Mean-field theories for the quantum phase transition in Josephson junction arrays

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Abstract. We compare two different approaches which both attempt to describe the zero-temperature superconductor-insulator transition in Josephson junction arrays through an effective Ginzburg-Landau functional. Both methods agree only for the special case of a diagonal capacitance matrix (self-charging limit). While the differences in the phase diagram are less significant, we find that the fourth order term of the "coarse-grained" functional changes its sign when the range of the interaction, λ, exceeds a few lattice constants. On the other hand, a variational ansatz predicts a positive sign with a coefficient which increases with increasing λ, which implies a large critical region and strong correlation between order parameter fluctuations.

1. Introduction

The superconductor-insulator transition in two-dimensional Josephson-junction arrays has recently attracted considerable attention, experimentally [1, 2] as well as theoretically [1, 3–9]. In short, with decreasing size of the superconducting grains and hence increasing charging energies, a transition from a superconducting to an insulating ground state is expected, and clear indications for this have been seen experimentally [2]. Of course, it is of major interest to understand precisely how the transition depends on the relevant parameters (e.g. the capacitances), on an external magnetic and/or charge frustration, and how the response behaves at and close to the transition point [9].

Based on the expectation that the phase transition is continuous, it appears natural to attempt, close to the transition, a description in terms of an effective Ginzburg-Landau functional, and to derive the parameters of this functional from the standard Josephson junction array model (which includes the charging and the Josephson coupling energies). In fact, most investigations [3, 6–9] have been based on what is known as coarse graining approach [10], which can be characterized as a certain path integral de-coupling procedure for the Josephson energies. Furthermore, it is often believed that this de-coupling procedure is equivalent to a variational ansatz: We point out in this article, which is largely based on the results of [11], that this believe can only be substantiated for a model which takes only into account a ground capacitance. Significant differences between the coarse graining and the variational approach become apparent when a nearest neighbor capacitance is included; and we have found that the coarse graining functional has serious shortcomings provided the range of the interaction exceeds a few lattice constants.

In Sect. 2, we briefly describe the model and discuss general aspects of the coarse graining vs. the variational approach. In Sect. 3, we present explicit results for the second order term in the Ginzburg-Landau functional (from which usually the phase diagram, as well as the response in the disordered region close to the transition, are derived). The results for the fourth order term are given in Sect. 4, and brief conclusions are presented in the final Sect. 5. Some technical details can be found in the appendix.

2. Coarse graining vs. variational approach

We consider the standard model Hamiltonian of a two-dimensional weakly coupled array of superconducting grains,

$$\mathscr{H} = \mathscr{H}^Q + \mathscr{H}^J \tag{1}$$

where $\mathscr{H}^Q$ and $\mathscr{H}^J$ denote the charging and the Josephson coupling energy, respectively. In particular,

$$\mathscr{H}^Q = \tfrac{1}{2} \sum_{k,l} Q_k \, \mathscr{C}_{kl}^{-1} \, Q_l \tag{2}$$

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with the convention $\mathscr{C}^{-1}_{kl} \equiv (\mathscr{C}^{-1})_{kl}$, where $\mathscr{C}$ is the capacitance matrix, and

$$\mathscr{H}^J = -\tfrac{1}{2} E_J \sum_{k,l} \gamma_{kl} \xi_k \xi_l^* . \tag{3}$$

Here E_J is the Josephson coupling energy between the nearest neighbour grains (taken to be uniform over the array); in view of later developments, we introduced γ_{kl} which is one for nearest neighbors and zero else, and $\xi_k \equiv \exp(i\varphi_k)$. The charge and the phase operators are conjugate variables such that

$$[Q_k, \varphi_l] = -2e \cdot i\delta_{k,l} . \tag{4}$$

An external charge can be included by the replacement $Q_k \to Q_k - 2eq_x$ in (2). We follow in notation [6–9]. Clearly, the eigenvalues and eigenfunctions of $\mathscr{H}^Q$ are easily determined; the latter are $\sim \exp\left(i \sum_k n_k \varphi_k\right)$, with $\{n_k\}$ integers, and will be used repeatedly below. Note that multiplication of an eigenstate by $\xi_l (\xi_l^*)$ increases (decreases) n_l by one.

