty(0,T;\mathbb{R}^m)$, $Z = L^2(0,T;\mathbb{R}^m)$, $K = L^2(0,T;\mathbb{R}^m_-)$. Further we define

$$\begin{aligned} f(y,u) &= \int_0^T \tilde{h}(y,u)dt, \\ e(y,u) &= (\dot{y} - Ay - B_0 u, y(0) - y_0), \\ b(y,u,p) &= \text{col}\,(p_1 u_1, \ldots, p_{m-1} u_{m-1}). \end{aligned}$$

Let us assume the existence of a solution $x_0 = (y_0, u_0)$ for $p = p_0$. Then, in the notation of Section 2,

$$\begin{aligned} E : H^1(0,T;\mathbb{R}^n) &\times L^2(0,T;\mathbb{R}^m) \to L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n, \\ B : H^1(0,T;\mathbb{R}^n) &\times L^2(0,T;\mathbb{R}^m) \to L^2(0,T;\mathbb{R}^{m-1}), \end{aligned}$$

are given by

$$\begin{aligned} E(h,v) &= (\dot{h} - Ah - B_0 v, h(0)), \\ B(h,v) &= \text{col}(p_1 v_1, \ldots, p_m v_m). \end{aligned}$$

Standard assumptions on $\tilde{h}$ can be made that guarantee the required smoothness properties of $f = \int_0^T \tilde{h} dt$ which imply (H2) and (H5). It is straightforward to verify the validity of (A1) for this example. Since there is no constraint described by g, Proposition 2.1 implies (H1) and (H4). Concerning the second order sufficient optimality conditions (H3) and (H7) we refer to [1, 8]. The polyhedricity assumption (H6) follows from Corollary 2 in [3].

We have shown that all results of Section 2 are applicable to (3.15). It is interesting to note that in the presence of the additional constraint

$$|u|_{L^2(0,T;\mathbb{R}^m)} \leq \gamma, \ \gamma > 0, \tag{3.16}$$

hypothesis (H4) cannot be guaranteed. However, (A.1) and (A.2) still hold, provided $u_0 \neq 0$ in $L^2(0,T;\mathbb{R}^m)$. In this case, Hölder continuity of the solution at p_0 can be derived from Theorem 2.1.

If the set of admissible parameters in (3.15) is replaced by

$$\begin{aligned} Q_{ad} = \{ & u \in L^2(0,T;\mathbb{R}^m) : |u|_{L^2(0,T;\mathbb{R}^m)} \leq \gamma, \\ & 0 \leq p_i(t)u_i(t) \leq z_i(t), \text{ a.e. on } (0,T), \text{ for } i = 1, \ldots, m-1\}, \end{aligned}$$

then the only hypotheses which require additional attention are (A2) and (H4). One can show that (A2) (and hence by Proposition 2.1 also (H1)) holds, provided that

$$\text{col}\left((u_0)_1, \ldots, (u_0)_{m-1}\right) \neq 0 \text{ in } L^2(0,T;\mathbb{R}^{m-1})$$

and (H4) is satisfied if

$$(u_0)_m \neq 0 \text{ in } L^2(0,T;\mathbb{R}).$$

Comparing the results on the optimal control problem of this section with [9] we obtain strong directional differentiability, whereas weak directional differentiability is obtained in [9].

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