In the next step, we introduce a mean-field approximation for $\mathscr{H}^J$, namely

$$\mathscr{H}^J_{mf} = -\sum_k (\xi_k \psi_k^* + \xi_k^* \psi_k) \tag{5}$$

and correspondingly define $\mathscr{H}_{mf} = \mathscr{H}^Q + \mathscr{H}^J_{mf}$. In order to have a general formulation which includes time-dependent situations, we change to a description in terms of path integrals (see e.g. [5–9]), and define Euclidean actions S, S^J, S^J_{mf}, etc., corresponding to $\mathscr{H}, \mathscr{H}^J, \mathscr{H}^J_{mf}$, etc. Then we introduce "partition functions" $\mathscr{Z}$ and $\mathscr{Z}_{mf}$,

$$\mathscr{Z} = \int \mathscr{D}(\text{paths})\, e^{-S} \leftrightarrow \operatorname{tr} e^{-\beta \mathscr{H}} \tag{6}$$

$$\mathscr{Z}_{mf} = \int \mathscr{D}(\text{paths})\, e^{-S_{mf}} \leftrightarrow \operatorname{tr} e^{-\beta \mathscr{H}_{mf}} \tag{7}$$

where β is the inverse temperature ($\hbar = k_B = 1$), and statistical averages $\langle(\cdot)\rangle_0$ and $\langle(\cdot)\rangle_{mf}$, taken with the "weight" $\exp(-S^Q)$ and $\exp(-S_{mf})$, respectively.

With these preliminaries, it is straightforward to summarize the results for the Ginzburg-Landau functional obtained through the coarse graining procedure (see e.g. [8]), $\mathscr{F}_{cg}$, and through the variational ansatz (see e.g. [12]), $\mathscr{F}_{\text{var}}$. These are

$$\mathscr{F}_{cg} = \sum_{k,l} \int \mathrm{d}\tau\, \psi_k^* \left(\frac{\gamma E_J}{2}\right)^{-1}_{kl} \psi_l - \ln \mathscr{Z}_{mf} \tag{8}$$

and

$$\mathscr{F}_{\text{var}} = \langle (S^J - S^J_{mf}) \rangle_{mf} - \ln \mathscr{Z}_{mf} . \tag{9}$$

For zero temperature and time-independent situations, the latter reduces to the standard expression

$$T=0: \quad E_{\text{var}} = \langle mf | \mathscr{H} | mf \rangle \tag{10}$$

where $|mf\rangle$ is the (normalized) ground state of $\mathscr{H}_{mf}$.

Assuming that $\mathscr{F}$ can be expanded with respect to the order parameter fields ψ_k, ψ_k^*, and omitting an unimportant constant, we arrive at expressions of the form $\mathscr{F} = \mathscr{F}^{(2)} + \mathscr{F}^{(4)} + \ldots$; explicitly

$$\mathscr{F}^{(2)}_{cg} = \sum_{k,l} \int \mathrm{d}\tau\, \psi_k^* \left(\frac{\gamma E_J}{2}\right)^{-1}_{kl} \psi_l - \tfrac{1}{2} \langle (S^J_{mf})^2 \rangle_0 \tag{11}$$

$$\mathscr{F}^{(2)}_{\text{var}} = \tfrac{1}{2} \langle (S^J + 1)(S^J_{mf})^2 \rangle_0 \tag{12}$$

and

$$\mathscr{F}^{(4)}_{cg} = -\tfrac{1}{24} \langle (S^J_{mf})^4 \rangle_0 + \tfrac{1}{8} \langle (S^J_{mf})^2 \rangle_0^2 \tag{13}$$

$$\mathscr{F}^{(4)}_{\text{var}} = \tfrac{1}{24} \langle S^J (S^J_{mf})^4 \rangle_0 + \tfrac{1}{8} [\langle (S^J_{mf})^4 \rangle_0 - \langle (S^J_{mf})^2 \rangle_0^2] \tag{14}$$

where in (14) we used that $\mathscr{F}^{(2)}_{\text{var}} = 0$ at the transition. [The fourth order term will only be evaluated in the low frequency, long wavelength limit.] No further general conclusions can be drawn at this point, except for the self-charging limit in which $\mathscr{C}^{-1}$ is diagonal (compare (2)): In this case, the expectation value in $\langle S^J \rangle_{mf}$ factorizes, such that $\langle \xi_k \xi_l^* \rangle_{mf} = \langle \xi_k \rangle_{mf} \langle \xi_l^* \rangle_{mf}$ for $k \neq l$, and it follows that the "mean-field" equations obtained by minimizing $\mathscr{F}_{cg}$ and $\mathscr{F}_{\text{var}}$, respectively, are consistent. Furthermore, in the expansion close to the transition, both approaches agree (see also below).

3. The phase diagram – $\mathscr{F}^{(2)}$

Considering the expressions (11) and (12), it is clear that $\mathscr{F}^{(2)}$ is of the form

$$\mathscr{F}^{(2)} = \sum_{k,l} \int \mathrm{d}\tau \mathrm{d}\tau'\, \psi_k^*(\tau) \chi_{k,l}(\tau, \tau') \psi_l(\tau') \tag{15}$$

where χ depends only on the difference variables (assuming a translation invariant capacitance matrix); hence we will use the Fourier representation, $\chi(\mathbf{q}, \omega)$.

Specifying now the capacitance matrix, $\mathscr{C}$, we put $\mathscr{C}_{kk} = C_0 + 4C_1$, $\mathscr{C}_{kl} = -C_1$ for kl nearest neighbors, and $\mathscr{C}_{kl} = 0$ else, i.e. $\mathscr{C}(\mathbf{q}) = C_0 + (4 - \gamma_q) C_1$, where $\gamma_q = 2(\cos q_x + \cos q_y)$ is the Fourier transform of γ_{kl}. Expanding for small $\mathbf{q}$, $\mathscr{C}(\mathbf{q}) \simeq C_0 + C_1 q^2$, we realize that $\lambda = (C_1/C_0)^{1/2}$ can be identified as the screening length (in units of the lattice constant, which is chosen to be unity).

Corresponding energies are denoted by $E_0 = e^2/2C_0$, $E_1 = e^2/2C_1$, and $E_C = e^2 \mathscr{C}^{-1}_{kk}/2$. The latter becomes logarithmically large for large λ; an approximate formula, interpolating between the two limits ($\lambda = 0, \lambda = \infty$), is given by

$$E_C \simeq E_1 \ln(1 + 4\pi\lambda^2)/4\pi . \tag{16}$$

As an illustration, consider the spectrum of $\mathscr{H}^Q$, and assume further that the external charge q_x is small enough, such that the ground state is given by $n_k = 0$ for all k. (For large λ, this requirement leads to the condition that – roughly – the external charge within the area λ^2 is smaller than e). Then define $\varepsilon_+ (\varepsilon_-)$ to be the energy of the state $n_k = +1$ ($n_k = -1$) for a single site, and accordingly $\varepsilon_{+-} \leftrightarrow (n_k = +1, n_l = -1)$ for kl nearest neighbors. Clearly

$$\varepsilon_{\pm} = 4E_C \mp 8q_x E_0 \tag{17}$$

and

$$\varepsilon_{+-} = 4e^2(\mathscr{C}_{kk}^{-1} - \mathscr{C}_{kl}^{-1}). \tag{18}$$

Explicitly,

$$\varepsilon_{+-} = e^2 \int \frac{d^2q}{(2\pi)^2} \frac{4-\gamma_q}{\mathscr{C}(\mathbf{q})} = \begin{cases} 8E_0 & \lambda = 0 \\ 2E_1 & \lambda = \infty . \end{cases} \tag{19}$$

Note that ε_{+-}, in contrast to E_C, has a finite limit for $\lambda \to \infty$ (since it is the energy of a charge-neutral state). In (19), the integration is over the first Brillouin zone (we consider a square lattice).

We restrict ourselves in the following to zero temperature, and small q_x as explained above, in which case the relevant correlation functions are easily evaluated. Defining for kl nearest neighbors the quantity

$$g(\tau - \tau') = \langle \xi_k(\tau)\, \xi_l^*(\tau') \rangle_0 \tag{20}$$

one finds [8]

$$\chi_{cg} = \frac{2}{\gamma_q E_J} - g(\omega) \tag{21}$$

and

$$\chi_{\mathrm{var}} = g(\omega) \left\{ 1 - \frac{\gamma_q E_J}{2} \times \left[g(\omega) + 2 \left(\frac{1}{\varepsilon_{+-}} - \frac{1}{\varepsilon_+ + \varepsilon_-} \right) \right] \right\} \tag{22}$$

where

$$g(\omega) = \frac{1}{\varepsilon_- - i\omega} + \frac{1}{\varepsilon_+ + i\omega} . \tag{23}$$

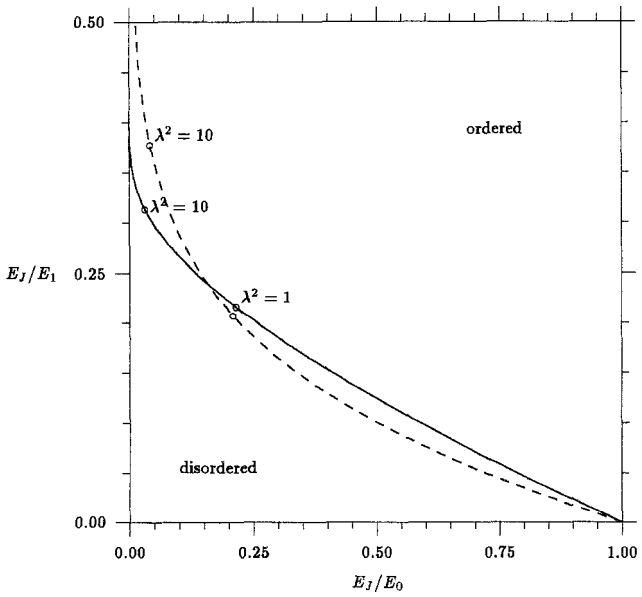

Fig. 1. Zero-temperature phase diagram for the capacitance model described at the beginning of Sect. 3, based on the variational (solid line) and the coarse graining (dashed line) approach

The transition point is determined from the condition $\chi(\mathbf{q}=0, \omega=0)=0$; and the phase diagram obtained on the basis of (21) has revealed fascinating features, in particular, as a function of the external charge [8, 9]. While we expect these features to persist qualitatively, there are quantitative differences between (21) and (22), as is apparent from Fig. 1 where we show the phase diagram for zero external charge [11]. These questions were also briefly discussed in [5], where an alternative plot of the present Fig. 1 is given as Fig. 4. Note the change of notation: The capacitances c and C [5] are denoted by C_0 and C_1 here.

As mentioned above, both methods agree in the self-charging limit, in which the transition point is given by $(E_J)^c = E_0$. On the other hand, for $\lambda \to \infty$, it follows from (21) that the system remains in the disordered phase even for very large values of E_J (i.e. $\chi_{cg} > 0$), while the variational calculation leads to the physically more sensible result of a finite $(E_J)^c$, namely $(E_J)^c = E_1/2$.

4. Ginzburg-Landau functional in fourth order

In order to complete the Ginzburg-Landau theory, especially close to and below (i.e. in the ordered region) the transition point, we consider now the fourth order contribution, $\mathscr{F}^{(4)}$, as given in (13) and (14), respectively. The calculations are lengthy, though straightforward, and we restrict ourselves to zero temperature and zero external charge. Furthermore, in agreement with the usual analysis of continuous phase transitions, the coefficient of the fourth order term is evaluated for zero frequencies and wavevectors; formally, this means that we may take the order parameter fields to be time (and space) independent, and thus apply standard time-independent perturbation theory. As a technical remark, we found it more convenient to consider (compare (8))

$$E_{mf} = \langle mf | \mathscr{H}_{mf} | mf \rangle \tag{24}$$

and the expression (10) for the variational energy. In particular, we need E_{mf} in fourth, and $|mf\rangle$ in third order in the expansion with respect to $\mathscr{H}_{mf}^J$.

Noting that presently $\varepsilon_+ = \varepsilon_- = 4E_C \equiv \varepsilon_1$ (compare (17) and (18)), we write the resulting static free energy functional in the form

$$\mathscr{F}/\beta = 2\varepsilon_1 \sum_k \left[a \frac{|\psi|^2}{\varepsilon_1^2} + b \frac{|\psi|^4}{\varepsilon_1^4} \right] \tag{25}$$

where a and b are dimensionless; clearly, $a = \chi(\mathbf{q}=0, \omega=0) \cdot \varepsilon_1/2$ (compare (15)), and this quantity was discussed in detail in the previous section.

In order to have a physically transparent presentation of the expression for b, see below and the appendix, we introduce the following notation for excitation energies of $\mathscr{H}^Q$, whose construction principle should be apparent from the list below ($|0\rangle$ denotes the ground state of $\mathscr{H}^Q$):

$$\varepsilon_{+-}^{kl} \leftrightarrow \xi_k\, \xi_l^* |0\rangle ; \quad \varepsilon_{++}^{kl} \leftrightarrow \xi_k\, \xi_l |0\rangle \tag{26}$$

$$\varepsilon^{kl}_{+2,-} \leftrightarrow \xi_k^2 \, \xi_l^* |0\rangle \tag{27}$$

$$\varepsilon^{klm}_{+--} \leftrightarrow \xi_k \, \xi_l^* \, \xi_m^* |0\rangle \tag{28}$$

where $k \neq l$ and $k \neq l \neq m$, respectively. Furthermore, if kl are nearest neighbors in (26) and (27), the superscript will be omitted. The energies are easily related to the elements of the inverse capacitance matrix, and various symmetries (e.g. $\varepsilon_{+--} = \varepsilon_{-++}$; recall that $q_x = 0$) are obvious.

Concerning b_{cg}, we obtain the following result:

$$b_{cg} = \tfrac{7}{4} + 2 \sum_k{}' \left(1 - \frac{\varepsilon_1}{\varepsilon^{kl}_{++}} - \frac{\varepsilon_1}{\varepsilon^{kl}_{+-}} \right) \tag{29}$$

which can also be written as

$$b_{cg} = \tfrac{7}{4} - 2 \sum_k{}' \frac{(\mathscr{C}^{-1}_{kl})^2}{(\mathscr{C}^{-1}_{kk})^2 - (\mathscr{C}^{-1}_{kl})^2} . \tag{30}$$

Here the prime indicates that $k = l$ has to be excluded. Clearly, $b_{cg}(\lambda = 0) = 7/4$. Using the fact that $\mathscr{C}^{-1}$ is a positive matrix, it also follows that $b_{cg} > 0$ for a nearest neighbor model for $\mathscr{C}^{-1}$. On the other hand, examples are easily found where $b_{cg} < 0$: Among these is the capacitance model introduced in Sect. 3, for which b_{cg} has been calculated numerically [11]. The result is given in Fig. 2, which shows that b_{cg} changes its sign for $\lambda^2 \simeq 2$.

The determination of the variational result, b_{var}, is considerably more involved and leads to the expression which is summarized in the appendix. Based on this expression, b_{var} was calculated numerically [11] for the capacitance matrix of Sect. 3 (ground and nearest neighbor capacitance), with the result shown in Fig. 3. From these data, we realize that, as a function of λ, b_{var} is practically constant for $\lambda^2 < 10$, i.e. $C_1 < 10\, C_0$, and then increases $\simeq 0.07\, \lambda^2$.

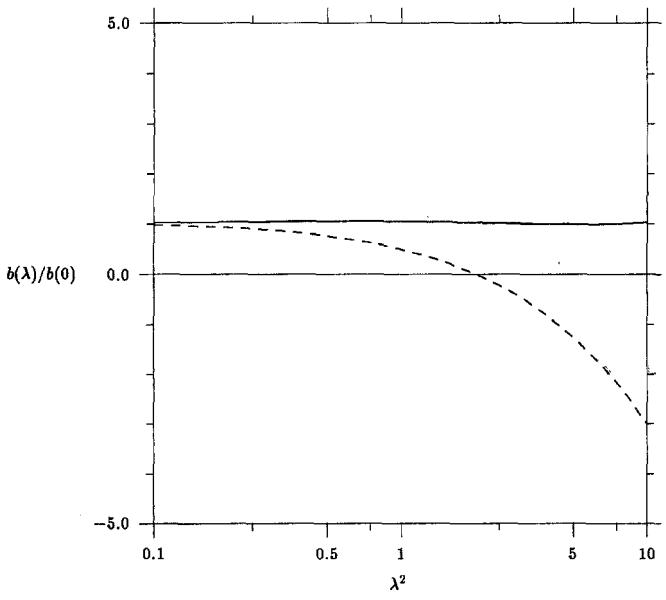

Fig. 2. The coefficient b, see (25), vs. λ^2, as obtained from the variational (solid line) and the coarse graining (dashed line) approach, normalized by $b(\lambda = 0)$

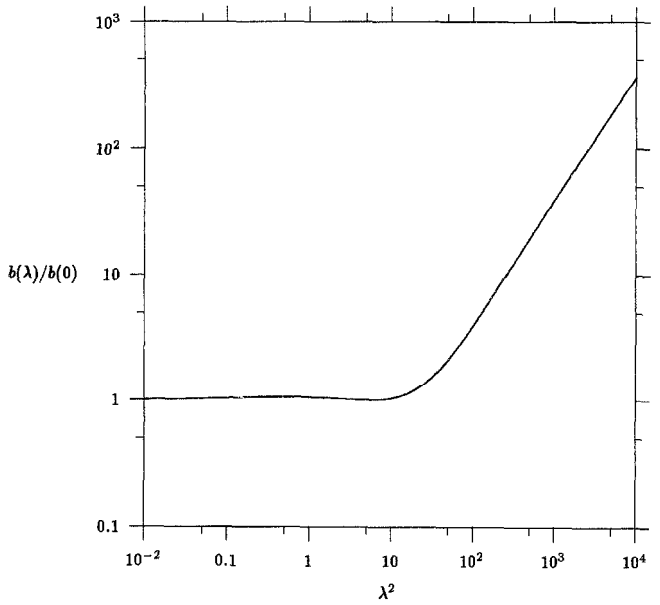

Fig. 3. The coefficient b, see (25), vs. λ^2, as obtained from the variational ansatz, normalized by $b(\lambda = 0)$, over a wider range compared to Fig. 2

The above results imply that the variational approach is in a sense more stable than the coarse graining procedure, in agreement what intuitively might be expected. However, we have found no simple criterion which ensures that $b_{\text{var}} > 0$ for general capacitance matrices.

We remark that our calculation of b_{cg} does not confirm the estimate presented in [3].

In summary, using the low frequency, long wavelength expansion of $\chi(\mathbf{q}, \omega)$, we find the following continuum limit form of the free energy functional ($T = 0$, $q_x = 0$):

$$\mathscr{F}_{\text{var}} = 2\varepsilon_1 \int \mathrm{d}\tau \int \mathrm{d}^2 r [a|\phi|^2 + b|\phi|^4 + (4E_J/\varepsilon_1^3)|\dot{\phi}|^2 + |\nabla\phi|^2/4] \tag{31}$$

where $\phi(\mathbf{r}, \tau) = \psi(\mathbf{r}, \tau)/\varepsilon_1$ is dimensionless; recall that $\varepsilon_1 = 4E_C$, and (compare (22)) $a = 1 - 2E_J(1/\varepsilon_1 + 2/\varepsilon_{+-})$, while b is given in Fig. 3. Finally, introducing $z^2 = \tau^2 \varepsilon_1^3 / 4E_J$, and rescaling the $x-y$-coordinates to remove the numerical factor $1/4$ in front of the gradient term, we obtain

$$\mathscr{F}_{\text{var}} = \left(\frac{E_J}{\varepsilon_1} \right)^{1/2} \int \mathrm{d}^3 r [a|\phi|^2 + b|\phi|^4 + |\nabla\phi|^2] \tag{32}$$

where ∇ is now the three-dimensional gradient.

The critical properties of the functional (32), which corresponds to the continuum limit of the classical 3-D $X-Y$ model, are well studied. We remark only that the critical region, as estimated from the Ginzburg criterion [13], is already of order one for $\lambda = 0$, and increases further $\sim \lambda$ for $\lambda \gg 1$.

5. Conclusion

In conclusion, we have compared two mean-field-like approaches to the quantum phase transition in a two-di-

mensional array of Josephson junctions, namely the coarse graining procedure vs. a variational ansatz, for the model which includes a ground and nearest neighbor capacitance. We have found minor differences in the phase diagram, and expect small quantitative, though presumably not qualitative, differences in the response (e.g. the conductivity, compare [9]) as well.

Surprisingly, within the coarse graining procedure, we also find that the coefficient of the fourth order term in the Ginzburg-Landau functional changes its sign when the screening length exceeds about two lattice constant. Given this fact alone, we could regard it as an indication of the possibility that the transition might be of first order, depending on the form of the capacitance matrix. However, this result is not confirmed by a variational ansatz, which turns out to be more robust such that it predicts a positive fourth order term for all λ. Furthermore, within the latter approach, we find that $b(\lambda)$ increases strongly for large λ, leading to strong correlations between order parameter fluctuations, which also indicates that results based on a Gaussian approximation may be of rather limited validity.

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Appendix

Here we list some results related to the variational energy, (10). We consider $T=0$, $q_x=0$ and, as explained in Sect. 4, take the order parameter time and site independent. Denoting by $|mf\rangle$ the ground state of $\mathscr{H}_{mf}=\mathscr{H}^Q+\mathscr{H}^J_{mf}$, compare (2) and (5), we have

$$E_{\text{var}}=\langle mf|\mathscr{H}^Q+\mathscr{H}^J|mf\rangle\,. \tag{A.1}$$

Clearly, these expectation values are proportional to N, the number of lattice sites. The notation with regard to excitation energies of $\mathscr{H}^Q$ has been explained in connection with (26)–(28); and, in the following, ij denote nearest neighbors. Using translation invariance, we may also write (A.1) as follows:

$$E_{\text{var}}=\langle mf|\mathscr{H}^Q|mf\rangle - 2NE_J\langle mf|\xi_i\xi_j^*|mf\rangle\,. \tag{A.2}$$

We find the following results in second order in ψ:

$$\langle mf|\mathscr{H}^Q|mf\rangle^{(2)}=N\cdot 2|\psi|^2/\varepsilon_1 \tag{A.3}$$

$$\langle mf|\xi_i\xi_j^*|mf\rangle^{(2)}=\frac{2|\psi|^2}{\varepsilon_1}\left(\frac{1}{\varepsilon_1}+\frac{2}{\varepsilon_{+-}}\right) \tag{A.4}$$

and in fourth order:

$$\langle mf|\mathscr{H}^Q|mf\rangle^{(4)} = N\frac{|\psi|^2}{\varepsilon_1^3}\left[-\tfrac{21}{2}-12\sum_k{}'\left(1-\frac{\varepsilon_1}{\varepsilon_{++}^{ki}}-\frac{\varepsilon_1}{\varepsilon_{+-}^{ki}}\right)\right] \tag{A.5}$$

where the prime indicates that $k=i$ has to be excluded. Furthermore

$$\langle mf|\xi_i\xi_j^*|mf\rangle^{(4)} = |\psi|^4\left[X_0+\sum_k{}'X_1+\sum_k{}''(X_2+X_3)\right] \tag{A.6}$$

where

$$X_1=\frac{8}{\varepsilon_1^2}\left[\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_{+-}}\right]\left[\frac{1}{\varepsilon_{++}^{ki}}+\frac{1}{\varepsilon_{+-}^{ki}}-\frac{1}{\varepsilon_1}\right] \tag{A.7}$$

$$X_2=\frac{8}{\varepsilon_1^2}\left[\frac{1}{\varepsilon_{++}^{ki}\varepsilon_{++}^{kj}}+\frac{1}{\varepsilon_{+-}^{ki}\varepsilon_{+-}^{kj}}-\frac{1}{2\varepsilon_1^2}\right] \tag{A.8}$$

$$X_3=\frac{8}{\varepsilon_1}\left[\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_{+-}}\right] \times\left[\frac{1}{\varepsilon_{-++}^{ijk}}\left(\frac{1}{\varepsilon_{+-}}+\frac{1}{\varepsilon_{+-}^{ki}}+\frac{1}{\varepsilon_{++}^{kj}}\right)-\frac{1}{\varepsilon_1\varepsilon_{+-}}\right]. \tag{A.9}$$

In (A.6), the double prime means that $k=i$ and $k=j$ have to be excluded. Concerning the summations, note that we subtracted the large k-limit of the expressions which are summed. In this connection, we remark that, for any α_k,

$$\sum_k{}'\alpha_k=\sum_k{}'(\alpha_k-\alpha_\infty)+(N-1)\alpha_\infty \tag{A.10}$$

and

$$\sum_k{}''\alpha_k=\sum_k{}''(\alpha_k-\alpha_\infty)+(N-2)\alpha_\infty\,. \tag{A.11}$$

Finally,

$$X_0=\frac{2}{\varepsilon_1^3}\left[\frac{1}{\varepsilon_{++}}-\frac{4}{\varepsilon_1}\right]+\frac{2}{\varepsilon_1}\left[\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_{++}}\right] \cdot\left[\frac{1}{2\varepsilon_1^2}+\left(\frac{1}{2\varepsilon_1}+\frac{4}{\varepsilon_{+-}}\right)\frac{1}{\varepsilon_{+2,-}}-\frac{4}{\varepsilon_1^2}-\frac{8}{\varepsilon_1^2\varepsilon_{+-}}\right]. \tag{A.12}$$

Further details can be found in [11].

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